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RESEARCH ARTICLE

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Periodic relativity: basic framework of the theory

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Abstract An alternative gravity theory is proposed which does not rely on Riemannian geometry and geodesic trajectories. The theory named periodic relativity (PR) does not use the weak field approximation and allows every two body system to deviate differently from the flat Minkowski metric. PR differs from general relativity (GR) in predictions of the proper time intervals of distant objects. PR proposes a definite connection between the proper time interval of an object and gravitational frequency shift of its constituent particles as the object travels through the gravitational field. PR is based on the dynamic weak equivalence principle which equates the gravitational mass with the relativistic mass. PR provides very accurate solutions for the Pioneer anomaly and the rotation curves of galaxies outside the framework of general relativity. PR satisfies Einstein's field equations with respect to the three major GR tests within the solar system and with respect to the derivation of Friedmann equation in cosmology. This article defines the underlying framework of the theory.

Keywords Time, Origin, Alternative gravity theory, Two-body problem, Cosmology

1 Introduction

In periodic relativity (PR), the deviation to the flat Minkowski metric in the presence of the gravitational field gets introduced in the form,

$$\left(\frac{dt}{d\tau}\right)^2 = \gamma^{2n} = (1-\beta^2)^{-n}, \tag{1.1}$$

$$d\tau = dt\left(1 - \frac{nv^2}{2c^2}\right), \tag{1.2}$$

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Fig. 1 Two-body system

Fig. 2 Bending of light around sun

to the first order accuracy for small values of v and n, where t is the coordinate time, τ the proper time of the orbiting body and n is a real number. The corresponding line element in polar coordinates is,

$$ds^2 = c^2dt^2 - ndr^2 - nr^2d\theta^2 - n(r^2\sin^2\theta)d\phi^2. \tag{1.3}$$

This presents an alternative to the weak field approximation. Here flat Minkowski metric is represented by $n = 1$, light trajectories deviate as $n = 0$ and solar system planets deviate as $n = 4$. For Pioneer trajectory [1],

$$n = \left(1 + \frac{(\delta/2)}{(\mu/r^2)}\right), \tag{1.4}$$

where $\delta = |\delta\hat{\boldsymbol{r}}|$ is the unit acceleration, $\mu = GM_0$, $G =$ gravitational constant, and M_0 is the solar mass. For rotation curves of Milky Way, various values of n are listed in [2].

As discussed later, the line element of Eq. (1.3) satisfies Einstein's field equation for the empty space $R_{\mu\nu} = 0$ for any constant value of n. In PR it is proposed that the proper time $d\tau$ of a massive object has a definite connection with the gravitational redshift of the massive particles of which the massive object is composed. This causes every two body system to deviate differently from the flat Minkowski metric. Weak field approximation of general relativity (GR) [3; 4; 5] does not allow such freedom. This could be a possible cause of difficulty in explaining the Pioneer anomaly in GR. Similarly in case of rotation curves of galaxies, PR uses velocities of the stars and the virial mass of the galaxy (which includes the mass due to cold dark matter) as the input parameters and obtains deviation factor n and the proper time of the star as the output parameters. This way it is possible to get the perfect fit for the rotation curves.

In PR the factor $(\cos\psi + \sin\psi)$ introduces geodesic like trajectories. Angle ψ is shown in Figs. 1 and 2. The field equations in the presence of matter are derived using the energy conservation law and proper use of the relativistic mass. This supplants the Riemannian geometry. The weak equivalence principle (WEP) is replaced by the dynamic WEP which states that the gravitational mass is equal to the relativistic mass. The main effect of having different deviation factors for different two body systems is that the proper time interval predictions of GR and PR are different, specially for distant objects. This is where the two theories could be tested for the soundness of their underlying logic of the meaning of time. Unfortunately, this test capability does not exist at present time.

1.1 Relativistic invariant

The relativistic invariant s^2 presented by Minkowski, Lorentz and Einstein relates two points in space-time by the expression,

$$s^2 = x^2 + y^2 + z^2 - c^2t^2. \tag{1.5}$$

Here $x^2 + y^2 + z^2$ represents three dimensional space, c is the velocity of light and t the ordinary linear time as we generally know. Einstein's relativity theory is founded upon this simple equation and a hypothesis based on the E*ö*tv*ö*s experiment which showed that the gravitational mass of a body is equal to its inertial mass. It is also well known that if we replace c with velocity $v < c$ for other massive particles, Eq. (1.5) no longer remain meaningful. One of the fundamental proposal of the present theory is to recognize continuity between the electromagnetic wave spectrum and the massive prticle wave spectrum. Such argument can be supported if we can supplant Eq. (1.5) by an equation which is not only applicable to velocity of light but also to velocity of all the other particles which travel at speeds less than that of light.

1.2 Periodic invariant

It is possible to propose one such equation which I call the periodic invariant. It can be written as

$$s^2 = \lambda^2 - V^2T^2, \tag{1.6}$$

where $\lambda = h/p$ is the associated de Broglie wavelength [6; 7], $V = c^2/v$ the phase velocity, v the particle velocity, and T the period of the wave. One can see that Eq. (1.6) does satisfy light particles as well as other massive particles. If we replace ordinary particle velocity with that of light, we get $v = c$, $V = c$ and

$$s^2 = \lambda^2 - c^2T^2, \tag{1.7}$$

If we multiply Eq. (1.7) for light with a real number n^2 and set $(n\lambda)^2$ equal to the cartesian distance $x^2 + y^2 + z^2$ and $(nT)^2$ equal to the linear time t^2, Eq. (1.7) becomes equivalent to Eq. (1.5). Therefore Eq. (1.5) is a special case of Eq. (1.6). Eq. (1.7) implies that space-time is not only curved but also wavy and that time does not flow in one direction but is strictly a periodic or cyclic phenomenon. Eqs. (1.5) and (1.7) both behave identically in the absence of gravitational field, but in the presence of g field, for astronomical distances, Eq. (1.7) remains null and Eq. (1.5) yields time like geodesics. In PR, both light as well as massive particles always travel along null paths. This makes it difficult to solve mundane problems of macroscopic proportions. This is why it becomes necessary to introduce approximations in the form of linear time and linear Euclidean distance in Eq. (1.6) which permits time like and space like geodesics for addressing the problems involving complex structures. However, for certain fundamental measurements such as gravitational redshift and deflection of light, Eq. (1.7) should yield more accurate results than that given by Eq. (1.5) because the reality does not get compromised.

We can also say that Eqs. (1.5) and (1.7) both behave identically even in the presence of gravitational field when the space-time interval involves atomic and sub-atomic distances. Thus the validity of the algebraic structure (Clifford and Lie Algebra) associated with Eq. (1.5) and the related gauge and spinor groups of particle physics is maintained. It is only at astronomical distances that the difference between two equations become perceptible and can affect any local symmetry formalism based on diffeomorphism. Therefore the validity of Dirac equation [8] is maintained with respect to Eqs. (1.6) and (1.7). Same is true for the algebra of Lorentz transformation when the space-time interval involves atomic and subatomic distances.

We can write Eq. (1.6) as

$$s^2 = (ch/cp)^2 - (c^2/v\nu)^2, \tag{1.8}$$

where h is Planck's constant, p the particle momentum and $\nu = 1/T$ is the frequency of the associated de Broglie wave. This period-frequency relation is the only fundamental and basic equation that relates the concept of time to the physical world in an objectively real manner. The relativistic invariant relates the space and time continuum on a macrocosmic scale. The periodic invariant does the same on a microcosmic scale. If we introduce the energy–momentum invariant

$$E^2 = E_0^2 + (cp)^2. \tag{1.9}$$

in Eq. (1.8), we get,

$$s^2 = ((hc)^2/(E^2 - E_0^2)) - (c^2/v\nu)^2, \tag{1.10}$$

where $E =$ total energy of the particle and $E_0 = m_0c^2$ is the rest energy of particle. Relativistic mass is little used by modern physicists. Notwithstanding the modern usage we will use m for relativistic mass and m_0 for rest mass throughout the article.

2 Quantum invariant

The invariant Eq. (1.10) has a general form applicable to all de Broglie particles. In relativity, the vanishing of the invariant s^2 given by Eq. (1.5) does not mean that the distance between two space-time points gets obliterated. It simply means that the two space-time points can be connected by a light signal in vacuum. The new invariant Eq. (1.10), however, can vanish in two different ways. First, in the characteristic relativistic sense implying that two points in space-time can be connected by a energy signal (which can be a light signal or a massive particle signal), and secondly in an absolute sense where both terms on the right also vanish individually like the Euclidean invariant. In the first case, we get the relation,

$$(E^2 - E_0^2)/\nu^2 = (h^2v^2)/c^2. \tag{2.1}$$

Substituting the photon parameters $E_0 = 0$ and $v = c$ into Eq. (2.1) gives the quantum hypothesis of Max Planck, $E = h\nu$. This provides sufficient reason to declare that Eq. (2.1) is a general form of Max Planck's quantum hypothesis applicable to both massless as well massive particles.

Essentially there is no difference between the relativistic invariant Eq. (1.5) and the invariant Eq. (1.10), other than the fact that the former defines the space-time continuum and the latter defines the energy–vibration continuum. The equivalence of both these continuums will become clear when we define the quantum invariant with the assumptions that, given sufficient energy, all particles having rest masses can disintegrate into particles with zero rest masses; and that all particles having zero rest masses will have a constant velocity in space regardless of the inertial frames of reference and equal to the velocity of light. These two assumptions would allow us to adopt the hypothesis that the creation begins with a vibration in the primal energy. We can introduce the photon parameters $E_0 = 0$ and $v = c$ in Eq. (1.10) to simulate the initial state of the universe. This gives,

$$s^2 = (hc/E)^2 - (c/\nu)^2. \tag{2.2}$$

And since the path of a massless particle is a null geodesic, for $s^2 = 0$, Eq. (2.2) can be further simplified to a form which is independent of the law of propagation of light. We shall call this form the Quantum Invariant.

$$s^2 = (h/E)^2 - (1/\nu)^2. \tag{2.3}$$

The quantum invariant can vanish in an absolute sense when $E \to \infty$ and $\nu \to \infty$. In this case the space-time continuum connecting two points gets completely obliterated and the resulting sub-quantic medium resembles a black hole singularity. Such a singularity suggests an equilibrated state of primal energy devoid of ripples which we shall call the unmanifest energy. This, however, is not a very accurate description of the unmanifest. The unmanifest could not be described as energy because there are no oscillations in the unmanifest which is motionless, whereas the energy is always associated with the oscillations. Hence the better way to put this is to say that the unmanifest is something which gives birth to both the energy and the oscillations which are two faces of the same coin. When one face disappears, the other automatically disappears with it. Since the unmanifest is not the energy, it does not gravitate. Similarly, the vibrating energy and the spacetime are two faces of the same coin. When the vibrating energy disappears, the spacetime superimposed on it disappears automatically. So is the unmanifest a perfect vacuum? Again the answer is no because the unmanifest is not the nothingness. So how do you describe the unmanifest? The unmanifest can only be described by negation. That is, you keep asking whether it is this or that and the answer is always no, because it is one of a kind in the whole universe and there is nothing else to compare it with.

So this repose of the equilibrated state of the unmanifest is disturbed when initial vibration sets off a chain reaction of creative processes. Following the first vibration in the unmanifest, several subtle and yet undetected forms of energies may have been created. Eventually certain gross form of vibrating energy of a very unified, fundamental and primal kind becomes manifest followed by what we call inflation [9; 10; 11; 12] in Lambda-CDM model [13; 14]. With the vibration comes the periodic phenomenon. Therefore time begins with the first vibration. Concept of proper time assumes linear time and distance scales whereas the true nature of reality is founded upon non-linear periods and wavelengths of the subatomic particles. Nevertheless to deal with a compound wave of a massive object such as a planet is not as simple as analyzing an individual particle. Thus the concept of proper time is useful in such cases as an approximation.

The vanishing of the quantum invariant leads one to conclude that energy and vibrations are not independent entities. Nowhere in the observable universe can one find any form of energy which is not in a state of vibration. The analogy of oneness of the waves and the ocean when former subsides will suffice to explain the vanishing of the quantum invariant. Another conclusion is that the space and energy are equivalent. There is nothing like empty space. All space is either filled with vibrating energy or with the unmanifest energy in equillibrium. One cannot conceive of space without associating it with some form of energy. In other words, space-time of Einstein's theory are mere imaginary artifacts superimposed on vibrating energy which is the only real substance.

3 Quantum energy equation

General form of Max Planck's quantum hypothesis (2.1) can be written in various alternate forms of which Eq. (3.3) is the most familiar.

$$E = \{E_0^2 + h^2\nu^2\beta^2\}^{1/2}, \tag{3.1}$$
$$E = m_0c^2\{1+\gamma^2\beta^2\}^{1/2}, \tag{3.2}$$
$$E = m_0c^2(1-\beta^2)^{-1/2}, \tag{3.3}$$
$$E = \pm\{(m_0c^2)^2 + (mc^2)(mv^2)\}^{1/2}, \tag{3.4}$$

where m is the relativistic mass, $\beta = v/c$ and $\gamma = m/m_0 = (1-\beta^2)^{-1/2}$. In PR, β and γ need not be constants, however, their instantaneous values are related with each other and with other parameters in the same manner as in special relativity.

3.1 True force

In order to come up with a truely invariant relationship between force and energy, we shall differentiate Eq. (3.1) w.r.t. time.

$$\frac{dE}{dt} = \frac{d}{dt}\{E_0^2 + h^2\nu^2\beta^2\}^{1/2} = \mathbf{vF}, \tag{3.5}$$
$$\mathbf{vF} = \frac{1}{2E}\left(2h^2\nu^2\beta\frac{d\beta}{dt} + 2h^2\beta^2\nu\frac{d\nu}{dt}\right), \tag{3.6}$$
$$\mathbf{vF} = \frac{1}{2E}\left(2E^2\frac{v}{c^2}\frac{dv}{dt} + 2Eh\frac{v^2}{c^2}\frac{d\nu}{dt}\right), \tag{3.7}$$

where $\mathbf{F}$ is the modified Lorentz force which we shall call the true force and $\mathbf{v}$ the velocity vector. Eq. (3.5) reduces to

$$\mathbf{vF} = v\left(ma + \frac{hv}{c^2}\frac{d\nu}{dt}\right), \tag{3.8}$$

where $a = dv/dt$ is the acceleration of the particle. From Eq. (3.8) we can deduce that the change in the energy of the particle is associated with two different changes in the state of the particle.

ffl The change in the velocity of the particle.
ffl The change in the frequency of the associated de Broglie wave.

When the second aspect is neglected, the invariant relationship between the force and the energy is lost. With respect to the massive particles, this second term on the right is comparable to the Doppler effect. Hence we will call it the de Broglie effect; and since this second term also has the units of force, we shall call this new force the de Broglie force. This shows that the true force consists of a sum of two forces, the Lorentz force and the de Broglie force. It can be shown that the true force $\mathbf{F}$ bears same relationship with tensorial Minkowski force, $\mathbf{F} = \gamma K^i$, which Minkowski force bears with Lorentz force, $K^i = \gamma F_i$.

Even though there are equations in Einstein's relativity for relating force and energy, it remains a fact that the relativity theory has failed to provide satisfactory quantification of force and energy. The principle reason in my opinion is the concept of time as adopted by the relativity theory which assumes that the time is linear and flow in one direction from past to present and from present to future.

This prevailing concept of time moving in one direction is a self-imposed illusion of the mind, just like imagining a blue sky which in reality is colorless, or riding a marry-go-round while all the time thinking that we are moving forward. Other authors have arrived at similar conclusion by analyzing the block universe concept [15]. Relativity no doubt unifies the space and the time continuum, but because of the adoption of the linear time scales, it becomes very convenient to compromise the invariance of force and energy.

Both the classical as well as the relativistic mechanics are founded upon the assumption that dv/dt is always zero for calculations involving force and energy. This is the very reason for which general relativity has failed to provide satisfactory quantification of force and energy. So whether one should hold on to the concept of proper time and linearity of time scales or adopt the view that the reality is based on non-linear periods? The answer to this question may not be very simple, but one thing is certain that if we assume $dv/dt = 0$, then the derivation of gravitational redshift discussed in [16] and the solution of Pioneer anomaly [17; 18; 19; 20] presented in [1] would not be possible.

4 Two-body problem and the gravitational field

4.1 Dynamic weak equivalence principle

In order to deal with the static spherically symmetric gravitational field produced by a spherically symmetric body at rest, we work in a single plane and base our formalism on following postulate which basically means that the orbital energy is constant and consists of sum of kinetic energy and gravitational potential energy.

> In empty space, the rate of change of kinetic energy of a particle is equal to the rate of change of potential energy influencing the particle.

In classical two-body problem, the gravitational field gets introduced as a single central potential acting radially, while the transverse component is assumed absent. In PR we will introduce gravitational field in terms of two components of a force acting normal and tangential to the particle trajectory, rather than as a resultant central force acting radially. This will allow us to account for the curvature of the trajectory. Moreover PR is not opposed to many of the conclusions of special and general relativity such as mass energy equivalence, distinction between rest mass and the relativistic mass, perihelic precession formalism, quadrupole formalism etc. Hence we will introduce another modification to Newton's inverse square law on following grounds. It is well recognized that the light is affected by the central gravitational potential due to rest mass of a body just like any other massive orbiting body. Making use of this observation we conclude that the mass of the orbiting body in the inverse square law should be relativistic mass and not the rest mass. This is because photon does not have any rest mass but only kinetic energy which can be correctly represented in terms of relativistic mass. And this relativistic mass is a variable parameter in gravitational field and not a constant. We shall adhere to this principle even while discussing massive bodies in orbit which also have some kinetic energy. Proof of the correctness of this assumption is evident in the derivation of the orbital period derivative of a binary star discussed in [21]. These two changes introduces two more variables in addition to the radial distance in the formalism of central potential. Hence it is not very straight forward to introduce the classical theories of gravitational potentials in PR. However, in PR also the central potential acts radially and the transverse component is assumed absent. Therefore when the second body has only radial motion as in the case of gravitational redshift of light, the two additional variables disappear and the potential reduces to classical Newtonian potential. However, this does not happen in case of bending of light.

How does the assumption that the gravitational attraction exists between the relativistic masses reflect on the equivalence principle? It appears that this assumption does not violate any of the three equivalence principles, the weak (WEP), Einstein (EEP) and the strong (SEP). "Universality of free fall" is based on Newton's weak equivalence principle (WEP) which states: "the property of a body called mass is proportional to the weight." In WEP Newton did not specify whether the mass is the rest mass or relativistic mass and no discussion of motion. In Einstein's notion, free fall indicates inertial mass as well as relativistic mass but again in consideration of E*ö*tv*ö*s experiment only inertial mass is equated to gravitational mass. Besides, E*ö*tv*ö*s experiment is a static experiment which does not involve moving masses. The present theory conforms with E*ö*tv*ö*s experiment when two gravitating bodies are in equilibrium and at rest in the same coordinate system. For example, if an object is thrown radially upwards in the coordinate system of the earth, it will attain a maximum height and then freely fall back to the earth. Momentarily when the object is at the maximum height, both the object and the earth are perfectly at rest in the same coordinate system. At this moment, the relativistic mass of the object is the same as its rest mass. Therefore at this moment of equilibrium, the gravitational mass is equal to inertial mass. Rest of the time it is the relativistic mass which is equal to the gravitational mass. This is the dynamic version of WEP we have introduced in this theory which states that the gravitational mass of a body is equal to its relativistic mass.

Whether one is working with the coordinate time of the central potential or with the proper time of the orbiting body, one of the two masses would certainly be the relativistic mass. So this factor needs to be considered. The effects of dynamic WEP gets absorbed in what is later defined as the deviation factor "n" and eventually shows up as a deviation in the proper time interval of the body. And proper time interval of the planet is not a part of the present day ephemerides [22; 23; 24; 25]. As long as the proper time interval is not included as one of the observables in ephemerides, there is no way to compare the GR predictions with the PR theory. The present day ephemerides are a three dimensional ephemerides. Introduction of proper time interval as a variable orbital parameter would introduce the fourth dimension to the ephemerides.

Figure 1 shows the radial vector $\mathbf{r}$ connecting the central mass M_0 with the particle in motion having rest mass m_0. θ is the polar angle and ψ is the angle between the radial vector $\mathbf{r}$ and the tangent vector $\widehat{\mathbf{T}}$. Here we are dealing with two coordinate systems, one centered on the central mass with polar parameters and another centered on the particle in motion with axes along the tangent and the normal to the trajectory. Both these coordinate systems are oriented w.r.t. each other in such a way that the normal vector and the radial vector make an angle equal to $((\pi/2) - \psi)$ between them, and ψ is a variable. Therefore in PR the transverse component is absent w.r.t. polar coordinate system of the central mass but not w.r.t. the coordinate system of the particle in motion. Hence, the rate of change of potential energy influencing the particle can be given by

$$\frac{dE_p}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v} =, \tag{4.1}$$

$$-\frac{\mu m}{r^2}(\cos\psi + \sin\psi)\,\hat{\boldsymbol{r}}\cdot\mathbf{v} = -\frac{\mu m}{r^2}\hat{\boldsymbol{r}}\frac{d\mathbf{r}}{dt}(\cos\psi + \sin\psi)\,. \tag{4.2}$$

where $\mu = GM_0$, $G =$ gravitational constant, $m =$ relativistic mass and following relations hold as usual.

$$\hat{\boldsymbol{r}}\cdot\frac{d\mathbf{r}}{dt} = \hat{\boldsymbol{r}}\cdot\mathbf{v} = \left(\frac{dr}{dt}\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}} + \frac{rd\theta}{dt}\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{r}}\right) = \frac{dr}{dt}.$$

Gravitational potential can be deduced from Eq. (4.2) as

$$-\int\frac{\mu\gamma}{r^2}(\cos\psi + \sin\psi)\,dr, \tag{4.3}$$

where $\gamma = m/m_0$ is a variable. γ and ψ both are functions of r.

4.2 Massless particles in gravitational field

For massless particles, the rate of change of kinetic energy can be given by Eq. (3.8) as described below. Following is applicable to all massless particles.

ffl The particles will have velocity equal to c. The particles cannot be accelerated along the direction of motion. The wavefront can, however, be accelerated normal to the direction of motion. The wave can be subjected to Doppler shift along the direction of motion.

4.2.1 Gravitational redshift in periodic relativity

In PR the light is a wave and does not have any inertial mass, only the relativistic mass which is equivalent to its kinetic energy. We can apply Eqs. (4.2) and (3.8) to the gravitational redshift problem [26; 27; 28; 29] involving a photon trvelling in a straight line from the sun to the earth along a path connecting their centers. In this case [16] we have $a = 0$ and $\psi = 0$, thus

$$c\left(ma+\frac{h}{c}\frac{d\nu}{dt}\right)(-\hat{\boldsymbol{r}})=-\frac{\mu m}{r^2}c\,(\cos\psi+\sin\psi)\,\hat{\boldsymbol{r}}, \tag{4.4}$$

$$\frac{\mu m}{r^2}=\frac{h}{c}\frac{d\nu}{dt}=\frac{h}{c}\frac{d\nu}{dr}\frac{dr}{dt}=h\frac{d\nu}{dr},$$

$$\frac{\mu}{r^2}=\frac{hc^2}{mc^2}\frac{d\nu}{dr},\quad \frac{\mu}{r^2}dr=\frac{h}{E}c^2d\nu=c^2\frac{1}{\nu}d\nu.$$

Integration over the entire trajectory gives

$$\int_{\Delta}^{\Delta+l}\frac{\mu}{r^2}dr=c^2\int_{\nu_s}^{\nu_\infty}\frac{1}{\nu}d\nu, \tag{4.5}$$

where Δ is the solar radius, l the distance travelled by photon, ν_s the frequency of light on the surface of the sun and ν_∞ the frequency of light on earth. This gives

$$\frac{\mu}{c^2}\frac{l}{(\Delta^2+\Delta l)}=\frac{\varphi_1-\varphi_2}{c^2}=\ln\left(1-\frac{\delta\nu}{\nu_s}\right), \tag{4.6}$$

$$-\frac{\delta\nu}{\nu_s}=\frac{1}{c^2}(\varphi_1-\varphi_2)+\frac{1}{c^4}\left(\frac{1}{2}\varphi_1^2-\varphi_1\varphi_2+\frac{1}{2}\varphi_2^2\right)+\mathscr{O}\left(\frac{1}{c^6}\right). \tag{4.7}$$

The first order term is exactly the same value predicted by general relativity in terms of gravitational potentials φ_1 and φ_2 at locations separated by distance l, and verified experimentally [26; 27; 28; 29] with a high level of accuracy. The second order term for general relativity is slightly different,

$$+\frac{1}{c^4}\left(-\frac{1}{2}\varphi_1^2-\varphi_1\varphi_2+\frac{3}{2}\varphi_2^2\right), \tag{4.8}$$

and below the accuracy [3; 27; 30; 31; 32] in the measurement of gravitational redshift. For $l > 100\Delta$, Eq. (4.6) approaches

$$-\frac{\delta\nu}{\nu_s}=\frac{\mu}{c^2\Delta}. \tag{4.9}$$

For $l < 0.01\Delta$, Eq. (4.6) approaches the formula for the Doppler effect [26], given by

$$-\frac{\delta\nu}{\nu_s}=\frac{\mu l}{c^2\Delta^2}=\frac{gl}{c^2}=\frac{\vartheta}{c}. \tag{4.10}$$

It should be noted here that if Eqs. (1.6) and (1.7) had nothing to do with Eq. (1.5), then it would be impossible to derive the gravitational redshift Eq. (4.6). It is mainly due to the periodic representation of Eq. (1.6) that the frequency term appears in Eqs. (2.1), (3.1), (3.8), and (4.4). This makes it possible to derive Eq. (4.6) without mentioning linear time or linear distance and without utilizing Riemannian geometry and geodesic trajectories.

The special theory of relativity assumes global coordinate systems and global invariance of speed of light as well as the equivalence of mass and energy. Einstein had to abandon the global coordinates and global invariance of speed of light while formulating general relativity because they were in conflict with the principle of equivalence of inertial mass and gravitational mass. However, he permitted the existence of a local system of inertial coordinates in a small region around any event. In PR we have gone one step further and restricted the local system of inertial coordinates to have only instantaneous existence. This makes the proper time a continuously variable phenomenon, which could now be identified with the continuously variable period of the body associated with the inertial coordinate. This has significant effect if the body is a fundamental particle such as a photon. When such instantaneous coordinate system is fixed on a massive body such as a planet, it acts exactly like the coordinate system of general relativity because the period of the associated wave of the planet does not change significantly from instance to instance and such is also the case with its proper time which remains practically constant. Similarly any fundamental particle travelling along a constant radial distance from a central massive body shall also have constant period and constant proper time. This is consistent with the general relativity definitions of tangential and radial velocities of light in a gravitational field [3] given in geometrical units ($c_0 = 1$) by ,

$$c_t = 1 + \varphi = \sqrt{1 + 2\varphi} = \sqrt{c_r}. \tag{4.11}$$

In general relativity, the rate of proper time at a fixed radial position in a gravitational field relative to the coordinate time can be obtained from general form of the metrical space time line element for a spherically symmetrical static field in polar coordinate and is given by

$$\frac{d\tau(r)}{dt} = \sqrt{g_{tt}(r)}. \tag{4.12}$$

From Schwarzschild metric we have $g_{tt}(r) = 1 + 2\varphi$. Now in general relativity, there is no explicit derivation of formula for gravitational redshift, but it is implicitly deduced from Eq. (4.12) and given by

$$\frac{\nu_2}{\nu_1} = \frac{\sqrt{1 + 2\varphi_1}}{\sqrt{1 + 2\varphi_2}}. \tag{4.13}$$

The only support this formula has is the experimental verification of the first order term. Hence we would not be violating any scientific law if we propose that the correct implication of Eq. (4.12) is

$$\frac{\nu_2}{\nu_1} = \frac{e^{\sqrt{1+2\varphi_1}}}{e^{\sqrt{1+2\varphi_2}}}. \tag{4.14}$$

Equation (4.14) also yields the same first order term, besides it can also be explicitly derived and is exactly the same formula (in geometrical units) given by Eq. (4.6). Therefore it is to be noted here that Eqs. (1.6) and (1.7) has this unique property that they remain null even in a gravitational field. This is not the case with Eq. (1.5). This would imply that in PR, for light waves in all inertial frames, we get,

$$\frac{d\tau(r)}{dt} = 1. \tag{4.15}$$

4.2.2 Bending of light in periodic relativity

We can apply equation Eq. (4.4) to the bending of light problem [27; 33; 34]. In this case [16] we have $d\nu/dt = 0$ (because the ray is equally blue shifted and then red shifted) and frequency shift is 0 at the limb of the sun. As shown in Fig. 2, ψ will vary from $\pi/2$ to 0 as the star light photon approaches earth from the limb of the sun.

$$\begin{aligned} ma(-\hat{\boldsymbol{r}}) &= m\left(\frac{d^2s}{dt^2}\widehat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\widehat{\mathbf{N}}\right) \\ &= -\frac{\mu m}{r^2}\left(\cos\psi + \sin\psi\right)\hat{\boldsymbol{r}}, \end{aligned} \tag{4.16}$$

where $ds/dt = c$, $d^2s/dt^2 = 0$ and $\kappa = d\phi/ds =$ curvature of the trajectory. Hence $ma(-\hat{\boldsymbol{r}}) = mc^2(d\phi/ds)\widehat{\mathbf{N}}$. Therefore,

$$mc^2\frac{d\phi}{ds} = \frac{\mu m}{r^2}\left(\cos\psi + \sin\psi\right). \tag{4.17}$$

As shown in Fig. 2, for half of the trajectory, as ψ changes from $\pi/2$ to 0, and s changes from 0 to ∞, angle ϕ will change from 0 to some value ϕ and this value of ϕ can be determined by integrating over half of the trajectory as follows. It is to be noted that both components of radial acceleration contributes to the curvature of trajectory but only cosine component contributes to the tangential velocity vector. This is true because when $d^2s/dt^2 = 0$ for light, one would expect the cosine component to be zero but this is not the case.

$$\int_0^{\phi}\int_{\frac{\pi}{2}}^{0} d\psi d\phi = \frac{\mu}{c^2}\int_0^{\infty}\int_{\frac{\pi}{2}}^{0}\frac{1}{r^2}\left(\cos\psi + \sin\psi\right)d\psi ds. \tag{4.18}$$

Substituting $r^2 = s^2 + \Delta^2$ and taking limits we get, $\phi = 2\mu/(c^2\Delta)$, and for the entire trajectory we get the same result for the lowest order term as predicted by general relativity, which is verified with a great deal of accuracy [27; 34; 35],

$$2\phi = \frac{4\mu}{c^2\Delta}. \tag{4.19}$$

As can be seen, the higher order terms does not exist in PR theory. The second order term in general relativity is below the accuracy in the measurement of deflection of light [3; 27; 30; 34; 35], and is given by

$$+\left(\frac{15\pi}{4}-4\right)\left(\frac{\mu}{c^2\Delta}\right)^2. \tag{4.20}$$

Experimental verification of the second order effect given by Eq. (4.20) is the principal goal of LATOR mission [36] planned for the year 2009. If this theory is correct, the experiment will yield null result. If the fundamental postulates of a theory, physical or mathematical, are built upon approximations, then there is a chance of appearance of pseudo terms resembling higher order terms in the end results. We have a reason to believe that the orbital energy equation in PR is exact in nature, and that is not the case with general relativity. In general relativity, Newtonian potential gets introduced into metric component g_{00} as a deviation to flat Minkowski metric. This constitutes the weak-field approximation. For this very reason Schwarszchild metirc does not remain null for light in the gravitational field as we have already discussed. Other competing theories modify Newtonian potential by way of Poisson's equation and multipole expansion. Another significant approximation in Schwarzschild solution is the assumption that the angle of deflection is subtended at the center of the sun. In PR, eventhough the schematic diagram shows the same arrangement, the calculations give us actual angle ϕ measured between the line $\theta = 0$ and the line normal to the velocity vector at the end of the trajectory. These factors add up to give different second order terms in these two theories. It is interesting to note that the higher order terms are in higher powers of Newtonian potential. It should also be noted that we have utilized Eq. (1.6) for the slow moving accelerating particle for determining the bending of light.

4.2.3 Black hole in periodic relativity

If we introduce circular trajectory parameters $dv/dt = 0$, $\psi = \pi/2$ and $r = r_l =$ *constant* in Eq. (4.17), and integrate over single orbit, we get,

$$d\phi = \frac{\mu}{c^2 r_l^2} ds, \tag{4.21}$$

$$\int_0^{2\pi} d\phi = \frac{\mu}{c^2 r_l^2} \int_0^{2\pi r_l} \mathrm{d}s. \tag{4.22}$$

This gives limiting radius of the event horizon which is half the value of Schwarzschild radius [3],

$$r_l = \frac{\mu}{c^2}. \tag{4.23}$$

This is a major difference between the PR and the general relativity which can be subjected to experimental verification. The black hole appears explicitly in PR and there is no ambiguous singularity to explain. The weak-field approximation in general relativity describes the gravitational field very far from the source of gravity. Near the limiting radius of a black hole, the gravitational field could no longer be considered weak. Therefore derivation of Schwarzschild radius does not have a very sound basis. The appearance of singularity could simply mean that the equations break down at this radius. In PR the gravitational potential is not introduced as a small deviation in the formalism. So when its value go very high near the event horizon, it does not affect our formalism.

Efforts to image the *event horizon* of a super-massive black hole such as the one in our own galaxy, Sgr A*, are already underway [37; 38; 39; 40; 41; 42].

4.3 Massive particles in gravitational field

From Eqs. (4.16) and (3.8) we get,

$$a(-\hat{\boldsymbol{r}}) = \left(\frac{d^2 s}{dt^2}\widehat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\widehat{\mathbf{N}}\right)$$

$$= \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}\left(\cos\psi + \sin\psi\right), \tag{4.24}$$

$$\mathbf{vF} = v\left(m\sqrt{\left(\frac{d^2 s}{dt^2}\right)^2 + \kappa^2\left(\frac{ds}{dt}\right)^4} + \frac{hv}{c^2}\frac{d\nu}{dt}\right). \tag{4.25}$$

So any conversion of acceleration between radial and tangential directions is accompanied by the conversion factor $(\cos\psi + \sin\psi)$. This factor acts as a single scalar quantity and does not get split into normal and tangential vector components. It also needs to be understood that $d^2\mathbf{r}/dt^2$ is a radial vector but $d\mathbf{r}/dt$ is not a radial vector which acts along the velocity vector $\mathbf{v}$. Therefore factor $(\cos\psi + \sin\psi)$ does not play any role in this expression of velocity $\mathbf{v} = d\mathbf{r}/dt$ which remains unaltered.

4.3.1 Perihelic precession of planets

We assume that the general relativity theory as applicable to solar system planets is valid in weak-field approximation. We also declare however, that the theory fails to predict accurate higher order terms for gravitational red-shift and deflection of light, because in introducing the weak-field approximation, it compromises the global invariance of speed of light in gravitational field. The weak-field approximation also leads to the value of limiting radius of the event horizon of a black hole which is twice the correct value. The weak field approximation does not account for second order velocity term in the kinetic energy. However, while Schwarzschild solution is sufficiently accurate in predicting the perihelic precession [3; 43; 44; 45] of the planets of the solar system, it may not be dependable for describing the photon trajectories in strong gravitational fields. Not only so, even trajectories of massive bodies in a strong gravitational field or extremely weak-gravitational field compared to Sun can also deviate significantly from the Schwarzschild solution. We can blend the general relativity theory [46; 47] with the PR theory in following manner.

From Eq. (3.2), the kinetic energy of a planet can be given by

$$(E - m_0 c^2) = m_0 c^2\left[1 + \gamma^2(v^2/c^2)\right]^{1/2} - m_0 c^2. \tag{4.26}$$

Differentiating w.r.t. time we get,

$$\frac{d}{dt}(E - m_0c^2) = m_0 v\frac{dv}{dt}\left(1 + \frac{3}{2}\frac{v^2}{c^2} - \frac{3}{2}\frac{v^4}{c^4} - \frac{v^6}{c^6}\right)$$
$$\approx m_0 v\frac{dv}{dt}\left(1 + \frac{3}{2}\frac{v^2}{c^2}\right). \tag{4.27}$$

Using vector notation we can equate Eqs. (4.2) and (4.27) as follows.

$$m_0 \mathbf{v}\frac{d\mathbf{v}}{dt}\left(1 + \frac{3}{2}\frac{v^2}{c^2}\right) = -\frac{\mu m}{r^2}\mathbf{v}\,(\cos\psi + \sin\psi)\,\hat{\boldsymbol{r}}. \tag{4.28}$$

Since the transverse component of the gravitational acceleration is absent in the polar coordinate system of the central potential, we can write

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - \frac{h^2}{r^3}\right)\hat{\boldsymbol{r}}. \tag{4.29}$$

From Eq. (4.24) we have the relation

$$\frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}\,(\cos\psi + \sin\psi)\,. \tag{4.30}$$

Substitution of $m = \gamma m_0$ in Eq. (4.28) gives,

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^2}\left(1 - \frac{v^2}{c^2}\right)\hat{\boldsymbol{r}}. \tag{4.31}$$

Here we introduce deviation to the flat Minkowski metric due to the gravitational field in the form,

$$\left(\frac{dt}{d\tau}\right)^2 = \gamma^{2n} = (1-\beta^2)^{-n}, \tag{4.32}$$

$$d\tau = dt\left(1 - \frac{nv^2}{2c^2}\right), \tag{4.33}$$

to the first order accuracy for small values of v and n, where t is the coordinate time, τ the proper time of the orbiting body and n is a real number. The corresponding line element in polar coordinates is,

$$ds^2 = c^2dt^2 - ndr^2 - nr^2d\theta^2 - n(r^2\sin^2\theta)d\phi^2. \tag{4.34}$$

Therefore,

$$\frac{d^2\mathbf{r}}{d\tau^2} = -\frac{\mu}{r^2}\left(1 + (n-1)\frac{v^2}{c^2}\right)\hat{\boldsymbol{r}}. \tag{4.35}$$

$$\left(\frac{d^2r}{d\tau^2} - \frac{h^2}{r^3}\right)\hat{\boldsymbol{r}} = -\frac{\mu}{r^2}\left(1 + (n-1)\frac{v^2}{c^2}\right)\hat{\boldsymbol{r}}. \tag{4.36}$$

Substitution of following gives a second order non-homogeneous, non-linear differential equation.

$$u = \frac{1}{r}, \quad \text{and} \quad \frac{d}{d\tau} = hu^2 \frac{d}{d\theta}. \tag{4.37}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\left(1 + (n-1)\frac{v^2}{c^2}\right). \tag{4.38}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + (n-1)\frac{\mu}{h^2}\frac{v^2}{c^2}. \tag{4.39}$$

Therefore the condition for PR to conform with the general relativity prediction is,

$$\frac{3\mu u^2}{c^2 \sin^2\psi} = (n-1)\frac{\mu}{h^2}\frac{v^2}{c^2}. \tag{4.40}$$

Here we have introduced the term $\sin^2\psi$ which does not occur in Schwarzschild solution but is necessary for n to satisfy Einstein's field equations. We will show that $\sin^2\psi$ does not contribute anything to the perihelic precession but further simplifies Schwarzschild solution. This gives

$$n = \left[1 + \frac{3h^2}{v^2 r^2 \sin^2\psi}\right]. \tag{4.41}$$

The vector constant called angular momentum per unit mass $\mathbf{h}$ is defined as

$$\mathbf{h} = \frac{\mathbf{L}}{m} = \frac{\mathbf{p}\times\mathbf{r}}{m} \equiv \frac{|\mathbf{p}||\mathbf{r}|\sin\psi}{m}\hat{\boldsymbol{h}}. \tag{4.42}$$

$$\mathbf{h} = \mathbf{r}\times\frac{d\mathbf{r}}{dt} = \mathbf{r}\times\left(\frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = r^2\frac{d\theta}{dt}\hat{\boldsymbol{h}}. \tag{4.43}$$

Therefore the scalar quantity $h^2 = v^2r^2\sin^2\psi$ and the deviation factor $n = 4 = constant$. The angle between the radial vector and the velocity vector ψ is defined as

$$\psi = \tan^{-1}\frac{r}{\dot{r}} \quad \text{where } \dot{r} = \frac{dr}{d\theta}. \tag{4.44}$$

$$\sin\psi = (r/\dot{r})/\sqrt{(r/\dot{r})^2 + 1}. \tag{4.45}$$

$$\sin^2\psi = (r/\dot{r})/\sqrt{(r/\dot{r})^2 + 1}. \tag{4.46}$$

$$\sin^2\psi = \frac{r^2}{r^2 + \dot{r}^2} = \left[1 + \frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2\right]^{-1}. \tag{4.47}$$

Substitution of Eqs. (4.40) and (4.47) in Eq. (4.39) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{3\mu u^2}{c^2} + \frac{3\mu}{c^2}\left(\frac{du}{d\theta}\right)^2. \tag{4.48}$$

The last term in Eq. (4.48) contributes two more terms to the Schwarzschild solution for u. These are

$$\frac{3\mu^3 e^2}{2c^2h^4} + \frac{\mu^3 e^2}{2c^2h^4}\cos 2\theta. \tag{4.49}$$

The last term in Eq. (4.49) cancells out same term with negative sign in Schwarzschild solution. Therefore the final solution of Eq. (4.48) is of the form

$$u = \frac{\mu}{h^2}[1 + k(1+e^2) + e\cos(\theta - k\theta)], \tag{4.50}$$

where $k = 3(\mu/ch)^2$. Another approach to solving the second order differential equation (4.48) is to directly simplify r.h.s. to the form

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}(1 + k(1+e^2) + 2ke\cos\theta), \tag{4.51}$$

Again the solution of Eq. (4.51) is same as Eq. (4.50). Equation (4.51) can also be solved in another way by writing

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} - \frac{\mu}{h^2}k(1-e^2) + 2ku, \tag{4.52}$$

$$\frac{d^2u}{d\theta^2} + (1-2k)u = 7\frac{\mu}{h^2}[1 - k(1-e^2)]. \tag{4.53}$$

$$\frac{1}{(1-2k)}\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\frac{[1-k(1-e^2)]}{(1-2k)}. \tag{4.54}$$

Equation (4.54) is a simple harmonic oscillator of the form $A\ddot{u} + u = B$ where A and B are constants. The general solution of this equation is

$$u = B[1 + C\cos(\theta/\sqrt{A})], \tag{4.55}$$

where C is a constant of integration. For a very small value of k we can have $1/\sqrt{A} = \sqrt{1-2k} \approx (1-k)$.

$$u = \frac{\mu}{h^2}[1 - k(1-e^2)](1+2k)[1 + C\cos(\theta - k\theta)]. \tag{4.56}$$

Ignoring terms of order k^2 we can write

$$u = \frac{\mu}{h^2}[1 + k(1+e^2)][1 + C\cos(\theta - k\theta)]. \tag{4.57}$$

If factor k was equal to zero, Eq. (4.57) could represent an ellipse and constant C would signify the eccentricity e and the constant μ/h^2 would represent the semilatus rectum. However, k is slightly greater than zero causing θ to go beyond 2π to complete one orbital cycle, consequently the axis of the ellipse precesses slightly. Value $k\theta$ is exactly same as that of the Schwarzschild solution. This solution (4.57) can be same as Eq. (4.50) if we select

$$C = e/[1 + k(1+e^2)]. \tag{4.58}$$

4.3.2 Proper time of a planet

We substitute Eq. (4.41) for constant n in Eq. (4.33) for proper time and for second order velocity term on the right we will introduce classical vis-viva equation for planetary velocity

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right). \tag{4.59}$$

$$d\tau = dt \left(1 - \frac{2\mu}{c^2} \left(\frac{2}{r} - \frac{1}{a} \right) \right). \tag{4.60}$$

Substituting $r = h^2/\mu(1 + e\cos\theta)$ to obtain

$$d\tau = dt \left(1 - \frac{2\mu(1 + 2e\cos\theta + e^2)}{c^2 a(1 - e^2)} \right). \tag{4.61}$$

For circular orbits $r = a$ and $e = 0$. This gives

$$d\tau = dt \left(1 - \frac{2\mu}{c^2 a} \right). \tag{4.62}$$

GR does not have a convenient way of expressing proper time equation such as Eq. (4.61) but it does provide expression for proper time in equatorial Keplerian circular orbits which is comparable to Eq. (4.62). This GR expression is given by

$$d\tau = dt \left(1 - \frac{3\mu}{2c^2 R} \right). \tag{4.63}$$

In order to highlight the differences between two theories we will compare the results of Eqs. (4.61) and (4.63) with respect to planet mercury. At perihelion the results are

$$d\tau = dt \left(1 - 7.74 \times 10^{-8} \right), \tag{4.64}$$

$$\text{and for GR} \qquad d\tau = dt \left(1 - 4.815 \times 10^{-8} \right) \tag{4.65}$$

This difference per second if measured over a period of 1 h with measurement split equally on both sides of the semi-major axis comes to approximately 105.3 μs. At aphelion the values are

$$d\tau = dt \left(1 - 3.3602 \times 10^{-8} \right), \tag{4.66}$$

$$\text{and for GR} \qquad d\tau = dt \left(1 - 3.1725 \times 10^{-8} \right) \tag{4.67}$$

The difference over a period of 1 h is only 6.757 μs.

For a stationary observer with a clock Eq. (4.34) reduces to $d\tau = dt$ which would imply that the proper time of his clock would equal coordinate time for him regardless of his location. As mentioned earlier, proper time interval of the planet is not a part of the present day ephemerides [22; 23; 24; 25]. As long as the proper time interval is not included as one of the observables in ephemerides, there is no way to compare the GR predictions with PR.

If we try to introduce photon parameter $v = c$, we find that Eq. (4.35) can be satisfied only if we put $n = 0$. This yields for light $d\tau = dt$ in agreement with the previous result Eq. (4.15). While Eq. (4.41) gives us same perihelic precession values for the planets of the solar system as given by general relativity, in case of light trajectories, we can conviniently deviate from general relativity by substituting $n = 0$. This would also eliminate the implied singularity of Lorentz transformation as v approaches c. Similarly n can be associated with many different functions related to other strong field or extremely weak field systems. General relativity would allow only one universal value for n given by Eq. (4.41), but in PR we can treat n as a system parameter or as an orbital parameter dependent on the gravitational field strength, the orbital velocity and the natural frequency (and hence the composition) of the orbiting body.

Few comments regarding the concept of strong and weak gravitational fields. Whether the g-field is strong or weak should be decided by the combined effect of the g-fields of both the bodies, the distance between them and the relative orbiting velocity between them. Superscript n introduced in Eq. (4.32) provides the measure of the strength of the g-field defined in this manner.

When we talk about gravitational force depending on the composition and natural frequency of the body we may feel inclined to think that it would violate the WEP of Newton and Einstein. If we look at the proposal from the point of view of the dynamic WEP which states that the gravitational mass is equal to the relativistic mass, we find that there is no violation because the gravitational frequency shift of massive body would cause its relativistic mass to alter and not its rest mass. This alteration of the relativistic mass is reflected in its motion, i.e. its velocity. If one is determined to test the static WEP with respect to the gravitational frequency shift proposal, one should have two very massive objects of different composition but exactly the same inertial (rest) mass subjected to a very large gravitational gradient over a distance of few AU or few kpc and then there is a chance that one would discover both objects experiencing different gravitational force.

4.4 Newton's theory of gravity

Here we will see how the proposed new theory reduces to Newtonian theory of gravity in the non-relativistic limit. If we ignore the higher order relativistic terms in Eq. (4.27) we find that this expression is simply the time derivative of the classical kinetic energy in Newtonian theory. Similarly replacing dynamic WEP by Newtonian WEP would mean substituting rest mass m_0 for the relativistic mass m in Eq. (4.28) which can now be written as

$$m_0 \mathbf{v} \frac{d\mathbf{v}}{dt} = -\frac{\mu m_0}{r^2} \mathbf{v} \left(\cos\psi + \sin\psi\right) \hat{\boldsymbol{r}}. \tag{4.68}$$

Substitution of Eq. (4.30) or (4.24) in Eq. (4.68) gives

$$\frac{d^2\mathbf{r}}{dt^2} \left(\cos\psi + \sin\psi\right) = -\frac{\mu}{r^2} \left(\cos\psi + \sin\psi\right) \hat{\boldsymbol{r}}. \tag{4.69}$$

Substitution of $\mu = GM_0$ gives the inverse square law of Newtonian gravity.

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_0}{r^2} \hat{\boldsymbol{r}}. \tag{4.70}$$

Laws of Newtonian gravity are based on the flat Minkowski metric. Hence we can get Newtonian gravity in one step from the final result Eq. (4.38) by putting $n = 1$ for the flat Minkowski metric.

$$\frac{d^2u}{d\theta^2} + u = \frac{GM_0}{h^2}. \tag{4.71}$$

Equation (4.71) has the solution

$$u = \frac{GM_0}{h^2} + B\cos(\theta - \theta_0), \tag{4.72}$$

$$r = \frac{h^2/GM_0}{1 + (Bh^2/GM_0)\cos(\theta - \theta_0)}, \tag{4.73}$$

where B and θ_0 are arbitrary constants. Defining eccentricity and focal parameter by

$$e = \frac{Bh^2}{GM_0} \quad \text{and} \quad p = a(1 - e^2) = \frac{h^2}{GM_0}, \tag{4.74}$$

gives polar equation for conic section

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)}, \tag{4.75}$$

where a is the semi major axis and θ_0 the argument of pericenter.

Following Eq. (4.24) we clarified that $d^2\mathbf{r}/dt^2$ is a radial vector but $d\mathbf{r}/dt$ is not a radial vector which acts along the velocity vector $\mathbf{v}$. Therefore factor $(\cos\psi + \sin\psi)$ does not play any role in this expression of velocity $\mathbf{v} = d\mathbf{r}/dt$ which remains unaltered. This is crucial in leaving untouched the vector constant $\mathbf{h}$ in Newton's theory which defines the angular momentum per unit mass.

$$\mathbf{h} = \frac{\mathbf{L}}{m} = \frac{\mathbf{r}\times\mathbf{p}}{m} = \mathbf{r}\times\frac{d\mathbf{r}}{dt} = r^2\frac{d\theta}{dt}\hat{\boldsymbol{h}}. \tag{4.76}$$

Since $\mathbf{h}$ is constant, scalar $h = r^2(d\theta/dt)$ is also constant and is equivalent to Kepler's third law of equal areas in equal times, $dA = (r^2 d\theta)/2$.

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h = constant. \tag{4.77}$$

4.5 Einstein's field equations

Now we are in a position to write Eq. (4.34) in a metric form as follows.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{4.78}$$

$$g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -n & 0 & 0 \\ 0 & 0 & -nr^2 & 0 \\ 0 & 0 & 0 & -n(r^2\sin^2\theta) \end{pmatrix}. \tag{4.79}$$

It is to be noted that θ in Eq. (4.37) corresponds to ϕ in Eq. (4.34). We can replace constant 3 in Eq. (4.41) by an orbital variable parameter ξ and adjust its value to match the observed value of the perihelic precession for individual planets. As a matter of fact all future strong field variations in general relativity could be explained by adjusting this parameter ξ or replacing the entire function n if need be. This kind of adjustment is not possible in general relativity and other metric theories because that will affect the predicted values of deflection of light, gravitational redshift and the limiting radius of event horizon. This factor ξ may have an internal structure dependent on the natural frequency and composition of the orbiting body (gravitational frequency shift of the constituent massive particles of the body). If we alter this factor ξ in Eq. (4.41) with any suitable constant then n will always remain a constant and as shown below it will always satisfy Einstein's field equations.

The theory developed here is a stand alone theory and need not satisfy Einstein's field equations $R_{\mu\nu} = 0$, but it will be interesting to see whether or not Eq. (4.34) satisfy Einstein's field equations. For this purpose it will be necessary to calculate Christoffel symbols $\Gamma^{\sigma}_{\mu\nu}$. At the same time the proper time interval should be experimentally verified because all deviations and variations get accumulated in the expression for proper time and any error in the theory would show up there as well.

The metric (4.79) is diagonal, so the non-zero components of the contravariant metric tensor are $g^{\sigma\sigma} = 1/g_{\sigma\sigma}$. Hence the diagonality of the metric allows us to simplify the definition of the Christoffel symbols to

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right), \tag{4.80}$$

where the suffixes assume four values 0, 1, 2, 3 and no summations are implied. We consider the case of static spherically symmetric field produced by a spherically symmetric body at rest. Line element given by Eq. (4.34) is compatible with spherical symmetry. Coordinate x^0 is taken to be time t, and the spatial coordinates may be taken to be spherical polar coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. We can determine the values of $g_{\mu\nu}$ from metric (4.79),

$$\begin{aligned} &g_{00} = c^2, \quad g_{11} = -n, \quad g_{22} = -nr^2, \quad g_{33} = -nr^2 \sin^2\theta, \\ &g^{\mu\nu} = 1/g_{\mu\nu} \quad \text{and} \quad g_{\mu\nu} = 0, \quad g^{\mu\nu} = 0 \quad \text{for } \mu \neq \nu. \end{aligned} \tag{4.81}$$

Inserting these values into Eq. (4.80) we find that the only non-vanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^1_{11} = \frac{1}{2n}\frac{\partial n}{\partial r} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \\ &\Gamma^1_{22} = -r - \frac{r^2}{2n}\frac{\partial n}{\partial r} \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} + \frac{1}{2n}\frac{\partial n}{\partial r} \\ &\Gamma^1_{33} = -r\sin^2\theta \left(1 + \frac{r}{2n}\frac{\partial n}{\partial r}\right) \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta. \\ &\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} + \frac{1}{2n}\frac{\partial n}{\partial r} \end{aligned} \tag{4.82}$$

The expression for the Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\,\nu} - \Gamma^{\alpha}_{\mu\nu,\,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{4.83}$$

Einstein's law of gravitation requires the Ricci tensor to vanish in empty space. We can now write the components of the Ricci tensor, each of which must vanish in order for the field equations to be satisfied. From symmetry arguments we can expect all the non-diagonal components to be zero. Hence the only components of interest in case of our line element are the diagonal elements. Substitution of Eq. (4.82) in Eq. (4.83) gives

$$R_{00} = 0, \tag{4.84}$$

$$R_{11} = \frac{1}{nr}\frac{\partial n}{\partial r} - \frac{1}{n^2}\left(\frac{\partial n}{\partial r}\right)^2 + \frac{1}{n}\frac{\partial^2 n}{\partial r^2}, \tag{4.85}$$

$$R_{22} = \frac{3r}{2n}\frac{\partial n}{\partial r} - \frac{r^2}{4n^2}\left(\frac{\partial n}{\partial r}\right)^2 + \frac{r^2}{2n}\frac{\partial^2 n}{\partial r^2}, \tag{4.86}$$

$$R_{33} = R_{22}\sin^2\theta. \tag{4.87}$$

The vanishing of Eq. (4.85) leads to

$$\frac{\partial^2 n}{\partial r^2} = \frac{1}{n}\left(\frac{\partial n}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial n}{\partial r}. \tag{4.88}$$

Substituting of Eq. (4.88) in Eq. (4.86) and equating it to zero gives the condition for vanishing of the Ricci tensor Eq. (4.83).

$$\left(\frac{r}{n}\frac{\partial n}{\partial r}\right)^2 + 4\left(\frac{r}{n}\frac{\partial n}{\partial r}\right) = 0. \tag{4.89}$$

This quadratic equation has two solutions.

$$\left(\frac{r}{n}\frac{\partial n}{\partial r}\right) = 0 \quad \text{and} \quad \left(\frac{r}{n}\frac{\partial n}{\partial r}\right) = -4. \tag{4.90}$$

This shows that any constant value of n will satisfy the first solution. Therefore Einstein's field equations are exactly satisfied for $n = 0$. This means that our derivation of gravitational redshift, deflection of light and the limiting radius of event horizon of a black hole discussed in [16] are exact solutions of Einstein's field equations. These solutions, however, are at variance with the Schwarzschild solution. The equations are also satisfied for $n = 4$ in case of planetary orbits and perihelic precession. One thing has to be noted, however, that if we introduce parameters of Keplerian ellipse in Eq. (4.41) and then calculate the value of n, we will find that the value will vary along the trajectory between 4 and $4 - 3e^2$. This is because the geodesic trajectories are not perfectly elliptical.

5 Field equations in PR in presence of matter

We have from Eqs. (4.3) and (4.26),

$$(E - m_0 c^2) = \Phi m_0 = -\int \frac{\mu m}{r^2} (\cos\psi + \sin\psi)\, dr. \tag{5.1}$$

For cosmological application we are only interested in radial motions hence we take $\psi = 0$. Secondly for small radial motions we assume $\gamma \approx const.$ which gives

$$mc^2 - m_0 c^2 = \frac{\mu}{r}\gamma m_0, \tag{5.2}$$

$$\{1 - (1/\gamma)\}c^2 = \frac{\mu}{r}, \tag{5.3}$$

The energy–momentum invariant Eq. (1.9) gives

$$\gamma = (m/m_0) = \pm\{1 - (v^2/c^2)\}^{-1/2}, \tag{5.4}$$

Here the $\pm$ sign is due to the positive and negative energies of Dirac's theory. Introduction of Eq. (5.4) in Eq. (5.3) gives

$$c^2(1 \mp \{1 - (v^2/c^2)\}^{1/2}) = \frac{\mu}{r}, \tag{5.5}$$

$$c^2 - v^2 = \left[\frac{\mu}{rc} - c\right]^2, \tag{5.6}$$

$$c^2 dt^2 - (dx^2 + dy^2 + dz^2) = \left[\left(\frac{\mu}{rc}\right)^2 + c^2 - \frac{2\mu}{r}\right] dt^2 = ds^2, \tag{5.7}$$

Equation (5.7) is simply the flat Minkowski metric given by Eq. (4.34) when $n = 1$, and this equation is based on the conservation of energy equation (5.1). For application in cosmology we can introduce deviation factor n in Eq. (5.7) and then assuming $(\mu/rc)^2$ to be negligibly small, the general line element satisfying the Weyl postulate and the cosmological principle can be given by

$$\begin{aligned} ds^2 &= c^2 dt^2 - na^2\left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\ &= \left[c^2 - \frac{2\mu}{ar}\right] dt^2, \end{aligned} \tag{5.8}$$

where $a(t)$ is the scale factor and parameter K is equal to $+1$ or 0 or -1 as in Friedmann model and decides the curvature of 3-surfaces. All the observable evidence indicate that the universe is near flat corresponding to $K = 0$, so we introduce this value in Eq. (5.8) at the outset. This and the fact that each galaxy has a constant set of coordinates (r, θ, ϕ), will considerably simplify the mathematics required for analyzing the model. For small and constant values of n, line element Eq. (5.8) does satisfy Einstein's field equation $R_{\mu\nu} = 0$ [16]. We can write this equation as

$$\frac{2\mu}{ar} dt^2 - na^2\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) = 0. \tag{5.9}$$

This can be transformed to metric form as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = 0, \qquad \text{where} \tag{5.10}$$

$$g_{\mu\nu} = \begin{pmatrix} 2\mu/ar & 0 & 0 & 0 \\ 0 & -a^2 n & 0 & 0 \\ 0 & 0 & -a^2 n r^2 & 0 \\ 0 & 0 & 0 & -a^2 n (r^2 \sin^2\theta) \end{pmatrix}. \tag{5.11}$$

where $dx^0 = dt$, $dx^1 = dr$, $dx^2 = d\theta$, $dx^3 = d\phi$. The metric Eq. (5.10) yields

$$\nabla^2[g_{\mu\nu}dx^{\mu}dx^{\nu}] = 0, \tag{5.12}$$

$$\text{where} \quad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{5.13}$$

In order to analyze the expanding universe scenario, we can use Eq. (5.9) for a small radial motion of the galaxy keeping θ and ϕ constant. This gives

$$\frac{\mu}{ar} = \frac{na^2}{2}\left(\frac{dr}{dt}\right)^2 = \frac{na^2v^2}{2} \equiv \frac{n(Hr)^2}{2}. \tag{5.14}$$

where $H = \dot{a}/a$ is Hubble parameter and deviation factor n associated with this system can conform to GR provided we select

$$n = -\frac{1}{2}\left(1 - \frac{\Lambda}{3H^2}\right). \tag{5.15}$$

Here dark energy [48; 49; 50; 51; 52; 53] associated with the cosmological constant Λ is presumed to cause deviation in the flat Minkowski metric. In GR Λ gets introduced through the action principle. Here n is a unitless number. By introducing this deviation factor we are proposing that the presence of uniformly distributed dark energy on a cosmological scale can alter the gravitational redshift of all the constituent particles of a galaxy. This is because the dark energy causes accelerated expansion of the universe which is bound to affect the gravitational redshift of every galaxy. This factor is not accounted by the weak field approximation and the corresponding deviation to the flat Minkowski metric in GR. Since PR relates the proper time of a body with the gravitational redshift of all the constituent particles of a body, we are justified in proposing the deviation factor Eq. (5.15) which only alters the proper time interval of a galaxy without introducing any curvature. This deviation factor n remains constant for any given epoch but varies from epoch to epoch because it is a function of the Hubble parameter. Therefore n satisfies Einstein's field equation $R_{\mu\nu} = 0$.

For the field point within the source of gravitation, in accordance with Poisson's equation we get from Eq. (5.14) and (5.13),

$$H^2 - \frac{\Lambda}{3} = \frac{8}{3}\pi G\rho. \tag{5.16}$$

This is same as the Friedmann equation for flat universe [50; 51; 53]. Hence the critical density in this model when $\Lambda = 0$ comes out to be same as the Friedmann model

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{5.17}$$

If we substitute $t = 1/H$ and

$$c^2\rho = (1/2)g\sigma T^4, \tag{5.18}$$

we get the time temperature relation

$$t = \left(\frac{3c^2}{16\pi G g\sigma}\right)^{1/2} T^{-2}, \tag{5.19}$$

where t is the time of the epoch, g the g factor, σ radiation constant, T the temperature.

If we take time derivative of Eq. (5.14), we get for constant Λ, the accelration equation

$$\left(\frac{\ddot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{2GM_0}{r^3}, \tag{5.20}$$

For a point particle on a homogeneous sphere of radius r and energy density ρ Eq. (5.20) reduces to

$$\left(\frac{\ddot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{8}{3}\pi G\rho. \tag{5.21}$$

Positive sign on the right imply accelerated expansion. Here we can introduce the equation of state $w = p/\rho$ through the relation

$$\rho \propto a^{-3(1+w)}. \tag{5.22}$$

which yields the relations

$$\dot{\rho} = -3H(\rho + p) \quad \text{and} \quad \dot{H} = -4\pi G(\rho + p). \tag{5.23}$$

If we compare Eqs. (5.21) and (5.16), we find that $\dot{H} = 0$, which means that Eq. (5.21) is valid for $w = -1$. Therefore for other values of w, we can introduce Eq. (5.23) in Eq. (5.21) which gives

$$\left(\frac{\ddot{a}}{a}\right)^2 - \frac{\Lambda}{3} = H^2 + \dot{H} - \frac{\Lambda}{3} = -\frac{4}{3}\pi G(\rho + 3p). \tag{5.24}$$

Therefore accelerated expansion occurs for $(\rho + 3p) < 0$. Since H is constant for $w = -1$, we get the inflationary exponential expansion.

$$a \propto e^{Ht}. \tag{5.25}$$

Looking at the above results we find that the theory is in conformance with the GR cosmology and the $\boldsymbol{\Lambda}$CDM model. For obtaining proper time interval of a galaxy we substitute Eq. (5.15) for constant n in Eq. (4.33) for the proper time interval where v is to be replaced by $av = Hr$. This gives

$$d\tau = dt\left(1 + \frac{r^2}{4c^2}\left(H^2 - \frac{\Lambda}{3}\right)\right), \tag{5.26}$$

$$d\tau = dt\left(1 + \frac{2\pi G\rho r^2}{3c^2}\right). \tag{5.27}$$

Equation (5.27) is valid for small values of v and this is where PR will differ from GR.

6 Conclusion

Physicists are gradually beginning to recognise that there are subtle forms of energy such as gravitational waves which are extremely difficult to detect even with highly sophisticated versions of Michelson-Morley type new generation of laser interferometer gravitational wave detectors such as Japanese TAMA [54], American LIGO [55], and European GEO and VIRGO [56]. These are higher frequency ground based detectors. The low frequency radiation is covered by the space based Laser Interferometer Space Antenna or LISA [57] which is expected to cover inspirals into massive black holes with primary mass $\leq 10^8 M_{\odot}$. Similar difficulty is also experienced in detecting dark matter and dark energy. Michelson Morley set out to detect ether with an assumption that if ether existed it would interact with the light waves in their interferometer. The very nature of this experiment imply that ether has properties of waves and therefore must be in a state of vibration. In PR we have proposed the unmanifest state of the primal energy which is devoid of any vibrations or motion and hence does not interact with any form of manifest energy. Many aspects of the theory given here are fully testable and that also without any additional effort. It is recognized that the second order term for the gravitational redshift may not be within the experimental accuracy limit in foreseeable future, but to verify the second order term for bending of light we only need the results of LATOR experiment. Similarly experiments for measuring the limiting radius of event horizon are already underway. Verification of these three predictions of the present theory will leave no doubt concerning the periodic nature of time. This verification would put to rest the notion of empty space and establish the idea that the universe began with a vibration in the unmanifest state of primal energy long before the big bang. In order to compare the present theory with the Schwarzschild solution with respect to the orbital motion of planets and other bodies, it is essential to intorduce the proper time interval of the orbiting body as one of the orbital parameter. This means an additional orbital parameter as a part of ephemerides. The present theory is developed in the classical tradition of Newtonian mechanics but it also satisfies Einstein's field equations so what we have here is an alternative to Schwarzschild solution. The theory shows that Einstein's field equations do provide some clue to the Pioneer anomaly but the solution is not very accurate. Hence the need to go beyond general relativity as discussed in [1]. Similarly in case of rotation curves of spiral galaxies [2], PR proposes that the flat Minkowski metric can deviate in different ways for different two body systems. This effect is more pronounced on galactic scale due to large variations in gravitational potential caused by non-uniform distribution of galactic matter which includes the cold dark matter. When the observed circular velocities of the stars of the Milky Way and their predicted virial masses are introduced in the PR formalism, they yield values of the deviation factor n and the corresponding proper time intervals of stars which are different than that predicted by the general relativity. Finally for cosmological application we have developed field equations in the presence of matter and assuming flat universe arrived at the Friedmann equation. The cosmological constant was introduced through the deviation factor n where dark energy associated with the cosmological constant is presumed to cause deviation in the flat Minkowski metric. PR theory is in conformance with the GR cosmology and the ΛCDM model.

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