SLAC-TN-66-41
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MULTIPLE ELASTIC SCATTERING OF MUONS WITH ENERGY LOSS

INTRODUCTION:

The root-mean-square lateral displacement and scattering angle due to multiple scattering of muons in iron and silicon dioxide are calculated for incident momenta of 5, 10, and 20 BeV/c. The defining equation is the Fermi diffusion equation[1] which has been solved by Eyges[2] with energy loss considered. A second order polynomial is fitted to existing range-momentum data,[4] and the integral expressions of Eyges are numerically integrated to obtain y_{rms} and θ_{rms}.

DISTRIBUTION FUNCTIONS:

The distribution function $F(t, y, \theta)$, which describes the multiple elastic scattering of charged particles as they pass through matter, can be obtained by solving the Fermi diffusion equation[1]

$$\frac{\partial F}{\partial t} = -\theta \frac{\partial F}{\partial y} + \frac{1}{W^2} \frac{\partial^2 F}{\partial \theta^2} \tag{1}$$

where $W = 2p\beta/E_s$

and where the notation and units are Rossi and Greisen's[1] (eg., t in radiation lengths.)

Consider a system of Cartesian coordinates with the origin at the point of incidence and the t-axis along the direction of motion of the incident particles. The other two axis will be the y and z axes, and we will consider the projection of motion of the particles on the (t, y) plane, so that $F(t, y, \theta)\, dy\, d\theta$ will be the number of particles at the thickness t having a lateral displacement (y, dy) and traveling at an angle $(\theta, d\theta)$ with the t axis. Because of symmetry, F also represents the distribution in the (t, z) plane, and the independent nature of the y and z orthogonal directions implies that $F(t, y, \theta_y) \cdot F(t, z, \theta_z)\, dy\, dz\, d\theta_y\, d\theta_z$ represents the general case in three dimensions.

Equation (1) is derived in Rossi and Greisen[1] under the assumption that θ is small, and is solved for the special case of a parallel and infinitely narrow beam of monoenergetic charge particles traversing some scattering substance with no energy loss.

Eyges[2] has treated the same problem by accounting for the energy loss. He assumes that W^2 is some known function of t and neglects the fact that a particle at t has traveled a somewhat greater distance than t due to deviations caused by scattering--a good approximation for high energy particles.

Eyges obtains the result*

$$F(t, y, \theta) = \frac{1}{4\pi\left[B(t)\right]^{1/2}} \exp\left[-\frac{\theta^2 A_2 - 2y\theta A_1 + y^2 A_o}{4B}\right] \tag{2}$$

where

$$B(t) = A_o A_2 - A_1^2 \tag{3}$$

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*Equation (14) of Eyges' does not agree with Eq. (2) above, although his other equations do agree with this paper.

and

$$A_0(t) = \int_0^t \frac{d\eta}{W^2(\eta)} \tag{4}$$

$$A_1(t) = \int_0^t \frac{(t-\eta)\, d\eta}{W^2(\eta)} \tag{5}$$

$$A_2(t) = \int_0^t \frac{(t-\eta)^2 d\eta}{W^2(\eta)} \tag{6}$$

and where η is an integration variable in length units.

If W^2 is constant, Eq. (2) reduces to the Fermi solution as given by Eq. (1.62) in Rossi and Greisen.[1]

If we integrate $F(t, y, \theta)$ either over y or over θ, we get for the angular and lateral distribution functions, respectively

$$\begin{aligned} G(t,\theta) &= \int_{-\infty}^{\infty} F(t, y, \theta)\, dy \\ &= \frac{1}{2(\pi A_0)^{1/2}} \exp\left(-\frac{\theta^2}{4A_0}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} H(t, y) &= \int_{-\infty}^{\infty} F(t, y, \theta)\, d\theta \\ &= \frac{1}{2(\pi A_2)^{1/2}} \exp\left(-\frac{y^2}{4A_2}\right) \end{aligned} \tag{8}$$

The fact that Eqs. (7) and (8) are Gaussian in θ and y is a result of the simplifications introduced in the derivation of Eq. (1).[3]

MEAN SQUARE SCATTERING ANGLES AND DISPLACEMENTS:

The mean square projected angle of scattering is easily obtained from Eq. (7) as follows:

$$\langle \theta^2 \rangle = \int_{-\infty}^{\infty} \theta^2 \, G(t, \theta) \, d\theta$$

$$= \frac{1}{2(\pi A_o)^{1/2}} \int_{-\infty}^{\infty} \theta^2 \exp\left(-\frac{\theta^2}{4A_o}\right) d\theta \qquad (9)$$

$$= 2A_o(t)$$

$$= 2 \int_o^t \frac{d\eta}{W^2(\eta)}$$

and similarly for the mean square lateral displacement

$$\langle y^2 \rangle = \int_{-\infty}^{\infty} y^2 \, H(t, y) \, dy$$

$$= 2A_2(t)$$

$$= 2 \int_o^t \frac{(t-\eta)^2 \, d\eta}{W^2(\eta)} \qquad (10)$$

CALCULATIONS:

The quantities $\langle \theta^2 \rangle$ and $\langle y^2 \rangle$ can be calculated from Eqs. (9) and (10) when the functional form of $W^2(\eta)$ is known and integrable.

Since

$$W \equiv \frac{2p\beta}{E_s} = \frac{2p\beta}{21.2 \text{ (MeV)}} \qquad (11)$$

a knowledge of $p\beta$ versus range is needed.

Range-energy functions have been developed by Barkas[4] and Fig. 1 plots $p\beta$ versus range for muons in Fe and SiO_2. The curves represent second order polynomial fits to Barkas data, which was extended to higher energies using Sternheimer's[5] recipe for ionization loss. Table I gives the various constants used in the calculations. Figure 1 is consistent with Fig. 11 of SLAC-TN-66-37.

The functional form of $p\beta$ is given by

$$\log_{10} p\beta = C_o + C_1 \log_{10} (R - \eta) + C_2 \left[\log_{10} (R - \eta) \right]^2 \tag{12}$$

where R is the maximum range possible for a given incident energy, and where C_o, C_1, and C_2 are given in Table II for $p\beta$ in MeV/c and residual range in centimeters.

The integrations in Eqs. (9) and (10) were performed numerically on the IBM-7090 using the FØRTRAN II subroutine SIMPN.[6] Each calculation was carried out to a residual range corresponding to a $p\beta$ of 100 MeV/c. Figures 2 and 3, respectively, plot $\sqrt{\langle\theta^2\rangle}$ and $\sqrt{\langle y^2\rangle}$ versus the distance into the scattering material.

If we neglect energy loss (i.e., $p\beta$ = constant), Eqs. (9) and (10) reduce to

$$\langle\theta^2\rangle = \frac{1}{2}\left(\frac{E_s}{p\beta}\right)^2 t \tag{13}$$

and

$$\langle y^2\rangle = \frac{1}{6}\left(\frac{E_s}{p\beta}\right)^2 t^3 \tag{14}$$

which correspond to Eqs. (1.67) and (1.68), respectively, in Rossi and Greisen.[1] This special case is compared in Fig. 2 and 3 for $p\beta$ = 20 BeV/c and for SiO_2.

References

1. B. Rossi and K. Greisen, Rev. Mod. Phys. 13, 240 (1941); B. Rossi, High-Energy Particles, Prentice-Hall, Englewood Cliffs, New Jersey, 1952.
2. L. Eyges, Phys. Rev. 74, 1534 (1948).
3. W. H. Barkas, Nuclear Research Emulsions, Volume I. Techniques and Theory, Academic Press, New York and London, 1963.
4. W. H. Barkas, UCRL-10292, August 21, 1962.
5. R. M. Sternheimer, Phys. Rev. 88, 851 (1952); Phys. Rev. 91, 256 (1953); Phys. Rev. 103, 511 (1956).
6. SIMPN by C. Rugge, UCLRL, July 24, 1963. Deck Number: DI EO SMPN.
7. O. I. Dovzhenko and A. A. Pomanskiĭ, Soviet Phys. -- JETP 18, 187 (1964).

TABLE I

	Fe	SiO_2
A	55.85	21.63
Z	26	10.8
I	288 eV	136.5 eV
ρ	7.85 g-cm^{-3}	2.30 g-cm^{-3}
$X_o^{(7)}$	13.9 g-cm^{-2}	27.4 g-cm^{-2}

TABLE II

	Fe	SiO_2
C_o	1.79722	1.62158
C_1	0.42107	0.31971
C_2	0.12263	0.11734

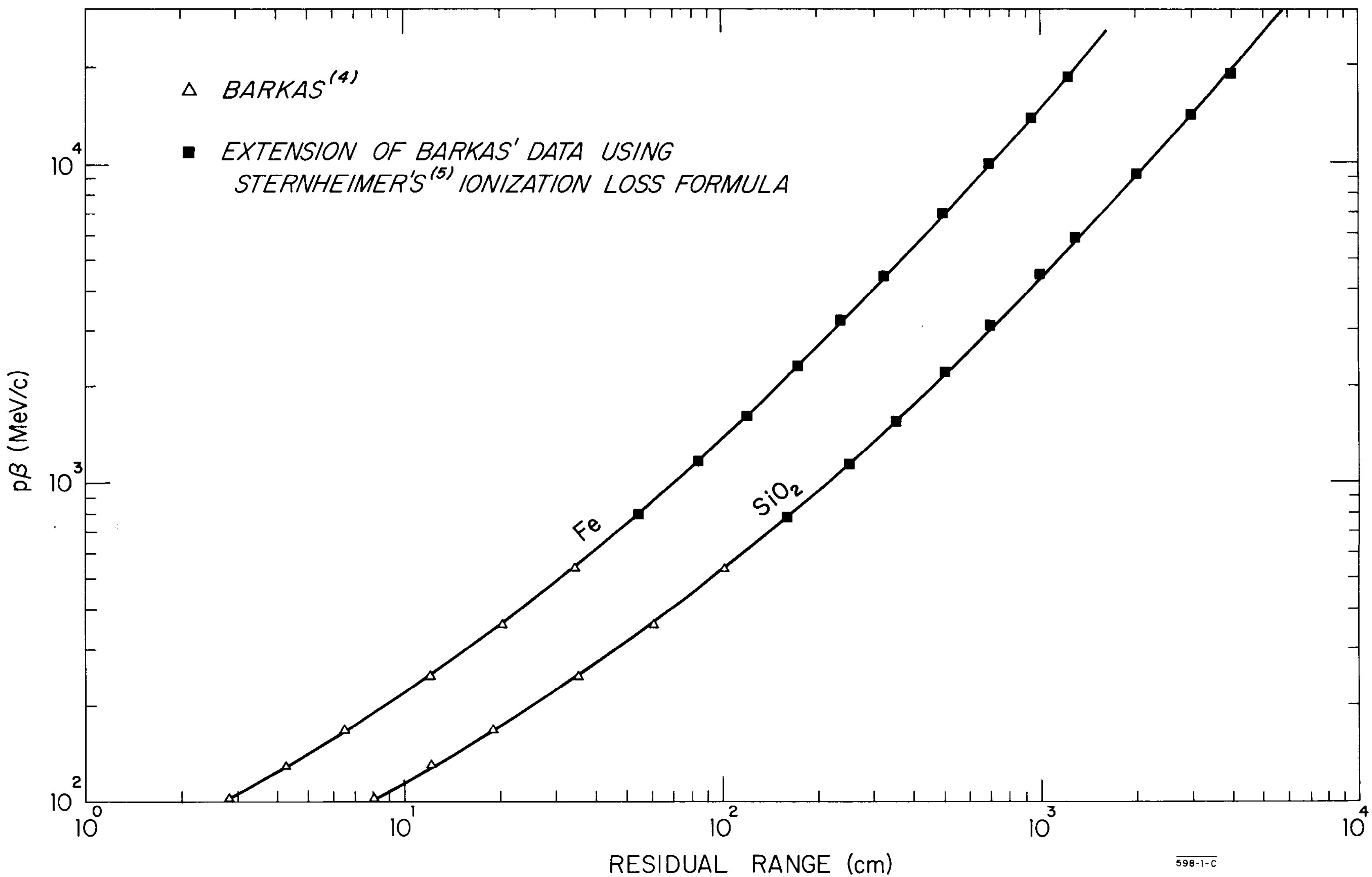

FIG. 1-- $p\beta$ vs RESIDUAL RANGE FOR MUONS IN Fe AND SiO_2

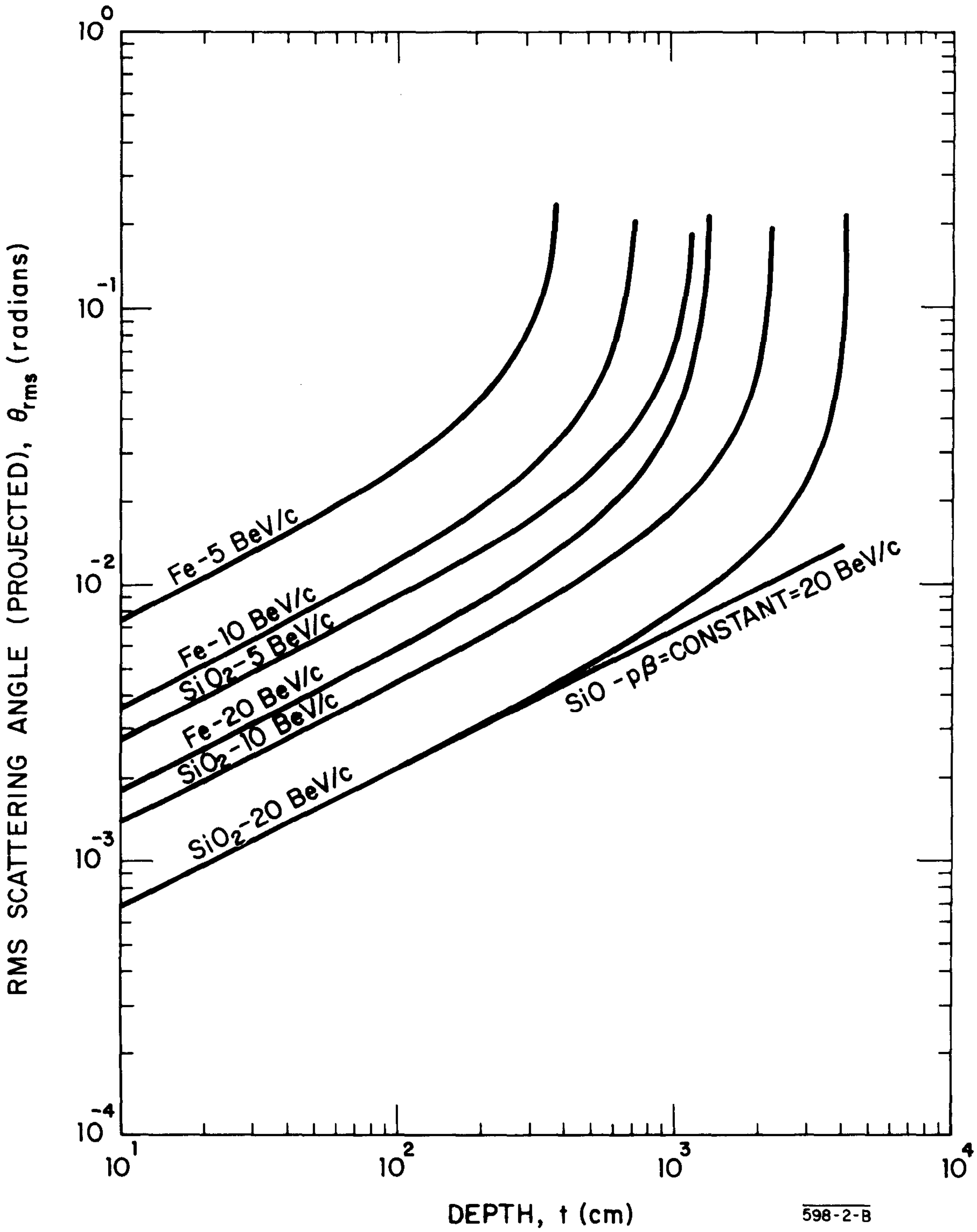

FIG. 2--RMS PROJECTED ANGLE DUE TO MULTIPLE ELASTIC SCATTERING OF MUONS OF VARIOUS INCIDENT MOMENTA IN IRON AND SILICON DIOXIDE.

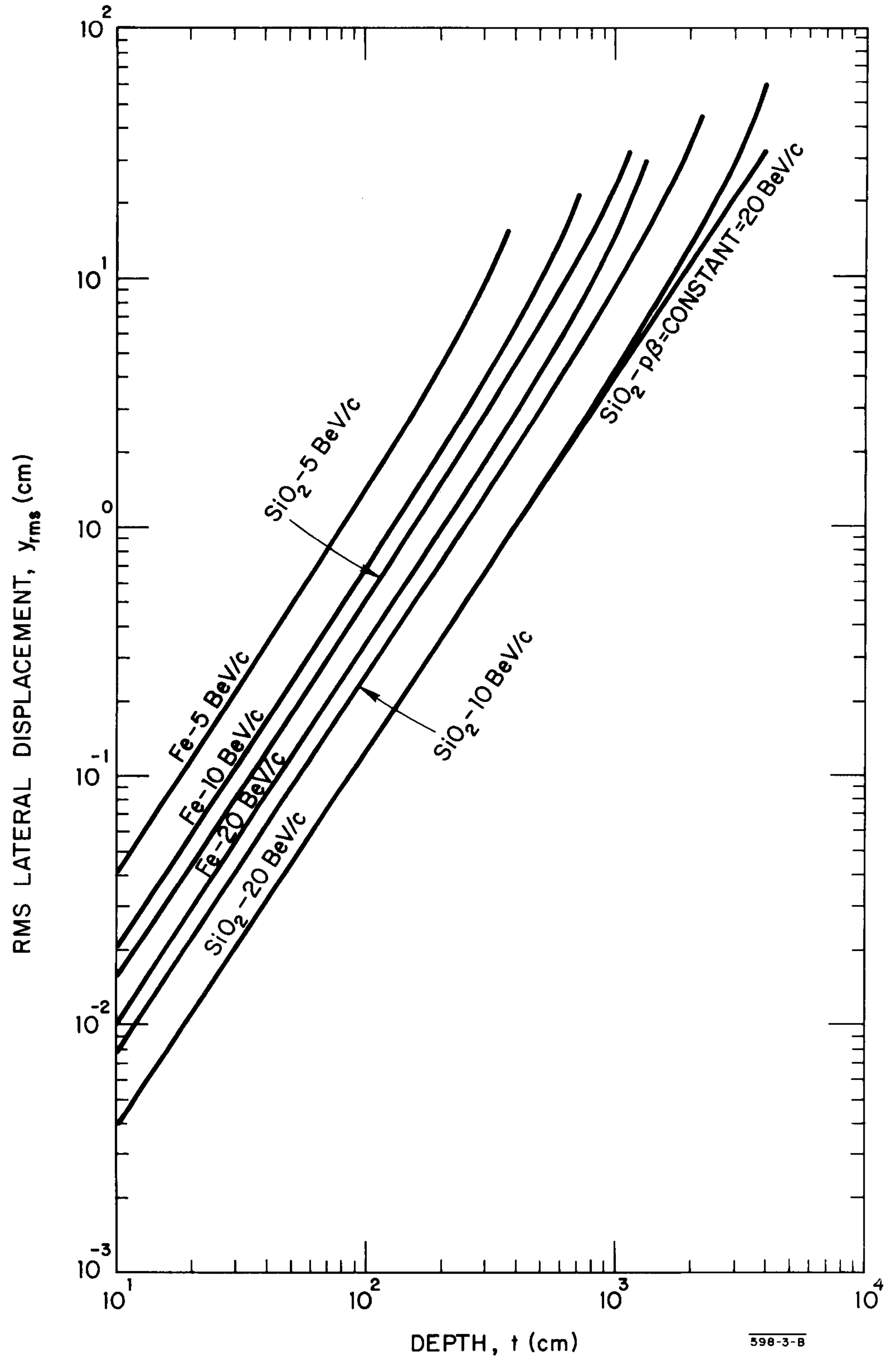

FIG. 3--RMS LATERAL DISPLACEMENT DUE TO MULTIPLE ELASTIC SCATTERING OF MUONS OF VARIOUS INCIDENT MOMENTA IN IRON AND SILICON DIOXIDE.