TREATMENT OF THREE BODY PROBLEMS IN COORDINATE SPACE

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In an attempt of solving the three-body problem in coordinate space, we face with three difficulties ; (1) to impose the elastic and the break-up boundary conditions at the same time, (2) to find the Fredholm solution of the system and (3) to find a practical way of solving the problem. In this paper, we will report how to overcome (1) and (2). Also the analytic structure of the solution at low energies is discussed. By analytic continuation of the low energy formula to the complex momentum plane, the presence of virtual state in ^2S and a resonance in ^4P are predicted. The Efimov effect is interpreted from the analytical view point.

§1. Correct asymptotic behavior

In solving the three body problem in coordinate space, we must impose the correct asymptotic behavior as the boundary condition. We designate by α the pair of particles 1 and 2, r_α their relative distance, ρ_α the distance of the particle 3 relative to the center of mass of the pair α. The asymptotic behavior of the three-body system reads

$$\phi \sim \sum_\alpha \varphi(r_\alpha)[e^{i\vec{k}\vec{\rho}_\alpha} + \frac{e^{ik\rho_\alpha}}{\rho_\alpha} f(\theta_\alpha)] + \frac{e^{i\sqrt{E}R}}{R^{5/2}} F(r/\rho) \ , \quad (1)$$

where $\varphi(r_\alpha)$ denotes the wave function of deuteron, k the wave number $\sqrt{(2\mu/\hbar^2)(E+|E_d|)}$ ($|E_d|$; The binding energy of deuteron), $f(\theta_\alpha)$ the scattering amplitude of neutron-deuteron channel, R the hyper-radius, r/ρ the hyper-angle and $F(r/\rho)$ the break-up amplitude. <u>The boundary condition (1) must be imposed not only for $E>0$ but also for $E<0$</u> , because the Schrödinger equation says nothing about the breaking of continuity of analyticity of the solution.

§2. How one can impose the correct boundary conditions

We demonstrate for the potential scattering how one can impose the correct boundary condition[1]. Let us introduce the functions $|u\rangle$ and $|w\rangle$ by

$$|u\rangle = \frac{1}{\sqrt{k}} \sin kr \ , \quad |w\rangle = \frac{1}{\sqrt{k}} e^{ikr} \quad . \tag{2}$$

The Lippmann-Schwinger equation reads

$$|\phi\rangle = |u\rangle + G_0 V|\phi\rangle \ , \tag{3}$$

where G_0 is the Green's function

$$G_0 = \frac{1}{E - H_0 + i\varepsilon} = - |w\rangle\langle u| + g. \tag{4}$$

In Eq.(4), g is the real Green's function which vanishes at large distances from the origin

$$g \ ; \ \text{real}, \quad g \xrightarrow[r\to\infty]{} 0 \quad . \tag{5}$$

Let T stand for the scattering amplitude defined by

$$T = \langle u|V|\phi\rangle \quad . \tag{6}$$

Eq.(3) reads then

$$|\phi\rangle = |u\rangle - |w\rangle T + gV|\phi\rangle = \omega\left[|u\rangle - |w\rangle T\right] \quad , \tag{7}$$

where

$$\omega = (1 - gV)^{-1} \quad . \tag{8}$$

Since

$$\omega \xrightarrow[r\to\infty]{} 1 \quad , \tag{9}$$

the asymptotic form of $|\phi\rangle$ is

$$|\phi\rangle \sim |u\rangle - |w\rangle T \quad . \tag{10}$$

To obtain the scattering amplitude in a solved form, we put Eq.(7) in Eq.(6)

$$T = \langle u|V|\phi\rangle = \langle u|V\omega|u\rangle - \langle u|V\omega|w\rangle T \quad .$$

Therfore,

$$T = \frac{\langle u|V\omega|u\rangle}{1 + \langle u|V\omega|w\rangle} = \frac{\mathrm{Im}\,J(k,0)}{J(k,0)} \quad . \tag{11}$$

$J(k,0)$ is known as the Jost function. $J(k,0)$ implies the properties.[2)]

(1) $J(k,0) = \det\left[1 - G_0 V\right]$. Therefore, Eq.(7) is the Fredholm solution of Eq.(3).

(2) $J(k,0)$ has the correct analytic property that it is the entire function on the upper half of the complex k-plane.

(3) The S-matrix defined by

$$S = 1 - 2iT \tag{12}$$

satisfies the unitarity.

(4) The iteration of Eq.(8) ;

$$= 1 + gV + gVgV + \cdots\cdots \qquad (13)$$

converges without regards to the magnitude of the potential, if it is local.

(5) Since $\omega|u\rangle$ or $\omega|w\rangle$ satisfies the Volterra equation, e.g.

$$\omega|u\rangle = |u\rangle - \frac{1}{k}\int_r^{\infty} \sin k(r-r')V(r')\omega|u(r')\rangle\, dr' \quad , \qquad (14)$$

it is very easily solved numerically.

§3. Faddeev equation and Alt-Grassberger-Sandhas equation

Faddeev[3] has shown that the amplitude of the process (three free particles) to (three free particles) satisfies the equation

$$T = T_\alpha + T_\beta + T_\gamma \quad ,$$

$$T_\alpha = t_\alpha + t_\alpha G_0(T_\beta + T_\gamma) \; . \qquad \text{(The Faddeev equation)}$$

Here t_α is the two-body scattering matrix in the three-body space. Also Faddeev has shown that the amplitude of the process (n-d) to (n,d) + (break up) is the residue of the Faddeev equation. The equation for (n,d) to (n,d) is

$$U = G_0^{-1}\,\bar{1} + \bar{1}tG_0U \quad , \qquad (15)$$

where

$$\bar{1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad , \qquad t = \begin{pmatrix} t_\alpha & 0 & 0 \\ 0 & t_\beta & 0 \\ 0 & 0 & t_\gamma \end{pmatrix}$$

The amplitude for (n,d) to (break up) is then

$$T = tG_0U \qquad (16)$$

Eqs.(15) and (16) are called A-G-S equation[4].

§4. Fredholm solution of three-body problem with correct boundary conditions

We adopt the same method as in §2 to the three-body problem.

(I) Preliminary Let $|\phi\rangle$ and V stand for matrices

$$|\phi\rangle = \begin{vmatrix} \phi_\alpha \\ \phi_\beta \\ \phi_\gamma \end{vmatrix} \quad , \qquad V = \begin{pmatrix} V_\alpha & 0 & 0 \\ 0 & V_\beta & 0 \\ 0 & 0 & V_\gamma \end{pmatrix} . \qquad (17)$$

Then the three-body Schrödinger equation reads

$$(E - H_0 - V)|\phi\rangle = V\bar{I}|\phi\rangle . \tag{18}$$

Let $|u\rangle$ ($|w\rangle$) be the product of the deuteron wave function and the plane wave (outgoing wave) of the incoming (outgoing) particle. Let G be

$$G = (E - H_0 - V + i\varepsilon)^{-1} . \tag{19}$$

We define $\tilde{t}$ by

$$t = -V|w\rangle\langle u|V + \tilde{t} . \tag{20}$$

We can show that $\tilde{t}$ satisfies

$$\tilde{t} = V + VG_0\tilde{t} , \tag{21}$$

$$G_0 t = -|w\rangle\langle u|V + G_0\tilde{t} . \tag{22}$$

Let $|f\rangle$ ($|h\rangle$) be the regular solution of $(E-H_0)|f\rangle = 0$ ($|h\rangle$'s outgoing wave, that is irregular) in the hyper-spherical coordinate. Then we can write G_0 as

$$G_0 = -|h\rangle\langle f| + G^0 , \tag{23}$$

Here G^0 plays the role of g in §2. G^0 is real, and vanishes at large distances from the origin in the hyper-spherical coordinate. We define a real matrix R by

$$R = V + VG^0R . \tag{24}$$

Then we can show that

$$\tilde{t} = R(1 - |h\rangle\langle f|\tilde{t}) . \tag{25}$$

(II) <u>Fredholm solution</u> We use Eqs.(22),(23) and (25) in the Lippmann-Schwinger type equation of (18) to obtain

$$|\phi\rangle = |u\rangle + G_0 t\bar{I}|\phi\rangle = \Omega\left[\, |u\rangle - |w\rangle T^{(e)} - (1+G^0R)|h\rangle T^{(B)}\right] \tag{26}$$

where

$$T^{(e)} = \langle u|V\bar{I}|\phi\rangle , \quad \text{the elastic amplitude} , \tag{27}$$

$$T^{(B)} = \langle f|\tilde{t}\bar{I}|\phi\rangle = \langle f|t\bar{I}|\phi\rangle , \tag{28}$$

and

$$\Omega = \frac{1}{1 - G^0R\bar{I}} , \quad \Omega \xrightarrow[R\to\infty]{} 1 . \tag{29}$$

By virtue of (26) and (29), we see that $|\phi\rangle$ satisfies the correct asymptotic behavior

$$|\phi\rangle \sim |u\rangle - |w\rangle T^{(e)} - |h\rangle T^{(B)} . \tag{30}$$

The sum of $T_\alpha^{(B)}$, $T_\beta^{(B)}$ and $T_\gamma^{(B)}$ makes up the break-up amplitude

$$\bar{T}^{(B)} = T_\alpha^{(B)} + T_\beta^{(B)} + T_\gamma^{(B)} . \tag{31}$$

If we use Eq.(26) in Eqs.(27) and (28), we obtain

$$T^{(e)} = \langle u|J\bar{I}|u\rangle - \langle u|J\bar{I}|w\rangle\ T^{(e)} - \langle u|J|h\rangle\,\bar{T}^{(B)} \qquad (32)$$

and

$$\bar{T}^{(B)} = \langle f|(1+\bar{I}^{-1})J\bar{I}|u\rangle - \langle f|(1+\bar{I}^{-1})J\bar{I}|w\rangle\,T^{(e)} - \langle f|(1+\bar{I}^{-1})J|h\rangle\,\bar{T}^{(B)}, \qquad (33)$$

where J satisfies the A-G-S type equation

$$J = G_0^{-1}\,\frac{1}{1-G^0\bar{I}R} = G_0^{-1} + \bar{I}RG^0J\ . \qquad (34)$$

We can show term by term that the denominator of $T^{(e)}$ and $\bar{T}^{(B)}$ yields the Fredholm determinant.

§5. Exchange Singularity

If we neglect the break-up channel, the n-d elastic amplitude is obtained from the solution of Eq.(32) as

$$T_{n,d} = T_{\alpha,\alpha} + T_{\beta,\alpha} + T_{\gamma,\alpha} \qquad (35)$$

For the partial wave ℓ it reads

$$T^{(\ell)}_{n,d} = (\mathrm{Im}\ \bar{J}_\ell)/\ \bar{J}_\ell = e^{i\ \delta_\ell}\sin\delta_\ell\ , \qquad (36)$$

From which we obtain

$$\cot\delta_\ell = -\ \mathrm{Re}\bar{J}_\ell\ /\mathrm{Im}\bar{J}_\ell\ . \qquad (37)$$

Here

$$\bar{J}_\ell = 1 + \langle u_\alpha|J|w_\alpha\rangle_\ell + 2\langle u_\alpha|J|w_\beta\rangle_\ell\ . \qquad (38)$$

We parametrize $\bar{J}_\ell$ as

$$\bar{J}_\ell = 1 + \bar{H}_\ell(g_\ell') + \lambda_\ell + a_\ell' z^{2\ell+1}(b_\ell' + c_\ell' z^2) \qquad (39)$$

where

$$\alpha = 0.2315\times10^{-13}\mathrm{cm}^{-1}\ ,\quad \alpha = \sqrt{(m/\hbar^2)|Ed|}\quad (m:\ \text{nucleon mass})$$

$$z = k/\alpha\ .$$

For S-wave $\bar{H}_0(g_0')$ reads

$$\bar{H}_0(g_0') = -(g_0'/z)\left[\tan^{-1}(3z/2) - \tan^{-1}(z/2) + 0.5\ i\ln(1+9z^2/4)/(1+z^2/4))\right]\ . \qquad (40)$$

For p-wave $H_1(g_1')$ is

$$\bar{H}_1(g_1') = -(g_1'/z)\left[\left\{(1/z^2)(1+5z^2/4)(\tan^{-1}(3z/2)-\tan^{-1}(z/2))-1/z\right\} + i\left\{(1/2z^2)(1+5z^2/4)\ln((1+9z^2/4)/(1+z^2/4))-1\right\}\right]. \qquad (41)$$

These terms are due to the exchange singlarity in $\langle u_\alpha|J|w_\beta\rangle$.

After renormalization of parameters

$$g_\ell = g'_\ell/(1 + \lambda_\ell), \quad a_\ell = a'_\ell/(1 + \lambda_\ell) \quad \text{etc.} \tag{42}$$

we obtain the low energy formula

$$k^{2\ell+1}\cot\delta_\ell = -\alpha^{2\ell+1}\frac{1 + \mathrm{Re}\,\overline{H}_\ell\,(g_\ell) + a_\ell z^2}{\frac{1}{z^{2\ell+1}}\,\mathrm{Im}\,\overline{H}_\ell(g_\ell) + b_\ell + C_\ell z^2} \tag{43}$$

On the real axis, we determine parameters form phase shifts as

	g	a	b	c
^2S	0.5374	-0.1015	0.6070	0.0451
^4P	4.0447	3.5275	-16.806	0.1229

The function $\overline{J}_\ell$ has the branch points of the logarithmic type at $-\frac{2}{3}\alpha i$ and $-2\alpha i$ and nowhere else. Solving the equation $\overline{J}_\ell = 0$, we have found the poles at $k = -0.1256i(10^{13}cm^{-1})$ for ^2S and at $k = 0.0326 - 0.0909\,i(10^{13}cm^{-1})$ for ^4P. Since the cut is due to the logarithmic singularity of $\overline{J}_\ell$, we have infinite number of poles, each one lying on each Riemann sheet. These poles accumulate at the origin when α tends to 0. This is the analytic explanation of the Efimov effect.[5)]

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