An Introduction to Free Higher-Spin Fields

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ABSTRACT. In this article we begin by reviewing the (Fang-)Fronsdal construction and the non-local geometric equations with unconstrained gauge fields and parameters built by Francia and the senior author from the higher-spin curvatures of de Wit and Freedman. We then turn to the triplet structure of totally symmetric tensors that emerges from free String Field Theory in the $\alpha' \to 0$ limit and its generalization to (A)dS backgrounds, and conclude with a discussion of the simple local compensator form of the field equations that displays the unconstrained gauge symmetry of the non-local equations.

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1   Introduction

This article reviews some of the developments that led to the free higher-spin equations introduced by Fang and Fronsdal [1, 2] and the recent constructions of free non-local geometric equations and local compensator forms of [3, 4, 5]. It is based on the lectures delivered by A. Sagnotti at the First Solvay Workshop, held in Brussels on May 2004, carefully edited by the other authors for the online Proceedings.

   The theory of particles of arbitrary spin was initiated by Fierz and Pauli in 1939 [6], that followed a field theoretical approach, requiring Lorentz invariance and positivity of the energy. After the works of Wigner [7] on representations of the Poincaré group, and of Bargmann and Wigner [8] on relativistic field equations, it became clear that the positivity of energy could be replaced by the requirement that the one-particle states carry a unitary representation of the Poincaré group. For massive fields of integer and half-integer spin represented by totally symmetric tensors $\Phi_{\mu_1\ldots\mu_s}$ and $\Psi_{\mu_1\ldots\mu_s}$, the former requirements are encoded in the Fierz-Pauli conditions

$$(\Box - M^2)\Phi_{\mu_1\ldots\mu_s} \;=\; 0 \;, \qquad\qquad (i\partial\!\!\!/ - M)\Psi_{\mu_1\ldots\mu_s} \;=\; 0 \;, \qquad (1)$$
$$\partial^{\mu_1}\Phi_{\mu_1\ldots\mu_s} \;=\; 0 \;, \qquad\qquad \partial^{\mu_1}\Psi_{\mu_1\ldots\mu_s} \;=\; 0 \;. \qquad (2)$$

The massive field representations are also irreducible when a (γ-)trace condition

$$\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\ldots\mu_s} = 0 \;, \qquad \gamma^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0 \;. \qquad (3)$$

is imposed on the fields.

   A Lagrangian formulation for these massive spin s-fields was first obtained in 1974 by Singh and Hagen [9], introducing a sequence of auxiliary traceless fields of ranks $s-2$, $s-3$, $\ldots 0$ or 1, all forced to vanish when the field equations are satisfied.

   Studying the corresponding massless limit, in 1978 Fronsdal obtained [1] four-dimensional covariant Lagrangians for massless fields of any integer spin. In this limit, all the auxiliary fields decouple and may be ignored, with the only exception of the field of rank $s-2$, while the two remaining traceless tensors of rank s and $s-2$ can be combined into a single tensor $\varphi_{\mu_1\ldots\mu_s}$ subject to the unusual "double trace" condition

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\varphi_{\mu_1\ldots\mu_s} \;=\; 0 \;. \qquad (4)$$

Fang and Fronsdal [2] then extended the result to half-integer spins subject to the peculiar "triple γ trace" condition

$$\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\psi_{\mu_1\ldots\mu_s} \;=\; 0 \;. \qquad (5)$$

   It should be noted that the description of massless fields in four dimensions is particularly simple, since the massless irreducible representations of the Lorentz group SO(3,1)$^\uparrow$ are exhausted by totally symmetric tensors. On the other hand, it is quite familiar from supergravity [10] that in dimensions $d > 4$ the totally symmetric tensor representations do not exhaust all possibilities, and must be supplemented by mixed ones. For the sake of simplicity, in this paper we shall confine our attention to the totally symmetric case, focussing on the results of [3, 4, 5]. The extension to the mixed-symmetry case was originally obtained in [11], and will be reviewed by C. Hull in his contribution to these Proceedings [12].

2   From Fierz-Pauli to Fronsdal

This section is devoted to some comments on the conceptual steps that led to the Fronsdal [1] and Fang-Fronsdal [2] formulations of the free high-spin equations. As a first step, we describe the salient features of the Singh-Hagen construction of the massive free field Lagrangians [9]. For simplicity, we shall actually refer to spin 1 and 2 fields that are to satisfy the Fierz-Pauli conditions (1)-(2), whose equations are of course known since Maxwell and Einstein. The spin-1 case is very simple, but the spin-2 case already presents the key subtlety. The massless limit will then illustrate the simplest instances of Fronsdal gauge symmetries. The Kaluza-Klein mechanism will be also briefly discussed, since it exhibits rather neatly the rationale behind the Fang-Fronsdal auxiliary fields for the general case. We then turn

to the novel features encountered with spin 3 fields, before describing the general Fronsdal equations for massless spin-s bosonic fields. The Section ends with the extension to half-integer spins.

2.1   Fierz-Pauli conditions

Let us first introduce a convenient compact notation. Given a totally symmetric tensor φ, we shall denote by $\partial\varphi$, $\partial \cdot \varphi$ and φ' (or, more generally, $\varphi^{[p]}$) its gradient, its divergence and its trace (or its p-th trace), with the understanding that in all cases the implicit indices are totally symmetrized.

Singh and Hagen [9] constructed explicitly Lagrangians for spin-s fields that give the correct Fierz-Pauli conditions. For spin 1 fields, their prescription reduces to the Lagrangian

$$\mathcal{L}_{spin1} = -\frac{1}{2}(\partial_\mu \Phi_\nu)^2 - \frac{1}{2}(\partial \cdot \Phi)^2 - \frac{M^2}{2}(\Phi_\mu)^2 \ , \tag{6}$$

that gives the Proca equation

$$\Box\Phi_\mu - \partial_\mu(\partial \cdot \Phi) - M^2\Phi_\mu = 0 \ . \tag{7}$$

Taking the divergence of this field equation, one obtains immediately $\partial^\mu\Phi_\mu = 0$, the Fierz-Pauli transversality condition (2), and hence the Klein-Gordon equation for Φ_μ.

In order to generalize this result to spin-2 fields, one can begin from

$$\mathcal{L}_{spin2} = -\frac{1}{2}(\partial_\mu \Phi_{\nu\rho})^2 + \frac{\alpha}{2}(\partial \cdot \Phi_\nu)^2 - \frac{M^2}{2}(\Phi_{\mu\nu})^2 \ , \tag{8}$$

where the field $\Phi_{\mu\nu}$ is traceless. The corresponding equation of motion reads

$$\Box\Phi_{\mu\nu} - \frac{\alpha}{2}\left(\partial_\mu\partial \cdot \Phi_\nu + \partial_\nu\partial \cdot \Phi_\mu - \frac{2}{D}\ \eta_{\mu\nu}\ \partial \cdot \partial \cdot \Phi\right) - M^2\Phi_{\mu\nu} = 0 \ , \tag{9}$$

whose divergence implies

$$\left(1 - \frac{\alpha}{2}\right)\Box\partial \cdot \Phi_\nu + \alpha\left(-\frac{1}{2} + \frac{1}{D}\right)\partial_\nu\partial \cdot \partial \cdot \Phi - M^2\partial \cdot \Phi_\nu = 0 \ . \tag{10}$$

Notice that, in deriving these equations, we have made an essential use of the condition that Φ be traceless.

In sharp contrast with the spin 1 case, however, notice that now the transversality condition is not recovered. Choosing $\alpha = 2$ would eliminate some terms, but one would still need the additional constraint $\partial \cdot \partial \cdot \Phi = 0$. Since this is not a consequence of the field equations, the naive system described by Φ and equipped with the Lagrangian $\mathcal{L}_{spin2}$ is unable to describe the free spin 2 field.

One can cure the problem introducing an auxiliary scalar field π in such a way that the condition $\partial \cdot \partial \cdot \Phi = 0$ be a consequence of the Lagrangian. Let us see how this is the case, and add to (8) the term

$$\mathcal{L}_{add} = \pi\ \partial \cdot \partial \cdot \Phi + c_1(\partial_\mu\pi)^2 + c_2\pi^2 \ , \tag{11}$$

where $c_{1,2}$ are a pair of constants. Taking twice the divergence of the resulting equation for $\Phi_{\mu\nu}$ gives

$$\left[(2 - D)\Box - DM^2\right]\partial \cdot \partial \cdot \Phi + (D - 1)\Box^2\pi = 0 \ , \tag{12}$$

while the equation for the auxiliary scalar field reduces to

$$\partial \cdot \partial \cdot \Phi + 2(c_2 - c_1\Box)\pi \ = \ 0 \ . \tag{13}$$

Eqs. (12) and (13) can be regarded as a linear homogeneous system in the variables $\partial \cdot \partial \cdot \Phi$ and π. If the associated determinant never vanishes, the only solution will be precisely the missing condition $\partial \cdot \partial \cdot \Phi = 0$, together with the condition that the auxiliary field vanish as well, $\pi = 0$, and as a result the transversality condition will be recovered. The coefficients c_1 and c_2 are thus determined by the

condition that the determinant of the system

$$\Delta \quad = \quad -2DM^2 c_2 + 2((2-D)c_2 + DM^2 c_1)\Box \tag{14}$$

$$\quad - \quad (2(2-D)c_1 - (D-1))\Box\Box \tag{15}$$

be algebraic, *i.e.* proportional to the mass M but without any occurrence of the D'Alembert operator $\Box$. Hence, for $D > 2$,

$$c_1 = \frac{(D-1)}{2(D-2)} \ , \qquad c_2 = \frac{M^2 D(D-1)}{2(D-2)^2} \ . \tag{16}$$

The end conclusion is that the complete equations imply

$$\pi = 0 \ , \qquad\qquad \partial \cdot \partial \cdot \Phi = 0 \ , \tag{17}$$

$$\partial \cdot \Phi_\nu = 0 \ , \qquad\qquad \Box\Phi_{\mu\nu} - M^2 \Phi_{\mu\nu} = 0 \ , \tag{18}$$

the Fierz-Pauli conditions (1) and (2), so that the inclusion of a single auxiliary scalar field leads to an off-shell formulation of the free massive spin-2 field.

2.2 "Fronsdal" equation for spin 2

We can now take the $M \to 0$ limit, following in spirit the original work of Fronsdal [1]. The total Lagrangian $\mathcal{L}_{spin2} + \mathcal{L}_{add}$ then becomes

$$\mathcal{L} \quad = \quad -\frac{1}{2} \left(\partial_\mu \Phi_{\nu\rho} \right)^2 + \left(\partial \cdot \Phi_\nu \right)^2 + \pi \partial \cdot \partial \cdot \Phi + \frac{D-1}{2(D-2)} \left(\partial_\mu \pi \right)^2 \ , \tag{19}$$

whose equations of motion are

$$\Box\Phi_{\mu\nu} - \partial_\mu \partial \cdot \Phi_\nu - \partial_\nu \partial \cdot \Phi_\mu + \frac{2}{D} \, \eta_{\mu\nu} \, \partial \cdot \partial \cdot \Phi + \partial_\mu \partial_\nu \pi = 0 \ , \tag{20}$$

$$\frac{D-1}{D-2} \Box\pi - \partial \cdot \partial \cdot \Phi = 0 \ . \tag{21}$$

The representation of the massless spin 2-gauge field via a traceless two-tensor $\Phi_{\mu\nu}$ and a scalar π may seem a bit unusual. In fact, they are just an unfamiliar basis of fields, and the linearized Einstein gravity in its standard form is simply recovered once they are combined in the unconstrained two-tensor

$$\varphi_{\mu\nu} = \Phi_{\mu\nu} + \frac{1}{D-2} \, \eta_{\mu\nu} \, \pi \ . \tag{22}$$

In terms of $\varphi_{\mu\nu}$, the field equations and the corresponding gauge transformations then become

$$\mathcal{F}_{\mu\nu} \equiv \Box\varphi_{\mu\nu} - (\partial_\mu \partial \cdot \varphi_\nu + \partial_\nu \partial \cdot \varphi_\mu) + \partial_\mu \partial_\nu \varphi' = 0 \ , \tag{23}$$

$$\delta\varphi_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \ , \tag{24}$$

that are precisely the linearized Einstein equations, where the "Fronsdal operator" $\mathcal{F}_{\mu\nu}$ is just the familiar Ricci tensor. The corresponding Lagrangian reads

$$\mathcal{L} = -\frac{1}{2} \left(\partial_\mu \varphi_{\nu\rho} \right)^2 + \left(\partial \cdot \varphi_\mu \right)^2 + \frac{1}{2} \left(\partial_\mu \varphi' \right)^2 + \varphi' \partial \cdot \partial \cdot \varphi \ , \tag{25}$$

and yields the Einstein equations $\mathcal{F}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{F}' = 0$, that only when combined with their trace imply the previous equation, $\mathcal{F}_{\mu\nu} = 0$.

The massless case is very interesting by itself, since it exhibits a relatively simple instance of gauge symmetry, but also for deducing the corresponding massive field equations via a proper Kaluza-Klein reduction. This construction, first discussed in [13], is actually far simpler than the original one of [9] and gives a rationale to their choice of auxiliary fields.

Let us content ourselves with illustrating the Kaluza-Klein mechanism for spin 1 fields. To this end, let us introduce a field[1] A_M living in $D+1$ dimensions, that decomposes as $A_M = (A_\mu(x,y), \pi(x,y))$, where y denotes the coordinate along the extra dimension. One can expand these functions in Fourier modes in y and a single massive mode corresponding to the D-dimensional mass m, letting for instance $A_M = (A_\mu(x), -i\pi(x)) \exp imy$ where the judicious insertion of the factor $-i$ will ensure that the field π be real. The $D+1$-dimensional equation of motion and gauge transformation

$$\Box A_M - \partial_M \partial \cdot A = 0 \ , \tag{26}$$

$$\delta A_M = \partial_M \Lambda \tag{27}$$

then determine the D-dimensional equations

$$\left(\Box - m^2\right) A_\mu - \partial_\mu \left(\partial \cdot A + m\pi\right) = 0,$$
$$\left(\Box - m^2\right) \pi + m \left(\partial \cdot A + m\pi\right) = 0 \ , \tag{28}$$
$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta \pi = -m\Lambda \ ,$$

where the leftover massive gauge symmetry, known as a Stueckelberg symmetry, is inherited from the higher dimensional gauge symmetry. Fixing the gauge so that $\pi = 0$, one can finally recover the Proca equation (7) for A_μ. The spin-2 case is similar, and the proper choice is $\varphi_{MN}(\varphi_{\mu\nu}, -i\varphi_\mu, -\varphi) \exp imy$, so that the resulting gauge transformations read

$$\delta \varphi_{MN} = \partial_M \Lambda_N + \partial_N \Lambda_M \ ,$$
$$\delta \varphi_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \ ,$$
$$\delta \varphi_\mu = \partial_\mu \Lambda - m\Lambda_\mu \ , \tag{29}$$
$$\delta \varphi = -2m\Lambda \ .$$

In conclusion, everything works as expected when the spin is lower than or equal to two, and the Fierz-Pauli conditions can be easily recovered. However, some novelties do indeed arise when then spin becomes higher than two.

Let us try to generalize the theory to spin-3 fields by insisting on the equations

$$\mathcal{F}_{\mu\nu\rho} \equiv \Box \varphi_{\mu\nu\rho} - (\partial_\mu \partial \cdot \varphi_{\nu\rho} + perm) + (\partial_\mu \partial_\nu \varphi'_\rho + perm) = 0 \ , \tag{30}$$

$$\delta \varphi_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu} \ , \tag{31}$$

that follow the same pattern, where *perm* denotes cyclic permutations of $\mu\nu\rho$. In this formulation, there are no auxiliary fields and the trace φ' of the gauge field does not vanish.

Let us first remark that, under a gauge transformation, $\mathcal{F}$ transforms according to

$$\delta \mathcal{F}_{\mu_1 \mu_2 \mu_3} = 3\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Lambda' \ . \tag{32}$$

Therefore, *$\mathcal{F}$ is gauge invariant if and only if the gauge parameter is traceless, $\Lambda' = 0$*. This rather unnatural condition will recur systematically for all higher spins, and will constitute a drawback of the Fronsdal formulation.

We can also see rather neatly the obstruction to a geometric gauge symmetry of the spin-3 Fronsdal Lagrangian along the way inspired by General Relativity. Indeed, the spin-3 Fronsdal equation differs in a simple but profound way from the two previous cases, since both for spin 1 and for spin 2 *all lower-spin constructs* built out of the gauge fields are present, while for spin 3 only constructs of spin 3 ($\varphi_{\mu\nu\rho}$), spin 2 ($\partial \cdot \varphi_{\mu\nu}$) and spin 1 ($\varphi'_\mu$) are present. Actually, de Wit and Freedman [14] classified long ago the higher-spin analogs of the spin-2 Christoffel connection $\Gamma_{mu;\nu_1\nu_2}$: they are a hierarchy of connections $\Gamma_{\mu_1...\mu_k;\nu_1\nu_2\nu_3}$, with $k = 1, \ldots (s-1)$, that contain k derivatives of the gauge field. They

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[1]Capital Latin letters denote here indices in $D+1$ dimensions, while Greek letters denote the conventional ones in D dimensions.

also noticed that the analog of the Riemann tensor, that for spin 3 would be $\Gamma_{\mu_1\mu_2\mu_3;\nu_1\nu_2\nu_3}$, would in general contain s derivatives of the gauge field, and *related the Fronsdal operator $\mathcal{F}$ to the trace of the second connection.*

One way to bypass the problem is to construct a field equation with two derivatives depending on the true Einstein tensor for higher spins, that however, as we have anticipated, contains s derivatives in the general case. This can be achieved, but requires that non-local terms be included both in the field equations and in the Lagrangian. This approach will be further developed in section 3. Another option is to compensate the non-vanishing term in the right-hand side of (32) by introducing a new field, a *compensator*. As we shall see, this possibility is actually suggested by String Theory, and will be explained in section 4.3.

But before describing these new methods, let us first describe the general Fronsdal formulation for arbitrary spin. In this way we shall clearly identify the key role of the *trace condition on the gauge parameter* that we already encountered for spin 3 and of the *double trace condition on the field*, that will first present itself for spin 4.

2.3   Fronsdal equations for arbitrary integer spin

We can now generalize the reasoning of the previous paragraph to arbitrary integer spins. Since we shall use extensively the compact notation of [3], omitting all indices, is it useful to recall the following rules:

$$
\begin{aligned}
(\partial^p \varphi)' &= \Box\,\partial^{p-2}\varphi + 2\partial^{p-1}\partial\cdot\varphi + \partial^p\varphi' , \\
\partial^p\,\partial^q &= \binom{p+q}{p}\partial^{p+q} , \\
\partial\cdot(\partial^p\varphi) &= \Box\,\partial^{p-1}\varphi + \partial^p\partial\cdot\varphi , \\
\partial\cdot\eta^k &= \partial\eta^{k-1}, \\
\left(\eta^k T_{(s)}\right)' &= k\,[\mathcal{D} + 2(s+k-1)]\,\eta^{k-1}T_{(s)} + \eta^k T'_{(s)} .
\end{aligned}
\tag{33}
$$

In this compact form, the generic Fronsdal equation and its gauge transformations read simply

$$
\mathcal{F} \equiv \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi' = 0 , \tag{34}
$$

$$
\delta\varphi = \partial\Lambda . \tag{35}
$$

In order to find the effect of the gauge transformations on the Fronsdal operators $\mathcal{F}$, one must compute the terms

$$
\begin{aligned}
\delta(\partial\cdot\varphi) &= \Box\Lambda + \partial\partial\cdot\Lambda , \\
\delta\varphi' &= 2\partial\cdot\Lambda + \partial\Lambda' ,
\end{aligned}
$$

and the result is, for arbitrary spin,

$$
\delta\mathcal{F} = \Box(\partial\Lambda) - \partial(\Box\Lambda + \partial\partial\cdot\Lambda) + \partial^2(2\partial\cdot\Lambda + \partial\Lambda') = 3\partial^3\Lambda' , \tag{36}
$$

where we used $-\partial(\partial\partial\cdot\Lambda) = -2\partial^2(\partial\cdot\Lambda)$, and $\partial^2\partial\Lambda' = 3\partial^3\Lambda'$. Therefore, in all cases where Λ' does not vanish identically, *i.e.* for spin $s \geq 3$, the gauge invariance of the equations requires that the gauge parameter be traceless

$$
\Lambda' = 0 . \tag{37}
$$

As second step, one can derive the Bianchi identities for all spins computing the terms

$$
\begin{aligned}
\partial\cdot\mathcal{F} &= \Box\partial\varphi' - \partial\partial\cdot\partial\cdot\varphi + \partial^2\partial\cdot\varphi', \\
\mathcal{F}' &= 2\Box\varphi' - 2\partial\cdot\partial\cdot\varphi + \partial^2\varphi'' + \partial\partial\cdot\varphi', \\
\partial\mathcal{F}' &= 2\Box\partial\varphi' - 2\partial\,\partial\cdot\partial\cdot\varphi + 3\partial^3\varphi'' + 2\partial^2\partial\cdot\varphi'.
\end{aligned}
$$

Therefore, the Fronsdal operator $\mathcal{F}$ satisfies in general the "anomalous" Bianchi identities

$$\partial \cdot \mathcal{F} - \frac{1}{2}\partial \mathcal{F}' = -\frac{3}{2}\partial^3 \varphi'' \ , \tag{38}$$

where the additional term on the right first shows up for spin $s = 4$. In the Fronsdal construction, one is thus led to impose the constraint $\varphi'' = 0$ for spins $s \geq 4$, since the Lagrangians would vary according to

$$\delta \mathcal{L} \ = \ \delta \varphi \left(\mathcal{F} - \frac{1}{2}\eta \mathcal{F}' \right) \ , \tag{39}$$

that does not vanish unless the double trace of φ vanishes identically. Indeed,

$$\partial \cdot \left(\mathcal{F} - \frac{1}{2}\mathcal{F}' \right) = -\frac{3}{2}\partial^3 \varphi'' - \frac{1}{2}\eta \partial \cdot \mathcal{F}' \ , \tag{40}$$

where the last term gives a vanishing contribution to $\delta \mathcal{L}$ if the parameter Λ is traceless. To reiterate, this relation is at the heart of the usual restrictions present in the Fronsdal formulation to traceless gauge parameters and doubly traceless fields, needed to ensure that the variation of the lagrangian

$$\delta \mathcal{L} = \delta \varphi \, \mathcal{G}, \tag{41}$$

where $\mathcal{G} = \mathcal{F} - \frac{1}{2}\eta \mathcal{F}'$, vanishes for $\delta \varphi = \partial \Lambda$.

We can also extend the Kaluza-Klein construction to the spin-s case. Given the double trace condition $\varphi'' = 0$, the reduction $\varphi_{D+1}^{(s)}$ from $D + 1$ dimensions to D dimensions gives rise to the tensors $\varphi_D^{(s)}, \ldots, \varphi_D^{(s-3)}$ of rank s to $s - 3$ only. In addition, the trace condition on the gauge parameter implies that only two tensors $\Lambda_D^{(s-1)}$ and $\Lambda_D^{(s-2)}$ are generated in D dimensions. Gauge fixing the Stueckelberg symmetries, one is left with only two traceful fields $\varphi_D^{(s)}$ and $\varphi_D^{(s-3)}$. But a traceful spin-s tensor contains traceless tensors of ranks s, $s - 2$, $s - 4$, etc. Hence, the two remaining fields $\varphi_D^{(s)}$ and $\varphi_D^{(s-3)}$ contain precisely the tensors introduced by Singh and Hagen [9]: a single traceless tensor for all ranks from s down to zero, with the only exception of the rank-$(s - 1)$ tensor, that is missing.

2.4   Massless fields of half-integer spin

Let us now turn to the fermionic fields, and for simplicity let us begin with the Rarita-Schwinger equation [15], familiar from supergravity [16]

$$\gamma^{\mu\nu\rho} \, \partial_\nu \psi_\rho = 0 \ , \tag{42}$$

that is invariant under the gauge transformation

$$\delta \psi_\mu = \partial_\mu \epsilon \ , \tag{43}$$

where $\gamma^{\mu\nu\rho}$ denotes the fully antisymmetric product of three γ matrices, normalized to their product when they are all different:

$$\gamma^{\mu\nu\rho} \ = \ \gamma^\mu \gamma^{\nu\rho} - \eta^{\mu\nu} \gamma^\rho + \eta^{\mu\rho} \gamma^\nu \ . \tag{44}$$

Contracting the Rarita-Schwinger equation with γ_μ yields

$$\gamma^{\nu\rho} \partial_\nu \psi_\rho = 0 \ , \tag{45}$$

and therefore the field equation for spin $3/2$ can be written in the alternative form

$$\not{\partial} \psi_\mu - \partial_\mu \not{\psi} = 0 \ . \tag{46}$$

Let us try to obtain a similar equation for a spin-$5/2$ field, defining the Fang-Fronsdal operator $S_{\mu\nu}$,

in analogy with the spin-3/2 case, as

$$S_{\mu\nu} \equiv i \ (\partial\!\!\!/\,\psi_{\mu\nu} - \partial_\mu\psi\!\!\!/_\nu - \partial_\nu\psi\!\!\!/_\mu) \ = \ 0 \ , \tag{47}$$

and generalizing naively the gauge transformation to

$$\delta\psi_{\mu\nu} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \ . \tag{48}$$

Difficulties similar to those met in the bosonic case for spin 3 readily emerge: this equation is not gauge invariant, but transforms as

$$\delta S_{\mu\nu} = -2i \ \partial_\nu\partial_\mu\epsilon\!\!\!/ \ , \tag{49}$$

and a similar problem will soon arise with the Bianchi identity. However, as in the bosonic case, following Fang and Fronsdal [2], one can constrain both the fermionic field and the gauge parameter ϵ so that the gauge symmetry hold and the Bianchi identity take a non-anomalous form.

We can now consider the generic case of half-integer spin $s + 1/2$ [2]. In the compact notation (33) the equation of motion and the gauge transformation read simply

$$S \ \equiv \ i \ (\partial\!\!\!/\,\psi - \partial\psi\!\!\!/) \ = \ 0 \ , \tag{50}$$

$$\delta\psi = \partial\epsilon \ . \tag{51}$$

Since

$$\delta S = -2i\partial^2\epsilon\!\!\!/ \ , \tag{52}$$

to ensure the gauge invariance of the field equation, one must demand that the gauge parameter be *γ-traceless*,

$$\epsilon\!\!\!/ = 0 \ . \tag{53}$$

Let us now turn to the Bianchi identity, computing

$$\partial \cdot S - \frac{1}{2}\partial S' - \frac{1}{2}\partial\!\!\!/ \ S\!\!\!/ \ , \tag{54}$$

where the last term contains the γ-trace of the operator. It is instructive to consider in detail the individual terms. The trace of S is

$$S' = i(\partial\!\!\!/\,\psi' - 2\partial \cdot \psi\!\!\!/ - \partial\psi\!\!\!/') \ , \tag{55}$$

and therefore, using the rules in (33)

$$-\frac{1}{2}\partial S' = -\frac{i}{2}(\partial\!\!\!/\,\partial\psi' - 2\partial\partial \cdot \psi\!\!\!/ - 2\partial^2\psi\!\!\!/') \ . \tag{56}$$

Moreover, the divergence of S is

$$\partial \cdot S = i(\partial\!\!\!/\,\partial \cdot \psi - \Box\psi\!\!\!/ - \partial\partial \cdot \psi\!\!\!/) \ . \tag{57}$$

Finally, from the γ trace of S

$$S\!\!\!/ = i(-2\partial\!\!\!/\,\psi\!\!\!/ + 2\partial \cdot \psi - \partial\psi') \ , \tag{58}$$

one can obtain

$$-\frac{1}{2}\partial\!\!\!/\,S\!\!\!/ = -\frac{i}{2}(-2\Box\psi\!\!\!/ + 2\partial\!\!\!/\,\partial \cdot \psi - \partial\partial\!\!\!/\,\psi') \ , \tag{59}$$

and putting all these terms together yields the Bianchi identity

$$\partial \cdot S - \frac{1}{2}\partial S' - \frac{1}{2}\partial\!\!\!/\,S\!\!\!/ = i\partial^2\psi' \ . \tag{60}$$

As in the bosonic case, this identity contains an "anomalous" term that first manifests itself for spin

$s = 7/2$, and therefore one is lead in general to impose the "triple γ-trace" condition

$$\psi' = 0 . \tag{61}$$

One can also extend the Kaluza-Klein construction to the spin $s+1/2$ in order to recover the massive Singh-Hagen formulation. The reduction from $D+1$ dimensions to D dimensions will turn the massless field $\psi_{D+1}^{(s)}$ into massive fields of the type $\psi_D^{(s)}, \psi_D^{(s-1)}$ and $\psi_D^{(s-2)}$, while no lower-rank fields can appear because of the triple γ-trace condition (61). In a similar fashion, the gauge parameter $\epsilon_{D+1}^{(s-1)}$ reduces only to a single field $\epsilon_D^{(s-1)}$, as a result of the γ-trace condition (53). Gauge fixing the Stueckelberg symmetries one is finally left with only two fields $\psi_D^{(s)}$ and $\psi_D^{(s-2)}$ that contain precisely the γ-traceless tensors introduced by Singh and Hagen in [9].

3   Free non-local geometric equations

In the previous section we have seen that is possible to construct a Lagrangian for higher-spin bosons imposing the unusual Fronsdal constraints

$$\Lambda' = 0, \qquad \varphi'' = 0 \tag{62}$$

on the fields and on the gauge parameters. Following [3, 4], we can now construct higher-spin gauge theories with unconstrained gauge fields and parameters.

3.1   Non-local Fronsdal-like operators

We can motivate the procedure discussing first in some detail the relatively simple example of a spin-3 field, where

$$\delta \mathcal{F}_{\mu\nu\rho} = 3 \, \partial_\mu \partial_\nu \partial_\rho \Lambda' . \tag{63}$$

Our purpose is to build a non-local operator $\mathcal{F}_{NL}$ that transforms exactly like the Fronsdal operator $\mathcal{F}$, since the operator $\mathcal{F} - \mathcal{F}_{NL}$ will then be gauge invariant without any additional constraint on the gauge parameter. One can find rather simply the non-local constructs

$$\frac{1}{3\Box} \left[\partial_\mu \partial_\nu \mathcal{F}'_\rho + \partial_\nu \partial_\rho \mathcal{F}'_\mu + \partial_\rho \partial_\mu \mathcal{F}'_\nu \right] , \tag{64}$$

$$\frac{1}{3\Box} \left[\partial_\mu \partial \cdot \mathcal{F}'_{\nu\rho} + \partial_\nu \partial \cdot \mathcal{F}'_{\rho\mu} + \partial_\rho \partial \cdot \mathcal{F}'_{\mu\nu} \right] , \tag{65}$$

$$\frac{1}{\Box^2} \partial_\mu \partial_\nu \partial_\rho \partial \cdot \mathcal{F}' , \tag{66}$$

but the first two expressions actually coincide, as can be seen from the Bianchi identities (38), and as a result one is led to two apparently distinct non-local fully gauge invariant field equations

$$\mathcal{F}_{\mu_1\mu_2\mu_3} - \frac{1}{3\Box} \left[\partial_{\mu_1}\partial_{\mu_2}\mathcal{F}'_{\mu_3} + \partial_{\mu_2}\partial_{\mu_3}\mathcal{F}'_{\mu_1} + \partial_{\mu_3}\partial_{\mu_1}\mathcal{F}'_{\mu_2} \right] = 0 , \tag{67}$$

$$\mathcal{F}_{\mu_1\mu_2\mu_3}^{new} \equiv \mathcal{F}_{\mu_1\mu_2\mu_3} - \frac{1}{\Box^2} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial \cdot \mathcal{F}' = 0 . \tag{68}$$

These equations can be actually turned into one another, once they are combined with their traces, but the second form, which we denote by $\mathcal{F}^{new}$, is clearly somewhat simpler, since it rests on the addition of the single scalar construct $\partial \cdot \mathcal{F}'$. From $\mathcal{F}^{new}$, one can build in the standard way

$$G_{\mu_1\mu_2\mu_3} \equiv \mathcal{F}_{\mu_1\mu_2\mu_3}^{new} - \frac{1}{2}(\eta_{\mu_1\mu_2}\mathcal{F}_{\mu_3}^{new\prime} + \eta_{\mu_2\mu_3}\mathcal{F}_{\mu_1}^{new\prime} + \eta_{\mu_3\mu_1}\mathcal{F}_{\mu_2}^{new\prime}) , \tag{69}$$

and one can easily verify that

$$\partial \cdot G_{\mu_1\mu_2} = 0 . \tag{70}$$

This identity suffices to ensure the gauge invariance of the Lagrangian. Moreover, we shall see that in all higher-spin cases, the non-local construction will lead to a similar, if more complicated, identity, underlying a Lagrangian formulation that does not need any double trace condition on the gauge field. It is worth stressing this point: we shall see that, modifying the Fronsdal operator in order to achieve gauge invariance without any trace condition on the parameter, the Bianchi identities will change accordingly and the "anomalous" terms will disappear, leading to corresponding gauge invariant Lagrangians.

Returning to the spin-3 case, one can verify that the Einstein tensor $G_{\mu_1\mu_2\mu_3}$ follows from the Lagrangian

$$
\begin{aligned}
\mathcal{L} = {} & -\frac{1}{2}(\partial_\mu \varphi_{\mu_1\mu_2\mu_3})^2 + \frac{3}{2}(\partial \cdot \varphi_{\mu_1\mu_2})^2 - \frac{3}{2}(\partial \cdot \varphi')^2 + \frac{3}{2}(\partial_\mu \varphi'_{\mu_1})^2 \\
& + 3\varphi'_{\mu_1} \partial \cdot \partial \cdot \varphi_{\mu_1} + 3\partial \cdot \partial \cdot \partial \cdot \varphi \frac{1}{\Box} \partial \cdot \varphi' - \partial \cdot \partial \cdot \partial \cdot \varphi \frac{1}{\Box^2} \partial \cdot \partial \cdot \partial \cdot \varphi \ ,
\end{aligned}
\tag{71}
$$

that is fully gauge invariant under

$$
\delta\varphi_{\mu_1\mu_2\mu_3} = \partial_{\mu_1}\Lambda_{\mu_2\mu_3} + \partial_{\mu_2}\Lambda_{\mu_3\mu_1} + \partial_{\mu_3}\Lambda_{\mu_1\mu_2} \ .
\tag{72}
$$

For all higher spins, one can arrive at the proper analogue of (67) via a sequence of pseudo-differential operators, defined recursively as

$$
\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\partial \cdot \mathcal{F}^{(n)} \ ,
\tag{73}
$$

where the initial operator $\mathcal{F}^{(1)} = \mathcal{F}$ is the classical Fronsdal operator. The gauge transformations of the $\mathcal{F}^{(n)}$,

$$
\delta\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\Lambda^{[n]} \ ,
\tag{74}
$$

involve by construction higher traces of the gauge parameter. Since the n-th trace $\Lambda^{[n]}$ vanishes for all $n > (s-1)/2$, the first corresponding operator $\mathcal{F}^{(n)}$ will be gauge invariant without any constraint on the gauge parameter. A similar inductive argument determines the Bianchi identities for the $\mathcal{F}^{(n)}$,

$$
\partial \cdot \mathcal{F}^{(n)} - \frac{1}{2n}\partial\mathcal{F}^{(n)\prime} = -\left(1 + \frac{1}{2n}\right)\frac{\partial^{2n+1}}{\Box^{n-1}}\varphi^{[n+1]} \ ,
\tag{75}
$$

where the *anomalous* contribution depends on the $(n+1)$-th trace $\varphi^{[n+1]}$ of the gauge field, and thus vanishes for $n > (s/2-1)$.

The Einstein-like tensor corresponding to $\mathcal{F}^{(n)}$

$$
G^{(n)} = \sum_{p=0}^{n-1} \frac{(-1)^p (n-p)!}{2^p n!} \eta^p \mathcal{F}^{(n)[p]} \ ,
\tag{76}
$$

is slightly more complicated than its lower-spin analogs, since it involves in general multiple traces, but an inductive argument shows that

$$
\partial \cdot G^{(n)} = 0 \ ,
\tag{77}
$$

so that $G^{(n)}$ follows indeed from a Lagrangian of the type

$$
\mathcal{L} \sim \varphi\, G^{(n)} \ .
\tag{78}
$$

We shall soon see that $G^{(n)}$ has a very neat geometrical meaning. Hence, the field equations, not directly the Lagrangians, are fully geometrical in this formulation.

3.2   Geometric equations

Inspired by General Relativity, we can reformulate the non-local objects like the Ricci tensor $\mathcal{F}^{(n)}$ introduced in the previous section in geometrical terms. We have already seen that, following de Wit and Freedman [14], one can define generalized connections and Riemann tensors of various orders in the derivatives for all spin-s gauge fields as extensions of the spin-2 objects as

$$\Gamma_{\mu;\nu_1\nu_2} \quad \Rightarrow \quad \Gamma_{\mu_1\ldots\mu_{s-1};\nu_1\ldots\nu_s} \ ,$$
$$R_{\mu_1\mu_2;\nu_1\nu_2} \quad \Rightarrow \quad R_{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s} \ ,$$

and actually a whole hierarchy of connections $\Gamma_{\mu_1\ldots\mu_k;\nu_1\ldots\nu_s}$ whose last two members are the connection and the curvature above. In order to appreciate better the meaning of this generalization, it is convenient to recall some basic facts about linearized Einstein gravity. If the metric is split according to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the condition that g be covariantly constant leads to the following relation between its deviation h with respect to flat space and the linearized Christoffel symbols:

$$\partial_\rho h_{\mu\nu} = \Gamma_{\nu;\rho\mu} + \Gamma_{\mu;\rho\nu} \ . \tag{79}$$

In strict analogy, the corresponding relation for spin 3 is

$$\partial_\sigma \partial_\tau \varphi_{\mu\nu\rho} \ = \ \Gamma_{\nu\rho;\sigma\tau\mu} + \Gamma_{\rho\mu;\sigma\tau\nu} + \Gamma_{\mu\nu;\sigma\tau\rho} \ . \tag{80}$$

It is possible to give a compact expression for the connections of [14] for arbitrary spin,

$$\Gamma^{(s-1)} = \frac{1}{s} \sum_{k=0}^{s-1} \frac{(-1)^k}{\binom{s-1}{k}} \ \partial^{s-k-1} \nabla^k \varphi \ , \tag{81}$$

where the derivatives ∇ carry indices originating from the gauge field. This tensor is actually the proper analogue of the Christoffel connection for a spin-s gauge field, and transforms as

$$\delta\Gamma_{\alpha_1\cdots\alpha_{s-1};\beta_1\cdots\beta_s} \ \sim \ \partial_{\beta_1}\cdots\partial_{\beta_s}\Lambda_{\alpha_1\cdots\alpha_{s-1}}. \tag{82}$$

That is a direct link between these expressions and the traces of non-local operators of the previous section. From this connection, one can then construct a gauge invariant tensor $\mathcal{R}_{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}$ that is the proper analogue of the Riemann tensor of a spin-2 field.

We can thus write in a more compact geometrical form the results of the iterative procedure. The non-local field equations for odd spin $s = 2n + 1$ generalizing the Maxwell equations $\partial^\mu F_{\mu;\nu} = 0$ are simply

$$\frac{1}{\Box^n}\partial \cdot \mathcal{R}^{[n]}_{;\mu_1\cdots\mu_{2n+1}} = 0 \ , \tag{83}$$

while the corresponding equations for even spin $s = 2n$ are simply

$$\frac{1}{\Box^{n-1}}\mathcal{R}^{[n]}_{;\mu_1\cdots\mu_{2n}} = 0 \ , \tag{84}$$

that reduce to $R_{;\mu\nu} = 0$ for spin 2.

The non-local geometric equations for higher-spin bosons can be brought to the Fronsdal form using the traces Λ' of the gauge parameters, and propagate the proper number of degrees of freedom. At first sight, however, the resulting Fronsdal equations present a subtlety [4]. The analysis of their physical degrees of freedom normally rests on the choice of de Donder gauge,

$$\mathcal{D} \equiv \partial \cdot \varphi - \frac{1}{2}\,\partial\varphi' = 0 \ , \tag{85}$$

the higher-spin analog of the Lorentz gauge, that reduces the Fronsdal operator to $\Box\varphi$, but *this is a proper gauge only for doubly traceless fields*. The difficulty one faces can be understood noting that,

in order to recover the Fronsdal equation eliminating the non-local terms, one uses the trace Λ' of the gauge parameter. The trace of the de Donder gauge condition, proportional to the double trace φ'' of the gauge field, is then in fact *invariant* under residual gauge transformations with a traceless parameter, so that the de Donder gauge cannot be reached in general. However, it can be modified, as in [4], by the addition of terms containing higher traces of the gauge field, and the resulting gauge fixed equation then sets to the zero the double trace φ'' on shell.

Following similar steps, one can introduce non-local equations for *fermionic* fields with unconstrained gauge fields and gauge parameters [3]. To this end, it is convenient to notice that the fermionic operators for spin $s + 1/2$ can be related to the corresponding bosonic operators for spin s according to

$$\mathcal{S}_{s+1/2} - \frac{1}{2} \frac{\partial \not{\partial}}{\Box} \not{\mathcal{S}}_{s+1/2} = i \frac{\not{\partial}}{\Box} \mathcal{F}_s(\psi) \ , \tag{86}$$

that generalize the obvious link between the Dirac and Klein-Gordon operators. For instance, the Rarita-Schwinger equation $\gamma^{\mu\nu\rho} \partial_\nu \psi_\rho = 0$ implies that

$$\mathcal{S} \equiv i(\not{\partial} \psi_\mu - \partial_\mu \not{\psi}) = 0 \ , \tag{87}$$

while (86) implies that

$$S_\mu - \frac{1}{2} \frac{\partial_\mu \not{\partial}}{\Box} \not{\mathcal{S}} \ = \ \frac{i\not{\partial}}{\Box} [\eta_{\mu\nu} - \partial_\mu \partial_\nu] \psi^\nu \ . \tag{88}$$

Non-local fermionic kinetic operators $\mathcal{S}^{(n)}$ can be defined recursively as

$$\mathcal{S}^{(n+1)} = \mathcal{S}^{(n)} + \frac{1}{n(2n+1)} \frac{\partial^2}{\Box} \mathcal{S}^{(n)'} - \frac{2}{2n+1} \frac{\partial}{\Box} \partial \cdot \mathcal{S}^{(n)} \ , \tag{89}$$

with the understanding that, as in the bosonic case, the iteration procedure stops when the gauge variation

$$\delta \mathcal{S}^{(n)} = -2i \, n \, \frac{2n}{\Box^{n-1}} \not{k}^{[n-1]} \tag{90}$$

vanishes due to the impossibility of constructing the corresponding higher trace of the gauge parameter. The key fact shown in [3,4] is that, as in the bosonic case, the Bianchi identities are similarly modified, according to

$$\partial \mathcal{S}^{(n)} \ - \ \frac{1}{2n} \partial \, \mathcal{S}^{(n)\,'} \ - \ \frac{1}{2n} \not{\partial} \not{\mathcal{S}}^{(n)} \ = \ i \, \frac{\partial^{2n}}{\Box^{n-1}} \psi^{[n]} \ , \tag{91}$$

and lack the anomalous terms when n is large enough to ensure that the field equations are fully gauge invariant. Einstein-like operators and field equations can then be defined following steps similar to those illustrated for the bosonic case.

4   Triplets and local compensator form

4.1   String field theory and BRST

String Theory includes infinitely many higher-spin massive fields with consistent mutual interactions, and it tensionless limit $\alpha' \to \infty$ lends itself naturally to provide a closer view of higher-spin fields. Conversely, a better grasp of higher-spin dynamics is likely to help forward our current understanding of String Theory.

Let us recall some standard properties of the open bosonic string oscillators. In the mostly plus convention for the metric, their commutations relations read

$$[\alpha_k^\mu, \alpha_l^\nu] = k \delta_{k+l,0} \eta^{\mu\nu} \ , \tag{92}$$

and the corresponding Virasoro operators

$$L_k = \frac{1}{2}\Sigma_{l=-\infty}^{+\infty}\alpha_{k-l}^{\mu}\alpha_{\mu l}, \tag{93}$$

where $\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu}$ and $p_{\mu} - i\partial_{\mu}$ satisfy the Virasoro algebra

$$[L_k, L_l] = (k-l)L_{k+l} + \frac{\mathcal{D}}{12}m(m^2-1) \ , \tag{94}$$

where the central charge equals the space-time dimension $\mathcal{D}$.

In order to study the tensionless limit, it is convenient to rescale the Virasoro generators according to

$$L_k \to \frac{1}{\sqrt{2\alpha'}}L_k, \qquad L_0 \to \frac{1}{\alpha'}L_0. \tag{95}$$

Taking the limit $\alpha' \to \infty$, one can then define the reduced generators

$$l_0 = p^2, \quad l_m = p \cdot \alpha_m \qquad (m \neq 0) \ , \tag{96}$$

that satisfy the simpler algebra

$$[l_k, l_l] = k\delta_{k+l,0}l_0 \ . \tag{97}$$

Since this contracted algebra does not contain a central charge, the resulting massless models are consistent in any space-time dimension, in sharp contrast with what happens in String Theory when α' is finite. It is instructive to compare the mechanism of mass generation at work in String Theory with the Kaluza-Klein reduction, that as we have seen in previous sections works for arbitrary dimensions. A closer look at the first few mass levels shows that, *as compared to the Kaluza-Klein setting, the string spectrum lacks some auxiliary fields, and this feature may be held responsible for the emergence of the critical dimension!*

Following the general BRST method, let us introduce the ghost modes C_k of ghost number one and the corresponding antighosts B_k of ghost number minus one, with the usual anti-commutation relations. The BRST operator [17, 18, 19]

$$\mathcal{Q} = \sum_{-\infty}^{+\infty}[C_{-k}L_k - \frac{1}{2}(k-l) : C_{-k}C_{-l}B_{k+l} :] - C_0 \tag{98}$$

determines the free string equation

$$\mathcal{Q}|\Phi\rangle = 0 \ , \tag{99}$$

while the corresponding gauge transformation is

$$\delta|\Phi\rangle = \mathcal{Q}|\Lambda\rangle \ . \tag{100}$$

Rescaling the ghost variables according to

$$c_k = \sqrt{2\alpha'}C_k \ , \quad b_k = \frac{1}{2\alpha'}B_k \ , \tag{101}$$

for $k \neq 0$ and as

$$c_0 = \alpha'C_0, \quad b_0 = \frac{1}{\alpha'}B_0 \tag{102}$$

for $k = 0$ allows a non-singular $\alpha' \to \infty$ limit that defines the *identically* nilpotent BRST charge

$$Q = \sum_{-\infty}^{+\infty}[c_{-k}l_k - \frac{k}{2}b_0c_{-k}c_k] \ . \tag{103}$$

Making the zero-mode structure manifest then gives

$$Q = c_0 l_0 - b_0 M + \tilde{Q} , \tag{104}$$

where $\tilde{Q} = \sum_{k \neq 0} c_{-k} l_k$ and $M = \frac{1}{2} \sum_{-\infty}^{+\infty} k c_{-k} c_k$, and the string field and the gauge parameter can be decomposed as

$$|\Phi\rangle = |\varphi_1\rangle + c_0 |\varphi_2\rangle , \tag{105}$$
$$|\Lambda\rangle = |\Lambda_1\rangle + c_0 |\Lambda_2\rangle . \tag{106}$$

It should be appreciated that *in this formulation no trace constraint is imposed on the master gauge field φ or on the master gauge parameter Λ.* It is simple to confine the attention to totally symmetric tensors, selecting states $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\Lambda\rangle$ that are built from a single string mode α_{-1},

$$|\varphi_1\rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\mu_1 \cdots \mu_s}(x) \alpha_{-1}^{\mu_1} \cdots \alpha_{-1}^{\mu_s} |0\rangle ,$$
$$+ \sum_{s=2}^{\infty} \frac{1}{(s-2)!} D_{\mu_1 \cdots \mu_{s-2}}(x) \alpha_{-1}^{\mu_1} \cdots \alpha_{-1}^{\mu_{s-2}} c_{-1} b_{-1} |0\rangle , \tag{107}$$

$$|\varphi_2\rangle = \sum_{s=1}^{\infty} \frac{-i}{(s-1)!} C_{\mu_1 \cdots \mu_{s-1}}(x) \alpha_{-1}^{\mu_1} \cdots \alpha_{-1}^{\mu_{s-1}} b_{-1} |0\rangle , \tag{108}$$

$$|\Lambda\rangle = \sum_{s=1}^{\infty} \frac{i}{(s-1)!} \Lambda_{\mu_1 \cdots \mu_{s-1}}(x) \alpha_{-1}^{\mu_1} \cdots \alpha_{-1}^{\mu_{s-1}} b_{-1} |0\rangle . \tag{109}$$

Restricting eqs. (99) and (100) to states of this type, the s-th terms of the sums above yield the *triplet* equations [20, 21, 4]

$$\begin{aligned}
\Box \varphi &= \partial C , \\
\partial \cdot \varphi - \partial D &= C , \\
\Box D &= \partial \cdot C ,
\end{aligned} \tag{110}$$

and the corresponding gauge transformations

$$\begin{aligned}
\delta \varphi &= \partial \Lambda , \\
\delta C &= \Box \Lambda , \\
\delta D &= \partial \cdot \Lambda ,
\end{aligned} \tag{111}$$

where φ is rank-s tensor, C is a rank-$(s-1)$ tensor and D is a rank-$(s-2)$ tensor. These field equations follow from a corresponding truncation of the Lagrangian

$$\mathcal{L} = \langle \Phi | Q | \Phi \rangle , \tag{112}$$

that in component notation reads

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \varphi)^2 + s \partial \cdot \varphi C + s(s-1) \partial \cdot C D + \frac{s(s-1)}{2} (\partial_\mu D)^2 - \frac{s}{2} C^2 , \tag{113}$$

where the D field, whose modes disappear on the mass shell, has a peculiar negative kinetic term. Note

that one can also eliminate the auxiliary field C, thus arriving at the equivalent formulation

$$
\begin{aligned}
\mathcal{L} \quad = \quad & -\sum_{s=0}^{+\infty} \frac{1}{s!} \Big[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{s}{2}(\partial \cdot \varphi)^2 + s(s-1)\partial \cdot \partial \cdot \varphi\, D \\
+ \quad & s(s-1)(\partial_\mu D)^2 + \frac{s(s-1)(s-2)}{2}(\partial \cdot D)^2 \Big] \; .
\end{aligned}
\tag{114}
$$

in terms of pairs (φ, D) of symmetric tensors, more in the spirit of [20].

For a given value of s, this system propagates modes of spin s, $s-2$, ..., down to 0 or 1 according to whether s is even or odd. This can be simply foreseen from the light-cone description of the string spectrum, since the corresponding physical states are built from arbitrary powers of a single light-cone oscillator α_1^i, that taking out traces produces precisely a nested chain of states with spins separated by two units. From this reducible representation, as we shall see, it is possible to deduce a set of equations for an irreducible multiplet demanding that the trace of φ be related to D.

If the auxiliary C field is eliminated, the equations of motion for the triplet take the form

$$
\mathcal{F} = \partial^2 (\varphi' - 2D) \; ,
\tag{115}
$$

$$
\Box D = \frac{1}{2}\partial \cdot \partial \cdot \varphi - \frac{1}{2}\partial\partial \cdot D \; .
\tag{116}
$$

4.2 (A)dS extensions of the bosonic triplets

The interaction between a spin 3/2 field and the gravitation field is described essentially by a Rarita-Schwinger equation where ordinary derivatives are replaced by Lorentz-covariant derivatives. The gauge transformation of the Rarita-Schwinger Lagrangian is then surprisingly proportional not to the Riemann tensor, but to the Einstein tensor, and this variation is precisely compensated in supergravity [10] by the supersymmetry variation of the Einstein Lagrangian. However, if one tries to generalize this result to spin $s \geq 5/2$, the miracle does not repeat and the gauge transformations of the field equations of motion generate terms proportional to the Riemann tensor, and similar problems are also met in the bosonic case. This is the Aragone-Deser problem for higher spins [22].

As was first noticed by Fradkin and Vasiliev [23], with a non-vanishing cosmological constant Λ it is actually possible to modify the spin $s \geq 5/2$ field equations introducing additional terms that depend on negative powers of Λ and cancel the dangerous Riemann curvature terms. This observation plays a crucial role in the Vasiliev equations [24], discussed in the lectures by Vasiliev [25] and Sundell [26] at this Workshop.

For these reasons it is interesting to describe the (A)dS extensions of the massless triplets that emerge from the bosonic string in the tensionless limit and the corresponding deformations of the compensator equations. Higher-spin gauge fields propagate consistently and independently of one another in conformally flat space-times, bypassing the Aragone-Deser inconsistencies that would be introduced by a background Weyl tensor, and this free-field formulation in an (A)dS background serves as a starting point for exhibiting the unconstrained gauge symmetry of [3,4], as opposed to the constrained Fronsdal gauge symmetry [1], in the recent form of the Vasiliev equations [24] based on vector oscillators [26].

One can build the (A)dS symmetric triplets from a modified BRST formalism [5], but in the following we shall rather build them directly deforming the flat triplets. The gauge transformations of φ and D are naturally turned into their curved-space counterparts,

$$
\delta\varphi \quad = \quad \nabla\Lambda \; ,
\tag{117}
$$

$$
\delta D \quad = \quad \nabla \cdot \Lambda \; ,
\tag{118}
$$

where the commutator of two covariant derivatives on a vector in AdS is

$$
[\nabla_\mu, \nabla_\nu]V_\rho = \frac{1}{L^2}(g_{\nu\rho}V_\mu - g_{\mu\rho}V_\nu) \; .
\tag{119}
$$

However, in order to maintain the definition of $C = \nabla \cdot \varphi - \nabla D$, one is led to deform its gauge variation,

turning it into

$$\delta C = \Box \Lambda + \frac{(s-1)(3-s-\mathcal{D})}{L^2}\Lambda + \frac{2}{L^2}g\Lambda' \ , \tag{120}$$

where $-1/L^2$ is the AdS cosmological constant and g is the background metric tensor. The corresponding de Sitter equations could be obtained by the formal continuation of L to imaginary values, $L \to iL$.

These gauge transformations suffice to fix the other equations, that read

$$\Box \varphi = \nabla C - \frac{1}{L^2}\left\{-8gD + 2g\varphi' - [(2-s)(3-\mathcal{D}-s) - s]\varphi\right\} \ , \tag{121}$$

$$C = \nabla \cdot \varphi - \nabla D, \tag{122}$$

$$\Box D = \nabla \cdot C - \frac{1}{L^2}\left\{-[s(\mathcal{D}+s-2)+6]D + 4\varphi' + 2gD'\right\} \ , \tag{123}$$

and as in the previous section one can also eliminate C. To this end, it is convenient to define the AdS Fronsdal operator, that extends (34), as

$$\mathcal{F} \equiv \Box \varphi - \nabla \nabla \cdot \varphi + \frac{1}{2}\{\nabla, \nabla\}\varphi' \ . \tag{124}$$

The first equation of (121) then becomes

$$\mathcal{F} \ = \ \frac{1}{2}\{\nabla, \nabla\}\left(\varphi' - 2D\right) + \frac{1}{L^2}\left\{8gD - 2g\varphi' + [(2-s)(3-\mathcal{D}-s)-s]\varphi\right\} \ . \tag{125}$$

In a similar fashion, after eliminating the auxiliary field C, the AdS equation for D becomes

$$\Box D + \frac{1}{2}\nabla \nabla \cdot D \ - \ \frac{1}{2}\nabla \cdot \nabla \cdot \varphi = -\frac{(s-2)(4-\mathcal{D}-s)}{2L^2}D - \frac{1}{L^2}gD'$$

$$+ \ \frac{1}{2L^2}\left\{[s(\mathcal{D}+s-2)+6]D - 4\varphi' - 2gD'\right\} \ . \tag{126}$$

It is also convenient to define the modified Fronsdal operator

$$\mathcal{F}_L = \mathcal{F} - \frac{1}{L^2}\left\{[(3-\mathcal{D}-s)(2-s)-s]\varphi + 2g\varphi'\right\} \ , \tag{127}$$

since in terms of $\mathcal{F}_L$ eq. (125) becomes

$$\mathcal{F}_L = \frac{1}{2}\{\nabla, \nabla\}(\varphi' - 2D) + \frac{8}{L^2}\, g\, D \ , \tag{128}$$

while the Bianchi identity becomes simply

$$\nabla \cdot \mathcal{F}_L - \frac{1}{2}\nabla \mathcal{F}_L' = -\frac{3}{2}\nabla^3 \varphi'' + \frac{2}{L^2}g\,\nabla\varphi'' \ . \tag{129}$$

4.3   Compensator form of the bosonic equations

In the previous sections we have displayed a non-local geometric Lagrangian formulation for higher-spin bosons and fermions. In this section we show how one can obtain very simple *local non-Lagrangian* descriptions that exhibit the unconstrained gauge symmetry present in the non-local equations and reduce to the Fronsdal form after a partial gauge fixing [3, 4, 5].

The key observation is that the case of a single propagating spin-s field can be recovered from the equations (115)-(116) demanding that all lower-spin excitations be pure gauge. To this end, it suffices to introduce a spin $s-3$ *compensator* α as

$$\varphi' - 2D = \partial\alpha \ , \tag{130}$$

that by consistency transforms as

$$\delta\alpha = \Lambda' \ . \tag{131}$$

Eq. (115) then becomes

$$\mathcal{F} = 3\partial^3\alpha \ , \tag{132}$$

while (116) becomes

$$\mathcal{F}' - \partial^2\varphi'' = 3\,\Box\partial\alpha \ + \ 2\,\partial^2\partial\cdot\alpha \ . \tag{133}$$

Combining them leads to

$$\partial^2\varphi'' \ = \ \partial^2(4\partial\cdot\alpha + \partial\alpha') \ , \tag{134}$$

and the conclusion is then that the triplet equations imply a pair of *local* equations for a single massless spin-s gauge field φ and a single spin-$(s-3)$ compensator α. Summarizing, the local compensator equations and the corresponding gauge transformations are

$$\begin{aligned}
\mathcal{F} &= 3\partial^3\alpha, \quad \varphi'' = 4\partial\cdot\alpha + \partial\alpha' \ , \\
\delta\varphi &= \partial\Lambda, \quad\ \ \delta\alpha = \Lambda' \ ,
\end{aligned} \tag{135}$$

and clearly reduce to the standard Fronsdal form after a partial gauge fixing using the trace Λ' of the gauge parameter . These equations can be regarded as the local analogs of the non-local geometric equations, but it should be stressed that *they are not Lagrangian equations*. This can be seen either directly, as in [4, 5], or via the corresponding BRST operator, that is not hermitian, as pertains to a reduced system that is not described by a Lagrangian [27]. Nonetheless, the two equations (135) form a consistent system, and the first can be turned into the second using the Bianchi identity.

One can also obtain the (A)dS extension of the spin-s compensator equations (135). The natural starting point are the (A)dS gauge transformations for the fields φ and α

$$\delta\varphi = \nabla\Lambda \ , \qquad \delta\alpha = \Lambda' \ . \tag{136}$$

One can then proceed in various ways to obtain the compensator equations

$$\begin{aligned}
\mathcal{F} &= 3\nabla^3\alpha + \frac{1}{L^2}\left\{-2g\varphi' + [(2-s)(3-\mathcal{D}-s) - s]\varphi\right\} - \frac{4}{L^2}g\nabla\alpha \\
\varphi'' &= 4\partial\cdot\alpha + \partial\alpha' \ ,
\end{aligned} \tag{137}$$

that, of course, again do *not* follow from a Lagrangian. However, lagrangian equations can be obtained, both in flat space and in an (A)dS background, from a BRST construction based on a wider set of constraints first obtained by Pashnev and Tsulaia [28, 5]. It is instructive to illustrate these results for spin 3 bosons.

In addition to the triplet fields φ, C and D and the compensator α, this formulation uses the additional spin-1 fields $\varphi^{(1)}$ and F and spin-0 fields $C^{(1)}$ and E, together with a new spin-1 gauge parameter μ and a new spin-0 gauge parameter $\Lambda^{(1)}$. The BRST analysis generates the gauge transformations

$$\begin{aligned}
\delta\varphi &= \partial\Lambda + \eta\,\mu \ , & \delta\alpha &= \Lambda' - \sqrt{2\mathcal{D}}\Lambda^{(1)} \ , \\
\delta\varphi^{(1)} &= \partial\Lambda^{(1)} + \sqrt{\tfrac{\mathcal{D}}{2}}\mu \ , & \delta C &= \Box\Lambda \ , \\
\delta D &= \partial\cdot\Lambda + \mu \ , & \delta C^{(1)} &= \Box\Lambda^{(1)} \ , \\
\delta E &= \partial\cdot\mu \ , & \delta F &= \Box\mu \ ,
\end{aligned} \tag{138}$$

and the corresponding field equations

$$\begin{aligned}
\Box\varphi &= \partial C + \eta F \ , & \Box\alpha C' - \sqrt{2\mathcal{D}}C^{(1)} \ , \\
\partial\cdot\varphi - \partial D - \eta E &= C \ , & \Box\varphi^{(1)} &= \partial C^{(1)} + \sqrt{\tfrac{\mathcal{D}}{2}}F \ , \\
\Box D &= \partial\cdot C + F \ , & \Box E &= \partial\cdot F \ , \\
\partial\alpha &= \varphi' - 2D - \sqrt{2\mathcal{D}}\varphi^{(1)} \ , & \partial\cdot\varphi^{(1)} - \sqrt{2\mathcal{D}}E &= C^{(1)} \ .
\end{aligned} \tag{139}$$

Making use of the gauge parameters μ and $\Lambda^{(1)}$ one can set $\varphi^{(1)} = 0$ and $C^{(1)} = 0$, while the other additional fields are set to zero by the field equations. Therefore, one can indeed recover the non-Lagrangian compensator equations gauge fixing this Lagrangian system. A similar, if more complicated analysis, goes through for higher spins, where this formulation requires $\mathcal{O}(s)$ fields.

The logic behind these equations can be captured rather simply taking a closer look at the gauge transformations (138). One is in fact gauging away φ', modifying the gauge transformation of φ by the μ term. This introduces a corresponding modification in the φ equation, that carries through by integrability to the C equation, and so on.

4.4   Fermionic triplets

We can now turn to the fermionic triplets proposed in [4] as a natural guess for the field equations of symmetric spinor-tensors arising in the tensionless limit of superstring theories. In fact, the GSO projection limits their direct occurrence to type-0 theories [29], but slightly more complicated spinor-tensors of this type, but with mixed symmetry, are present in all superstring spectra, and can be discussed along similar lines [5].

The counterparts of the bosonic triplet equations and gauge transformations are

$$
\begin{aligned}
\not{\partial}\psi &= \partial\chi \, , & \delta\psi &= \partial\epsilon \, , \\
\partial\cdot\psi - \partial\lambda &= \not{\partial}\chi \, , & \delta\lambda &= \partial\cdot\epsilon \, , \\
\not{\partial}\lambda &= \partial\cdot\chi \, , & \delta\chi &= \not{\partial}\epsilon \, .
\end{aligned}
\tag{140}
$$

It can be shown that this type of system propagates spin-$(s+1/2)$ modes and *all* lower half-integer spins. One can now introduce a spin-$(s-2)$ compensator ξ proceeding in a way similar to what we have seen for the bosonic case, and the end result is a simple non-Lagrangian formulation for a single spin-s field,

$$
\begin{aligned}
\mathcal{S} &= -2\,i\,\partial^2\xi \, , & \delta\psi &= \partial\epsilon \, , \\
\psi' &= 2\,\partial\cdot\xi + \partial\xi' + \not{\partial}\xi \, , & \delta\xi &= \not{\epsilon} \, .
\end{aligned}
\tag{141}
$$

These equations turn into one another using the Bianchi identity, and can be extended to (A)dS background, as in [5]. However, a difficulty presents itself when attempting to extend the fermionic triplets to off-shell systems in (A)dS, since the BRST analysis shows that the operator extension does not define a closed algebra.

4.5   Fermionic compensators

One can also extend nicely the fermionic compensator equations to an (A)dS background. The gauge transformation for a spin-$(s+1/2)$ fermion is deformed in a way that can be anticipated from supergravity and becomes in this case

$$
\delta\psi = \nabla\epsilon + \frac{1}{2L}\gamma\epsilon \, ,
\tag{142}
$$

where L determines again the (A)dS curvature and ∇ denotes an (A)dS covariant derivative. The commutator of two of these derivatives on a spin-1/2 field η reads

$$
[\nabla_\mu, \nabla_\nu]\eta = -\frac{1}{2L^2}\,\gamma_{\mu\nu}\eta \, ,
\tag{143}
$$

and using eqs (119)-(143) one can show that the compensator equations for a spin-s fermion ($s = n+1/2$) in an (A)dS background are

$$
\begin{aligned}
(\slashed{\nabla}\psi - \nabla\slashed{\psi}) \quad &+ \quad \frac{1}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{1}{2L}\gamma\slashed{\psi} \\
&= \quad -\{\nabla, \nabla\}\xi + \frac{1}{L}\gamma\nabla\xi + \frac{3}{2L^2}g\xi \ , \\
\psi' \quad &= \quad 2\nabla \cdot \xi + \slashed{\nabla}\slashed{\xi} + \nabla\xi' + \frac{1}{2L}[\mathcal{D} + 2(n-2)]\xi - \frac{1}{2L}\gamma\xi'.
\end{aligned}
\tag{144}
$$

These equations are invariant under

$$
\begin{aligned}
\delta\psi &= \nabla\epsilon \ , \tag{145} \\
\delta\xi &= \slashed{\epsilon} \ , \tag{146}
\end{aligned}
$$

with an unconstrained parameter ϵ. Eqs. (144) are again a pair of non-lagrangian equations, like their flat space counterparts (141).

As in the flat case, eqs (144) are nicely consistent, as can be shown making use of the (A)dS deformed Bianchi identity (60)

$$
\nabla \cdot \mathcal{S} - \frac{1}{2}\nabla\mathcal{S}' - \frac{1}{2}\slashed{\nabla}\slashed{\mathcal{S}} = \frac{i}{4L}\gamma\mathcal{S}' + \frac{i}{4L}[(\mathcal{D} - 2) + 2(n-1)]\slashed{\mathcal{S}} \tag{147}
$$

$$
+ \frac{i}{2}\left[\{\nabla, \nabla\} - \frac{1}{L}\gamma\nabla - \frac{3}{2L^2}\right]\psi' \ , \tag{148}
$$

where the Fang-Fronsdal operator $\mathcal{S}$ is also deformed and becomes

$$
\mathcal{S} = i(\slashed{\nabla}\psi - \nabla\slashed{\psi}) + \frac{i}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{i}{2L}\gamma\slashed{\psi} \ . \tag{149}
$$

Acknowledgments

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