B-mode CMB spectrum estimation using a pure pseudo cross-spectrum approach

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I describe the pure pseudo-spectrum formalism for the estimation of the Cosmic Microwave Background polarized power spectra, as proposed by [Smith 2006] and extended to incorporate cross-spectra of the multiple maps of the same sky area by [Grain et al. 2009]. I summarize the performance of the method as compared to other existing algorithms and their implementations. In particular, I show that the statistical uncertainty of the estimated B-mode spectra is typically found to be within a factor ~ 2 of the variance derived from the most optimistic Fisher matrix predictions accounting only for the sampling and noise uncertainty of the B-modes alone. I conclude that the presented formalism thanks to its speed and efficiency can provide an interesting alternative to the CMB polarized power spectra estimators based on the optimal methods.

1 Introduction

The reliable characterization and scientific exploitation of the polarized Cosmic Microwave Background (CMB) Anisotropy signal is one of the main challenges facing the CMB research at the present. We discuss the CMB power spectrum estimation via the pure pseudo spectrum technique as published in [Grain et al. 2009][1]. Pseudo-C_ℓ algorithms provide a computationally quick and flexible framework for estimating the power spectra. However, it has been long recognized[2] that a straightforward application of the pseudo spectrum technique to cut-sky polarized CMB maps leads to the so-called "E-to-B" leakage, or power aliasing. A consequence of which the cosmologically important information contained in the CMB B-modes is overwhelmed by the statistical uncertainty of the (much larger) E-modes. [Grain et al. 2009][1] propose an approach that relies on a suitably chosen sky apodization to remove from the map harmonic modes which are neither solely E nor B as introduced by [Smith 2006][3].

2 Angular Power Spectrum Estimation

Maps of I, Q and U components of the CMB signal are decomposed into spherical harmonics $a^T_{\ell m}$, $a^E_{\ell m}$ and $a^B_{\ell m}$. From these coefficients, one can construct the 6 angular power spectrum : C_ℓ^{TT}, C_ℓ^{EE}, C_ℓ^{BB}, C_ℓ^{TE}, C_ℓ^{TB} and C_ℓ^{EB}. Systematic effects need to be taken into account in this process. In particular, beam smoothing effects or partial coverage of the sky must be accounted for. Even for full sky missions, foreground residuals usually still dominate the noise in the Galactic plane. To avoid any contamination of the angular power spectra, a mask is applied to suppress pixels in which parasitic signal are strong, leading to less than full-sky effective coverage. Angular power spectra estimators can be separated in two main categories:

- Maximum Likelihood methods[5,7,6] which estimate angular power spectra using the angular correlation function M by maximizing the probability of C_ℓ considering the maps T:

$$\mathcal{P}(C_\ell|T) \propto \exp\left[-\frac{1}{2}\left(T^T M^{-1} T + Tr(\ln M)\right)\right].$$

  The algorithm scales as $\mathcal{O}(N_{bins} N_{pix}^3)$ for CPU time and $\mathcal{O}(N_{pix}^2)$ for memory. This implies that maximum likelihood methods are not well adapted to surveys such as Planck which should deliver high resolution maps with more than 10^7 pixels.

- Pseudo-C_ℓ methods[8,9,10,11] compute the angular power spectra directly from the observed maps before correcting for instrumental effects such as beam smoothing effects (B_ℓ), partial sky coverage ($M_{\ell\ell'}$ which is computed analytically using the spherical transform of the weight mask) or filtering of data (F_ℓ). The biased spectrum (called pseudo-spectrum) $\widetilde{C}_\ell$ rendered by the direct spherical harmonics transform of a partial sky map is different from the full sky angular spectrum C_ℓ but their ensemble average are linked by :

$$\langle \tilde{C}_\ell \rangle = \sum_{\ell'} M_{\ell\ell'} F_{\ell'} B_{\ell'}^2 \langle C_{\ell'} \rangle + \langle \tilde{N}_\ell \rangle.$$

  $\langle \tilde{N}_\ell \rangle$ is the noise contribution to the estimated pseudo-spectra and vanishes, whenever the noise in the two data sets is not correlated, as for example, in a case of two data sets produced by two different experiments or two uncorrelated detectors of a single experiment. This emphasizes one of the biggest advantages of the cross-spectrum based estimators, which do not need such a correction. Pseudo-C_ℓ estimators make use of the fast spherical harmonics transform that scales in $\mathcal{O}(N_{pix}^{3/2})$. Moreover they are often nearly optimal in practice (at least for temperature). Nevertheless, they need a precise description of the instrument (beam, filtering, noise) that requires a large number of Monte-Carlos.

3 E-B mixing

Due to the limited sky coverage, and non-uniform, pixel-dependent weights, the above pseudo-C_ℓ estimator is biased and its average over CMB realizations, $\left\langle \tilde{C}_\ell^X \right\rangle$, involves a mixing between different ℓ modes (or bins) and polarization states ($X = E, B$). The latter can be described by a so-called *mixing kernel* $M_{\ell\ell'}$. The unbiased estimator C_ℓ^X is thus obtained by inverting the following linear system,

$$\begin{pmatrix} M_{\ell\ell'}^{diag} & M_{\ell\ell'}^{off} \\ M_{\ell\ell'}^{off} & M_{\ell\ell'}^{diag} \end{pmatrix} \begin{pmatrix} C_{\ell'}^E \\ C_{\ell'}^B \end{pmatrix} = \begin{pmatrix} \tilde{C}_\ell^E - N_\ell^E \\ \tilde{C}_\ell^B - N_\ell^B \end{pmatrix}. \quad (1)$$

In the ensemble average sense the above expression is unbiased as a result of a subtle cancellation of the E mode power present in the pseudo-B and E spectra. Such a cancellation does not however apply to the variance of the estimator and as a result the variance of the spectra of one type will include a contribution from the other preventing any detection of the primordial B-mode (figure 1).

4 Pure pseudo-C_ℓ estimators for cross-spectrum

Using the two differential operators $\mathbf{D}_s^{E(B)}$ which generalize to arbitrary spin the operators used in [Bunn et al. 2003][2], we can write the harmonic representation of the field $\mathbf{P} = (Q, U)$ in the

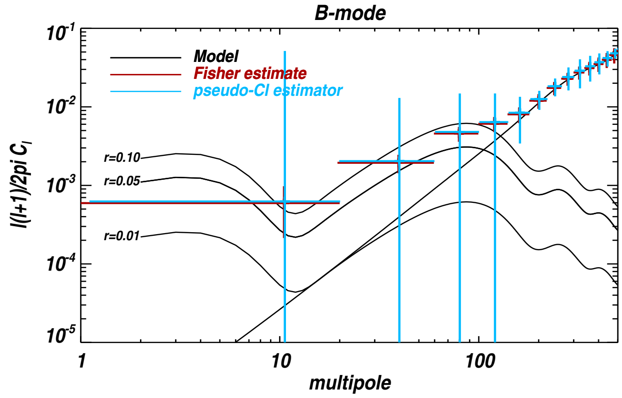

Figure 1: B-mode reconstruction with error bars from a Fisher analysis (*red*) and a pseudo-C_ℓ estimator (*blue*) based on a B-mode with r=0.05. The shown B-mode spectrum (*black*) is decomposed into primordial (plotted for three different values of r=0.01, 0.05, 0.1) and lensing parts.

E/B subspace. We introduce the sky coverage setting $W = 1$ for pixel in the sky area and $W = 0$ elsewhere. In particular, for the B mode

$$\begin{aligned} a^B_{\ell m} &= \int_{4\pi} W \cdot (Q,U) \times D^B Y_{\ell m} \\ &= \int_\Omega D^B(Q,U) \times Y_{\ell m} + \oint_{C_\Omega} (Q,U)\partial Y_{\ell m} + \oint_{C_\Omega} \partial(Q,U) Y_{\ell m} \end{aligned} \quad (2)$$

The two contour integrals represent the so-called "ambiguous" modes that are responsible for the E-to-B leakage. Pseudo-C_ℓ estimators of the polarization power spectra which do not mix E and B modes can be constructed in projecting the polarization fields on the "pure" E and B subspaces [2]. Pure B multipoles on a partial sky are defined as follows [3]:

$$\begin{aligned} a^B_{\ell m} &= \int_{4\pi} (Q,U) \times D^B(W Y_{\ell m}) \\ &= \int_\Omega W \cdot D^B(Q,U) \times Y_{\ell m} + \oint_{C_\Omega} (Q,U) \times \partial(W Y_{\ell m}) + \oint_{C_\Omega} \partial(Q,U) \times W Y_{\ell m} \end{aligned} \quad (3)$$

We choose W a spin-0 window function in order to satisfy the Dirichlet and Neumann conditions on the boundary of the observed sky region. Such conditions on the window function are optimized for the estimated multipoles to be free of a E/B leakage due to partial sky either analytically or numerically [3,4,1] (figure 2). This translates into vanishing mixing matrices

$$M^{off}_{\ell\ell'} = 0.$$

Our numerical implementation proceeds in two steps [1]:

1. from the spin-0 window function W_0, we define two spin-weighted windows

$$W_1 = \eth W \text{ and } W_2 = \eth^2 W$$

2. we calculate the pure multipoles $a^{(s)}_{\ell m}$ of the s-spin fields $\tilde{P}_s = W_s^\dagger (Q + iU)$. The pure estimated multipoles coefficients then reads as linear combinations of the $a^{(s)}_{\ell m}$

$$\mathcal{A}^B_{\ell m} = a^{(2)}_{\ell m} + \lambda_{1,\ell} a^{(1)}_{\ell m} + \lambda_{0,\ell} a^{(0)}_{\ell m} \quad (4)$$

The code is fully parallel (both in CPU time and memory) and very fast (less than 30min for 1000 simulations on 1024 procs) using the *pureS2HAT* library[12]. We can recover the B-mode angular power spectrum without bias and with a significant improvement in the level of error bars compared to "standard" pseudo-C_ℓ methods (figure 3).

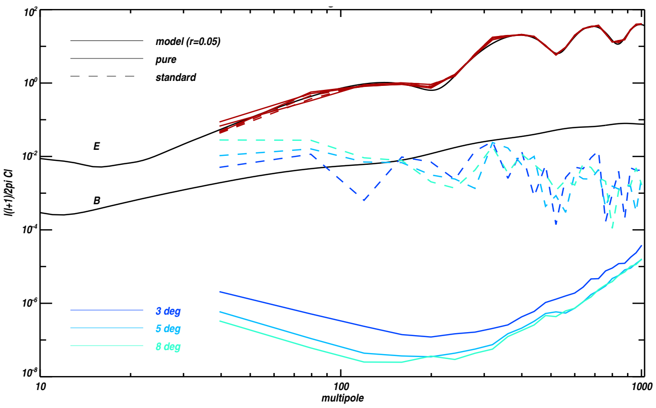

Figure 2: Leakage from E to B in the case of standard estimator (*dashed line*) and pure estimator (*solid line*) using three different apodization length : 3, 5 and 8 deg. Remaining leakage at high multipoles is induced by pixel effects.

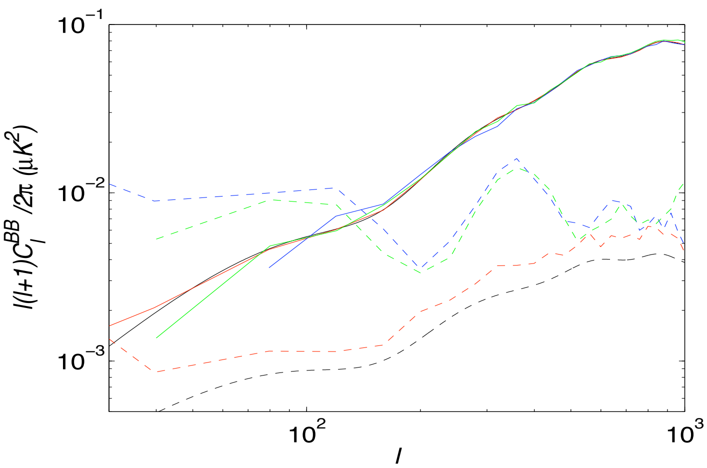

Figure 3: B-mode angular power spectrum estimation (*solid lines*) and their MC-estimated variance as compared with the input (*black solid*) and Fisher variance (*black dashed*) from various estimators : Xpol standard estimator (*blue*), SpicePol standard estimator (*green*), Xpure pure estimator (*red*). figure by H. Nishino (KEK)

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