The Hidden Kerr/CFT Conjecture

by

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to my brother

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Chapter 1

Introduction

At the turn of the twentieth century, classical mechanics was at its peak. Everything but two known phenomena were to be explained via Newtonian physics. In order to account for the photoelectric effect and blackbody radiation, we needed to resort to a completely new framework, which significantly altered our understanding of the world. Interestingly enough, the theory of gravity follows a similar historic route. Classical general relativity gives a remarkably precise description of our telescopically observable universe, and up to 70's solving the Einstein's equations and computing linearized perturbations was a technical difficulty that was rather slowly but steadily being resolved. The twist in our understanding was induced by Bekenstein and Hawking, who taught us black holes had entropy and were radiating particles. The central observation, leading us on this rocky road towards quantum gravity, is that black hole event horizons exhibit thermal properties of a quantum field in curved spacetime. Even 40 years after the Bekenstein-Hawking breakthrough, we are still having hard time understanding this strongly coupled quantum theory.

However, for free quantum theories our intuition serves us well and the black hole can be shown to behave very much like a blackbody at Hawking temperature T_H. Even at weak coupling, we would still expect that this picture remains qualitatively true, within the realm of perturbation theory. As we further increase the coupling, we arrive to unchartered territories, in which defining even the relevant degrees of freedom becomes a challenge. Unfortunately, at strong coupling the qualitative understanding is completely lacking, emphasizing the need of novel methods that can access strongly coupled regimes of field theories. A promising way out is delivered via Anti-de Sitter/conformal field theory duality, or AdS/CFT, in which we map the dynamics of conformal field theory to classical gravity in negatively curved spacetimes. The beauty of this correspondence lies in the fact that where one theory fails the other one succeeds, enabling us to explore one side of the duality to learn more about the other. For example, we can use classical gravity to provide insight into strongly coupled field theories. On the other hand, there are many longstanding open problems one hopes to better address with the help of field theory methods, such as emergence of space and time, resolving spacetime singularities and information release from black holes.

The universal relation between the entropy and black hole horizon area, $S = A/4G$, is one of those puzzles. The geometric computation does not give any insight in the nature of the microstates this entropy is counting. In order to make progress, we need a concrete quantum theory, such as supplied by AdS/CFT. Maybe the simplest example is provided with the black hole in three dimensional negatively curved space, the BTZ black hole. In that case the geometric calculation is precisely matched with the Cardy formula, counting the states of thermal gauge bosons in a 2-dimensional CFT. The remarkable fact is that the agreement is independent of the details of the correspondence, making it a robust technique for general spacetimes. Thus, the AdS/CFT correspondence has been shown to be a fruitful one, and it has made impact on a broad range of areas in physics.

Chapter 2

Preliminaries

Ten years after we have learnt about classical black hole solutions in a 4-dimensional asymptotically flat spacetime, we found that the black hole behaves as a thermal object, a blackbody radiating thermal spectrum, and as a result has entropy associated with the area of its horizon. From the point of view of thermodynamics this is a striking phenomenon, as the entropy scales as area instead of volume. Thus, we learn that black holes behave in a rather peculiar way for classical objects. This thermodynamic behavior should have statistical interpretation in a quantum theory. Furthermore, this quantum theory should be of an unusual kind, holographic in its nature, representing all that happens in bulk spacetime by some degrees of freedom on a screen of area A. If this were true, we would expect to observe hidden structure in our equations which govern physics in black hole backgrounds. In that sense, we can think of black holes as windows to a quantum theory of gravity.

Figure 2.0.1: Spiral galaxy NGC 1365, Credit: European Southern Observatory, http://www.eso.org/

2.1 The Kerr black hole

As astrophysical objects, black holes are very simple; they are completely determined by their mass M and angular momentum J. Every galaxy, such as one in Fig. 2.0.1, is supposed to have one at its center. If a black hole passes through a cloud of interstellar gas, it will draw it inward in a process known as accretion. Just as planets orbiting around stars, matter will circle around a black hole until it loses its angular momentum. If we consider a particle going around the black hole, we can determine the black hole's mass by solving a simple Newtonian problem. In order for the particle to travel in a circular orbit, the gravitational pull of the black hole should be balanced with the centrifugal force. By measuring the velocity of the particle and radius of the orbit, we can estimate the black hole's mass. Furthermore, this is not the only measurement of the black hole's mass that can be done, and by comparing results across different measurements we can get a reliable estimate.

Angular momentum is a little more difficult to measure. As the object spins around the black hole, there is the innermost stable circular orbit, beyond which accretion disk terminates and gas falls freely into the black hole. The size of this orbit strongly depends on the black hole's angular momentum, and for a non-rotating, Schwarzschild black hole it is at $r = 6M$, whereas for the maximally rotating, extreme Kerr, it is at $r = M$. By measuring the inner radius of the accretion disk, such as one illustrated in Fig. 2.1.1, we can work out the value of the rotation parameter. For the spiral galaxy in the Fig. 2.0.1, recent experimental evidence [1] strongly suggests there is a black hole at its center, spinning at more than 84% of its mass.

Figure 2.1.1: Illustration of a black hole accretion disk, Credit: Prague Relativistic Astrophysics, http://astro.cas.cz/

However, all these measurements in their most simplest form come with some assumptions about how the spacetime looks near the black hole horizon. The astrophysicists have so far found a handful of black hole candidates that have their mass ranging from several solar masses to a few milion solar masses [2]. But in order to prove these objects truly are black holes, one would need to experimentally verify the existence of an event horizon, which is a surface along which the causal structure of the spacetime changes. Measuring this and testing the metric of the spacetime near the black hole are two major challenges in black hole astrophysics.

Nevertheless, metric is something we already know – it is an exact solution of Einstein equations in general relativity. It is just a function that gives us distance between two points in spacetime, and in Boyer-Lindquist coordinates $x^\mu = \{t, r, \theta, \phi\}$ it takes the following form:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= (1 - 2Mr/\Sigma) dt^2 + \left(4Mar \sin^2\theta/\Sigma\right) dt d\phi - (\Sigma/\Delta)\, dr^2 \\ &\quad - \Sigma d\theta^2 - \sin^2\theta \left(r^2 + a^2 + 2Ma^2 r \sin^2\theta/\Sigma\right) d\phi^2 \end{aligned}$$

where t is the timelike coordinate and three coordinates parametrizing the 3-dimensional space are radial coordinate r and two angles, θ and ϕ. We also used units in which $c = G = 1$ and where M is the black hole mass, $J = aM$ is the angular momentum, and two functions appearing in the metric are given with $\Sigma = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. The outer and inner horizons sit at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

Hawking temperature is

$$T_H = \frac{\sqrt{M^2 - a^2}}{4\pi M r_+}$$

and Bekenstein-Hawking entropy

$$S = \frac{A_{\text{horizon}}}{4} = 2\pi M r_+ \,.$$

2.2 The Teukolsky equation

Armed with a metric, we can compute response to small perturbations sent from infinity into the curved spacetime of the black hole. Let us consider the wave equation for massless fields ψ_s in this background, which in its most simplest notational form is $\Box\psi_s = 0$. Soon enough we will show some of less simple ways of writing this equation. Most of the physics that we know and understand in the background of a Kerr black hole will be supplied by the solutions of this equation.

Teukolsky found that it was possible to separate the wave equation for general spin massless fields [3]. He wrote down a master equation for different spin-weight fields. The spin-weight should be thought of as a label to distinguish different types of solutions for different fields. For example, the spin weight zero would correspond to a massless scalar field ψ_0, and the Teukolsky equation would reduce to a Klein-Gordon equation in a curved background,

$$\frac{1}{\sqrt{-\det g_{\mu\nu}}}\partial_\mu \left(\sqrt{-\det g_{\mu\nu}}\, g^{\mu\nu}\partial_\nu\psi_0\right) = 0\,.$$

In the Newman-Penrose formalism, we supply each spacetime point with a basis spanned by four null vectors, l, n, m and $\bar{m}$, given by

$$
\begin{aligned}
l^\mu &= \left\{\tfrac{r^2+a^2}{\Delta}, 1, 0, \tfrac{a}{\Delta}\right\},\ n^\mu = \left\{\tfrac{r^2+a^2}{2\Sigma}, -\tfrac{\Delta}{2\Sigma}, 0, \tfrac{a}{2\Sigma}\right\}, \\
m^\mu &= \left\{\tfrac{ia\sin\theta}{r+ia\cos\theta}, 0, \tfrac{1}{r+ia\cos\theta}, \tfrac{i}{(r+ia\cos\theta)\sin\theta}\right\}/\sqrt{2}\,.
\end{aligned}
$$

This allows us to encode the six independent components of the Maxwell tensor $F_{\mu\nu}$ into three scalars via contractions of the stress tensor with two null directions at each point, recasting the well known Maxwell equations $dF = 0$ and $d\star F = 0$ into four equations for Newman-Penrose complex scalars. For spin weight ± 1 the Teukolsky master equation is for two components of the Maxwell tensor, corresponding to ingoing and outgoing waves with respect to infinity. As an example, for $s = -1$, which corresponds to outgoing waves at infinity, we have $\psi_{-1} \propto F_{\mu\nu}\bar{m}^\mu n^\nu$.

Furthermore, in the notation of Teukolsky, the 10 independent components of the Weyl tensor are encoded into 5 scalars, obtained by contracting the Weyl tensor along those 4 null directions. For the sourceless equation, which is the only one we will consider, these correspond to perturbations of the Riemann tensor. For example, in the transverse traceless gauge there are only two nonvanishing components of the linearized metric perturbations $\delta g_{\mu\nu} = h_{\mu\nu}$, conveniently written in terms of two polarizations, so called plus and cross, which completely encode the properties of gravitational waves in empty space. Then the black hole response to metric perturbations outgoing at infinity is simply given with the spin-weight $s = -2$ field:

$$
\psi_{-2} \propto \frac{1}{2}\frac{\partial^2}{\partial t^2}\left(h_{\theta\theta} - h_{\phi\phi} + ih\theta\phi\right) = \frac{\partial^2}{\partial t^2}\left(-h_+ + h_\times\right).
$$

A master equation, spanning different physical processes in black hole backgrounds is a perfect candidate to exhibit a hidden structure due to some underlying more fundamental microscopic description. This is the main reason we study the Teukolsky equation, given with

$$\begin{aligned}\left[\frac{\left(r^2+a^2\right)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi_s}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi_s}{\partial t\partial\phi}+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi_s}{\partial\phi^2}\\ -\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi_s}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_s}{\partial\theta}\right)-2s\left[\frac{a\left(r-M\right)}{\Delta}+i\frac{\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi_s}{\partial\phi}\\ -2s\left[\frac{M\left(r^2-a^2\right)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi_s}{\partial t}+\left(s^2\cot^2\theta-s\right)\psi_s &= 0\end{aligned}$$

With the following ansatz for the solutions,

$$\psi_s=e^{-i\omega t}e^{im\phi}S(\theta)R(r) \tag{2.2.1}$$

the master equation separates into the angular part

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left(a^2\omega^2\cos^2\theta-2a\omega s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+E-s^2\right)S=0 \tag{2.2.2}$$

and the radial equation

$$\begin{aligned}\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right)+\Big(\Big[\left(r^2+a^2\right)^2\omega^2-4aMr\omega m-2iM(r^2-a^2)\omega s\\ +a^2m^2+2ia(r-M)ms\Big]\Delta^{-1}+2ir\omega s-E+s(s+1)-a^2\omega^2\Big)R &= 0\end{aligned} \tag{2.2.3}$$

where the separation constant E is constrained by the requirement that S be regular at $\theta = 0, \pi$. For the special case $a\omega = 0$ this may be computed exactly $E = \ell(\ell+1)$. For general ω this may be computed numerically, or as a series expansion. For both of these equations the exact solutions in a closed form are unknown.

2.3 Holographic methods

More than 25 years ago Brown and Henneaux have shown that the asymptotic symmetry group of a quotient of a 3-dimensional Anti-de Sitter (AdS) space consists of two copies of a 2-dimensional conformal group, whose generators close under a Virasoro algebra, with central charge $c = 3\ell_{AdS}/2G_3$ [4]. This enables us to count the microscopic degrees of freedom in the conformal field theory (CFT) via Cardy formula [5]. These microscopic degrees of freedom at finite temperature give rise to thermodynamic phenomena in the bulk. Surprisingly enough, even before we knew about black holes in AdS_3, we could count their entropy! A subsequent discovery of BTZ black hole [6] has put this hint of a duality on a firmer footing: the conformal field theory at the AdS boundary exactly reproduces the classical entropy of the black hole. The details of the duality have since been further explored within the framework of AdS/CFT correspondence, by mapping scattering amplitudes in the bulk to correlation functions in the CFT, with hope that this better understood lower dimensional cousin of a Kerr black hole can give us insight into the more complicated 4-dimensional gravity.

The AdS/CFT is an equivalence between classical gravity scattering amplitudes and CFT correlation functions. The correspondence is achieved via a bulk-to-boundary map, by which we identify bulk fields in AdS with operators defined in the lower dimensional theory on the asymptotic boundary of AdS. As the statement is that a lower dimensional quantum theory fully captures the physics of higher dimensional gravity, the correspondence is holographic in nature. Maybe the simplest and most instructive way [7] to write down this correspondence is

$$\left\langle \exp \int_{\partial M} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z(\phi_0)\,,$$

where ϕ_0 is the boundary value of the bulk field ϕ, to which we couple a conformal operator $\mathcal{O}$. The right hand side is the gravity partition function on manifold M, computed by evaluating the classical action for ϕ.

Bardeen and Horowitz have realized early on that extremal Kerr black hole admits a near horizon scaling, under which the geometry takes a form of $U(1)$ fiber over AdS_3 [8]. In these near horizon coordinates Guica et al. propose a duality between a two-dimensional chiral CFT and an extremal near horizon region of a Kerr black hole (NHEK), which is referred to as the Kerr/CFT correspondence [9]. The NHEK geometry shares some of the isometries with quotients of AdS_3, so the correspondence is made by finding commutators of vector fields that generate an asymptotic symmetry group of NHEK, whose algebra closes under Virasoro, with central charge $c = 12J$.

The result they obtained was not surprising in view of the Brown and Henneaux computation, except for the curious detail that the central extension comes from the enhancement of $U(1)$, not the asymptotic symmetry group of AdS_3. Later on, alternative realizations were found enhancing the other symmetry group [10, 11]. The entropy on the CFT side can be computed by using $S = \frac{\pi^2}{3} cT$, and it is shown to precisely match the Bekenstein-Hawking entropy of the Kerr black hole. The conjecture is these results can be extended away from extremal case. As the extremal

Kerr black hole has zero Hawking temperature, it can be thought of as a ground state in the CFT dual, with non-extremal black holes completing the spectrum.

In many ways this framework resembles to Brown and Henneaux's, whose observation paved a way to a duality between a conformal field theory on the boundary and a gravity theory with a negative cosmological constant in the bulk. However, the Kerr/CFT is still far from being promoted to a well defined CFT dual for asymptotically flat non-extremal geometries. This tantalizing analogy points out to a far richer structure deeply hidden within our equations of motion. So we will take our motivation from there, while searching for the possible manifestations of the well hidden conformal symmetry. A suggestive piece of evidence was recently supplied by Castro et al., where the authors identify a hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry in the low frequency near region scalar field equation in Kerr background [12].

The conformal structure of scattering amplitudes may be considered as a first and necessary step towards defining a conformal field theory. Whether or not this field theory can be mathematically well-defined and provide relevant information for bulk scattering amplitudes is a separate question, which we seek to answer with this thesis. Subsequently, we were able to show in [13] that the well-defined bulk to boundary map softly breaks the conformal symmetry, still allowing us to organize bulk fields in terms of higher dimension operators within the CFT. However, the fixed point where the hidden conformal symmetry becomes exact is flat spacetime. This indicates that if there is an exact CFT underlying the dynamics of Kerr, there is not a smooth geometric limit connecting the low frequency limit of general Kerr with the dynamics of extremal Kerr. Furthermore, upon closer inspection of the low frequency regime we show in [14] that the low energy physics does not provide access to information that would fully fix the theory, but makes it prominent there should exist a theory that admits a smooth limit to the static Schwarzschild black hole. We find in [15] that the hidden structure in our equations can even persist at finite frequencies, but in that case we are confronted with a daunting task of defining a holographic dual at a finite radius. All these clues point us to note that in order to have a well defined holographic map, we have to give up on geometric interpretation – perhaps even leading us to a deeper understanding of quantum theories of gravity.

But in some sense, a geometric interpretation is also missing in Kerr/CFT. How is physics in NHEK background related to Kerr, what kind of states is the CFT counting and what are the CFT n-point functions dual to in full Kerr background[1] are still some of unanswered questions. Can we extend this reasoning beyond extremal black holes, to a non-chiral CFT? A search for a quantum theory that accounts for scarce hints of quantum nature of gravity has been active on many fronts.

If the hidden CFT viewpoint can be put on solid footing, these techniques would lead to a radically new way to treat the quantum physics of the entire class of Kerr black holes, including the Schwarzschild limit. In addition to accounting for the quantum entropy of the black hole, it would provide an efficient mechanism for computing of scattering amplitudes. Moreover, if the central charge can be computed from the gravity side, this proposal would yield dramatic new insight into the physics of the black hole microstates that account for the Bekenstein-Hawking entropy.

[1]There is a sense in which a CFT 2-point function matches a massless scalar field scattering amplitude in a near-NHEK region, near the superradiant bound [16].

2.4 Fundamental results

Being able to simply relate the entropy formula for a generic Kerr black hole to one of a two-dimensional CFT at finite temperature as a holographic dual [12] is a most striking result, especially because we have no knowledge of the mechanism by which these two quantities bear resemblance. This has motivated attempts to determine the properties of such a conformal field theory. In the near-extremal limit there has been some success in this direction [9, 16, 17, 18, 19]. Clearly, this is a very important problem, as a complete description of the holographic theory could lead to an exact quantum description of black holes beyond the semiclassical limit commonly studied, or the exact descriptions found in special limits in string theory.

Statistical derivations of Bekenstein-Hawking entropy from weakly coupled fundamental string theory have elucidated the origin of black hole microstates, for a large class of 5- and 4-dimensional extremal[2] black holes, in terms of a low energy 2-dimensional conformal field theory [20, 21, 22, 23]. The extremality is a necessity, rather than a convenience, as it admits a weakly coupled description.

For example, here we outline the result that can be obtained in $N = 2$ supergravity as an effective low energy 4-dimensional theory of the Type IIA string theory compactified on Calabi-Yau. A nice pedagogic review was given by Frederic Denef in his 2010 TASI lectures [24]. In the weakly coupled regime, D-branes wrapped on compact cycles are described by pointlike particles in $\mathbb{R}^3$, interacting with each other via exchange of light stretched open string modes. When the coupling gets tuned to larger values, the interactions are described by massless closed strings, i.e. scalars, photons and gravitons. For large charges, the D-branes are manifested through black

[2]and slightly non-extremal

hole solutions. The (BPS) solution to the equations of motion has static and spherically symmetric metric of the form

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}d\vec{x}^2$$

The equations of motion reduce to first order flow equations for central charge [25], which acquires nonzero minimum value $Z_\star$ for $e^{-2U(r)} = |Z_\star|^2 / r^2$, describing $AdS_2 \times S^2$ metric with S^2 horizon of area $A = 4\pi |Z_\star|^2$. Writing down the partition function for the D4/D0 system, we can derive the number of states in the effective 2-dimensional conformal field theory, exactly reproducing the Bekenstein-Hawking formula, $S = A/4$.

This framework has been shown to extend beyond the regime in which the counting of microscopic degrees of freedom can be done, signaling that there may be a universal effective description that does not depend on the details of the microscopic theory [26]. The striking similarity between the geometry presented here and the near horizon scaling of extremal Kerr has motivated a wealth of research trying to bridge the gap between the low energy effective string description and 4-dimensional black holes as astrophysical objects. This will provide us with sufficient motivation to study the hidden Kerr/CFT.

2.5 Outline of the thesis

In Chapter 3 we briefly review some of the aspects of 2-dimensional conformal field theories, which we will frequently encounter in the remainder of the thesis. Care has been taken to introduce not just notation, but also intuition, as this will serve as a starting point in building the hidden Kerr/CFT interpretation.

Chapters 4, 5 and 6 are sourced from the contents of the three papers [13, 14, 15]. The results of all three chapters rely on careful rewriting of the Teukolsky equation, so each of these will start from the same point and evolve into a different realization of the symmetry. In Chapter 4 we introduce the hidden conformal symmetry in the solution space of the massless low energy scalar equation in Kerr background and build the dictionary between bulk fields and CFT operators. This framework makes it possible to realize bulk observables in terms of CFT correlation functions, giving rise to a low energy effective theory.

The natural question to ask is if this framework can survive the Schwarzschild limit and can it reconstruct the quasinormal mode spectrum of the Kerr black hole, i.e. the typical response of the black hole to perturbations from infinity. In Chapter 5 we tackle those problems from the low energy standpoint. We find the Schwarzschild limit does exist and the theory is capable of reconstructing highly damped quasinormal mode frequencies. In order to achieve this, we deformed the initial wave equation at inner horizon, generating a one-parameter family of solutions.

In Chapter 6 we consider finite frequencies and general spin. We determine that the solutions decompose into non-unitary representations of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry group. Curiously enough, the non-unitarity is of exactly the same kind as in the case of the BTZ black hole. This leads us to postulate a correspondence between gravity in Kerr background and conformal field theory at finite temperatures.

We conclude with Chapter 7, in which we present some open questions and propose directions for future research.

Chapter 3

A short review of 2-dimensional CFTs

Most of the content of this thesis will rely on some properties of 2-dimensional conformal field theories. Here we provide a brief review of the conformal group in two dimensions. The literature on this topic is vast, in addition to the standard reference [27], many lecture notes have become available. Here we draw material from [27, 28, 29, 30]. This is not supposed to be a thorough review, but will serve to introduce notation and provide sufficient background for the remainder of the thesis.

3.1 Conformal group on the sphere

Let us consider a real d-dimensional space $\mathbb{R}^d$ supplied with coordinates x^μ and flat metric $g_{\mu\nu}(x)$. The Lorentz-invariant line element is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Under the coordinate transformation $x \to x'$, the metric transforms as $g_{\mu\nu} \to \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}$. We define the conformal group as the group of coordinate transformations that leave metric invariant up to a local scale factor:

$$\Omega(x) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\,.$$

In the 2-dimensional case we can introduce coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. Then the conformal group is generated by analytic coordinate transformations

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z})$$

which then translate to

$$ds^2 \to \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2\,.$$

The generators of infinitesimal conformal transformations $z \to f(z) = z + \epsilon(z)$ are obtained by expanding $\epsilon(z)$ in a Laurent series for small deformations away from z:

$$\epsilon(z) = \sum_n \alpha_n z^{n+1}\,,$$

and similarly for $\bar{z}$. Acting on scalar functions, we observe the generators

$$L_n = -z^{n+1}\frac{\partial}{\partial z}$$

recover the n-th coefficient α_n. The L_n's and $\bar{L}_n$'s close under the infinite Lie algebra,

$$[L_m, L_n] = (m-n)L_{m+n}, \quad \left[L_m, \bar{L}_n\right] = 0, \quad \left[\bar{L}_m, \bar{L}_n\right] = (m-n)\bar{L}_{m+n}\,. \tag{3.1.1}$$

The action of the conformal group in two dimensions naturally factorizes in two independent local algebras, so we continue z and $\bar{z}$ beyond the domain in which they are conjugate to each other. The generators L, $\bar{L}$ are not well defined for all points on the Riemann sphere $\mathbb{S}^2$, specifically for $z = \infty$. A subset of these is well defined globally, consisting of $\{L_-, L_0, L_+\} \times \{\bar{L}_-, \bar{L}_0, \bar{L}_+\}$ generators. They form $SL(2,\mathbb{C})/\mathbb{Z}_2$ algebra, which is isomorphic to the algebra of the proper orthochronous component of $SO(3,1)$, one we usually refer to as Lorentz group. The restriction of $SL(2,\mathbb{C})$ to real algebra is to either $SU(2)$, or $SL(2,\mathbb{R})$.

The global conformal algebra, which is a closed subalgebra of (3.1.1), is the one we use to characterize physical states. Since the operator $L_0 + \bar{L}_0$ is identified with the Hamiltonian, we prefer to work in the basis of the eigenstates $\phi(z,\bar{z})$ of L_0 and $\bar{L}_0$, with eigenvalues h and $\bar{h}$. These eigenvalues will be referred to as conformal weights, and the eigenvalue of the Hamiltonian, $h+\bar{h}$, as the conformal dimension of the state. Such a state will transform under a conformal transformation $z \to f(z)$ as

$$\phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{3.1.2}$$

and will be called a primary field. All fields will be characterized by their conformal weights $\left(h,\bar{h}\right)$ and their relation to primaries.

The transformation property (3.1.2) imposes strong restrictions on correlation functions of primary fields. Conformal invariance requires that the observables be expressed in invariants of the theory, fixing the 2-point function to

$$\langle \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}\,, \tag{3.1.3}$$

where C_{12} is a constant depending on the normalization of the fields. We construct the fields descending from primaries via repeated action of weight-lowering operators, L_{-n}. Schematically, $\phi^{(-n)} = L_{-n}\phi$, where $\phi^{(-n)}$ is a descendant field of primary ϕ, carrying conformal weights $\left(h+n, \bar{h}\right)$. The stress tensor $T(z)$ is one such field of weight $(2,0)$, obtained by the action of L_{-2} on the identity operator. Its correlation function takes a simple form,

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}\,.$$

The constant c is known as the central charge.

Including the stress tensor in the theory enhances the Witt algebra (3.1.1) to a quantum Virasoro, given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad (3.1.4)$$

similarly for barred generators and zero otherwise. From the form of the central extension in (3.1.4) it is visible the global conformal group generated by $\{L_-,\, L_0,\, L_+\}\times\left\{\bar{L}_-,\, \bar{L}_0,\, \bar{L}_+\right\}$ remains unaltered.

3.2 Conformal group on the cylinder and central charge

Mapping the theory on the complex plane onto a cylinder of radius R is achieved via the exponential map, $z = \exp(w/R)$, where z are the coordinates on the complex plane and w are the coordinates on the cylinder, symmetric under $w \to w + 2\pi R$. Invoking (3.1.2), we find that the fields transform according to $\phi(w,\bar{w}) = R^{-\Delta} z^h \bar{z}^{\bar{h}} \phi(z,\bar{z})$. The stress tensor in addition picks up a contribution from a Schwartzian derivative, a consequence of the non-vanishing central charge, giving

$$T(w) = R^{-2}\left[z^2 T(z) - c/24\right] .$$

This is perhaps the simplest example where the soft breaking of the conformal symmetry occurs due to imposing periodic boundary conditions. Zero point energy on the cylinder is shifted with respect to the plane by

$$L_0^{\mathrm{cyl}} - L_0 = -\frac{c}{24R^2} .$$

To give the central charge a physical interpretation, we can calculate the free energy of the system described by the theory on the cylinder. We easily integrate $\delta F \propto \int d^2x \sqrt{g}\delta g_{\mu\nu} T^{\mu\nu}$ to get

$$F = -\frac{c}{12R} . \tag{3.2.1}$$

From the transformation property of primaries we can infer how the correlation functions behave, for example the 2-point function

$$\begin{aligned} \langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle &= \left|\frac{dw_1}{dz_1}\right|^{-2h} \left|\frac{d\bar{w}_1}{d\bar{z}_1}\right|^{-2\bar{h}} \langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\rangle \\ &\propto R^{-2\Delta}\left[4\sinh\frac{w}{2R}\sinh\frac{\bar{w}}{2R}\right]^{-\Delta} , \end{aligned} \tag{3.2.2}$$

is fully fixed by the conformal invariance, where we defined $w = w_1 - w_2$. For $w \gg R$ the correlation function exponentially decays

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle \propto R^{-2\Delta} \exp\left(-\frac{(w+\bar{w})\Delta}{2R}\right) ,$$

from which we read off the characteristic correlation length $\xi = R/\Delta$. For the finite system with periodic boundary conditions this would signal the existence of a mass gap; for the infinite system at finite temperature $T = (2\pi R)^{-1}$, it would measure coherence length over which thermal fluctuations are suppressed. In this case we can rewrite the free energy (3.2.1) per unit length, $f = F/2\pi R$, as

$$f = -\frac{c}{24\pi R^2} = -\frac{\pi c}{6}T^2\,,$$

from which we get the entropy per unit length,

$$S = -\frac{\partial F}{\partial T} = \frac{\pi}{3}cT\,.$$

This formula is to supply us with the intuition that the central charge counts the number of degrees of freedom in the theory.

3.3 Torus partition function

Mapping the plane onto a cylinder, we arrive at a torus by imposing two discrete identifications of w. Mapping onto a torus preserves all local operators, but only L_0 and $\bar{L}_0$ survive as generators of the global conformal group. Alternatively said, the $SL(2,\mathbb{C})$ gets broken down to $U(1)\times U(1)$ generated by $\{L_0\}\times\left\{\bar{L}_0\right\}$. On the complex plane, $L_0 \pm \bar{L}_0$ generate dilatations and rotations; here that role is inherited by $L_0^{\text{cyl}} \pm \bar{L}_0^{\text{cyl}}$. The identifications in w can be parametrized with two complex numbers, giving the most general way of defining the torus

$$w \sim w + 2\pi n_1\alpha_1 + 2\pi n_2\alpha_2, \quad n_{1,2}\in\mathbb{Z}$$

We define the complex structure on the torus as $\tau = \alpha_2/\alpha_1 = \tau_1 + i\tau_2$, parametrizing the family of distinct tori with $w \sim w + 2\pi n_1 + 2\pi n_2 \tau$. The isometry group acting on this parameter is the torus modular group $SL(2,\mathbb{Z})/\mathbb{Z}_2$. For example, we can take $\tau \to \tau + 1$, which we will call the T-transformation, or $\tau \to -1/\tau$, which will be referred to as S-transformation.

Now we conventionally define $Re(w)$ as space direction and $Im(w)$ as time, and observe the imaginary time translation by $2\pi\tau_2$ is accompanied by translation in space by $2\pi\tau_1$. Then the torus partition function can be written as:

$$\begin{aligned} Z = \int e^{-S} &= tr\left(e^{2i\pi\tau_1 P} e^{-2\pi\tau_2 H}\right) \\ &= tr\left[q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}\right], \end{aligned} \tag{3.3.1}$$

where we used $H = L_0^{\text{cyl}} + \bar{L}_0^{\text{cyl}}$ and $P = L_0^{\text{cyl}} - \bar{L}_0^{\text{cyl}}$, $q = e^{2i\pi\tau}$ and the trace is taken over all the states in the Hilbert space. Note the modular parameter plays the role of inverse temperature in the canonical partition function.

3.4 Cardy formula

Natural question to ask is how does the modular invariance constrain the theory given by the partition function (3.3.1)? Under T-transformation, we thus require $h - \bar{h} \in \mathbb{Z}$. Under S-transformation we are free to take $q \to q' = e^{-2\pi i/\tau}$. In order to simplify things, let us consider a limit in which $L_0 = \bar{L}_0$, with $q \to 1$, $q \in \mathbb{R}$. We can rewrite the partition function in terms of the density $\rho(\Delta)d\Delta$ of fields that have conformal dimension in the range $[\Delta, \Delta + d\Delta]$, where we set $q = 1 - \epsilon$, with $\epsilon \ll 1$

$$
\begin{aligned}
Z \propto \int \rho(\Delta) q^{\Delta} d\Delta &= \int \rho(\Delta) e^{\Delta \log q} d\Delta \\
&\approx \int \rho(\Delta) e^{-\Delta\epsilon} d\Delta
\end{aligned}
$$

We may now transform this to get the formula for the density of states

$$
\rho(\Delta) \propto \int Z e^{\Delta\epsilon} d\epsilon \,. \tag{3.4.1}
$$

On the other hand, as $q' \to 0$, the partition function is $Z \sim q'^{-c/12} \approx e^{(2\pi)^2 c/12\epsilon}$. We can now evaluate the integral (3.4.1) at a saddle point as we let $\Delta \gg 1$

$$
\begin{aligned}
\rho(\Delta) &\propto \int e^{\epsilon\Delta + (2\pi)^2 c/12\epsilon} d\epsilon \\
&\sim e^{4\pi\sqrt{\frac{c}{6}h}}
\end{aligned}
$$

Taking the logarithm of the density of states we get an estimate for the entropy, $S \sim 4\pi\sqrt{\frac{c}{6}h}$. The result is known as Cardy formula.

3.5 Thermal 2-point functions

In this section we will briefly touch on the finite temperature 2-point function (3.2.2), in order to rewrite it into a form we will come across in this thesis. We assign different periodicities to w and $\bar{w}$, differentiating the left movers from the right movers along the compact direction. The correlation function (3.2.2) can then be written as

$$G_2(x,t) = \langle \phi(w,\bar{w})\phi(0)\rangle \propto \frac{(\pi T_L)^{2h_L}(\pi T_R)^{2h_R}}{\sinh^{2h_L}(\pi T_L x_+)\sinh^{2h_R}(\pi T_R x_-)}\,, \tag{3.5.1}$$

where we have set $w = ix_-$, $\bar{w} = ix_+$ and $x_\pm = t \pm x$. By taking the imaginary part of the Fourier transform of the correlator $G_2(x,t)$, we arrive at

$$\begin{aligned}\sigma &= \mathrm{Im}\int dt dx e^{i\omega t + ikx} G_2(x,t)\\ &\propto \frac{(2\pi T_L)^{2h_L-1}(2\pi T_R)^{2h_R-1}}{\Gamma(2h_L)\Gamma(2h_R)} \sinh\left(\frac{p_+}{2T_L} + \frac{p_-}{2T_R}\right) \times\\ &\quad \left|\Gamma\left(h_L + i\frac{p_+}{2\pi T_L}\right)\Gamma\left(h_R + i\frac{p_-}{2\pi T_R}\right)\right|^2 .\end{aligned} \tag{3.5.2}$$

where $p_\pm = (\omega \mp k)/2$. This result was independently obtained in [31] and [32].

Chapter 4

Hidden conformal symmetry

Here we study the hidden conformal symmetry of the Kerr black hole in the low frequency limit by developing a non-geometric holographic map relating the bulk modes to an expansion within the conformal field theory in terms of higher dimension operators. We find that the dual CFT must contain infinite towers of quasi-primary operators with positive conformal weights. However, the full Kerr geometry softly breaks the conformal symmetry, and induces a nontrivial running of the scaling dimensions of these operators.

In the following we briefly review the low frequency approach to solving the spin weight s massless wave equations in Kerr background, and present an exact solution in a low frequency expansion [33]. Then we build on the observation of [12], by identifying conformal weights of operators in the CFT dual to bulk modes. We conclude with a discussion of the newly identified symmetry; even though it displays some peculiarities in the bulk, it seems to be well defined from the point of view of a dual field theory.

4.1 Low frequency expansion

Exact low frequency solutions to the equations (2.2.2) and (2.2.3) can be obtained by following [33, 34, 35]. Let us begin by going back to the angular equation (2.2.2):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left(a^2\omega^2\cos^2\theta-2a\omega s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+E-s^2\right)S=0 \tag{4.1.1}$$

As shown in [34], the solution in a small ω expansion is written in terms of an infinite series of Jacobi polynomials $P_j^{(\alpha,\beta)}(y)$

$$S=e^{a\omega x}\left(\frac{1-y}{2}\right)^{|m+s|/2}\left(\frac{1+y}{2}\right)^{|m-s|/2}{}_sU_{\ell m}(y)$$

where $y=\cos\theta$ and

$${}_sU_{\ell m}=\sum_{j=0}^{\infty}c_jP_j^{(|m+s|,|m-s|)}(y)\,. \tag{4.1.2}$$

The expansion for U will be well-defined if each c_j is determined and finite. Inserting (4.1.2) into (4.1.1) leads to a 3-term recurrence relation for the coefficients c_j:

$$\begin{aligned} B_0c_0+A_0c_1 &= 0\\ A_jc_{j+1}+B_jc_j+C_jc_{j-1} &= 0,\quad j=1,2,\ldots \end{aligned}$$

where

$$
\begin{aligned}
A_j &= 2a\omega \frac{(j+|m+s|+1)(j+|m-s|+1)(2j+|m+s|+|m-s|+2-2s)}{(2j+|m+s|+|m-s|+2)(2j+|m+s|+|m-s|+3)} \\
B_j &= E+(a\omega)^2-\frac{1}{2}(2j+|m+s|+|m-s|)(2j+|m+s|+|m-s|+2) \\
&\quad +2a\omega \frac{s(|m+s|-|m-s|)(|m+s|+|m-s|)}{(2j+|m+s|+|m-s|)(2j+|m+s|+|m-s|+2)} \\
C_j &= -2a\omega \frac{j(j+|m+s|+|m-s|)(2j+|m+s|+|m-s|+2s)}{(2j+|m+s|+|m-s|-1)(2j+|m+s|+|m-s|)}
\end{aligned}
$$

From the coefficients A_j, B_j, and C_j we can observe that for large j the coefficient c_j either blows up as

$$
c_j \propto (a\omega)^{-j}\,\Gamma\left(j+\frac{|m+s|+|m-s|+1}{2}+s\right),
$$

or uniformly converges

$$
c_j \propto \frac{(-a\omega)^j}{\Gamma\left(j+\frac{|m+s|+|m-s|+3}{2}-s\right)}.
$$

Convergence will require that eigenvalue E satisfies a transcendental equation, most conveniently expressed as a continued fraction using the recurrence relations satisfied by coefficients c_j. We will study such continued fractions in more detail when we get to the radial equation, but the same approach may be applied here. The solution to this transcendental equation in a low frequency regime admits a power series expansion in $a\omega$, with the result

$$E = \ell(\ell+1) - \frac{2s^2 ma\omega}{\ell(\ell+1)} + \mathcal{O}(a\omega)^2 \,. \tag{4.1.3}$$

Now we turn to the radial equation (2.2.3):

$$\begin{aligned}\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \Big(\Big[\left(r^2+a^2\right)^2\omega^2 - 4aMr\omega m - 2iM(r^2-a^2)\omega s \\ +a^2m^2 + 2ia(r-M)ms\Big]\Delta^{-1} + 2ir\omega s - E + s(s+1) - a^2\omega^2\Big)R \;&=\; 0\end{aligned} \tag{4.1.4}$$

which may be tackled in a similar way as studied in [33, 36]. Analogously to the solutions of angular equation, $R(r)$ is expressed as a series of hypergeometric functions. Defining a rescaled radial coordinate $x = \omega(r_+ - r)/\epsilon\kappa$, and the constants $\epsilon = 2M\omega$, $\kappa = \sqrt{1-(a/M)^2}$ and $\tau = (\epsilon - ma/M)/\kappa$, the radial function is factored as

$$R_s(x) = e^{i\epsilon\kappa x}(-x)^{-s-\frac{i}{2}(\epsilon+\tau)}(1-x)^{\frac{i}{2}(\epsilon-\tau)}P(x)\,. \tag{4.1.5}$$

The function $P(\rho)$ then satisfies the differential equation

$$\begin{aligned}x(1-x)P''(x) + \left[1+s-i\epsilon-i\tau-(2-2i\tau)\,x\right]P'(x) + (\nu+i\tau)\,(\nu+1-i\tau)\,P(x) \\ = 2i\epsilon\kappa\left[-x\,(1-x)\,P'(x) + (1-s+i\epsilon-i\tau)\,xP(x)\right] \\ + \left[-E + \epsilon q - \tfrac{1}{4}\epsilon^2 q^2 + \nu\,(\nu+1) + \epsilon^2 - i\epsilon\kappa\,(1-2s)\right]P(x)\end{aligned}$$

where a term $\nu\,(\nu+1)\,P(x)$ was added to both sides of the equation. Then $P(x)$ admits the series expansion

$$P(x) = \sum_{n=-\infty}^{\infty} a_n\,{}_2F_1(n+\nu+1-i\tau, -n-\nu-i\tau; 1-s-i\epsilon-i\tau; x) \tag{4.1.6}$$

where the coefficients a_n satisfy a three-term linear recursion relation

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \tag{4.1.7}$$

with

$$\begin{aligned}
\alpha_n &= \frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)} \\
\beta_n &= -E + 2\epsilon^2 - q^2\epsilon^2/4 + (n+\nu)(n+\nu+1) + \frac{\epsilon(\epsilon-mq)(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)} \\
\gamma_n &= -\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)},
\end{aligned}$$

This is a direct consequence of the following identities satisfied by the hypergeometric functions:

$$\begin{aligned}
xF_{n+\nu} &= -\frac{(n+\nu+1-s-i\epsilon)\,(n+\nu+1-i\tau)}{2\,(n+\nu+1)\,(2n+2\nu+1)}F_{n+\nu+1} \\
&\quad +\frac{1}{2}\left[1+\frac{i\tau\,(s+i\epsilon)}{(n+\nu)\,(n+\nu+1)}\right]F_{n+\nu} \\
&\quad -\frac{(n+\nu+s+i\epsilon)\,(n+\nu+1-i\tau)\,(n+\nu+i\tau)}{2\,(n+\nu)\,(2n+2\nu+1)}F_{n+\nu-1}
\end{aligned}$$

and

$$\begin{aligned}
x(1-x)F'_{n+\nu} &= \frac{(n+\nu+1-s-i\epsilon)(n+\nu+1-i\tau)(n+\nu+i\tau)}{2(n+\nu+1)(2n+2\nu+1)}F_{n+\nu+1} \\
&+\frac{1}{2}(s+i\epsilon)\left[1+\frac{i\tau(1-i\tau)}{(n+\nu)(n+\nu+1)}\right]F_{n+\nu} \\
&-\frac{(n+\nu+s+i\epsilon)(n+\nu+1-i\tau)(n+\nu+i\tau)}{2(n+\nu)(2n+2\nu+1)}F_{n+\nu-1}
\end{aligned}$$

where we used $F_{n+\nu}$ as shorthand for ${}_2F_1(n+\nu+1-i\tau,-n-\nu-i\tau;1-s-i\epsilon-i\tau;x)$.

There exist standard methods for solving such recursion relations, as discussed in [37]. The general solution can be expressed as a linear combination of two independent solutions since, for example, one can choose arbitrary initial values for a_0 and a_1. One solution has coefficients that diverge as $|n| \to \infty$ and is called the dominant solution. The other solution, of most interest for the present work, is the minimal solution, where the a_n converge[1] at large $|n|$. This solution must be obtained by tuning a_1 with respect to a_0.

Furthermore, if the a_n converge, a continued fraction equation may be set up to determine the value of the eigenvalue ν. This may be arranged by solving for ν in two different ways: by setting $a_0 = 1$ and evolving the minimal solution to $n = \infty$ or by evolving the minimal solution to $n = -\infty$. To see this we define the ratios

$$R_n = \frac{a_n}{a_{n-1}}\,, \qquad L_n = \frac{a_n}{a_{n+1}}$$

so that R_n converges as $n \to \infty$ and L_n converges as $n \to -\infty$. Then, the three-term

[1] or at least diverge less rapidly

recurrence relation (4.1.7) may be rewritten in terms of raising and lowering ratios

$$R_n = -\frac{\gamma_n}{\beta_n + \alpha_n R_{n+1}}\,, \qquad L_n = -\frac{\alpha_n}{\beta_n + \gamma_n L_{n-1}}$$

which may then be developed as convergent continued fractions. From the low frequency expansion of R_n we obtain $a_n \propto \epsilon^n$ for $n \geq 1$ and $a_0 = 1$. For negative values of n we may use L_n to deduce the behavior of coefficients. We find $a_n \propto \epsilon^{|n|}$ for $n \geq -\ell$ and with at least $\mathcal{O}(\epsilon^\ell)$ or subleading corrections beyond that. These continued fractions yield the equation

$$R_1 L_0 = 1$$

which generates a transcendental equation for ν. This equation may be solved as a low frequency expansion in ϵ, yielding a solution to the Teukolsky equation infalling on the future outer horizon. The leading terms in the expansion for ν are

$$\begin{aligned} \nu \ &= \ \ell - \frac{\epsilon^2}{2\ell+1}\left[2 + \frac{s^2}{\ell(\ell+1)} + \frac{\left(\ell^2 - s^2\right)^2}{(2\ell-1)2\ell(2\ell+1)} \right. \\ & \qquad \left. - \frac{\left((\ell+1)^2 - s^2\right)^2}{(2\ell+1)(2\ell+2)(2\ell+3)}\right] + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{4.1.8}$$

4.2 Bulk field/CFT operator map

In a low frequency expansion, the exact solution (4.1.5) may be expanded as a regular series in ϵ. The leading term is

$$R_s^0(x) = e^{i\epsilon\kappa x}(-x)^{-s-\frac{i}{2}(\epsilon+\tau)}(1-x)^{\frac{i}{2}(\epsilon-\tau)}\,{}_2F_1(\nu+1-i\tau, -\nu-i\tau; 1-s-i\epsilon-i\tau; x)\,. \tag{4.2.1}$$

Castro, Maloney and Strominger (CMS) [12] considered the scalar, $s = 0$, case of the wave equation and noticed that in a low frequency expansion $\omega M \ll 1$ the leading order term in the radial equation in the near-region, where $\omega r \ll 1$, reduces to a hypergeometric equation. They then showed that the full solution in this limit transformed as a representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, broken to $U(1) \times U(1)$ when the periodic identification of $\phi \sim \phi + 2\pi$ is taken into account. This led them to propose a hidden Kerr/CFT duality, with a scalar mode with angular momentum ℓ being identified with a CFT operator of conformal weight $(h_L, h_R) = (\ell, \ell)$. If one further speculates that the hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ extends to a left-right Virasoro algebra with central charges $(c_L, c_R) = 12J$, then the Cardy formula for the CFT entropy agrees exactly with the Kerr horizon entropy $S = \mathrm{A}/4$.

In the following we will expand on [12] by showing that the entire set of higher order frequency corrections can be organized into a CFT-like expansion. The precise statement is that the scaling dimensions run with frequency, which implies the CFT is deformed away from its exact conformal fixed point. Unfortunately we will see that the exact fixed point is dual to the $M = 0$ solution, that is flat spacetime.

We may perform a transformation $x \to \frac{x}{x-1}$ in the argument of the hypergeometric function to give

$$\begin{aligned} R_s^0(x) &= e^{i\epsilon\kappa x}(-x)^{-s-\frac{i}{2}(\epsilon+\tau)}(1-x)^{\frac{i}{2}(\epsilon+\tau)-\nu-1} \times \\ & \quad {}_2F_1\left(\nu+1-i\tau, 1-s-i\epsilon+\nu; 1-s-i\epsilon-i\tau; \frac{x}{x-1}\right), \end{aligned}$$

which agrees with eqn. (6.1) in [12], upon replacing ν with its low frequency limit ℓ, setting $s = 0$, and dropping the first factor, as appropriate for the near-region. The argument of [12] proceeds by noting that (4.2.1) solves the equation

$$\mathcal{H}^2\psi_0 = \bar{\mathcal{H}}^2\psi_0 = \ell(\ell+1)\psi_0\,,$$

where $\mathcal{H}^2$ and $\bar{\mathcal{H}}^2$ are the Casimir operators of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ algebra generated by

$$\begin{aligned}
H_1 &= ie^{-2\pi T_R\phi}\left(\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\Delta^{1/2}}\partial_t\right)\\
H_0 &= \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t \qquad (4.2.2)\\
H_{-1} &= ie^{2\pi T_R\phi}\left(-\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\Delta^{1/2}}\partial_t\right)
\end{aligned}$$

and

$$\begin{aligned}
\bar{H}_1 &= ie^{-2\pi T_L\phi+\frac{t}{2M}}\left(\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right)\\
\bar{H}_0 &= -2iM\partial_t \qquad (4.2.3)\\
\bar{H}_{-1} &= ie^{2\pi T_L\phi-\frac{t}{2M}}\left(-\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right)
\end{aligned}$$

which obey

$$[H_0, H_{\pm1}] = \mp iH_{\pm1}\,, \qquad [H_{-1}, H_1] = -2iH_0 \qquad (4.2.4)$$

and likewise for the others. Here we define left and right temperatures

$$T_L = \frac{M^2}{2\pi J}\,, \quad T_R = \frac{\sqrt{M^4-J^2}}{2\pi J}\,.$$

For convenience, we identify $L_n = -iH_n$ and $\bar{L}_n = -i\bar{H}_n$ so that the L_n's satisfy the standard form of the Witt algebra

$$[L_n, L_m] = (n - m)L_{n+m} \,.$$

We begin by investigating bulk modes that satisfy a lowest weight condition, which should be dual to primary operators in the CFT. Imposing the equations $L_1\psi(r, t, \phi) = \bar{L}_1\psi(r, t, \phi) = 0$ yields the solution

$$\psi(r, t, \phi) \propto \left(rr_+ - a^2\right)^{-iam/r_+} e^{im\phi - i\omega t}$$

and the condition

$$\omega = am/\left(2Mr_+\right) \,. \tag{4.2.5}$$

The conformal weights are

$$(h_L, h_R) = (\frac{iam}{r_+}, \frac{iam}{r_+}) \,.$$

So this will solve the scalar field equation of motion in Kerr if we further identify the Casimir with $\ell(\ell + 1)$

$$\left(L_0^2 - \frac{1}{2}\left(L_1L_{-1} + L_{-1}L_1\right)\right)\psi(r, t, \phi) = \ell(\ell + 1)\psi(r, t, \phi) \,, \tag{4.2.6}$$

which implies $h_L(h_L - 1) = \ell(\ell + 1)$ and $h_L = \ell + 1$ for the positive solution.

The inner product of these primaries is rather different from the usual Klein-Gordon norm in the Kerr background. The inner product of the CFT must yield conjugation that switches $L_1 \leftrightarrow L_{-1}$ and leaves L_0 invariant. This is accomplished by

Hermitian conjugation, followed by $\phi \to -\phi$ and $t \to -t$. This suggests the symmetry may be interpreted directly as acting in an analytic continuation of the Kerr geometry where $\phi \to i\phi$ and $t \to it$.

We find that the large r falloff $r^{-(h_L+h_R)/2}$ of a mode allows us to read off the conformal weight of the dual CFT operator $\Delta = h_L + h_R$. It is worth mentioning that in the usual AdS/CFT correspondence, the radial fall-off in Poincare coordinates is instead of the form $\tilde{r}^{-(h_L+h_R)}$. Thus if one inferred some effective AdS metric from the Kerr Laplacian in the near region, the relation between coordinates is of the form $r \sim \tilde{r}^2$ at large r.

4.3 Conformal weights for general spin

From (4.2.1) we can generalize the above to the higher spin s fields, and include the higher powers of $M\omega$ on the right hand side of (4.2.6) by using the expansion (4.1.3). In the near region, the large r falloff of (4.2.1) takes the form $r^{-s-\nu-1}$. To extract the behavior of the primary field we must also take into account the normalization of the component vectors used to set up the Teukolsky equation. In order to do that, we have to go back to the Teukolsky assignment of spin-s fields and identify solutions of the Teukolsky equation with physical quantities. The details are presented in [3]; here we will only refer to the outgoing components of the vector and tensor perturbations, $F_{\mu\nu}\bar{m}^\mu n^\nu = \psi_{-1}\Sigma$ and $R_{\alpha\beta\gamma\delta}n^\alpha\bar{m}^\beta n^\gamma\bar{m}^\delta = \psi_{-2}\Sigma^2$. Thus, we extract the large r behavior of the outgoing component of the vector potential A_μ for spin -1, and the behavior of the outgoing component of the graviton $g_{\mu\nu}$ for spin -2, in asymptotically Minkowski coordinates. This leads us to identify the conformal weight of a higher spin mode with

$$\Delta = -2|s| + 2\nu + 2\,. \tag{4.3.1}$$

Therefore, at leading order in $M\omega$, all the massless bulk fields have $\Delta = 2$ for the lowest nontrivial modes of angular momentum.

When the higher order $M\omega$ terms in the Teukolsky equation are included, the $SL(2,\mathbb{R})$ symmetry associated with the L_n's is softly broken. We expect the bulk scalar fields to be dual to CFT operators involving a sum of higher dimension operators. The conformal dimensions of these operators may be read off by examining the large r falloff of the expansion for the exact radial mode function (4.1.6), yielding a prediction for the dimensions of other CFT operators that must be present

$$\begin{aligned} \Delta &= -2|s| + 2n + 2\nu + 2 \qquad n > -\nu - 1 \\ &= 2|s| - 2n - 2\nu \qquad n < -\nu \end{aligned}$$

which are again all positive.

In this way, each term in (4.1.6) can be interpreted as a higher dimension correction in the mapping between the bulk mode and CFT operators. Because (4.1.6) reproduces the exact mode function for any finite r, one may deduce the exact two point function for scattering of massless modes off Kerr, including all higher $M\omega$ corrections, generalizing the lead-order matching noted in [12].

4.4 A few comments on the CFT dual

There are now a number of puzzles we need to address. The $\bar{L}_n$ generators act on functions that may be written using the basis $R(r)e^{im\phi-i\omega t}$, but as we see from the exponential prefactors in (4.2.3), the $\bar{L}_1$ and $\bar{L}_{-1}$ generators shift the m and ω eigenvalues by imaginary amounts. The shift in ω means the low frequency approximation leading to (4.2.1) can no longer be trusted, so that the whole $SL(2,\mathbb{R})$ associated with these generators is strongly broken down to the $U(1)$ subgroup generated by $\bar{L}_0$.

This leads us to the unfamiliar situation, where the eigenvalue m must be analytically continued to imaginary values to construct a mode dual to a primary CFT operator. However this is not entirely unexpected, since the space of infalling modes is a superset containing the quasi-normal modes of Kerr, studied, for example in [38]. Likewise, the case of quasi-normal modes of the 3-dimensional black hole have been studied in [39]. These modes have complex eigenvalues for ω so it is perhaps not too surprising we also wind up with complex eigenvalues for m prior to imposing periodicity of ϕ. This phenomenon is encountered in a similar context in [40].

However, as indicative from (4.2.5), the primary modes with respect to L_n do take us out of the low frequency limit, as the frequency condition for a primary field (4.2.5) becomes

$$\omega = \frac{\ell+1}{2iM}\,. \tag{4.4.1}$$

We may still use the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ representations of the leading order wave equation to organize the expansion of the higher order corrections. A priori we have no reason to expect convergence when we relate the associated CFT operators with bulk operators, but nevertheless, the mode function expansion (4.1.6) happens to converge for all finite r.

Of course, as noted in [12], $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is explicitly broken once ϕ is periodically identified, which projects out all noninteger m modes. So in this way, we see how this can be a low frequency symmetry of the Kerr modes prior to periodic identification, without it being realized manifestly in the spectrum. For example, such $SL(2,\mathbb{R})$ towers are not observed in the numerically determined quasi-normal mode spectrum [38].

The parameter ν serves a purpose of renormalized angular momentum, which can be interpreted as arising from the soft $SL(2,\mathbb{R})$ symmetry breaking terms in the low frequency wave equation. Having found a set of scaling dimensions associated with the exact solution of the Teukolsky equation, we are confronted with the problem that $\nu = \ell + \mathcal{O}(M^2\omega^2)$ as shown in (4.1.8). This means the scaling dimensions run with frequency – another sign that conformal symmetry is broken away from the $M\omega = 0$ fixed point. At first sight this seems rather disappointing: if we wish to study the conformal fixed point, we are forced to set $M = 0$. To retain a smooth geometry, this limit must be taken with $a < M$ which takes us to flat spacetime.[2] To keep the generators (4.2.2) and (4.2.3) well-defined, one must also rescale the time coordinate, keeping $\tilde{t} = t/M$ finite and the dimensionless temperatures T_L and T_R fixed. Thus the metric becomes $\mathbb{R}^3$ times a null direction $\tilde{t}$

$$ds^2 = 0d\tilde{t}^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,. \tag{4.4.2}$$

Certainly we see no sign of a nontrivial central charge $c = 12J$ associated with an exact CFT dual to flat spacetime. One might have expected extremal Kerr to

[2] One may also consider the $M \to 0$ limit with either fixed a or with fixed J. In each case, the limit of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators are not Killing vectors of the limiting metric, but rather conformal symmetries of the massless field equations.

emerge at the fixed point, but this simply does not occur in a small $M\omega$ limit. The generators (4.2.2) and (4.2.3) are not isometries of the metric (4.4.2). Moreover, in the extremal limit, they do not match the asymptotic symmetry generators of the NHEK geometry found in [9, 10, 11][3]. Rather they correspond to conformal transformations of (4.4.2) that leave the massless wave equations invariant. Thus the hidden Kerr/CFT correspondence does not seem easily generalized to massive modes.

Of course our scaling dimension computations are only valid at "strong" coupling where the gravitational solution is smooth. There could still be a nontrivial CFT with $c = 12J$ with conformal dimensions that match those obtained here when its strong coupling limit is taken. Studies of the near super-radiant modes of extremal Kerr provide strong evidence for such a conformal field theory [9, 16]. While the two limits do not seem to be smoothly connected within the realm of smooth gravity solutions[4], they may well be connected within the exact microscopic CFT. Similar phenomena are observed in the duality between D1,D5-brane backgrounds and CFT.

[3]For further work in this direction see [41, 42].

[4]For example, the super-radiant modes do not satisfy the low frequency limit needed to obtain the symmetry studied here.

Chapter 5

Generalizations and Schwarzschild limit

In the previous chapter we showed how to organize the low frequency expansion of spin weight s bulk fields by using the hidden $SL(2,\mathbb{R})$, providing a map between bulk modes and higher dimension operators in the CFT. The CFT interpretation has been shown to be far from complete, raising more questions than providing answers. Nevertheless, it is a promising novel approach to a longstanding problem.

The idea we wish to further pursue is that the underlying hidden conformal field theory description may be studied via low frequency scattering, rather than by simply looking for geometric isometries. This opens up the possibility that even the Schwarzschild black hole may have a CFT dual. By studying deformations of the low frequency scalar wave equation in Kerr background, we test the robustness of the low frequency symmetry and provide insight into its possible manifestations.

In this chapter, the hidden CFT generators (4.2.2), (4.2.3) are generalized to a one-parameter family. For special values of the parameter, a contraction to a single

$SL(2,\mathbb{R})$ factor generates symmetries of the scalar field equation in Schwarzschild black hole background. We find agreement with Schwarzschild symmetry generators found in [43]. Moreover, if we assume that the $SL(2,\mathbb{R})$ factors are enhanced to full Virasoro symmetries underlying the CFT, state counting in the CFT is able to reproduce the exact Kerr entropy. Finally we speculate on the connection between this hidden CFT and a more fundamental CFT describing the black hole. In the large damping limit, hints of CFT structure also emerge [44], and we are able to reproduce the spectrum of the Kerr quasinormal modes from a particular choice of our free parameter.

5.1 Deforming the wave equation

We start from the equation for a massless scalar field $\psi(t,r,\theta,\phi)$ propagating in Kerr background. As before, with the following ansatz

$$\psi \sim e^{im\phi - i\omega t} S(\theta) R(r),$$

the spin zero Teukolsky equation is separated into an angular and radial part. In the low frequency limit $M\omega \ll 1$, the angular equation reduces to a Laplacian on S^2, with eigenfunctions being spherical harmonics and eigenvalues $K_\ell \approx \ell(\ell+1)$, $\ell = 0, 1, \ldots$ We will not focus any further on the angular equation and its properties beyond zeroth order solution; this was dealt with in more depth in the previous chapter. For the radial equation, we may rewrite (2.2.3) in a somewhat more suggestive form [12]

$$\begin{aligned}\Big[\partial_r\left(\Delta\partial_r\right)+\tfrac{(2Mr_+\omega-am)^2}{(r-r_+)(r_+-r_-)}-\tfrac{(2Mr_-\omega-am)^2}{(r-r_-)(r_+-r_-)}&\\ +(r^2+2M(r+2M))\omega^2-K_l\Big]\,R(r) &= 0\,. \end{aligned}\tag{5.1.1}$$

We wish to consider a low frequency, near region limit

$$r\omega \ll 1\,, \qquad M\omega \ll 1, \tag{5.1.2}$$

following [12]. This allows one to drop the $(r^2+2M(r+2M))\omega^2$ term in (5.1.1), at which point the equation reduces to hypergeometric form.

If one is interested in Kerr black holes far from extremality, another interesting possibility arises. Namely, one can demand that $r-r_-$ be sufficiently large that the order ω and higher terms coming from the pole near $r\to r_-$ in (5.1.1) be subleading. That is, we may introduce the deformation parameter κ and deform (5.1.1) to

$$\left[\partial_r\left(\Delta\partial_r\right)+\frac{\left(2Mr_+\omega-am\right)^2}{\left(r-r_+\right)\left(r_+-r_-\right)}-\frac{\left(2M\kappa r_+\omega-am\right)^2}{\left(r-r_-\right)\left(r_+-r_-\right)}\right]R(r)=l(l+1)R(r), \tag{5.1.3}$$

This leaves the low frequency limit unchanged, as long as the two constraints

$$\begin{aligned}\frac{\kappa M^2 am\omega}{(r-r_-)(r_+-r_-)} &\ll 1,\\ \frac{\kappa^2M^4\omega^2}{(r-r_-)(r_+-r_-)} &\ll 1,\end{aligned}\tag{5.1.4}$$

are satisfied. A version of this deformation for the Schwarzschild black hole was considered in [43]. It should be noted that these conditions are implied by the near-region condition (5.1.2) as long as $r - r_-$ does not vanish. Thus, these conditions are a rather weak modification of the near-region limit.

Furthermore, one might consider deforming the positions of the singularities and coefficients in (5.1.3) in an arbitrary way, such that the wave equation reduces to (5.1.1) as $\omega \to 0$. However, if the coefficient involving the singularity at $r = \infty$ or at $r = r_+$ is deformed, the low energy solutions to the wave equation are changed in a drastic way, since the coefficients control the divergence of the solutions at the singular points. Shifting the positions of these singularities produces a deformation that could only be explained by an action involving more than two time derivatives, which we choose not to consider in the present work. However, the inner horizon $r = r_-$ is a special case, because we expect the full nonlinear solution for a perturbation of Kerr to become singular there. Including back-reaction is expected to yield an asymptotically null spacelike singularity capping the would-be inner horizon [45]. Since the low energy linearized wave equation is not relevant at this singular surface near $r = r_-$ it is natural to explore deformations of the wave equation near this point. The κ deformation in (5.1.3) is the unique such deformation of the linearized equation of motion, yielding an equation of motion second order in time derivatives.

5.2 Constructing the $SL(2,\mathbb{R})$

We start with a set of vector fields

$$\begin{aligned} L_\pm &= e^{\pm\alpha t \pm \beta\phi} \left(g_\pm(r)\partial_r + h_\pm(r)\partial_\phi + k_\pm(r)\partial_t \right), \qquad (5.2.1) \\ L_0 &= \gamma\partial_t + \delta\partial_\phi \,, \end{aligned}$$

that are to satisfy an $SL(2,\mathbb{R})$ algebra,

$$\begin{aligned}[L_+, L_-] &= 2L_0, \\ [L_\pm, L_0] &= \pm L_\pm \,.\end{aligned}$$

The requirement that L_0 is an eigenvector of a state $\psi \sim e^{im\phi - i\omega t}R(r)S(\theta)$ sets γ and δ to constants. We also demand the quadratic Casimir reproduces the scalar field wave equation in the Kerr background in the near region low frequency approximation (5.1.2) subject to the additional constraints (5.1.4):

$$L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) \;=\; \partial_r(\Delta\partial_r) + f(r)\,,$$

where $f(r)$ is a function that involves no single derivatives in t or ϕ, except for $\partial_t\partial_\phi$.

Given these constraints, we claim the following is the most general functional form of such generators:

$$\begin{aligned}L_\pm &= e^{\pm\alpha t \pm \beta\phi}\left(\mp\sqrt{\Delta}\partial_r + \frac{C_2 - \delta r}{\sqrt{\Delta}}\partial_\phi + \frac{C_1 - \gamma r}{\sqrt{\Delta}}\partial_t\right), \\ L_0 &= \gamma\partial_t + \delta\partial_\phi\,,\end{aligned} \tag{5.2.2}$$

with constraints on parameters arising from imposing the $sl(2,\mathbb{R})$ algebra:

$$\begin{aligned} \alpha C_1 + \beta C_2 + M &= 0, \\ 1 + \alpha\gamma + \beta\delta &= 0. \end{aligned} \tag{5.2.3}$$

These determine α and β. The last three equations we impose are the ones identifying appropriate terms in the quadratic Casimir with ∂_ϕ^2, $\partial_t\partial_\phi$ and ∂_t^2 terms in the wave equation (5.1.3). The ∂_ϕ^2 term gives us the branches:

$$\begin{array}{ll} \delta = \pm a/\sqrt{M^2 - a^2} & \qquad \delta = 0 \\ C_2 = M\delta & \qquad C_2 = \pm a\,. \end{array}$$

The differing signs simply generate automorphisms of the algebra, so may be dropped in the following. Examining the two remaining terms gives:

$$\gamma\delta a^2 - C_1C_2 - r(2M\gamma\delta - C_2\gamma - C_1\delta) = -\frac{2Mr_+a}{r_+ - r_-}\left[r(1-\kappa) - (r_- - \kappa r_+)\right],$$

and

$$\gamma^2 a^2 - C_1^2 - 2r\gamma(M\gamma - C_1) = -\frac{4M^2r_+^2}{r_+ - r_-}\left[r(1-\kappa^2) - (r_- - \kappa^2 r_+)\right]\ .$$

The two possible branches are shown in Table 5.2.1 describing a one-parameter family of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators labeled by κ. The solution for the generators is

$\gamma = \frac{2Mr_+}{r_+ - r_-}(\kappa + 1)$	$\gamma = \frac{2Mr_+}{r_+ - r_-}(\kappa - 1)$
$\delta = \frac{2a}{r_+ - r_-}$	$\delta = 0$
$C_1 = \frac{2Mr_+}{r_+ - r_-}(\kappa r_+ + r_-)$	$C_1 = \frac{2Mr_+}{r_+ - r_-}(\kappa r_+ - r_-)$
$C_2 = M\delta$	$C_2 = a$

Table 5.2.1: Two branches of solutions for the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators.

$$\begin{aligned} L_\pm &= e^{\mp 2\pi T_R \phi}\Big[\mp\sqrt{\Delta}\partial_r - \tfrac{1}{2\pi T_H}\tfrac{r-M}{\sqrt{\Delta}}\left(\Omega\partial_\phi + \partial_t\right) \\ &\qquad + \tfrac{1}{2\pi\Omega(T_L+T_R)}\tfrac{r-r_+}{\sqrt{\Delta}}\partial_t\Big], \\ L_0 &= \frac{1}{2\pi T_H}\left(\Omega\partial_\phi + \partial_t\right) - \frac{1}{2\pi\Omega(T_L+T_R)}\partial_t, \end{aligned} \tag{5.2.4}$$

and

$$\begin{aligned} \bar{L}_\pm &= e^{\pm 2\pi\Omega(T_L+T_R)t \mp 2\pi T_L \phi}\Big[\mp\sqrt{\Delta}\partial_r + \tfrac{2Mr_+}{\sqrt{\Delta}}\left(\Omega\partial_\phi + \partial_t\right) \\ &\qquad + \tfrac{1}{2\pi\Omega(T_L+T_R)}\tfrac{r-r_+}{\sqrt{\Delta}}\partial_t\Big], \\ \bar{L}_0 &= -\frac{1}{2\pi\Omega(T_L+T_R)}\partial_t, \end{aligned} \tag{5.2.5}$$

where $T_H = \frac{\sqrt{M^2-a^2}}{4\pi M r_+}$ is Hawking temperature and $\Omega = \frac{a}{2Mr_+}$ angular velocity at the outer horizon, and we have introduced "CFT" temperatures as

$$T_R = \frac{\sqrt{M^2-a^2}}{2\pi a} \tag{5.2.6}$$

and

$$T_L = T_R\frac{1+\kappa}{1-\kappa}. \tag{5.2.7}$$

5.3 Connection with previous results

We note the generators (5.2.5) shift the frequency of a mode by the imaginary amount $2\pi\Omega(T_L+T_R)$. This means we only stay within the low frequency limit (5.1.2) provided we are close to the extremal Kerr limit, so that $T_R \ll 1$ (assuming κ is fixed). Outside this limit the descendants of some primary operator are no longer mapped to eigenfunctions of the low frequency scalar field equation. The other set of generators (5.2.4) do not suffer from this additional constraint.

Higher order corrections to the Teukolsky equation in the low frequency limit give rise to soft breaking of the conformal symmetry, and running of the anomalous dimensions, as described in the previous chapter. In addition, global identifications on the solution space by $\phi \to \phi + 2\pi$ explicitly breaks the symmetry algebra down to $U(1) \times U(1)$ generated by $(L_0, \bar{L}_0)$.

If we set $\kappa = r_-/r_+$ our results match (4.2.2), (4.2.3). For this choice of κ the pole term in (5.1.3) is exact, so the subsidiary constraints (5.1.4) may be dropped.

Furthermore, let us comment on two special cases where the general solutions (5.2.4) and (5.2.5) do not hold. For $\kappa = 1$ and general rotation parameter a, the right branch in Table 5.2.1 fails to yield a consistent solution to the constraint equations, so only the left branch generates an $SL(2,\mathbb{R})$. Since we only find one $SL(2,\mathbb{R})$ for general a we are unable to carry through the conjecture that the theory should be dual to a 2-dimensional CFT, so we do not pursue this case further.

The Schwarzschild case, $a = 0$, should likewise be treated as a special case. Here we find the constraint equations are inconsistent unless $\kappa = \pm 1$. For both values of κ, the same single copy of $SL(2,\mathbb{R})$ is found. This may be read off, for example, from the right branch of Table 5.2.1 by setting $\kappa = -1$ and $a = 0$. We find a single set of

$SL(2,\mathbb{R})_{Sch}$ generators

$$L_{\pm} = -e^{\pm t/4M}\left(\pm\sqrt{\Delta}\partial_r - \frac{4M(r-M)}{\sqrt{\Delta}}\partial_t\right), L_0 = -4M\partial_t. \tag{5.3.1}$$

Our results match those of Bertini, Cacciatori and Klemm [43], up to the algebra automorphism $L_{\pm} \to -L_{\pm}$ and $L_0 \to L_0$.

5.4 Quasinormal modes

It is well known that classical black holes are characterized by a discrete set of complex frequencies, named quasinormal modes. The quasinormal modes correspond to a certain set of boundary conditions, with waves purely outgoing at infinity and ingoing at the horizon. Two observations immediately jump to mind: a quantum theory of gravity should reproduce this spectrum; and if the states in this quantum theory are fully characterized by quasinormal modes, studying semiclassical physics outside the black hole horizon should teach us about this quantum theory.

The connection between quasinormal modes for three-dimensional black holes and CFT states has been made precise in [39] by looking at linearized perturbations[1] and showing an explicit agreement between quasinormal frequencies and the poles of the retarded correlation function in the CFT.

Likewise for the Kerr black hole, there is a discrete spectrum of quasinormal modes [43]. At large damping the imaginary part of the frequency increases approximately linearly with mode number, while the real part approaches a constant.

[1] The previous definition of quasinormal modes via ingoing flux at infinity does not make it if we put the system in a box. The way quasinormal modes were defined in asymptotically AdS backgrounds was to impose either Dirichlet boundary conditions at asymptotic infinity, or a vanishing flux $\mathcal{F} \sim \sqrt{-g}\,(R^*\partial_\mu R - c.c.)$. Both choices lead to same spectrum.

According to Keshet and Neitzke [44], at large damping is where one expects the CFT description to emerge, as the transmission and reflection amplitudes take a familiar CFT-like form. Moreover, a step towards this understanding has been made in [44], where the authors have obtained the quasinormal mode spectrum via a WKB approximation to the wave equation.

An interesting observation was made in [43] – the descendant states $(L_-)^n\,\psi(t,r,\phi)$ reproduce the large damping quasinormal spectrum of the Schwarzschild black hole. We speculate this might be the case with generators (5.2.5) as well, giving a connection between the low frequency hidden CFT, and some more fundamental underlying CFT that correctly describes the quasinormal modes.

Keshet and Hod [46] compute the quasinormal mode spectrum at large damping and obtain

$$\omega = -m\hat{\omega} - 2\pi i T_0(n+1/2), \tag{5.4.1}$$

to leading order in n, where $T_0 = T_H\, f(a/M)$ and $f(a/M)$ is a smooth, slowly varying function of angular momentum with $f(0) = 1$, that may be expressed in general using elliptic integrals.

We can choose the value of $\kappa(a)$ by solving

$$T_0 = \Omega T_R \frac{2}{1-\kappa}\,.$$

Then by defining the lowest weight state via

$$\begin{aligned} \bar{L}_0\Phi^{(0)} &= \bar{h}\Phi^{(0)}, \\ \bar{L}_+\Phi^{(0)} &= 0, \end{aligned}$$

it is easy to check the descendants $\Phi^{(n)} = \left(\bar{L}_-\right)^n \Phi^{(0)}$ reproduce the large n behavior of the spectrum (5.4.1).

5.5 Entropy

Following [12], we propose that the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry is promoted to a full left and right-moving Virasoro symmetry in the full quantum theory. The Cardy formula gives

$$S = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right) . \tag{5.5.1}$$

The T_L and T_R appearing in (5.2.6) and (5.2.7) may be matched with the left and right-moving CFT temperatures. The CFT inherits periodic identifications in the imaginary left and right moving directions, from the periodic identification of ϕ, and the action of the Virasoro generators.

The Bekenstein-Hawking entropy is

$$S_{BH} = 2\pi M r_+, \tag{5.5.2}$$

which agrees exactly with (5.5.1) provided we identify the central charge as

$$c_L = c_R = \frac{6a(1-\kappa)Mr_+}{\sqrt{M^2 - a^2}} . \tag{5.5.3}$$

We emphasize the identification of the central charge (5.5.3) is not an independent computation. Such a computation may well be possible, but would require a reworking of the Brown-Henneaux calculation [4] within the hidden CFT/low frequency framework.

The central charge depends in a nontrivial way on the deformation parameter κ. This indicates the dual description is actually a family of conformal field theories with a conformal deformation parameter. While we do not yet have enough data to specify the CFTs at hand in detail, there are many examples of such families of conformal field theories. A simple example is the level number of a conformal field theory based on an affine Lie algebra. A much more general set of examples, including continuous deformations of CFTs that change the central charge, appears in [47].

The formula (5.5.3) reduces to the result of [12] when $\kappa = r_-/r_+$ where $c_L = c_R = 12J$. There one can argue that in the extremal limit, the central charge follows from standard geometric argument [9]. However, a closer look shows that the low frequency symmetry generators of hidden conformal symmetry do not smoothly match the isometry generators of [18]. Thus, the fixed point of the low frequency conformal symmetry does not coincide with extremal Kerr [13]. Moreover, a strong argument for the non-renormalization of the central charges away from the extremal point is lacking for the case considered in [12]. The addition of the extra parameter κ in (5.5.3) does not help this situation.

In the Schwarzschild limit we only retain a single $SL(2,\mathbb{R})$ symmetry, which might be associated with conformal quantum mechanics. If we assume this alone is promoted to a full Virasoro symmetry, we find periodicity with respect to ϕ no longer fixes the CFT temperature. Rather we must return to the generators (5.3.1), where we can read off the temperature

$$T_{CFT} = T_H = \frac{1}{8\pi M}\,.$$

The central charge for conformal quantum mechanics dual to Schwarzschild is then predicted to be

$$c_{Sch} = 96M^3\,, \tag{5.5.4}$$

an apparently new result.

Chapter 6

Conformal symmetry at finite frequencies

In the previous chapter we found that the hidden conformal symmetry in a low frequency limit has a rich structure, generic for a wide class of black holes. These results are in a way both encouraging and disappointing, as they enable study of the hidden symmetry at low frequencies for a whole class of geometries that exhibit a horizon (even for positively curved de Sitter, as in [48]), but don't give precise information on the underlying physics. In what follows, we extend the treatment of the hidden symmetry to finite frequencies, to find the 1-parameter family of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ algebras singles out a particular value of the parameter, which may be indicative of the precise nature of the map between bulk fields and CFT primaries.

In this chapter we study in more detail the symmetry structure of the equations of motion for massless fields of general spin in a generic Kerr background without taking any limits. We find a hidden $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ symmetry structure which matches that of the global conformal group of a two-dimensional conformal field theory.

In order to construct the parameters that characterize the representations that appear, we solve an eigenvalue problem that may be expressed as a continued fraction equation, exhibiting solutions in a low frequency expansion and by numerical computation for finite frequency modes. We develop the numerical solution of the eigenvalue problem for quasinormal modes to determine the associated representations.

The Kerr mode functions lead to non-unitary representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ for the mode functions which reflect the non-invariance of the coordinate patch under this group. However the representations that appear match exactly what one expects of the BTZ black hole, for which the correspondence between gravity and a CFT at finite left/right temperatures is well-understood [39]. These results provide useful clues and constraints on the structure of the holographic dual to a generic Kerr black hole.

6.1 The $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ and massless fields in Kerr

We start by revisiting the Teukolsky equation for spin weight s fields. The solution takes the following form for angular quantum numbers ℓ and m

$$\psi = e^{-i\omega t} e^{im\phi} S_\ell^m(\theta) R_{\omega\ell m}(r)\,,$$

with S_ℓ^m being a spin weighted spheroidal harmonic, dependent on the spin weight s. This satisfies the angular equation

$$\left(\frac{d}{dy}(1-y^2)\frac{d}{dy} + \frac{1}{4}q^2\epsilon^2 y^2 - sq\epsilon y - \frac{m^2+s^2+2msy}{1-y^2} + E\right) S_\ell^m(y) = 0\,, \qquad (6.1.1)$$

with angular eigenvalue E, where $y = \cos\theta$, and $\epsilon = 2M\omega$, $q = a/M$ as before. For finite frequencies the eigenvalue is not known, but that does not prevent us from organizing the solution in terms of an expansion in Jacobi polynomials, just as we did in (4.1.2). The only thing we need to enforce is that the choice of E leads to a convergent series, which is done by solving a continued fraction equation.

As an aside, Jacobi polynomials $P_j^{(|m+s|,|m-s|)}(y)$ are often occurring when the underlying equation has $SU(2)$ symmetry. They are connected with the irreducible representations of $SU(2)$, the Wigner representations

$$d^j_{m,s}(\theta) \propto \left(\sin\frac{\theta}{2}\right)^{|m+s|}\left(\cos\frac{\theta}{2}\right)^{|m-s|} P_{j-m}^{(|m+s|,|m-s|)}(\cos\theta)\,.$$

In what follows, we will build our "Wigner" representations for the hidden conformal symmetry. The radial equation takes the form

$$\left(\Delta^{-s}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr} + [V_0(r) + V_{\omega\ell m}(r)]\right) R_{\omega\ell m}(r) = 0\,, \tag{6.1.2}$$

where the potentials V_0 and $V_{\omega\ell m}$ are defined as

$$\begin{aligned} V_0(r) &= \frac{r_+ - r_-}{r - r_+}\omega_+(\omega_+ - is) - \frac{r_+ - r_-}{r - r_-}\omega_-(\omega_- + is) \\ V_{\omega\ell m}(r) &= s(s+1) - E + \epsilon(\epsilon - is) + r^2\omega^2 + r\omega(\epsilon + 2is)\,, \end{aligned}$$

and we have introduced

$$\omega_\pm = \frac{2Mr_\pm\omega - am}{r_+ - r_-}\,.$$

Around a low frequency limit the radial equation solutions may be expanded in terms of hypergeometric functions, (4.1.6):

$$
\begin{aligned}
R_{\omega\ell m}(x) &= e^{i\epsilon\kappa x}(-x)^{-s-\frac{i}{2}(\epsilon+\tau)}(1-x)^{\frac{i}{2}(\epsilon-\tau)} \\
&\sum_{n=-\infty}^{\infty} a_n^\nu \, {}_2F_1\left(n+\nu+1-i\tau, -n-\nu-i\tau; 1-s-i\epsilon-i\tau; x\right) ,
\end{aligned}
\tag{6.1.3}
$$

where the solution is chosen to satisfy ingoing boundary conditions at the horizon. Here we find it convenient to use $x = \omega(r_+ - r)/\epsilon\kappa$, $\kappa = \sqrt{1-q^2}$, $\tau = (\epsilon - mq)/\kappa$. The parameter ν will ultimately determine the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ representations that appear in the mode function. In a low frequency expansion, $\nu = \ell + \mathcal{O}(\epsilon^2)$. For sufficiently small ϵ usually a single term in the expansion survives at leading order. Remarkably, it was found the series converges for all $r < \infty$ beyond the small ϵ limit.

If we define

$$
B_{\omega\ell m}(x) = (-x)^{s/2}(1-x)^{s/2}e^{-i\kappa\epsilon x}R^\nu_{\omega\ell m}(x) \,, \tag{6.1.4}
$$

we transform the radial equation into

$$
\begin{aligned}
\left(\frac{d}{dx}\Lambda\frac{d}{dx} + \left[\frac{\hat{\omega}_+^2}{x} + \frac{\hat{\omega}_-^2}{1-x} - \nu(\nu+1)\right]\right) B_{\omega\ell m} &= \\
= -\left[2i\kappa\epsilon\Lambda\frac{d}{dx} - i\kappa\epsilon(s-1)\frac{d\Lambda}{dx} - \epsilon^2\kappa\frac{d\Lambda}{dx} - E + \nu(\nu+1) + \frac{\epsilon^2}{4}\left(7+\kappa^2\right)\right] B_{\omega\ell m}
\end{aligned}
\tag{6.1.5}
$$

where

$$
\Lambda = -x(1-x), \qquad \hat{\omega}_\pm = \frac{s}{2} \pm i\omega_\pm \,.
$$

Now the first line of (6.1.5) is (up to a trivial modification) what is known as the q-form of the hypergeometric equation. This form is useful for exhibiting the $SL(2,\mathbb{R})$ structure of the solution. There are two ways to rewrite this first line as a quadratic Casimir of $SL(2,\mathbb{R})$, so all together we find a $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ structure. The generators are

$$
\begin{aligned}
J^{\pm} &= e^{\pm 2\pi T_R \phi}\left[-\sqrt{\Lambda}\partial_x \pm \frac{M}{T_R}\frac{T_L\Lambda' - T_R}{\sqrt{\Lambda}}\partial_t \pm \frac{1}{4\pi T_R}\frac{\Lambda'}{\sqrt{\Lambda}}\partial_\phi \pm \frac{s}{2\sqrt{\Lambda}}\right], \\
J^3 &= 2M\frac{T_L}{T_R}\partial_t + \frac{1}{2\pi T_R}\partial_\phi\,, \qquad (6.1.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{J}^{\pm} &= e^{\pm(2\pi T_L \phi - t/2M)}\left[-\sqrt{\Lambda}\partial_x \pm \frac{M}{T_R}\frac{T_L - T_R\Lambda'}{\sqrt{\Lambda}}\partial_t \pm \frac{1}{4\pi T_R}\frac{1}{\sqrt{\Lambda}}\partial_\phi \pm \frac{s\Lambda'}{2\sqrt{\Lambda}}\right], \\
\bar{J}^3 &= -2M\partial_t + s\,, \qquad (6.1.7)
\end{aligned}
$$

where we have introduced the parameters

$$
T_R = \frac{r_+ - r_-}{4\pi a}, \qquad T_L = \frac{r_+ + r_-}{4\pi a}\,. \qquad (6.1.8)
$$

These generators satisfy the $SL(2,\mathbb{R})$ algebra in the form

$$
\begin{aligned}
\left[J^+, J^-\right] &= 2J^3\,, \\
\left[J^\pm, J^3\right] &= \mp J^\pm\,,
\end{aligned}
$$

and likewise for barred generators. The barred and unbarred generators commute. This generalizes the proposed algebra of [12] to general spin weight s, up to algebra isomorphism. The Casimir operators

$$C = J^+J^- + J^3J^3 - J^3, \qquad \bar{C} = \bar{J}^+\bar{J}^- + \bar{J}^3\bar{J}^3 - \bar{J}^3,$$

each reproduce the term in (6.1.5)

$$C = \bar{C} = \frac{d}{dx}\Lambda\frac{d}{dx} + \frac{\hat{\omega}_+^2}{x} + \frac{\hat{\omega}_-^2}{1-x}\,.$$

It is worth pointing out that differential generators (6.1.6) and (6.1.7) appear in the literature on the relation between representations of $SL(2,\mathbb{R})$ and hypergeometric functions [49]. A related $SU(2)$ algebra may also be defined for the angular equation, and was used in [34] to give a compact derivation of the Press-Teukolsky identities [50]. All together, the expansion for a mode function in the Kerr geometry exhibits a hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SU(2) \times SU(2)$ symmetry, which curiously matches the near-horizon isometry of the string theory black holes for which the microscopic entropy counting is well-understood.

From (6.1.3) we observe $R_{\omega\ell m}(x)$ is invariant under the exchange $\nu \to -\nu - 1$, $n \to -n$. In order to build representations of $SL(2,\mathbb{R})$ under the action of (6.1.6) and (6.1.7), we decompose $B_{\omega\ell m}(x)$ into two independent modes:

$$B_{\omega\ell m}(x) = \sum_{n=-\infty}^{\infty} \left[\tilde{a}_n^{\nu} B_{\omega\ell m}^{n+\nu}(x) + \tilde{a}_n^{-\nu-1} B_{\omega\ell m}^{n-\nu-1}(x)\right]\,,$$

where

$$\begin{aligned} B_{\omega\ell m}^{n+\nu}(x) &= (-x)^{n+\nu-\frac{s}{2}-\frac{i}{2}(\epsilon-\tau)}(1-x)^{\frac{s}{2}+\frac{i}{2}(\epsilon-\tau)} \\ &\quad {}_2F_1\left(-n-\nu-i\tau, -n-\nu+s+i\epsilon; -2n-2\nu; 1/x\right) \end{aligned} \qquad (6.1.9)$$

and

$$\tilde{a}_n^\nu = a_n^\nu \frac{\Gamma(1-s-i\epsilon-i\tau)\,\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\,\Gamma(n+\nu+1-s-i\epsilon)}\,.$$

The invariance of $B_{\omega\ell m}(x)$ under $\{\nu \to -\nu-1, n \to -n\}$ is enforced via the assignment $a_0^\nu = a_0^{-\nu-1}$.

One can define the irreducible $SL(2,\mathbb{R})$ representations we will be interested in as realized on a set of basis functions $f_{m_0}^u$ via

$$\begin{aligned} J^3 f_{m_0}^u &= m_0 f_{m_0}^u\,, \\ J^\pm f_{m_0}^u &= (-u \pm m_0) f_{m_0\pm 1}^u\,. \end{aligned}$$

With the help of the hypergeometric identity

$$\frac{d}{dz}\left[z^a F(a,b,c;z)\right] = a z^{a-1} F(a+1,b,c;z)\ ,$$

where $z = 1/x$, we conclude that $e^{-i\omega t+im\phi}B_{\omega\ell m}^{n+\nu}(x) = f_{-i\tau}^{n+\nu}$ under the action of (6.1.6). For barred generators (6.1.7), we swap the first two arguments in the hypergeometric function and use the same identity to show $e^{-i\omega t+im\phi}B_{\omega\ell m}^{n+\nu}(x) = f_{s+i\epsilon}^{n+\nu}$.

In the notation $D(u,m_0)$ of [49], with respect to the product algebra $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ the representation associated with the n-th term of the expansion (6.1.4) is a direct product

$$D(n+\nu, s+i2M\omega)_L \times D\left(n+\nu, -i\frac{2M^2\omega - am}{\sqrt{M^2-a^2}}\right)_R, \tag{6.1.10}$$

which is a non-unitary representation of the algebra, with Casimir $(|n|+\nu)(|n|+\nu+1)$.

Later we will see that ν will be real provided ω is real, but will be complex for quasinormal modes. With respect to the generators defined above, the weights of representations are $ik_L - 2M\omega$, $ik_R - \frac{2M^2\omega - am}{\sqrt{M^2-a^2}}$, where $k_{L,R}$ is an integer. The condition that the representations collapse to a highest weight or lowest weight representation is that $\nu + i2M\omega$ or $\nu + i\frac{2M^2\omega - am}{\sqrt{M^2-a^2}}$ is an integer. This is generally not satisfied for real non-vanishing frequencies or momenta.

The original motivation for the expansion (6.1.3) was as a low frequency expansion. As we have seen, this is equivalent to organizing the expansion according to the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry, which moreover leads to a convergent expansion for general frequencies.

This expansion straightforwardly produces low frequency scattering amplitudes. In particular, the results of Page/Starobinsky [51, 52] can easily be recovered by retaining only the $n = 0$ term in (6.1.3), as shown in [36].

We will find the parameter ν becomes a function of frequency, determined by picking out a convergent solution to (6.1.3). We solve for this parameter in various different limits. But first we will investigate in more detail the connection to conformal field theory.

6.2 CFT/gravity mapping

In the above we have shown how a general mode function decomposes into irreducible representations of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ algebra. To make the meaning of these representations more clear it is helpful to compare to the analogous computation for the three-dimensional black hole in asymptotically anti-de Sitter spacetime [6], where the same symmetry structure appears, and the holographic dictionary is well-known.

6.2.1 BTZ example

Quasinormal modes have been studied in this context in [39] so we find it helpful to follow their notation. The BTZ metric can be written in the form

$$ds^2 = \frac{dz^2}{4z(1-z)^2} + \frac{1}{1-z}\left(-r_- dt + r_+ d\phi\right)^2 - \frac{z}{1-z}\left(r_+ dt - r_- d\phi\right)^2 , \qquad (6.2.1)$$

where infinity is $z = 1$ and the outer horizon sits at $z = 0$. Here r_+ and r_- are the radii of the outer and inner horizons. These are related to left and right temperatures in the CFT via (6.1.8) by replacing $a \to 1/2$:

$$T_R = \frac{r_+ - r_-}{2\pi}, \qquad T_L = \frac{r_+ + r_-}{2\pi} . \qquad (6.2.2)$$

For a scalar field of mass $\tilde{m}$, or a vector field of mass $\tilde{m}$ (with spin parameter $s = \pm 1$), an analog of the Teukolsky solution is

$$\Phi = e^{-ik_+ x^+ - ik_- x^-} B(z)$$

where

$$x^+ = r_+ t - r_- \phi, \qquad x^- = r_+ \phi - r_- t$$

and

$$\begin{aligned} k_+ + k_- &= \frac{\omega - k}{2\pi T_R} \\ k_+ - k_- &= \frac{\omega + k}{2\pi T_L} . \end{aligned}$$

The radial equation takes the hypergeometric form

$$z(1-z)\frac{d^2B}{dz^2}+(1-z)\frac{dB}{dz}+\left[\frac{k_+^2}{4z}-\frac{k_-^2}{4}-\frac{\tilde{m}^2+2s\tilde{m}}{4(1-z)}\right]B=0 \tag{6.2.3}$$

with

$$B(z)=z^{\alpha}(1-z)^{\beta}\,{}_2F_1(a,b;c;z) \tag{6.2.4}$$

where

$$\alpha=-\frac{ik_+}{2},\qquad \beta=\frac{1}{2}\left(1-\sqrt{1+\tilde{m}^2+2\tilde{m}s}\right),$$

$$a=\frac{k_+-k_-}{2i}+\beta,\qquad b=\frac{k_++k_-}{2i}+\beta,\qquad c=1+2\alpha\,.$$

The radial equation (6.2.3) is of the same form as the first line in (6.1.5) with the replacement $B(z)\to(1-z)^{1/2}\tilde{B}(z)$. However, one subtlety we should address is that a given hypergeometric equation has 24 different equivalent ways of writing the solution in terms of hypergeometric functions. These different solutions are related by Kummer transformations. From the viewpoint of the black hole, these presumably correspond to mode functions in different coordinate patches. To compare with the form of the solution (6.1.9), where x runs over the range $(-\infty,0)$ we perform a $z\to y=1-1/z$ Kummer transformation which maps the solution (6.2.4) to

$$B(x)=(-y)^{c-a-b+\beta}(1-y)^{b-\alpha-\beta}\,{}_2F_1(c-a,1-a;c+1-a-b;y)\,.$$

From here we may read off the parameters of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ representation. In the notation $D(\nu, m_0)$ of [49] we find

$$D\left(\beta - 1, i\frac{k_+ - k_-}{2}\right)_L \times D\left(\beta - 1, -i\frac{k_+ + k_-}{2}\right)_R \tag{6.2.5}$$

for which the quadratic Casimir is

$$C = \bar{C} = \frac{\tilde{m}^2 + 2\tilde{m}s}{4} \, .$$

These are non-unitary representations of $SL(2,\mathbb{R})$, although the Casimir agrees with what we expect for a discrete highest weight irreducible representation with conformal weight $h = \beta$. The non-unitarity may be attributed to the fact that the external coordinate patch of the BTZ black hole (6.2.1) is not invariant under global AdS isometries. When one takes a pure AdS limit, the coordinates (6.2.1) only cover one of an infinite number of patches needed to cover AdS (or more precisely the covering space of AdS). Thus, while modes on the covering space of AdS transform as a unitary highest weight representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, they are related to a nontrivial composition of non-unitary representations on different patches[1]. At the level of the mode functions, the mapping between the global mode functions and the BTZ patch mode functions will (and consequently the group representations will) follow analogous results in [54] for Rindler versus Minkowski spacetime. We conclude that the non-unitary representations (6.2.5) for a single BTZ patch can be viewed as descending from a unitary highest weight representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ corresponding to a primary operator in the CFT with conformal dimension β.

[1]This is treated explicitly for AdS_4 in [53], and the result for AdS_3 follows by decomposing the highest weight representations of $SO(3,2)$ into $SO(2,2)$.

6.2.2 Conjecture for Kerr/CFT

We conclude that the representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ for BTZ modes (6.2.5) match exactly those of Kerr (6.1.10). This leads us to conjecture that an exact mode of Kerr may be reconstructed from some more fundamental primary conformal field of weight ν and its descendants.

Let us note that the expansion for a Kerr mode (6.1.3) may be expressed as a sum from $n = 0$ to ∞ simply by using the symmetry of the hypergeometric function under its first two arguments. This swaps $\nu \to -\nu - 1$ and $n \to -n$. As is well-known in the AdS/CFT correspondence, both terms may be viewed as originating from correlators of a conformal primary of weight ν in the presence of a source term [7].

A new issue that arises in Kerr/CFT is that of the proper normalization of the modes with physical boundary conditions at spatial infinity. Certainly one may use the expansion (6.1.3) to compute scattering correlators on some surface of large fixed r, but to properly impose incoming/outgoing boundary conditions at spatial infinity, another expansion must be used that is convergent at $r = \infty$. Such an expansion is crucial for obtaining quasinormal mode boundary conditions, or computing genuine scattering amplitudes from past infinity to future infinity. To achieve this at the level of mode functions, one must instead expand in terms of a different set of basis functions. In [33, 35, 36] this is chosen to be a set of Coulomb functions, though other choices are possible [38].

6.2.3 Absorption probability

The expansion in Coulomb functions proceeds along the same lines as the expansion in hypergeometric functions that we have described in detail in Chapter 4. The Coulomb functions are

$$R_C(x) = (-\epsilon\kappa x)^{-1-s}\,(1-1/x)^{\frac{i}{2}(\epsilon-\tau)} \sum_{n=-\infty}^{\infty} b_n f_{n+\nu}(x),$$

where

$$\begin{aligned} f_{n+\nu}(x) \quad &= \quad e^{i\epsilon\kappa x}\,(-2\epsilon\kappa x)^{n+\nu}\,(-\epsilon\kappa x)\,\frac{\Gamma\,(n+\nu+1-s+i\epsilon)}{\Gamma\,(2n+2\nu+2)} \\ &\quad\; {}_1F_1\,(n+\nu+1-s+i\epsilon, 2n+2\nu+2; -2i\epsilon\kappa x) \end{aligned}$$

and where ${}_1F_1(a,b;x)$ is a regular confluent hypergeometric function. The coefficients b_n satisfy a similar three-term recurrence relation, and upon inspecting the convergence of the infinite continued fractions we find they converge as fast as a_n, but the mode expansion converges for all $r > r_+$. By matching the solutions obtained via an expansion in Coulomb functions[2] convergent at $r = \infty$ to the expansion (6.1.3) valid at all finite r, and with the help of Teukolsky-Starobinsky identities and a few other simplifying relations, Mano and Eiichi [36] write down the exact Kerr absorption rate for all finite frequencies:

$$\sigma_{\rm abs} \quad = \quad (2\epsilon\kappa)^{2\nu+1}\,\frac{e^{\pi\epsilon}}{\pi}\,\sinh\pi\,(\epsilon+\tau)\,\frac{D_\nu}{|N_\nu|^2}\,, \qquad (6.2.6)$$

[2]Via matching in the region where both solutions are valid $1 < x < \infty$, the hypergeometrics inherit a norm from infinity.

where

$$
\begin{aligned}
N_\nu &= 1 + \frac{i}{\pi}\left(2\epsilon\kappa\right)^{2\nu+1}\left(-1\right)^{2s} e^{i\pi\nu}\sin\pi\left(\nu - i\tau\right)\left(\frac{\sin\pi\left(\nu - s - i\epsilon\right)}{\sin 2\pi\nu}\right)^2 D_\nu \\
D_\nu &= \left|\frac{\Gamma\left(\nu+1-i\tau\right)\Gamma\left(\nu+1-s+i\epsilon\right)\Gamma\left(\nu+1+s+i\epsilon\right)}{\Gamma\left(2\nu+1\right)\Gamma\left(2\nu+2\right)}\right|^2 d_\nu \\
d_\nu &= \left|\sum_{n\leq 0}\frac{\left(-1\right)^n}{\left(-n\right)!\left(2\nu+2\right)_n}\frac{\left(\nu+1+s-i\epsilon\right)_n}{\left(\nu+1-s+i\epsilon\right)_n}a_n^\nu\right|^2 \\
&\quad\times\left|\sum_{n\geq 0}\frac{\left(2\nu+1\right)_n}{n!}\frac{\left(\nu+1+s-i\epsilon\right)_n}{\left(\nu+1-s+i\epsilon\right)_n}a_n^\nu\right|^{-2}.
\end{aligned}
$$

In the small frequency approximation, $\epsilon \ll 1$, the absorption rate properly reproduces the Page formula [51, 52]. By keeping $\mathcal{O}(\epsilon)$ terms in (6.2.6), we can set $E = \ell(\ell+1) + \mathcal{O}(\epsilon^2)$ in (6.1.5). Choosing the integer shift in ν so that $\nu = \ell + \mathcal{O}(\epsilon)$ makes the $n = 0$ term the leading term in the series expansion. To leading order in ϵ, the formula (6.2.6) reduces to:

$$
\sigma^\epsilon_{\text{abs}} = \left(2\epsilon\kappa\right)^{2\ell+1}\frac{e^{\pi\epsilon}}{\pi}\frac{\omega_R}{2T_R}\prod_{k=1}^{\ell}\left[k^2 + \left(\frac{\omega_R}{2\pi T_R}\right)^2\right]\times\left(1+\mathcal{O}(\epsilon)\right)
$$

where we introduced the frequencies

$$
\omega_L = \frac{2M^2\omega}{a}, \quad \omega_R = \frac{2M^2\omega}{a} - m\,.
$$

In the $\epsilon \to 0$ limit the dependence on the left-moving frequency $\omega_L \propto \epsilon$ is rather trivial, however there is a highly non-trivial dependence on the right-moving frequency which remains finite $\omega_R = -m$. This is a familiar behavior of the effective string absorption cross section for massless bosons in $N = 4$ supergravity [32]. With the representations surviving this limit we associate conformal weight $h_R = \ell + 1$, compatible with results of [13] and conformal weight identification of [32].

6.2.3.1 Connection with extremal Kerr/CFT

Encouraged with our findings, here we speculate on a possible connection with results obtained in the extremal limit $a \to M$. More precisely, we search for regime in which we find representations corresponding to near horizon extreme Kerr (NHEK) modes near the superradiant bound. In [16] it was found that the absorption probability of the modes saturating the superradiant bound in the near-extremal Kerr background corresponds to a thermal CFT 2-point function:

$$\sigma \ \propto T_H^{2\beta} e^{\pi m} \sinh \pi \left(m + \frac{\omega - m/2M}{2\pi T_H} \right) \left| \Gamma \left(\frac{1}{2} + \beta + im \right) \right|^4 \left| \Gamma \left(\frac{1}{2} + \beta + i\frac{\omega - m/2M}{2\pi T_H} \right) \right|^2 \tag{6.2.7}$$

where $\beta^2 = \frac{1}{4} - 2m^2 + \bar{A}_{lm}$, and $\bar{A}_{lm}$ is the angular eigenvalue evaluated for $a\omega = m/2$.

We do not observe this truncation of (6.2.6) for quasinormal mode excitations (QNM). In terms of the transmission and reflection coefficients, the QNMs correspond to frequencies at which both T and R develop poles, in such way that $|T| \approx |R|$. There is also another set of modes compatible with the purely outgoing boundary condition at infinity, called total reflection modes (TRM). These correspond to frequencies at which transmission coefficient vanishes, making them standing waves at $r = \infty$. We observe that total reflection modes with exact frequencies given by [44]

$$\omega_{TRM} = m\Omega - 2\pi i T_H (n - s), \qquad n \in \mathbb{N}$$

correctly reproduce the near-superradiant frequencies in the extremal limit.

Here $\Omega = \frac{a}{r_+^2+a^2}$ is the angular velocity at the outer horizon and $T_H = \frac{r_+-r_-}{4\pi\left(r_+^2+a^2\right)}$ is the Hawking temperature.

Taking the extremal limit on the parameters of our $SL(2,\mathbb{R})$ representations,

$$\lim_{a\to M} i\frac{2M^2\omega_{TRM} - ma}{\sqrt{M^2-a^2}} = n - s\,,$$

we arrive at an interesting analogy: just as the BTZ scattering amplitudes for quasinormal mode frequencies reproduce the pole structure of a CFT 2-point function [39], so do the hidden Kerr/CFT representations in the case of extremal total reflection modes. As $\kappa \to 0$, the denominator in (6.2.6) is exactly equal to 1, and the Kerr absorption cross section reproduces (6.2.7), provided we identify $\nu + 1$ with $\beta + 1/2$.

The absorption cross section accounts for poles of a chiral 2-dimensional CFT 2-point function; we suspect the present understanding of NHEK asymptotic boundary conditions, providing an enhancement to one Virasoro only, can then be recast in terms of the monodromy analysis in the highly damped regime.

The fact that the total reflection modes have no bulk degrees of freedom corroborates with the findings of [55], which makes the extremal Kerr/CFT a topological theory. As we know how to count the microscopic degrees of freedom for AdS_3 quotients at any value of the angular momentum [4], it is reasonable to assume same can be achieved in Kerr/CFT. We expect that the two hidden $SL(2,\mathbb{R})$'s enhance to a full $vir_L \times vir_R$, with central charges given by $c_L = c_R = 12J$. As originally hinted in [12], the Cardy formula with this value of central charges, together with temperatures (6.1.8), exactly reproduces the classical Bekenstein-Hawking entropy of the Kerr black hole.

6.3 Eigenvalue equation

In order to solve the eigenvalue equation, we review the convergence criterion introduced in Chapter 4. The expansion of the mode functions (6.1.3) converges only for special values of the parameter ν. To find this parameter a continued fraction is set up. Similar methods are used to construct the angular eigenvalue, and the frequency of quasinormal modes. To proceed, one expresses the radial equation (6.1.5) as a three-term recurrence relation (4.1.7):

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \tag{6.3.1}$$

where

$$\begin{aligned}
\alpha_n &= \frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)} \\
\beta_n &= -\lambda - s(s+1) + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon - mq) \\
&\quad + \frac{\epsilon(\epsilon-mq)(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)} \\
\gamma_n &= -\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}
\end{aligned}$$

and

$$\lambda = E - s(s+1) - \epsilon q + \frac{1}{4}\epsilon^2 q^2 \,.$$

The eigenvalue equation is expressed using the continued fractions

$$R_n = \frac{a_n}{a_{n-1}} = -\frac{\gamma_n}{\beta_n + \alpha_n R_{n+1}}, \qquad L_n = \frac{a_n}{a_{n+1}} = -\frac{\alpha_n}{\beta_n + \gamma_n L_{n-1}} \,. \tag{6.3.2}$$

For general values of ν the solution to the recurrence relation (6.3.1) will diverge as $|n| \to \infty$. To avoid this and find the so-called minimal solution, one must demand that ν solve an additional eigenvalue equation

$$R_n L_{n-1} = 1\,. \tag{6.3.3}$$

There is an equivalence of solutions under $\nu \to \nu + k$ where k is an integer (apparently not noticed in [33, 36]). One convention, useful for real frequencies, is to choose the integer shift in ν such that $E - \nu(\nu+1)$ term on the right-hand side of (6.1.5) is minimized, in order that the $n = 0$ term tends to be the leading order term in the expansion. Another convention, that will be useful when discussing quasinormal modes, will be to simply shift the real part of ν into the range $[0, 1/2)$ using the combined symmetries of the expansion under $\nu \to -\nu - 1$ and $\nu \to \nu + k$.

6.3.1 Convergence at finite frequencies

Still, there is a priori no reason to expect the continued fractions to converge for finite frequencies, even more so as the coefficients are in general complex. However, for n much larger than any other constant in the problem, the coefficients α_n, β_n and γ_n are all dominated by their n-dependence in the following way

$$\begin{aligned}
\alpha_n &\sim n\frac{i\epsilon\kappa}{2}, \\
\beta_n &\sim n^2, \\
\gamma_n &\sim -n\frac{i\epsilon\kappa}{2}.
\end{aligned}$$

Assuming that (6.3.3) is satisfied, we may evaluate R_n and L_n at large n:

$$\lim_{n\to\infty} nR_n = -\lim_{n\to-\infty} nL_n = \frac{i\epsilon\kappa}{2}\,.$$

This leads to convergence of the infinite continued fractions for all finite frequencies.

Similarly we show convergence of the series (6.1.3) for all finite values of the radial coordinate, by evaluating [33]:

$$\begin{aligned}
\lim_{n\to\infty} n\frac{a_{n+1}F_{n+\nu+1}}{a_nF_{n+\nu}} &= \lim_{n\to-\infty} -n\frac{a_{n-1}F_{n+\nu-1}}{a_nF_{n+\nu}} \\
&= \frac{i\epsilon\kappa}{2}\left[1-2x+2\sqrt{x(x-1)}\right],
\end{aligned}$$

where $F_{n+\nu} = {}_2F_1(n+\nu+1-i\tau, -n-\nu-i\tau; 1-s-i\epsilon-i\tau; x)$. From this we observe that the n-th mode in the expansion (6.1.3) at finite frequencies will be given as a sum of at least order n descendants before the sum converges.

6.3.2 Low frequency expansion

In a low frequency expansion, the solution of (6.3.3) is

$$\begin{aligned}
\nu &= \ell - \epsilon^2\frac{\ell(\ell+1)(-11+15\ell(1+\ell))+6\left(-1+\ell+\ell^2\right)s^2+3s^4}{2\ell(1+2\ell)(1+\ell)(-1+2\ell)(3+2\ell)} \\
&\quad +\epsilon^3\frac{mq}{\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)}\left[5\ell(1+\ell)-3+\frac{s^2\left(3\ell(\ell+1)\left(\ell^2+\ell-3\right)+11\right)}{(-1+\ell)\ell(1+\ell)(2+\ell)}\right. \\
&\quad \left. +\frac{s^4\left(-16+3\ell(1+\ell)+5s^2\right)}{(-1+\ell)\ell(1+\ell)(2+\ell)}\right]+\cdots
\end{aligned}$$

which is obtained by substituting the low frequency expansion for E of [34] into (6.3.3). Note for real frequencies ν is also real. As we will see in the following, ν

becomes complex for quasinormal modes, as does the angular eigenvalue. The first two terms match the expression in [33] and the next term is new. As we have seen in section 6.1, with ν and ω given, the irreducible representations that appear in the mode function are fully determined.

6.3.3 Numerical solutions

The solution of closely related continued fraction eigenvalue equations has been considered for spheroidal harmonics in [34, 56]. Quasinormal modes have been studied using similar techniques in [35, 38]. In this case one imposes quasinormal mode boundary conditions on solutions of the radial equation. Imposing the purely outgoing boundary condition at infinity requires a different expansion of the radial solution near infinity. This is then matched to the boundary condition that the mode be purely infalling on the future horizon, leading to a radial eigenvalue problem that determines the quasinormal mode frequencies.

One method that is helpful in improving the convergence of such algorithms has been developed in [57]. When one numerically computes a continued fraction, such as (6.3.2), some cutoff on n is needed. By choosing an initial value for $R_{n_{\mathrm{cutoff}}}$ judiciously, it is possible to improve the convergence of the continued fraction, as well as its domain of convergence. In our numerical code, we apply this method at high order in both the determination of the quasinormal mode frequencies, as well as the determination of the ν eigenvalues. The numerical determination of quasinormal modes has been studied more recently in [58, 59, 60, 61, 62] and our results for the frequencies match well with those found there.

6.3.3.1 Numerical recipe

Here we present a brief numerical recipe. The solution to the angular or radial eigenvalue equation can be expressed as an expansion in polynomials[3], with coefficients satisfying a recurrence equation [38] of the form:

$$
\begin{aligned}
\alpha_0^{\theta,r} a_1^{\theta,r} + \beta_0^{\theta,r} a_0^{\theta,r} &= 0, \\
\alpha_n^{\theta,r} a_{n-1}^{\theta,r} + \beta_n^{\theta,r} a_n^{\theta,r} + \gamma_n^{\theta,r} a_{n-1}^{\theta,r} &= 0,
\end{aligned}
$$

where superscript θ, r is to denote one or the other equation. The angular separation constant E and quasinormal mode frequency ω can be determined as roots of the corresponding continued fraction equations:

$$
\beta_0^{\theta,r} = \frac{\alpha_0^{\theta,r}\gamma_1^{\theta,r}}{\beta_1^{\theta,r}-} \frac{\alpha_1^{\theta,r}\gamma_2^{\theta,r}}{\beta_2^{\theta,r}-} \cdots \frac{\alpha_n^{\theta,r}\gamma_{n+1}^{\theta,r}}{\beta_{n+1}^{\theta,r}-} \cdots \tag{6.3.4}
$$

Numerical evaluation of this problem requires truncation to a finite number of terms. To improve convergence, we implement the Nollert algorithm [57] to evaluate the remainder of the continued fraction,

$$
R_N^{\theta,r} \simeq \frac{\gamma_{N+1}^{\theta,r}}{\beta_{N+1}^{\theta,r}-} \frac{\alpha_{N+1}^{\theta,r}\gamma_{N+2}^{\theta,r}}{\beta_{N+2}^{\theta,r}-} \cdots \tag{6.3.5}
$$

for some large N. For $N \gg 1$ the remainder $R_N^{\theta,r}$ will be well approximated with a large N expansion,

[3] In the case of the radial equation, the expansion is of the form $R = e^{i\omega r} (r - r_+)^{-1-s+i\omega+i\sigma} (r - r_-)^{-s-i\sigma} \sum_{n\geq 0} a_n^r \left(\frac{r-r_+}{r-r_-}\right)^n$ where $\sigma = \frac{\omega r_+ - am}{r_+ - r_-}$, whereas the solution to the angular equation may be given as $S_{\ell m} = e^{a\omega y} (1+y)^{\frac{1}{2}|m-s|} (1-y)^{\frac{1}{2}|m+s|} \sum_{j\geq 0} a_j^\theta (1+y)^j$, where $y = \cos\theta$.

$$R_N^{\theta,r}(\omega, A_{lm}) \simeq C_0^{\theta,r} + C_1^{\theta,r} N^{-1/2} + C_2^{\theta,r} N^{-1} + \ldots + C_{2k}^{\theta,r} N^{-k} + \ldots .$$

By rewriting (6.3.5) in an implicit form, $R_N^{\theta,r} - \gamma_{N+1}^{\theta,r}/(\beta_{N+1}^{\theta,r} - \alpha_{N+1}^{\theta,r} R_{N+1}^{\theta,r})$ and expanding for large N, we read off the coefficients $C_k^{\theta,r}(\omega, E)$. For the radial continued fraction we impose $Re(C_1) > 0$ following [57]; for the angular continued fraction all half-integer powers vanish.

At low overtones, we assume an initial value for ω_0, E_0 around which we look for solutions, and evaluate the remainder $R_N(\omega_0, E_0)$ for a set number of terms where we apply cutoff, $N = n_{\text{cutoff}}$. We utilize an iterative procedure, separately computing radial and angular continued fractions (6.3.4), and feeding solutions back into the next iteration. We increase n_{cutoff} with each iteration, until results converge.

At high overtones a much more reliable method is to replace (6.3.4) with its inversion [38],

$$\beta_0^r - \frac{\alpha_{n-1}^r \gamma_n^r}{\beta_{n-1}^r -} \frac{\alpha_{n-2}^r \gamma_{n-1}^r}{\beta_{n-2}^r -} \cdots \frac{\alpha_0^r \gamma_1^r}{\beta_0^r} = \frac{\alpha_n^r \gamma_{n+1}^r}{\beta_{n+1}^r -} \frac{\alpha_{n+1}^r \gamma_{n+2}^r}{\beta_{n+2}^r -} \cdots$$

for any positive integer n. We find convergence for the number of terms in the inverted continued fraction of the order of the overtone number.

6.3.3.2 Low order quasi-normal modes

In the case of low order quasinormal modes the spheroidal eigenvalue equation and the frequency equation must be solved simultaneously. Towards that end we adopt the technique of Leaver [38], with the Nollert improvement of continued fractions [57], for both the radial and spheroidal eigenvalue equation. This is implemented using a high precision iterative method in Mathematica, as described above.

a/M	ω	A_{lm}	ν
0.0	$0.747343 - 0.177925i$	$4.000000 + 0.000000i$	$0.271625 - 0.233965i$
0.1	$0.748064 - 0.177796i$	$3.999309 + 0.000348i$	$0.272325 - 0.234104i$
0.2	$0.750248 - 0.177401i$	$3.997216 + 0.001395i$	$0.274456 - 0.234519i$
0.3	$0.753970 - 0.176707i$	$3.993667 + 0.003142i$	$0.278128 - 0.235204i$
0.4	$0.759363 - 0.175653i$	$3.988560 + 0.005596i$	$0.283541 - 0.236149i$
0.5	$0.766637 - 0.174138i$	$3.981738 + 0.008757i$	$0.291024 - 0.237327i$
0.6	$0.776108 - 0.171989i$	$3.972969 + 0.012620i$	$0.301108 - 0.238673i$
0.7	$0.788259 - 0.168905i$	$3.961901 + 0.017153i$	$0.314673 - 0.240044i$
0.8	$0.803835 - 0.164313i$	$3.947997 + 0.022256i$	$0.333269 - 0.241085i$
0.9	$0.824009 - 0.156965i$	$3.930384 + 0.027633i$	$0.359906 - 0.240780i$

Table 6.3.1: Numerical results for $s = -2$, $\ell = 2$ and $m = 0$ quasinormal modes

Once the quasinormal mode frequency is known to high precision, we determine ν by using similar numerical methods to solve the equation (6.3.3) for $n = 1$. In this case both rising and lowering infinite continued fractions R_n and L_n need to be truncated to some n_{cutoff}, where we use the Nollert approximation as described to estimate the remainder of the continued fraction.

Some representative results are shown in Tables 6.3.1 and 6.3.2, for $s = -2$, $\ell = 2$ and $m = 0, 1$ quasinormal modes. The quasinormal mode frequencies precisely agree with results presented in [38]. Furthermore, some sample computations of ν appear in [63], and we have checked our numerics correctly reproduces those results. We conventionally normalize $2M = 1$. For clarity of presentation we utilize the invariance under $\nu \to -\nu - 1$ and $\nu \to \nu + k$, $k \in \mathbb{Z}$ to map our ν numeric values into the range $0 < Re(\nu) < 1/2$.

6.3.3.3 Highly damped quasi-normal modes

In the highly damped regime the simultaneous solution of the spheroidal eigenvalue equation and the frequency equation becomes numerically unstable. To make

a/M	ω	A_{lm}	ν
0.0	$0.747343 - 0.177925i$	$4.000000 + 0.000000i$	$0.271625 - 0.233965i$
0.1	$0.760865 - 0.177597i$	$3.948480 + 0.012231i$	$0.281465 - 0.236237i$
0.2	$0.776496 - 0.176977i$	$3.893150 + 0.025197i$	$0.293393 - 0.238774i$
0.3	$0.794661 - 0.176000i$	$3.833210 + 0.038881i$	$0.307963 - 0.241559i$
0.4	$0.815958 - 0.174514i$	$3.767570 + 0.053241i$	$0.326020 - 0.244553i$
0.5	$0.841265 - 0.172346i$	$3.694740 + 0.068178i$	$0.348933 - 0.247684i$
0.6	$0.871937 - 0.169128i$	$3.612470 + 0.083470i$	$0.379077 - 0.250847i$
0.7	$0.910243 - 0.164170i$	$3.517150 + 0.098594i$	$0.420976 - 0.254022i$
0.8	$0.960461 - 0.155910i$	$3.402280 + 0.112173i$	$0.484170 - 0.258022i$
0.9	$1.032583 - 0.139609i$	$3.253450 + 0.119510i$	$0.590665 - 0.266455i$

Table 6.3.2: Numerical results for $s = -2$, $\ell = 2$ and $m = 1$ quasinormal modes

progress, we follow the method of [59] and use a conjectured asymptotic expansion for the high order eigenvalues of spheroidal equation

$$E = (2L + 1)iq\epsilon/2 + \mathcal{O}(\epsilon^0)$$

where $L = \min(\ell - |m|, \ell - |s|)$. We obtain the quasinormal mode frequencies by numerically searching for a solution of an inverted continued fraction equation, as proposed by Leaver [38]. At high damping, the numeric solutions can be shown to follow the analytic result $\omega \simeq \omega_0 + 4\pi i T_0 (N + 1/2)$, where ω_0, T_0 can also be computed within the WKB approximation [44], and the overtone number N takes integer values.

We numerically solve for ν by finding solutions of the equation quadratic in the continued fractions, (6.3.3), where we find better convergence if we shift index n by the imaginary part of the quasinormal mode frequency. As a rule of thumb, we use the number of terms in the continued fraction of the order of the overtone number and implement the Nollert improvement for the remainder of the fraction. With each overtone we increase computational precision until the result converges. The numerics appear most stable when a starting value $\nu \sim \mathcal{O}(N/2)$ is used. As the solutions to (6.3.3) are determined up to integer shift, $\nu + k$, and reflection $\nu \to -\nu - 1$, we can always find a solution such that $Re(\nu) < 1/2$. We use this symmetry when we plot our data.

In Figures 6.3.1 and 6.3.2 we show quasinormal mode frequencies at high damping for different values of s, and m with $\ell = 2$. For m fixed, we observe the real part of the frequency converges to the same value at high overtones, irrespective of spin weight. The results are compatible with numerical computations of Berti et al. [59].

In Figures 6.3.3 and 6.3.4 we display our numerical solutions for $s = -2$, $\ell = 2$, and $m = 2$ modes. We plot quasinormal mode data as a function of a/M and for fixed overtone number (Figure 6.3.3). We find the frequencies approach the expected asymptotic behavior with increasing overtones. Our data for the 400th overtone is comparable to the extrapolated asymptotic curve computed in [59], with the plot in Figure 6.3.3 suggesting the overtones above 240 already give a qualitatively good estimate of the asymptotic regime. The improvement in convergence for ν equation happens roughly around this value of frequency. The plot of ν values displays two strands of solutions corresponding to even and odd valued overtones, which persist at all values of quasinormal frequency (6.3.4). This arises from the approximate half-integer spacing between frequencies at high damping, analytically computed in [44], and also numerically confirmed in [59]. In Figures 6.3.5 and 6.3.6 we compare ν values corresponding to a single strand, for fixed $\ell = 2$ and independently varied $s = 0$, -2 and $m = 1$, 2, where we map out the solutions of (6.3.3) up to 1000th overtone.

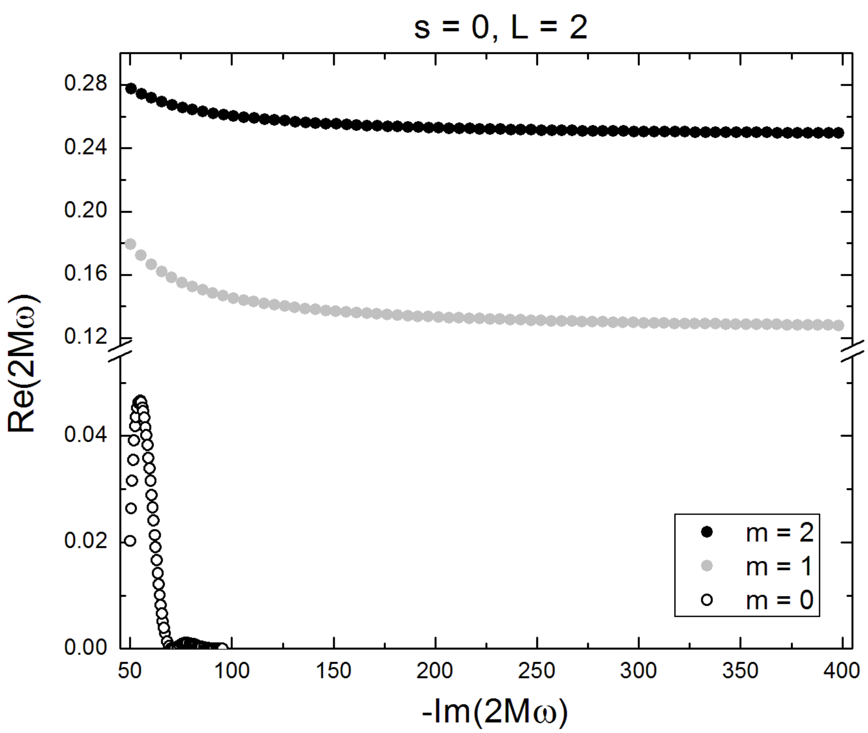

Figure 6.3.1: Quasinormal frequencies for $s = 0$, $\ell = 2$ modes and $a/M = 0.2$ with m varied.

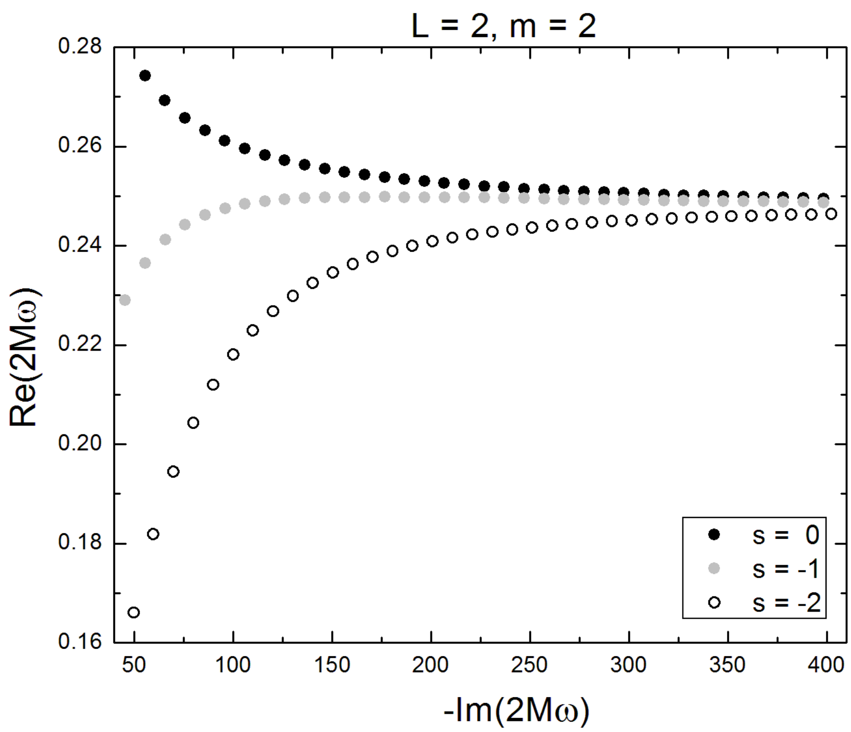

Figure 6.3.2: Quasinormal frequencies for $\ell = 2$, $m = 2$ modes and $a/M = 0.2$ with s varied.

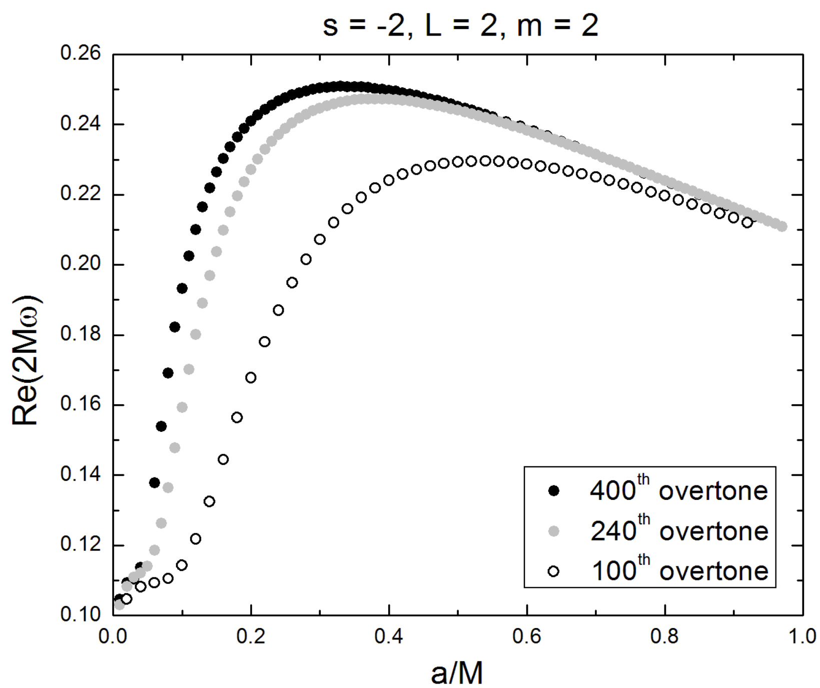

Figure 6.3.3: Real part of $s = -2,\ \ell = 2,\ m = 2$ quasinormal mode frequencies for 100th, 240th and 400th overtone as a function of a/M.

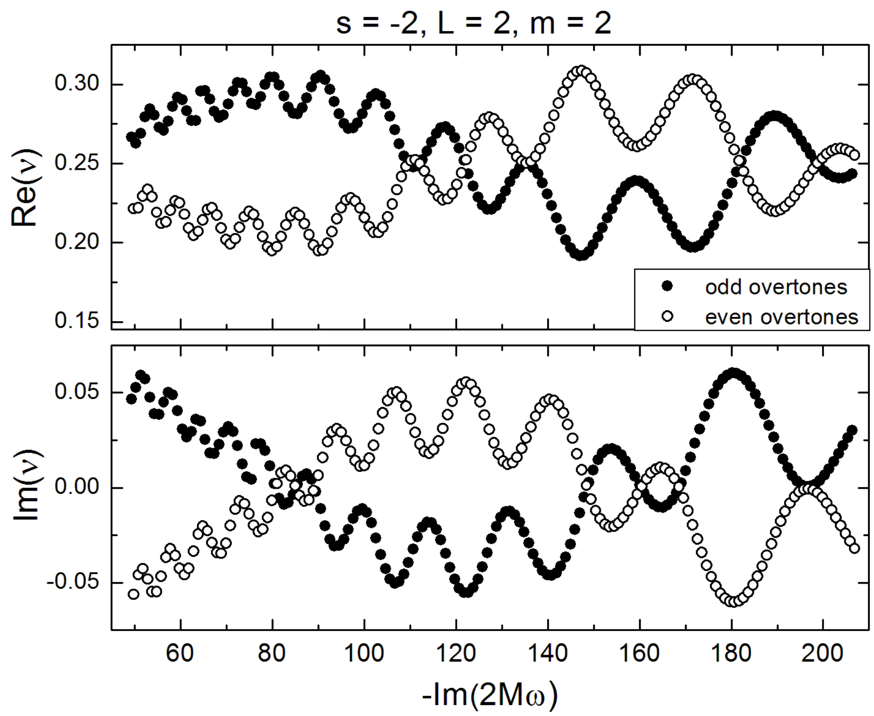

Figure 6.3.4: Real and imaginary ν values at high overtones for $s = -2,\ \ell = 2,\ m = 2$ mode and $a/M = 0.2$.

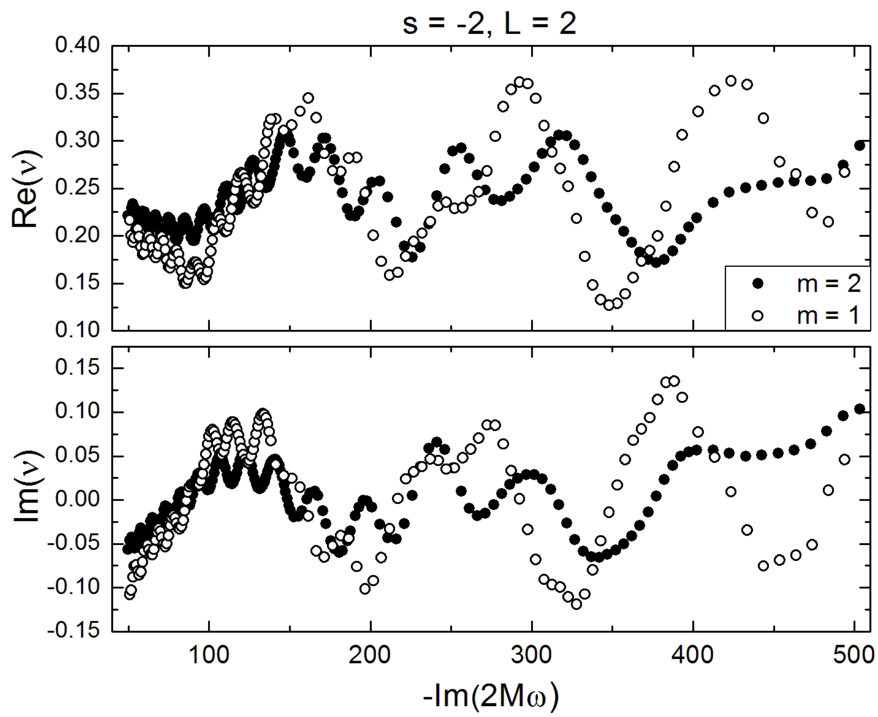

Figure 6.3.5: Comparison of real and imaginary ν values at high overtones for $m = 1$ and $m = 2$ and fixed $s = 2$, $\ell = 2$ at $a/M = 0.2$.

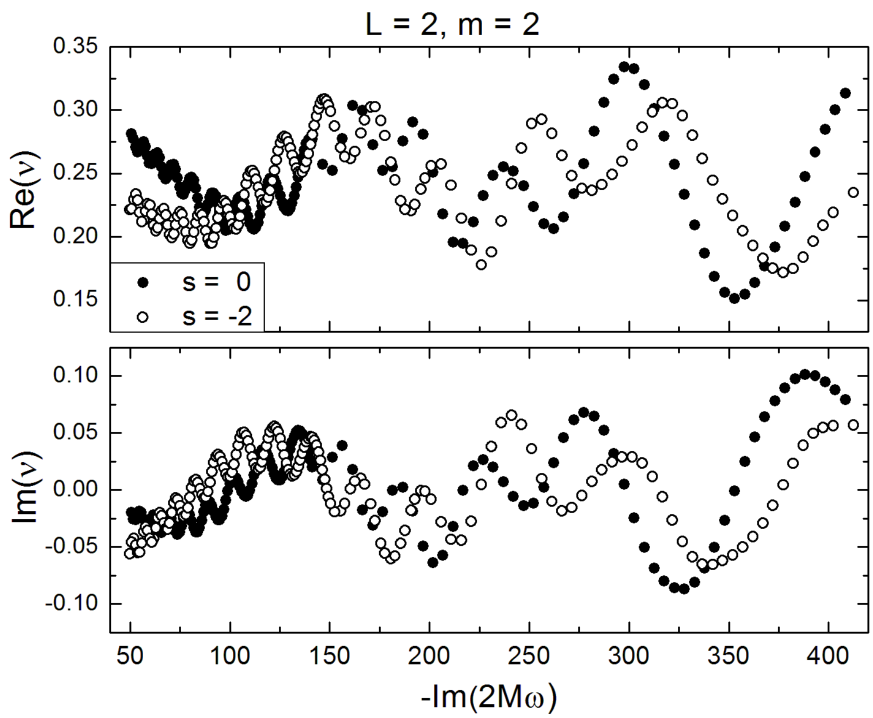

Figure 6.3.6: Comparison of real and imaginary ν values at high overtones for $s = 0$ and $s = -2$ with fixed $\ell = 2,\ m = 2$.

Chapter 7

Outlook and loose ends

The hidden Kerr/CFT proposal has drawn a good amount of attention since its formulation, and in the past years has been applied to a number of gravitational backgrounds. The allure of the proposal lies in the notion that conformal symmetries need not be manifest symmetries of the geometry in order to consider a conformal field theory description of low frequency scattering processes in Kerr background. This should be contrasted with the usual geometric approach where AdS/CFT methods may be applied for near-extremal black holes with throat geometries containing AdS subspaces, or for more general black holes in asymptotically AdS spacetimes.

Low frequency physics in black hole backgrounds has already proved its fruitfulness in [26] by observing that the scalar low energy decay spectrum shows characteristic behavior seen in CFT correlation functions. Similar results have also been obtained in [64]. The guiding principle was that by studying this low frequency limit, we learn about the underlying conformal field theory conjectured to provide a holographic description of the full quantum Kerr black hole. For example, by building the holographic dictionary between low energy physics in Kerr background and the conjectured CFT dual, we can hope to access relevant information in the CFT that would simplify our understanding of scattering in Kerr background, but also provide hints of properly addressing these novel ways of arriving at holographic dualities.

However, we find low frequency treatment not to be necessary, nor sufficient. As opposed to AdS, where the symmetries of the wave equations are generated by spacetime isometries, in Kerr we do not have the luxury of utilizing dualities that rest on geometric grounds. Nevertheless, we find an interesting $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SU(2) \times SU(2)$ structure at finite frequencies, which shares the symmetries with the generators of isometries of 5-dimensional extremal black holes in string theory. This may as well be a good starting point in the endeavor of understanding a holographic dual to asymptotically flat spacetimes.

For a general massless excitation in a Kerr black hole background it is possible to compute a universal function $\nu(\omega, l, m, s)$ which determines a sequence of irreducible representations of the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which gives an exact expression for the mode function. These representations for general black hole rotation parameter a do not correspond to highest/lowest weight representations as originally conjectured in the Kerr/CFT literature, but are nevertheless expected to arise from the typical primary representations of the conformal group, i.e. lowest weight representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, upon restriction to a non-global coordinate patch.

From the CFT point of view, one may view the evolution from $r = \infty$ to finite r as a renormalization group flow from a purported holographic dual to the Kerr/CFT structure that takes over at finite values of the radius. However, we know little about the holographic dual of asymptotically flat spacetime from which the theory is flowing into the infrared. Without such a description we cannot formulate the proper boundary conditions in the holographic dual that would allow us to resolve detailed scattering amplitudes from past null infinity to future null infinity in the Kerr background. Some related works that study the problem of holographic duals to asymptotically flat spacetime include [65, 66, 67].

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