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Investigation of Spin-Independent CP Violation in Neutron and Nuclear Radiative β Decays

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Daheng He, Student

Dr. Susan Gardner, Major Professor

Dr. Tim Gorringe, Director of Graduate Studies

Investigation of Spin-Independent CP Violation in Neutron and Nuclear Radiative β Decays

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

> By Daheng He Lexington, Kentucky

Director: Dr. Susan Gardner, Professor of Physics Lexington, Kentucky 2013

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ABSTRACT OF DISSERTATION

Investigation of Spin-Independent CP Violation in Neutron and Nuclear Radiative β Decays

CP violation is an important condition to explain the preponderance of baryons in our universe, yet the available CP violation in the Standard Model (SM) via the so-called Cabibbo-Kobayashi-Maskawa mechanism seems to not provide enough CP violation. Thus searching for new sources of CP violation is one of the central tasks of modern physics. In this thesis, we focus on a new possible source of CP violation which generates triple-product correlations in momenta which can appear in neutron and nuclear radiative β decay. We show that at low energies such a CP violating correlation may arise from the exotic coupling of nucleon, photon and neutrino that was proposed by Harvey, Hill, and Hill (HHH). One specialty of such an exotic HHH coupling is that it does not generate the well-known CP-violating terms such as "D-term", "R-term", and neutron electric dipole moment, in which particle's spins play critical role. We show that such a new HHH-induced CP violating effect is proportional to the imaginary part of $c_5 g_V$, where g_V is the vector coupling constant in neutron and nuclear β decay, and c_5 is the phenomenological coupling constant that appears in chiral perturbation theory at $\mathcal{O}(M^{-2})$ with M referring to the nucleon or nuclear mass. We consider a possible non-Abelian hidden sector model, which is beyond the SM and may yield a nontrivial $Im(c_5)$. The available bounds on both $\operatorname{Im}(c_5)$ and $\operatorname{Im}(q_V)$ are considered, and a better limit on $\operatorname{Im}(c_5)$ can come from a direct measurement in radiative beta decay. We calculate the competitive effect that arises from the general parameterization of the weak interaction that was proposed by Lee and Yang in 1956. We also show that in the proposed measurements, the CPviolating effect can be mimicked by the SM via final-state interactions (FSI). For a better determination of the bound of $\text{Im}(c_5)$, we consider the FSI-induced mimicking effect in full detail in $\mathcal{O}(\alpha)$ as well as in leading recoil order. To face ongoing precision measurements of neutron radiative β decay of up to 1% relative error, we sharpen our calculations of the CP conserving pieces of neutron radiative β decay by considering the largest contributions in $\mathcal{O}(\alpha^2)$: the final-state Coulomb corrections as well as the contributions from two-photon radiation.

KEYWORDS: CP violation, chiral perturbation theory, neutron radiative β decay,

final-state interactions, hidden sector interactions.

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Date: November 19, 2013

Investigation of Spin-Independent CP Violation in Neutron and Nuclear Radiative β Decays

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Director of Dissertation: Susan Gardner

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Chapter 1 Introduction

One of the greatest forward steps in modern physics during the past half century is to realize the essential role of symmetries, including many approximate symmetries. Symmetries are related to invariances of a system. Finding a symmetry can often allow for substantial constraints on the form of a dynamical description. The case of a broken symmetry is often even more important, because a broken symmetry can reveal the existence of new forces and/or of new particles, or of fundamental properties of the vacuum. In this sense, investigation of symmetries, especially broken symmetries, provides us an excellent tool to find "new physics".

In general, symmetries can fall into one of two categories:

- 1. continuous symmetries, such as translational or rotational symmetry;
- 2. discrete symmetries, such as charge conjugation C, parity P, and time reversal T these 3 fundamental symmetries are especially of our interest.

Until the middle of last century, people believed that all these symmetries are universally valid, based on rather irrational statements of belief from everyday experiences in the macroscopic world. In 1956, the discovery of parity violation in weak decays [3] shook our earlier unconditional belief in the absolute validity of these discrete symmetries in the quantum world. As a compromise, one tried to argue that although P by itself is not conserved in neutron and nuclear β decay, the combined symmetry of charge conjugation and parity, CP, should still hold. In 1964, soon after the discovery of the P violation, CP as a combination was also experimentally confirmed to be violated [4] [5] [6]. With these continuing failures of assuming validity of symmetries, the lesson is that on quantum level, symmetries should not be assumed to hold a priori, but have to be subjected to determined experimental scrutiny.

As mentioned above, finding a broken symmetry is often an exciting thing since it often suggests the existence of new particles and new physics that are still beyond the scope of our current knowledge. CP violation is an outstanding example. The most important thing about CP violation is that it helps provide us a solution to a very important question: how we managed to come to be.

One of the most striking achievements of cosmology of the 20th century is the "Big Bang" scenario to explain the origin of the universe. Such a theory naturally sews modern cosmology and modern particle physics together. Such a scenario is supported by many cosmological observations, and thus is widely accepted as valid. A big challenge that remains for us is to explain the so-called baryogenesis problem. More explicitly, according to the standard point of view at the very origin of the big bang, matter and antimatter were supposed to be produced in equal amounts at incredibly high temperature. Due to the quick expansion, the universe cooled down, with most of the matter and antimatter annihilating into radiations such as photons that pervade everywhere in the universe. Based on this picture, one immediately

draws the conclusion that the number of photons greatly exceeds the number of baryons, and this is in fact supported by cosmological observations. Defining the baryon-to-photon abundance parameter $\eta_{10} \equiv 10^{10} n_{bar}/n_{\gamma}$, where n_{bar} and n_{γ} refer to the number number density of baryons and number density of photons respectively, it can be shown that [7]

$$\eta_{10} \approx 274 \Omega_B h^2, \tag{1.1}$$

where $\Omega_B \equiv \rho_{bar}/\rho_c$, with $\rho_c = 3H_0^2/8\pi G_N$ referring to the present critical mass density, H_0 is the present value of the Hubble parameter, which is defined via $v = H_0D$ with v and D referring to the speed and the distance of a galaxy relative to us, and G_N is the Newton constant. The quantity h is defined as the present value of the Hubble parameter in units of $100 \text{km}s^{-1}\text{Mpc}^{-1}$. The current global average value from a variety of cosmic microwave background (CMB) experiments [8] is $\Omega_B h^2 =$ 0.0223 ± 0.0007 , corresponding to $\eta_{10} = 6.11 \pm 0.20$. This implies

$$\frac{n_{bar}}{n_{\gamma}} \approx 6.11 \times 10^{-10},\tag{1.2}$$

This seems satisfactory, but a big problem arises from the imbalance between the observed numbers of baryons and antibaryons, which would otherwise have been assumed to be equal. Up to date, actual cosmological observations suggest that in our universe the number of baryons is far beyond the number of antibaryons, this brings people to the very theoretical challenge of the very important question "how did we manage to beat out our antimatter twins?". For many people in olden times, such a subtle question may seem purely philosophical or even of a religious nature, but in fact modern physics already has some clue. Sakharov in 1967 [9] listed three ingredients that are essential for the baryogenesis to happen. Let us here briefly present the three conditions with a little interpretation for each.

1. There must exist transitions that are baryon number violating. This is a fundamental condition for baryogenesis, allowing baryon number to vary in time.

2. C and CP symmetry breaking has to appear. This condition is based on the condition 1. With N and \overline{N} referring to the baryon and antibaryon states, let us imagine that the baryon number violating transitions such as $N \to f$ do occur, then their CP conjugate partners $\overline{N} \to \overline{f}$ would happen as well with exactly the same strengths if CP symmetry is strictly respected. If so, the numbers of N and \overline{N} would decrease at the same rate, and so cannot yield baryogenesis.

3. The baryon number violating and CP violating transitions must proceed out of thermal equilibrium. To understand this condition, we just need to recall that an equilibrium state does not vary in time, so that the T transformation becomes irrelevant in equilibrium and has to be symmetric in any transition. If so, the necessary requirement of CP violation would then suggest CPT combined violation. As will be discussed later, CPT combined symmetry is based on very firm footing, and has been confirmed by many experiments up to date, thus allowing for CPT symmetry breaking is a way too big price to pay. So, instead of violating the CPT theorem, we'd better stick with the validity of condition 3.

From the Sakharov's conditions above, one knows that, instead of being a disaster, CP violation is actually a very important ingredient in understanding the origin of the universe, and thus it deserves thorough investigation. As will be seen in later chapters, even though the SM does allow for the existence of CP violation via the so-called Cabibbo-Kobayashi-Maskawa (CKM) mechanism, it does not seem to yield a sufficient "amount" of CP violation. This forces us to broaden our scope to search for more possible sources of CP violation that are beyond the Standard Model (BSM), which is the main motivation of the projects to be discussed in my thesis.

Another motivation to measure CP-violating processes is that many extensions of the Standard Model provides new sources of CP violation. These sources often allow for significant deviations from the Standard Model predictions. Consequently, CP violation provides an excellent probe of new physics.

To make the arrangement of the thesis more transparent, here let me make a brief outline of the thesis.

Chapter 2 serves as an introduction, which aims to introduce the basic definitions and the fundamental properties of P, C, and T transformations. Within each section of Chapter 1, I shall discuss the relevant contents according to the chronological steps of the progress in physics, that is, first in classical mechanics, then in nonrelativistic quantum mechanics, and finally in quantum field theory (QFT). These theoretical backgrounds are mainly based on [10]. I am hoping that with these theoretical preparations the more detailed discussions in later chapters could be more easily understood.

Chapter 3 will touch on some formal analysis of CP violation. In this chapter, I will discuss the discovery of CP violation, and the theoretical mechanisms of CP violation in the SM. In the same chapter, I will also take a brief review of the on-going experimental works in searching for CP violation, such as searches for permanent electric dipole moments, and the so-called "D-term" and "R-term" searches in β decay. At the end of this chapter, I will discuss a new possible source of CP violation. Such a new source is realized via a triple-product correlation in momenta in radiative β decay, which is one of the core tasks of the thesis. More theoretical details about this will be explicitly presented in the next few chapters.

Chapter 4 is to discuss some detailed $\mathcal{O}(\alpha)$ calculations of the branching ratios in the special case of neutron radiative β decay. Although in this chapter we will just focus on the CP-even part, such work is still important because it is directly coupled to our later work on CP violation. In this chapter, I will also briefly discuss some $\mathcal{O}(\alpha^2)$ corrections to the branching ratio result since such higher order corrections can be relevant for currently on-going precision measurements of the decay rate.

In Chapter 5, I will briefly discuss the fundamental ideas of chiral perturbation theory (ChPT), which serves as an effective field theory of the fundamental QCD theory at sufficiently low energies, because our concrete work is dependent on a specific part of ChPT. I am hoping with the very brief discussion on ChPT in Chapter 4, at least the fundamental ideas of ChPT can be conveyed and our later work would

not seem too abrupt.

Following the introduction of ChPT, in Chapter 6 there will be detailed analysis of how a CP-violating triple-product correlation in momenta arises, and what kind of BSM model could give rise to such a new source of CP violation.

In Chapter 7, I will turn to another possible BSM model that is based on Lee and Yang's 1956 work [3], which is still popular today. This would be an interesting reinvention of their original work that we have shown being able to give rise to a possible CP violation via the similar triple-product correlation in momenta that is controlled by new combinations of coefficients that were introduced in Lee and Yang's original paper.

In Chapter 8, I will make a full investigation of the T-odd mimicking effect, which arises in the SM due to final-state interactions (FSI), and serves as a background in the future measurements of the CP violation that is proposed in Chapter 6. In order to obtain better-informed constraints of the possible mechanism in Chapter 6, we need to figure out the exact size of such a FSI-induced T-odd mimicking effect.

Chapter 9 is to perform an extension from neutron radiative β decay to more general nuclear cases, which allow us much more and perhaps better candidates for the relevant experiments. We shall show that such an extension requires minimal adjustments of the formulae that we have developed in the neutron case.

Finally, Chapter 10 is to summarize the content of the whole thesis, and to discuss some other work that awaits us in the near future.

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Chapter 2 Fundamentals of C, P, and T Discrete Symmetries

2.1 P, C, and T in Classical Mechanics

Following the treatment of Bigi and Sanda's textbook [10], we shall begin with the explicit definitions of parity (P), charge conjugation (C), and time reversal (T). The fundamental definition of parity transformation (P) is to change a space coordinate \mathbf{x} into $-\mathbf{x}$, which can be realized by a mirror reflection followed by a rotation of 180° around some relevant axis. The fundamental definition of charge conjugation (C) is to reverse the sign of a charge and flip spin. The fundamental definition of time reversal is to reflect the time t to -t while leaving \mathbf{x} unchanged. Followed by these basic definitions, we have the following parity and time reversal transformations for 3-momentum \mathbf{p} and angular momentum $\mathbf{l} \equiv \mathbf{r} \times \mathbf{p}$:

$$\begin{array}{ll} \mathbf{p} \xrightarrow{\mathrm{P}} -\mathbf{p}, & \mathbf{l} \xrightarrow{\mathrm{P}} \mathbf{l} \\ \mathbf{p} \xrightarrow{\mathrm{T}} -\mathbf{p}, & \mathbf{l} \xrightarrow{\mathrm{T}} -\mathbf{l}. \end{array}$$
(2.1)

In the examples of Eq. (2.1), we see that some vectors like \mathbf{p} change their signs under parity transformation, such P-odd vectors are called polar vectors; on the other hand, some vectors like \mathbf{l} do not change signs under parity transformation, such Peven vectors are called axial vectors or, in many books, pseudovectors. Apparently, objects like $\mathbf{p}_1 \cdot \mathbf{p}_2$, where \mathbf{p}_1 and \mathbf{p}_2 are both 3-momentum, are P-even and are called scalars; objects like $\mathbf{p} \cdot \mathbf{l}$ are P-odd and are called pseudoscalars.

The opinion of parity being a symmetry in classical mechanics first appears as an intuition from everyday experience. Take me for instance, to have a healthy life style, I push myself to go to the gym of University of Kentucky every day. There are huge mirrors covering the whole wall right in front of the running machine that I usually use everyday. This allows me to, while I was running, enjoy watching the TV that is actually behind me. The TV show I see in the mirror look exactly same as in the real one. This happens simply because everything I see in the mirrors with left and right switched still agrees with my everyday experiences in the real world, thus I have no way to distinguish them.

Such a mirror-reflection equivalence is called parity conservation or parity invariance in classical mechanics, and can be mathematically summarized as the invariance of the fundamental Newton's equation of motion:

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2},\tag{2.2}$$

together with the basic definition of the force \mathbf{F} :

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.\tag{2.3}$$

As one can easily check, by following the mentioned transformation laws of position \mathbf{x} and momentum \mathbf{p} , Newton's equation of motion is invariant under a parity transformation.

Of course, the situation changes when subtitles appear in the television program. The subtitles I see in the mirror are left-right flipped and very hard to follow, in this way can I tell immediately whether I am watching the TV show in the mirror or not. In other words, the existence of subtitles "breaks" the left-right symmetry of the classical world, giving us a way to distinguish between the real world and the one in a mirror. As we will see in later sections, in the Standard Model (SM), weak interaction plays a role similar to that of subtitles in breaking left-right symmetry in the quantum world.

In electrodynamics, which is governed by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.$$
(2.4)

Here C, P, and T are all involved. First of all, Eq. (2.4) is invariant under the following manifest C transformations:

$$\begin{array}{ll}
\rho \xrightarrow{\mathcal{C}} -\rho, & \mathbf{J} \xrightarrow{\mathcal{C}} -\mathbf{J} \\
\mathbf{E} \xrightarrow{\mathcal{C}} -\mathbf{E}, & \mathbf{B} \xrightarrow{\mathcal{C}} -\mathbf{B}.
\end{array}$$
(2.5)

It can be easily imagined when two opposite charges switch their positions, corresponding to a parity transformation, the electric field \mathbf{E} changes its sign, similar conclusion can be drawn for the electric current vector \mathbf{J} , thus the following parity transformations should hold:

$$\begin{array}{ll}
\rho \xrightarrow{\mathrm{P}} \rho, & \mathbf{J} \xrightarrow{\mathrm{P}} -\mathbf{J} \\
\mathbf{E} \xrightarrow{\mathrm{P}} -\mathbf{E}, & \mathbf{B} \xrightarrow{\mathrm{P}} \mathbf{B},
\end{array} \tag{2.6}$$

note here that the magnetic field **B** does not change its sign under P. This is because $\mathbf{B} = \nabla \times \mathbf{A}$, where **A** is the vector potential. Thus **B** is said to be an axial vector, and it does not change sign under P. One can easily check that Maxwell's equations, Eq. (2.4), are P invariant, in other words, electrodynamics respects parity symmetry!

As for time reversal T, it is natural to expect the vector current \mathbf{J} and the magnetic field \mathbf{B} to reverse direction, whereas charge and the electric field \mathbf{E} remain invariant, then we have:

$$\begin{array}{ll}
\rho \xrightarrow{\mathrm{T}} \rho, & \mathbf{J} \xrightarrow{\mathrm{T}} -\mathbf{J} \\
\mathbf{E} \xrightarrow{\mathrm{T}} \mathbf{E}, & \mathbf{B} \xrightarrow{\mathrm{T}} -\mathbf{B}.
\end{array}$$
(2.7)

Under the combined T transformations, Maxwell's equations (2.4) are found to be T invariant.

2.2 P, C, T in Nonrelativistic Quantum Mechanics

In the quantum world, the states of a microscopic system are described by the wave functions $|a\rangle$, $|b\rangle$, ..., which are often directly referred to as the quantum states. Starting from the beginning of the last century, to deal with the problems in the microscopic system, people finally developed a successful theoretical framework: quantum mechanics, which serves as the foundation of the modern physics. A very basic concept of quantum mechanics is the superposition principle, which says that if the states $|a\rangle$ and $|b\rangle$ are vectors in a Hilbert space, so are the states $|\psi\rangle = \alpha |a\rangle + \beta |b\rangle$ and $|\psi'\rangle = \alpha' |a\rangle + \beta' |b\rangle$. An operation acting on the quantum states is represented as an operator \mathcal{O} with:

$$|\psi\rangle \xrightarrow{\mathcal{O}} \mathcal{O} |\psi\rangle; \quad \langle\psi| \xrightarrow{\mathcal{O}} \langle\psi| \mathcal{O}^{\dagger}.$$
 (2.8)

If such an operation represents a symmetry, we should expect the following equality to hold:

$$|\langle \psi'| \mathcal{O}^{\dagger} \mathcal{O} |\psi\rangle|^{2} = |\langle \psi'|\psi\rangle|^{2}, \qquad (2.9)$$

such that no outcome of a measurement is influenced. Apparently, Eq. (2.9) can be satisfied if:

$$\langle \psi' | \mathcal{O}^{\dagger} \mathcal{O} | \psi \rangle = \langle \psi' | \psi \rangle \implies \mathcal{O}^{\dagger} \mathcal{O} = \mathbf{I},$$
 (2.10)

in such a case, \mathcal{O} is called unitary operator, satisfying $\mathcal{O}^{\dagger} = \mathcal{O}^{-1}$ according to Eq. (2.10), and parity P is in fact one of this kind. Of course, as one may have realized, the Eq. (2.10) is not necessarily the only solution to Eq. (2.9). There exist another solution:

$$\langle \psi' | \mathcal{O}^{\dagger} \mathcal{O} | \psi \rangle = \langle \psi' | \psi \rangle^{*}.$$
 (2.11)

An operator that satisfies Eq. (2.11) is called anti-unitary operator, which acts on a quantum state $|\psi\rangle = \alpha |a\rangle + \beta |b\rangle + \dots$ in such a way:

$$\mathcal{O} |\psi\rangle = \alpha^* \mathcal{O} |a\rangle + \beta^* \mathcal{O} |b\rangle + \dots$$
(2.12)

It turns out that both P and C are unitary operators, satisfying Eq. (2.10), but T is anti-unitary, satisfying Eq. (2.11), as will be seen in a later section. Here, taking P for instance, nonrelativistic quantum mechanics can be described by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi; t\rangle = H |\psi; t\rangle$$
$$H = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}), \qquad (2.13)$$

where $\hat{\mathbf{p}}$ and $\hat{\mathbf{x}}$ are the momentum and position operators, respectively. In the framework of quantum mechanics, if an operator is to represent a symmetric operation that leaves the system invariant, this operator is said to be conserved, and should commute with H. In regards to P conservation, we expect:

$$[\mathbf{P}, H] = 0. \tag{2.14}$$

Thus the principle of quantum mechanics tells us that if $|\psi; t\rangle$ is a solution of Eq. (2.13), so is P $|\psi; t\rangle$. For this conclusion to be correct, we naturally need the following relation to hold:

$$\mathbf{P}^{-1}i\mathbf{P} = i,\tag{2.15}$$

P being an unitary operator then stands.

The simple way to obtain the behavior of observables under P is to apply the correspondence principle and to require the expectation value of the position operator $\hat{\mathbf{x}}$ to change sign under P:

$$\langle \psi; t | \hat{\mathbf{x}} | \psi; t \rangle \xrightarrow{\mathbf{P}} \langle \psi; t | \mathbf{P}^{\dagger} \hat{\mathbf{x}} \mathbf{P} | \psi; t \rangle = - \langle \psi; t | \hat{\mathbf{x}} | \psi; t \rangle.$$
(2.16)

This is guaranteed to happen if:

$$\mathbf{P}^{\dagger}\hat{\mathbf{x}}\mathbf{P} = -\hat{\mathbf{x}} \quad \text{or} \quad \{\hat{\mathbf{x}}, \mathbf{P}\} = 0, \tag{2.17}$$

where the curly bracket in Eq. (2.17) denotes the anticommutator. Apparently, if we apply the parity operation in Eq. (2.17) twice, we should get:

$$\mathbf{P}^2 = 1 \implies \mathbf{P}^\dagger = \mathbf{P}^{-1} = \mathbf{P}.$$
 (2.18)

Besides the operator $\hat{\mathbf{x}}$, the behavior of the momentum operator $\hat{\mathbf{p}}$ is also important, and can be easily identified via the following reasoning. In both classical and quantum mechanics, the momentum operator is understood as the generator of infinitesimal spatial translations $d\mathbf{x}$:

$$\mathcal{T}(d\mathbf{x}) \simeq 1 + \frac{i}{\hbar} \hat{\mathbf{p}} \cdot d\mathbf{x}.$$
 (2.19)

Obviously, a translation that is followed by a parity transformation is equivalent to a parity reflection that is followed by a translation in the opposite direction, say, $\mathcal{T}(-d\mathbf{x})$:

$$P\mathcal{T}(d\mathbf{x}) = \mathcal{T}(-d\mathbf{x})P, \qquad (2.20)$$

inserting Eq. (2.19) into Eq. (2.20) then gives:

$$P(1 + \frac{i}{\hbar}\hat{\mathbf{p}} \cdot d\mathbf{x}) \simeq (1 - \frac{i}{\hbar}\hat{\mathbf{p}} \cdot d\mathbf{x})P, \qquad (2.21)$$

thus we have

$$\mathbf{P}^{\dagger}\hat{\mathbf{p}}\mathbf{P} = -\hat{\mathbf{p}} \quad \text{or} \quad \{\hat{\mathbf{p}}, \mathbf{P}\} = 0, \tag{2.22}$$

With Eq. (2.17) and Eq. (2.22), it is not hard to show that the quantization condition:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{2.23}$$

is invariant, which is encouraging because the uncertainty relation Eq. (2.23) serves as the very basis of quantum mechanics and is not supposed to change under our construction of the parity operator. With Eq. (2.17) and Eq. (2.22), one can get the behavior of the angular momentum $\hat{\mathbf{J}} \equiv \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ under parity transformation:

.

$$\mathbf{P}^{\dagger} \hat{\mathbf{J}} \mathbf{P} = \hat{\mathbf{J}} \quad \text{or} \quad [\hat{\mathbf{J}}, \mathbf{P}] = 0.$$
(2.24)

Another important conclusion that is worth knowing is that for the Hamiltonian in Eq. (2.13), if the potential part $V(\mathbf{x})$ is parity even such that:

$$V(-\mathbf{x}) = V(\mathbf{x}),\tag{2.25}$$

and assuming the state $\psi(\mathbf{x}, t)$ is a solution to Eq. (2.13), then it can be shown that $\psi(-\mathbf{x}, t)$ is also a solution. This means that the superposition $\psi_{\pm}(\mathbf{x}, t) \equiv \psi(\mathbf{x}, t) \pm \psi(-\mathbf{x}, t)$ is also a solution, i.e. for a parity even potential we can express all solutions as eigenstates of parity: $\psi_{+}(\mathbf{x}, t)$ for parity even and $\psi_{-}(\mathbf{x}, t)$ for parity odd.

For elementary particles or fields one can define also an intrinsic parity. If the relevant forces, strong forces for instance, are experimentally confirmed to conserve parity, one can then assign an intrinsic parity as determined by the strong forces that produce the particle in question, and then use it to constrain or test other kinds of forces that may be also involved.

Let us now look at the charge conjugation C. The electromagnetic field can enter the Schrödinger equation, Eq. (2.13), via the so-called minimal electromagnetic coupling. In the presence of the electromagnetic field, the Hamiltonian of a charged particle can be written as:

$$\hat{H} = \frac{(\hat{\mathbf{p}} - e\hat{\mathbf{A}})^2}{2m} + e\phi, \qquad (2.26)$$

where we are following the Natural Unit System, in which the light speed "c" is taken as unity, and e in Eq. (2.13) is the charge of the particle. Note that **A** is the vector potential, and ϕ is the electric potential. Under C, we have the following combined transformations:

$$e \xrightarrow{\mathrm{C}} -e, \quad \mathbf{A} \xrightarrow{\mathrm{C}} -\mathbf{A}, \quad \phi \xrightarrow{\mathrm{C}} -\phi.$$

Obviously the Hamiltonian, Eq. (2.26), is invariant under C. Again, just like the case of P, applying C twice should get us back to the original charge configuration. Thus, with proper choice of phase convention, we have:

$$C^2 = 1 \implies C^{\dagger} = C^{-1} = C. \tag{2.27}$$

Finally, let us turn to the case of time reversal T, which is not a trivial repetition of the previous P and C discussion. With the fundamental definition of T and the analogous discussion of Eq. (2.17), it is not hard to deduce the following properties of T:

$$[\hat{\mathbf{x}}, \mathbf{T}] = 0, \ \{\hat{\mathbf{p}}, \mathbf{T}\} = 0, \ \{\mathbf{J}, \mathbf{T}\} = 0,$$
 (2.28)

These at first seem to be OK, but a serious problem arises after more careful inspection. Apparently, after applying T to the commutation relation of Eq. (2.23), it seems to be destroyed! As mentioned before, the quantization condition (2.23) is accepted as the most important foundation of quantum mechanics, it is not supposed to be perturbed by our definition of the T operation. The whole problem lies in our naive assumption of T being unitary. If we instead require T being anti-unitary, such that, according to (2.12):

$$\mathbf{T}^{\dagger}i\mathbf{T} = -i, \tag{2.29}$$

then the quantization condition (2.23) is restored. One more satisfactory thing about T being anti-unitary is that we can then easily confirm that the Schrödinger equation (2.13) is T invariant. From now on, we shall keep in mind that P and C are classified as unitary transformations, but T is classified as an anti-unitary transformation. One test of T invariance in the microscopic world can be realized by seeing if a reaction:

$$a + b \to c + d, \tag{2.30}$$

and its reverse

$$c + d \to a + b, \tag{2.31}$$

occur with equal probability in nature. Such a test is called detailed balance.

Apparently, in high–energy particle processes such as Eqs. (2.30), (2.31), creation and annihilation of different kinds of particles are always involved, and nonrelativistic quantum mechanics is not powerful enough to deal with such cases. This brings us to the discussion of quantum field theory, which has been developed for this purpose.

2.3 P, C, T in Quantum Field Theory

As mentioned in the previous section, the frame work of quantum field theory has been developed since the 1930's to handle processes with high–energy particles. The foundation of quantum field theory is based on

1. the principle of special relativity;

2. the extension of the principle of quantum mechanics from a single particle to classical fields with infinite degrees of freedoms.

In canonical quantization, one introduces relevant creation and annihilation operators, and they satisfy the same commutation relations as the usual quantum theory. These creation and annihilation operators represents the fields that are created or annihilated in the physical vacuum, which people have realized possesses much richer structure than a trivial state of "nothing". The validity of the framework of quantum field theory has been confirmed by various modern particle experiments, and quantum field theory is the standard language in which the Standard Model is described. After almost a century's hard work by generations of the most gifted physicists, the SM has been established and proven to be a self-consistent, successful theory which yields many satisfactory theoretical predictions in perfect agreement with experiments.

Despite the many successes of SM, people are also aware of its limits and incapabilities, in which CP violation is an important component. We shall talk more about this in a later chapter. As for P, C, and T in quantum field theory, this thesis is not aiming to repeat all of the details, since many of those are not closely related to my projects. One can go to Bigi and Sanda's textbook [10] for more a detailed and thorough introduction.

In the SM the fundamental fields are:

1. fermions - the matter fields such as e^{\pm} , μ^{\pm} , ν_e , ν_{μ} , ...;

2. gauge bosons - the particles that convey forces such as γ , W^{\pm} , Z^{0} , and g;

3. Higgs boson - the long proposed but only newly discovered particle that serves as an important ingredient in the SM to give rise to other particles masses via the mechanism of spontaneous symmetry breaking, which can give all particles mass without breaking gauge invariance.

Among these different fields in the SM, to save some space and time, I may only focus on the fermion case 1, because they are the building blocks of matter. The logics and procedures in the bosons cases 2 and 3 are not that different with 1.

As we know, the discussion of spin-1/2 fermion fields naturally involves representation of the Dirac matrices. Although these choices do not affect the final results of physical observables, I would still like to make this part explicit before—hand for later convenience. Our choices can be listed as follows: we adopt the Natural Unit System, in which we have set both the speed of light and the reduced Planck's constant to unity:

$$c = \hbar = 1. \tag{2.32}$$

Following the convention of Ref. [10], we write 4-vectors with upper and lower indices, with the metric tensor $g_{\mu\nu}$ defined by:

The 4-vectors with upper or lower indices in our convention take the following general form:

$$v^{\mu} = (v^{0}, \mathbf{v}), \quad v_{\mu} = g_{\mu\nu}v^{\nu} = (v^{0}, -\mathbf{v}),$$
 (2.34)

which gives, as an explicit example, the 4-momentum $p^{\mu} = (E, \mathbf{p})$ and $p_{\mu} = (E, -\mathbf{p})$. The covariant derivatives are defined as follows:

$$\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \quad \partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right). \tag{2.35}$$

Following the standard principle in quantum mechanics, one makes the replacement $p^{\mu} \rightarrow i\partial^{\mu}$ in coordinate space for quantization. As for the Dirac matrices, we choose the Dirac-Pauli representation:

$$\gamma^{0} = \begin{pmatrix} \mathbf{I} & 0\\ 0 & -\mathbf{I} \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i}\\ -\sigma^{i} & 0 \end{pmatrix}, \quad (2.36)$$

where I refers to the 2 by 2 unit matrix, and σ^i (i = 1, 2, 3) refer to the Pauli matrices:

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
 (2.37)

Finally, besides the four γ matrices γ^{μ} ($\mu = 0, 1, 2, 3$), there is another very important matrix γ^{5} with:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} = \gamma_5.$$
(2.38)

It can be easily shown that γ_5 has the following properties:

$$\gamma_5 = \gamma_5^{\dagger}$$

 $(\gamma^5)^2 = 1,$
 $\{\gamma^5, \gamma^{\mu}\} = 0.$ (2.39)

As one has learned, the special matrix γ^5 plays an important role in quantum field theory.

In quantum field theory, a physical system is described by a corresponding Lagrangian. For a free spin-1/2 fermion field, the relevant Lagrangian reads:

$$\mathcal{L} = \overline{\psi}(t, \mathbf{x})(i\gamma_{\mu}\partial^{\mu} - m)\psi(t, \mathbf{x}), \qquad (2.40)$$

which yields the expected Dirac equation, and the associated Dirac current is given by

$$J^{\mu} = \overline{\psi}(t, \mathbf{x})\gamma^{\mu}\psi(t, \mathbf{x}).$$
(2.41)

Upon performing canonical quantization in quantum field theory, the fermion spinor field $\psi(t, \mathbf{x})$ in Eq. (2.40) and Eq. (2.41) is no longer a wave function as in the nonrelativistic quantum mechanics, but is promoted to an operator, which contains both creation and annihilation operators in order to create and annihilate fermions. More explicitly we have:

$$\psi(t, \mathbf{x}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \sum_{s=\pm} [\mathbf{b}(\mathbf{p}, s)u(\mathbf{p}, s)e^{-i\mathbf{p}\cdot x} + \mathbf{d}^{\dagger}(\mathbf{p}, s)v(\mathbf{p}, s)e^{i\mathbf{p}\cdot x}], \quad (2.42)$$

where $\mathbf{b}[\mathbf{b}^{\dagger}]$ and $\mathbf{d}[\mathbf{d}^{\dagger}]$ denote annihilation [creation] operators, respectively, for fermions. The commutation relations of $\mathbf{b}[\mathbf{b}^{\dagger}]$ and $\mathbf{d}[\mathbf{d}^{\dagger}]$ as well as the normalization constants are arranged such that the following postulated anticommutation relations hold:

$$\{\psi_{\alpha}(t, \mathbf{x}), \psi_{\beta}^{\dagger}(t, \mathbf{y})\} = \delta^{3}(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} \{\psi_{\alpha}(t, \mathbf{x}), \psi_{\beta}(t, \mathbf{y})\} = \{\psi_{\alpha}^{\dagger}(t, \mathbf{x}), \psi_{\beta}^{\dagger}(t, \mathbf{y})\} = 0.$$
 (2.43)

Just by following a thinking similar to that in nonrelativistic quantum mechanics, the anticommutation relations in Eq. (2.43) are taken as the most fundamental conditions in the quantization of fermion fields, so that they are not disturbed by the introduction of the operators P, C, and T into the system.

The Dirac spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ are solutions to the Dirac equation in momentum space for a particle and antiparticle respectively:

$$(\not p - m)u(\mathbf{p}, s) = 0 (\not p + m)v(\mathbf{p}, s) = 0.$$

$$(2.44)$$

Solving the Dirac equation in Eq. (2.44) yields the solutions of $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$:

$$u(\mathbf{p},+) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u(\mathbf{p},-) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m} \end{pmatrix}, \quad (2.45)$$
$$v(\mathbf{p},+) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0 \end{pmatrix}, \quad v(\mathbf{p},-) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1 \end{pmatrix}, \quad (2.46)$$

here the polarization direction has been chosen to be along the z axis. Furthermore, it turns out that the following identities are very useful for the later discussion of P,C, and T in this section. Let me just present them here directly; the proofs are pretty straightforward by just following the γ matrices defined in Eq. (2.36) together with Eq. (2.37), and the solutions of the Dirac equation we have given. These identities are:

$$u(-\mathbf{p}, s) = \gamma_0 u(\mathbf{p}, s)$$

$$v(-\mathbf{p}, s) = -\gamma_0 v(\mathbf{p}, s)$$

$$i \gamma^2 u(\mathbf{p}, s)^* = s v(\mathbf{p}, s)$$

(2.47)

$$i\gamma^{2}u(\mathbf{p},s)^{*} = sv(\mathbf{p},-s)$$
$$i\gamma^{2}v(\mathbf{p},s)^{*} = -su(\mathbf{p},-s)$$
(2.48)

$$\gamma_1 \gamma_3 u(\mathbf{p}, s)^* = -su(-\mathbf{p}, -s)$$

$$\gamma_1 \gamma_3 v(\mathbf{p}, s)^* = -sv(-\mathbf{p}, -s)$$
(2.49)

We now turn to the consideration of P, C, and T. Let us first consider parity P. Following the classical definition of P – changing the spatial variable \mathbf{x} to $-\mathbf{x}$, together with the expected behavior of a 4-vector – reversing its sign of its spatial components, one then naturally expects the following identity for the fermion vector current of Eq. (2.41) to hold:

$$\mathbf{P}J^{\mu}(t,\mathbf{x})\mathbf{P}^{\dagger} = J_{\mu}(t,-\mathbf{x}), \qquad (2.50)$$

which can be easily confirmed with the help of Eq. (2.34). To make this happen, one notices the following special identity:

$$\gamma_0 \gamma^\mu \gamma_0 = \gamma_\mu, \tag{2.51}$$

and introduces the following parity transformation P of the fermion field operator accordingly:

$$\psi^{\mathrm{P}}(t,\mathbf{x}) \equiv \mathrm{P}\psi(t,\mathbf{x})\mathrm{P}^{\dagger} = \gamma_{0}\psi(t,-\mathbf{x}), \qquad (2.52)$$

which then correspondingly yields:

$$\overline{\psi}^{\mathrm{P}}(t,\mathbf{x}) \equiv \mathrm{P}\overline{\psi}(t,\mathbf{x})\mathrm{P}^{\dagger} = \overline{\psi}(t,-\mathbf{x})\gamma_{0}.$$
(2.53)

Apparently the introduction of P in such a manner is consistent with the natural expectation of Eq. (2.50), and it also keeps the anticommutation relations of Eq. (2.43) unchanged. Thus we have just obtained an pretty convincing transformation law for P. Next we shall apply the operation that we have just obtained on the explicit form of the fermion field operator $\psi(t, \mathbf{x})$, which is expanded in terms of the creation and annihilation operators, as shown in Eq. (2.42). Here the logic is that we want to see how the creation and annihilation operators transform if we impose the parity transformations of $\psi(t, \mathbf{x})$, Eq. (2.52) and Eq. (2.53). It is a little tedious but straightforward to show that, using Eq. (2.42), Eq. (2.47), Eq. (2.52), and Eq. (2.53), the following relations have to hold:

$$P\mathbf{b}(\mathbf{p},s)P^{\dagger} = \mathbf{b}(-\mathbf{p},s), \quad P\mathbf{d}(\mathbf{p},s)P^{\dagger} = -\mathbf{d}(-\mathbf{p},s), \quad (2.54)$$

from which one can immediately observe a very interesting thing: the fermions and antifermions carry opposite intrinsic parity.

One important property here is that without any interaction presented in the Lagrangian, the free-fermion states are parity conserving:

$$P\mathcal{L}(t, \mathbf{x})P^{\dagger} = \mathcal{L}(t, -\mathbf{x}).$$
(2.55)

It turns out that even when the interaction of QED or QCD are included via the principle of local gauge invariance, the total Lagrangian is still invariant under P. In other words, parity conservation is respected in both the electromagnetic and strong interactions. The actual proofs are based on procedures similar to what we have shown here, and one can show that the quanta of spin-1 boson fields, such as photon, must have odd intrinsic parity. Although it can be theoretically shown that E&M and strong interactions are P conserving, the weak interaction is NOT – we shall turn to this issue in the next chapter.

As for the charge conjugation operator C, which changes a particle into its antiparticle, one would naturally require the fermion current $J^{\mu}(t, \mathbf{x})$, which is often charged, to change its sign:

$$CJ^{\mu}(t, \mathbf{x})C^{\dagger} = -J^{\mu}(t, \mathbf{x}).$$
(2.56)

Just like what we did for P, we now need to "invent" some explicit form of the operation to represent C such that Eq. (2.56) is satisfied. The following C transformation is a satisfactory option:

$$\psi^{\rm C}(t,\mathbf{x}) \equiv {\rm C}\psi(t,\mathbf{x}){\rm C}^{\dagger} = i\gamma^2\gamma^0\overline{\psi}^{\rm tr}(t,\mathbf{x}), \qquad (2.57)$$

where "tr" refers to "transpose". Although obtaining Eq. (2.57) is less obvious than in the case of parity, Eq. (2.52) and Eq. (2.53), the validity of Eq. (2.57) can be confirmed quickly by inserting it back into the basic fermion current definition, Eq. (2.41),

and making use of Eq. (2.48). With this introduced C transformation for fermion fields, one can also easily check that the fundamental anticommutation relations in Eq. (2.43) are unchanged, which convinces us that Eq. (2.57) is an ideal option for the charge–conjugation transformation of fermion fields. The next step is also similar to the case of P: we shall enforce the validity of Eq. (2.57), and see how the creation and annihilation operators in Eq. (2.42) behave under C. The derivation is just as straightforward as in the case of P, by inserting Eq. (2.57) into Eq. (2.56), and making use of (2.48), one finally obtains:

$$C\mathbf{b}(\mathbf{p},s)C^{\dagger} = s\mathbf{d}(\mathbf{p},-s), \qquad (2.58)$$

which tells us under charge conjugation, a fermion is changed into its antipartner with opposite spin configuration. Also, it can be checked immediately that without the presence of interactions, the Lagrangian of fermion fields is invariant under C:

$$C\mathcal{L}(t, \mathbf{x})C^{\dagger} = \mathcal{L}(t, \mathbf{x}).$$
(2.59)

It can be further checked that both electromagnetic and strong interactions respect C invariance.

Let us finally concentrate on the time reversal case, which, as we have discussed before, should be anti-unitary. Following the steps of P and C, we now shall find an explicit operation T such that for the T transformed fermion field operator $\psi(t, \mathbf{x})$:

$$\psi^{\mathrm{T}}(t, \mathbf{x}) \equiv \mathrm{T}\psi(t, \mathbf{x})\mathrm{T}^{-1} = U\psi(-t, \mathbf{x}), \qquad (2.60)$$

the electromagnetic current transforms as:

$$TJ^{\mu}(t, \mathbf{x})T^{-1} = J_{\mu}(-t, \mathbf{x}),$$
 (2.61)

which follows the fundamental definitions of T and the 4-vectors. Inserting $\psi^{T}(t, \mathbf{x})$ into such an expected equation above, we have:

$$TJ^{\mu}(t, \mathbf{x})T^{-1} = \overline{\psi}(-t, \mathbf{x})U^{\dagger}\gamma^{\mu*}U\psi(-t, \mathbf{x}).$$
(2.62)

Thus we must search for a matrix U to yield:

$$U^{\dagger}\gamma^{\mu*}U = \gamma_{\mu}. \tag{2.63}$$

It takes a few guesses, and one finds that an special identity:

$$U = \gamma^1 \gamma^3 \tag{2.64}$$

does the job, and the fundamental quantization conditions of Eq. (2.43) remain unchanged as well. This shows the validity of our construction of the T operation. Enforcing the validity of Eq. (2.64), we can show that the creation and annihilation operators have to have the following behavior:

$$T\mathbf{b}(\mathbf{p},s) = s\mathbf{b}(-\mathbf{p},-s), \quad T\mathbf{d}(\mathbf{p},s) = s\mathbf{d}(-\mathbf{p},-s), \tag{2.65}$$

which tells us that under time reversal transformation, both a fermion and its antiparticle change the sign of its 3-momentum and spin polarization. Furthermore, it can be checked immediately that without the presence of interactions, the Lagrangian of fermion fields is invariant under T:

$$T\mathcal{L}(t, \mathbf{x})T^{-1} = \mathcal{L}(-t, \mathbf{x}).$$
(2.66)

It can be further checked that, as one would expect, both electromagnetic and strong interactions respect T invariance:

$$T\mathcal{L}(t, \mathbf{x})T^{-1} = \mathcal{L}(-t, \mathbf{x}).$$
(2.67)

Note here we have not considered the possible θ -term in the QCD Lagrangian. In principle such a term can appear in the QCD Lagrangian, and breaks the T invariance of strong interaction. I postpone more detailed discussion on this issue in a later chapter.

This brings us to the end of the first chapter. To sum up, in this chapter we defined the discrete transformations of parity P, charge conjugation C, and time reversal T. We discussed some of the important properties of P, C, and T in classical mechanics, nonrelativistic quantum mechanics, and quantum field theory. We mean to stress that, within the framework of SM, one is able to show strictly that both the electromagnetic and the strong interactions respect P, C, and T symmetries simultaneously. However, just as mentioned at the beginning of this chapter, one shall not take the P, C, and T symmetries, single or combined, for granted all the time. Sometime nature "disobeys" our expectations or simple thinking in an unexpected, fantastic way! When such disagreements happen, it is a good news because then we have an opportunity to realize a better and deeper understanding of nature! This brings us to Chapter 3.

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Chapter 3 CP Violation in the SM and Beyond

3.1 Discovery of CP Violation in Kaon Decay

As mentioned in Chapter 1, the intuition of parity being a symmetry arises from our everyday experiences, which are dominated by classical mechanics. One is also allowed to confirm such intuition theoretically by checking the invariance of Newton's equation of motion, Eq. (2.2), and Maxwell's equations, Eq. (2.4). Based on this doubly confirmed belief in P symmetry, people asserted that parity conservation is always respected unconditionally! Such a robust assertion turned out to be broken by nature through the weak interaction, about which physicists didn't gain real knowledge until the middle of the 1950s. The story of the discovery of parity violation has become a classic of modern physics. Thus I think it wouldn't be a waste of time to review very briefly this part of history, and reappreciate the wonderful era of the middle of the 20^{th} century.

Shortly after the beginning of the 20th century, along with the rapid development of the cloud chamber, attributed to Charles T. Wilson (1869–1959) [11], designed to record tracks of many kinds of incoming particles or their decay products, via the so-called V-shaped tracks. Thus in many high energy physics (HEP) experiments physicists are then able to obtain a lot of important information about particles under study. In October 1946, instead of carrying out a new HEP experiment in their lab, Rochester and Butler decided to expose their cloud chamber directly to the sky, where people now know is full of many kinds of cosmic rays. After carefully inspecting and analyzing the information that was recorded in their cloud chamber, they realized that they had observed, from the cosmic rays, the decay of a new kind of particle with a mass of 435 MeV into two lighter particles with equal masses around 100 MeV, or in today's language:

$$K^0 \to \pi^+ \pi^-. \tag{3.1}$$

Although at the time Rochester and Butler were carrying their observations, neither the kaon nor pion was known to physicists, it actually did not take too long to discover the charged pion in another experiment, in May 1947! Following the identification of pion in the same year, Rochester and Butler reported their observation, including another exotic process besides (3.1):

$$K^+ \to \pi^+ \pi^0. \tag{3.2}$$

These early objects were studied again at the Brookhaven National Lab (BNL) in 1953 and at the Berkeley Lab in 1955. Besides the observations of the processes in Eqs. (3.1) and (3.2), more observations were made, such as:

$$\Lambda \to \pi^- p, \tag{3.3}$$

where the Λ is now known as one of hyperons, with a mass a bit larger than the proton. The most important reason I am mentioning the Λ decay (3.3) here is that

the observations of the process in (3.3) led to the introduction of a new quantum number called "strangeness," which is carried by a whole family of particles called "strange particles." Kaons, and the Λ we mentioned are just members of this family. More explicitly, sets of experiments on Λ hyperon showed a puzzle: the production rate for Λ seemed to greatly exceed its decay rate. It was later deduced, attributed to Pais in 1952 [12], that the production of Λ is via the strong interaction, which has a very big reaction cross section, and the decay of Λ is via the weak interaction, which has a much smaller cross section. The puzzle is then solved with a price of introducing a new quantum number "strangeness," which is conserved in the strong interactions but not in the weak interaction! Thus any process that changes the total strangeness will have to be a weak process. By the way, we are spending a little time to introduce the background of kaons because they play an important role in the later discussion of CP symmetry breaking.

Following the introduction of strangeness, the second period of study is characterized by the so-called $\theta - \tau$ puzzle. Two kinds of decays were observed:

$$\begin{array}{l} \theta^+ & \to & \pi^+ \pi^0, \\ \tau^+ & \to & \pi^+ \pi^+ \pi^-. \end{array}$$

$$(3.4)$$

In analyzing sets of θ^+ and τ^+ production data, people deduce that both θ^+ and τ^+ has strangeness +1, and we also know that pion does not have strangeness. Thus the total strangeness changes in the processes in Eq. (3.4). This suggests that the two decay processes in Eq. (3.4) have to be via the weak interaction. Here the puzzle is very easy to describe: it can be shown or experimentally confirmed that in the two decays in Eq. (3.4), the decayed products (pions) are in a configuration of zero angular momentum, which means the angular momentum part contribute trivially to the parity consideration. Thus with the known fact that pion has internal parity -1, one knows the two particles θ and τ have opposite internal parities based on the requirement of parity conservation. With different parities, θ and τ had to be classified as two different particles. Normally this would not be a big issue, but the problem arose when other very precise measurements failed to find any significant difference in mass and lifetime of the two particles $- \infty$ hich is unlikely for two distinct particles. This unpleasant situation was called $\theta - \tau$ puzzle.

Ever since the emergence of the $\theta - \tau$ puzzle, people had tried to solve it without success until 1956, when T. D. Lee and C. N. Yang first brought people's attention to the right track [3]. Simply speaking, their key of success was to doubt the assumption of parity conservation in the weak interaction. Although parity conservation had been confirmed in quantum electro-dynamics (QED) theoretically and experimentally, parity conservation had never been explicitly checked in any experiment at that time. In other words, if they were right, the puzzle could be solved rapidly: parity conservation should not be a requirement in the weak decays, and the so-called " θ^{+} " and " τ^{+} " particles merely represent two allowed decay modes of the same particle, which was later identified as the K^+ meson which is mentioned at the beginning of this chapter. To confirm this guess, in their 1956 paper Lee and Yang suggested a direct experiment in nuclear β decay, a type of weak interaction, to test the parity conservation, or equivalently, the left-right symmetry. Based on their assertion of parity violation, Lee and Yang also proposed an analytic Lagrangian to describe the weak interaction, in the low energy region that was accessible at the time. Here could have been a proper place to present their proposed result, but I shall postpone these details to the next section for convenience. Shortly after Lee and Yang's 1956 paper was published, a famous experiment, using polarized ⁶⁰Co as the source of β decay, was quickly carried out by C. S. Wu and her collaborators at NIST [13]. This experiment followed the same picture proposed by Lee and Yang, as shown in Fig. (3.1), and



Figure 3.1: An illustration of the experiment by Ambler et al. on parity violation in 60 Co β decay, selected from Ref. [1].

was designed to give a fundamental test of the parity symmetry in weak interaction. The result shocked many people at that time, because the experiment confirmed that in weak decays the parity conservation was violated. It was also confirmed in the same experiment that charge conjugation symmetry was also broken. We refer to the original paper [13] for more details and appreciate this amazing era of history. Here we take a quick look at another simple example, serving as a quick exhibition of symmetry breaking of P and C separately, in, namely, pion weak decay [14]:

$$\pi^+ \to \mu^+ \nu_{L,\mu}, \tag{3.5}$$

where $\nu_{L,\mu}$ refers to muon left-handed neutrino. If we apply the parity transformation P on the decay process in Eq. (3.5), we would have expected a corresponding process such as:

$$\pi^+ \to \mu^+ \nu_{R,\mu},\tag{3.6}$$

A problem lies in that in any particle experiment that involves neutrinos up to now, only the left-handed neutrino ν_L , and the right handed antineutrino $\bar{\nu}_R$ are present – nothing like $\nu_{R,\mu}$ ever been found. Thus the expected process in Eq. (3.6), deduced from the assumption of P symmetry, does not actually happen – P symmetry is violated. One could use very similar reasoning to confirm that C symmetry is also violated. For the first time, after thousands of years of accumulated science, people finally entered a new region, where some seemingly valid principles, purely based on people's intuitions and past experiences, should not be taken for granted. Instead, one should use caution, and rely on strict experimental tests before asserting the validity of a symmetry.

As one may have noticed in the π^+ weak decay to illustrate P and C symmetry breaking in last section, there seems to be a combined symmetry of CP – performing the P and C transformations simultaneously yields an expected weak decay process:

$$\pi^- \to \mu^- \bar{\nu}_{R,\mu},\tag{3.7}$$

which does happen in nature, and such kind of phenomena can also be confirmed in many other weak processes! Shall we then assert that although neither P nor C is a symmetry in weak decay, the combined symmetry transformation CP is still respected in weak interaction? The quick answer is still NO! To see more details about this, we shall now look at the case where CP symmetry is violated – kaon nonleptonic weak decays.

Corresponding to the neutral kaon weak decay, Eq. (3.1), there also exists another process:

$$\overline{K^0} \to \pi^- \pi^+, \tag{3.8}$$

where $\overline{K^0}$ refers to the antiparticle of the K^0 meson. Since K^0 and $\overline{K^0}$ carry opposite strangeness, they cannot be the same type of particle, but as one can see in Eqs. (3.1) and (3.8), the decay products are identical! Here an interesting question arises naturally: how can we tell K^0 and its antipartner $\overline{K^0}$ apart? Such a theoretical challenge was taken up successfully by Gell-Mann and Pais through careful quantum mechanical reasoning [15]. Their reasoning starts with temporarily "turning off" the weak interaction, thus K^0 and $\overline{K^0}$ can neither decay nor transform into each other. One can then technically treat them as two orthonormal quantum bases, and construct a wave function:

$$\Psi(t) = a(t) |K^0\rangle + b(t) |\overline{K^0}\rangle \equiv \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \qquad (3.9)$$

which varies according to the free Schrödinger equation for $\Psi(t)$:

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi, \qquad (3.10)$$

where H denotes the free Hamiltonian operator when any additional interaction term is absent:

$$H = \begin{pmatrix} M_K & 0\\ 0 & M_K \end{pmatrix}. \tag{3.11}$$

Now imagine we "turn on" the weak interaction, so that the decay processes in Eqs. (3.1) and (3.8) both come back accordingly, and an additional interaction term that mixes K^0 and $\overline{K^0}$ via the chain $K^0 \to \pi^+\pi^- \to \overline{K^0}$ can enter. We do not really need to know the explicit form of such a mixing term, but only denote it as Δ .

Since we are aware that Δ arises from weak interaction, it is certainly much smaller than the kaon mass term. With the introduction of the mixing term Δ , treated as a perturbation to H, we have:

$$H = \begin{pmatrix} M_K & \Delta \\ \Delta & M_K \end{pmatrix}. \tag{3.12}$$

Using the standard perturbative treatment of quantum mechanics, one can easily check that now the new eigenstates become:

$$|K_S\rangle \equiv \frac{1}{\sqrt{2}} (|K^0\rangle + |\overline{K^0}\rangle),$$

$$|K_L\rangle \equiv \frac{1}{\sqrt{2}} (|K^0\rangle - |\overline{K^0}\rangle).$$
(3.13)

With

$$CP | K^0 \rangle = | \overline{K^0} \rangle, \qquad (3.14)$$

one immediately gets $\operatorname{CP} |K_S\rangle = + |K_S\rangle$, and $\operatorname{CP} |K_L\rangle = - |K_L\rangle$. It is also known that $\operatorname{CP} |\pi^+\pi^-\rangle = + |\pi^+\pi^-\rangle$, thus, assuming CP conservation, only $K_S \to \pi^+\pi^-$ is allowed, whereas $K_L \to \pi^+\pi^-$ is forbidden. The leading decay process for K_L then has to be:

$$K_L \to \pi \pi \pi,$$
 (3.15)

where we note here that the phase space for the decay process in Eq. (3.15) is very restricted. Explicitly, the parent particle K_L has a mass of ~ 500MeV, and the total mass of the decay products is about $3 \cdot M_{\pi} \sim 420$ MeV. This leads to a much smaller allowed phase space than in K_S decay. Thus we expect that the K_S decays much faster than the K_L does. Indeed this is why we call them K_S and K_L , respectively, in the first place, with the subscript "S" denoting "short lived" and "L" denoting "long lived".

The point here is that the unexpected processes $K_L \to \pi^+\pi^-$ does happen, and has been confirmed in several experiments independently [4], [5], [6]. Thus CP conservation is also broken by nature!

3.2 Mechanisms of CP Violation in the SM and Beyond

As mentioned in the last section, the observations of $K_L \to \pi^+\pi^-$ in a series of independent experiments demonstrate that CP symmetry is broken. Is this a disaster? Is it really allowed by the SM? The answer is that CP violation is allowed by the SM. In quantum field theory, the γ matrices play a critical role in the discrete symmetries P, C, and T. As we have seen, the fermion vector current in Eq. (2.41), carrying a factor of " γ^{μ} ", transforms as a 4-vector current. With the introduction of the special matrices " γ^{5} " as shown in Eq. (2.38), we can also have another type of current:

$$J^{\mu 5} = \overline{\psi}(t, \mathbf{x}) \gamma^{\mu} \gamma^{5} \psi(t, \mathbf{x}).$$
(3.16)

With the property of γ^5 , as shown in Eq. (2.39), and the proposed transformation of P, Eq. (2.52), one can show that the current in Eq. (3.16) transforms as a pseudovector.

Based on Eq. (2.41) and Eq. (3.16), one can build the so-called left-handed and righthanded currents (a more thorough analysis of the left-handed and right-handed fields and currents will be discussed in later chapters when we are concentrating on the axial anomaly in QCD):

$$J_L^{\mu} = \overline{\psi}(t, \mathbf{x})\gamma^{\mu} \frac{1 - \gamma^5}{2} \psi(t, \mathbf{x}), \quad J_R^{\mu} = \overline{\psi}(t, \mathbf{x})\gamma^{\mu} \frac{1 + \gamma^5}{2} \psi(t, \mathbf{x}).$$
(3.17)

Using the transformation of P, one can easily check that under P, the left-handed and right-handed currents transform into each other. Apparently the linear superposition of J_L^{μ} and J_R^{μ} with equal weight, $J_L^{\mu} + J_R^{\mu} = J^{\mu}$, yields a vector current, and leaves the fermionic dynamical system symmetric under P, but either J_L^{μ} or J_R^{μ} by itself is not. Based on this simple observation, in order to describe the semi-leptonic neutron and nuclear weak decay, Lee and Yang in their 1956 paper proposed the most general and straightforward effective Hamiltonian of the interaction that meets the basic requirement of parity violation [3]:

$$H_{\text{int}} = (\overline{\psi}_{p}\psi_{n})(C_{S}\overline{\psi}_{e}\psi_{\nu} - C_{S}'\overline{\psi}_{e}\gamma_{5}\psi_{\nu}) + (\overline{\psi}_{p}\gamma^{\mu}\psi_{n})(C_{V}\overline{\psi}_{e}\gamma_{\mu}\psi_{\nu} - C_{V}'\overline{\psi}_{e}\gamma_{\mu}\gamma_{5}\psi_{\nu}) + (\overline{\psi}_{p}\gamma_{5}\psi_{n})(C_{P}\overline{\psi}_{e}\gamma_{5}\psi_{\nu} - C_{P}'\overline{\psi}_{e}\psi_{\nu}) - (\overline{\psi}_{p}\gamma^{\mu}\gamma_{5}\psi_{n})(C_{A}\overline{\psi}_{e}\gamma_{\mu}\gamma_{5}\psi_{\nu} - C_{A}'\overline{\psi}_{e}\gamma_{\mu}\psi_{\nu}) + \frac{1}{2}(\overline{\psi}_{p}\sigma^{\mu\nu}\psi_{n})(C_{T}\overline{\psi}_{e}\sigma_{\mu\nu}\psi_{\nu} - C_{T}'\overline{\psi}_{e}\sigma_{\mu\nu}\gamma_{5}\psi_{\nu}) + \text{h.c.}, \qquad (3.18)$$

where, following their original notation, h.c. refers to the hermitian conjugate, and the coefficients $C_S^{(\prime)}$, $C_P^{(\prime)}$, $C_V^{(\prime)}$, $C_A^{(\prime)}$, and $C_T^{(\prime)}$ refer to the phenomenological coupling constants of scalar, pseudoscalar, vector, axial vector, and tensor type, respectively. Equation (3.18) treats the neutron field ψ_n , and the proton field ψ_p as the fundamental degrees of freedom in the neutron β decay. In the 1950's, neither quarks nor the electroweak theory were known yet, so that even though Lee and Yang's parameterization in Eq. (3.18) was not written in terms of fundamental degrees of freedom, it was really the best thing people could do at that time. Most importantly, it works at sufficiently low energy. Now in modern terms, such an effective theory is embedded in a more fundamental theory - the SM, in which electromagnetic interaction and weak interaction are unified through the $SU(2)_L \times U(1)_Y$ gauge theory. In the SM, the only nonvanishing terms in the parameterization of Eq. (3.18) are the $C_V^{(\prime)}$ and $C_A^{(\prime)}$ terms with $C_V = C_V^{\prime}$, and $C_A = C_A^{\prime}$, whose absolute values can also be obtained (in principle) from the more fundamental theory – quantum chromodynamics (QCD). Regardless, we now have an effective Hamiltonian for the semi-leptonic weak decay at sufficiently low energy:

$$H_{\text{int}} = (\overline{\psi}_p \gamma^\mu \psi_n) (C_V \overline{\psi}_e \gamma_\mu \psi_\nu - C'_V \overline{\psi}_e \gamma_\mu \gamma_5 \psi_\nu) - (\overline{\psi}_p \gamma^\mu \gamma_5 \psi_n) (C_A \overline{\psi}_e \gamma_\mu \gamma_5 \psi_\nu - C'_A \overline{\psi}_e \gamma_\mu \psi_\nu) + \text{h.c.},$$
(3.19)

where this is a specific description of neutron β decay. Based on Eq. (3.19), we now restrict ourselves to consideration of a simplified case of the weak interaction, as a toy model, whose generic structure is still embedded in that of Eq. (3.19). Such a toy model allows us a quick demonstration of how CP violation can arise. We have

$$H = aV_{\mu}^{+}(t, \mathbf{x})V^{\mu,-}(t, \mathbf{x}) + bA_{\mu}^{+}(t, \mathbf{x})A^{\mu,-}(t, \mathbf{x}) + cV_{\mu}^{+}(t, \mathbf{x})A^{\mu,-}(t, \mathbf{x}) + c^{*}A_{\mu}^{+}(t, \mathbf{x})V^{\mu,-}(t, \mathbf{x}),$$
(3.20)

where the V^{\pm}_{μ} and A^{\pm}_{μ} represent the charged vector current of Eq. (2.41) and the axial vector current of Eq. (3.16), respectively, and the coefficients $a, b, c^{(*)}$ are coupling constants, which act just as C_V and C_A , etc. do in Eq. (3.19). Under CP transformation, one has:

$$CPHCP^{\dagger} = aV_{\mu}^{-}(t, -\mathbf{x})V^{\mu, +}(t, -\mathbf{x}) + bA_{\mu}^{-}(t, -\mathbf{x})A^{\mu, +}(t, -\mathbf{x}) + cV_{\mu}^{-}(t, -\mathbf{x})A^{\mu, +}(t, -\mathbf{x}) + c^{*}A_{\mu}^{-}(t, -\mathbf{x})V^{\mu, +}(t, -\mathbf{x}).$$
(3.21)

Comparing Eq. (3.20) with its CP-transformed case Eq. (3.21), one finds that CP is conserved if the coupling constant c is real. Although Eq. (3.20) is just a simplified example, it turns out that the conclusion is pretty general. That is, it is generally true that for a Hamiltonian H and its subsets of local terms H_i :

$$H = \sum_{i} a_i H_i + \text{h.c.}, \qquad (3.22)$$

with $CPH_iCP^{\dagger} = H_i^{\dagger}$, CP is conserved if all the coefficients a_i are real. In other words, in many cases, if not all, searching for sources of CP violations is associated with finding complex phases of certain coupling constants. We should keep this in mind, because our later work just serves as an exhibition of this case.

Finally, let us continue to look at the simplified example of Eq. (3.20); we can very easily confirm that even when c is complex, Eq. (3.20) remains invariant under the combined transformations of CP together with T, which, as discussed before, is an anti-unitary operator. This CPT invariance turns out to be true not only in our simplified toy model of Eq. (3.20), but in completely general cases. Strictly speaking, we are introducing the so-called CPT theorem, which states local quantum field theories respect CPT, i.e.,

$$CPT\mathcal{L}(t, \mathbf{x})(CPT)^{-1} = \mathcal{L}(-t, -\mathbf{x}).$$
(3.23)

The CPT theorem (3.23) can be proven rigorously based on the following assumptions [16]:

- 1. Lorentz invariance;
- 2. the existence of a unique vacuum state;
- 3. weak local commutativity obeying the "right" statistics.

It can be shown that the CPT theorem demands the equality of masses and total widths of lifetimes for particles and their antipartners. As we have been emphasizing, in the quantum world one should not take validity of any symmetry for granted, but should rather rely on actual experimental tests. Some very accurate experiments to test the validity of CPT symmetry have been carried out and no significant violation has ever been identified at least up to now. Throughout our later works we assume the validity of CPT symmetry.

One apparent consequence of the CPT theorem Eq. (3.23) is that it tells us that CP violation is always connected with the T violation simultaneously, since the combined CPT transformation is required to be a symmetry. Thus T violating observables also serve as an important way to study sources of CP violation. In fact, this is exactly the track that we are following in our work to be presented in the following chapters.

Based on the above discussion, asking if the SM can handle CP violation is more or less the same as asking whether complex coupling constants can enter in the SM in a natural manner. As we know, the answer is yes. A complex coupling can enter via the so-called Cabibbo-Kobayashi-Maskawa mechanism, which was introduced to deal with weak transitions between different quark flavors. In this subsection, we make a very brief review of how this works. Here we are not aiming at an exact historical description, so that some of the reasoning simply does not represent the actual sequence of footprints in the history of physics.

Even long before the development of the fundamental electroweak unified theory, it had been a known experimental fact that many different types of weak processes appear with similar strength, being universally controlled by the Fermi constant G_F . Such a broadly supported universality made people believe that there must be an universal underlying description behind all the observed weak processes.

It is now known that hadrons are made of quarks, which are bound by strong color forces. So far people have identified 6 different types of quarks: u, d, s, c, b, and t. These quarks are grouped, for strong physical reasons, into 3 generations:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \text{ and } \begin{pmatrix} t \\ b \end{pmatrix}.$$
 (3.24)

In fact, the heaviest t quark was confirmed only very lately – in 1995. As the first step of our discussion, we temporarily focus on the 2-generation case. The underlying process of neutron β decay, $n \rightarrow p + e^- + \overline{\nu}_e$, is a weak transition from a d quark to u quark. Apparently, this transition happens within the same generation. When the second generation of quarks (c, s) are introduced, one would naively think, out of an assumption of simplicity, there would only be a weak transition within this generation, between c and s quarks. But such a naive extension is vetoed by the observed kaon weak decay process: $K^+ \rightarrow \mu^+ \nu_{\mu}$. More explicitly, we know that K^+ meson is a bound state of u quark and \bar{s} quark, which disappears in the final state, so that there must also exist a weak coupling between the u quark and s quark so that the $u-\bar{s}$ annihilation is permitted. At first glance, it might seem that one would want to introduce a new weak coupling constant to accommodate this, but such a track is obviously not appealing because it is neither economical nor accordant with our wish of having an universal description of weak interaction. Consequently, instead of stuffing more and more phenomenological coupling constants into our theory, Cabibbo
in 1963 [17] first proposed the following idea: one shall assume the charged weak current only couples "rotated" generations of quark states:

$$\left(\begin{array}{c} u\\d'\end{array}\right), \quad \left(\begin{array}{c} c\\s'\end{array}\right), \quad (3.25)$$

where

$$d' = d\cos\theta_c + s\sin\theta_c$$

$$s' = -d\sin\theta_c + s\cos\theta_c,$$
(3.26)

or equivalently in the matrix form,

$$\begin{pmatrix} d'\\s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c\\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d\\s \end{pmatrix}$$
(3.27)

By following this scenario, one is then able to maintain an universal description of the weak interaction, with a minimal introduction of parameters – the quark mixing angle θ_c is usually called the Cabibbo angle. Such a mechanism has been confirmed to be valid by a series of experiments, and the beauty of this mechanism is that it can be easily extended to an arbitrary number of generations of quarks. The compact form of the extension can be understood as N generations of left-handed "rotated" quark doublets:

$$\begin{pmatrix} u_i \\ d'_i \end{pmatrix} \text{ with } i = 1, 2, ..., N$$
(3.28)

where d'_i refer to mixtures of the "down-type" mass eigenstates d_i :

$$d'_{i} = \sum_{j}^{N} V_{ij} d_{j}.$$
 (3.29)

Here the rotation matrix V is a $N \times N$ unitary matrix to be determined by the flavor-changing weak processes. In the two-generation case, we already see that V contains only one observable parameter – the Cabibbo angle θ_c . How many observable parameters does V contain in the case of an arbitrary number of generation of quarks? A simple reasoning is as follows: each of the N quark states can tolerate a phase change independently without affecting the physics, therefore V contains $N^2 - 2N + 1$ real parameters. Note here that the one overall phase change which leaves V invariant is omitted. When we have N = 2 generations, the number of parameter is 1 - the Cabibbo angle. What is more interesting is that since an orthogonal $N \times N$ matrix can contain at most N(N-1)/2 real parameters, one will have to allow for at least:

$$N^{2} - 2N + 1 - \frac{1}{2}N(N-1) = \frac{1}{2}(N-1)(N-2)$$
(3.30)

remaining phase factors. For the case of N = 2, there is no phase factor, thus no complex coupling can enter. Now it is a good time for us to look at the actual

case in the real world; we have three generations of quarks, as shown in Eq. (3.24). For N = 3, the matrix V is usually called the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and now there is one phase factor " $e^{i\delta}$ " allowed to enter V, and we have obtained a complex coupling constant in the SM from parameter counting. This is to say that CP violation can enter the SM in a natural manner if the number of generations of quarks is larger than 3. In fact, CKM mixing is the only source of CP violation in the weakly-interacting quark sector of the SM.

Unlike the simple 2-generation case with the Cabibbo angle " θ_c " as the only free parameter, the realistic 3-generation case with three "down-type" quarks d, s, and b calls for 3 real angles and 1 complex phase factor. There are in principle infinite numbers of equivalent ways to parametrize the CKM matrix V_{ij} with i, j = 1, 2, 3. In the original work of Kobayashi and Maskawa [18], one representation has been proposed:

$$V_{ij} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (3.31)$$

where $c_{ij} \equiv \cos \theta_{ij}$, and $s_{ij} \equiv \sin \theta_{ij}$, with θ_{12} , θ_{13} , and θ_{23} referring to the 3 real weak angles in total analogy with the Cabibbo angle θ_c .

Before going further, let us be clear on the two types of relations that arise from the requirement of unitarity of CKM matrix:

$$\sum_{i=1}^{3} |V_{ij}|^2 = 1; \quad j = 1, ..., 3$$
(3.32)

$$\sum_{i=1}^{3} V_{ji} V_{ki}^{*} = \sum_{i=1}^{3} V_{ij} V_{ik}^{*} = 0; \quad j, k = 1, ..., 3, \ j \neq k.$$
(3.33)

Equation (3.33) is more relevant to us here, because it suggests that with the existence of a nonvanishing CP-violating phase the products of CKM matrix elements as shown in Eq. (3.33) in the complex plane form triangles, and we shall see that these triangles appear to have very different shapes according to the experimental data of the CKM matrix elements. Based on data from the 2012 compilation of the Particle Data Group (PDG) [19], we have

$$|V_{ij}| = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97427 & 0.22534 & 0.00351 \\ 0.2252 & 0.97344 & 0.0412 \\ 0.00867 & 0.0404 & 0.999146 \end{pmatrix}.$$
 (3.34)

From these experimentally determined values of $|V_{ij}|$, we see that the CKM matrix V_{ij} possesses an intriguing structure. That is,

1. the diagonal elements $|V_{ud}|$, $|V_{cs}|$, and $|V_{tb}|$ are all close to unity, and they are much larger than the off-diagonal elements;

2. as the elements are further from diagonal, they get smaller and smaller.

Such a special structure is directly related to the unexpected long life time of B mesons [20], and it suggests that although the original representation in Eq. (3.31) is theoretically correct, it is not convenient for practical purposes. To account for these special structures, people prefer another representation, which is due to Wolfenstein [21]:

$$V_{ij} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}.$$
 (3.35)

As can be seen in Eq. (3.35), there are four free parameters in the CKM matrix elements. Note that λ is a small parameter, for which the current world average value is $\lambda = 0.22535 \pm 0.00065$ [19], and we are only keeping terms through $\mathcal{O}(\lambda^3)$. The other free parameters |A|, $|\rho|$, and $|\eta|$ are all of order unity. Let us also keep in mind that the CKM phase factor can still be changed into different places, and it will rotate the whole triangle in the complex plane. With such a representation, it is now convenient to discuss the different patterns of the CKM triangles that are formed by the products of CKM matrix elements in the complex plane. Working out the details of Eq. (3.33) one gets 6 triangles with different patterns [21]:

$$V_{ud}^* V_{us}[\mathcal{O}(\lambda)] + V_{cd}^* V_{cs}[\mathcal{O}(\lambda)] + V_{td}^* V_{ts}[\mathcal{O}(\lambda^5)] = 0$$
(3.36)

$$V_{ud}V_{cd}^*[\mathcal{O}(\lambda)] + V_{us}V_{cs}^*[\mathcal{O}(\lambda)] + V_{ub}V_{cb}^*[\mathcal{O}(\lambda^5)] = 0$$
(3.37)

$$V_{us}^* V_{ub}[\mathcal{O}(\lambda^4)] + V_{cs}^* V_{cb}[\mathcal{O}(\lambda^2)] + V_{ts}^* V_{tb}[\mathcal{O}(\lambda^2)] = 0$$
(3.38)

$$V_{td}V_{cd}^{*}[\mathcal{O}(\lambda^{4})] + V_{ts}V_{cs}^{*}[\mathcal{O}(\lambda^{2})] + V_{tb}V_{cb}^{*}[\mathcal{O}(\lambda^{2})] = 0$$
(3.39)

$$V_{td}V_{ud}^{*}[\mathcal{O}(\lambda^{3})] + V_{ts}V_{us}^{*}[\mathcal{O}(\lambda^{3})] + V_{tb}V_{ub}^{*}[\mathcal{O}(\lambda^{3})] = 0$$
(3.40)

$$V_{ud}V_{ub}^{*}[\mathcal{O}(\lambda^{3})] + V_{cd}V_{cb}^{*}[\mathcal{O}(\lambda^{3})] + V_{td}V_{tb}^{*}[\mathcal{O}(\lambda^{3})] = 0, \qquad (3.41)$$

where the square brackets denote the rough lengths of sides of each triangle in powers of the small parameter λ . Apparently, these triangles in the complex plane can be basically categorized into 2 cases:

1. Equations (3.36) through (3.39) describe triangles containing two sides that are much longer than the third due to the relative suppression factor $\lambda^4 \sim 10^{-3}$ and $\lambda^2 \sim 10^{-2}$. Such a badly "squashed" pattern refers to the weak transitions between

the neighboring generations of quarks, and it is roughly presented as the first plot in Fig. (3.2).

2. Equations (3.40) and (3.41) describe triangles in which all three sides have roughly the same order, with each of the three angles being \sim a few×10°. Such a pattern refers to the weak transitions between the first and the third generations, and is roughly presented as the second plot in Fig. (3.2).

The second case is especially of interest to us, because with all sides the same



Figure 3.2: Rough exhibition of CKM triangles. The plot (a) with two sides of the triangle much longer are accessible in kaon decays; the plot (b) with all the three sides roughly of the same order denotes the transitions between accessible in B-meson decays.

size tests of the relationships between the matrix elements become experimentally accessible. Note that among Eq. (3.40) and (3.41) in the second case, Eq. (3.40) is closely related to the t quark decays, but t quark is the known heaviest quark flavor and is very short lived. Thus it is not an ideal material for experiments. It is Eq. (3.41) that is most important to us: it is closely related to the b quark decays with b quark much lighter than t quark, and the B meson has long life. All the information of the triangle can be determined by experiments - the sides of the triangle can be determined by measuring decay rates, and the three CP-violating angles can be measured by various asymmetries in B meson decays. Such an area has become a broad and popular area usually called "B physics", shortly after the beauty quark and B meson were discovered in 1977 [22]. Also, this is the motivation of the projects behind the construction of "B-factories", which were designed to produce a large number of B mesons. The Belle experiment at the KEKB collider in Tsukuba, Japan, and the BaBar experiment at the PEP-II collider at SLAC laboratory in California, USA, completed data collection in 2010 and 2008, respectively. The B-factories yielded rich results, including the first observation of CP violation outside of the kaon system. For a better understanding of the experimental outcomes of the B-factory era, let us look at the triangle in Fig. (3.3) described by Eq. (3.41) more closely. In the complex



Figure 3.3: CKM triangle of Eq. (3.41) in the complex plane.

plane, it can be found that

$$\phi_{1} = \pi - \arg\left(\frac{V_{tb}^{*}V_{td}}{V_{cb}^{*}V_{cd}}\right),$$

$$\phi_{2} = \arg\left(\frac{V_{tb}^{*}V_{td}}{-V_{ub}^{*}V_{ud}}\right),$$

$$\phi_{3} = \arg\left(\frac{V_{ub}^{*}V_{ud}}{-V_{cb}^{*}V_{cd}}\right).$$
(3.42)

Note in many papers another notation is used, in which the three CKM angles are specified as α , β , and γ . One has $\alpha = \phi_2$, $\beta = \phi_1$, and $\gamma = \phi_3$. All of these CKM phases can directly measured via the asymmetries appearing in different modes of B-meson decays. For example, ϕ_1 can be obtained via the CPviolating asymmetry realized from the interference of $B^0 - \overline{B^0}$ mixing and the direct decay $B^0 \rightarrow$ charmonium $K_{S,L}$ [23] [24]; ϕ_2 can be similarly obtained via $B \rightarrow$ $\pi\pi$, $\rho\rho$, $\rho\pi$ [25] decays; and ϕ_3 can be obtained via the interference of $B^- \rightarrow D^0 K^$ and $B^- \rightarrow \overline{D^0} K^-$ [25] [26]. According to the 2012 PDG [19], the central values of the world average values of the angles in Eq. (3.42) are:

$$\phi_1 \approx 21.4^\circ, \quad \phi_2 \approx 89^\circ, \quad \phi_3 \approx 68^\circ, \tag{3.43}$$

which can be summed to give a total inner angle $\phi_1 + \phi_2 + \phi_3 \approx 178^\circ$, very close to the standard expectation of 180°. On combining all the experimental data, one can also obtain the fit for the Wolfenstein parameters:

$$\lambda = 0.22535 \pm 0.00065, \quad A = 0.811^{+0.022}_{-0.012}, \quad \bar{\rho} = 0.131^{+0.026}_{-0.013}, \quad \bar{\eta} = 0.345^{+0.013}_{-0.014}. \quad (3.44)$$

With the discussion thus far, we confirm that the SM does allow for CP violation naturally in the manner of CKM mechanism, and this has been experimentally verified. The problem is that detailed theoretical analysis reveals that CP violation due to the CKM mechanism is not sufficient to explain the puzzle of BAU [27] [28] [29] [30] [31] [32] [33] [34]. This in turn reveals that there must exist additional mechanism that is BSM, in which some nontrivial complex coupling constants can appear.

I think it might also be worth spending a little time to mention briefly another interesting CP-violating source in QCD – the so-called " θ term". Besides the CKM scenario above, SM could have permitted, in principle, another possible and economical way to violate CP symmetry in the strong interaction, but it does not operate though we do not know why – this is the "strong CP problem." Since this topic is not so relevant to our work, here I shall just very brief picture without going into a lot of detail.

In the QCD sector of the SM, the QCD Lagrangian as a theoretical extension of Yang-Mills theory [35] is written as:

$$\mathcal{L}_{\text{QCD}} = \overline{Q}_f i \not\!\!D Q_f - \overline{Q}_{fR} M Q_{fL} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \text{h.c.}, \qquad (3.45)$$

where the subscript f denotes the six quark flavors u, d, c, s, t, and b. For each of these flavors, there are 3 color degrees of freedom as well:

$$Q_f = \begin{pmatrix} Q_{fr} \\ Q_{fg} \\ Q_{fb} \end{pmatrix}$$
(3.46)

with the indices r, g, and b denoting the color "red", "green", and "blue" respectively. Note in QCD theory, the number of colors N_c is three to account for the observed baryon states $\Delta^{++}(uuu)$, $\Delta^{-}(ddd)$, and $\Omega^{-}(sss)$, which contain three identical fermions and thus the minimum of $N_c = 3$ is required. As will be discussed later, $N_c = 3$ is necessary in understanding the pi^0 lifetime, and it also plays an important role in the anomaly cancellation of the SM. Furthermore, M is the quark mass matrix, and D_{μ} is defined as the covariant derivative:

$$D_{\mu} \equiv \partial_{\mu} - igt^a A^a_{\mu}, \qquad (3.47)$$

with t^a referring to the eight SU(3) generators, A^a_{μ} the gluon field, and g the strong coupling constant. $G^a_{\mu\nu}$ denotes the gluon-field strength tensor:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \qquad (3.48)$$

where f^{abc} is referred to as the structure constant of SU(3) gauge group. The whole construction of \mathcal{L}_{QCD} , Eq. (3.45), is based on the fundamental requirement of local gauge invariance of SU(3)_c. It can be shown, as stressed before, that the standard QCD Lagrangian of Eq. (3.45) conserves all the C, P, and T symmetries separately – and of course CP is conserved accordingly. However, there could have been, by all means, another possible term, which reads:

$$\mathcal{L}_{\text{QCD}}^{eff} = \mathcal{L}_{\text{QCD}} + \frac{\theta g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} G^a_{\mu\nu} G^a_{\alpha\beta}.$$
 (3.49)

where θ is a constant, representing the strength of the additional term " $\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta}$." Let us emphasize that one can show [36] [37] that, in the framework of the U(1) axial anomaly, allowing for a possible imaginary part of the quark mass in the QCD Lagrangian also leads to a term of the similar form $\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta}$ that is proportional to Arg [Det (M_q)], where M_q refers to the quark mass matrix. Thus it is really the $\bar{\theta} \equiv \theta + \text{Arg}$ [Det (M_q)] that serves as the physical observable. With the additional term added, Lorentz invariance as well as SU(3) gauge invariance are still respected. Thus, based on the fundamental spirit of constructing a local gauge–invariant quantum field theory, there is no reason to exclude such an additional term, so that strictly speaking, we should have included such a term in the QCD Lagrangian from the very beginning, and if so, we have a CP–violating term in QCD. It can be easily checked [38] that the original gluon dynamical term $G^a_{\mu\nu}G^{\mu\nu,a}$ leads to:

$$G^{a}_{\mu\nu}G^{\mu\nu,a} \propto \sum_{a} (|\mathbf{E}_{a}|^{2} + |\mathbf{B}_{a}|^{2}),$$
 (3.50)

where $|\mathbf{E}_a|^2$ and $|\mathbf{B}_a|^2$ are called color electric and color magnetic fields, which transform under P, C, and T in a way similar to the electromagnetic fields, such as Eq. (2.6) and Eq. (2.7). Thus we have:

$$\mathbf{E}_a \xrightarrow{\mathrm{P}} -\mathbf{E}_a, \quad \mathbf{E}_a \xrightarrow{\mathrm{T}} \mathbf{E}_a,$$
 (3.51)

$$\mathbf{B}_a \xrightarrow{\mathbf{P}} \mathbf{B}_a, \quad \mathbf{B}_a \xrightarrow{\mathbf{T}} -\mathbf{B}_a, \tag{3.52}$$

from which one immediately sees that the original gluon dynamical term, " $G^a_{\mu\nu}G^{\mu\nu,a}$ ", is definitely symmetric under P and T. Based on the similar argument, however, one can also show:

$$\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta} \propto \sum_a \mathbf{E}_a \cdot \mathbf{B}_a, \qquad (3.53)$$

which is obviously both P-violating and T-violating!

As one may have noticed, a similar CP-violating term such as $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ could also enter the QED Lagrangian; if so, do we expect a CP violation in QED also? The quick answer is NO. Essentially, in both QED and QCD, the CP-violating term can be written as a total divergence. In the case of QED, because the associated U(1)gauge group possesses very simple topological structure of the ground, or vacuum state, such a CP-violating term can be argued away by an allowed choice of $A_{\mu} = 0$ at infinity. In the case of QCD, however, the complex topological structure of the QCD ground state stops one from setting $A^a_{\mu} = 0$, so that the T-violating term cannot be simply argued away [39] [40].

To sum up, the term " $\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta}$ " in QCD could have been chosen by nature to serve as a very economical source of CP violation, but interestingly it was not. In fact, detailed analysis reveals a connection between the nEDM and the term " $\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta}$ " such that $d_n \approx 10^{-16}\overline{\theta} \text{ cm}$ [41] [37]. With the current experimental limit of d_n [42]:

$$d_n < 2.9 \times 10^{-26} \quad (90\% \text{ C.L.}),$$
 (3.54)

one finds a bound on $\bar{\theta}$ such that:

$$\overline{\theta} \lesssim 10^{-10},\tag{3.55}$$

which suggests no evidence of the existence of such a term in QCD. Is there any hidden theory or symmetry making the " $\overline{\theta}$ term" forbidden? The question why the seemingly allowed " $\overline{\theta}$ term" is so tiny remains an open question, and we are sure that the strong CP problem provides a important clue to new physics. The discussions on this issue have lead to several proposed BSM models such as models which have a massless u-quark as reviewed by Ref. [43], the Peccei-Quinn mechanism [44] [45], as well as various supersymmetry (SUSY) and string theory compactifications where CP is an exact gauge symmetry and must be spontaneously broken [46] [47]. Roughly speaking, the Peccei-Quinn mechanism assumes that the Standard Model may be augmented by appropriate additional fields and admits a symmetry $U(1)_{PQ}$, which acts on states charged under $SU(3)_C$. Assuming this symmetry is spontaneously broken at some necessarily high-scale f_a , a pseudo-scalar Goldstone boson – the axion – results. Symmetry dictates that the essential components of the axion Lagrangian:

$$\mathcal{L}_a = \frac{1}{2} \partial_\mu a \partial^\mu a + \frac{a(x)}{f_a} \frac{\alpha_s}{8\pi} G \tilde{G}, \qquad (3.56)$$

where a(x) refers to the proposed axion field, and $\tilde{G} \equiv \epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$ – the $G^{\mu\nu}$ dual. Apparently, Eq. (3.56) provides a field-dependent shift of $\bar{\theta}$:

$$\overline{\theta} \to \overline{\theta} + \frac{a(x)}{f_a}.$$
 (3.57)

Furthermore, one finds that below the QCD scale $U(1)_{PQ}$ is explicitly broken by the chiral anomaly, and thus the axion is in reality a pseudo-Goldstone boson and acquires a potential

$$\mathcal{L}_{a}^{eff} = \frac{1}{2} \partial_{\mu} a \partial^{\mu} a - \frac{1}{2} \chi(0) \left(\overline{\theta} + \frac{a(x)}{f_{a}}\right)^{2}, \qquad (3.58)$$

where $\chi(0)$ is a constant, known as the topological susceptibility. We see from Eq. (3.57) that the vacuum expectation value of the axion field $\langle a(x) \rangle$ renormalizes the value of $\overline{\theta}$ so that all observables depend on the combination $\overline{\theta} + \frac{\langle a(x) \rangle}{f_a}$. At the same time, such a combination must vanish in the vacuum as it minimizes the value of the axion potential in Eq. (3.58). The strong CP problem is then solved in the Peccei-Quinn mechanism, which is independent of the initial value of $\overline{\theta}$ and thus seems very appealing. However, the Peccei-Quinn mechanism calls for the confirmation of the axion as a new particle that is beyond the SM. One can show that the excitations around $\langle a \rangle$ correspond to a massive axion particle with a mass

$$m_a \sim \frac{1}{f_a} |\chi(0)|^2.$$
 (3.59)

For large f_a the axion should be very light and thus should have had significant phenomenological consequences. Yet the direct and indirect searches for "invisible axions" up to date yield null results [48] [49] [50] [51] [19], setting the lower bound of $f_a > 10^{10}$ GeV. Currently, axion theory is still an active area.

There are also other ways to solve the strong CP problem. People suggest that perhaps CP (maybe even both P and CP) is an exact symmetry of nature at some very high energy scale. One can then declare that the θ term must vanish as a result of symmetry at that very high energy scale. To account for the CP violation that has been observed in the SM via the CKM mechanism, one has to assume that the CP symmetry is spontaneously broken at a particular scale $\Lambda_{(P,CP)}$. The theoretical challenge here is that one is forced to ensure that such a spontaneously broken CP symmetry allows for a significant CP-violating CKM phase $\delta_{\rm CKM}$, yet keeps the $\overline{\theta}$ term very small. Among the attempts of this type, a scenario in SUSY [52] [53] [54] exists, it states that there exist two very distinct symmetry-breaking scales, one for CP, the other for SUSY, and the CP-breaking scale $\Lambda_{\rm CP}$ is much higher than the SUSYbreaking scale Λ_{SUSY} . Were this true, strong interactions in the CP-breaking sector can then generate a large CKM phase, while a SUSY nonrenormalization theorem ensures that $\overline{\theta}$ is not generated until down to the much lower scale Λ_{SUSY} where SUSY is broken. One expects the corrections to $\overline{\theta}$ to be highly suppressed by power(s) of the small ratio $\Lambda_{\rm SUSY}/\Lambda_{\rm CP}$.

3.3 Experiments at Low Energies to Probe CP Violation

Besides the observed CP violation in both kaon and B-meson decays, CP violation could also arise in other possible places. Here we focus on the case of neutron observables. In the following short sections, we shall briefly mention other possible probes of CP violation – the neutron electric dipole moment (nEDM), the D and R term in neutron β decay, and a triple-product correlation in momenta in radiative β decay, which is the main goal we are pursuing.

3.3.1 Electric Dipole Moments

The basic definition of an EDM \mathbf{d} in a composite system is:

$$\mathbf{d} \equiv \sum_{i} \mathbf{r}_{i} Q_{i},\tag{3.60}$$

where "*i*" refers to a particular constituent inside the particle under discussion, not to be confused with a vector's component indices. According to Eq. (3.60), a nonvanishing value of **d** of an object signals an asymmetric charge distribution. Such an asymmetric charge distribution can be permanent or induced due to the presence of an external electric field. For a system, such as an elementary particle, an atom, or a molecule placed in a weak electric field **E**, the energy shift, $\Delta \mathcal{E}$, due to the external electric field can be, in general, expanded in a power series in **E**:

$$\Delta \mathcal{E} = \mathbf{d} \cdot \mathbf{E} + \mathbf{d}_{ij} E_i E_j + \dots, \qquad (3.61)$$

where \mathbf{d} in the first term is the permanent EDM, which leads to an energy shift linear in \mathbf{E} , and that of the second term is the induced EDM, which leads to an energy shift quadratic in \mathbf{E} . This is easy to understand because the electric field \mathbf{E} must be used to induce an EDM. In such a way, one is able to distinguish between a permanent EDM and an induced EDM. Obviously, a non-vanishing expectation value of the dipole moment operator for a certain particle implies that the dynamics acting on the particle violates parity symmetry. This is such a simple fact that people had realized it a long time ago. In fact, Purcell and Ramsey used this argument to test parity conservation in nuclear forces in the early 1950's. It can also be shown that a permanent EDM also violates T symmetry [55]. It can be understood in the following way: for a nondegenerate ground state of a particle, say a neutron, one assumes that no other internal quantum numbers but the spin \mathbf{S} serves as the only 3-component object characterizing a free static neutron, thus if the nEDM is present, the following proportionality then inevitably holds:

$$\langle n, s | \mathbf{d} | n, s \rangle = C_s \langle n, s | \mathbf{S} | n, s \rangle, \qquad (3.62)$$

where C_s is just a constant. With presence of an external electric field **E** as well as T transformation, the scalar product $\mathbf{E} \cdot \mathbf{d}$ is not to be affected, but $\mathbf{E} \cdot \mathbf{S}$ is. With the spin polarization **S** transforming as angular momentum, one has:

$$\mathbf{TE} \cdot \mathbf{dT}^{-1} = \mathbf{E} \cdot \mathbf{d}, \qquad (3.63)$$

$$\mathbf{TE} \cdot \mathbf{ST}^{-1} = -\mathbf{E} \cdot \mathbf{S}. \tag{3.64}$$

Thus we know that nEDM has to vanish if T is conserved, an observation of nonvanishing nEDM unambiguously signals T violation, as well as CP violation according to the CPT theorem.

Up to date, searching for non-vanishing permanent nEDM has been an important and ongoing area in nuclear physics. The current record of the accepted upper limit of $|\mathbf{d}| \equiv d_n$, obtained from [42], is:

$$d_n < 2.9 \times 10^{-26} \text{ e cm} (90\% \text{ C. L.}).$$
 (3.65)

New projects have been proposed, and are already on their way to push the limit to the level of 10^{-28} e cm [56], allowing us better understanding of CP violation. More details of the proposals for the future experiments can be found in Ref. [57] [58] [59] [60] [61].

3.3.2 T-odd Decay Correlations in Neutron β Decay

As the second example of the possible places where CP violation could enter, we now look at the polarized neutron β decay:

$$\vec{n} \rightarrow p + e^- + \overline{\nu}_e,$$
 (3.66)

which can be described with the Feynman diagram of Fig. (3.4), in which p_n , p_p , l_e , and l_{ν} refer to the 4-momentum of the neutron, proton, electron, and electron antineutrino respectively. Within this subsection, we are assuming that the neutron is at rest in the lab frame, and polarized along a direction chosen to be the z direction. The component " \bigotimes " in Fig. (3.4) represents an effective coupling vertex at sufficiently



Figure 3.4: Neutron β decay, $n \rightarrow p + e^- + \overline{\nu}_e$.

low energy. Roughly speaking, the momentum transfer in neutron β decay is only about 1 MeV, which is much less than the W^{\pm} -boson mass of ~ 80 GeV. In this case, the W^{\pm} -boson propagator "shrinks" to a point, controlled by Fermi constant G_F . Following the standard Feynman rules, one can readily write down the amplitude for the decay process (3.66) at tree level:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \overline{u}_e \gamma^\mu (1 - \gamma_5) u_\nu \overline{u}_p (g_V \gamma_\mu - g_A \gamma_\mu \gamma_5) \frac{1 + \gamma_5 \$}{2} u_n, \qquad (3.67)$$

where u_e , u_ν , u_p , and u_n refer to the Dirac spinors of the electron, neutrino, proton, and neutron, respectively. The coupling g_V and g_A are of vector and axial vector character, respectively. Apparently, Eq. (3.67) agrees with Eq. (3.19) with only some minor changes in notation. The insertion of the operator " $(1 + \gamma_5 \$)/2$ " is to account for the neutron spin polarization, with S^{μ} the spin polarization operator. To obtain the decay rate Γ – a physical observable, one just needs to take the absolute square of \mathcal{M} in Eq. (3.67), and employ the Clifford (γ -matrices) algebra. A straightforward calculation [62] yields:

$$d\Gamma \propto |\mathcal{M}|^2 \propto 1 + a \frac{\mathbf{l}_e \cdot \mathbf{l}_\nu}{E_e E_\nu} + b \frac{m_e}{E_e} + A \frac{\mathbf{S} \cdot \mathbf{l}_e}{E_e} + B \frac{\mathbf{S} \cdot \mathbf{l}_\nu}{E_\nu} + D \frac{\mathbf{S} \cdot (\mathbf{l}_e \times \mathbf{l}_\nu)}{E_e E_\nu}, \qquad (3.68)$$

with the following T and P transformation properties of \mathbf{S} , \mathbf{l}_e , and \mathbf{l}_{ν} :

$$\begin{array}{cccc}
 \mathbf{S} \xrightarrow{\mathrm{P}} \mathbf{S}; & \mathbf{S} \xrightarrow{\mathrm{T}} -\mathbf{S}; \\
 \mathbf{l}_{e} \xrightarrow{\mathrm{P}} -\mathbf{l}_{e}; & \mathbf{l}_{e} \xrightarrow{\mathrm{T}} -\mathbf{l}_{e}; \\
 \mathbf{l}_{\nu} \xrightarrow{\mathrm{P}} -\mathbf{l}_{\nu}; & \mathbf{l}_{\nu} \xrightarrow{\mathrm{T}} -\mathbf{l}_{\nu}.
 \end{array}$$
(3.69)

Thus, the "A" and "B" related terms are parity-violating, and they account for the observed parity violation in the weak interaction. What is more interesting is that the so-called "D" term, which is proportional to the triple product $\mathbf{S} \cdot (\mathbf{l}_e \times \mathbf{l}_{\nu})$, is a T-odd correlation, and can signal the presence of CP violating physics if $D \neq 0$. Here

let us keep in mind that such a T-odd correlation is not really a true test of T but only probes motional invariance. Thus it can be mimicked by final-state interactions (FSI). In fact, such a FSI mimicking effect is universal to many T-odd correlations; we will see more details about this in neutron radiative β decay in a later chapter. One can work out the actual analytic form of D [62], which reads:

$$D \propto \operatorname{Im}(g_V g_A^*). \tag{3.70}$$

Currently the most precise measurement [63] yields $D = [-0.96 \pm 1.89(\text{stat}) \pm 1.01(\text{sys})] \times 10^{-4}$. One can show that the contribution from the FSI mimicking effect is only of order 10^{-5} [64].

In the discussion of the "D-term" thus far, we consider the polarized neutron but leave the final electron unpolarized. Many other possible CP-violating observables have been discussed in Ref. [62]. Among these possibilities one more observable that could also be experimentally viable comes from the case where both neutron and electron are polarized. One finds a T-odd correlation in the differential decay rate in such a case:

$$|\mathcal{M}|_{\mathrm{T-odd}}^{2} = R \frac{\hat{\mathbf{S}}_{n} \times \mathbf{p}_{e} \times \hat{\mathbf{S}}_{e}}{E_{e}}, \qquad (3.71)$$

where to distinguish the polarizations of neutron and electron, we denote them as \mathbf{S}_n and \mathbf{S}_e respectively, and the hat symbol means the unit vector. The coefficient R is a T odd, P odd correlation. On applying the Lee and Yang's general Hamiltonian of Eq. (3.18), and ignoring the effect of final-state interactions, one finds [62]:

$$R = -0.128 \operatorname{Im}\left(\frac{C_S + C'_S}{C_V}\right) + 0.335 \operatorname{Im}\left(\frac{C_T + C'_T}{C_A}\right).$$
(3.72)

A recent measurement [65] yields $R = 0.004 \pm 0.012 (\text{stat}) \pm 0.005 (\text{sys})$. Inspecting both the "D-term" and the "R-term", we see that to allow for a non-vanishing CP violation in neutron β decay, at least some of the coupling constants must be complex. This is in agreement with our earlier discussion of the conditions for CP violation in a simplified model, Eq. (3.20). As will be seen in later chapters, this is in fact the track we are following in searching for T-odd correlations.

3.3.3 T-odd Decay Correlations in Neutron Radiative β Decay

Although we are presenting this topic as a short subsection, there is really much to say. We are postponing a general discussion to later chapters. Here we will only motivate that later study.

As seen in the discussions so far, the searches for CP violation in both the nEDM and the T-odd decay correlations in neutron β decay are inevitably involved with the neutron spin. Now suppose the spin polarization is "turned off", or, say, we consider unpolarized neutron β decay, are we still able to track down a CP-violating source? The quick answer is YES, because even without a spin **S**, one can still, in principle, find another kind of correlation, a triple-product correlation in momenta, $\sim \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)$. This kind of correlation is not available in regular neutron β decay, because it does not have enough number of independent 3-momenta. To allow for a nontrivial triple-product correlation in momenta, one needs at least three independent momenta in the final state, but regular neutron β decay only has two due to energy-momentum conservation. Thus, to have enough independent momenta, neutron radiative β decay is needed.

Simply speaking, neutron radiative β decay refers to the regular neutron β decay accompanied by the emission of one or more photons. In neutron decay, the electron and proton decay products are both charged particles, photon(s) can be emitted due to the presence of these charged particles, and this has been observed recently [66]. In the language of QED perturbation theory, the contribution of multiple-photon emission is suppressed by higher powers in the fine-structure constant $\alpha \sim 1/137$, so that for now we just consider the single photon case. In the very end of the thesis, we will turn to a detailed analysis of the case of two photons emission, since this issue can be important for soft photon momenta.

More precisely, we describe radiative neutron β decay as:

$$n \rightarrow p + e^- + \overline{\nu}_e + \gamma,$$
 (3.73)

which can be realized via standard QED bremsstrahlung due to the electron and proton, and some other possible processes which depend on hadron structure. For now, we just focus on the contribution of QED bremsstrahlung, which is unambiguously described in the SM. One obtains the relevant Feynman diagrams, Fig. (3.5), where we denote the final states with the momenta of the relevant particles. The processes



Figure 3.5: Neutron radiative β decay, $n \rightarrow p + e^- + \overline{\nu}_e + \gamma$.

shown in Fig. (3.5) serve as a starting point for our later work, and the SM already provides us with all the essential information. Following the Feynman rules, one is able to obtain the amplitude and to carry out all the relevant calculations in just the same way as in the usual neutron β decay case. The only difference is that because the electron and proton lines are now dressed with the photon emission, more complicated spin sums are involved. The calculations are more tedious, but still doable by hand, one just needs more patience and time. Alternatively, a symbolic manipulation form, such as FORM, can be employed. As stressed before, in this section we just quickly point out that in neutron radiative β decay, we are now able to form a nontrivial triple-product correlation in momenta, $\mathbf{l}_e \cdot (\mathbf{l}_{\nu} \times \mathbf{k})$. More details will be presented in the later chapters.

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Chapter 4 CP-Conserving Observables in Neutron Radiative β Decay

In this chapter, we discuss the SM calculations on the neutron radiative β decay [67] [68]. For now we are assuming, with great confidence, that the main contribution to the electromagnetic radiation is the QED bremsstrahlung, which has been understood very well in the SM. We can then calculate the differential and total radiative decay rate as a function of the lowest detected photon energy ω_{\min} . The decay rate for a fixed ω_{\min} can also be converted into a branching ratio (BR), and it can be used to compare with high-precision experimental measurements [69]. contributions that are BSM if any significant discrepancy identified. As we will see in later chapters, another newly proposed mechanism can also contribute to the radiative β decay.

Throughout the chapter, our calculations are restricted to leading recoil order (LRO) for simplicity. By LRO we mean that all the contributions of $\mathcal{O}(E/M_N)$ are ignored, where $E \sim 1$ MeV refers to the typical energy scale of the electron mass and the kinetic energies of the decay products, and $M_N \sim 10^3$ MeV refers to the average of the proton and neutron masses. The relative error that arises from the LRO approximation can then be evaluated as $E/M_N \sim 10^{-3}$, which has been confirmed by the explicit calculations in [68] and is far beyond the precision that experiments have been able to achieve up to date! Thus we can ignore the recoil–order terms without losing any predictive power. Throughout the rest of this thesis, the Natural Unit System, in which the speed of light c and Plank constant h are both set to unity, is assumed unless stated otherwise.

We consider the neutron case only in this chapter. The more general case of nuclear radiative β decay is discussed in a later chapter.

4.1 Neutron Radiative β Decay in Leading Order

We start with neutron radiative β decay with a single photon detected. As we have noted, here only the bremsstrahlung process is needed. One can show that the contributions that are beyond bremsstrahlung are beyond LRO [68]. In this section, we work in leading order, which means that we are only keeping the contributions of LRO as well as of $\mathcal{O}(\alpha)$. Furthermore, as a rough estimate, the typical radius of both neutron and the proton, R_N , has been experimentally determined to be less than 1 fm [19], so that:

$$R_N \lesssim \frac{1}{197} \mathrm{MeV}^{-1}.$$
(4.1)

As for the wavelength of the photon, λ_{γ} , since the typical energy carried by the emitted photon, E_{γ} , is surely less than 1 MeV, we have:

$$\lambda_{\gamma} = \frac{ch}{E_{\gamma}} > 1 \mathrm{MeV}^{-1} \gg R_N.$$
(4.2)

Equations (4.1) and (4.2) shows us we can take the long-wavelength limit and treat the neutron as well as the proton as point-like objects. Or more simply speaking, at

sufficiently low energy, the emitted photon "sees" no structure, such as quarks and gluons, inside the neutron and proton. This justifies our earlier approximation of treating the neutron and proton as fundamental spin-1/2 fermions. The interactions resulting from the deeper stucture of nucleon can be treated systematically in chiral perturbation theory [68], which we will discuss in a later chapter.

With this preparation, we are now ready to carry out the calculations. In Chapter 3, we have shown the relevant Feynman diagrams in Fig. (3.5) with bremsstrahlung. The scattering amplitude reads:

$$\mathcal{M}_0 = \frac{eg_V G_F}{\sqrt{2}} (\mathcal{M}_{01} - \mathcal{M}_{02}), \qquad (4.3)$$

where \mathcal{M}_{01} and \mathcal{M}_{02} describe the bremsstrahlung diagrams (01) and (02) in Fig. (3.5). There can be other possible contributions that are beyond bremsstrahlung, but they are only subleading. We postpone the discussions of such contributions to a later section.

For the diagram (01) in Fig. (3.5) we have, following Ref. [70] for all conventions:

$$\mathcal{M}_{01}(l_e, k, p_p) = \bar{u}_e(l_e) \frac{2l_e \cdot \epsilon^* + \not\!\!\!\!/ \epsilon^* \not\!\!\!\!/ k}{2l_e \cdot k} \gamma_\rho(1 - \gamma_5) v_\nu(l_\nu) \bar{u}_p(p_p) \gamma^\rho(1 - \lambda\gamma_5) u_n(p_n), \quad (4.4)$$

and for the diagram (02) in Fig. (3.5) we have:

$$\mathcal{M}_{02}(l_e, k, p_p) = \bar{u}_e(l_e)\gamma_\rho (1 - \gamma_5) v_\nu(l_\nu) \bar{u}_p(p_p) \frac{2p_p \cdot \epsilon^* + \not \epsilon^* \not k}{2p_p \cdot k} \gamma^\rho (1 - \lambda\gamma_5) u_n(p_n), \quad (4.5)$$

where $u_e(l_e)$, $v_\nu(l_\nu)$, $u_p(p_p)$ and $u_n(p_n)$ are Dirac spinors describing the asymptotic states of the electron, neutrino, proton, and neutron, respectively, with their momentum dependence shown explicitly – we do this for later convenience. In Eq. (4.3), e refers to the unit of electric charge, which, following Heaviside-Lorentz convention as per [70], satisfies $e^2/4\pi = \alpha \approx 1/137$; g_V refers to the vector current coupling constant; G_F is the Fermi constant; λ is defined as $\lambda \equiv g_A/g_V = 1.2701$ [19] with g_A the axial vector current coupling constant.

In Eq. (4.3), \mathcal{M}_{01} and \mathcal{M}_{02} differ by a minus sign because the electron and proton carry opposite electric charges. Such a relative minus sign is important for the Ward identity – a fundamental requirement of QED gauge invariance which states that the total amplitude has to vanish on the replacement of a photon's polarization vector ϵ_{μ} with its momentum k_{μ} [70]. Replacing ϵ_{μ} with k_{μ} in both Eq. (4.4) and Eq. (4.5) combined with the identity:

$$kk = k \cdot k = m_{\gamma}^2 = 0, \qquad (4.6)$$

shows that the Ward Identity:

$$\mathcal{M}_0|_{\epsilon_\mu \to k_\mu} = 0 \tag{4.7}$$

holds manifestly.

With the amplitude \mathcal{M}_0 , Eq. (4.3), in association with Eq. (4.4) and Eq. (4.5), one is able to obtain much important information for the neutron radiative β decay. The decay rate Γ as a physical observable is our aim in this section. To get the decay rate, we first need to calculate $|\mathcal{M}_0|^2$, which of contains:

$$|\mathcal{M}_0|^2 = \frac{e^2 g_V^2 G_F^2}{2} \left(|\mathcal{M}_{01}|^2 + |\mathcal{M}_{02}|^2 - 2\operatorname{Re}(\mathcal{M}_{01}\mathcal{M}_{02}^*) \right),$$
(4.8)

Before starting the detailed calculations, it is worth pointing out that the term " $(\not \in k)/(2p_p \cdot k)$ " in Eq. (4.5) only gives a contribution of recoil order, which, based on the LRO approximation, is negligible. So we shall simply drop this term throughout our calculations. We just work out the terms in Eq. (4.8) one by one. Although the procedures to work out $|\mathcal{M}_0|^2$ are standard, I am thinking that since I had a hard time handling the analytic calculations of $|\mathcal{M}_0|^2$, both as a basic training and as a preparation for later projects, it still seems worthwhile to go a little bit further than simply saying something like "it is easy to show...". In fact, the details here are also useful for later discussion of the extension of the neutron to nuclear cases.

Take the calculation of $|\mathcal{M}_{01}|^2$ for example, one has:

$$|\mathcal{M}_{01}|^{2} = \mathcal{M}_{01} \cdot \mathcal{M}_{01}^{*} = \bar{u}_{e} \frac{2l_{e} \cdot \epsilon^{*} + \not\epsilon^{*} \not k}{2l_{e} \cdot k} \gamma_{\rho} (1 - \gamma_{5}) v_{\nu} \bar{u}_{p} \gamma^{\rho} (1 - \lambda \gamma_{5}) u_{n} \cdot \bar{v}_{\nu} (1 + \gamma_{5}) \gamma_{\delta} \frac{2l_{e} \cdot \epsilon + \not k \not\epsilon}{2l_{e} \cdot k} u_{e} \bar{u}_{n} (1 + \lambda \gamma_{5}) \gamma^{\delta} u_{p},$$

$$(4.9)$$

which can be reorganized into a product of two tensors, which we classify as the baryonic tensor $H^{\rho\delta}$:

$$H^{\rho\delta} \equiv \bar{u}_p \gamma^{\rho} (1 - \lambda \gamma_5) u_n \bar{u}_n (1 + \lambda \gamma_5) \gamma^{\delta} u_p$$

= $\operatorname{Tr} \left[(\not{p}_p + m_p) \gamma^{\rho} (1 - \lambda \gamma_5) (\not{p}_n + m_n) (1 + \lambda \gamma_5) \gamma^{\delta} \right], \qquad (4.10)$

where m_n and m_p refer to neutron and proton mass, respectively. The leptonic tensor $L_{\rho\delta}^{ee}$ reads:

such that we have:

$$|\mathcal{M}_{01}|^2 = \frac{1}{(l_e \cdot k)^2} L_{\rho\delta}^{\text{ee}} \cdot H^{\rho\delta}.$$
(4.12)

The remaining jobs are just to work out $L^{ee}_{\rho\delta}$ and $H^{\rho\delta}$, with great patience, by following the trace theorems that are usually available in many QFT textbooks, such as [70]. One more issue that might be worth of mentioning here concerns the manipulation of the photon polarization term " $\epsilon^*_{\mu}\epsilon_{\nu}$," which will certainly appear in the detailed calculations. Let me just say a few words on this issue here. It can be shown that, on summing over all the photon polarizations, if the Ward identity Eq. (9.43) is satisfied, then the following replacement:

$$\sum_{s} \epsilon^*_{\mu} \epsilon_{\nu} \to -g_{\mu\nu} \tag{4.13}$$

is valid. In Peskin and Schroeder's QFT textbook [70], Eq. (4.13) is called the "gauge replacement trick," which helps simplify our calculation.

With the short statements of the general procedures we have listed, we are now ready to present the final results, which can also be found in Refs. [68] [71]:

$$L_{\rho\delta}^{\rm ee}H^{\rho\delta} = -64m_n m_p \left(m_e^2 - l_e \cdot k\right) \left((1+3\lambda^2)E_\nu(E_e+\omega) + (1-\lambda^2)(\mathbf{l}_e \cdot \mathbf{l}_\nu + \mathbf{l}_\nu \cdot \mathbf{k})\right),$$
(4.14)

The rest of the terms in Eq. (4.8) can be processed in pretty much the same way. We have:

$$|\mathcal{M}_{02}|^2 = \frac{1}{m_p^2 \omega^2} L^{\rho\delta} \cdot H_{\rho\delta}^{\text{ee}}, \qquad (4.15)$$

with

$$L^{\rho\delta} \cdot H^{\rm ee}_{\rho\delta} = -64m_n m_p^3 \left((1+3\lambda^2) E_\nu E_e + (1-\lambda^2) \mathbf{l}_e \cdot \mathbf{l}_\nu \right), \qquad (4.16)$$

and finally the mixing term:

$$2\operatorname{Re}(\mathcal{M}_{01}\mathcal{M}_{02}^*) = \frac{1}{m_p \omega(l_e \cdot k)} M_{\operatorname{mix}}^{\operatorname{ee}}$$
(4.17)

with

$$M_{\rm mix}^{\rm ee} = -64m_n m_p^2 \left((1+3\lambda^2) E_\nu (2E_e^2 + E_e\omega - l_e \cdot k) + (1-\lambda^2) E_e (2\mathbf{l}_e \cdot \mathbf{l}_\nu + \mathbf{l}_\nu \cdot \mathbf{k}) \right).$$
(4.18)

Apparently, according to Eq. (4.14), Eq. (4.16), and Eq. (4.18), $|\mathcal{M}_0|^2$ only contains terms that are both P-even and T-even. In later chapters we will see how T-odd correlations arise. Also, the result for $|\mathcal{M}_0|^2$ is obtained by summing over all the particles' spin orientations, thus in the following calculation of decay rate Γ , we shall average over the neutron's spin polarization by associating $|\mathcal{M}_0|^2$ with an extra factor of 1/2.

With $|\mathcal{M}_0|^2$ obtained, one can then calculate the physical observable, the decay rate Γ . In QFT, an unpolarized decay rate Γ in the particle rest frame can be computed according to the following formula [70]:

$$\Gamma = \frac{1}{2m_A} \int \prod_i \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2E_i} \left(\frac{1}{2} \sum_{\text{spin}} |\mathcal{M}|^2\right) (2\pi)^4 \delta^4 \left(p_A - \sum_i p_i\right), \quad (4.19)$$

where m_A refers to the mass of the parent particle, and the index *i* runs over all the decayed products. For neutron radiative β decay, we have:

$$\Gamma = \frac{1}{(2\pi)^8 2m_n} \int \frac{d^3 \mathbf{p}_p}{2E_p} \frac{d^3 \mathbf{l}_e}{2E_e} \frac{d^3 \mathbf{l}_\nu}{2E_\nu} \frac{d^3 \mathbf{k}}{2\omega} \left(\frac{1}{2}|\mathcal{M}_0|^2\right) \delta^4 \left(p_n - p_p - l_e - l_\nu - k\right).$$
(4.20)

The actual phase space integrations could have been very complicated, but the LRO approximation helps reduce the complexity greatly. Simply speaking, we work in the neutron rest frame, so that p_n only has its zeroth component m_n . We can also drop the momentum \mathbf{p}_p in the heavy particle limit, so that in the actual evaluations only the zeroth components m_n and m_p contribute. In this way, many degrees of freedom are removed. Our specific choice of the coordinate system is shown in Fig.(4.1): we choose the direction of the electron 3-momentum \mathbf{l}_e as the z direction, and we can also let the two vectors, \mathbf{l}_e and \mathbf{k} , determine the x-z plane. The convenience in such a



Figure 4.1: The coordinate system for the calculation of neutron radiative β decay.

specific coordinate system is that some integrations become trivial, for instance, the integration over the electron solid angle Ω_e simply gives 4π , and the integration over the photon azimuthal angle ϕ_k simply yields 2π . A bit more calculations yields the ultimate expression for neutron radiative β decay rate:

$$\Gamma(\omega^{\min}) = \frac{1}{16m_n^2(2\pi)^6} \int_{\omega^{\min}}^{\omega^{\max}} \omega d\omega \int_{m_e}^{E_e^{\max}(\omega)} |\mathbf{l}_e| dE_e \int_{-1}^1 dx_k \int_{-1}^1 dx_\nu \int_0^{2\pi} d\phi_\nu E_\nu \times \left(\frac{1}{2}|\mathcal{M}_0|^2\right) \Big|_{p_p, E_\nu} (4.21)$$

where $x_k \equiv \cos(\theta_k)$, $x_\nu \equiv \cos(\theta_\nu)$, and $E_e^{\max}(\omega) = m_n - m_p - \omega$. In Eq. (4.21), the conditions on $|\mathcal{M}_0|^2|$ refer to the constraint of the energy-momentum conservation

 $p_n-p_p-l_e-l_\nu-k=0$, as demanded by the δ function in Eq. (4.20), which can be shown to yield the second condition $E_\nu=m_n-m_p-E_e-\omega$ in LRO approximation. The highest possible value of ω is $\omega_{\max}=m_n-m_p-m_e\approx 0.782 {\rm MeV}$, whereas ω_{\min} refers to the lowest detectable photon energy that is decided by experimental consideration. Note here we are not allowing ω to go to zero, because a photon of zero energy cannot be detected. Thus there is no infrared divergence to worry about. Although Eq. (4.21) looks quite ready for numerical evaluation, it can be further simplified still. One is able to work out the integrations of $\int d\phi_\nu$ as well as $\int dx_\nu$ analytically, thus leaving at most 3 integrations to numerical evaluation, which can be done efficiently.

As suggested by Eq. (4.21), Γ depends on ω_{\min} , which refers to the minimum detectable photon energy that is determined by the sensitivity threshold of the experimental devices. As for the electron energy integration, we have for each chosen ω , $E_e^{\max}(\omega) = m_n - m_p - \omega$, and therefore when the electron is at rest, ω reaches its maximum $\omega_{\max} = m_n - m_p - m_e \approx 0.782$ MeV.

With Γ obtained, we translate it into a branching ratio (BR), which is a more accessible experimental quantity in the case of neutron decay because knowledge of the initial neutron flux is not needed. In general, the BR for a particular decay mode is defined as the ratio of the number of atoms decaying by that decay mode to the number decaying in total. In terms of decay rate, BR can also be equivalently obtained by:

$$BR_i = \frac{\Gamma_i}{\Gamma_{tot}},\tag{4.22}$$

where Γ_i refers to the *i*th allowed decay mode. For the total decay rate Γ_{tot} , which may include many decay modes, some of which may have not discovered yet in experiments for being tiny, it can be deduced from the measured total lifetime since $\Gamma_{\text{tot}} = 1/\tau$, where τ refers to the total lifetime of the parent particle. In the most recent PDG [19], one finds for neutron $\tau_n = 880.1 \pm 1.1$ s, which needs to be converted into the natural unit system for consistency. We have:

$$1 \,\mathrm{s} = \frac{10^{22}}{6.582122} \,\mathrm{MeV^{-1}}.\tag{4.23}$$

To be more transparent for possible double checking, I am listing our actual numerical inputs here; that is, we employ the following: neutron mass $m_n = 939.56533$ MeV; proton mass $m_p = 938.272$ MeV; electron mass $m_e = 0.511$ MeV; fine-structure constant $\alpha = 1/137.036$; Fermi constant $G_F = 1.16639 \times 10^{-11}$ MeV⁻²; $\lambda \equiv g_A/g_V = 1.2701$; $g_V = \sqrt{1 + \Delta_r} V_{ud}$ with the $V_{ud} = 0.974$, and $\Delta_r = 0.024$, representing a small radiative correction. The branching ratios at some specified values of ω_{\min} are listed in Tab. (4.1), where BR⁰ means that the final-state Coulomb correction is not included. In a later section, we will discuss the final-state Coulomb correction in more detail. As a specific comparison between the theoretical calculation and a recent measurement of the branching ratios of neutron radiative β decay, we obtain that in the photon energy range $\omega \in [0.015, 0.340]$ MeV, the theoretical prediction of the BR value is BR⁰_{theo} $\approx 2.85 \times 10^{-3}$. As will be seen in a later section, the correction due to the Coulomb correction will be a few percent. The experimental value is

Table 4.1: Branching ratios of neutron radiative β decay for various ω_{\min} , where the final-state Coulomb correction has not been included.

$\omega_{\min}({ m MeV})$	$\mathrm{BR}^0(n)$
0.01	3.45×10^{-3}
0.05	1.41×10^{-3}
0.1	$7.19 imes 10^{-4}$
0.3	8.60×10^{-5}

 $BR_{exp} = (3.09 \pm 0.32) \times 10^{-3}$ [69]. Nevertheless, we see they agree well within the error bar.

There is another important reason for us to have the discussion of CP-conserving part here. We discuss the T-even part first to get prepared for the later work. In later chapters where we will talk about a T-odd asymmetry, and we will define the physical observable \mathcal{A} :

$$\mathcal{A} = \frac{\Gamma_+ - \Gamma_-}{\Gamma_+ + \Gamma_-},\tag{4.24}$$

where Γ refers to the radiative β decay rate. The subscript "+" and "-" refer to the positive and negative hemisphere of the phase space integral with respect to the sign of the triple-product correlation in momenta $\mathbf{l}_{\nu} \cdot (\mathbf{l}_e \times \mathbf{k})$. Clearly, the CP-conserving part of Γ that we have obtained in this chapter plays a dominant role in the denominator of Eq. (4.24). With the CP-conserving part we have computed in this chapter, we will be able to compute the asymmetry \mathcal{A} provided that the T-odd part is obtained. This is just what we will be doing in later chapters.

Also, as one may have noticed, in this section we do not consider the effect of finalstate Coulomb correction, but we know that since the decayed proton and electron are both charged particles, right after they appear in the decay they certainly feel a nonnegligible attractive Coulomb force between them as they fly apart. The influence of such a process starts as a higher order term in α . A careful work should not simply ignore such an effect even if it is a small correction; this and other like corrections will be discussed in full detail to give a sharper prediction for the experimental energy spectrum of neutron radiative β decay.

4.2 Higher–Order Corrections

From the discussion in the previous section, we see that the SM prediction agrees well with the experimental result within the range of uncertainty, but we also see that the precision of the direct measurement of the neutron radiative β decay branching ratio is currently only up to $\mathcal{O}(10\%)$. The RDK-II collaboration is in the process of raising the precision of the BR measurement to $\mathcal{O}(1\%)$ [72]. We are then expecting to obtain much more information on the decay. As emphasized at the beginning of the section, we are considering single-photon radiation, and we have been taking the leading order contributions as the desired accuracy. In facing an upcoming higher precision measurement at the level of 1%, such $\mathcal{O}(\alpha)$ approximation may not be sufficient any more. In order to sharpen the theoretical predictions against the sharper experimental data in the future, we need to consider the higher-order contributions as well. In the following sections, we will try to consider two important factors that may turn out to be significant corrections to our earlier calculations. One concerns the final-state Coulomb interaction, which, strictly speaking, is not of fixed order but starts in $\mathcal{O}(\alpha^2)$, between the electron and proton in neutron radiative β decay. The other concerns the possibility of double photon emission, which is usually thought to be suppressed by an additional factor of α compared to the single photon emission that we have been assuming thus far. Although this is true, as we will show a little bit later, there can be other compensating effects that can make such higher order corrections give noticeable contribution.

4.2.1 Final-State Coulomb Interaction

As has been suggested above, the calculations thus far have not taken final-state Coulomb corrections into consideration. But since the decay products, proton and electron, are both charged particles, a long ranged, attractive Coulomb force between them is always present, and its effect can be greatly enhanced for very low electron kinetic energy. Thus to be on the safe side, such a final-state Coulomb interaction deserves being investigated in detail, and it is what we present in this section.

The Coulomb correction can be described as a "distortion" of the wave functions of the outgoing electron and proton due to the Coulomb force between them. This means that when calculating the decay rate of neutron β decay, instead of simply assuming that the final-state wave functions (Dirac spinors) of the electron and proton are free plane waves, one should have solved for the asymptotic states using the Dirac equation with the spherically symmetric Coulomb field included. In the limit of heavy nucleon mass, such an influence on the proton side is only of recoil order, thus it can be ignored in our approximation. The influence on the electron wave function can be more significant. One can show that, on inserting the absolute square of the distorted electron wave function into the scattering amplitude square $|\mathcal{M}|^2$, the influence of the distortion of the electron wave function on the total decay rate can be represented as a modification of the final phase space integration

$$\int d\rho \to \int F(Z, E_e) d\rho, \qquad (4.25)$$

where the inserted additional factor $F(Z, E_e)$ is referred to as the Fermi function because this work was first carried out by Fermi [73] [74]. Adopting the conventions of Wilkinson [75], we have:

$$F(Z, E_e) \equiv 2(1+\gamma)(2|\mathbf{l}_e|R)^{-2(1-\gamma)} \frac{|\Gamma(\gamma+iy)|^2}{\Gamma^2(2\gamma+1)} e^{\pi y}, \qquad (4.26)$$

where y is defined as:

$$y \equiv \frac{\alpha Z E_e}{|\mathbf{l}_e|},\tag{4.27}$$

Table 4.2: Coulomb corrections (CC) to the branching ratios of neutron radiative β decay, the subscript "0" means that the Coulomb corrections is not considered, the subscript "CC" means that the Coulomb corrections is considered.

$\omega_{\min}(MeV)$	$BR_0(n)$	$BR_{CC}(n)$
0.01	3.45×10^{-3}	3.56×10^{-3}
0.05	1.41×10^{-3}	1.45×10^{-3}
0.1	$7.19 imes 10^{-4}$	7.43×10^{-4}
0.3	$8.60 imes 10^{-5}$	$8.96 imes 10^{-5}$

and γ is defined as:

$$\gamma \equiv \sqrt{1 - (\alpha Z)^2}.\tag{4.28}$$

We note \mathbf{l}_e and E_e refer to the 3-momentum and total energy of the emitted electron; Z is the electric charge of the daughter nucleus; R is the charge radius of the daughter nucleus; $\alpha \approx 1/137$ is the fine-structure constant. The general expansion of $|\Gamma(\gamma+iy)|$ is obtained from the following equation [75]:

$$\log\left(|\Gamma(\gamma+iy)|^{2}\right) = \log\left(\frac{y_{1}^{2}}{\gamma^{2}+y^{2}}\right) + \log\left(\frac{\pi}{y_{1}\sinh(\pi y_{1})}\right) + \log(1+y_{1}^{2}) + (1-\gamma)$$

$$\times \left[2 - \log[(1+\gamma)^{2}+y^{2}] + \frac{2y}{1+\gamma}\arctan\left(\frac{y}{1+\gamma}\right) + \frac{1}{(1+\gamma)^{2}+y^{2}}\frac{1}{6a}\right] - 3\log(a),$$
(4.29)

which is followed by the definitions:

$$a \equiv \frac{2}{1+\gamma},\tag{4.30}$$

$$y_1 \equiv \frac{a\alpha E_e}{|\mathbf{l}_e|} = \frac{2}{1+\gamma}y.$$
(4.31)

Equations (4.26) through (4.31) are valid for any nuclear β decay. In our discussions, we simply assume that the nucleus is point-like, there will be corrections due to the finite size of nucleus [76] [77]. In this section we are just considering the neutron case with Z = 1. The proton charge radius R_p is roughly assumed to be ≈ 1 fm. Here we are just evaluating the correction at the "edge" of the charge distribution. It turns out that the Fermi function Eq. (4.26) only depends on the charge radius of the daughter nucleus very weakly, so one doesn't really need a very precise value of R_p . Inserting Eq. (4.26) as an additional multiplicative factor into the phase space integration of Eq. (4.21), and then proceeding to make the numerical evaluation, we obtain the Coulomb-corrected branching ratios, BR_{CC}, as shown in Table (4.2).

4.2.2 Influence of Double Photon Radiation on Neutron Radiative β Decay

In the case of single-photon radiation, we have considered the contributions from the tree-level Feynman diagrams in Figure (3.5), which contain one factor of e. On taking

the absolute square of the corresponding scattering amplitudes, we obtain the dominant contribution of $\mathcal{O}(\alpha)$. As for the $\mathcal{O}(\alpha^2)$ contribution, it receives contributions from two classes of diagrams. The first class refers to the ones shown in Figure (4.2) and Figure (4.3), which are of $\mathcal{O}(e^3)$ since they contain an additional photon loop. It is the interference between the amplitudes obtained from Figure (3.5) and the ones from Figure (4.2) as well as Figure (4.3) that yields a correction of $\mathcal{O}(\alpha^2)$. Such



Figure 4.2: Neutron radiative β decay with one loop, part I.

 $\mathcal{O}(\alpha^2)$ corrections contain unphysical infrared divergences¹ arising from the loop integration, and thus is incomplete. In order to cancel the infrared divergence to get the physical value of $\mathcal{O}(\alpha^2)$ correction, one is forced to include the second class of contribution shown in Figure (4.5), where one of the k_1 and k_2 is set to be equal to k, and the other is integrated out up to the lowest detectable energy ω_{\min} , e.g., and is thus undetectable. Following the standard procedures in Ref. [70], one can manage to cancel the infrared divergence and get an infrared finite $\mathcal{O}(\alpha^2)$ correction. Thus we see that the two-photon radiation process serves as an important subset of the processes to give an $\mathcal{O}(\alpha^2)$ correction.

The process of two-photon radiation in neutron radiative β decay is usually thought to be suppressed by a factor of α , and thus can be ignored within certain expectations of accuracy. In the following discussion, however, we will consider one situation where such a process may not be simply ignored, and evaluate its contribution explicitly.

In our calculations so far, we have been working on the case of only single photon emission. This approximation follows the general argument that although the electromagnetic field radiation in the real world is always composed of multiple photon

¹Counterterms will also be required to control their ultraviolet behavior.



Figure 4.3: Neutron radiative β decay with one loop, part II.

radiation, as shown in Fig. (4.4), the single photon case is dominant because each



Figure 4.4: An example of a higher order graph of neutron radiative β decay.

extra photon emission brings an extra factor of e in the scattering amplitude, making it smaller than the preceding one. Normally such an argument would be valid as long as we are only seeking for an numerical accuracy up to $\mathcal{O}(\alpha)$. In some special situations, however, the argument breaks down. In QED bremsstrahlung, it is well known that as the energy of an emitted photon becomes lower and lower, the corresponding cross section increases logarithmically. For a sufficiently low photon energy, the extra α does not mean a safe suppression any more – the increases due to the lowering the minimum photon energy may outstrip the $\mathcal{O}(\alpha)$ suppression to a great degree. Such a situation may prove especially important to an on-going experiment at National Institute of Standards and Technology (NIST) [72].

Now let us consider the T-even part of the two-photon emission process as described in Fig. (4.5). We should be aware of the situation that since now the final state contains two photons, we should make the amplitude symmetric by including the Feynman diagrams in which the γ_1 and γ_2 switch places. We have the total



Figure 4.5: Two photons emission in neutron radiative β decay. We should keep in mind the 3 diagrams as shown are accompanied by 3 additional ones with γ_1 and γ_2 switched due to the symmetry of two-boson system.

amplitude:

$$\mathcal{M}^{\gamma\gamma} = \frac{e^2 g_V G_F}{\sqrt{2}} (\mathcal{M}^{\rm ee} + \mathcal{M}^{\rm pp} - \mathcal{M}^{\rm ep}), \qquad (4.32)$$

where the superscripts of \mathcal{M} on the right hand side of Eq. (4.32) denote the locations of the sources of the photon emissions. Again, one should keep in mind that \mathcal{M}^{ee} , \mathcal{M}^{pp} , and \mathcal{M}^{ep} should all be understood as symmetric arrangements under the exchange of γ_1 and γ_2 . Following the Feynman rules one can write down the scattering amplitudes, which can be readily obtained by dressing the single photon case with an additional photon radiation, which has been described by Eq. (4.4) and Eq. (4.5). One has:

$$\mathcal{M}^{\text{ee}} = \frac{1}{2} \left(\bar{u}_e \phi_2^* \frac{\vec{l}_e + \vec{k}_2 + m_e}{2l_e \cdot k_2} \phi_1^* \frac{\vec{l}_e + \vec{k}_0 + m_e}{(l_e + k_0)^2 - m_e^2} \gamma_\rho (1 - \gamma_5) v_\nu \bar{u}_p \gamma^\rho (1 - \lambda \gamma_5) u_n + \text{term}^{\text{ee}} (1 \leftrightarrow 2) \right)$$

$$(4.33)$$

with "term^{ee}(1 \leftrightarrow 2)" referring to the similar expression as the first term only with the exchanges of $\epsilon_1 \leftrightarrow \epsilon_2$ as well as $k_1 \leftrightarrow k_2$, and $k_0 \equiv k_1 + k_2$. Similarly, one can also obtain \mathcal{M}^{pp} and \mathcal{M}^{ep} :

$$\mathcal{M}^{\rm pp} = \frac{1}{2} \bigg(\bar{u}_e \gamma_\rho (1 - \gamma_5) v_\nu \bar{u}_p \epsilon_2^* \frac{p_p + k_2 + m_p}{2p_p \cdot k_2} \epsilon_1^* \frac{p_p + k_0 + m_p}{(p_p + k_0)^2 - m_p^2} \gamma^\rho (1 - \lambda \gamma_5) u_n + \operatorname{term}^{\rm pp}(1 \leftrightarrow 2) \bigg),$$

$$(4.34)$$

and

$$\mathcal{M}^{\text{ep}} = \frac{1}{2} \left(\bar{u}_e \not{\epsilon}_1^* \frac{\not{l}_e + \not{k}_1 + m_e}{2l_e \cdot k_1} \gamma_\rho (1 - \gamma_5) v_\nu \bar{u}_p \not{\epsilon}_2^* \frac{\not{p}_p + \not{k}_2 + m_p}{2p_p \cdot k_2} \gamma^\rho (1 - \lambda \gamma_5) u_n + \text{term}^{\text{ep}} (1 \leftrightarrow 2) \right).$$

$$(4.35)$$

It is easy to check that $\mathcal{M}^{\gamma\gamma}$ satisfies Ward identity. With Eq. (4.33), Eq. (4.34) and Eq. (4.35), one can obtain $|\mathcal{M}^{\gamma\gamma}|^2$ by following the same procedures as before – separating the γ matrices and the fermion spinors into leptonic and hadronic traces and then work out the traces by following the standard algebra of γ matrices. But we are here dealing with an $\mathcal{O}(\alpha^2)$ correction, the detailed calculations are for sure much more complicated than the leading order calculations – and the number of terms generated explodes in higher orders. Analytic calculations with brute force will simply go on for ever. To make our lives easier, we employ the FORM [78] code to handle the detailed calculations of $|\mathcal{M}^{\gamma\gamma}|^2$. The detailed FORM code can be found in Appendix B. It turns out that the output of $|\mathcal{M}^{\gamma\gamma}|^2$ contains over 250 terms, which are not even manageable for reproducing in this thesis. The numerical work is still manageable, however.

What we are interested in here is to see its influence on the measurement of neutron radiative β decay. As mentioned before, normally one would naively argue that the contribution of two-photon bremsstrahlung is suppressed by an extra factor of α comparing to the single-photon contribution, thus within a certain requirement of precision, it is negligible. This looks quite convincing, and in fact it is true in most of the occasions. But as the photon energy ω getting lower and lower, the decay rate increases as $\log(\omega)$, thus the suppression α may not be sufficient for us to ignore them. We should consider the detailed contributions in such a situation carefully. In ongoing neutron radiative β decay experiments at NIST, the Monte Carlo programs, which simulate the experiment, have thus far assumed that only single photon emission occurs [79]. Our goal here is to compute the two-photon emission rate and to provide some way to help tell the two-photon events apart from the one-photon ones. Here we focus on the angular correlation, that is, we want to know at certain selected energies of the two photons, what is the differential decay rate as a function of the two-photon opening angle $\theta_{k_1k_2}$? This would provide us some useful information as to whether the photon signal is really from the expected single-photon events, or just from some unexpected two-photon signal. For this goal, the original choice of coordinate system, where \mathbf{l}_e are chosen to be parallel to z-axis, is not convenient; we shall make a little different choice: we make \mathbf{k}_1 as the z-axis, and let \mathbf{l}_e sit in the x-z plane. This way helps expose the dependence of the differential decay rate on the relative angle between \mathbf{k}_1 and \mathbf{k}_2 . Fig. (4.6) and Fig. (4.7) show the differential decay rate $d\Gamma_{\gamma\gamma}/d\omega_1 d\omega_2 d\cos(\theta_{k_1k_2})$ vs $\cos\theta_{k_1k_2}$. From



Figure 4.6: $d\Gamma_{\gamma\gamma}/d\omega_1 d\omega_2 d\cos(\theta_{k_1k_2})$ (in natural unit) versus $\cos(\theta_{k_1k_2})$ when $\omega_1 = 0.01 \text{ MeV}, \omega_2 = 0.5 \text{ MeV}.$

Fig. (4.6) and Fig. (4.7), we see that the influence of the two-photon contribution to the differential decay rate gets more and more significant for smaller ω_1 and ω_2 . Also, at lower energies, the two emitted photons show a clear preference for being collinear. Such a preference in turn makes its inclusion important for the interpretation of the experimental data. With our numerical analysis and analytic results, we believe we have a better theoretical description of the experimental results.

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Figure 4.7: $d\Gamma_{\gamma\gamma}/d\omega_1 d\omega_2 d\cos(\theta_{k_1k_2})$ (in natural unit) versus $\cos(\theta_{k_1k_2})$ when $\omega_1 = 0.02 \,\mathrm{MeV}, \,\omega_2 = 0.1 \,\mathrm{MeV}.$

Chapter 5 Topics in Low-Energy Physics for BSM Searches (in Charged–Current Processes)

After briefly discussing the T-even correlations of the neutron radiative β decay in Chapter 4, we now turn to one of the major topics of this thesis – the possibility of a triple-product correlation in momenta, serving as a T-odd and thus CPviolating source. Such a T-odd correlation arises from the interference between the SM-induced coupling and the exotic coupling that is proposed by Harvey, Hill, and Hill [80] [81] [82]. From this point onwards, I will refer to their theoretical work as "HHH." Understanding their work requires discussion of the following topics:

1. axial anomaly in gauge field theory (QED and QCD).

2. the chiral effective theory.

Briefly introducing these topics is the main job in this chapter, so that we have a context for the discussion of HHH in the next chapter.

5.1 Introduction to Axial Anomaly Theory

discussions, We now make a very quick review of the problem of the axial anomaly in QED and QCD in order to set the stage for later discussion. revealing the fundamental physical idea that is going to be suggestive to the later contents of this chapter.

In Chapter 3 of the general discussions of parity violation, I mentioned the idea of the left-handed current J_L^{μ} and the right-handed current J_R^{μ} by introducing the lefthanded and right-handed projection operator P_L and P_R . In this chapter I include more details on this issue [70] [83]. Let us return to the original Dirac equation for a free fermion field ψ with mass m:

$$(i\partial - m)\psi = 0, \tag{5.1}$$

which, in the chiral limit, or, equivalently, in the limit of vanishing fermionic mass, clearly gives:

$$i\partial\!\!\!/\psi = 0. \tag{5.2}$$

On multiplying γ_5 on both sides of Eq. (5.2) and applying the commutation relation $\{\gamma_5, \gamma_\mu\} = 0$, we have:

$$i\partial\!\!\!/\gamma_5\psi = 0. \tag{5.3}$$

We can then superpose Eq. (5.2) and Eq. (5.3) to form the combinations:

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi \equiv P_L\psi, \quad \psi_R = \frac{1}{2}(1 + \gamma_5)\psi \equiv P_R\psi.$$
 (5.4)

As mentioned before, ψ_L and ψ_R are two independent solutions of the massless Dirac equation with definite chirality (i.e., handedness). For a massless fermion moving

with precise momentum, these solutions correspond respectively to the spin being anti-aligned (left-handed) and aligned (right-handed) relative to the momentum. Eq. (5.2) and Eq. (5.3) can also be easily obtained from the left-right decoupled fermion Lagrangian:

$$\mathcal{L} = i\bar{\psi}_L \partial\!\!\!/ \psi_L + i\bar{\psi}_R \partial\!\!\!/ \psi_R. \tag{5.5}$$

Apparently, here \mathcal{L} is invariant under the separate global chiral transformations:

$$\psi_L \to \psi'_L = e^{-i\alpha_L} \psi_L,$$

$$\psi_R \to \psi'_R = e^{-i\alpha_R} \psi_R,$$
(5.6)

where α_L and α_R refer to left-handed and right-handed real-valued rotations. According to Noether's theorem, such global chiral transformations yield two separate conserved currents:

$$J_{L,R}^{\mu} \equiv \bar{\psi}_{L,R} \gamma^{\mu} \psi_{L,R}, \qquad (5.7)$$

$$\partial_{\mu}J^{\mu}_{L,R} = 0, \qquad (5.8)$$

which can be readily reorganized into the conserved vector current V^{μ} , and the conserved axial-vector current A^{μ} :

Such a straightforward derivation is based on the classical point of view and can be trivially extended to the case of the quark fields. For simplicity, let us only focus on the SU(2) flavor $(SU(2)_f)$ group, so that only the lightest quarks u and d, are considered. Similar considerations exist for $SU(3)_f$, which concerns the three lightest quark flavors u, d, and s. In the chiral limit we set the small masses of the u and dquarks to zero. Thus they form left-handed and right-handed quark isospin doublets:

$$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad Q_R \equiv \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{5.11}$$

following which there would then be four separate symmetric unitary global transformations, both

$$Q_L \to \exp(i\theta_L^{(s)})Q_L, \quad Q_R \to \exp(i\theta_R^{(s)})Q_R$$
 (5.12)

with the superscript "(s)" meaning the isosinglet, where $\theta_{L,R}^{(s)}$ are two independent parameters, and

$$Q_L \to \exp(i\tau^a \theta_L^a) Q_L, \quad Q_R \to \exp(i\tau^a \theta_R^a) Q_R$$
 (5.13)

where τ^a refer to the SU(2) group generators with a = 1, 2, 3, and $\theta^a_{L,R}$ are independent transformation parameters. The four independent symmetric transformations

above are commonly denoted as $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$ symmetry, and lead to the following 4 conserved left- and right- handed currents:

$$J_{L,R}^{\mu} = \bar{Q}_{L,R} \gamma^{\mu} Q_{L,R}, \qquad (5.14)$$

$$J_{L,R}^{\mu a} = \bar{Q}_{L,R} \gamma^{\mu} \tau^{a} Q_{L,R}, \qquad (5.15)$$

which, on using $Q = Q_L + Q_R = P_L Q + P_R Q$, leads to the following two conserved axial-vector currents:

$$J^{\mu 5} = \bar{Q}\gamma^{\mu}\gamma^{5}Q,$$

$$\partial_{\mu}j^{\mu 5} = 0,$$

$$I^{\mu 5 a} = \bar{Q}\gamma^{\mu}\gamma^{5}\sigma^{a}Q$$
(5.16)

$$J^{\mu 5 a} = Q \gamma^{\mu} \gamma^{5} \tau^{a} Q,$$

$$\partial_{\mu} J^{\mu 5 a} = 0.$$
(5.17)

The conservation of the axial-vector currents shown in Eq. (5.16) and Eq. (5.17) seem to be on a firm footing; however, more careful investigations that are strictly based on the fundamental principles of QFT, however, reveal that the seemingly flawless conclusion in Eq. (5.16) does not hold if a gluon gauge field present. In fact, even in the simpler Abelian U(1) case, such a problem is also present. Historically, such a result that is apparently against the expectation of classical field theory gained the unpleasant name "anomaly." It was later realized that the failure of Eq. (5.16)is because of quantum corrections. One finds that generally the conservation of the axial vector current in QED and QCD is incompatible with gauge invariance, and the so-called anomaly actually plays a very important role, without which some special observed processes cannot be explained.

There can be many ways to reveal the nonconservation of $J^{\mu 5}$. First let us recall the fundamental QCD Lagrangian, Eq. (3.45), with only u and d quarks in the presence of a gluon field. On ignoring the quark masses, we have:

$$\mathcal{L}_{\text{QCD}} = \overline{Q} i \not\!\!\!D Q - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}, \qquad (5.18)$$

where a refers to the color index, the covariant derivative D_{μ} is defined by Eq. (3.47), and the gluon field strength tensor $G^a_{\mu\nu}$ is defined by Eq. (3.48). One may perform detailed and careful calculations of the so-called triangular diagrams as shown in Fig. (5.1) to reveal the nonconservation of $J^{\mu 5}$. Note there is one subtlety here: the evaluation of Fig. (5.1) involves dimensional regularization, which introduces extra dimensions into the loop integrals, but we know that γ_5 is an intrinsically fourdimensional object, thus an extension of the definition of γ_5 into d dimensions is needed [84]. The actual procedures are rather tedious and are unneeded later, so that let us directly quote the results [70]:

$$\partial_{\mu}J^{\mu5} = -\frac{g_s^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} \neq 0, \qquad (5.19)$$

$$\partial_{\mu}J^{\mu5a} = -\frac{g_s^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}G^c_{\mu\nu}G^d_{\rho\sigma}\operatorname{Tr}(\tau^a)\operatorname{Tr}(t^c t^d) = 0, \qquad (5.20)$$



Figure 5.1: Summing over the two Feynman diagrams yields $\partial_{\mu}J^{\mu5} \neq 0$. The internal lines refer to massless quark propagators, and the two curvy lines refer to the gluon fields.

where t^c and t^d are the color matrices with $\text{Tr}(t^c t^d) \propto \delta^{cd}$. We stress that similar relation as in Eq. (5.19) also holds for QED, where the gluon field strength $G^a_{\mu\nu}$ is to be replaced by the electromagnetic field strength $F_{\mu\nu}$ with the color factor removed. We focus on the QCD case in what follows.

From Eq. (5.19) we see that the isosinglet axial-vector current $J^{\mu 5}$ is not conserved in the first place due to QCD quantum effects. The isospin nonsinglet axial-vector current $J^{\mu 5a}$, however, remains conserved because in Eq. (5.20) we have $\text{Tr}(\tau^a) = 0$. Note that this conclusion only holds in strict QCD; the axial anomaly still plays a role in non-singlet currents because of QED effects. Eq. (5.19) and Eq. (5.20) tell us that if the Lagrangian of QCD that only contains the lightest quark flavors of u and d, it only possesses the symmetry $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_V$ – the $\text{U}(1)_A$ part does not represent a symmetry in the first place. Let us keep this conclusion in mind since it will be important for the later discussions in this chapter.

There is yet a more elegant and illuminating way to reveal the existence of the axial anomaly, which is based on the path integral treatment. It was first introduced by Fujikawa [85]. The detailed derivations are not quite relevant to us here, let us directly jump to the essence. In the standard path integral treatment, a physical massless quark system can be described by the generating functional

$$W[A^a_{\mu}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(i \int d^4x \mathcal{L}_{\text{QCD}}(\psi, \bar{\psi}, A^a_{\mu})\right)$$
(5.21)

in the absence of any source term. Here ψ and $\bar{\psi}$ represent two Grassman fields, which are characterized by the anticommutative nature, which is in accordance with the fundamental property of spin-1/2 fermion. In the language of path integrals, in order to get a conserved $J^{\mu 5}$, $W[A^a_{\mu}]$ must remain invariant under the transformation of Eq. (5.6). We have already shown the invariance of Lagrangian $\mathcal{L}_{\text{QCD}}(\psi, \bar{\psi}, A^a_{\mu})$ under Eq. (5.6), but we need to be more careful here because a chiral transformation of the Grassman field results in the following identity on the measure sector:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \to \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' \mathcal{J}^2, \qquad (5.22)$$

where \mathcal{J} is the Jacobian accompanying the change of variables, thus $W[A^a_{\mu}]$ cannot be said to be invariant unless one can also show that the Jacobian $\mathcal{J} = 1$. Interestingly, detailed analysis that is based on the special nature of Grassman algebra tells us that, in the presence of the gauge field A_{μ} , $\mathcal{J} \neq 1$! Thus our quick conclusion that the conserved $J^{\mu 5}$ is conserved is false, it is really not conserved at the quantum level, and detailed calculations in path integral method [70] can also reproduce the results of Eq. (5.19) and Eq. (5.20).

We have so far considered the U(1) axial anomaly with the lightest two quark flavors u and d in QCD. In fact, similar idea can also apply when these quarks couple to the electromagnetic field. People have learned that the mechanism of anomalies has wide applications in modern physics. A very typical example is the so-called isovector axial current $J^{\mu 5(3)}$, which transforms as the third component of an SU(2) flavor doublet

$$J^{\mu 5(3)} = \bar{u} \gamma^{\mu} \gamma_5 u - \bar{d} \gamma^{\mu} \gamma_5 d, \qquad (5.23)$$

contains an anomaly when the light quarks are coupled to an external electromagnetic field. Such an anomaly plays a critical role in describing the neutral pion decay process $\pi^0 \to \gamma\gamma$. In other words, without the axial anomaly, one cannot explain the observed decay of $\pi^0 \to \gamma\gamma$.

Let us summarize the gist of this section on the axial anomaly. That is, even though the existence of some symmetries seem quite convincing based on Lagrangian– level arguments, they may receive quantum corrections and be explicitly broken. In such cases, the language of path integrals turns out to be especially illuminating: it tells us that even if a seemingly symmetric transformation does leave the relevant Lagrangian or action invariant – the classical condition for a symmetry to hold – it may still lead to a Jacobian which deviates from unity, and thus signifies an explicit breaking of the symmetry. Instead of being a disaster, such anomaly effects indeed play critical roles in understanding certain particle processes. Anomalies happen not only in the fundamental QCD theory, but also in its low-energy effective theories such as chiral perturbation theory, which will be discussed later.

There is an additional important area where an anomaly can enter: it provides a possible mechanism in the electroweak gauge theory to allow for baryon number violation. So far we have briefly discussed the picture of chiral anomaly arising from the global chiral transformations such as the one in Eq. (5.13) in QED and QCD, where the fundamental couplings between matter fields and gauge fields are always of vector type. In the Weinberg-Salam model [86] [87], on the other hand, a special chiral nature has been built in the electroweak Lagrangian– left-handed fermions couple differently from right-handed fermions. In the left-handed sector, matter fields interact with charged gauge fields via the "V-A" law. Evaluations of the triangular quark loops that are very similar to Figure (5.1), only with gluons replaced by W^{\pm} , would generate another anomaly: the anomalous baryon number current, denoted by $J_{\rm Bar}^{\mu}$ [10]:

$$\partial_{\mu}J^{\mu}_{\text{Bar}} = \partial_{\mu}\sum_{q} (\overline{q}_{L}\gamma^{\mu}q_{L}) = \frac{g^{2}_{W}}{16\pi^{2}}\text{Tr}G_{\mu\nu}\tilde{G}_{\mu\nu}, \qquad (5.24)$$

where q refers to all the quark flavors, g_W denotes the $SU(2)_L$ gauge coupling constant, and $G_{\mu\nu}$ the electroweak field strength tensor

$$G_{\mu\nu} = \tau^a (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu)$$
(5.25)

with the τ^a being SU(2) generators. Equation (5.24) gives rise to baryon number violation, which, as mentioned in Chapter 1, is another necessary condition to generate baryogenesis. In this sense, SM can provide a possible mechanism, via an anomaly in the electroweak gauge theory, to allow for baryon number violation. However one encounters the similar problem here as in the case of CP violation: SM in principle does allow for baryon number violation to understand baryogenesis, but it is not sufficient due to the smallness of the electroweak coupling [88] [89]. It is argued that although the anomaly-induced baryon number violation is badly suppressed at low energies, the suppression could be greatly relaxed at sufficiently high energies (or temperatures), and baryon number violation can become operative [90] [91].

To end this section, let us briefly discuss the issue of gauge anomaly cancellation. Generally speaking, if an anomaly is associated with a global symmetry, then it does not indicate any inconsistency of the theory, and can often have important physical consequences – the anomalies we have discussed before are of this type. On the other hand, if an anomaly is associated with a local (gauge) symmetry, then it indicates a fundamental inconsistency of the theory and must vanish. Thus one must find a way within the framework of SM to cancel the anomaly. Remarkably, this is indeed achieved in SM – the gauge anomaly arising from the quark sector gains a total cancellation with the one arising from the lepton sector that transforms via the electroweak symmetry group SU(2)_L × U(1)_Y. Here the necessary conditions are [70] [92]:

1. there are three colors in QCD;

2. there are an equal number of generations of quark and lepton, and in fact the anomaly cancellation occurs generation by generation;

3. specific assigned couplings of the quarks and leptons.

As discussed before, we convince ourselves that the number of color degrees of freedom is three. Also, three generations of quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix},$$
(5.26)

and three generations of leptons:

$$\left(\begin{array}{c}\nu_e\\e\end{array}\right), \quad \left(\begin{array}{c}\nu_\mu\\\mu\end{array}\right), \quad \left(\begin{array}{c}\nu_\tau\\\tau\end{array}\right) \tag{5.27}$$

have been observed. We say that the SM is gauge anomaly free.

5.2 Introduction to Chiral Perturbation Theory

As another important part of this chapter, we consider chiral perturbation theory, which is an important example of an effective field theory. implementation of the quantum ladder. In constructing theories to describe the real world, we have to accept that even our most well-considered theoretical framework, the SM, is probably an effective field theory in the sense that it is a low energy approximation of some underlying, more "fundamental" theory. We should keep in mind that the SM is theoretically consistent up to very high energy scales, thus the "new physics" energy scale can also be very high [93]. In general, as the energy increases and smaller distances are probed, new degrees of freedom become relevant that must be included in the theory. At the same time, other fields may lose their status as fundamental fields if the corresponding states are recognized as bound states of new degrees of freedom. On the other hand, as the energy is lowered, some degrees of freedom are frozen out and disappear from the accessible spectrum of states. To construct the effective field theory at low energies, we rely on the symmetries of the fundamental, or underlying, theory. The Lagrangian must contain all terms allowed by the symmetries of the fundamental theory for the given set of fields, so that the effective theory can be the low-energy limit of the fundamental theory [94].

As an explicit example, chiral perturbation theory (ChPT) serves as a systematically improveable method of analyzing strong interaction processes at low-energies, following the symmetries of quantum chromodynamics (QCD). To understand the origin of chiral perturbation theory, we first return to the fundamental theory of the strong interaction, QCD [95] [96][97]. Here let us recall the QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \sum_{f=u,d,s,c,b,t} \overline{Q}_f i \not\!\!\!D Q_f - \overline{Q}_{f,R} M Q_{f,L} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \text{h.c.}, \qquad (5.28)$$

where Q_f refers to a quark state of flavor f, noting f can be either a u, d, s, c, b, or t quark. The matrix M is the quark mass matrix, and D_{μ} is the covariant derivative:

$$D_{\mu} \equiv \partial_{\mu} - igt^a A^a_{\mu}, \tag{5.29}$$

with t^a referring to any of eight SU(3) generators, A^a_{μ} the gluon field, and g the strong coupling constant. The tensor $G^a_{\mu\nu}$ denotes the gluon field strength tensor:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{5.30}$$

with f^{abc} the structure constant defined by

$$\left[\frac{t^a}{2}, \frac{t^b}{2}\right] = i f^{abc} \frac{t^c}{2}.$$
(5.31)

Such a theoretical construction of the strong interaction is based on the assumption of a local $SU(3)_C$ gauge symmetry. Comparing Eq. (5.28) with the standard theory of QED, we see that QCD provides a language similar to QED in describing the strong interactions: just as the interactions between electrically charged particles are
mediated by the exchange of virtual photons, the strong interactions between quarks are mediated by the exchange of virtual particles – the gluons, of which there are eight in number in a SU(3) theory. is now established that So far everything looks quite beautiful, but $SU(N)_{C}$ gauge theory has a special behavior called "asymptotic freedom" in the interaction between quarks [98] [99]. It means that when the momentum transfer between the quarks gets smaller and smaller, or, equivalently, when the separation of any two quarks in a hadron gets larger and larger apart, the coupling constant q in QCD increases faster and faster. Roughly speaking, it is a feature of the non-Abelian group that gives rise to such a special property: unlike the QED case, where the gauge particle, photon, does not carry electric charge, the QCD gauge particle, gluon, also carries color charge just like the quarks do, and thus interact with gluons themselves. QCD and QED use a similar QFT language to describe interactions, but their structures are fundamentally very different. For example, the resulting large coupling constant q in QCD at low energies vetoes the validity of the traditional method of perturbative expansion in powers of the coupling strength. The latter works very well in the case of QED because it has a relatively much smaller coupling strength and possesses no behavior of asymptotic freedom.

The nonperturbative nature of QCD at low momentum transfer makes any precise numerical analysis of a hadronic process a big challenge. People have been working on different methods for handling the strong interaction at low energies. Among these frameworks, ChPT is a successful and widely accepted one. There have been a variety of papers and review articles on this topic [100] [101] [102] [103] [104] [105] [106] [107] [108] [109]. We now review the basic aspects of chiral perturbation theory, which is developed as a theoretical tool to deal with the strong-coupling problem in the event that only light quarks (u, d, s) involved.

Being an example of effective field theory, the basic idea of ChPT is that one does not need to know everything in order to make a sensible description of the physical process that one is interested in. Instead of directly solving the underlying more fundamental theory, low-energy physics may be described with a set of variables that is suited for the particular energy region we are interested in. ChPT can then be used to calculate physical quantities in terms of an expansion in p/Λ , where p stands for momenta or masses that are supposed to be much smaller than a certain momentum or mass scale Λ . In this way, one is able to retain the basic spirit of perturbation theory.

5.2.1 QCD in the Chiral Limit

The core spirit of constructing ChPT is that the fundamental symmetries of the underlying fundamental theory should be kept in writing down the effective Lagrangian [110]. Only in this way can the validity of the low energy extension be placed on a firm footing. To see the relevant symmetry of the fundamental QCD theory, let us return to Eq. (5.28), and consider the case of u, d, and s quarks only in the limit in which their masses are negligible relative to all the other energy scales in the problem. This limit is called the chiral limit. With the recently updated values of quark masses $m_u \approx 3$ MeV, $m_d \approx 5$ MeV, and $m_s \approx 100$ MeV [19], we conclude that

light quark masses are much smaller than the masses of the mesons and baryons of comparable flavor content. We note that the pion is comprised of u and d quarks, whereas the kaon contains either s and d or s and u quarks, depending on its electric charge. Let us check the masses of the lightest baryons that are composed of u, d, and s quarks. For the proton, which is made of two u quarks and one d quark:

$$m_p \approx 938 \,\mathrm{MeV} \gg 2m_u + m_d,\tag{5.32}$$

and whereas for the Λ^0 , which is made of u, d, and s quarks:

$$m_{\Lambda^0} \approx 1116 \,\mathrm{MeV} \gg m_u + m_d + m_s.$$
 (5.33)

These observations tell us that the masses of light quarks play only a minor role in the masses of the light hadrons; the same conclusion does not hold, however, for the much heavier c, b, and t quarks. Thus, within a certain expectation of precision, the mass terms of the light quarks u, d and s may be ignored at the starting point of the discussion. Based this observation, we set $m_u = m_d = m_s = 0$, and the QCD effective Lagrangian in the light quarks reads:

$$\mathcal{L}_{\text{QCD}}^{0} = \sum_{f=u,d,s} \overline{Q}_{f} i \not{\!\!D} Q_{f} - \frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu,a} + \text{h.c.}$$
(5.34)

In the first section of the chapter, we have already mentioned the left and right currents, Eq. (3.17). The left and right projection operators are defined as:

$$P_L \equiv \frac{1}{2}(1 - \gamma_5); \quad P_R \equiv \frac{1}{2}(1 + \gamma_5),$$
 (5.35)

with the apparent properties:

$$P_L + P_R = 1,$$

 $(P_L)^2 = P_L,$
 $(P_R)^2 = P_R,$
 $P_L P_R = P_R P_L = 0.$ (5.36)

The operators P_L and P_R act on the general quark fields Q to yield the so-called "left" and "right" quark fields $Q_{L,R}$:

$$Q_L \equiv P_L Q; \quad Q_R \equiv P_R Q. \tag{5.37}$$

With Eq. (5.36) and the commutation relation between γ^{μ} and γ_5 as shown in Eq. (2.39), one can easily check that

$$\mathcal{L}_{\text{QCD}}^{0} = \sum_{f=u,d,s} (\overline{Q}_{L,f} i \not\!\!D Q_{L,f} + \overline{Q}_{R,f} i \not\!\!D Q_{R,f}) - \frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu,a} + \text{h.c.}.$$
(5.38)

Equation (5.38) shows that in the chiral limit, the left-handed and right-handed quark fields are completely decoupled.

Furthermore, because the covariant derivative in Eq. (5.29) is blind to quark flavor, and the masses of u, d, and s have been set equal to 0, one is then allowed to organize the three light quark flavors into a 3-component spinor such as:

$$Q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \tag{5.39}$$

and one can also easily show that Eq. (5.38) is invariant under the following four independent global transformations:

$$Q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to Q'_L = e^{-i\alpha^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}, \qquad (5.40)$$

and

$$Q_R = \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to Q'_R = e^{-i\alpha^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}, \qquad (5.41)$$

which are $U(1)_L$ and $U(1)_R$ global transformations, respectively, with α^L and α^R two independent constants. Besides these, we also have the $SU(3)_L$ as well as $SU(3)_R$ transformations

$$Q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to Q'_L = V_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i\sum_{a=1}^8 \Theta_a^L \frac{\lambda_a}{2}\right) \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}, \quad (5.42)$$

and

$$Q_R = \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to Q'_R = V_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i\sum_{a=1}^8 \Theta_a^R \frac{\lambda_a}{2}\right) \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}, \quad (5.43)$$

where Θ_a^L and Θ_a^R refer to two distinct sets of eight arbitrary angular parameters. The λ_a refer to the eight generators of the SU(3) group, speaking of which, let us be very clear on what we are doing here: the SU(3) group transformation here acts on the flavor space, it stems from the approximate treatment of setting the masses of the three lightest flavors u, d and s zero; on the other hand, the SU(3) gauge group in the real QCD theory applies to the color space, based on the fundamental assumption of each quark flavor carrying 3 possible quantum numbers of color, usually referred to as "red", "green", and "blue". In one words, the SU(3) group symmetry in ChPT and the one in fundamental QCD are totally different.

With the invariance under the combined transformations as in Eq. (5.40), Eq. (5.41), Eq. (5.42), and Eq. (5.43), we say that $\mathcal{L}^0_{\text{QCD}}$ possesses a $\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_L \times \text{U}(1)_R$ symmetry. There is actually an alternative way to express such a combined symmetry: one may simply recombine the left-handed and right-handed transformations U_L and U_R to yield:

$$V_V = V_L + V_R, \quad V_A = V_L - V_R,$$
 (5.44)

which are understood as vector and axial-vector transformations. Correspondingly, the symmetry of \mathcal{L}_{QCD}^0 can also be equivalently presented as $SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$. So far so good! However, let us not forget our important discussion of the U(1) axial anomaly in the last section: even in the massless limit, where the U(1)_A symmetry is supposed to hold in the classical point of view, it is still explicitly broken by quantum effects, as can be seen in Eq. (5.19). We conclude that the so-called "U(1)_A" symmetry really does not exist, and there is no associated isosinglet Nambu-Goldstone boson. Thus, the more quantitative conclusion is that in the chiral limit \mathcal{L}_{QCD}^0 possesses a $SU(3)_V \times SU(3)_A \times U(1)_V$, or, equivalently, a $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry, which is to be inherited by ChPT in constructing the low-energy effective Lagrangian.

5.2.2 Symmetry Breaking and Nambu-Goldstone Bosons

The next step in understanding the ChPT is to understand the mechanism of symmetry breaking. Imagine a Lagrangian of a physical system that possesses a certain symmetry, there are two ways to break the symmetry:

1. by introducing terms that do not respect the symmetry into the Lagrangian; this is called explicit symmetry breaking;

2. the ground state of the Lagrangian does not have the same symmetry as that of the Lagrangian – this is called spontaneous symmetry breaking.

Some general conclusions can be drawn in each of the two cases: in the second case, if a global continuous symmetry is broken spontaneously, that is, if the Lagrangian is globally symmetric under a group G of order n_G , but the ground state of the Lagrangian is only symmetric under a smaller group **H** of order n_H , then, according to the Nambu-Goldstone theorem, $n_G - n_H$ massless bosons will appear in the theory. These are usually called Goldstone (or Nambu-Goldstone) bosons [111] [112] [113]. If the continuous symmetry is only approximate, so that the first case also happens simultaneously, the massless bosons will then acquire non-zero mass; in this event they are usually called pseudo-Goldstone bosons. It turns out that ChPT employs both mechanisms. Since detailed discussions of ChPT is not our main goal in this thesis, in the following discussions, I will just make a very brief statement of its fundamental logic and its most important conclusions. Based on the experimental observation of the low-lying hadron spectrum, namely, that there are eight "light" pseudoscalar mesons, there are strong indications that spontaneous symmetry breaking does happen in the low energy QCD. The original global symmetry $SU(3)_L \times SU(3)_R$ is spontaneously broken [114]. The remaining $SU(3)_V$ can never be broken in the case of chiral limit, or more strictly, $m_u = m_d = m_s$, because symmetries which are not

broken by quark mass are not spontaneously broken either [115]. The $U(1)_V$ also remains a valid symmetry and serves as the basic condition for the baryon number conservation. One thus expects the spontaneous symmetry breaking:

$$\operatorname{SU}(3)_L \times \operatorname{SU}(3)_R \times \operatorname{U}(1)_V \to \operatorname{SU}(3)_V \times \operatorname{U}(1)_V$$
 (5.45)

following which there would arise eight pseudoscalar (0^-) mesons as massless Nambu-Goldstone bosons. They are π^+ , π^0 , π^- , K^+ , K^- , K^0 , $\overline{K^0}$, and η . Although a strict theoretical understanding of spontaneous symmetry breaking as shown in Eq. (5.45) is not clearly known yet due to the very complicated nonperturbative nature of QCD theory, one is still able to show that a nonvanishing singlet scalar quark condensate

$$\langle 0 | \,\overline{q}q \, | 0 \rangle \neq 0 \tag{5.46}$$

serves as a sufficient but not necessary condition to realize Eq. (5.45) [116]. The most practical way to reveal the existence of Eq. (5.46) may be through the tool of lattice QCD, which has yielded promising results [117] [118]. In the real world these eight pseudoscalar (0⁻) mesons are considerably lighter in mass than the low-lying vector (1⁻⁻) mesons or baryons, but they are clearly not massless. The non-vanishing masses of the eight pseudoscalar (0⁻) mesons are widely believed to arise from the non-zero quark masses terms in the QCD Lagrangian [119]. Note that an alternative scenario, which enlarges the framework of the standard ChPT and thus is called generalized chiral perturbation theory, exists [120]. In that case it is argued that spontaneous symmetry breaking only occurs if quark masses are not zero, and thus the masses are quadratic in the explicit symmetry breaking parameter. Note that such a scenario is disfavored by π - π scattering lengths in K_{l4} by the DIRAC experiment [121] [122].

5.2.3 ChPT for Pseudoscalar Mesons

Now let us turn to the ChPT for pseudoscalar mesons. The following discussions follow the flow of the introduction in Ref. [109].

Due to the asymptotic freedom property of QCD, at low energies only the colorneutral bound states of quarks, known as hadrons, can be the actual physical observables. The more fundamental and very complicated color interactions among the quarks and gluons are buried deep inside the hadrons and are not relevant to the theoretical description of the low-energy hadron processes. As a metaphor, a tape measure is a perfect tool to measure the size of a desk or even a house, but it would be definitely a bad idea to use a tape measure to measure the distance between Lexington and New York, although one is allowed to do so in principle. The eight pseudoscalar mesons $(\pi^+, \pi^0, \pi^-), (K^+, K^-), (K^0, \overline{K^0})$, and η are the actual degrees of freedom of ChPT in the meson sector. Note that there is the ninth 0⁻ pseudoscalar meson η' , but it is not treated as a Goldston boson since it is so heavy which is believed to be related to axial anomaly, it is not a degree of freedom of ChPT.

As mentioned before, the principle of constructing the effective Lagrangian of ChPT is that with light hadrons as the fundamental degrees of freedom one still keeps the original $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry of the underlying \mathcal{L}^0_{QCD} , and

the ground state should only possess the $SU(3)_V \times U(1)_V$ symmetry. The effective Lagrangian of the pseudoscalar meson sector can be constructed as follows. First we introduce the dynamical variables in the SU(3) matrix U(x):

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right),\tag{5.47}$$

where F_0 is temporarily understood as a free parameter that is to be determined later. The function $\phi(x)$ is defined as a special collection of color singlets of pseudoscalar mesons:

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}K^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}.$$
 (5.48)

The lowest massless effective Lagrangian of ChPT of the pseudoscalar mesons sector then reads [100] [101]:

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{F_0^2}{4} \text{Tr} \left(\partial_{\mu} U \partial^{\mu} U^{\dagger} \right).$$
(5.49)

It can be shown [109] that the $SU(3)_L \times SU(3)_R$ transformation corresponds to:

$$U(x) \to U'(x) = V_R U(x) V_L^{\dagger}, \qquad (5.50)$$

where the left-handed and right-handed SU(3) transformations V_L and V_R are defined as in Eq. (5.42) and Eq. (5.43) respectively. One can easily show that Eq. (5.49) is invariant under such a combined global transformation. Note here the superscript "2" stresses that it contains 2 factors of the 4-derivative, which in momentum space is to be replaced by 4-momentum p^{μ} , serving as a small quantity in the low energy expansion. Thus $\mathcal{L}_{\text{eff}}^{(2)}$ is of $\mathcal{O}(p^2)$. Equation (5.49) is organized in such a way that to lowest order one can restore the familiar form of the kinematic term " $\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi$ ", which can be easily confirmed by expanding U(x) up to the linear terms in ϕ . An expansion of U(x) to higher order of ϕ would result in extra factors of $1/4\pi F_0$, which is roughly of $\mathcal{O}(M_p^{-1})$, and thus is negligible at sufficiently low energies. Let us also emphasize that up to the order of only two derivatives, Eq. (5.49) is the only possible choice.

Apparently, Eq. (5.49) does not contain mass terms, so that it is not quite complete to the lowest order. To account for the relatively small masses of the pseudoscalar mesons, one may add to Eq. (5.49) a mass term. As mentioned before, the nonvanishing masses of the pseudoscalar mesons are supposed to arise from the nonvanishing light-quark mass matrix:

$$M = \begin{pmatrix} m_u & 0 & 0\\ 0 & m_d & 0\\ 0 & 0 & m_s \end{pmatrix},$$
 (5.51)

which adds explicit symmetry breaking terms to \mathcal{L}^0_{QCD} . Thus, to the lowest order, the mass term in $\mathcal{L}^{(2)}_{eff}$ should be arranged to be linearly dependent on the quark masses

matrix M. Forcing the following transformation of the quark masses matrix M:

$$M \to M' = V_R M V_L^{\dagger}, \tag{5.52}$$

we can construct the mass-related term for ChPT to the lowest order [100] [101]:

$$\mathcal{L}_M = \frac{F_0^2 B_0}{2} \text{Tr} \left(M U^{\dagger} + U M^{\dagger} \right), \qquad (5.53)$$

which is also $SU(3)_L \times SU(3)_R \times U(1)_V$ invariant. For a better understanding of the validity of the construction of \mathcal{L}_M , we can just expand U(x) to the second power in ϕ . Keeping in mind that M is in fact a real matrix, we find:

$$\mathcal{L}_M \approx -\frac{B_0}{2} \text{Tr}\left(\phi^2 M\right). \tag{5.54}$$

On substituting the concrete form of ϕ as shown in Eq. (5.48), we have

$$\operatorname{Tr}(\phi^{2}M) = 2(m_{u} + m_{d})\pi^{+}\pi^{-} + 2(m_{u} + m_{s})K^{+}K^{-} + 2(m_{d} + m_{s})K^{0}\overline{K^{0}} + (m_{u} + m_{d})\pi^{0}\pi^{0} + \frac{2}{\sqrt{3}}(m_{u} - m_{d})\pi^{0}\eta + \frac{m_{u} + m_{d} + 4m_{s}}{3}\eta^{2}.$$
 (5.55)

Within the limit of SU(2) isospin symmetry, one has $m_u = m_d \equiv m$, and Eq. (5.55) leads to the following results:

$$M_{\pi}^{2} = 2B_{0}m;$$

$$M_{K}^{2} = B_{0}(m + m_{s});$$

$$M_{\eta}^{2} = \frac{2}{3}B_{0}(m + 2m_{s}),$$
(5.56)

which in turn yields the following relation:

$$4M_K^2 = 3M_\eta^2 + M_\pi^2, (5.57)$$

which is the well-known Gell-Mann-Okubo relation [123] [124] [125].

Thus far we have obtained the lowest order effective chiral Lagrangian that possesses a global $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry [100] [101]:

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{F_0^2}{4} \text{Tr} \left(\partial_{\mu} U \partial^{\mu} U^{\dagger} \right) + \frac{F_0^2 B_0}{2} \text{Tr} \left(M U^{\dagger} + U M^{\dagger} \right), \qquad (5.58)$$

where F_0 and B_0 serve as two free parameters to be determined. Equation (5.58) represents the lowest order mesonic ChPT Lagrangian. The most general chiral Lagrangian describing the dynamics of the Goldstone bosons is organized as a string of terms with an increasing number of derivatives and quark mass terms:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm eff}^{(2)} + \mathcal{L}_{\rm eff}^{(4)} + \mathcal{L}_{\rm eff}^{(6)} + ...,$$
(5.59)

where the superscripts refer to the order in a momentum and quark mass expansion. The index "4", for example, denotes either four derivatives or two quark mass terms. We have obtained the complete form of $\mathcal{L}_{eff}^{(2)}$. As for $\mathcal{L}_{eff}^{(4)}$ and higher orders, the constructions still follow exactly the same logic, which is to simply sum over all the possible combinations of U(x) and M with the proper number of derivatives to yield the most general Lagrangian that is invariant under $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry. For each linearly independent term, one assigns a new free parameter. As one can imagine, as the order of the ChPT gets higher and higher, the number of the allowed terms and so the number of free parameters increases very fast. Since here we are only aiming at demonstrating the fundamental idea of ChPT, we do not need to show the concrete forms of the higher order terms. Let me just stress that these free parameters arising from the chiral expansions order by order are generally believed to be deducible from the underlying QCD theory, but so far this connection has only been explored in a limited way because QCD is very hard to solve. Irrespective of this, these free parameters can be determined by directly confronting ChPT with relevant experimental data. Once all the free parameters have been fixed in this way, one can then use ChPT to solve other hadron-related problems to a precision determined by the size of the neglected subleading terms.

To solve more realistic problems, there is an additional important point that needs to be mentioned here. Equation (5.59) is only invariant under a global $SU(3)_L \times$ $SU(3)_R \times U(1)_V$ symmetry and contains only hadron degrees of freedom. Apparently, Eq. (5.59) is only suitable for solving problems involving only pseudoscalar mesons, such as strong meson-meson scattering problems. Clearly this does not show the full power of ChPT. Indeed, ChPT can be much more powerful by promoting the global $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry to a local $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry, which means that the space-time-independent constants Θ_a^L and Θ_a^R in the chiral rotations defined in Eq. (5.42) and Eq. (5.43) are now forced to be local: $\Theta_a^L \to \Theta_a^L(x)$ and $\Theta_a^R \to \Theta_a^R(x)$. invariance and the interactions naturally arise from the introduction of covariant derivatives. The principle of constructing the covariant derivative is to make the covariant derivative $D_{\mu}U(x)$ transform in the same way as U(x) under the local group transformation. In addition, by including external sources of differing Lorentz character, one can build effective theories in which the hadrons interact via electroweak forces, in addition to strong forces. Through these methods, one can also build terms which include quark mass effects. Following the theoretical work of Gasser and Leutwyler [100] [101], we introduce the covariant derivative in the following way. For any object U(x) transforming as $V_R U(x) V_L^{\dagger}$, the covariant derivative D_{μ} is defined as:

$$D_{\mu}U(x) \equiv \partial_{\mu}U(x) - ir_{\mu}U(x) + iU(x)l_{\mu}, \qquad (5.60)$$

where r_{μ} and l_{μ} are right-handed and left-handed external fields, respectively. Correspondingly, the first term in Eq. (5.58) becomes:

$$\mathcal{L}_{\rm eff}^{(2)} = \frac{F_0^2}{4} \text{Tr} \left(D_{\mu} U D^{\mu} U^{\dagger} \right) + \frac{F_0^2 B_0}{2} \text{Tr} \left(M U^{\dagger} + U M^{\dagger} \right).$$
(5.61)

These fields are introduced to cancel the additional terms arising from the local symmetry transformations. To restore the local $SU(3)_L \times SU(3)_R \times U(1)_V$ invariance,

 r_{μ} and l_{μ} are associated with the following chiral transformation rules:

$$\begin{aligned} r_{\mu} \to r'_{\mu} &= V_R r_{\mu} V_R^{\dagger} + i V_R \partial_{\mu} V_R^{\dagger}, \\ l_{\mu} \to l'_{\mu} &= V_L l_{\mu} V_L^{\dagger} + i V_L \partial_{\mu} V_L^{\dagger}. \end{aligned}$$
(5.62)

In constructing the most general chiral Lagrangian that is invariant under the local $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry in the presence of external fields, there are two other possible external sources: the scalar field s and the pseudo-scalar field p appearing as a combination of $\mathcal{L}_{\text{eff}} \chi \equiv 2B_0(s + ip)$, which by itself is not chiral invariant, so that χ is to be accompanied by a factor of U(x). With the introduction of the external fields r_{μ} , l_{μ} , s, and p, one is now able to deal with many hadron-involved problems beyond strictly strong-interaction processes. As a simple sample application of ChPT, let us briefly look at pseudoscalar meson weak decay. Following the SM, we promote the external fields to the charged weak gauge boson field W^{\pm} :

$$l_{\mu} = -\frac{g}{\sqrt{2}} (W_{\mu}^{+} T_{+} + \text{h.c.}),$$

$$r_{\mu} = 0, \qquad (5.63)$$

with

$$T_{+} = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
(5.64)

where V_{ud} and V_{us} are CKM matrix elements that were mentioned in the earlier chapter. Such an external field could couple to the muonic lepton current $\mu\gamma^{\mu}(1 - \gamma_5)\overline{\nu}_{\mu}$ + h.c. also. On substituting l_{μ} in Eq. (5.61) and expanding U(x) to the linear term in ϕ , one is able to get meson weak decay amplitudes. For example, after a bit of straightforward computation with the heavy degree of freedom of W^+ integrated out implicitly, one is able to find the amplitude of pion weak decay:

The standard treatment yields a theoretical expression for the pion weak decay rate:

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 |V_{ud}|^2}{4\pi} F_0^2 M_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{M_\pi^2}\right)^2, \qquad (5.66)$$

which gives the free parameter F_0 , which in this order is just the pion decay constant F_{π} , which is about 92.4 MeV.

With the very brief introduction of mesonic ChPT I have given, I am hoping to have conveyed at least the general ideas. As emphasized before, since this chapter is not really designed to be a complete reference on ChPT, we have skipped several important topics, such as the actual form of the chiral Lagrangian in higher orders and the regularization and renormalization procedures because they are not relevant to our concrete projects.

5.2.4 ChPT for Baryons

Thus far we have considered the purely mesonic sector involving the interactions of Goldstone bosons with each other and with external fields. The effective dynamics of light baryons that are the bound states of the light u, d and s quarks can also be analyzed in ChPT at sufficiently low energies [119] [126] [127] [128] [129]. The fundamental principle is still to follow the constraints of $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry. The construction of a ChPT in the baryonic sector is much more complicated than in the mesonic sector, and in fact there are different scenarios in this new context, though the theoretical foundations are just the same. The following discussions follow Ref. [109]. To show the basic idea, we will just work within the smaller chiral symmetry group $SU(2)_L \times SU(2)_R \times U(1)_V$, where only proton and neutron appear as the lightest baryons. One can put the proton and neutron in a nucleon isospin doublet:

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{5.67}$$

where p and n represent proton and neutron fields, respectively, both of which are four-component Dirac spinors. The core task is that, for nucleons being spin-1/2 fermions, one expects to construct the most general ChPT Lagrangian of nucleons in the form of $\bar{N}ON$, noting here that \hat{O} refers to the collection of operators to be constructed and keeps the object $\bar{N}ON$ invariant under the transformation $SU(2)_L \times$ $SU(2)_R \times U(1)_V$.

Just as we did in the meson sector, we first need to set up the nonlinear realization of the $SU(2)_L \times SU(2)_R$ symmetry on the nucleon doublet N. against the Nambu-Goldstone theorem. A thorough analysis reveals a special way to realize the ideal transformation, which actually induces a coupling between pseudoscalar mesons and nucleons. Let me stress again that since here we only concentrate on the SU(2) subgroup, the hadronic degrees of freedom only include the nucleon and pion. Following Gasser's procedures [126], one first introduces a special field u(x) as the square root of the pseudoscalar meson collection field U(x) as shown in Eq. (5.48):

$$u(x) \equiv \sqrt{U(x)},\tag{5.68}$$

where, as has been mentioned, U(x) transforms as $V_R U(x) V_L^{\dagger}$. Following this realization, one then introduces a special quantity called compensator field K that is $V_{L,R}$ and U(x) dependent via the following identity:

$$u(x) \to u'(x) = V_R u(x) K^{-1} = K u(x) V_L^{-1}.$$
 (5.69)

It can be shown that the following nonlinear realization of $SU(2)_L \times SU(2)_R$ transformation on the nucleon sector reads:

$$N \to N' = K(V_L, V_R, U)N. \tag{5.70}$$

Apparently, such a nonlinear realization induces a direct coupling between the pion and nucleon. The next general step is to define a covariant derivative D_{μ} such that $D_{\mu}N$ transforms in the same way as N under the local $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ transformation. One finds that such a covariant derivative can do the job:

$$D_{\mu}N = (\partial_{\mu} + \Gamma_{\mu} - iv_{\mu}^{(s)})N, \qquad (5.71)$$

where Γ_{μ} is defined as:

$$\Gamma_{\mu} \equiv \frac{1}{2} \left[u^{\dagger} (\partial_{\mu} - ir_{\mu}) u + u (\partial_{\mu} - il_{\mu}) u^{\dagger} \right], \qquad (5.72)$$

where r_{μ} and l_{μ} are the same external sources as introduced in the meson sector, they transform according to Eq. (5.62), and $v_{\mu}^{(s)}$ as an additional source to take care of the U(1)_V part, with the corresponding local gauge transformation:

$$v_{\mu}^{(s)} \to v_{\mu}^{\prime(s)} = v_{\mu}^{(s)} - \partial_{\mu}\Theta,$$
 (5.73)

where Θ refers to the arbitrary rotation under $U(1)_V$. It is easy to see that the field Γ_{μ} is $\mathcal{O}(p)$. Besides Γ_{μ} which is even under parity P, there can also exist another Hermitian building block:

$$A_{\mu} \equiv \left[u^{\dagger} (\partial_{\mu} - ir_{\mu}) u - u (\partial_{\mu} - il_{\mu}) u^{\dagger} \right], \qquad (5.74)$$

which is a P-odd axial field.

With these building blocks, one can obtain the lowest O(p) ChPT Lagrangian that contains the pion-nucleon couplings [126]. We have:

$$\mathcal{L}_{\pi N} = \bar{N}(i\not\!\!D - m_N + \frac{1}{2}g_A\gamma_\mu\gamma_5 A_\mu)N, \qquad (5.75)$$

where g_A is called axial-vector coupling constant, which can be inferred from neutron β decay. As one can check, the pion-nucleon interaction described in Eq. (5.75) reproduces the result of a simple model called the linear sigma model [130], which represents an early trial of constructing an effective and operational theory to describe the interactions between the nucleon and pion.

The construction of the baryonic ChPT Lagrangian to higher orders is more complicated than the mesonic case. Unlike the relatively simpler mesonic case, the power counting in the baryonic sector requires one to use more caution. For example, in the mesonic case, the pseudoscalar mesons' masses vanish in the strict chiral limit, thus the factor $D_{\mu}U$ serves as an O(p) building block. On the baryonic sector, however, the nucleon mass does not vanish in the chiral limit, thus " $D_{\mu}N$ " cannot be simply taken as being O(p) because the $\partial_0 N$ part gives a large contribution to $O(p^0)$. Observing that the baryonic chiral expansion always takes the bilinear form " $\bar{N}\Gamma N$ " with " Γ " referring to possible combinations of Dirac γ matrices, detailed analysis yields the following general conclusions of power counting [109]:

$$N, \ \bar{N} \sim \mathcal{O}(p^0), \ D_{\mu}N \sim \mathcal{O}(p^0), \ (i\not\!\!D - m_N)N \sim \mathcal{O}(p), 1, \ \gamma_{\mu}, \ \gamma_{\mu}\gamma_5, \ \sigma_{\mu\nu} \sim \mathcal{O}(p^0), \ \gamma_5 \sim \mathcal{O}(p),$$
(5.76)

which, with the external source terms r_{μ} and l_{μ} still as $\mathcal{O}(p)$ building blocks, serve as the basic building blocks in constructing a baryonic ChPT. The concrete construction of baryonic ChPT can take different forms in different scenarios, such an area is still under development. In the pseudoscalar meson ChPT, we have chosen " $4\pi F_{\pi}$ ", which are close to 1 GeV, as the large scale suppression factor. In the later part of the thesis, we are following Richard Hill's work [129], where the nucleon mass M serves as the large scale suppression factor – the higher order terms are thus $\mathcal{O}(M^{-1})$ suppressed. As will be seen later, we will present Hill's work, which performed analyzed baryonic ChPT on the full $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ electroweak gauge symmetry up to $\mathcal{O}(M^{-2})$. After this, we will discuss in detail our theoretical work on a possible new source of CP violation contributing to certain terms in the chiral Lagrangian.

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Chapter 6 New Sources of CP Violation at Low Energies

6.1 Chiral Effective Theory for Electroweak Processes

Although the preceding discussion of axial anomalies and ChPT in the last chapter are rather simplified, I am still hoping that at least the main ideas are transparent.

1. A symmetry that is expected to hold on the classical level may not hold on the quantum level due to quantum corrections. This means that the expected symmetry simply is not there. Such an explicit symmetry-breaking effect is called an anomaly, and its existence can be established via the Jacobian that is associated with a transformation of the fermionic fields. The lesson here is that one should use caution before asserting the validity of any symmetry based only on classical arguments.

2. In the discussion of ChPT, we first showed that the fundamental QCD Lagrangian possesses the chiral symmetry $SU(3)_L \times SU(3)_R$ in the chiral limit, and we see the chiral symmetry plays the most important role in constructing the low-energy ChPT Lagrangian, which has the low-lying pseudoscalar meson octet and nucleons as its fundamental degrees of freedom.

The juxtaposition of items 1 and 2 immediately suggests an obvious question: since the chiral symmetry $SU(3)_L \times SU(3)_R$ we have been using in constructing the Lagrangian of ChPT is deduced from classical arguments, is it affected by the axial anomaly? A careful investigation that was first carried out by Wess and Zumino in 1971 [131] reveals that the local $SU(3)_L \times SU(3)_R$ chiral symmetry is explicitly violated by the anomaly at the fundamental QCD level. This tells us that the effective Lagrangian we have discussed in last chapter is based on a semi-classical footing, not on the full quantum field theory footing. If the local chiral symmetry is explicitly broken on the fundamental QCD level, such an anomalous effect must also be present in its low energy effective description, so that our earlier constructions of ChPT Lagrangian are not complete yet; we are forced to adopt an additional term into the ChPT Lagrangian to account for such an anomaly. The introduction of such an additional term turns out to be complicated. In fact, Wess and Zumino noted that the result could not even be expressed as a single local effective Lagrangian, but only as a Taylor expansion representation, known as the Wess-Zumino contribution [131]. It was Witten who subsequently gave an elegant representation of the Wess-Zumino contribution as an integral over a five-dimensional space whose boundary is physical four-dimensional spacetime [132]. This is usually called Wess-Zumino-Witten (WZW) term, which appears as an effective functional. Although the basic idea is very similar to the path integral method we talked about in the section on anomalies in the last chapter – one needs to find the Jacobian in association with the transformations of the nucleon fields, which are also spin-1/2 fermions. The actual derivation of the WZW term is rather formal and complicated. Here we just quote the final result,

which was originally presented in [132] and can also be found in [106]:

$$\Gamma[U,\ell,r]_{\rm WZW} = -\frac{iN_c}{240\pi^2} \int d^5 x \epsilon^{ijklm} \operatorname{Tr} \left(\Sigma_i^L \Sigma_j^L \Sigma_k^L \Sigma_l^L \Sigma_m^L \right) - \frac{iN_c}{48\pi^2} \int d^4 x \, \epsilon_{\mu\nu\alpha\beta} \left[W(U,l,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},l,r)^{\mu\nu\alpha\beta} \right], \quad (6.1)$$

with the five dimensional antisymmetric Levi-Civita tensor chosen to satisfy $\epsilon^{50123} = +1$. The parameter N_c refers to the number of colors in QCD, and l and r refer to the left-handed and right-handed external fields as discussed in Chapter 4. Also, we have

$$W(U,l,r)_{\mu\nu\alpha\beta} = \operatorname{Tr}(Ul_{\mu}l_{\nu}l_{\alpha}U^{\dagger}r_{\beta} + \frac{1}{4}Ul_{\mu}U^{\dagger}r_{\nu}Ul_{\alpha}U^{\dagger}r_{\beta} + iU\partial_{\mu}l_{\nu}l_{\alpha}U^{\dagger}r_{\beta} + i\partial_{\mu}r_{\nu}Ul_{\alpha}U^{\dagger}r_{\beta} - i\Sigma_{\mu}^{L}l_{\nu}U^{\dagger}r_{\alpha}Ul_{\beta} + \Sigma_{\mu}^{L}l_{\nu}\partial_{\nu}l_{\alpha}l_{\beta} - i\Sigma_{\mu}^{L}\Sigma_{\nu}^{L}U^{\dagger}r_{\alpha}Ul_{\beta} + \Sigma_{\mu}^{L}l_{\nu}\partial_{\alpha}l_{\beta} + \Sigma_{\mu}^{L}\partial_{\nu}l_{\alpha}l_{\beta} - i\Sigma_{\mu}^{L}l_{\nu}l_{\alpha}l_{\beta} + \frac{1}{2}\Sigma_{\mu}^{L}l_{\nu}\Sigma_{\alpha}^{L}l_{\beta} - i\Sigma_{\mu}^{L}\Sigma_{\nu}^{L}\Sigma_{\alpha}^{L}l_{\beta}) - (L \leftrightarrow R), \qquad (6.2)$$

where $\Sigma_{\mu}^{L} = U^{\dagger} \partial_{\mu} U$, $\Sigma_{\mu}^{R} = U \partial_{\mu} U^{\dagger}$, and $(L \leftrightarrow R)$ stands for the interchanges $U \leftrightarrow U^{\dagger}$, $l_{\mu} \leftrightarrow r_{\mu}$ and $\Sigma_{\mu}^{L} \leftrightarrow \Sigma_{\mu}^{R}$.

Let us not worry about the complicated appearance of the WZW term as shown in Eq. (6.1) and Eq. (6.2) too much. The important things for us to know are that (1) the new terms are not strictly chirally invariant but their variation under the broken symmetry is a spacetime derivative, so that the action associated with the new term remains invariant [133]; (2) it allows for the description of hadronic processes with an odd number of mesons. As one can check, the "primitive" building blocks in the original ChPT Lagrangian in last chapter only contain an even number of hadronic degrees of freedom in any order of power expansion; thus they are incapable of describing processes involving an odd number of hadrons such as $\pi^0 \to \gamma\gamma$, $\gamma \to \pi\pi\pi$, and $K^0\overline{K^0} \to \pi\pi\pi$. It was the WZW term that came to rescue! For instance, the process $K^0\overline{K^0} \to \pi\pi\pi$ in which five pseudoscalar mesons participate in the WZW term is contained in the first line of Eq. (6.1). On substituting $\Sigma_{\mu}^L = U^{\dagger}\partial_{\mu}U$ with U expanded to the lowest order, one finds the relevant Lagrangian that concerns the process $K^0\overline{K^0} \to \pi\pi\pi$:

$$\mathcal{L} = \frac{N_c}{240\pi^2 F_{\pi}^5} \epsilon^{\mu\nu\alpha\beta} \operatorname{Tr} \left(\phi \partial_{\mu} \phi \partial_{\nu} \phi \partial_{\alpha} \phi \partial_{\beta} \phi \right), \qquad (6.3)$$

where ϕ is defined by Eq. (5.48). Note in Eq. (6.3), four derivatives enter, thus we say the process $K\overline{K} \to 3\pi$ occurs in $\mathcal{O}(p^4)$ in ChPT. Based on this discussion, we draw the conclusion that the WZW term generates an essential part of the physics, and it should be placed on the same footing as the other terms in the ChPT Lagrangian.

So far so good! In 2008, however, Jeffrey Harvey, Christopher Hill, and Richard Hill pointed out that more attention need to be paid to the gauging of the WZW term under the full electroweak $SU(2)_L \times U(1)_Y$ gauge symmetry group [80] [81] [82]. In

their paper, they carefully investigated a special situation: with a set of fundamental (generic) electroweak gauge fields W^{\pm} , Z^{0} , and γ , if we turn on a classical background vector field B_{μ} that couples to the axial vector current J^5_{μ} with $B^{\mu}J^5_{\mu}$, then the vector current is no longer conserved, but develops a mixed anomaly $\sim \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_{EW}F^{\rho\sigma}_B$ that arises from the WZW term. This would cause serious problem because the conservation of the gauge vector current is closely connected to the gauge invariance and thus the renormalizability. So it is essential to maintain the gauge vector current conservation with the presence of any background field. To ensure this, a usual way is to incorporate a necessary counterterm into the WZW term of ChPT Lagrangian to cancel the new unpleasant gauge vector current anomaly. The remarkable result of their theoretical work is that, with the promotion of the classical background vector field B_{μ} to the physical spin-1 vector meson fields such as ω and ρ at low energies, the necessary counterterm gives rise to a contact interaction of the pseudo Chern-Simons (pCS) form, for example, $\sim \epsilon^{\mu\nu\rho\sigma}\omega_{\mu}Z_{\nu}F_{\rho\sigma}$, as well as the charged current analogy $\sim \epsilon^{\mu\nu\rho\sigma}\rho_{\mu}W^{\pm}_{\nu}F_{\rho\sigma}$, where ω and ρ refer to the omega and rho mesonic background fields, Z the Z boson and $F_{\mu\nu}$ the photon field strength. According to HHH, on the fundamental quark level such exotic pCS term can be induced via the triangle coupling as shown in Fig. (6.1).



Figure 6.1: A possible triangle diagram on the fundamental quark level that could give rise to HHH's exotic coupling. Here we demonstrate the case of a charged current.

The ω or ρ mesons in turn can couple to the nucleon current, and Z to the lepton current. On integrating out the heavy degrees of freedoms of ρ mesons and W with the presence of nucleon current at low energies, we can obtain the following form of contact coupling for the case of charged weak current [80]:

$$\mathcal{L} = \kappa \,\varepsilon^{\sigma\mu\nu\rho} \bar{p} \gamma_{\sigma} n \bar{\psi}_{eL} \gamma_{\mu} \psi_{\nu_e L} F_{\nu\rho}, \tag{6.4}$$

where ψ_e and ψ_{ν_e} refers to the final electron and neutrino asymptotic states, and we have $2\psi_{eL} = (1 - \gamma_5)\psi_e$. The *n* and *p* spinors refer to the initial neutron and the final proton states, and $F_{\nu\rho}$ is the electromagnetic field strength tensor. Equation (6.4) contains a phenomenological coupling constant κ , and it denotes the strength of an exotic contact interaction of the nucleon, leptons, and photon. Let us recall our earlier discussions of neutron radiative β decay, in which the QED bremsstrahlung was taken as the only source of the radiated photon. Equation (6.4) reveals an additional possible contribution and demands an adjustment of our original thinking.

Based on the earlier discussion of anomalies, we know that the appearance of a gauge anomaly in a gauge field theory threatens the renormalizability of the theory, and thus it must be cancelled exactly, usually by introducing in the Lagrangian a certain counterterm, which cancels the deviation of the Jacobian that arises from the corresponding local transformation of fermion fields from unity. This is the motivation of the recent work by Richard Hill in chiral effective theory [129]. Upon including the necessary counterterm of the electroweak gauge anomaly of WZW term with explicit choices of the external currents, he proposed the following low energy effective Lagrangian of a ChPT that is complete up to $\mathcal{O}(M^{-2})$ in the nucleon sector:

$$\mathcal{L} = M\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \frac{1}{M}\mathcal{L}^{(2)} + \frac{1}{M^2}\mathcal{L}^{(3)} + \dots, \qquad (6.5)$$

where M refers to the nucleon mass. With the nucleon doublet N defined as:

$$N \equiv \left(\begin{array}{c} p\\n \end{array}\right),\tag{6.6}$$

the detailed expansions are:

$$\mathcal{L}^{(0)} = -c^{(0)}\bar{N}N,\tag{6.7}$$

$$\mathcal{L}^{(1)} = \bar{N} [c_1^{(1)} i D - c_2^{(1)} A \gamma_5] N.$$
(6.8)

Apparently, we see the form of $M\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ is in agreement with Eq. (5.75) provided that the phenomenological low energy coupling constants $c^{(0)} = c_1^{(1)} = 1$, and $c_2^{(1)} = g_A$. Assuming the SM and T symmetry, the rest of the expansion reads:

$$\mathcal{L}^{(2)} = \bar{N} \left[-c_1^{(2)} \frac{i}{2} \sigma^{\mu\nu} \text{Tr}([iD_{\mu}, iD_{\nu}]) - c_2^{(2)} \frac{i}{2} \sigma^{\mu\nu} \tau^a \text{Tr}(\tau^a [iD_{\mu}, iD_{\nu}]) + \dots \right] N, \quad (6.9)$$

$$\mathcal{L}^{(3)} = \bar{N} \Biggl[c_1^{(3)} \gamma^{\nu} [iD_{\mu}, \operatorname{Tr}([iD^{\mu}, iD_{\nu}])] + c_2^{(3)} \gamma^{\nu} [iD_{\mu}, \tau^a \operatorname{Tr}(\tau^a [iD^{\mu}, iD_{\nu}])] + c_3^{(3)} \gamma^{\nu} \gamma_5 [iD_{\mu}, [iD^{\mu}, A_{\nu}]] + c_4^{(3)} i \epsilon^{\mu\nu\rho\sigma} \gamma_{\sigma} \operatorname{Tr}(\{A_{\mu}, [iD_{\nu}, iD_{\rho}]\}) + c_5^{(3)} i \epsilon^{\mu\nu\rho\sigma} \gamma_{\sigma} \tau^a \operatorname{Tr}(\tau^a \{A_{\mu}, [iD_{\nu}, iD_{\rho}]\}) + c_6^{(3)} \gamma^{\nu} \gamma_5 [[iD_{\mu}, iD_{\nu}], A^{\mu}] + c_7^{(3)} \frac{1}{4M^2} \gamma^{\nu} \gamma_5 \{ [[iD_{\mu}, iD_{\nu}], A_{\rho}], \{iD^{\mu}, iD^{\rho}\}\} + \dots \Biggr] N.$$
(6.10)

In the chiral expansions above, τ^a , with a = 1, 2, 3, refers to the SU(2) group generators, and the covariant derivative D_{μ} is defined as:

$$iD_{\mu} \equiv i\partial_{\mu} + V_{\mu}', \tag{6.11}$$

where V'_{μ} represents a collection of electroweak source fields:

$$V'_{\mu} = \frac{g_2}{4} \begin{pmatrix} \frac{1}{c_W} (1 - 4s_W^2) Z_{\mu} & \sqrt{2} W_{\mu}^+ \\ \sqrt{2} W_{\mu}^- & -\frac{1}{c_W} Z_{\mu} \end{pmatrix} + e \begin{pmatrix} A_{\mu}^{\text{em}} & 0 \\ 0 & 0 \end{pmatrix} + \dots,$$
(6.12)

and the axial-vector source field A_{μ} reads:

$$A_{\mu} = \frac{g_2}{4} \begin{pmatrix} \frac{1}{c_W} Z_{\mu} & \sqrt{2} W_{\mu}^+ \\ \sqrt{2} W_{\mu}^- & -\frac{1}{c_W} Z_{\mu} \end{pmatrix} - \frac{1}{2f_{\pi}} \begin{pmatrix} \partial_{\mu} \pi^0 & \sqrt{2} \pi^+ \\ \sqrt{2} \pi^- & -\partial_{\mu} \pi^0 \end{pmatrix} + \dots, \quad (6.13)$$

where dots denote terms with two or more fields. Also, $s_W \equiv \sin\theta_W$, and $c_W \equiv \cos\theta_W$, with $s_W^2 \approx 0.231$. We see from Eq. (6.10), Eq. (6.12), and Eq. (6.13) that there is a "HHH" term of comparable strength to be found in various processes, e.g., parity violating $\pi \pi$ photoproduction from a proton as well as in muon radiative capture on a nucleon.

6.2 Spin-Independent CP Violation and Constraints Thereon

Now let us apply the chiral Lagrangian of Hill [129], discussed in the previous section, to neutron radiative β decay [134]. Here we are interested in the " c_5 "-dependent term, which appears in the $\mathcal{O}(M^{-2})$ chiral expansion. Let us take this term out and present it here for later convenience:

$$\mathcal{L}(c_5) = \frac{c_5}{M^2} \bar{N} i \epsilon^{\mu\nu\rho\sigma} \gamma_{\sigma} \tau^a \text{Tr}(\tau^a \{ A_{\mu}, [iD_{\nu}, iD_{\rho}] \}) N, \qquad (6.14)$$

where the superscript "(3)" indicating the order of chiral expansion has been dropped to minimize confusion. Equation (6.14) has the following properties:

1. it allows for a contact coupling among the nucleons N, weak gauge boson W^{\pm} , and electromagnetic field $F_{\mu\nu}$; such a contact coupling could serve as an extra contribution to the neutron radiative β decay in addition to the standard QED bremsstrahlung; 2. it contains a special component " $\epsilon^{\mu\nu\rho\sigma}$," which makes the interaction of pseudo-

2. It contains a special component $e^{-r_{e}}$, which makes the interaction of pseudo– Chern–Simons form [80]. It is the interference between such a pseudo Chern-Simon term and the standard bremsstrahlung process which gives rise to the triple-product correlation in momenta, which is P-odd and T-odd.

To obtain the explicit form of the interaction relevant to the process $n(p_n) \rightarrow p(p_p) + e^-(l_e) + \overline{\nu}_e(l_{\nu}) + \gamma(k)$, we substitute the covariant derivative D_{μ} , defined by Eq. (6.12), as well as the axial-vector field A_{μ} , defined by Eq. (6.13), and work out the (anti)commutations carefully, and we find:

$$\mathcal{L}(c_5) = -\frac{4c_5}{M^2} \frac{eG_F V_{ud}}{\sqrt{2}} \varepsilon^{\sigma\mu\nu\rho} \bar{p}\gamma_\sigma n \bar{\psi}_{eL} \gamma_\mu \psi_{\nu_e L} F_{\nu\rho}.$$
(6.15)

where the heavy degree of freedom of W^- has been integrated out to yield the lowenergy weak coupling constant G_F . As one can easily check, Eq. (6.14) reproduces Eq. (6.4) with $\kappa = -4c_5 e G_F V_{ud}/\sqrt{2}M^2$. An analogous interference term is possible in neutral weak current processes, which is described by the $c_4^{(3)}$ -dependent term in $\mathcal{L}^{(3)}$ [80]. We make the factor of V_{ud} associated with the physical nucleon basis explicit. Thus the baryon weak vector current can mediate parity violation on its own, through the interference of the leading vector amplitude mediated by

$$\frac{G_F V_{ud}}{\sqrt{2}} g_V \bar{p} \gamma^\mu n \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_{\nu_e} , \qquad (6.16)$$

dressed by bremsstrahlung from the charged particles, with the c_5 term. On including the contact coupling of Eq. (6.15), we have the Feynman diagrams in Fig. (6.2), where the diagrams (01) and (02) refer to the standard bremsstrahlung, and (03) refers to the contact coupling from the HHH term, Eq. (6.16).



Figure 6.2: Contributions to neutron radiative β decay, $n \rightarrow p + e^- + \overline{\nu}_e + \gamma$, with the HHH contribution, Eq. (6.15), denoted by (03).

Noting Figure (6.2), we have:

$$|\mathcal{M}|^{2} = (\mathcal{M}_{0} + \mathcal{M}_{03}) \cdot (\mathcal{M}_{0}^{\dagger} + \mathcal{M}_{03}^{\dagger}) = |\mathcal{M}_{0}|^{2} + 2\operatorname{Re}(\mathcal{M}_{0} \cdot \mathcal{M}_{03}^{\dagger}) + |\mathcal{M}_{03}|^{2}, \qquad (6.17)$$

where \mathcal{M}_0 is defined by Eq. (4.3), and the calculation of $|\mathcal{M}_0|^2$ has been worked out in full detail in Chapter 3. The contribution $|\mathcal{M}_{03}|^2$ is proportional to $(c_5/M^2)^2$, which is assumed to be subleading with respect to the other terms, though we should emphasize that $|\mathcal{M}_{03}|^2$ can still play a role in determining a possible constraint on $|\text{Im}(c_5/M^2)|$ based on the current precision measurement of the branching ratio of neutron radiative β decay. We will discuss this further later in this section. The most interesting part is the interference $2\text{Re}(\mathcal{M}_0 \cdot \mathcal{M}_{03}^{\dagger})$, defined as $|\mathcal{M}|_{c_5}^2$. Following the standard procedures of the Feynman rules, we find after computing the needed spin sums and average:

$$|\mathcal{M}|_{c_5}^2 = 256e^2 G_F^2 |V_{ud}|^2 \operatorname{Im} (c_5 g_V) \frac{E_e}{l_e \cdot k} (\mathbf{l}_e \times \mathbf{k}) \cdot \mathbf{l}_\nu, \qquad (6.18)$$

Table 6.1: T-odd asymmetries arising from Eq. (6.18), in units of $\text{Im}\left[(c_5/M^2)g_V\right]$, for the neutron. The branching ratios are reported as well as per Table (4.1).

$\omega_{\min}(MeV)$	$\mathcal{A}^{\mathrm{HHH}}(n)$	BR(n)
0.01	-5.61×10^{-3}	3.45×10^{-3}
0.05	-1.30×10^{-2}	1.41×10^{-3}
0.1	-2.20×10^{-2}	$7.19 imes10^{-4}$
0.3	-5.34×10^{-2}	8.60×10^{-5}

which is obviously P-odd and T-odd due to the triple-product correlation in momenta $(\mathbf{l}_e \times \mathbf{k}) \cdot \mathbf{l}_{\nu}$. As shown in Eq. (6.18), such a T-odd correlation probes the imaginary part of $c_5 g_V$, which we define as Im C_{HHH} ; thus it represents a real CP violating effect. And just as pointed out at the beginning of this thesis, such a new source of CP violation does not depend on the particles' spins; this distinguishes the HHH-induced CP violation from the other low-energy sources of CP violation which are under experimental investigation presently.

Working in units of Im $[(c_5/M^2)g_V]$ and defining $\xi \equiv (\mathbf{l}_e \times \mathbf{k}) \cdot \mathbf{l}_{\nu}$, we partition phase space into regions of definite sign, so that we form an asymmetry as per Eq. (4.24):

$$\mathcal{A}(\omega_{\min}) \equiv \frac{\Gamma_{+}(\omega_{\min}) - \Gamma_{-}(\omega_{\min})}{\Gamma_{+}(\omega_{\min}) + \Gamma_{-}(\omega_{\min})}, \qquad (6.19)$$

where Γ_{\pm} contains an integral of the (spin-averaged) $|\mathcal{M}|_{c_5}^2$ over the region of phase space with $\xi \gtrless 0$, respectively, neglecting corrections of recoil order. We work out the asymmetries $\mathcal{A}^{\text{HHH}}(n)$ at some selected photon energy thresholds ω_{\min} as listed in Table (6.1). As one can check from Eq. (6.18), and confirm from Table (6.1), the HHH-induced T-odd asymmetry increases at larger photon threshold energy.

One question naturally arises here: what are the possible constraints that are based on current experimental data on Im $C_{\rm HHH}$? To address this question, we consider existing empirical constraints on the coefficients of Eq. (6.18). For the part of $|{\rm Im}(c_5/M^2)|$, the best and perhaps only constraint comes from the precision measurement of the branching ratio of neutron radiative β decay, which has a contribution which goes as $|c_5|^2$. We note $|{\rm Im}(c_5/M^2)| < 12 \,{\rm MeV}^{-2}$ at 68% CL from the most recent measurement of the branching ratio for neutron radiative β decay [69], for which $\omega \in [15, 340]$ keV. The constraint is poor because the radiative decay rate is driven by the contributions from the lowest photon energies, for which $|\mathcal{M}|^2$ is proportional to ω^{-2} . If one could measure the photon energy spectrum, e.g., close to its endpoint, then the constraint could be much stronger. That is, in the event that one could measure the BR to within 1% of its SM value for $\omega_{\rm min} = 100 \,{\rm keV}$, or for $\omega_{\rm min} \approx$ $\omega_{\rm max} = 782 \,{\rm keV}$, one would find at 68% CL the limits $|{\rm Im}(c_5/M^2)| < 0.88 \,{\rm MeV}^{-2}$ and $|{\rm Im}(c_5/M^2)| < 0.15 \,{\rm MeV}^{-2}$, respectively. As for the value of ${\rm Im}(g_V)$, it can be bounded from the deviation of the empirical CKM unitarity test, the most recent data in PDG [19] shows that $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99995 \pm 0.00061$ from unity, to yield $\text{Im}(g_V) < 0.024$ at 68% CL, the limit from the *D* term is much sharper, however: $\text{Im}(g_V) < 7 \times 10^{-4}$ at 68% CL [135].

6.3 Possible Model to Yield $Im(c_5)$

After calculating the numerical results of the T-odd asymmetry induced by the HHH theory, let us address another important issue: how could the $Im(c_5)$ coefficient of Eq. (6.18) be generated? In this section we consider this, and we illustrate some possibilities in Fig. (6.3), which include mixing with new degrees of freedom, which possess a hidden gauge symmetry that is decoupled from the ones in the SM. Such a scenario is motivated by searches for the particle origin of the dark matter which pervades the Universe. There can be different ways for the known matter to couple to a hidden sector, see Ref. [136] for a recent review. Here we will just pick one way – a "hidden sector" ρ' , though complex phases associated with the production of known nucleon resonances, or N^* 's are also possible [134]. Here we are more attracted to the mechanism of $\rho - \rho'$ kinetic mixing. We now develop a rudimentary model in which the ρ' helps mediate a difference in the radiative n and $\bar{n} \beta$ decay rates. The notion of a hidden sector of strongly coupled matter is of some standing [137] [138], and has more recently been discussed in the context of models which provide a common origin to baryons and dark matter [139] [140], though the mechanism need not be realized through strong dynamics [141] [142] – we note Ref. [143] for a recent review. Intriguing astrophysical anomalies have prompted the study of hidden sector models which permit couplings to SM leptons; specifically, the visible and hidden sectors are connected through the kinetic mixing of the gauge bosons of their respective U(1)symmetries, notably through a SM hypercharge $U(1)_Y$ portal [144] [145] [146] [147]. Constraints on long-range interactions between dark-matter particles are sufficiently severe [148] [149] [150] that in such models the dark gauge symmetries are also broken through some dark Higgs sector [147].

In fact there can be other kinds of portals between the visible and hidden sectors. In addition to the Abelian portal that was mentioned above, there can be non-Abelian ones, too. As an explicit example, we consider a non-Abelian portal, mediated, e.g., by heavy scalars Φ which transform under the adjoint representation of the group;



Figure 6.3: Processes which could give rise to the $\text{Im}(c_5)$ -dependent interaction of Eq. (6.18). We use "N*" to denote a nucleon resonance, and " \otimes " for $\rho - \rho'$ mixing.

such an interaction can also be realized through kinetic mixing, generalizing from Ref. [147], through $\text{Tr}(\Phi F_{\mu\nu}) \text{Tr}(\tilde{\Phi} \tilde{F}^{\mu\nu})$, as well as $\epsilon^{\mu\nu\rho\sigma} \text{Tr}(\Phi F_{\mu\nu}) \text{Tr}(\tilde{\Phi} \tilde{F}_{\rho\sigma})$, where $F^{a\,\mu\nu}$ is the SM SU(3)_c field strength, and $\tilde{\Phi}^a$ and $\tilde{F}^{a\,\mu\nu}$ are fields and field strengths of a hidden strongly-coupled sector, nominally based on SU(3)_c. We anticipate that the dark matter candidate is a color singlet, so that there are no dark long-range forces to negate. The connector is not a marginal operator, but the appearance of QCD-like couplings should make it more important in the infrared. To build a pertinent model at low energies we recall the hidden local symmetry model of QCD [151] [152], in which the ρ mesons function as effective gauge bosons of the strong interaction. Upon including electromagnetism this becomes a vector-meson dominance model [153] [154], noting "VMD1" of Ref. [154], which we adapt to this case as

$$\mathcal{L}_{\rm mix} = -\frac{1}{4}\rho^a_{\mu\nu}\rho^{a\,\mu\nu} - \frac{1}{4}\rho^{\prime a\,\mu\nu}_{\mu\nu}\rho^{\prime a\,\mu\nu} + \frac{\epsilon}{2}\rho^a_{\mu\nu}\rho^{\prime a\,\mu\nu} + \frac{m^2_{\rho}}{2}\rho^a_{\mu}\rho^{a\,\mu} + \frac{m^2_{\rho'}}{2}\rho^{\prime a\,\mu}_{\mu}\rho^{\prime a\,\mu} + g_{\rho}J^{\mu\,a}\rho^a_{\mu} \quad (6.20)$$

where $J^{a\,\mu}$ denotes the baryon vector current and $\rho^{(\prime)\,a}$ are the gauge bosons of a hidden local SU(2) symmetry — though $\rho^{(\prime)\,a}_{\mu\nu} = \partial_{\mu}\rho^{(\prime)\,a}_{\nu} - \partial_{\nu}\rho^{(\prime)\,a}_{\mu}$ [154]. Our model resembles those in Refs. [144] [145] [146] [147] but contains two massive vector fields. With $J^{\pm}_{\mu} = J^{1}_{\mu} \pm i J^{2}_{\mu}$ and $\rho^{\pm}_{\mu} = (\rho^{1}_{\mu} \mp i \rho^{2}_{\mu})/\sqrt{2}$, the charged current pieces, dropping the mass terms, become

$$\mathcal{L}_{\rm mix}^{\pm} = -\frac{1}{4}\rho^{+\,\mu\nu}\rho_{\mu\nu}^{-} - \frac{1}{4}\rho^{\prime+\,\mu\nu}\rho_{\mu\nu}^{\prime-} + \frac{\epsilon}{2}\left(\rho^{+\,\mu\nu}\rho_{\mu\nu}^{\prime-} + \rho^{-\,\mu\nu}\rho_{\mu\nu}^{\prime+}\right) + \frac{g_{\rho}}{\sqrt{2}}(\rho_{\mu}^{+}J^{+\,\mu} + \rho_{\mu}^{-}J^{-\,\mu}).$$
(6.21)

The kinetic mixing term can be removed through the field redefinition $\tilde{\rho}^{\pm}_{\mu} = \rho^{\pm}_{\mu} - \epsilon \rho'^{\pm}_{\mu}$, thus yielding a coupling of the baryon vector current to ρ' , as illustrated in the first panel of Fig. (6.3), mimicking the role of the "dark photon" in fixed target experiments [155]. The ρ'^{\pm} does not couple to photons; indeed, the particles of the hidden sector couple only to strongly interacting particles — we refer to Ref. [143] for discussion of models with generalized conserved charges. We consider $m_q \sim \mathcal{O}(m_{q'})$ but with confinement scales $\Lambda' < \Lambda$ so that $m_{\rho'} < m_{\rho}$, noting that dark and baryonic matter can have a common origin even if the dark matter candidate is lighter than the proton in mass [156]. Unlike related "quirk" models [157], the collider signatures of our scenario are minimal and are hidden within hadronization uncertainties. However, if $m_{\rho'} \lesssim 1 \,\mathrm{MeV}$ it can be constrained by other low-energy experiments and observations; e.g., it can appear as a mismatch in the value of the neutron lifetime inferred from counting surviving neutrons from that inferred from counting SM decay products. It is also possible to build a model with additional hidden-sector portals. With a $U(1)_Y$ portal, e.g., the hidden quarks are allowed to have a milli-electric charge if the dark-matter particle is an electrically neutral composite [158]. This possibility is illustrated in the "mixed basis" in the central panel of Fig. (6.3). Limits on the $SU(2)_L$ and $U(1)_{em}$ couplings follow, e.g., from studies of the W^{\pm} width and the running of α and are significant; for simplicity we set this possibility aside. Thus limits on the T-odd asymmetry, for which a statistical error of $\mathcal{O}(10^{-3})$ could be achievable [159], limits $\operatorname{Im}(c_5/M^2) = 2\epsilon \operatorname{Im}\epsilon g_{\rho^0}^2 / (16\pi^2 m_{\rho'}^2)$ with $g_{\rho^0} \sim 3.3$ [160].

Chapter 7 CP Violation Arising from Lee and Yang's Weak Hamiltonian

In the discussion of the possible new spin-independent source of CP violation that may arise from the " c_5 " term in HHH's theory, we see that the current constraints on $\text{Im}(c_5/M^2)$ that comes from the highest precision measurement of the branching ratio of neutron radiative β decay is very poor. In fact, a branching ratio measurement is not a very efficient way to bound $\text{Im}(c_5/M^2)$ because one can only bound the square of $\text{Im}(c_5)$. Maybe the most efficient way for us is to perform a direct measurement of the T-odd asymmetry in neutron or nuclear radiative β decay. Thus it is necessary to address an important question: in proposing a direct measurement of $\text{Im}(g_V c_5/M^2)$ dependent new physics, can the possible T-odd asymmetry be generated in any other way? In other words, if in a future experiment one did observe a nonzero T-odd asymmetry, how likely is it that it is really contributed by the HHH term?

Our analysis shows that the competitive effects do exist – the T-odd asymmetry arising from a triple-product correlation in momenta could have multiple origins, which, in general, can be categorized into the following two cases:

1. some mechanism gives rise to a triple-product correlation in momenta, and represents a real source of CP violation;

2. some mechanism does NOT represent a real CP violation, but could still generate a T-odd correlation acting as a "noise".

We will investigate both of these cases in this thesis. In this chapter, we focus on the first case. The second case will be fully addressed in later chapters.

As the beginning of this short chapter, let us first recall the weak interaction Hamiltonian that was proposed by Lee and Yang in 1956 [3], Eq. (3.18). As explained before, Lee and Yang proposed the existence of parity violation in the weak interaction to explain the so-called $\theta - \tau$ puzzle. The standard theory of the electroweak unification was proposed by Glashow in 1963 [161] and later revised by Salam and Weinberg independently [86] [87], but at the time of 1950's, people had no idea about it. Thus Lee and Yang proposed the most general decomposition form of Eq. (3.18)that is consistent with their assumption of parity violation and of a semileptonic decay. The V-A part in Eq. (3.18) is confirmed by the standard electroweak theory at low energies, but there is in fact no strong reasons to reject other terms, because they may as well exist due to some unknown mechanism that is beyond the SM. In our research, we realize that the application of the Lee and Yang's general decomposition Eq. (3.18) in the neutron radiative β decay results in a real CP violation, which is also signaled by a triple-product correlation in momenta as in the "HHH" case. Here to describe the additional radiation that was not present in Eq. (3.18), we are simply assuming the bremsstrahlung as the leading order contribution and thus dress the charged particles states with a photon radiation as per the standard treatment of QED; the other parts remains unaffected. Such a treatment results in the following interaction for the neutron radiative β decay:

$$H_{\rm int}^{\rm nRDK} = eH_{\rm int}^{e\gamma} - eH_{\rm int}^{p\gamma},\tag{7.1}$$

where the superscripts $e\gamma$ and $p\gamma$ imply the source of photon bremsstrahlung:

$$H_{\text{int}}^{e\gamma} = (\overline{\psi}_{p}\psi_{n})(C_{S}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\psi_{\nu}-C_{S}^{\prime}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{5}\psi_{\nu})+(\overline{\psi}_{p}\gamma^{\mu}\psi_{n})$$

$$(C_{V}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{\mu}\psi_{\nu}-C_{V}^{\prime}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{\mu}\gamma_{5}\psi_{\nu})+(\overline{\psi}_{p}\gamma_{5}\psi_{n})$$

$$(C_{P}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{5}\psi_{\nu}-C_{P}^{\prime}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\psi_{\nu})-(\overline{\psi}_{p}\gamma^{\mu}\gamma_{5}\psi_{n})$$

$$(C_{A}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{\mu}\gamma_{5}\psi_{\nu}-C_{A}^{\prime}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\gamma_{\mu}\psi_{\nu})+\frac{1}{2}(\overline{\psi}_{p}\sigma^{\mu\nu}\psi_{n})$$

$$(C_{T}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\sigma_{\mu\nu}\psi_{\nu}-C_{T}^{\prime}\overline{\psi}_{e}\frac{2l_{e}\cdot\epsilon^{*}+\underline{\ell}^{*}\underline{k}}{2l_{e}\cdot k}\sigma_{\mu\nu}\gamma_{5}\psi_{\nu}),$$

$$(7.2)$$

and

$$H_{\rm int}^{p\gamma} = \frac{p_p \cdot \epsilon^*}{p_p \cdot k} H_{\rm int},\tag{7.3}$$

where H_{int} is defined in Eq. (3.18). Here we are only taking the leading recoil order contribution, so that the proton term has been neglected and only the $p_p \cdot \epsilon^* / p_p \cdot k$ part survives. In the original work of Lee and Yang, the Fermi constant G_F is hidden in the sets of coupling constants. For example, the C_V and C'_V have the correspondence in SM:

$$C'_{V} = C_{V} = -\frac{G_{F}V_{ud}g_{V}}{\sqrt{2}}.$$
(7.4)

To be consistent with the standard convention, here we will redefine the original coupling constants by abstracting a factor of $\frac{G_F V_{ud}}{\sqrt{2}}$ out of each one the original coupling constants such that:

$$C_i^{(\prime)} \equiv G_F V_{ud} \tilde{C}_i^{(\prime)} / \sqrt{2}. \tag{7.5}$$

With the interaction given by Eq. (7.1), Eq. (7.2), and Eq. (7.3), one easily obtains:

$$|\mathcal{M}|_{\mathrm{T-odd,LY}}^{2} = 16e^{2}G_{F}^{2}|V_{ud}|^{2}M\mathbf{l}_{\nu}\cdot(\mathbf{l}_{e}\times\mathbf{k})\frac{1}{l_{e}\cdot k}\mathrm{Im}[\tilde{C}_{T}(\tilde{C}_{S}^{\prime*}+\tilde{C}_{P}^{\prime*})+\tilde{C}_{T}^{\prime}(\tilde{C}_{S}^{*}+\tilde{C}_{P}^{*})],$$
(7.6)

where M refers to the nucleon mass which is the average of the proton and neutron massses – in leading recoil order, the effect of the difference between neutron and proton mass is subleading. In Eq. (7.6) we see that the T-odd asymmetry is proportional to the imaginary part of the special combination $\tilde{C}_T(\tilde{C}_S^* + \tilde{C}_P^*) + \tilde{C}_T'(\tilde{C}_S^* + \tilde{C}_P^*)$; such a complex-phase dependence suggests that such a T-odd correlation leads to a genuine CP violation. We can also see in Eq. (7.6) that the T-odd asymmetry is of recoil order, since in our definition of the asymmetry factor \mathcal{A} , Eq. (4.24), the denominator that is always controlled by the T-even part of neutron radiative β decay, which is of order M^2 .

As per the definition of \mathcal{A} in Eq. (4.24), we have worked out the numerical results for the T-odd asymmetry induced by Lee and Yang's general Hamiltonian. The tabulated results are in units of $\operatorname{Im}[\tilde{C}_T(\tilde{C}_S'^*+\tilde{C}_P'^*)+\tilde{C}_T'(\tilde{C}_S^*+\tilde{C}_P^*)] \equiv \operatorname{Im}(\mathcal{C}_{\mathrm{LY}})$. Tab. (7.1) confirms the estimation that the T-odd asymmetry induced by Lee and Yang's general

Table 7.1: T-odd asymmetry in neutron radiative β decay induced by Lee and Yang's general Hamiltonian dressed by photon bremsstrahlung, Eq. (7.2) and Eq. (7.2), in units of Im(C_{LY}).

$\omega_{\min}(MeV)$	$\mathcal{A}^{\mathrm{LY}}(n)$
0.01	4.98×10^{-7}
0.05	1.16×10^{-6}
0.1	1.98×10^{-6}
0.3	5.21×10^{-6}

Hamiltonian in neutron radiative β decay is only to recoil order. This, combined with existing constraints on the \tilde{C} 's [162], means that a measurement at anticipated levels of sensitivity would not be generated by this mechanism. In future measurements, such a real source of CP violation is not able to compete with another mechanism, which is in the SM and gives a more significant mimicking effect of the T-odd tripleproduct correlation in momenta. Here by "mimicking" we mean the SM can also give rise to a T-odd correlation, which is in analogy with the ones we have obtained in both HHH's and Lee and Yang's mechanisms, but let us just say such a SM-induced T-odd correlation does not represent a real CP violation. It only acts as a background noise in the measurements of the T-odd asymmetry searching for new sources of CP violation. This brings us to the next chapter.

Chapter 8 Final–State–Interaction–Induced Triple–Product Momentum Correlation in Radiative β Decay in the SM

In the previous chapters, we have discussed in detail the possibility of obtaining a triple-product correlation in momenta, $\mathbf{l}_e \cdot (\mathbf{l}_{\nu} \times \mathbf{k})$, in radiative β decay from the interference between the tree level QED bremsstrahlung mechanism in the SM and terms generated through new sources of CP violation from BSM physics. The latter include not only the "HHH" term we have discussed in Chapter 6 and 7, but also the non-V-A terms in the Lee-Yang beta-decay Lagrangian, dressed by QED bremsstrahlung effects. The T-odd and P-odd correlation arising from both of the mechanisms signifies possible new sources of CP violation which are controlled by either the coefficient $\operatorname{Im}(q_V c_5)$ or $\operatorname{Im}(\mathcal{C}_{LY})$. But as per our detailed analysis, the mechanism of Lee and Yang's general decompositions can only yield a T-odd asymmetry that is only of recoil order; it has to be very small and thus does not concern us that much. Current nuclear data has already set a pretty severe bound on $\text{Im}(q_V)$ - the best constraint on Im g_V comes from the recent D term measurement [63] [135], to yield Im $q_V < 7 \times 10^{-4}$ at 68% CL [135]. The bound on Im(c_5), on the other hand, is poor. According to our earlier analysis, it seems that the most efficient way to set a bound for $Im(c_5)$ is to measure the size of the triple-momentum correlation in neutron and/or nuclear radiative β decay directly. Such a measurement gives us a window on possible new CP-violating physics at low energies. So far this picture looks quite nice and simple, but a serious problem exists – as mentioned in last chapter, such "T-odd" decay correlations can be mimicked by the SM. It happens because of the final-state interactions (FSI) between the charged particles in the final state [10]. In such a situation, even if in a future experiment, a sizable T-odd correlation is observed, it may have been due to the SM mimicking effect, which actually neither signifies the real CP violation nor helps us set a real bound on $\text{Im}(c_5)$. In this sense, such an "evil" SM background effect has to be tracked down and well understood, and we will do this in this chapter. Unlike the calculations of the HHH-induced T-odd term in the previous chapter, finding the exact behavior of the mimicking T-odd effect is much more complicated, because of the following:

1. The mimicking effect arises from the interference of tree with loop diagrams, which give us many terms to handle.

2. Some special mathematical tricks are involved.

3. One has to treat subtle effects such as the cancellation of infrared divergences carefully.

With these requirements, making the whole work more logically accessible is a challenge. A similar kind of problem has been investigated in the context of $K_{l3\gamma}^+$ decay [163]. The T-odd asymmetry computed in [163] from electromagnetic final-state interactions has recently been recalculated and is in significant disagreement with the earlier result [164].

Following the similar path of [163] and [164], we have managed to solve such a problem in neutron radiative β decay [165]. We shall split this chapter into four sections. The first section is to discuss how such a mimicking effect arises due to the final-state interactions, with the "Cutkosky cut" [166] being the major theoretical tool. The second section is to introduce a nice mathematical tool that turns out critical for our later calculations. In the third section, both of the tools that are explained in the first two sections are to be applied for the full calculations of the final-state-interaction-induced mimicking effect, and, as will be seen, a "phantom" of infrared divergence appears and haunts us. The final section is to show the "phantom" that haunted us in the past section is only an illusion, we are really infrared finite.

8.1 Imaginary Part of Looped Diagrams and Cutkosky Rules

As has been shown in many examples so far, the nonvanishing of imaginary parts of coupling constants play crucial roles in generating T-odd correlations. In the technical sense, the triple product correlation in momenta arises from the interference between the vector current, which contains no γ_5 , and the axial vector current pieces containing a γ_5 , which in doing the spin sums always results in a factor of the Levi-Civita tensor " $\epsilon_{\mu\nu\rho\sigma}$ " which also gives rise to a triple-product correlation in momenta. According to the basic definition of γ_5 , Eq. (2.38), one sees that γ_5 by itself contains a factor of i, thus for tree-level $|\mathcal{M}|^2$ being real, the mixing terms between vector and axial vector pieces have to vanish unless we can find an extra factor of i to make the final result of $|\mathcal{M}|^2$ real. This is the actual reason for the appearance of $\mathrm{Im}(g_V c_5)$ and $\mathrm{Im}(\mathcal{C}_{\mathrm{LY}})$ in Chapter 6 and Chapter 7. We shall show, however, that the SM also allows for the appearance of such an extra factor of i, arising from the imaginary part of \mathcal{M} , which can be obtained from the final-state interactions in the loop diagrams.

As per the general definition of S-matrix in QFT, S = 1 + iT, where T is related to the scattering amplitude \mathcal{M} by $\mathcal{M} = \langle f | T | i \rangle (2\pi)^4 \delta^{(4)} (\sum p_i - \sum p_f)$, where the δ function is to guarantee the 4-momentum conservation. The operator S is required to be unitary, such that $S^{\dagger}S = 1$, from which one obtains:

$$-i(T - T^{\dagger}) = T^{\dagger}T. \tag{8.1}$$

On sandwiching Eq. (8.1) between the initial state $|i\rangle$ and final state $|f\rangle$ as well as insertion of the complete set of intermediate states in the right hand side, one finally obtains [70]:

$$-i(\mathcal{M} - \mathcal{M}^{*}) = 2\mathrm{Im}(M)$$

= $\sum_{n} \prod_{i=1}^{n} \int \frac{d^{3}q_{i}}{(2\pi)^{3}2E_{i}} M^{*}(i \to q_{i}) M(f \to q_{i})(2\pi)^{4} \delta^{(4)} \left(\sum p_{f} - \sum q_{i}\right),$ (8.2)

where $\sum p_f$ and $\sum q_i$ refer to the sum of the four-momenta in the final state and intermediate state, respectively. Equation (8.2) shows us that, although the insertion of the complete set of intermediate states usually represents loops in the QFT perturbative calculations in QFT, calculating the imaginary part of \mathcal{M} puts these intermediate states on their mass shells – the phase space integrations on the right hand side of Eq. (8.2) takes exactly the same form as a real physical process has. To gain better understanding of this issue, let us recall that in the very original introduction of a propagator in QFT, there is a factor of " $i\epsilon$ " in its denominator. As long as the denominator of a propagator does not vanish, or equivalently, the intermediate particle is off-shell, the factor " $i\epsilon$ " is always irrelevant, and thus \mathcal{M} is just real. The situation is totally different when an intermediate particle is put to be on-shell, in this case the denominator of the propagator vanishes and now the factor " $i\epsilon$ " matters. As one remembers, such a factor represents a branch cut across the real axis, starting at the threshold energy of the intermediate particle, and the discontinuity reads:

$$\operatorname{Disc}(M) = 2i\operatorname{Im}(M). \tag{8.3}$$

Combining Eq. (8.2) and Eq. (8.3) gives us the explicit way of calculating a discontinuity Disc(M):

$$\operatorname{Disc}(M) = 2i \operatorname{Im}(M) = i \sum_{n} \prod_{i=1}^{n} \int \frac{d^{3}q_{i}}{(2\pi)^{3} 2E_{i}} M^{*}(i \to q_{i}) M(f \to q_{i}) (2\pi)^{4} \delta^{(4)} \left(\sum p_{f} - \sum q_{i}\right),$$
(8.4)

Cutkosky proved that the form of Eq. (8.4) is completely general [166]. One can quickly reproduce Eq. (8.4) by performing the Cutkosky rules [166] [70], which states: 1. Cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell. The thus-constrained cut propagators represent physical scattering.

2. For each cut, perform the replacement " $1/(p^2 - m^2 + i\epsilon) \rightarrow -2\pi i \delta(p^2 - m^2)$ " in each cut propagator, then perform the standard loop integrals.

3. Sum the contributions of all possible cuts.

We can now apply the Cutkosky rules as shown above to the neutron radiative β decay. It turns out that due to the very low energy release, the only relevant loops are the photon loops, representing the electromagnetic final-state interactions. Performing the Cutkosky cuts gives us the imaginary part of the scattering amplitude \mathcal{M} . In mathematical terms this can be understood as:

$$|\mathcal{M}|^{2} = |\mathcal{M}_{\text{tree}}|^{2} + \mathcal{M}_{\text{tree}} \cdot \mathcal{M}_{\text{loop}}^{*} + \mathcal{M}_{\text{loop}} \cdot \mathcal{M}_{\text{tree}}^{*} + \mathcal{O}(\alpha^{2})$$

$$\approx |\mathcal{M}_{\text{tree}}|^{2} + 2\text{Re}(\mathcal{M}_{\text{tree}} \cdot \mathcal{M}_{\text{loop}}^{*}), \qquad (8.5)$$

where the meanings of $\mathcal{M}_{\text{tree}}$ and $\mathcal{M}_{\text{loop}}$ are assumed to be transparent. As discussed before, the imaginary part can only arise from the loop diagrams, and it is the imaginary part of $\mathcal{M}_{\text{loop}}$ that makes the mixing terms between the vector and axial vector piece nontrivial. As an explicit example of the application of Cutkosky rules, we take a look at one example in the actual calculation, as shown in Figure (8.1), which is only one of many possible cuts. I hope with the following detailed sample procedure, the ideas and procedures in all the rest of cuts, which are similar, become



Figure 8.1: An example of a Cutkosky cut in neutron radiative β decay. The two crosses denote a physical two-particle cut.

clear. In Figure (8.1), the cross symbols " \times " on the internal electron and photon lines represent the Cutkosky cuts, which, according to the Cutkosky rules, imply the following simultaneous replacements of the denominators of the cut electron and photon propagators with delta functions:

$$\frac{1}{l_e'^2 - m_e^2} \to -2\pi i \delta(l_e'^2 - m_e^2), \quad \frac{1}{k'^2} \to -2\pi i \delta(k'^2), \tag{8.6}$$

which are also associated with the relevant loop integrals over l_e' and k':

$$\int \frac{d^4 l'_e}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4} (2\pi)^4 \delta^{(4)} (l'_e + k' - l_e - k).$$
(8.7)

After the cut in Figure (8.1), let us look at the left hand side of the cuts, such a "partial" diagram looks the same as the tree level diagram of neutron radiative β decay, which is seen as the diagram (02) in Figure (3.5), with only some minor changes of arguments: $l_e \rightarrow l'_e$ and $k \rightarrow k'$. We thus can directly write down its amplitude $\mathcal{M}_{02}(l'_e, k', p_p)$, as per Eq. (4.5). As for the right hand side of the cuts in Figure (8.1), the partial diagram looks like Compton scattering, which can be described with the Feynman diagrams of Figure (8.2). Obviously, the example we are discussing here only concerns the first diagram in Figure (8.2), and let us temporarily denote it as \mathcal{M}_{com} . With this preparation, we are ready to write down the expression for Im(\mathcal{M}_{loop}) in our specific example of Figure (8.1):

$$\operatorname{Im}\mathcal{M}_{\operatorname{loop}} = \sum_{s} \int \frac{d^{4}l'_{e}d^{4}k'}{2(2\pi)^{2}} \delta^{(4)}(l'_{e} + k' - l_{e} - k)\delta(l'^{2}_{e} - m^{2}_{e})\delta(k'^{2})\mathcal{M}_{\operatorname{com}}^{*}\mathcal{M}_{02}(l'_{e}, k', p_{p})$$
$$= \frac{1}{8\pi^{2}} \sum_{s} \int \frac{d^{3}\mathbf{l}'_{e}}{2E'_{e}} \frac{d^{3}\mathbf{k}'}{2\omega'} \mathcal{M}_{\operatorname{com}}^{*}\mathcal{M}_{02}(l'_{e}, k', p_{p})\delta^{(4)}(l'_{e} + k' - l_{e} - k), \qquad (8.8)$$



Figure 8.2: Compton scattering arising from Cutkosky cuts.

which is in principle solvable on the substitution of \mathcal{M}_{com}^* , and $\mathcal{M}_{02}(l'_e, k', p_p)$.

Thus far we have given a complete example of how to hunt down the imaginary part of a scattering amplitude, arising from a loop diagrams, and which generates a final-state interaction. As stressed before, we have so far considered only one of the many possible cuts. In the future section, we shall list all the cuts. Since the logics and procedures are similar to what we have seen in this example, in the future discussions, we will not repeat all these details. I am hoping this detailed sample will provide a clear blueprint for the rest of the work.

Let us end this section by emphasizing a technical difficulty that arises from the intermediate phase space integrations, which are involved with quite entangled angular integrations. Even if one managed to solve this simple case, let us not forget that we still have many other cases with totally different angular integrations! Handling them one by one would be painful and slow. There has got to be a better way to handle it, and in fact there is! Thanks to Veltman and Passarino, we have a very nice trick, which is now usually referred to as Veltman-Passarino reduction [167], to reduce the many seemingly complicated intermediate phase space integrations down to only a few simpler ones. This brings us to the next section.

8.2 Veltman-Passarino Reduction

The Veltman-Passarino reduction (VPR) [167] was first introduced to handle very complicated loop integrations. It is based on completely general principles, so that it works equally well in our problem. It serves as an important mathematical tool to deal with the very complicated intermediate phase space integrations that one encounters in finding the imaginary part of \mathcal{M}_{loop} . Because of the importance of this theoretical tool, I am thinking it deserves its own section for a quick review. For technical reasons, I will present the idea of Veltman-Passarino reduction by considering a very concrete example, which should suffice in helping to understand our later work.

We consider a sample loop diagram that can appear in the scalar ϕ^3 theory. The triangular diagram contains three vertices, as shown in Figure (8.3). To expose the essence of the VPR more easily, we shall assume the 3 internal propagators in Figure (8.3) are massless. The presence of masses would not affect the method, but would just make the illustration more involved. Figure (8.3) has 3 external lines, carrying momentum p_1 , p_2 , and p_3 respectively, but due to the 4-momentum conservation on a closed loop, only two of them are independent. We can randomly



Figure 8.3: A sample loop diagram that can appear in scalar ϕ^3 theory with l as the internal 4-momentum.

choose p_1 and p_2 as the two independent momenta, p_3 is then determined as $p_3 = -(p_1 + p_2)$. Imagine we are calculating some scattering amplitude that contains such a 3-point loop, the denominator that is contributed by this loop is always of form $l^2(l + p_1)^2(l + p_1 + p_2)^2$ – if the particles are massless. The numerator, on the other hand, can be categorized into different cases as per the power of l^{μ} it contains. Let us see the simplest cases that are relevant to our future calculations:

1. it contains zero power of l^{μ} – the scalar case, and the resulting loop integration reads:

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2} \equiv I_3;$$
(8.9)

2. it contains one power of l^{μ} - the vector case, the resulting form of loop integration reads:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu}}{l^2(l+p_1)^2(l+p_1+p_2)^2};$$
(8.10)

3. it contains two powers of l^{μ} – the tensor case, and the resulting loop integration reads:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu}l^{\nu}}{l^2(l+p_1)^2(l+p_1+p_2)^2}.$$
(8.11)

Cases with even higher powers of l^{μ} are possible and become more and more complicated, but fortunately they are not relevant to us. Among the 3 equations above, Eq. (8.9) has to be solved either exactly or numerically, but Eq. (8.10) and Eq. (8.11) do not have to be. To handle Eq. (8.10), we argue that since in the loop diagram of Fig. (8.3) the only independent external degrees of freedom are p_1 and p_2 , the following decomposition identity for Eq. (8.10) must hold:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu}}{l^2(l+p_1)^2(l+p_1+p_2)^2} = C_1 p_1^{\mu} + C_2 p_2^{\mu}.$$
(8.12)

We just need to find the coefficients C_1 and C_2 . Contracting both sides with $p_{1\mu}$ gives:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l \cdot p_1}{l^2(l+p_1)^2(l+p_1+p_2)^2} = C_1 p_1^2 + C_2 p_1 \cdot p_2.$$
(8.13)

Using

$$l \cdot p_1 = \frac{1}{2} \left((l+p_1)^2 - l^2 - p_1^2 \right)$$
(8.14)

we have

$$p_1^2 C_1 + p_1 \cdot p_2 C_2 = \frac{1}{2} (I_1 - I_2 - p_1^2 I_3),$$
 (8.15)

where I_3 is already defined as in Eq. (8.9). We note I_1 and I_2 are defined as:

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2(l+p_1+p_2)^2} \equiv I_1, \tag{8.16}$$

and

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l+p_1)^2(l+p_1+p_2)^2} \equiv I_2.$$
(8.17)

Likewise, we can also contract with $p_{2\mu}$ and have:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l \cdot p_2}{l^2(l+p_1)^2(l+p_1+p_2)^2} = C_1 p_1 \cdot p_2 + C_2 p_2^2.$$
(8.18)

Using

$$l \cdot p_2 = \frac{1}{2} \left((l + p_1 + p_2)^2 - (l - p_1)^2 + p_1^2 - (p_1 + p_2)^2 \right)$$
(8.19)

we have

$$p_1 \cdot p_2 C_1 + p_2^2 C_2 = \frac{1}{2} (I_4 - I_1 - (2p_1 \cdot p_2 + p_2^2)I_3), \qquad (8.20)$$

where I_4 is defined as:

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2(l+p_1)^2} \equiv I_4.$$
(8.21)

What have we actually done so far? With the manipulations listed above, instead of painfully solving each unique loop integral, we only need to solve the sets of equations Eq. (8.15) and Eq. (8.20) to find C_1 and C_2 . The only loop integrals that really need to be computed are I_1 , I_2 , I_3 , and I_4 , but such integrals contain no l^{μ} in their numerators. Thus the angular integrations are much more straightforward, and are usually much easier to handle. Even in the worst case, one can still solve I_1 through I_4 numerically.

The manipulation of the loop integrals of the tensor case, Eq. (8.11), proceeds along a very similar track. As one may have already guessed, we are still seeking a decomposition – a tensor decomposition. Since the numerator in Eq. (8.11) is symmetric on exchanging the indices μ and ν , we are then seeking a complete tensor decomposition that is symmetric about μ and ν . Out of the available degrees of freedom p_1^{μ} and p_2^{μ} , one can build the linearly independent tensors: $p_1^{\mu}p_1^{\nu}, p_2^{\mu}p_2^{\nu}, (p_1^{\mu}p_2^{\nu} + p_2^{\mu}p_1^{\nu})$. Besides these, we'd better not forget about an additional one: $g^{\mu\nu}$. With these building blocks, we have the decomposition:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu}l^{\nu}}{l^2(l+p_1)^2(l+p_1+p_2)^2} = D_{00}g^{\mu\nu} + D_{11}p_1^{\mu}p_1^{\nu} + D_{22}p_2^{\mu}p_2^{\nu} + D_{12}(p_1^{\mu}p_2^{\nu} + p_2^{\mu}p_1^{\nu}).$$
(8.22)

The remaining tasks are similar to the vector case. One just needs to contract each of the tensor terms on the right hand side of Eq. (8.22) in turn to obtain four linear equations to solve the coefficients D_{00} , D_{11} , D_{22} , and D_{12} . Just like the vector case, there will be some fundamental integrations to be solved, but again these integrations usually possess much simpler geometric structures, and thus are not hard to handle. The beauty of this part is that in the tensor case some of the fundamental integrations are I_1 , I_2 , I_3 , and I_4 , which have already been solved in the vector case. Out of space limitations, we shall skip the detailed procedures of the tensor case, but I still hope that with the quick demonstrations thus far in this section, the basic picture has been conveyed with minimal confusion. In neutron radiative β decay, the situation is certainly much more complicated than the examples discussed in this section, but the logic is still exactly same. That is, instead of working on each loop integration, we first take complete decompositions for both the vector and tensor cases, then contract them with independent degrees of freedom to build sufficient, independent linear equations to solve for the coefficients of the decomposition. In doing this, some remaining fundamental scalar integrations cannot be reduced further, and thus have to be solved, but these remaining integrations are usually much easier to handle.

With the Cutkosky rules discussed in last section, and the VPR discussed in this section, we now should be ready to demonstrate the actual calculations of the T-odd mimicking effect due to the final-state interactions in the neutron radiative β decay, which brings us to the next section.

8.3 Calculating the FSI-induced T-odd Correlation in Neutron Radiative β Decay

We are now ready to turn to the real calculations of the T-odd mimicking effect in neutron radiative β decay [165]. As shown before, the critical step in finding the T-odd mimicking correlation $\mathbf{l}_{\nu} \cdot (\mathbf{l}_e \times \mathbf{k})$ is to find the imaginary part of $\mathcal{M}_{\text{loop}}$. Based on the Cutkosky rules and in analogy to Eq. (8.8), we have:

$$\operatorname{Im}(\mathcal{M}_{\operatorname{loop}}) = \frac{1}{8\pi^2} \sum_{n} \int d\rho_n \sum_{s_n} \mathcal{M}_{fn} \mathcal{M}_{in}^* = \frac{1}{8\pi^2} \sum_{n} \int d\rho_n \sum_{s_n} \mathcal{M}_{fn} \mathcal{M}_{ni}, \quad (8.23)$$

where n refers to all the possible cut intermediate states. In the neutron case only QED loops are relevant due to the low energy release. The integral $\int d\rho_n$ refers to

the intermediate phase space integration over the relevant intermediate momenta for each possible cut. The Cutkosky cuts put the internal lines on shell, which makes the resulting cut diagrams behave exactly the same as the two physical ones, which allow us to write down the scattering amplitude according to the standard Feynman rules for each part of the cut diagrams. This is reflected by the terms \mathcal{M}_{fn} and \mathcal{M}_{ni} , referring to the two tree-level diagrams after a physical cut. We have seen this in the example of Figure (8.1), where \mathcal{M}_{ni} describes the tree-level nRDK process (02) of Figure (3.5), and \mathcal{M}_{fn} describes the tree-level Compton scattering process described in Figure (8.2). Also, the spin sums for the cut internal lines are included manifestly.

As one can double check, there are multiple loop diagrams, and associated with each loop diagram there are multiple possible two-body cuts. It turns out that there are 14 allowed cuts in all, which are summarized and numbered in Fig. (8.4), where an " \times " on the internal lines represents a Cutkosky cut. First of all, one question arises naturally: since there are so many loops in association with the cuts, are we sure we have collected all of them? The quick answer is yes, because we can use QED gauge invariance as a standard check – as long as all the possible loops and cuts are collected, the Ward identity of Eq. (9.43) must hold. The cuts that are collected in Fig. (8.4) indeed respect the Ward identity Eq. (9.43). Thus we convince ourselves that the loops and cuts are complete without omission. We will see this more easily shortly.

Looking at Fig. (8.4) more closely, we find that we can categorize the cuts as per the sorts of processes involved. That is, \mathcal{M}_{fn} describes the manner in which selected particles rescatter, so that we can have Compton scattering or electronproton scattering, the latter with or without the emission of an additional photon. The family of diagrams given by (1), (2), (5.1), and (6.2) contain Compton scattering from the electron, as illustrated in Figure (8.2), whereas the family comprised of (3), (4), (7.2), and (8.3) contain Compton scattering from the proton. In these families \mathcal{M}_{fn} is captured by one of the following expressions that follow the conventions in Ref. [70]:

$$\mathcal{M}_{\gamma e}^{\rm d}(l'_{e},k',l_{e},k) = -e^{2}\bar{u}_{e}(l_{e})\frac{2l_{e}\cdot\epsilon^{*} + \not\epsilon^{*}\not k}{2l_{e}\cdot k}\not\epsilon' u_{e}(l'_{e}), \qquad (8.24)$$

$$\mathcal{M}_{\gamma e}^{c}(l'_{e},k',l_{e},k) = e^{2}\bar{u}_{e}(l_{e})\frac{2l_{e}\cdot\epsilon'-\epsilon'k'}{2l'_{e}\cdot k}\epsilon^{*}u_{e}(l'_{e}), \qquad (8.25)$$

$$\mathcal{M}^{d}_{\gamma p}(p'_{p},k',p_{p},k) = -e^{2}\bar{u}_{p}(p_{p})\frac{2p_{p}\cdot\epsilon^{*} + \not\epsilon^{*}k}{2p_{p}\cdot k}\not\epsilon' u_{p}(p'_{p}), \qquad (8.26)$$

or

$$\mathcal{M}_{\gamma p}^{c}(p_{p}', k', p_{p}, k) = e^{2} \bar{u}_{p}(p_{p}) \frac{2p_{p} \cdot \epsilon' - \epsilon' k'}{2p_{p}' \cdot k} \epsilon'^{*} u_{p}(p_{p}'), \qquad (8.27)$$

where $\epsilon' \equiv \epsilon(k')$, and the superscript "c" and "d" refer to the direct and cross Compton scattering. Correspondingly, \mathcal{M}_{ni} is given by the tree-level neutron radiative β -decay amplitude, as per the form of \mathcal{M}_{01} and \mathcal{M}_{02} , defined in Eq. (4.4) and Eq. (4.5), with only some of the arguments changed. Technically we define a "family" to be those contributions to the T-odd correlation which cancel amongst themselves to



Figure 8.4: All two-particle cut contributions to $n(p_n) \to p(p_p) + e^-(l_e) + \bar{\nu}_e(l_\nu) + \gamma(k)$ which appear in $\mathcal{O}(\alpha)$ up to corrections of recoil order.

yield zero when we replace ϵ or ϵ^* by k or ϵ' or ϵ'^* by k' as per the Ward-Takahashi identities. Here we can quickly confirm that the Ward-Takahashi identities do hold for the " $\gamma - e$ " and " $\gamma - p$ " families respectively. Furthermore, let us clearly define the relevant intermediate phase space integrations over the kinematically allowed phase space. For $\gamma - e$ scattering we have

$$\int d\rho_{\gamma e} \equiv \int \frac{d^3 \mathbf{l}'_e}{2E'_e} \frac{d^3 \mathbf{k}'}{2\omega'} \delta^{(4)}(l'_e + k' - P_{0e})$$
(8.28)

with $P_{0e} \equiv l_e + k$, whereas for $\gamma - p$ scattering we have

$$\int d\rho_{\gamma p} \equiv \int \frac{d^3 \mathbf{p}'_p}{2E'_p} \frac{d^3 \mathbf{k}'}{2\omega'} \delta^{(4)}(p'_p + k' - P_{0p})$$
(8.29)



Figure 8.5: Diagrams which appear in $\text{Im}(\mathcal{M}_{\text{loop}})$ for e - p scattering with electron bremsstrahlung. We denote the two graphs by $\mathcal{M}_{ep\gamma}^{\text{ef}}(l'_e, p'_p, l_e, k, p_p)$ and $\mathcal{M}_{ep\gamma}^{\text{ei}}(l'_e, p'_p, l_e, k, p_p)$, respectively. The diagrams and amplitudes appropriate to proton bremsstrahlung follow from exchanging electron and proton variables.

with $P_{0p} \equiv p_p + k$. Collecting the pieces, we have

$$\operatorname{Im}(\mathcal{M}_{1}) = \frac{1}{8\pi^{2}} \int d\rho_{\gamma e} \sum_{s_{\gamma e}} \mathcal{M}_{\gamma e}^{d}(l'_{e}, k', l_{e}, k) \mathcal{M}_{01}(l'_{e}, k', p_{p}), \qquad (8.30)$$

$$\operatorname{Im}(\mathcal{M}_2) = \frac{1}{8\pi^2} \int d\rho_{\gamma e} \sum_{s_{\gamma e}} \mathcal{M}^c_{\gamma e}(l'_e, k', l_e, k) \mathcal{M}_{01}(l'_e, k', p_p), \qquad (8.31)$$

$$\operatorname{Im}(\mathcal{M}_{5.1}) = \frac{1}{8\pi^2} \int d\rho_{\gamma e} \sum_{s_{\gamma e}} \mathcal{M}^d_{\gamma e}(l'_e, k', l_e, k) \mathcal{M}_{02}(l'_e, k', p_p), \qquad (8.32)$$

$$\operatorname{Im}(\mathcal{M}_{6.2}) = \frac{1}{8\pi^2} \int d\rho_{\gamma e} \sum_{s_{\gamma e}} \mathcal{M}^c_{\gamma e}(l'_e, k', l_e, k) \mathcal{M}_{02}(l'_e, k', p_p), \qquad (8.33)$$

for the " $\gamma - e$ " cuts, and

$$\operatorname{Im}(\mathcal{M}_3) = \frac{1}{8\pi^2} \int d\rho_{\gamma p} \sum_{s_{\gamma p}} \mathcal{M}^d_{\gamma p}(p'_p, k', p_p, k) \mathcal{M}_{02}(l_e, k', p'_p), \qquad (8.34)$$

$$\operatorname{Im}(\mathcal{M}_4) = \frac{1}{8\pi^2} \int d\rho_{\gamma p} \sum_{s_{\gamma p}} \mathcal{M}^c_{\gamma p}(p'_p, k', p_p, k) \mathcal{M}_{02}(l_e, k', p'_p), \qquad (8.35)$$

$$\operatorname{Im}(\mathcal{M}_{7.2}) = \frac{1}{8\pi^2} \int d\rho_{\gamma p} \sum_{s_{\gamma p}} \mathcal{M}^d_{\gamma p}(p'_p, k', p_p, k) \mathcal{M}_{01}(l_e, k', p'_p), \qquad (8.36)$$

$$\operatorname{Im}(\mathcal{M}_{8.3}) = \frac{1}{8\pi^2} \int d\rho_{\gamma p} \sum_{s_{\gamma p}} \mathcal{M}^c_{\gamma p}(p'_p, k', p_p, k) \mathcal{M}_{01}(l_e, k', p'_p), \qquad (8.37)$$

for the " $\gamma - p$ " cuts.

In addition to the families of Compton cuts, there are cuts in which \mathcal{M}_{fn} is determined by electron-proton scattering either with and without bremsstrahlung,

and, correspondingly, \mathcal{M}_{ni} is determined by either nonradiative or radiative β -decay. Referring to Figure (8.4), we see for cuts in which the electron and proton scatter with bremsstrahlung that diagrams (5.2) and (6.1) comprise the family associated with electron bremsstrahlung, as shown in Figure (8.5), and (7.1) and (8.1) comprise the family associated with proton bremsstrahlung. In these families \mathcal{M}_{fn} is given by one of the following:

$$\mathcal{M}_{ep\gamma}^{\text{ef}}(l'_{e}, p'_{p}, l_{e}, k, p_{p}) = -e^{3}\bar{u}_{e}(l_{e})\frac{2l_{e}\cdot\epsilon^{*} + \not\epsilon^{*}\not k}{2l_{e}\cdot k}\gamma^{\mu}u_{e}(l'_{e})\frac{g_{\mu\nu}}{(p'_{p} - p_{p})^{2}}\bar{u}_{p}(p_{p})\gamma^{\nu}u_{p}(p'_{p}),$$
(8.38)

$$\mathcal{M}_{ep\gamma}^{\rm ei}(l'_{e},p'_{p},l_{e},k,p_{p}) = e^{3}\bar{u}_{e}(l_{e})\gamma^{\mu}\frac{2l'_{e}\cdot\epsilon^{*}-k\!\!\!/\epsilon^{*}}{2l'_{e}\cdot k}u_{e}(l'_{e})\frac{g_{\mu\nu}}{(p'_{p}-p_{p})^{2}}\bar{u}_{p}(p_{p})\gamma^{\nu}u_{p}(p'_{p}),$$
(8.39)

$$\mathcal{M}_{ep\gamma}^{\rm pf}(l'_{e},p'_{p},l_{e},k,p_{p}) = e^{3}\bar{u}_{p}(p_{p})\frac{2p_{p}\cdot\epsilon^{*} + {\not\!\!\!\!/} {e^{*} \not\!\!\!\!/} {k}}{2p_{p}\cdot k}\gamma^{\mu}u_{p}(p'_{p})\frac{g_{\mu\nu}}{(l'_{e}-l_{e})^{2}}\bar{u}_{e}(l_{e})\gamma^{\nu}u_{e}(l'_{e}),$$
(8.40)

or

$$\mathcal{M}_{ep\gamma}^{\mathrm{pi}}(l'_{e}, p'_{p}, l_{e}, k, p_{p}) = -e^{3}\bar{u}_{p}(p_{p})\gamma^{\mu} \frac{2p'_{p} \cdot \epsilon^{*} - \not\!\!\! k \not\!\!\! \epsilon^{*}}{2p'_{p} \cdot k} u_{p}(p'_{p}) \frac{g_{\mu\nu}}{(l'_{e} - l_{e})^{2}} \bar{u}_{e}(l_{e})\gamma^{\nu} u_{e}(l'_{e}) \,.$$
(8.41)

Moreover, \mathcal{M}_{ni} is given by neutron β -decay, which has been shown in Fig. (3.4). Up to recoil-order corrections, we have:

$$\mathcal{M}_{\rm DK}(l'_e, p'_p) = \frac{g_V G_F}{\sqrt{2}} \bar{u}_e(l'_e) \gamma_\rho (1 - \gamma_5) v_\nu(l_\nu) \bar{u}_p(p'_p) \gamma^\rho (1 - \lambda \gamma_5) u_n(p_n), \qquad (8.42)$$

which is similar to Eq. (3.67), only with the spin polarization projection operator " $(1 + \gamma_5 \beta)/2$ " dropped since we are considering unpolarized neutron decay. Thus we find

$$\operatorname{Im}(\mathcal{M}_{5.2}) = \frac{1}{8\pi^2} \int d\rho_{ep\gamma} \sum_{s_{ep}} \mathcal{M}_{ep\gamma}^{\text{ef}}(l'_e, p'_p, l_e, k, p_p) \mathcal{M}_{\text{DK}}(l'_e, p'_p), \quad (8.43)$$

$$\operatorname{Im}(\mathcal{M}_{6.1}) = \frac{1}{8\pi^2} \int d\rho_{ep\gamma} \sum_{s_{ep}} \mathcal{M}_{ep\gamma}^{\mathrm{ei}}(l'_e, p'_p, l_e, k, p_p) \mathcal{M}_{\mathrm{DK}}(l'_e, p'_p), \quad (8.44)$$

and

$$\operatorname{Im}(\mathcal{M}_{7.1}) = \frac{1}{8\pi^2} \int d\rho_{ep\gamma} \sum_{s_{ep}} \mathcal{M}_{ep\gamma}^{\text{pf}}(l'_e, p'_p, l_e, k, p_p) \mathcal{M}_{\text{DK}}(l'_e, p'_p) , \quad (8.45)$$

$$\operatorname{Im}(\mathcal{M}_{8.1}) = \frac{1}{8\pi^2} \int d\rho_{ep\gamma} \sum_{s_{ep}} \mathcal{M}_{ep\gamma}^{\text{pi}}(l'_e, p'_p, l_e, k, p_p) \mathcal{M}_{\text{DK}}(l'_e, p'_p) \qquad (8.46)$$


Figure 8.6: Diagram which appears in $\text{Im}(\mathcal{M}_{\text{loop}})$ for e-p scattering. We denote the graph by $\mathcal{M}_{ep}(l'_e, p'_p, l_e, p_p)$.

for the " $e - p - \gamma$ " cuts. The relevant intermediate phase space integration for the " $e - p - \gamma$ " cuts reads:

$$\int d\rho_{ep\gamma} \equiv \int \frac{d^3 \mathbf{l'}_e}{2E'_e} \frac{d^3 \mathbf{p}'_p}{2E'_p} \delta^{(4)} (l'_e + p'_p - P_{0e} - p_p).$$
(8.47)

Let us also note that the Ward-Takahashi identities holds for the " $e - p - \gamma$ " family.

The last family of cuts is given by (6.3) and (8.2) in Fig. (8.4). In this case \mathcal{M}_{fn} is given by e - p scattering, shown as in Fig. (8.6), and we have:

$$\mathcal{M}_{ep}(l'_e, p'_p, l_e, p_p) = -e^2 \bar{u}_e(l_e) \gamma^{\mu} u_e(l'_e) \frac{g_{\mu\nu}}{(l'_e - l_e)^2} \bar{u}_p(p_p) \gamma^{\nu} u_p(p'_p) \,. \tag{8.48}$$

The corresponding \mathcal{M}_{ni} is given by $\mathcal{M}_{01}(l'_e, k, p'_p)$ for (6.3) and $\mathcal{M}_{02}(l'_e, k, p'_p)$ for (8.2). We thus have:

$$\operatorname{Im}(\mathcal{M}_{6.3}) = \frac{1}{8\pi^2} \int d\rho_{ep} \sum_{s_{ep}} \mathcal{M}_{ep}(l'_e, p'_p, l_e, p_p) \mathcal{M}_{01}(l'_e, k, p'_p), \qquad (8.49)$$

$$\operatorname{Im}(\mathcal{M}_{8.2}) = \frac{1}{8\pi^2} \int d\rho_{ep} \sum_{s_{ep}} \mathcal{M}_{ep}(l'_e, p'_p, l_e, p_p) \mathcal{M}_{02}(l'_e, k, p'_p)$$
(8.50)

for the "e - p" cuts. The intermediate phase space integration for the "e - p" cuts reads:

$$\int d\rho_{ep} \equiv \int \frac{d^3 \mathbf{l}'_e}{2E'_e} \frac{d^3 \mathbf{p}'_p}{2E'_p} \delta^{(4)} (l'_e + p'_p - l_e - p_p).$$
(8.51)

Again, one can readily check that the Ward-Takahashi identities does hold for the "e-p" family. This completes our earlier claim that the Ward identity is respected

within each type of the cuts, thus our exhaustive list of loops and cuts in Fig. (8.4) is complete.

Based on the listed building blocks collected from all of the loop diagrams, we thus obtain the spin-averaged T-odd mimicking correlation from the SM physics as per:

$$\overline{|\mathcal{M}|^2}_{\mathrm{T-odd}}^{\mathrm{(SM)}} \equiv \frac{1}{2} \sum_{\mathrm{spins}} |\mathcal{M}|^2_{\mathrm{T-odd}}^{\mathrm{(SM)}} = \frac{1}{2} \sum_{\mathrm{spins}} (2\mathrm{Re}(\mathcal{M}_{\mathrm{tree}}i\mathrm{Im}\mathcal{M}_{\mathrm{loop}}^*)), \qquad (8.52)$$

Further calculations can be carried out by following the steps similar to those demonstrated in Chapter 3. That is, one can make spin sums and recombine the involved spinors with the γ matrices into traces of leptonic and baryonic components. The big challenge then is that we are now facing much more complicated trace calculations – the complexity grows tremendously fast with the number of γ matrices involved. Usually the calculation of a trace containing six γ matrices is already a time consuming job, but in Eq. (8.52) one is challenged by traces containing up to twelve γ matrices, which is too difficult to handle by hand! We had to handle the calculations in Eq. (8.52) by computer. We used the program "FORM" to compute the traces [168]. In what follows we will directly report the results of the T-odd correlation in $\mathcal{O}(\alpha)$ up to corrections of recoil order. We organize the results as per the various gauge-invariant families we describe in the main body of the text, employing the subscript convention which follows the labeling in Fig. (3.5) and Fig. (8.4). The coefficients that appear in the following results are introduced by the applications of the VPR, which is demonstrated in last section. The details of finding the required linear equations as well as solving the relevant fundamental integrations follow exactly the ideas as explained in the last section. They are just much more tedious. We reserve these details for Appendix A.

With ξ referring to the triple-product in momenta, $\xi \equiv \mathbf{l}_{\nu} \cdot (\mathbf{l}_e \times \mathbf{k})$, we here present the results directly. We first define the T-odd piece in Eq. (8.5) via

$$2\operatorname{Re}(\mathcal{M}_{\operatorname{tree}} \cdot \mathcal{M}_{\operatorname{loop}}^{*}) \equiv \overline{|\mathcal{M}|^{2}}_{\operatorname{T-odd}}^{(\operatorname{SM})}.$$
(8.53)

Since $\overline{|\mathcal{M}|^2}_{T-odd}^{(SM)}$ is obtained from the interference between the tree level processes, which are labeled as (01) and (02) in Figure (3.5), and the loop diagrams which are labeled as (1) through (8.3) in Figure (8.4), we denote each different family in such a way: the labels of loop diagrams followed by a "dot" and then the labels of tree level

diagrams. The result for the $\gamma - e$ family is

$$\begin{split} \overline{|\mathcal{M}|^{2}}_{\mathrm{T-odd}}^{\mathrm{(SM)}} &[1.01+1.02+2.01+2.02+5.1.01+5.1.02+6.2.01+6.2.02] \\ = -\alpha^{2}g_{V}^{2}G_{F}^{2}\xi 64M^{2}(1-\lambda^{2}) \left(\frac{m_{e}^{2}}{(l_{e}\cdot k)^{2}\omega}a_{1}+\frac{m_{e}^{2}}{(l_{e}\cdot k)^{2}\omega}J_{1}+\frac{1}{l_{e}\cdot k\,\omega}c_{2}+\frac{1}{l_{e}\cdot k\,\omega}a_{1}\right) \\ &-\frac{1}{l_{e}\cdot k\,\omega}J_{1}+\frac{m_{e}^{2}}{l_{e}\cdot k\,\omega}b_{2}+\frac{m_{e}^{2}}{l_{e}\cdot k\,\omega}a_{2}-\frac{m_{e}^{2}}{l_{e}\cdot k\,\omega}J_{2}+\frac{ME_{e}}{\omega}k_{6.2}+\frac{ME_{e}}{\omega}g_{6.2}-\frac{ME_{e}}{\omega}b_{6.2} \\ &-\frac{2ME_{e}}{\omega}a_{6.2}+\frac{ME_{e}}{\omega}J_{6.2}+\frac{ME_{e}}{l_{e}\cdot k\,\omega}b_{5.1}-\frac{ME_{e}}{l_{e}\cdot k\,\omega}J_{5.1}-\frac{ME_{e}}{2l_{e}\cdot k}g_{6.2}+\frac{ME_{e}}{2l_{e}\cdot k}f_{6.2} \\ &+\frac{ME_{e}}{l_{e}\cdot k}a_{6.2}+\frac{M^{2}}{\omega}i_{6.2}-\frac{M^{2}}{2\omega}c_{6.2}-\frac{M^{2}E_{e}}{l_{e}\cdot k}c_{6.2}+\frac{M^{2}E_{e}}{l_{e}\cdot k}h_{6.2}+\frac{M^{2}}{2l_{e}\cdot k\,\omega}a_{5.1} \\ &+\frac{M^{3}}{2l_{e}\cdot k}e_{6.2} \\ \end{pmatrix}, \end{split}$$

$$(8.54)$$

where M is defined as $M \equiv (M_n + M_p)/2$, the average value of M_n and M_p . Using M suggests that we are ignoring the small relative difference between M_n and M_p for simplification of the theoretical computations, and the validity of this is justified since we work in LRO approximation. The coefficients a_1, \ldots in Eq. (8.54) and also in all the rest of families are just the parameters that are introduced in performing the VPR, and they are solved in Appendix A.

The result for the $\gamma - p$ family is

$$\overline{|\mathcal{M}|^2}_{T-odd}^{(SM)} [3.01 + 3.02 + 4.01 + 4.02 + 7.2.01 + 7.2.02 + 8.3.01 + 8.3.02]$$

$$= -\alpha^2 g_V^2 G_F^2 \xi 64 M^3 (1 - \lambda^2) \left(\frac{E_e}{l_e \cdot k \,\omega} a_{7.2} + \frac{E_e}{l_e \cdot k \,\omega} J_{7.2} - \frac{1}{l_e \cdot k \,\omega^2} J_3 - \frac{M E_e}{l_e \cdot k} b_{8.3} - \frac{M E_e}{l_e \cdot k} J_{8.3} + \frac{M}{2l_e \cdot k \,\omega} b_{7.2} + \frac{M}{l_e \cdot k \,\omega} J_4 - \frac{M^2}{2l_e \cdot k} c_{8.3} \right)$$

$$= 0 + \mathcal{O}(M). \qquad (8.55)$$

Note here we have determined that the contribution to the " $\gamma - p$ " family vanishes in leading order in M based on the solutions of the relevant coefficients, which are discussed in Appendix A. The results for the $e - p - \gamma$ families are

$$\begin{aligned} \overline{|\mathcal{M}|^{2}}_{\mathrm{T-odd}}^{\mathrm{(SM)}} \left[5.2.01 + 5.2.02 + 6.1.01 + 6.1.02 \right] \\ &= -\alpha^{2} g_{V}^{2} G_{F}^{2} \xi 64 M^{3} (1 - \lambda^{2}) \left(\frac{2E_{e}}{\omega} k_{6.1} + \frac{2M}{\omega} i_{6.1} - \frac{M}{\omega} c_{6.1} - \frac{2E_{e}}{l_{e} \cdot k} \frac{1}{\omega} a_{5.2} - \frac{2m_{e}^{2}}{l_{e} \cdot k} k_{6.1} \right. \\ &+ \frac{m_{e}^{2}}{l_{e} \cdot k} f_{6.1} + \frac{m_{e}^{2}}{l_{e} \cdot k} J_{6.1} - \frac{M}{l_{e} \cdot k} \frac{1}{\omega} c_{5.2} - \frac{2ME_{e}}{l_{e} \cdot k} i_{6.1} + \frac{2ME_{e}}{l_{e} \cdot k} h_{6.1} + \frac{2ME_{e}}{l_{e} \cdot k} c_{6.1} \\ &+ \frac{M^{2}}{l_{e} \cdot k} e_{6.1} \right), \end{aligned}$$

$$(8.56)$$

and

$$\overline{|\mathcal{M}|^2}_{\mathrm{T-odd}}^{\mathrm{(SM)}} [7.1.01 + 7.1.02 + 8.1.01 + 8.1.02]$$

$$= -\alpha^2 g_V^2 G_F^2 \xi 64 M^3 (1 - \lambda^2) \left(\frac{2E_e}{l_e \cdot k \,\omega} a_{7.1} + \frac{M}{l_e \cdot k \,\omega} b_{7.1} - \frac{2ME_e}{l_e \cdot k} a_{7.1} - \frac{2M}{\omega} b_{7.1} + \frac{2ME_e}{l_e \cdot k} b_{7.1} - \frac{M^2}{l_e \cdot k} c_{8.1} \right) = 0 + \mathcal{O}(M), \qquad (8.57)$$

where we have determined that the contribution to this family also vanishes in leading order in M, which can be easily double checked with the solutions of the relevant coefficients, which are discussed in Appendix A. Finally, the result for the e - pfamily is

$$\overline{|\mathcal{M}|^{2}}_{T-odd}^{(SM)} \left[6.3.01 + 6.3.02 + 8.2.01 + 8.2.02 \right] \\
= -\alpha^{2} g_{V}^{2} G_{F}^{2} \xi 64 M^{3} (1 - \lambda^{2}) \left(\frac{2m_{e}^{2}}{l_{e} \cdot k} k_{6.3} - \frac{2E_{e}}{\omega} k_{6.3} - \frac{2E_{e}}{\omega} a_{6.3} - \frac{2M}{\omega} i_{6.3} - \frac{M}{\omega} c_{6.3} \right) \\
- \frac{m_{e}^{2}}{l_{e} \cdot k} f_{6.3} + \frac{2m_{e}^{2}}{l_{e} \cdot k} a_{6.3} - \frac{m_{e}^{2}}{l_{e} \cdot k} J_{6.3} + \frac{2ME_{e}}{l_{e} \cdot k} a_{8.2} + \frac{2ME_{e}}{l_{e} \cdot k} i_{6.3} - \frac{2ME_{e}}{l_{e} \cdot k} h_{6.3} \\
+ \frac{M^{2}}{l_{e} \cdot k} c_{8.2} - \frac{M^{2}}{l_{e} \cdot k} e_{6.3} \right).$$
(8.58)

The coefficients that appear in Eqs. (8.54) through (8.58) can obtained by solving the sets of linear equations as listed in detail in Appendix A.

As one final technical point, let me stress that it is most convenient to choose a restricted range in the $\gamma - e$ opening angle. As one can see from the formulae in Appendix A, the solutions to the Passarino-Veltman equations become invalid if the opening angle $\theta_{e\gamma}$ between the outgoing electron and the photon is exactly equal to 0 or to π . There is no physical divergences. Rather, the spatial components of the vector and tensor equations to determine the relevant coefficients become degenerate at such a boundary. Potentially one could remove this difficulty by solving the equations for infinitesimal values of $\theta_{e\gamma}$ or $(\theta_{e\gamma} - \pi)$ and then interpolating the solutions to the needed $\theta_{e\gamma} = 0$ and π points. In our present work, we simply choose a restricted range $x_k \equiv \cos \theta_{e\gamma} \in [-0.9, 0.9]$, which spans the angular range over which the neutron radiative decay rate is largest [169].

8.4 Cancellation of Infrared Divergences

Before presenting the final results, let us first briefly discuss an important issue – the control of infrared divergences. As mentioned at the beginning of this chapter, we encountered an infrared divergence in our calculations. In the case of e - p cuts, the intermediate momenta satisfy $l'_e + p'_p = l_e + p_p$. Obviously, the diagrams (6.3) and (8.2) are each infrared divergent when $l'_e = l_e$ since they both contain the term

 $1/(l'_e - l_e)^2$ arising from the photon propagator as shown in Fig. (8.6). Such an infrared divergent problem bothered us for a while. But as expected by the so-called KLN theorem [170] [171], the observed process should be infrared finite. Thus we are sure that the divergence under consideration should eventually cancel. We analyzed this problem carefully and have shown that the infrared divergences do cancel. The key to this problem lies in the fundamental requirement of the Ward identity within the e-p family. To guarantee the Ward identity, the diagrams (6.3) and (8.2) in Fig. (8.4) has to be considered together – either (6.3) or (8.2) by itself does not have any physical meaning. To allow the calculation of e - p family to proceed, we can first regularize the divergence by giving the intermediate photon propagator a fictitious mass m_{γ} . After finishing all the relevant integrals, the infrared divergence in the $m_{\gamma} \to 0$ limit can be isolated as a common factor of $\log (m_{\gamma}^2/4|\mathbf{l}_e|^2)$. On taking the large nucleon mass limit $M \to \infty$, the detailed calculations of $the large 1000 m_{\gamma}^2/4|\mathbf{l}_e|^2$.

8.5 Results

We can now present our results for $A_{\xi}^{(\text{SM})}$. As can be seen explicitly, all of the contributions to $\overline{|\mathcal{M}|^2}_{\text{T-odd}}^{(\text{SM})}$ are found to be proportional to $(1 - \lambda^2)$, so that the resulting asymmetry goes as $(1 - \lambda^2)/(1 + 3\lambda^2)$, up to recoil-order corrections. The dependence on λ in $\overline{|\mathcal{M}|^2}_{\text{T-odd}}^{(\text{SM})}$ stems from the special nature of the T-odd correlation. It is a real triple product in momenta arising from the interference of a tree-level diagram with an imaginary part of an one-loop diagram after summing over the particles' spins. To leading order in M, the only surviving contribution is obtained from the product of the symmetric part of the lepton tensor, which is determined by a trace containing γ_5 , namely, $l_{\nu}^{\rho} \epsilon^{\alpha\beta\gamma\delta} + l_{\nu}^{\delta} \epsilon^{\alpha\beta\gamma\rho} - g^{\rho\delta} l_{\nu\mu} \epsilon^{\alpha\beta\gamma\mu}$, where α, β , and γ refer to photon or lepton indices, with the symmetric part of the hadron tensor. The latter is proportional to $(1 + \lambda^2)p_{\rho}p_{\delta} - \lambda^2 M^2 g_{\rho\delta}$, where p is a baryon momentum and $p^2 = M^2$. As one can easily check, this special combination generates an overall $(1 - \lambda^2)$ coefficient. We use $\lambda = -1.2701 \pm 0.0025$ [19] in our numerical evaluation. For definiteness, let us restate the remaining input parameters we employ are $m_e =$ $0.510999 \text{ MeV}, M_n = 939.565 \text{ MeV}, M_p = 938.272 \text{ MeV}, \text{ and } \alpha^{-1} = 137.0360 - \text{these}$ quantities can be regarded as exact for our current purposes [19]. We show our results for the T-odd asymmetry in neutron radiative β -decay in Table (8.1) and Fig. (8.7). We see that the asymmetry is rather smaller than α .

As of this point we have completed our analysis of the T-odd mimicking effect due to the final-state interaction in SM for neutron radiative β decay. One important thing we should know about this mimicking effect is that although it gives a T-odd signal, but it actually does not represent a true source of CP violation [10]. Such a mimicking effect stems from the CP conserving Lagrangian if we assume all the coefficients real. A quick way to distinguish this mimicking effect is that for the presence of a real CP violating source, when all the involved particles are changed to their antipartners, the total decay rate will also change, revealing a real difference

$\omega_{\min}(MeV)$	A_{ξ}
0.01	1.76×10^{-5}
0.05	3.86×10^{-5}
0.1	6.07×10^{-5}
0.2	9.94×10^{-5}
0.3	1.31×10^{-4}
0.4	1.54×10^{-4}
0.5	1.70×10^{-4}
0.6	1.81×10^{-4}
0.7	$1.89 imes 10^{-4}$

Table 8.1: T-odd asymmetry as a function of ω_{\min} for neutron radiative β -decay.



Figure 8.7: The asymmetry A_{ξ} versus the smallest detectable photon energy ω_{\min} in neutron radiative β -decay. Note that the asymmetry is determined from integrals over the complete phase space with the constraints $\omega > \omega_{\min}$ and $x_k \in [-0.9, 0.9]$.

between the properties of matter and antimatter; for the mimicking effect, on the other hand, changing all the involved particles by their antipartners does not affect the final decay rate.

Furthermore, as has been stressed, the mimicking T-odd asymmetry is controlled by the factor $(1 - \lambda^2)$, and an interesting question naturally arises: if in some cases, say for some nuclear radiative β decays, if a similar structure also exists, characterized by a λ_{eff} which happens to be close to unity, then the T-odd mimicking effect can be greatly suppressed. Such a system might serve as an ideal experimental candidate in the search for real sources of CP violation, such as the HHH interaction. This is a part of our work – extending all of our calculations to the case of nuclear decay. As will be seen in later sections, detailed analysis shows that such an extension is quite practical with scarcely new mathematical manipulations.

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Chapter 9 Extending from Neutron to Nuclear Radiative β Decay

In principle, nuclear β decays need not be very different from neutron β decay. Nucleus are bound systems of nucleons, which are held together by strong forces. At sufficiently low energies the details of nuclear structure may not matter; the decay may still be effectively viewed as that of a point-like object. The situation here is similar to the neutron case. As mentioned before, we know the neutron is a hadronic bound state of one u quark and two d quarks, and when it undergoes weak decay, it is really one of its d quarks that undergoes weak decay, into a u quark, as shown in Fig. (9.1). At sufficiently low energies, however, one does not detect the fundamental degree of freedom of quarks, thus we effectively view the neutron as the fundamental particle that undergoes weak decay. The transition from the description on the fun-



Figure 9.1: Real picture of neutron β decay on the fundamental quarks level.

damental quark level to the effective description on the neutron level is represented by the effective axial-vector coupling constant $\lambda = 1.2701$, which is otherwise equal to unity on the quark level. In the mean time, we should keep in mind that treating nuclei as point-like objects is only a crude picture due to the larger sizes of nuclei.

There is a big difference, however, between the neutron and nuclear β decay. In the isolated neutron case, since neutron has a bit higher mass than proton, then only the case of neutron decaying into proton is allowed, since the proton as the lightest and the most stable baryon cannot undergo a weak decay into a neutron. In the nuclear case, on the other hand, due to the complicated strongly interacting environment within the nucleus, the inverse process $p \to n + e^+ + \nu_e$ becomes possible. Thus in nuclear β decays, there are two possible types of weak decay as per the sign of the charge of the lepton decay electronic product. That is, $N(Z) \to N'(Z \pm 1) + e^{\mp} + \bar{\nu}_e(\nu_e)$, where Z refers to the total number of protons in the parent nucleus N.

Based on our calculations of the HHH-induced T-odd correlation and the FSIinduced T-odd mimicking effect in the neutron radiative β decay, we observed that

1. the HHH-induced T-odd correlation as a real CP violation gets larger when the energy release gets higher;

2. the T-odd mimicking effect due to final-state interactions is modulated by the overall factor $(1 - \lambda^2)$, where $\lambda \equiv g_A/g_V = 1.2701$.

These two observations made us wonder whether a nuclear radiative decay might be better suited to an experimental search. That is, if in some radiative β decay the energy release does get enhanced and the associated λ_{eff} happens to be very close to unity, then we can expect a much larger HHH-induced T-odd effect along with a greatly suppressed FSI-induced mimicking effect. We expect such a candidate may exist among the very rich nuclear β decays, and we have found some suitable choices. This is our motivation in performing the extension.

In this chapter we extend our investigations of the spin-independent T-odd correlation from the neutron radiative β decay into the more general nuclear radiative β decays. To do this, we will first look at the special case, ¹⁹Ne radiative β decay: ¹⁹Ne \rightarrow ¹⁹ F + $e^+ + \nu_e + \gamma$. The specialty of such a nuclear weak decay lies in the fact that both the parent nucleus ¹⁹Ne and the daughter nucleus ¹⁹F are spin-1/2 particles, in complete analogy with the neutron and proton. In such a situation, we can translate what we have done in the neutron case to the nuclear case rather directly. This helps us expose the similarities as well as the differences between the neutron and nuclear radiative β decays, and illuminates the way to perform the extension. It turns out, as we will see in the end, there are great similarities between the neutron and nuclear cases, and the difference is minimal as long as we work only in leading recoil order, so that only minor adjustments in the formulae we have developed so far are ultimately needed.

9.1 ¹⁹Ne Radiative β Decay as a Total Analogy of the Neutron Case

In trying to extend our treatment from neutron to nuclear radiative β decay, it is the ¹⁹Ne case that first came into our scope. From the nuclear data table, we know that the nucleus ¹⁹Ne is a spin-1/2, isospin-1/2 particle. Moreover, it can undergo the decay ¹⁹Ne \rightarrow ¹⁹ F + e⁺ + ν_e , with ¹⁹F also being a spin-1/2, isospin-1/2 particle. In comparison with the neutron case, ¹⁹Ne could seemingly be viewed as a "fat neutron," and allowing us to apply the previous formulae directly. This is false, however, because there is a major difference between the cases: in the ¹⁹Ne case both the parent and daughter nuclei are charged particles. Thus, when dressing the weak processes with QED bremsstrahlung to describe the ¹⁹Ne radiative β decay, there will be more Feynman diagrams. The leading order ¹⁹Ne radiative β decay can be described by three Feynman diagrams (01), (02), and (02') as shown in Fig. (9.2). Comparing with the neutron case, Fig. (3.5), we see that the diagrams (01) and (02) in Fig. (9.2) are just same as in the neutron case, but the diagram (02') is new. The



Figure 9.2: QED bremsstrahlung processes in ¹⁹Ne(P) \rightarrow ¹⁹F(P') + $e^+(l_e) + \nu_e(l_\nu) + \gamma(k)$.

additional diagram (02') appears due to the fact that the parent nucleus ¹⁹Ne also carries electric charge, thus it can also couple to the electromagnetic field. It seems that we are now forced to deal with totally different calculations and to develop new and more complicated sets of formulae. It turns out that the situation is not that bad as long as we constrain ourselves within the leading recoil order contribution; many of the new contributions cancel.

To be more explicit, let us first write down the scattering amplitude as per the Feynman diagrams of Fig. (9.2):

$$\mathcal{M} = \frac{eg_V^{eff}G_F}{\sqrt{2}}(\mathcal{M}_{01} + \mathcal{M}_{02} + \mathcal{M}_{02'}), \qquad (9.1)$$

where the overall coupling constants $eg_V^{eff}G_F/\sqrt{2}$ has been taken out, and we have:

$$\mathcal{M}_{01} = \bar{u}_e(l_e) \frac{2l_e \cdot \epsilon^* + \not \epsilon^* \not k}{2l_e \cdot k} \gamma_\rho (1 - \gamma_5) v_\nu(l_\nu) \bar{u}_{\rm F} \gamma^\rho (1 - \lambda^{eff} \gamma_5) u_{\rm Ne}, \qquad (9.2)$$

$$\mathcal{M}_{02} = Z \bar{u}_e(l_e) \gamma_{\rho} (1 - \gamma_5) v_{\nu}(l_{\nu}) \bar{u}_{\rm F} \frac{2P' \cdot \epsilon^* + \epsilon^* k}{2P' \cdot k} \gamma^{\rho} (1 - \lambda^{eff} \gamma_5) u_{\rm Ne}, \qquad (9.3)$$

$$\mathcal{M}_{02'} = (Z+1)\bar{u}_e(l_e)\gamma_{\rho}(1-\gamma_5)v_{\nu}(l_{\nu})\bar{u}_{\rm F}\gamma^{\rho}(1-\lambda^{eff}\gamma_5)\frac{2P\cdot\epsilon^*-k_{\rm F}}{-2P\cdot k}u_{\rm Ne}, \ (9.4)$$

where Z = 9 is the total electric charge of the ¹⁹F nucleus. Since we are working at low energies, we are treating both the parent nucleus ¹⁹Ne and the daughter nucleus ¹⁹F as spin-1/2 point-like particles, just as we did in the neutron case. We see that \mathcal{M} possesses the following properties:

1. on replacing the photon polarization vector ϵ and its complex conjugate ϵ^* with its 4-momentum k, we have

$$\mathcal{M}|_{\epsilon,\,\epsilon^*\to k} = 0,\tag{9.5}$$

so that QED gauge invariance is satisfied. This allows us to perform the gauge replacement trick $\sum_{s} \epsilon_{\mu}^{*} \epsilon_{\nu} \rightarrow -g_{\mu\nu}$. 2. In determining the leading recoil order contribution, we have $1/P \cdot k \sim 1/P' \cdot k \sim 1/M$, where M is defined as the average mass of the ¹⁹Ne and ¹⁹F, $M \equiv (M_{^{19}Ne} + M_{^{19}F})/2$. We can ignore the subleading term $\notin k$ in Eq. (9.3) and Eq. (9.4).

On combining these properties, we get

$$\mathcal{M}_{02} = Z\epsilon_{\mu}\bar{u}_{e}(l_{e})\gamma_{\rho}(1-\gamma_{5})v_{\nu}(l_{\nu})\bar{u}_{\mathrm{F}}\frac{P'^{\mu}}{P'\cdot k}\gamma^{\rho}(1-\lambda^{eff}\gamma_{5})u_{\mathrm{Ne}},\tag{9.6}$$

and

$$\mathcal{M}_{02'} = (Z+1)\epsilon_{\mu}\bar{u}_{e}(l_{e})\gamma_{\rho}(1-\gamma_{5})v_{\nu}(l_{\nu})\bar{u}_{\mathrm{F}}\gamma^{\rho}(1-\lambda^{eff}\gamma_{5})\frac{P^{\mu}}{-P\cdot k}u_{\mathrm{Ne}}.$$
(9.7)

The difference between P and P' is only to the recoil order, so that we can just set P' = P. Thus, to leading recoil order, we have:

$$\mathcal{M}_{02} + \mathcal{M}_{02'} = -\epsilon_{\mu} \bar{u}_e(l_e) \gamma_{\rho} (1 - \gamma_5) v_{\nu}(l_{\nu}) \bar{u}_{\rm F} \gamma^{\rho} (1 - \lambda^{eff} \gamma_5) \frac{P^{\mu}}{P \cdot k} u_{\rm Ne}.$$
(9.8)

Comparing Eq. (9.8) with Eq. (4.5), we find that even though the ¹⁹Ne radiative β decay does receive an additional contribution $\mathcal{M}_{02'}$, the sum of $\mathcal{M}_{02'}$ and \mathcal{M}_{02} still returns a result similar to the neutron case if we only work to leading recoil order. Thus, at least for the QED bremsstrahlung contribution to ¹⁹Ne radiative β decay, one can almost directly transplate with only minor adjustments in the actual numerical evaluations – we just need to replace the nucleons' masses with the masses of ¹⁹Ne and ¹⁹F, also, the factor $\lambda = 1.2701$ in the neutron case should be replaced by λ_{eff} , which is defined by $\lambda_{\text{eff}} = \rho/\sqrt{3}$ and ρ is called the Gamow-Teller-to-Fermitransition parameter [172]. The values of ρ are nuclear–structure dependent and vary a lot among different nuclei. It is challenging to calculate theoretically, but it can be deduced from the measured $\mathcal{F}t$ values of nuclei. We found that for the ¹⁹Ne radiative β decay $\lambda_{\text{eff}} \approx 0.920$ according to the measured value of $\rho = 1.5933 \pm 0.0030$ [172]. With these replacements, we can easily obtain the branching ratio for ¹⁹Ne radiative β decay. We will leave a summary of numerical results to tables at the very end of the chapter.

As discussed before, in weak decays, the Coulomb correction due to the Coulomb interaction between the charged decay products deserves to be considered carefully, and in the nuclear cases, due to the larger charges carried by the daughter nucleus, the Coulomb correction can be much more enhanced than in the neutron case. We can also easily perform the calculation of the Coulomb correction in the ¹⁹Ne case; the only changes here are that we have Z = 9 for ¹⁹F, and the effective electric charge radius of ¹⁹F is R = 2.85(9) fm [173].

For the calculations of HHH-induced T-odd asymmetry in the ¹⁹Ne radiative β decay, the transition is almost trivial – the only thing that needs to be replaced is the nucleon vector current $\bar{p}\gamma_{\mu}n$ in Eq. (6.15), to be replaced with the nuclear case $\bar{u}_{\rm F}\gamma_{\mu}u_{\rm Ne}$.

After showing the intimate connection between neutron and ¹⁹Ne radiative β decay, there is still another important issue to be addressed: what about the SM

FSI-induced T-odd mimicking effect in the ¹⁹Ne case? As one can imagine, there are new kinds of loop diagrams and associated Cutkosky cuts due to the presence of the electric charges of the parent nucleus. Can we still take advantage of the formulae that were developed in the neutron case, without having to perform any new and tedious manipulations? Fortunately, YES! We find that even in the very complicated FSI-induced T-odd mimicking effect, we are still able to apply most of the neutron formulae. To show this, let us see all the one-loop diagrams for the ¹⁹Ne radiative β decay. There are many more diagrams than in the neutron case. To make the discussions more logically accessible, we shall present the loop diagrams with cuts as per the different types of post-cut processes, which describe the \mathcal{M}_{fn} and \mathcal{M}_{in} that were introduced in Eq. (8.23). Just as we did in the neutron case, we shall organize the cuts into different groups, within each of which QED gauge invariance is manifestly satisfied. Following this guideline, in the Feynman diagrams we will label each diagram as per the convention of the neutron case. That is, if the loop diagram has the same look as in the neutron case, we will label it with the same number as in Fig. (8.4); on the other hand, if a diagram of a certain family is brand new compared to the earlier neutron case, we will label it with a primed number.

First is the positron Compton scattering as shown in Fig. (9.3). On the two



Figure 9.3: The positron Compton rescattering cuts in ¹⁹Ne(P) \rightarrow ¹⁹F(P') + $e^+(l_e)$ + $\nu_e(l_\nu) + \gamma(k)$.

sides of the cuts, which are denoted by the cross symbols, we get the diagrams of positron Compton scattering, describing \mathcal{M}_{fn} , and the ones for \mathcal{M}_{in} that describes the tree-level ¹⁹Ne radiative β decay process, as shown in Fig. (9.2). Apparently the \mathcal{M}_{fn} part is identical to the neutron case since it represents charged lepton Compton scattering. As for the M_{in} part, based on the earlier discussions, although the tree-level diagrams for ¹⁹Ne radiative β decay receives an extra diagram, the leading recoil order approximation immediately allows for a cancellation and yields a structure identical to the case of tree-level neutron radiative β decay. Thus we draw the conclusion that in the positron Compton rescattering cuts of ¹⁹Ne radiative β decay, the formal structure remains unchanged, and the relevant formulae we developed in the neutron case directly apply with only some corresponding modifications needed in the final phase space integration, such as the nucleons' masses being replaced with the nuclei's masses.

Next, let us check the nucleus Compton scattering as shown in Fig. (9.5). This family resembles the positron Compton rescattering cuts, only with positron being replaced by nucleus. After the cuts, on the two sides of cross symbols, one gets the tree-level ¹⁹Ne radiative β decay and the nucleus Compton scattering. Again,



Figure 9.4: Nucleus Compton rescattering cuts in ¹⁹Ne(P) \rightarrow ¹⁹ F(P') + $e^+(l_e)$ + $\nu_e(l_{\nu}) + \gamma(k)$, part I.



Figure 9.5: Nucleus Compton rescattering cuts in ¹⁹Ne(P) \rightarrow ¹⁹ F(P') + $e^+(l_e)$ + $\nu_e(l_\nu) + \gamma(k)$, part II.

following the very similar analysis as in the positron Compton scattering cuts, we find

that the nucleus Compton scattering cuts in the ¹⁹Ne case has a structure identical to that of the proton Compton scattering cuts in the neutron case. It is easy to show (see Appendix A) that the cuts of the nucleus Compton scattering do not give a leading recoil contribution, thus we do not need to consider such cuts any further.

The third family of cuts are those with \mathcal{M}_{fn} describing the positron-nucleus rescattering with photon emission, as shown in Fig. (9.6). Comparing Fig. (9.6) with the



Figure 9.6: Feynman diagrams of the positron-nucleus-photon rescattering cuts in ${}^{19}\text{Ne}(P) \rightarrow {}^{19}\text{F}(P') + e^+(l_e) + \nu_e(l_{\nu}) + \gamma(k).$

neutron case, we see that this family shares a structure similar to the neutron case. The only difference here is that in the positron-nucleus rescattering the positron feels a stronger electromagnetic field because the daughter nucleus ¹⁹F has Z = 9, yet in the neutron case, the electron only experiences the electromagnetic field generated by the single proton. A straightforward analysis shows that the formulae we developed in the neutron case for this type of cuts are almost directly applicable, but we should just keep in mind that \mathcal{M}_{fn} in this family should come with an extra multiplicative factor of Z, which is the total charge of ¹⁹F.

The last family are cuts with \mathcal{M}_{fn} describing the positron-nucleus rescattering without photon emission, as shown in Fig. (9.7). This family also shares a structure



Figure 9.7: The positron-nucleus rescattering cuts in ¹⁹Ne(P) \rightarrow ¹⁹F(P') + $e^+(l_e)$ + $\nu_e(l_\nu) + \gamma(k)$.

similar to the same family in the neutron case. We note \mathcal{M}_{in} describes tree-level

¹⁹Ne radiative β decay, and \mathcal{M}_{fn} describes the positron-nucleus rescattering without photon emission. Just as in the positron-nucleus-photon rescattering cuts, here the positron also experiences the electromagnetic field that is generated by the Z charges of ¹⁹F, so that we can directly apply the formulae of e - p cuts in the neutron case, with an extra multiplicative factor of Z in the calculation of \mathcal{M}_{fn} . Also, as discussed in the neutron case, such a positron-nucleus rescattering cuts come with the infrared divergence problem, we need to keep this whole family together in the whole calculations so that the infrared divergence arising from the diagram (6.3) is finally cancelled by the combination of the diagrams (8.2) and (8.2') in Fig. (9.7).

Let us summarize what we have concluded so far:

1. in the case of ¹⁹Ne radiative β decay, ¹⁹Ne \rightarrow ¹⁹ F + $e^+ + \nu_e + \gamma$, the parent nucleus ¹⁹Ne radiative β and the daughter nucleus ¹⁹F happen to share the same spin-1/2 quantum number with the neutron and proton;

2. in very low energies processes, the structure of the nucleus is not really relevant to us, so that we can treat both of the nuclei as point-like particles.

Combining the two conditions allows us to draw similar Feynman diagrams and follow the standard Feynman rules to find that even though the ¹⁹Ne radiative β decay has more diagrams than the neutron case does, there are cancellations due to the leading recoil order approximation, and eventually the formulae we have developed for the neutron case are just enough for us to perform the evaluations of the ¹⁹Ne case, with only some minor adjustments such as the replacements of nucleus masses and the extra factor of Z = 9 in the diagrams of Fig. (9.6) and Fig. (9.7). This encourages us to believe that the T-odd calculations in the nuclear cases should not differ from the neutron one very much.

Of course ¹⁹Ne radiative β decay is just a very special case that allows us to use the language of quantum field theory and apply standard Feynman rules to demonstrate our analysis and justify our claim. For more general nuclear cases with general spin configurations, however, we may not be able to enjoy such convenience any more. Even so, as will be seen in the next section, the conclusion in this section still holds, that is, the formulae that we have developed for the neutron radiative β decay are really sufficient for the more general nuclear cases.

9.2 Alternative Treatments of Nuclear β Decay

In last section, we took the ¹⁹Ne radiative β decay as a simple model to demonstrate how we can translate, almost directly, the formulae we have developed for the neutron radiative β decay to the nuclear case. The main theoretical language we used was the standard Feynman rules, which were allowed because both the parent nucleus ¹⁹Ne and the daughter ¹⁹F are spin-1/2 point-like particles. For more general nuclear cases, such a tricky method may not apply. We need to figure out a more general way to deal with the problem. In this section, we very briefly discuss the general idea. Cutkosky cuts suggests that after performing the cuts the loop diagrams can be viewed as two physical processes described by \mathcal{M}_{fn} and \mathcal{M}_{in} . This holds for all kinds of spin configurations. Let us first take a closer look at the scattering amplitudes in Eq. (9.2), Eq. (9.6), and Eq. (9.7), where the leading recoil order approximation has been considered. The three scattering amplitudes can be simply denoted as:

$$\mathcal{M}_{01} = l_{1\rho} h_1^{\rho}, \tag{9.9}$$

$$\mathcal{M}_{02} = l_{2\rho} h_2^{\rho}, \tag{9.10}$$

$$\mathcal{M}_{02'} = l_{2\rho} h_{2'}^{\rho}, \tag{9.11}$$

where l^{ρ} and h^{ρ} are defined as the leptonic and hadronic current respectively, and we have:

$$l_{1\rho} = \bar{u}_e(l_e) \frac{2l_e \cdot \epsilon^* + \not \epsilon^* \not k}{2l_e \cdot k} \gamma_\rho(1 - \gamma_5) v_\nu(l_\nu), \qquad (9.12)$$

$$l_{2\rho} = \bar{u}_e(l_e)\gamma_\rho(1-\gamma_5)v_\nu(l_\nu), \qquad (9.13)$$

$$h_1^{\rho} = \bar{u}_{\rm F} \gamma^{\rho} (1 - \lambda^{eff} \gamma_5) u_{\rm Ne}, \qquad (9.14)$$

$$h_2^{\rho} = \epsilon_{\mu} \bar{u}_{\rm F} \frac{P^{\mu}}{P' \cdot k} \gamma^{\rho} (1 - \lambda^{eff} \gamma_5) u_{\rm Ne}, \qquad (9.15)$$

$$h_{2'}^{\rho} = \epsilon_{\mu} \bar{u}_{\rm F} \gamma^{\rho} (1 - \lambda^{eff} \gamma_5) \frac{P^{\mu}}{-P \cdot k} u_{\rm Ne}.$$
(9.16)

which are of course very easily obtained by following the Feynman rules. But the scattering amplitudes by themselves are not directly related with the physical observable - the decay rate, it is their absolute squares that matters. Thus we have, for example,

$$\begin{aligned} |\mathcal{M}_{01}|^2 &= l_{1\rho} l_{1\delta}^* h_1^{\rho} h_1^{\delta *} = l_{11\rho\delta} h_{11}^{\rho\delta}, \\ |\mathcal{M}_{02}|^2 &= l_{2\rho} l_{2\delta}^* h_2^{\rho} h_2^{\delta *} = l_{22\rho\delta} h_{22}^{\rho\delta}, \\ |\mathcal{M}_{02'}|^2 &= l_{2\rho} l_{2\delta}^* h_{2'}^{\rho} h_{2'}^{\delta *} = l_{22\rho\delta} h_{2'2'}^{\rho\delta}. \end{aligned}$$
(9.17)

Thus it is really the tensors in Eq. (9.17) that determine the final result of $|\mathcal{M}|^2$. If we can find the expressions of the currents that can yield the same results of the tensors as the ones obtained by following the Feynman rules, we then do not really have to rely on the Feynman rules any more. We are seeking for such kind of alternative path because the spin configurations of general nuclei can be multiple and QED-like Feynman rules may not apply in many cases.

In fact, since on the leptonic current side the electron (positron) as well as the (anti)neutrino are universally present in any nuclear β decay, the Feynman rules always work well on this side! The major job for us is to find alternative and more general path to describe the hadronic currents that interact with electroweak gauge bosons such as W^{\pm} and photon.

Let us start with the regular nuclear β decay, that is, with no photon radiation involved. For definiteness, we consider the case of β decay. In general we have the diagram depicted in Fig. (9.8) for a nuclear β decay. We define



Figure 9.8: A nuclear β decay $\alpha(p_1) \to \beta(p_2) + e^-(l_e) + \bar{\nu}_e(l_{\nu})$. The double lines refer to the nuclei.

$$P = p_1 + p_2; \quad q = p_1 - p_2 = l_e + l_{\nu};$$

$$M_N = \frac{1}{2}(M_{\alpha} + M_{\beta}); \quad \Delta = M_{\alpha} - M_{\beta}.$$
 (9.18)

The scattering amplitude \mathcal{M} can be generally written as the product of the nuclear current and the leptonic current:

$$\mathcal{M} = \frac{G_F V_{ud}}{\sqrt{2}} \left\langle \beta \right| V_{\mu} + A_{\mu} \left| \alpha \right\rangle l^{\mu}, \qquad (9.19)$$

where l^{μ} refers to the leptonic current, which can always be obtained easily by following the Feynman rules:

$$l^{\mu} = \bar{u}_e(l_e)\gamma^{\mu}(1-\gamma_5)v_{\nu}(l_{\nu}).$$
(9.20)

Following the basic requirement of parity violation in weak decay, the nuclear current in Eq. (9.19) contains both vector and axial-vector components. Based on rotational invariance and the parity consideration, several equivalent representations of the nuclear weak current have been proposed, such as in Ref. [174] [175] [176]. Here let us just take Holstein's representations [174] [175]. That is, we suppose the parent nucleus has spin J with its third component M, and the daughter nucleus has spin J' with its third component M'. Translational invariance dictates that the nuclear current is only dependent on the momentum transfer \mathbf{q} , and we have [175]:

$$\langle \beta | V_0 | \alpha \rangle = \sum_{j=\text{even}} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M} \left(\frac{4\pi}{2j+1} \right)^{\frac{1}{2}} Y_j^m(\hat{q}) F_j^V(q^2) \left(\frac{|\mathbf{q}|}{2M_N} \right)^j,$$

$$\langle \beta | \mathbf{V} | \alpha \rangle = \sum_{l=\text{odd}} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M} \left(\frac{4\pi}{2l+1} \right)^{\frac{1}{2}} \mathbf{T}_{jl}^m(\hat{q}) F_{jl}^V(q^2) \left(\frac{|\mathbf{q}|}{2M_N} \right)^l,$$
(9.21)

for the nuclear vector current, and

$$\langle \beta | A_0 | \alpha \rangle = \sum_{j=\text{odd}} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M} \left(\frac{4\pi}{2l+1} \right)^{\frac{1}{2}} Y_j^m(\hat{q}) F_j^A(q^2) \left(\frac{|\mathbf{q}|}{2M_N} \right)^j,$$

$$\langle \beta | \mathbf{A} | \alpha \rangle = \sum_{l=\text{even}} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M} \left(\frac{4\pi}{2l+1} \right)^{\frac{1}{2}} \mathbf{T}_{jl}^m(\hat{q}) F_{jl}^A(q^2) \left(\frac{|\mathbf{q}|}{2M_N} \right)^l,$$
(9.22)

for the nuclear axial-vector current. In Eq. (9.21) and Eq. (9.22), Y_j^m and \mathbf{T}_{jl}^m refer to the spherical harmonics and vector spherical harmonics, defined in Rose's book [177]. We note $C_{J'j;J}^{M'm;M}$ is the Clebsch-Gordan coefficient, and $F_j^V(q^2)$, $F_{jl}^V(q^2)$, $F_j^A(q^2)$, and $F_{il}^A(q^2)$ are form factors.

The expressions in Eq. (9.21) and Eq. (9.22) look quite complicated since they contain all the information apropos the nuclear vector and axial-vector currents. Here, however, we are only interested in the leading recoil order contributions, so that we can simply drop all the nonleading terms, which greatly simplifies the nuclear currents. For example, we see that for the zeroth component of the nuclear vector current, $\langle \beta | V_0 | \alpha \rangle$ in Eq. (9.21), the only leading order term is such that j = 0, so that m = 0; otherwise, we have $\langle \beta | \mathbf{V} | \alpha \rangle = 0$. Similarly the only leading order contribution in the axial-vector current comes from the vector current vanishes: $\langle \beta | A_0 | \alpha \rangle = 0$. Based on such simplifications, one finds that for the allowed nuclear weak decay with $\Delta J = 0, \pm 1$, the leading recoil order pieces of the nuclear weak current can be rewritten the following form:

$$V_0 = (-1)^{J'-M'} \sqrt{2J+1} \begin{pmatrix} J' & 0 & J \\ -M' & 0 & M \end{pmatrix} F_0^V, \qquad (9.23)$$

$$\mathbf{V} = 0, \tag{9.24}$$

and

$$A_0 = 0,$$
 (9.25)

$$\mathbf{A} = \sum_{m} (-1)^{J'-M'} \sqrt{2J+1} \begin{pmatrix} J' & 1 & J \\ -M' & m & M \end{pmatrix} \sqrt{4\pi} \mathbf{Y}_{10}^{m*} F_{10}^{A}, \qquad (9.26)$$

where, following the general convention, F_0^V corresponds to g_V , and F_{10}^A corresponds to $\sqrt{3}g_A$ for the neutron β decay. We know \mathbf{Y}_{Jl}^M are called "vector spherical harmonics," defined as:

$$\mathbf{Y}_{Jl}^{M} = \sum_{m,q} Y_{lm} \langle l, m, 1, q | l, 1, J, M \rangle \, \hat{e}_{q}, \quad (q = -1, 0, +1), \tag{9.27}$$

where \hat{e}_q is unit vector. This yields the result for \mathbf{Y}_{10}^{m*} :

$$\mathbf{Y}_{10}^{m*} = \sum_{q} Y_{00} \langle 0, 0, 1, q | 0, 1, 1, m \rangle \, \hat{e}_{q} = \frac{1}{\sqrt{4\pi}} \hat{e}_{m}.$$
(9.28)

Furthermore, we see that the so-called "3-jm symbol" appears in the expressions for the nuclear weak currents. The 3-jm symbol plays an important role in dealing with the geometry of angular momenta. It is closely connected to the Clebsch-Gordan (CG) coefficients such that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} = (-1)^{-j_1+j_2-m_3} \frac{1}{\sqrt{2j_3+1}} \langle j_1, m_1, j_2, m_2 | j_3, m_3 \rangle, \quad (9.29)$$

where $\langle j_1, m_1, j_2, m_2 | j3, m_3 \rangle$ refers to the C-G coefficient. It is not hard to see that following the definition of 3-jm symbol, we have:

$$\begin{pmatrix} J' & 0 & J \\ -M' & 0 & M \end{pmatrix} = (-1)^{J-M'} \frac{1}{\sqrt{2J'+1}} \langle 0, 0, J, M | J', M' \rangle ,$$
$$= (-1)^{J'-M'} \frac{1}{\sqrt{2J+1}} \delta_{JJ'} \delta_{MM'},$$
(9.30)

which yields the final spin-independent result for the vector current:

$$V_0 = F_0^V \delta_{JJ'} \delta_{MM'}. \tag{9.31}$$

Following a similar procedure, it is also easy to find the non-trivial piece of the axialvector current:

$$\mathbf{A} = \sum_{m=-1}^{+1} (-1)^{J'-M'} \sqrt{2J+1} \begin{pmatrix} J' & 1 & J \\ -M' & m & M \end{pmatrix} F_{10}^{A} \hat{e}_{m}.$$
(9.32)

With Eq. (9.31) and Eq. (9.31), together with $\mathbf{V} = 0$, $A_0 = 0$, we are ready to calculate the hadronic (nuclear) tensor that is the actual quantity that determines the scattering amplitude square:

$$h_{\mu\nu} = (V_{\mu} + A_{\mu})(V_{\nu} + A_{\nu})$$

= $V_{\mu}V_{\nu} + V_{\mu}A_{\nu} + A_{\mu}V_{\nu} + A_{\mu}A_{\nu}.$ (9.33)

There are basically three cases:

1. $\mu = 0$ and $\nu = 0$, we have:

$$h_{00} = V_0 V_0 = (F_0^V)^2 \delta_{JJ'} \tag{9.34}$$

2. $\mu = 0$ and $\nu = i$ for i = 1, 2, 3, or the other way around, we have:

$$h_{0i} = h_{i0} = V_0 A_i \propto (-1)^{J'-M'} \sqrt{2J+1} \begin{pmatrix} J' & 1 & J \\ -M' & m & M \end{pmatrix} \delta_{JJ'} \delta_{MM'} = (-1)^{J-M} \sqrt{2J+1} \begin{pmatrix} J & 1 & J \\ -M & m & M \end{pmatrix}.$$
(9.35)

The 3-jm symbol above can be nontrivial only when m = 0 and J = 1/2, which means $M = \pm 1/2$. Since we are considering unpolarized processes, so we should sum over the nucleus' polarization M, therefore we have

$$V_0 A_i \propto (-1)^0 \begin{pmatrix} 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix} + (-1)^1 \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & -1/2 \end{pmatrix} = 0, \qquad (9.36)$$

which tells us that $h_{0i} = h_{i0} = 0$. Finally we have the $\mu = i$, $\nu = j$, or the other way around, we have:

$$h_{ij} = h_{ji} = A_i A_j, \tag{9.37}$$

which on using the orthogonality relation of the 3-jm symbol:

$$(2J+1)\sum_{m,n} \begin{pmatrix} J' & 1 & J \\ -M' & m & M \end{pmatrix} \begin{pmatrix} J' & 1 & J \\ -M' & n & M \end{pmatrix} = \delta_{mn}$$
(9.38)

we get

$$h_{ij} = (F_{10}^A)^2 \text{ (if } i = j) h_{ij} = 0 \text{ (if } i \neq j),$$

$$(9.39)$$

which, following the standard convention, we have $(F_0^V)^2 = g_V^2$ and $(F_{10}^A)^2 = 3g_A^2$.

Now let us compare $h_{\mu\nu}$ we have got so far with the one which we have got for the neutron case by applying the Feynman rules for spin-1/2 fermions:

$$h_{\mu\nu} = h_{\mu}h_{\nu}^{*} = \bar{u}_{p}\gamma_{\mu}(g_{V} - \lambda g_{A})u_{n}\bar{u}_{n}(g_{V} + \lambda g_{A})\gamma_{\nu}u_{p}$$

$$= \operatorname{Tr}\left[(\not{p}_{p} + m_{p})\gamma_{\mu}(g_{V} - \lambda g_{A})(\not{p}_{n} + m_{n})(g_{V} + \lambda g_{A})\gamma_{\nu}\right]$$

$$= 8(P_{\mu}P_{\nu}g_{V}^{2} + P_{\mu}P_{\nu}g_{A}^{2} - g_{\mu\nu}M^{2}g_{A}^{2}), \qquad (9.40)$$

where the leading recoil order approximation has been taken in the last line, with $P^{\mu} \equiv p_{p}^{\mu} + p_{n}^{\mu}$, and $M = P^{0} \equiv (m_{p} + m_{n})/2$. From Eq. (9.40), we see that $h_{00} \sim g_{V}^{2}$, $h_{0i} = h_{i0} = 0$, $h_{ij} \sim 3g_{A}^{2}$ if i = j, and $h_{ij} = 0$ if $i \neq j$. Thus we find the same tensor structure as the one that is obtained for the spin-1/2 fermions by following the standard Feynman rules. In other words, the concrete spin configurations of different nuclei really do not affect our calculations of the nuclear β decays in the LRO approximation.

So far we have shown that in nuclear β decays, the formulae can be translated from the neutron case with only some minor modifications, is this still true in the case of radiative β decay? The answer is still yes.

In the case of QED bremmstrahlung, the only difference between the nuclear radiative β decay and the neutron radiative β decay is that the electrically charged currents are dressed with photon bremmstrahlung. We do not worry about the case where photon couples to the electron since we have standard Feynman rules for the lepton current, and there is an alternative representation of the nuclear electroweak current with different spin configurations playing no role in LRO approximation. When the photon couples to the nuclear current, the situation becomes a little bit more complicated. After the Cutkosky cuts, as we have seen in both neutron and ¹⁹Ne radiative β decays, the loop diagrams are split into two parts, represented by \mathcal{M}_{fn} and \mathcal{M}_{in} such that the \mathcal{M}_{in} describes tree-level regular or radiative β decay, and the \mathcal{M}_{fn} describes either Compton-like processes or electron-nucleus rescattering processes. The following questions arise:

1. What is the alternative representation of the nuclear electric current in the corresponding \mathcal{M}_{fn} ?

2. When the emitted photon couples to the nuclear sector via the bremmstrahlung processes, how does it appear in the current?

The first question is more or less trivial since the earlier discussion on the alternative representation of nuclear electroweak current already provides us some clue. The electromagnetic coupling is a vector coupling, and there is no axial vector piece. When the electric nuclear current is not dressed with photon bremmstrahlung, we can just pick up the vector piece from Eq. (9.19) and thus Eq. (9.31) in LRO – with the coupling F_0^V being replaced by the electric charge of the involved nucleus. Consequently, we find the nuclear electric tensor $h_{\rm EM}^{00} = V^0 V^{0*} = (Ze)^2$ This is in fact consistent with the electric current obtained by following the QED Feynman rule in the proton case, where we have the proton electromagnetic current

$$h_{\rm EM}^{\mu} = e\bar{u}_p \gamma^{\mu} u_p, \qquad (9.41)$$

which gives the proton electromagnetic tensor

$$h_{\rm EM}^{00} = 8M^2 e^2 \tag{9.42}$$

with all other components vanishing in LRO. The extra factor of $8M^2$ can be trivially absorbed into a choice of convention of the final phase space integrations.

Let us discuss the second question listed above – how should we handle the nuclear photon bremsstrahlung? From the considerations of both neutron and the ¹⁹Ne cases, we see that in LRO, such a process only brings an extra factor of $Ze\epsilon_{\mu}P'^{\mu}/P' \cdot k$ or $-(Z+1)e\epsilon_{\mu}P^{\mu}/P \cdot k$ into the regular radiative β decay. It turns out that such a property is universal in any LRO nuclear bremsstrahlung because it is closely related to the Ward-Takahashi identity that holds up to arbitrary order. In LRO, the Ward-Takahashi identity in QED reads:

$$q^{\mu}\Gamma_{\mu}(p,q,p+q) = Q\left(S_F^{-1}(p+q) - S_F^{-1}(p)\right)$$
(9.43)

with a relevant diagram shown in Figure (9.9), which is to represent the case where the photon couples to the daughter nucleus. On multiplying $S_F(p+q)$ (from left) and $S_F(p)$ (from right) on both sides of Eq. (9.43), we have

$$q^{\mu}S_{F}(p+q)\Gamma_{\mu}(p,q,p+q)S_{F}(p) = Q\left(S_{F}(p) - S_{F}(p+q)\right).$$
(9.44)



Figure 9.9: A demonstrative diagram to show the Ward-Takahashi identity. S_F refers to the renormalized propagator of a charged matter field with certain 4-momentum and electric charge Q (=Ze), the q refers to the 4-momentum of photon. The grayed blob denotes any other processes that connect to the bremsstrahlung.

Defining $S_F(p+q)\Gamma_{\mu}(p,q,p+q) \equiv J_{\mu}$, which represents the influence of an insertion of photon bremsstrahlung into the diagram. We have then

$$q^{\mu}J_{\mu}S_F(p) = Q\left(S_F(p) - S_F(p+q)\right).$$
(9.45)

For arbitrary p, we have

$$q^{\mu}J_{\mu} = Q. (9.46)$$

In our consideration, the only two independent degrees of freedom are p and q, thus we must have $J_{\mu} = ap_{\mu} + bq_{\mu}$. The q_{μ} part is irrelevant since it will eventually be combined with the photon polarization vector ϵ^{μ} and have $\epsilon \cdot q = 0$. Thus only the ap_{μ} is left. To determine a, we multiply q^{μ} to J_{μ} , and find $a = Q/p \cdot q$, thus we show that

$$J_{\mu} = Q \frac{p^{\mu}}{p \cdot q},\tag{9.47}$$

which is the expected result we have seen in both neutron and ¹⁹Ne radiative β decays. The bremsstrahlung that dresses the parent nucleus can also be derived by following very similar way, and gives

$$J_{\mu} = -Q' \frac{p'^{\mu}}{p' \cdot q} \tag{9.48}$$

with Q' = (Z+1)e and p' denoting the charge and 4-momentum carried by the parent nucleus. Apparently, in LRO approximation, the sum of Eq. (9.47) and Eq. (9.48) would yield a net contribution of $-eP^{\mu}/P \cdot k$, which is just what we see in the neutron and ¹⁹Ne case. In one words, what we have shown here is that in LRO approximation the insertion of a photon bremsstrahlung will only add the similar factor of $Ze\epsilon_{\mu}P'^{\mu}/P' \cdot k$ or $-(Z+1)e\epsilon_{\mu}P^{\mu}/P \cdot k$ into the nonradiative amplitudes for any spin configurations. Thus it does not alter any of our earlier formulae that are developed in the neutron case. To sum up, we have shown that in the nuclear case in LRO neither the electroweak nor the electromagnetic current depends on the spin configurations, and we can draw the conclusion that even without applying the standard Feynman rules for spin-1/2 fermions, the general nuclear radiative β decay with more general spin-configurations still have the same structure as the spin-1/2 fermions do, so that the formulae that we have developed for spin-1/2 case can be applied to the more general nuclear cases. We should stress that due to the daughter nucleus charge factor Z, the scattering amplitudes for the lepton-nucleus rescattering cuts should be given an extra multiplicative factor of Z accordingly.

9.3 Numerical Results

In this section, we present the numerical results of the neutron radiative β decay as well as the nuclear cases of ⁶He, ¹⁹Ne, and ³⁵Ar. We have computed the branching ratios, HHH-induced CP-violating asymmetries in units of Im C_{HHH} , and SM FSIinduced T-odd mimicking effects. Due to the larger electric charges carried by the nuclei than proton, the Coulomb correction can be more significant, so that we also evaluated the Coulomb corrections to these cases.

The decay rate is proportional to the overall factor " $(1 + \rho^2)$ " with ρ defined as Gamow-Teller-to-Fermi-transition parameter [172], which can be determined from both the "A" correlation in the decay rate and the lifetime measurements. The FSIinduced T-odd mimicking effect is controlled by the overall factor " $(1 - \rho^2/3)(1 + \rho^2)$ ", thus the parameter ρ is important. In the neutron case we have $\rho[n] = \sqrt{3\lambda} = \sqrt{3}g_A/g_V$. The branching ratios with and without the Coulomb correction have been presented in Table (4.2). Here we just present the comparison of the numerical results of \mathcal{A}^{HHH} versus \mathcal{A}^{FSI} for the neutron, to be seen in Table (9.1), where the subscript "0" implies the evaluations are without Coulomb corrections and "CC" implies that Coulomb corrections have been included.

For ⁶He radiative β decay, ⁶He \rightarrow ⁶Li + $e^- + \overline{\nu}_e + \gamma$, we have ρ [He] = 2.74913, which is deduced from the lifetime measurements of 6 He [178], and the electric charge radius $R^{[6}$ Li] $\approx 2.589 \pm 0.039$ fm [179]. With these numerical inputs, we evaluate the BR, to be seen in Table (9.2) and the HHH-induced asymmetry \mathcal{A}^{HHH} as well as the FSI-induced T-odd mimicking effect \mathcal{A}^{FSI} , to be seen in Table (9.3). The differential decay rate of ⁶He radiative β decay has been measured years ago [2]. This allows us a chance to check if our formulae works for the general nuclear radiative β decay by comparing our theoretical results with the experimental results. Without being offered the actual data, we have to read off the approximate data from the original plot. Note the original measurements on the differential decay rate was taken in arbitrary unit, we need to fix a point for the overall multiplicative factor, and compare the rest of the points. The theory versus experiment plot is presented in Fig. (9.10). We see that they agree fairly well, and this supports the validity of our procedure. One specialty here is that in ⁶He radiative β decay, the parent nucleus ⁶He is spin-0, yet the daughter nucleus ⁶Li is spin-1, thus the initial spin Jand the final spin J' are not equal. Following the earlier discussion of Eq. (9.31), we assert that there is no vector weak current piece in the 6 He case – this means that

Table 9.1: Summary of the numerical results for $\mathcal{A}^{\text{HHH}}(n)$ and $\mathcal{A}^{\text{FSI}}(n)$ in neutron decay at some selected ω_{min} . \mathcal{A}^{HHH} is in unit of Im \mathcal{C}_{HHH} .

$\omega_{min}(MeV)$	$\mathcal{A}_0^{\mathrm{FSI}}(n)$	$\mathcal{A}_{CC}^{\mathrm{FSI}}(n)$	$\mathcal{A}_0^{\mathrm{HHH}}(n)$	$\mathcal{A}_{CC}^{\mathrm{HHH}}(n)$
0.01	1.76×10^{-5}	1.78×10^{-5}	-0.561×10^{-2}	-0.565×10^{-2}
0.05	3.86×10^{-5}	3.88×10^{-5}	-1.30×10^{-2}	-1.31×10^{-2}
0.1	6.07×10^{-5}	6.11×10^{-5}	-2.20×10^{-2}	-2.20×10^{-2}
0.3	1.31×10^{-4}	1.31×10^{-4}	-5.34×10^{-2}	-5.33×10^{-2}



Figure 9.10: Differential decay rate of ⁶He radiative β decay, where the line represents the theoretical prediction and the dots with error bars represents experimental data [2].

we have a pure Gamow-Teller transition, so that no HHH-induced effect in this case simply because such an asymmetry is proportional to $\text{Im}(c_5 g_V)$. In this sense, the ⁶He radiative β decay serves as an ideal candidate for a clean test of the FSI-induced T-odd mimicking effect.

For the ¹⁹Ne radiative β decay, ¹⁹Ne \rightarrow ¹⁹F+ $e^+ + \nu_e + \gamma$, ρ can be deduced from the measurements of the nuclear lifetimes. We find that ρ [Ne] = -1.5933 ± 0.0030 [172]. For the numerical evaluation of the Coulomb correction based on Eq. (4.26), we have

Table 9.2: Summary of the numerical results for the BR in ⁶He decay at some selected ω_{min} .

$\omega_{min}(MeV)$	$BR_0(^6He)$	$BR_{CC}(^{6}He)$
0.01	8.02×10^{-3}	8.64×10^{-3}
0.05	4.77×10^{-3}	5.14×10^{-3}
0.1	3.44×10^{-3}	$3.71 imes 10^{-3}$
0.3	1.59×10^{-3}	1.72×10^{-3}

Table 9.3: Summary of the numerical results for $\mathcal{A}^{\text{HHH}}(^{6}\text{He})$ and $\mathcal{A}^{\text{FSI}}(^{6}\text{He})$ in ^{6}He decay at some selected ω_{min} . \mathcal{A}^{HHH} is in unit of Im \mathcal{C}_{HHH} .

$\omega_{min}(MeV)$	$\mathcal{A}_0^{\mathrm{FSI}}(^6\mathrm{He})$	$\mathcal{A}_{CC}^{\mathrm{FSI}}(^{6}\mathrm{He})$	$\mathcal{A}_0^{\mathrm{HHH}}(^6\mathrm{He})$	$\mathcal{A}_{CC}^{\mathrm{HHH}}(^{6}\mathrm{He})$
0.01	6.98×10^{-5}	7.08×10^{-5}	0	0
0.05	1.14×10^{-4}	$1.16 imes 10^{-4}$	0	0
0.1	1.52×10^{-4}	1.54×10^{-4}	0	0
0.3	2.63×10^{-4}	2.66×10^{-4}	0	0

Table 9.4: Summary of numerical results for the BR in ¹⁹Ne decay at some selected ω_{min} .

$\omega_{min}(MeV)$	$BR_0(^{19}Ne)$	$BR_{CC}(^{19}Ne)$
0.01	4.82×10^{-2}	6.10×10^{-2}
0.05	2.82×10^{-2}	$3.57 imes 10^{-2}$
0.1	2.01×10^{-2}	2.55×10^{-2}
0.3	0.886×10^{-2}	1.14×10^{-2}

the electric charge radius $R^{[19}\text{F}]$, which we find $R^{[19}\text{F}] \approx 2.85(9)\text{fm}$ [180]. With these numerical input, we evaluate the BR, to be seen in Table (9.4), and the HHH-induced asymmetry \mathcal{A}^{HHH} with the FSI-induced T-odd mimicking effect \mathcal{A}^{FSI} , to be seen in Table (9.5).

For the ³⁵Ar radiative β decay, ³⁵Ar \rightarrow ³⁵Cl + $e^+ + \nu_e + \gamma$, we have $\rho[\text{Ar}] = 0.2841 \pm 0.0025$ [172], and the electric charge radius $R[^{35}\text{Cl}] \approx 3.335(18)\text{fm}$ [180]. With these numerical inputs, we evaluate the BR, to be seen in Table (9.6); and the HHH-induced asymmetry \mathcal{A}^{HHH} with the FSI-induced T-odd mimicking effect \mathcal{A}^{FSI} , to be seen in Table (9.7).

From the numerical results listed in Table 9.1, Table 9.5, Table 9.7, we see that the ¹⁹Ne case has the higher ratio of the absolute size of $\mathcal{A}_{CC}^{\text{HHH}}/\mathcal{A}_{CC}^{\text{FSI}}$, which implies a good candidate. The ³⁵Ar may be a good candidate as well simply because the

Table 9.5: Summary of numerical results for $\mathcal{A}^{\text{HHH}}(^{19}\text{Ne})$ and $\mathcal{A}^{\text{FSI}}(^{19}\text{Ne})$ in ¹⁹Ne decay at some selected ω_{min} . \mathcal{A}^{HHH} is in unit of Im \mathcal{C}_{HHH} .

$\omega_{min}(MeV)$	$\mathcal{A}_0^{\mathrm{FSI}}(^{19}\mathrm{Ne})$	$\mathcal{A}_{CC}^{\mathrm{FSI}}(^{19}\mathrm{Ne})$	$\mathcal{A}_0^{\mathrm{HHH}}(^{19}\mathrm{Ne})$	$\mathcal{A}_{CC}^{\mathrm{HHH}}(^{19}\mathrm{Ne})$
0.01	-2.86×10^{-5}	-2.99×10^{-5}	-3.60×10^{-2}	-3.64×10^{-2}
0.05	-4.76×10^{-5}	-4.97×10^{-5}	-6.13×10^{-2}	-6.18×10^{-2}
0.1	-6.40×10^{-5}	-6.68×10^{-5}	-8.46×10^{-2}	-8.52×10^{-2}
0.3	-1.14×10^{-4}	-1.18×10^{-4}	-16.5×10^{-2}	-16.5×10^{-2}

Table 9.6: Summary of numerical results for the BR in 35 Ar decay at some selected ω_{min} .

$\omega_{min}(MeV)$	$BR_0(^{35}Ar)$	$BR_{CC}(^{35}Ar)$
0.01	6.55×10^{-2}	10.2×10^{-2}
0.05	4.24×10^{-2}	$6.63 imes 10^{-2}$
0.1	3.28×10^{-2}	5.13×10^{-2}
0.3	$1.85 imes 10^{-2}$	2.91×10^{-2}

Table 9.7: Summary of numerical results for $\mathcal{A}^{\text{HHH}}(^{35}\text{Ar})$ and $\mathcal{A}^{\text{FSI}}(^{35}\text{Ar})$ in ^{35}Ar decay at some selected ω_{min} . \mathcal{A}^{HHH} is in unit of Im \mathcal{C}_{HHH} .

$\omega_{min}(MeV)$	$\mathcal{A}_0^{\mathrm{FSI}}(^{35}\mathrm{Ar})$	$\mathcal{A}_{CC}^{\mathrm{FSI}}(^{35}\mathrm{Ar})$	$\mathcal{A}_0^{\mathrm{HHH}}(^{35}\mathrm{Ar})$	$\mathcal{A}_{CC}^{\mathrm{HHH}}(^{35}\mathrm{Ar})$
0.01	-0.835×10^{-3}	-0.875×10^{-3}	-0.280	-0.283
0.05	-1.26×10^{-3}	-1.32×10^{-3}	-0.431	-0.435
0.1	$-1.60 imes10^{-3}$	-1.67×10^{-3}	-0.556	-0.559
0.3	-2.55×10^{-3}	-2.65×10^{-3}	-0.943	-0.945

effect is enhanced among the nuclear examples to reveal the possible existence of the Im(c_5), which in turn signifies the underlying BSM contributions as discussed in Chapter 6. This happens mainly because in the ¹⁹Ne radiative β decay the effective factor $\rho = 1.5933 \rightarrow \rho^2/3 \approx 0.85$, which is pretty close to unity and thus provides higher suppression on the FSI-induced T-odd mimicking effect via its overall factor " $(1 - \rho^2/3)(1 + \rho^2)$ " than in the other nuclear examples. We are taking these nuclear examples to demonstrate the basic principles involved in selecting the most ideal candidate to search for a nontrivial Im(c_5). Due to the very special analytic structure " $(1 - \rho^2/3)(1 + \rho^2)$ " in the FSI-induced mimicking effect in SM, we are looking for a nuclear candidate with its effective ρ very close to unity, which consequently greatly suppresses the SM baseline and makes the exotic CP-violating signal more visible.

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Chapter 10 Summary and Outlook

10.1 What We Have Learned

Let me briefly summarize the work I have done. In this thesis, we focus on the study of CP violation which appears in the form of a triple-product correlation in momenta, in both the neutron and nuclear radiative β decays. CP violation is an important and necessary element in explaining the BAU problem – if there were no CP violation, all the matter and antimatter that were created in the Big Bang would have totally annihilated when the universe cooled during its expansion. With the fundamental introduction of P, C, and T transformations in this thesis, we see that to allow for CP violation in the framework of quantum field theory, at least some coupling constants have to be complex, so that the Hamiltonian or equivalently Lagrangian can be different under the combined CP transformation. In the spirit of the CPT theorem, which is on a firm footing, CP violation is equivalent to T violation.

To understand CP violation, people first checked the SM, which has been confirmed as a very successful theoretical framework of modern physics. As discussed in the thesis, CP violation can be allowed to arise in two ways:

1. the so-called " θ -term" which could have entered the QCD Lagrangian;

2. the complex weak phase(s) that are introduced in the CKM mechanism in a natural manner when there are more than 2 quarks generations.

In the first way, we saw that even though such a mechanism could have been an economical and even an ideal way to allow for CP violation, that nature somehow rejected this option. The reason why this happens remains an open question to modern physics. In the second way, the nontrivial weak phase for 3 generations of quarks (u, d), (c, s), (t, b) has been confirmed. Due to the special CKM matrix structure, which is largely correlated with the unexpected long lifetime of b quark, we discuss that the CP-violating effects in B-mesons-relevant weak processes are expected to be greatly enhanced in other cases, and the B-factories, Belle and BaBar, have achieved rich results. It turns out, however, that the CKM-induced CP violation is still not sufficient to solve the BAU problem. This clearly reminds us that the SM is not complete, people are seeking for answers in theories that are beyond the SM, this is also one of the main goals of modern physics.

There are many on-going searches for CP-violating signals. In this thesis, we talked about several possible directions:

- 1. neutron EDM;
- 2. "D-term" and "R-term" in neutron radiative β decay;

3. triple-product correlation in momenta.

The SM prediction that is based on the CKM mechanism states that the neutron can possess a nontrivial EDM that is of $\mathcal{O}(10^{-33}) e - cm$, whereas current experimental data reveals null result up to the order of $\mathcal{O}(10^{-26}) e - cm$. People are now constructing an experiment to push the precision down to $\mathcal{O}(10^{-28}) e - cm$. We also see that in both the neutron EDM and the "D-term"/"R-term" measurements, the spin polarization plays an essential role. Yet the third direction that we are mainly focusing on in this thesis has no dependence on neutron spin. We have demonstrated that such a T-odd and P-odd correlation in terms of $(\mathbf{l}_e \times \mathbf{k}) \cdot \mathbf{l}_{\nu}$ can arise from the neutron and nuclear radiative β decay via three possible mechanisms:

1. the general parameterization of weak interaction Hamiltonian that was first proposed by Lee and Yang [3];

2. the HHH-induced exotic contact coupling between the baryon current and the weak charged current at sufficiently low energies [80] [81] [129];

3. a T-odd mimicking effect that arises from final-state interactions in the SM. As has been emphasized before, unlike the first two cases which are generated by real CP violating effects, the FSI-induced T-odd mimicking effect does not represent a real CP violating effect.

We analyzed both of the two cases in detail, and showed that the Lee and Yang's general parameterization leads to a possible triple-product correlation in momenta only in recoil order, thus in the future experiments we are suggesting, such a contribution should be relatively negligible in comparison with the FSI-induced T-odd mimicking effect. We showed that the HHH-induced T-odd correlation is controlled by the overall factor $\text{Im}\left[(c_5/M^2)q_V\right]$, and it serves as a new candidate in searching for CP violation in theories beyond the SM. We also discussed a possible model that could give rise to a nontrivial imaginary part of the exotic coupling c_5 – via the possible hidden sector " ρ " meson interaction. In the measurements of such a possible T-odd, CP-violating correlation in the future experiments, we have to face the challenge of the FSI-induced T-odd mimicking effect, which behaves pretty much in the same way as the HHH contribution but does not really signify a real CP violation. Such a SM mimicking effect forms a baseline that calls for detailed investigation. We have analyzed this effect in full detail up to recoil order effects. It has been shown that the mimicking effect in neutron radiative β decay is only of $\mathcal{O}(10^{-4})$, which leaves an optimal window to constrain $\operatorname{Im}\left[(c_5/M^2)g_V\right)$. It has also been shown that the mimicking effect is controlled by the overall factor " $(1 - \rho^2/3)(1 + \rho^2)$ ", where $\rho = \sqrt{3}\lambda = \sqrt{3}g_A/g_V$ in the neutron case. This suggests that if in some nuclear radiative β decay with ρ happens to be very close to $\sqrt{3}$, the FSI-induced mimicking effect can be greatly suppressed, making a much better candidate for detecting the HHH-induced CP violation. We show that in leading recoil order the nuclear cases are very similar to the neutron case with only minor adjustments, and we have

considered ⁶He, ¹⁹Ne, and ³⁵Ar as examples.

Furthermore, in the currently on-going precision measurements on the neutron radiative β decay at NIST, people are pushing the error down to the level of 1%. Following the new upcoming record of precision, one is forced to consider more contributions that previously have been ignored. In this thesis, we considered two major corrections that can affect the numerical evaluations at the order of 1%. The first is the final-state Coulomb corrections that arise from electromagnetic interactions between the charged decay products. In the neutron case, we found that such correction has about a 3% contribution to the branching ratio. Second, as has been argued, the contribution of two photon radiation in neutron and nuclear weak decay processes can be important in the regions of phase space where the photons are of low energy because the radiative decay rate is logarithmically enhanced in these regions and the additional $\mathcal{O}(\alpha)$ suppression does not guarantee a negligible contribution. Currently the analysis of neutron radiative β decay at NIST has not taken this contribution into account. We have analyzed its behavior and impact, hoping that this can be helpful to the experimentalists in improving data analysis.

10.2 Outlook

In the whole thesis our numerical evaluations have centered around the T-odd correlations associated with radiative weak processes, particularly those due to the charged electroweak currents induced by the exchange of W^{\pm} gauge bosons, which gives rise to neutron and nuclear β decay. In fact, the potential exotic interactions contained in Eq. (6.10) are very rich. There are many other possible channels that are worth investigating. Taking another look at the chiral Lagrangian in Eq. (6.10), we see that inserting the external source terms V'_{μ} and A_{μ} defined in Eq. (6.12) and Eq. (6.13) and carefully taking care of the (anti)commutations in $\mathcal{L}^{(3)}$, there are also exotic couplings among nucleons, the photon, and the Z^0 boson or pions. The neutral current pieces arise from the $c_4^{(3)}$ -related terms and the charged current pieces arise from the $c_5^{(3)}$ related terms. The phenomenological coupling constants $c_4^{(3)}$ and $c_5^{(3)}$ are supposed to be of the same order. For example, in keeping the leading order terms, we find the exotic couplings:

$$\mathcal{L}_{pnW\gamma}^{(3)} \sim c_5^{(3)} i \epsilon^{\mu\nu\rho\sigma} \bar{p}\gamma_\sigma n W_\mu F_{\nu\rho}, \qquad (10.1)$$

$$\mathcal{L}^{(3)}_{pZ^0\gamma} \sim c_4^{(3)} i \epsilon^{\mu\nu\rho\sigma} \bar{p} \gamma_\sigma p Z^0_\mu F_{\nu\rho}, \qquad (10.2)$$

$$\mathcal{L}^{(3)}_{p\pi^0\gamma} \sim c_4^{(3)} i \epsilon^{\mu\nu\rho\sigma} \bar{p}\gamma_\sigma p \partial_\mu \pi^0 F_{\nu\rho}.$$
(10.3)

We have discussed some of the applications of Eq. (10.1) in neutron radiative β decay. In fact one can also apply such an effective low energy coupling to Eq. (10.1) in the muon radiative capture process: $\mu^- + p \rightarrow n + \gamma + \overline{\nu}_{\mu}$, which has been investigated in next leading order in heavy-baryon ChPT [181]. In comparing with the experimental result [182], we are expecting to be able to set a sharper constraint on the absolute size of $|c_5|$ and possibly offer an explanation for the observed differences in the ordinary and radiative muon capture results. The exotic coupling given in Eq. (10.3) can be applied in the γ -proton scattering process: $\gamma + p \rightarrow \pi^0 + p$ as well as $\gamma + p \rightarrow \pi^+ + \pi^- + p$. Due to the special tensor structure that are proportional to $\epsilon_{\mu\nu\rho\sigma}$, these exotic interactions could also give rise to additonal parity-violating effects. From the experience we have gained in neutron and nuclear radiative β decays, we know that the T-odd, P-odd effects we have studied are generally proportional to the total energy release. In the neutron and nuclear radiative β decays, the total energy release in mainly determined by the parent and daughter particles mass differences, which is out of our control. In collision processes, however, one is able to control the energies of the incoming particles, thus one can hope to find larger signals of these T-odd, P-odd interactions.

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Chapter A Intermediate Phase Space Integrals

This appendix is taken from [165]. The computation of the imaginary parts of the loop diagrams requires an integration over the allowed phase space of the intermediate momenta as fixed by the momenta of the final-state particles and energy-momentum conservation. In this Appendix we report the integrals which appear in the diagrams of Fig. 8.4 and label them as per the diagrams in that figure. For diagrams with cuts which yield Compton scattering from electrons our results can be compared to, and agree with, those of Refs. [163] and [183]. In what follows we report the integrals which arise from $\gamma - e$ cuts: (1), (2), (5.1), and (6.2), and then the integrals which arise from the cutting of electron and proton lines to generate physical $ep \rightarrow ep\gamma$ scattering, namely, (5.2) and (6.1), and $ep \rightarrow ep$ scattering, (6.3) and (8.2). The integrals associated with the rest of the cuts in Fig. 8.4 are not given explicitly because they do not contribute in leading order in the recoil expansion, as we note in the main body of the text. Nevertheless, we note the relationships between these integrals which appear in the large M_p limit in order to make the cancellations associated with these terms transparent.

From diagram (1), defining $P_{0e} \equiv l_e + k$, we have

$$J_{1} \equiv \int \frac{d^{3}\mathbf{l}'_{e}}{2E'_{e}} \frac{d^{3}\mathbf{k}'}{2\omega'} \delta^{(4)}(l'_{e} + k' - P_{0e}) \equiv \int d\rho_{\gamma e}$$
$$= \frac{\pi}{2} \left(1 - \frac{m_{e}^{2}}{P_{0e}^{2}}\right), \qquad (A.1)$$

as well as

$$K_1^{\mu} \equiv \int d\rho_{\gamma e} k'^{\,\mu} = a_1 P_{0e}^{\mu} \tag{A.2}$$

with

$$a_1 = \frac{\pi}{4} \left(1 - \frac{m_e^2}{P_{0e}^2} \right)^2$$

From diagram (2) we have

$$J_2 \equiv \int d\rho_{\gamma e} \frac{1}{l_e \cdot k'} = \frac{\pi}{2l_e \cdot k} \log\left(\frac{P_{0e}^2}{m_e^2}\right) \,. \tag{A.3}$$

We apply the Passarino-Veltman reduction method to compute integrals which contain additional powers of the intermediate momenta [167]. That is, writing

$$K_2^{\mu} = \int d\rho_{\gamma e} \frac{k'^{\mu}}{l_e \cdot k'} = a_2 l_e^{\mu} + b_2 P_{0e}^{\mu}, \qquad (A.4)$$

the values of a_2 and b_2 are fixed by the solution of the set of equations

$$J_1 = a_2 m_e^2 + b_2 l_e \cdot P_{0e} ,$$

$$l_e \cdot k J_2 = a_2 l_e \cdot P_{0e} + b_2 P_{0e}^2 .$$

Moreover,

$$L_2^{\mu\nu} = \int d\rho_{\gamma e} \frac{k'^{\mu} k'^{\nu}}{l_e \cdot k'} = c_2 g^{\mu\nu} + d_2 l_e^{\mu} l_e^{\nu} + e_2 P_{0e}^{\mu} P_{0e}^{\nu} + f_2 (l_e^{\mu} P_{0e}^{\nu} + P_{0e}^{\mu} l_e^{\nu}), \qquad (A.5)$$

where c_2 , d_2 , e_2 , and f_2 are given by the solution of the set of equations

$$0 = 4c_2 + d_2m_e^2 + e_2P_{0e}^2 + 2f_2l_e \cdot P_{0e},$$

$$0 = c_2 + d_2m_e^2 + f_2l_e \cdot P_{0e},$$

$$a_1 = e_2l_e \cdot P_{0e} + f_2m_e^2,$$

$$l_e \cdot kb_2 = c_2 + e_2P_{0e}^2 + f_2l_e \cdot P_{0e}.$$

For integrals which depend on M_p we report their form in the large M_p limit for subsequent use. Note that M rather than M_p appears in the limiting form because the n-p mass difference itself is of higher order in the recoil expansion. From diagram (5.1) we have

$$J_{5.1} = \int d\rho_{\gamma e} \frac{1}{p_p \cdot k'} = \frac{\pi}{2I_{0e}} \log\left(\frac{p_p \cdot P_{0e} + I_{0e}}{p_p \cdot P_{0e} - I_{0e}}\right) , \qquad (A.6)$$

with $I_{0e} = \sqrt{(p_p \cdot P_{0e})^2 - M_p^2 P_{0e}^2}$, noting

$$J_{5.1} \sim \frac{\pi}{2M|\mathbf{k} + \mathbf{l}_{\mathbf{e}}|} \log \left(\frac{E_e + \omega + |\mathbf{k} + \mathbf{l}_{\mathbf{e}}|}{E_e + \omega - |\mathbf{k} + \mathbf{l}_{\mathbf{e}}|} \right)$$
(A.7)

as $M_p \to \infty$. In addition

$$K_{5.1}^{\mu} = \int d\rho_{\gamma e} \frac{k'^{\mu}}{p_p \cdot k'} = a_{5.1} p_p^{\mu} + b_{5.1} P_{0e}^{\mu} , \qquad (A.8)$$

where $a_{5.1}$ and $b_{5.1}$ are given by the solution of the set of equations

$$J_1 = a_{5.1}M_p^2 + b_{5.1}p_p \cdot P_{0e} ,$$

$$l_e \cdot kJ_{5.1} = a_{5.1}p_p \cdot P_{0e} + b_{5.1}P_{0e}^2 .$$

In the large M_p limit $b_{5.1} \sim 1/M$ and $a_{5.1} \sim 1/M^2$. We postpone discussion of the integrals from diagrams (5.2) and (6.1) to consider the integrals from the remaining diagrams with Compton cuts. From diagram (6.2) we have

$$J_{6.2} = \int d\rho_{\gamma e} \frac{1}{(l_e \cdot k')(p_p \cdot k')} = \frac{\pi}{2(l_e \cdot k)I_e} \log\left(\frac{p_p \cdot l_e + I_e}{p_p \cdot l_e - I_e}\right), \quad (A.9)$$

with $I_e = \sqrt{(p_p \cdot l_e)^2 - M_p^2 m_e^2}$ and
 $J_{6.2} \sim \frac{\pi}{2M |\mathbf{l}_e| k \cdot l_e} \log\left(\frac{E_e + |\mathbf{l}_e|}{E_e - |\mathbf{l}_e|}\right).$ (A.10)

as $M_p \to \infty$. In addition

$$K_{6.2}^{\mu} = \int d\rho_{\gamma e} \frac{k'^{\mu}}{(l_e \cdot k')(p_p \cdot k')} = a_{6.2} P_{0e}^{\mu} + b_{6.2} l_e^{\mu} + c_{6.2} p_p^{\mu}, \qquad (A.11)$$

where $a_{6.2}$, $b_{6.2}$, and $c_{6.2}$ are given by the solution to the set of equations

$$J_{2} = a_{6.2}p_{p} \cdot P_{0e} + b_{6.2}p_{p} \cdot l_{e} + c_{6.2}M_{p}^{2},$$

$$J_{5.1} = a_{6.2}l_{e} \cdot P_{0e} + b_{6.2}m_{e}^{2} + c_{6.2}p_{p} \cdot l_{e},$$

$$l_{e} \cdot kJ_{6.2} = a_{6.2}P_{0e}^{2} + b_{6.2}l_{e} \cdot P_{0e} + c_{6.2}p_{p} \cdot P_{0e}.$$

and in the large M_p limit $a_{6.2}, b_{6.2} \sim 1/M$ and $c_{6.2} \sim 1/M^2$. Finally

$$L_{6.2}^{\mu\nu} = \int d\rho_{\gamma e} \frac{k'^{\mu}k'^{\nu}}{(l_e \cdot k')(p_p \cdot k')} = d_{6.2}g^{\mu\nu} + e_{6.2}p_p^{\mu}p_p^{\nu} + f_{6.2}l_e^{\mu}l_e^{\nu} + g_{6.2}P_{0e}^{\mu}P_{0e}^{\nu} + h_{6.2}(p_p^{\mu}l_e^{\nu} + l_e^{\mu}p_p^{\nu}) + i_{6.2}(p_p^{\mu}P_{0e}^{\nu} + P_{0e}^{\mu}p_p^{\nu}) + k_{6.2}(l_e^{\mu}P_{0e}^{\nu} + P_{0e}^{\mu}l_e^{\nu}), \qquad (A.12)$$

where the coefficients which appear are given by the solution to set of the equations

$$\begin{aligned} &4d_{6.2} + e_{6.2}M_p^2 + f_{6.2}m_e^2 + g_{6.2}P_{0e}^2 + 2h_{6.2}p_p \cdot l_e + 2i_{6.2}p_p \cdot P_{0e} + 2k_{6.2}l_e \cdot P_{0e} = 0 , \\ &d_{6.2} + f_{6.2}m_e^2 + h_{6.2}p_p \cdot l_e + k_{6.2}l_e \cdot P_{0e} = 0 , \\ &e_{6.2}p_p \cdot l_e + h_{6.2}m_e^2 + i_{6.2}l_e \cdot P_{0e} = a_{5.1} , \\ &g_{6.2}l_e \cdot P_{0e} + i_{6.2}p_p \cdot l_e + k_{6.2}m_e^2 = b_{5.1} , \\ &d_{6.2} + e_{6.2}M_p^2 + h_{6.2}p_p \cdot l_e + i_{6.2}p_p \cdot P_{0e} = 0 , \\ &g_{6.2}p_p \cdot P_{0e} + i_{6.2}M_p^2 + k_{6.2}p_p \cdot l_e = b_2 , \\ &d_{6.2} + g_{6.2}P_{0e}^2 + i_{6.2}p_p \cdot P_{0e} + k_{6.2}l_e \cdot P_{0e} = l_e \cdot ka_{6.2} . \end{aligned}$$

Note that the equations have been chosen to yield a self-consistent solution for the six coefficients.

The integrals associated with the $\gamma - p$ cuts can be found if necessary by replacing the intermediate momentum l'_e by p'_p as well as l_e by p_p in the $\gamma - e$ integrals we have provided. Specifically we note

$$J_3 \equiv \int \frac{d^3 \mathbf{p}'_p}{2E'_p} \frac{d^3 \mathbf{k}'}{2\omega'} \delta^{(4)}(p'_p + k' - P_{0p}) \equiv \int d\rho_{\gamma p} \,,$$

where $P_{0p} \equiv p_p + k$, and

$$J_4 = \int d\rho_{\gamma p} \frac{1}{p_p \cdot k'} \tag{A.13}$$

so that

$$J_4 \sim \frac{1}{M\omega} J_3 \sim \mathcal{O}\left(\frac{1}{M^2}\right)$$
 (A.14)

as $M_p \to \infty$. Moreover,

$$J_{7.2} = \int d\rho_{\gamma p} \frac{1}{l_e \cdot k'} \tag{A.15}$$

and

$$K_{7.2}^{\mu} = \int d\rho_{\gamma p} \frac{k'^{\mu}}{l_e \cdot k'} = a_{7.2} l_e^{\mu} + b_{7.2} p_p^{\mu}, \qquad (A.16)$$

whereas

$$J_{8.3} = \int d\rho_{\gamma p} \frac{1}{(l_e \cdot k')(p_p \cdot k')}$$
(A.17)

and

$$K_{8.3}^{\mu} = \int d\rho_{\gamma p} \frac{k'^{\mu}}{(l_e \cdot k')(p_p \cdot k')} = a_{8.3}k^{\mu} + b_{8.3}l_e^{\mu} + c_{8.3}p_p^{\mu}, \qquad (A.18)$$

so that

$$J_{8.3} \sim \frac{1}{M\omega} J_{7.2} \sim \mathcal{O}\left(\frac{1}{M^2}\right) \quad ; \quad a_{8.3} \sim 0 + \mathcal{O}\left(\frac{1}{M^3}\right) ,$$

$$b_{8.3} \sim \frac{1}{M\omega} a_{7.2} + \mathcal{O}\left(\frac{1}{M^3}\right) \quad ; \quad c_{8.3} \sim \frac{1}{M\omega} b_{7.2} + \mathcal{O}\left(\frac{1}{M^4}\right) \qquad (A.19)$$

as $M_p \to \infty$.

The integrals in the remaining diagrams of Fig. 8.4 arise from cutting the electron and proton lines to generate physical $ep \rightarrow ep\gamma$ or $ep \rightarrow ep$ scattering. The intermediate phase space integrals in these cases are more complicated than those associated with the Compton cuts; fortunately, closed-form expressions for the integrals in the large M_p limit suffice to leading order in the recoil expansion. With $P_0 \equiv p_p + l_e + k$, we note for diagram (5.2)

$$I_{5.2} = \int \frac{d^3 \mathbf{l}'_e}{2E'_e} \frac{d^3 \mathbf{p}'_p}{2E'_p} \delta^{(4)}(l'_e + p'_p - P_0) \equiv \int d\rho_{ep\gamma}$$

$$= \frac{\pi}{2P_0^2} \sqrt{(P_0^2 - M_p^2 + m_e^2)^2 - 4P_0^2 m_e^2} \sim \frac{\pi}{M} \sqrt{(E_e + \omega)^2 - m_e^2} \quad (A.20)$$

as $M_p \to \infty$. Moreover,

$$J_{5.2} = \int d\rho_{ep\gamma} \frac{1}{(p'_p - p_p)^2}$$
(A.21)

and

$$J_{5.2} \sim \frac{\pi}{4M|\mathbf{l}_e + \mathbf{k}|} \log \left(\frac{m_e^2 + l_e \cdot k - (E_e + \omega)^2 + \sqrt{(E_e + \omega)^2 - m_e^2} \, |\mathbf{l}_e + \mathbf{k}|}{m_e^2 + l_e \cdot k - (E_e + \omega)^2 - \sqrt{(E_e + \omega)^2 - m_e^2} \, |\mathbf{l}_e + \mathbf{k}|} \right)$$
(A.22)

as $M_p \to \infty$. In addition,

$$K_{5.2}^{\mu} = \int d\rho_{ep\gamma} \frac{l_e^{\prime\mu}}{(p_p^{\prime} - p_p)^2} = a_{5.2} P_{0e}^{\mu} + c_{5.2} p_p^{\mu}, \qquad (A.23)$$

where $a_{5,2}$ and $c_{5,2}$ are given by the solution to

$$(m_e^2 + l_e \cdot k)J_{5.2} - \frac{I_{5.2}}{2} = a_{5.2}P_{0e}^2 + c_{5.2}p_p \cdot P_{0e},$$

$$p_p \cdot P_{0e}J_{5.2} + \frac{1}{2}I_{5.2} = a_{5.2}p_p \cdot P_{0e} + c_{5.2}M_p^2,$$

so that in the large M_p limit $a_{5.2} \sim 1/M$ and $c_{5.2} \sim 1/M^2$. Turning to the integrals from diagram (6.1) we have

$$I_{6.1} = \int d\rho_{ep\gamma} \frac{1}{(l'_e \cdot k)}, \qquad (A.24)$$

so that as $M_p \to \infty$

$$I_{6.1} \sim \frac{\pi}{2M\omega} \log \left(\frac{E_e + \omega + \sqrt{(E_e + \omega)^2 - m_e^2}}{E_e + \omega - \sqrt{(E_e + \omega)^2 - m_e^2}} \right),$$
(A.25)

as well as

$$I'_{6.1} = \int d\rho_{ep\gamma} \frac{(p'_p - p_p)^2}{(l'_e \cdot k)}, \qquad (A.26)$$

where as $M_p \to \infty$

$$I'_{6.1} \sim 2(m_e^2 + l_e \cdot k)I_{6.1} - 2I_{5.2} - 2\tilde{I}_{6.1}$$
(A.27)

with

$$\tilde{I}_{6.1} = \frac{\pi \mathbf{k} \cdot \mathbf{l}_e}{M\omega^2} \left(\sqrt{(E_e + \omega)^2 - m_e^2} + \frac{(E_e + \omega)k \cdot l_e}{2\mathbf{k} \cdot \mathbf{l}_e} \log \left(\frac{E_e + \omega + \sqrt{(E_e + \omega)^2 - m_e^2}}{E_e + \omega - \sqrt{(E_e + \omega)^2 - m_e^2}} \right) \right).$$
(A.28)

Moreover,

$$J_{6.1} = \int d\rho_{ep\gamma} \frac{1}{(l'_e \cdot k)(p_p - p'_p)^2}, \qquad (A.29)$$

so that as $M_p \to \infty$

$$J_{6.1} \sim \frac{\pi}{4M|\mathbf{l}_e|k \cdot l_e} \left(\log\left(\frac{A_+}{A_-}\right) - \log\left(\frac{B_+}{B_-}\right) \right) , \qquad (A.30)$$

where

$$A_{\pm} = m_e^2 + l_e \cdot k - (E_e + \omega)^2 \pm |\mathbf{l}_e + \mathbf{k}| \sqrt{(E_e + \omega)^2 - m_e^2}$$
(A.31)

 $\quad \text{and} \quad$

$$B_{\pm} = |\mathbf{l}_{e}|^{2}(l_{e} \cdot k)^{2} - (\omega^{2}m_{e}^{2} - E_{e}\omega(l_{e} \cdot k)) A_{\pm} + |\mathbf{l}_{e}|(l_{e} \cdot k) \left((E_{e} + \omega)\omega|\mathbf{l}_{e} + \mathbf{k}| \mp (\omega^{2} + \mathbf{l}_{e} \cdot \mathbf{k})\sqrt{(E_{e} + \omega)^{2} - m_{e}^{2}} \right) (A.32)$$

In addition,

$$K_{6.1}^{\mu} = \int d\rho_{ep\gamma} \frac{l_e^{\prime\mu}}{(l_e^{\prime} \cdot k)(p_p - p_p^{\prime})^2} = a_{6.1}l_e^{\mu} + b_{6.1}k^{\mu} + c_{6.1}p_p^{\mu}, \qquad (A.33)$$

where the undetermined coefficients are fixed by the solution to

$$J_{5.2} = a_{6.1}l_e \cdot k + c_{6.1}p_p \cdot k ,$$

$$(m_e^2 + l_e \cdot k)J_{6.1} - \frac{I_{6.1}}{2} = a_{6.1}(m_e^2 + l_e \cdot k) + b_{6.1}l_e \cdot k + c_{6.1}p_p \cdot P_{0e} ,$$

$$p_p \cdot P_{0e}J_{6.1} = a_{6.1}p_p \cdot l_e + b_{6.1}p_p \cdot k + c_{6.1}M_p^2 ,$$

so that in the large M_p limit $a_{6.1}, b_{6.1} \sim 1/M$ and $c_{6.1} \sim 1/M^2$. Also

$$L_{6.1}^{\mu\nu} = \int d\rho_{ep\gamma} \frac{l_e^{\prime\mu} l_e^{\prime\nu}}{(l_e^{\prime} \cdot k)(p_p - p_p^{\prime})^2} = d_{6.1}g^{\mu\nu} + e_{6.1}p_p^{\mu}p_p^{\nu} + f_{6.1}l_e^{\mu}l_e^{\nu} + g_{6.1}k^{\mu}k^{\nu} + h_{6.1}(p_p^{\mu}l_e^{\nu} + l_e^{\mu}p_p^{\nu}) + i_{6.1}(p_p^{\mu}k^{\nu} + k^{\mu}p_p^{\nu}) + k_{6.1}(l_e^{\mu}k^{\nu} + k^{\mu}l_e^{\nu}),$$

where the undetermined coefficients are fixed by the solution to

$$\begin{split} &4d_{6.1} + e_{6.1}M_p^2 + f_{6.1}m_e^2 + 2h_{6.1}p_p \cdot l_e + 2i_{6.1}p_p \cdot k + 2k_{6.1}l_e \cdot k = m_e^2 J_{6.1} \,, \\ &d_{6.1} + e_{6.1}M_p^2 + h_{6.1}p_p \cdot l_e + i_{6.1}p_p \cdot k = p_p \cdot P_{0e}c_{6.1} \,, \\ &g_{6.1}p_p \cdot k + i_{6.1}M_p^2 + k_{6.1}p_p \cdot l_e = p_p \cdot P_{0e}b_{6.1} \,, \\ &f_{6.1}p_p \cdot l_e + h_{6.1}M_p^2 + k_{6.1}p_p \cdot k = p_p \cdot P_{0e}a_{6.1} \,, \\ &e_{6.1}p_p \cdot k + h_{6.1}l_e \cdot k = c_{5.2} \,, \\ &f_{6.1}l_e \cdot k + h_{6.1}p_p \cdot k = a_{5.2} \,, \\ &d_{6.1}P_{0e}^2 + e_{6.1}(p_p \cdot P_{0e})^2 + f_{6.1}(l_e \cdot P_{0e})^2 + g_{6.1}(l_e \cdot k)^2 + 2h_{6.1}p_p \cdot P_{0e}l_e \cdot P_{0e} \\ &+ 2i_{6.1}p_p \cdot P_{0e}l_e \cdot k + 2k_{6.1}l_e \cdot P_{0e}l_e \cdot k = (m_e^2 + l_e \cdot k)^2 J_{6.1} - (m_e^2 + l_e \cdot k)I_{6.1} + \frac{I_{6.1}'}{4} \,. \end{split}$$

For the remaining $e - p - \gamma$ cuts we have

$$J_{7.1} = \int d\rho_{ep\gamma} \frac{1}{(l'_e - l_e)^2} \sim \mathcal{O}\left(\frac{1}{M}\right)$$
(A.34)

and

$$K_{7.1}^{\mu} = \int d\rho_{ep\gamma} \frac{l_e^{\prime\mu}}{(l_e^{\prime} - l_e)^2} = a_{7.1} l_e^{\mu} + b_{7.1} p_p^{\mu}, \qquad (A.35)$$

whereas

$$J_{8.1} = \int d\rho_{ep} \frac{1}{(p'_p \cdot k)(l'_e - l_e)^2}$$
(A.36)

 $\quad \text{and} \quad$

$$K_{8.1}^{\mu} = \int d\rho_{ep} \frac{l_e^{\prime \mu}}{(p_p^{\prime} \cdot k)(l_e^{\prime} - l_e)^2} = a_{8.1}l_e^{\mu} + b_{8.1}k^{\mu} + c_{8.1}p_p^{\mu}, \qquad (A.37)$$

so that

$$J_{8.1} \sim \frac{1}{M\omega} J_{7.1} \sim \mathcal{O}\left(\frac{1}{M^2}\right) \quad ; \quad b_{8.1} \sim 0 + \mathcal{O}\left(\frac{1}{M^3}\right) ,$$
$$a_{8.1} \sim \frac{1}{M\omega} a_{7.1} + \mathcal{O}\left(\frac{1}{M^3}\right) \quad ; \quad c_{8.1} \sim \frac{1}{M\omega} b_{7.1} + \mathcal{O}\left(\frac{1}{M^4}\right) \qquad (A.38)$$
as $M_p \to \infty$.

The integrals for the e - p cuts follow from those we have just analyzed under the replacement of P_0 with $\tilde{P}_0 \equiv l_e + p_p$. In this case, however, there is an added complication because the integrals become infrared divergent when $p'_p = p_p$. This divergence cancels once we construct an observable quantity; nevertheless, we regulate the integrals as they stand by adding a fictitious photon mass m_{γ}^2 – this will allow us to track the infrared divergences through the course of the calculation, so that we can demonstrate the divergence cancellation manifestly. In what follows we set m_{γ}^2 to zero in all terms which are finite in the $m_{\gamma}^2 \rightarrow 0$ limit. We have

$$I_{8.2} = \int \frac{d^{3}\mathbf{l}'_{e}}{2E'_{e}} \frac{d^{3}\mathbf{p}'_{p}}{2E'_{p}} \delta^{(4)}(l'_{e} + p'_{p} - \tilde{P}_{0}) \equiv \int d\rho_{ep} \\ \sim \frac{\pi |\mathbf{l}_{e}|}{M}$$
(A.39)

as $M_p \to \infty$. In addition,

$$J_{8.2} = \int d\rho_{ep} \frac{1}{p'_{p} \cdot k} \frac{1}{(p'_{p} - p_{p})^{2} - m_{\gamma}^{2}}$$
(A.40)
 $\sim \frac{\pi}{4|\mathbf{l}_{e}|\omega M^{2}} \log\left(\frac{m_{\gamma}^{2}}{4|\mathbf{l}_{e}|^{2}}\right)$

as $M_p \to \infty$. Thus we see that $J_{8,2}$ vanishes in this limit save for the infrared divergent piece, which we define as $J_{8,2}^{div}$. In addition,

$$K_{8.2}^{\mu} = \int d\rho_{ep} \frac{1}{p'_p \cdot k} \frac{l'_e^{\mu}}{(p'_p - p_p)^2 - m_{\gamma}^2} = a_{8.2} l_e^{\mu} + b_{8.2} k^{\mu} + c_{8.2} p_p^{\mu}.$$
(A.41)

The coefficients are given by the solution to

$$m_e^2 J_{8.2} - \frac{1}{2} \tilde{I}_{8.2} = a_{8.2} m_e^2 + b_{8.2} l_e \cdot k + c_{8.2} p_p \cdot l_e ,$$

$$p_p \cdot l_e J_{8.2} + \frac{1}{2} \tilde{I}_{8.2} = a_{8.2} p_p \cdot l_e + b_{8.2} p_p \cdot k + c_{8.2} M_p^2 ,$$

$$(l_e + p_p) \cdot k J_{8.2} - I'_{8.2} = a_{8.2} l_e \cdot k + c_{8.2} p_p \cdot k ,$$

where

$$\tilde{I}_{8.2} = \int d\rho_{ep} \frac{1}{p'_p \cdot k} \quad ; \quad I'_{8.2} = \int d\rho_{ep} \frac{1}{(p'_p - p_p)^2 - m_\gamma^2} \,. \tag{A.42}$$

In the large M_p limit we note that

$$K_{8.2}^{\mu} \sim \frac{1}{M\omega} I_{8.2}^{\prime}$$
 (A.43)

so that $b_{8.2} \sim 0$, and we need only solve

$$m_e^2 J_{8.2} - \frac{I_{8.2}}{2M\omega} = a_{8.2}m_e^2 + c_{8.2}ME_e, E_e J_{8.2} = a_{8.2}E_e + c_{8.2}M$$
(A.44)

to determine the leading-order expressions for $a_{8,2}$ and $c_{8,2}$. We can track the infrared divergence in $J_{8,2}$ in $a_{8,2}$ and $c_{8,2}$ by solving these equations with $I_{8,2} = 0$ and $J_{8,2} = J_{8,2}^{div}$, which yields $a_{8,2}^{div} \sim J_{8,2}^{div}$ and $c_{8,2}^{div} \sim 0$ in leading order.

The integrals from diagram (6.3) are

$$I_{6.3} = \int d\rho_{ep} \frac{1}{(l'_e \cdot k)}$$

$$\sim \frac{\pi}{2\omega M} \log \left(\frac{E_e + |\mathbf{l}_e|}{E_e - |\mathbf{l}_e|} \right)$$
(A.45)

as $M_p \to \infty$ and

$$I'_{6.3} = \int d\rho_{ep} \frac{(p'_p - p_p)^2}{(l'_e \cdot k)}$$

$$\sim 2m_e^2 I_{6.3} - 2\tilde{I}_{6.3}$$
(A.46)

with

$$\tilde{I}_{6.3} \sim \frac{\pi}{2M\omega} \left((E_e^2 - E_e |\mathbf{l}_e| \cos \theta_e) \log \left(\frac{E_e + |\mathbf{l}_e|}{E_e - |\mathbf{l}_e|} \right) + 2|\mathbf{l}_e|^2 \cos \theta_e \right)$$
(A.47)

as $M_p \to \infty$. We define $\mathbf{k} \cdot \mathbf{l}_e \equiv |\mathbf{k}| |\mathbf{l}_e| \cos \theta_e$. Moreover,

$$J_{6.3} = \int d\rho_{ep} \frac{1}{(l'_e \cdot k)} \frac{1}{(p_p - p'_p)^2 - m_\gamma^2}$$
(A.48)
 $\sim \frac{\pi}{4|\mathbf{l}_e|(l_e \cdot k)M} \left(\log \frac{m_\gamma^2}{4|\mathbf{l}_e|^2} + \log \frac{m_e^2 \omega^2}{(l_e \cdot k)^2}\right)$

as $M_p \to \infty$. In this case we see that $J_{6.3}$ has both infrared finite and divergent pieces in the $M_p \to \infty$ limit – the latter we define as $J_{6.3}^{\text{div}}$. Finally

$$K_{6.3}^{\mu} = \int d\rho_{ep} \frac{1}{(l'_e \cdot k)} \frac{l'_e^{\mu}}{(p_p - p'_p)^2 - m_{\gamma}^2} = a_{6.3} l_e^{\mu} + b_{6.3} k^{\mu} + c_{6.3} p_p^{\mu} , \qquad (A.49)$$

where the undetermined coefficients are fixed by the solution to

$$p_p \cdot kJ_{8.2} = a_{6.3}l_e \cdot k + c_{6.3}p_p \cdot k ,$$

$$m_e^2 J_{6.3} - \frac{I_{6.3}}{2} = a_{6.3}m_e^2 + b_{6.3}l_e \cdot k + c_{6.3}p_p \cdot l_e ,$$

$$p_p \cdot l_e J_{6.3} = a_{6.3}p_p \cdot l_e + b_{6.3}p_p \cdot k + c_{6.3}M_p^2 .$$

Also

$$\begin{split} L_{6.3}^{\mu\nu} &= \int d\rho_{ep} \frac{1}{(l'_e \cdot k)} \frac{l'_e^{\mu} l'^{\nu}_e}{(p_p - p'_p)^2 - m_{\gamma}^2} = d_{6.3} g^{\mu\nu} + e_{6.3} p_p^{\mu} p_p^{\nu} + f_{6.3} l_e^{\mu} l_e^{\nu} + g_{6.3} k^{\mu} k^{\nu} \\ &+ h_{6.3} (p_p^{\mu} l_e^{\nu} + l_e^{\mu} p_p^{\nu}) + i_{6.3} (p_p^{\mu} k^{\nu} + k^{\mu} p_p^{\nu}) + k_{6.3} (l_e^{\mu} k^{\nu} + k^{\mu} l_e^{\nu}) \,, \end{split}$$

where the undetermined coefficients are fixed by the solution to

$$\begin{split} &4d_{6.3} + e_{6.3}M_p^2 + f_{6.3}m_e^2 + 2h_{6.3}p_p \cdot l_e + 2i_{6.3}p_p \cdot k + 2k_{6.3}l_e \cdot k = m_e^2 J_{6.3} \,, \\ &d_{6.3} + e_{6.3}M_p^2 + h_{6.3}p_p \cdot l_e + i_{6.3}p_p \cdot k = p_p \cdot l_e c_{6.3} \,, \\ &g_{6.3}p_p \cdot k + i_{6.3}M_p^2 + k_{6.3}p_p \cdot l_e = p_p \cdot l_e b_{6.3} \,, \\ &f_{6.3}p_p \cdot l_e + h_{6.3}M_p^2 + k_{6.3}p_p \cdot k = p_p \cdot l_e a_{6.3} \,, \\ &e_{6.3}p_p \cdot k + h_{6.3}l_e \cdot k = p_p \cdot k c_{8.2} \,, \\ &f_{6.3}l_e \cdot k + h_{6.3}p_p \cdot k = p_p \cdot k a_{8.2} \,, \\ &d_{6.3}m_e^2 + e_{6.3}(p_p \cdot l_e)^2 + f_{6.3}m_e^4 + g_{6.3}(l_e \cdot k)^2 + 2h_{6.3}p_p \cdot l_e m_e^2 + 2i_{6.3}p_p \cdot l_e l_e \cdot k \\ &+ 2k_{6.3}m_e^2 l_e \cdot k = m_e^4 J_{6.3} - m_e^2 I_{6.3} + \frac{I_{6.3}'}{4} \,. \end{split}$$

We can track the infrared divergence in $J_{6.3}$ in the solutions for the vector and tensor coefficients by solving the equations in the large M_p limit with $I_{6.3} \sim I'_{6.3} \sim 0$ and $J_{6.3} \sim J^{\text{div}}_{6.3}$, with $a_{8.2} \sim a^{\text{div}}_{8.2}$, which yields $a^{\text{div}}_{6.3} \sim f^{\text{div}}_{6.3} \sim J^{\text{div}}_{6.3}$ with all other coefficients zero in this limit.

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Chapter B FORM Code for the Calculation of $|\mathcal{M}^{\gamma\gamma}|^2$

Off Statistics; Vectors P, Le, Lv, K, K1, K2, K0; Symbols M, Mt, Me, Ee, Ek, Ek1, Ek2, Ev, lambda, K0denominator, K0denominator2, int01, int02, int11, int12, int21, int22, Pi; Indices mu, nu, rho, delta, alpha, beta;

*This box is to calculate the nRDK with double-photon emission.

*eeee terms:

- Local [Meeee11] = ($g_{(1,Le)}+Me_{)*g_{(1,mu)}*(g_{(1,Le)}+g_{(1,K1)})+Me_{)}/(2*Le.K1_)*g_{(1,nu)}*(g_{(1,Le)}+g_{(1,K0)}+Me_{)*g_{(1,K1)}})*(1-g5_{(1)})*$
- $\begin{array}{c} g_{(1,Lv)*(1+g_{(1,Le)})*g_{(1,Me)}*(g_{(1,Le)})+g_{(1,Me)})*g_{(1,Me)}*g_{(1,Me$
- $(g_{2,P})+M)*g_{2,rho}*(1-lambda*g5_{2})*(g_{2,P})+M)*(1+lambda*g5_{2})*g_{2,delta};$
- $g_{(1,Lv)} * (1+g_{(1,Le)}) * g_{(1,delta)} * (g_{(1,Le)}+g_{(1,K0)}) + Me) * g_{(1,Mu)} * (g_{(1,Le)}+g_{(1,K2)}) + Me) / (2*Le.K2) * g_{(1,Nu)} * (2*Le.K2) * g_{(1,N$
- $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta);$

 $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta);$

Local [Meeee] = ([Meeee11] + [Meeee12] + [Meeee21] + [Meeee22]) *K0denominator2/4;

*eeep terms:

- $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta)*P(nu)/(P.K2);$

- $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta)*P(nu)/(P.K2);$

- $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta)*P(mu)/(P.K1);$
- Local [Meeep] = ([Meeep11] + [Meeep12] + [Meeep21] + [Meeep22]) *K0denominator / 4;

*eepp terms:

- $g_{1}(1,Lv)*(1+g_{1}(1))*g_{1}(1,delta)*$

- Local [Meepp] = ([Meepp11] + [Meepp12] + [Meepp21] + [Meepp22]) *K0denominator / 4;

*epep terms:

- Local [Mepep11] = ($g_{1}(1, Le) + Me$) * $g_{1}(1, mu)$ *($g_{1}(1, Le) + g_{1}(1, K1) + Me$)/(2*Le.K1) * $g_{1}(1, rho)$ *(1- $g_{1}(1)$) *
- $(g_(2,P)+M)*g_(2,rho)*(1-lambda*g5_(2))*(g_(2,P)+M)*(1+lambda*g5_(2))*g_(2,delta)*P(nu)/(P.K2)*P(nu)/(P.K2);$
- $\begin{array}{l} \text{Local} \left[\text{Mepep12} \right] = \left(\begin{array}{c} \text{g}_{(1,\text{Le})+\text{Me}} \right) \ast \text{g}_{(1,\text{mu})} \ast \left(\begin{array}{c} \text{g}_{(1,\text{Le})+\text{g}_{(1,\text{K1})}} \right) \\ \text{He} \end{array} \right) / \left(\begin{array}{c} 2 \ast \text{Le.K1} \end{array} \right) \ast \text{g}_{(1,\text{rho})} \ast \left(\begin{array}{c} 1 \text{g5}_{(1)} \end{array} \right) \ast \\ \text{g}_{(1,\text{Lv})} \ast \left(\begin{array}{c} 1 + \text{g5}_{(1)} \end{array} \right) \ast \text{g}_{(1,\text{delta})} \ast \left(\begin{array}{c} \text{g}_{(1,\text{Le})+\text{g}_{(1,\text{K2})}} \right) \\ \text{He} \end{array} \right) / \left(\begin{array}{c} 2 \ast \text{Le.K1} \end{array} \right) \ast \text{g}_{(1,\text{delta})} \ast \left(\begin{array}{c} 1 \text{g5}_{(1)} \end{array} \right) \ast \\ \text{g}_{(1,\text{Lv})} \ast \left(\begin{array}{c} 1 + \text{g5}_{(1)} \end{array} \right) \ast \text{g}_{(1,\text{delta})} \ast \left(\begin{array}{c} 1 \text{g5}_{(1,\text{Le})} \right) \\ \text{He} \end{array} \right) / \left(\begin{array}{c} 2 \ast \text{Le.K1} \end{array} \right) \ast \text{g}_{(1,\text{delta})} \ast \left(\begin{array}{c} 1 \text{g5}_{(1,\text{Le})} \right) \\ \text{He} \end{array} \right) \ast \text{g}_{(1,\text{Le})} \\ \text{He} \end{array} \right) \ast \text{g}_{(1,\text{Le})} = \left(\begin{array}{c} 1 \text{g5}_{(1,\text{Le})} \right) \\ \text{He} \end{array} \right) \ast \text{g}_{(1,\text{Le})} \\ \text{He} \end{array} \right) = \left(\begin{array}{c} 1 \text{g}_{(1,\text{Le})} \right) \ast \text{g}_{(1,\text{Le})} \\ \text{He} \end{array} \right)$
 - (1, nu)/(-2*Le.K2)*

Local [Mepep] = ([Mepep11] + [Mepep12] + [Mepep21] + [Mepep22]) / 4;

*eppp terms:

Local [Meppp] = ([Meppp11] + [Meppp12] + [Meppp21] + [Meppp22])/4;

*pppp terms:

- Local [Mpppp11] = $(g_(1, Le) + Me) *g_(1, rho) *(1-g5_(1)) *g_(1, Lv) *(1+g5_(1)) *g_(1, delta) *$
- $(g_{2,P})+M \approx g_{2,nu} *(g_{2,P})+g_{2,K0} *0+M)/(2*P.K0)*g_{2,R0} *g_{2,R0} *0+M)$

Local [Mpppp] = ([Mpppp11] + [Mpppp12] + [Mpppp21] + [Mpppp22]) /4;

Now sum up all the components above and take trace: Local $[M_2gamma] = [Meeee] - 2[Meeep] + 2*[Meepp] + [Mepep] - 2*[Meppp] + [Mpppp];$

Trace4,1; Trace4,2; id K0=K1+K2; id K1.K1=0; id K2.K2=0; id Le.Le=Me^2; id P.P=M^2; id P.K1=M*Ek1; id P.K2=M*Ek2; id P.Le=M*Ee;

*

id P.Lv=M*Ev; id P.K1^-1=M^(-1)*Ek1^(-1); id P.K2^-1=M^(-1)*Ek2^(-1); id P.K0^-1=M^(-1)*(Ek1+Ek2)^(-1);

 $. \operatorname{sort}$

*-----

id e_(mu?,nu?,rho?,delta?)=0;

Bracket e_,M; Format 140; Print [M_2gamma]; .end Copyright[©] Daheng He, 2013.

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