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CASIMIR AND BEYOND: LONG-RANGE FORCES BETWEEN NEUTRAL AND CHARGED PARTICLES*

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In quantum field theory the concept of a Lagrangian interaction density, expressed in terms of fields, is primary. Forces between two particles are regarded as arising primarily from the exchange of quanta of the bosonic fields. Thus, in contrast to nonrelativistic quantum mechanics, the concept of a two-body potential is secondary. Potentials are not given a priori but must be defined. I review the issues involved when such definitions are made, with particular emphasis on the implications for the concept of "long-range force", of relevance to the Casimir effect. I give a survey of results obtained with dispersion-theoretic methods for the forces arising from photon exchange between both neutral and charged particles, including particles with spin-1/2. I emphasize that, in contrast to the methods employing the concept of zero-point energy, no cutoff-dependent quantities need to be evaluated in this approach.

Keywords: Casimir; long-range forces; charged particles; analyticity; unitarity

1 INTRODUCTION

The celebrated paper of Casimir and Polder concerning the electromagnetic forces between neutral systems appeared in the February 15, 1948 issue of the *Physical Review*, 50 years and 5 weeks ago [1]. In this paper there appears a famous formula [Eq. 56], describing the asymptotic form of the interaction energy of two hydrogenic atoms A and B:

$$E_L(R \to \infty) = -(23/4\pi)(\hbar c/R^7)\alpha(A)\alpha(B). \qquad (1)$$

Here the α's denote the static electric polarizabilities of the atoms. There also appears another famous formula [Eq. 25] for the interaction energy of

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two parallel infinitely conducting plates. The final sentence of the paper reads:

"The very simple form of Eq. (56) and the analogous formula (25) suggest that it might be possible to derive these expressions, perhaps apart from numerical factors, by more elementary considerations. This would be desirable since it would give a more physical background to our result, a result which in our opinion is rather remarkable. So far we have not been able to find such a simple argument."

In view of this anniversary, I thought it would be appropriate to describe an approach to these problems which I believe comes close to meeting the desire expressed in this quote. Here I interpret the word "elementary" in the sense of "fundamental", as in "fundamental principles." In particular, I will sketch how a generalization of (1) may be obtained which is based only on general principles of relativistic qantum field theory (RQFT), without any recourse to "models" of the systems involved and without the need to evaluate any cutoff-dependent quantities.

The concept of potential in both classical and nonrelativistic quantum mechanics, including the passage from one to the other, is a familiar one. However, if the classical forces are velocity dependent the relationship is less clear. As a relevant example, consider U_D, the v^2/c^2 correction to the Coulomb interaction U_C between charged particles, first obtained by Darwin from classical electrodynamics [2],

$$U_D = -(\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \hat{\mathbf{r}}\mathbf{v}_2 \cdot \hat{\mathbf{r}})U_C/2c^2. \tag{2}$$

On writing $\mathbf{v}_i = \mathbf{p}_i/m_i$ and then replacing $\mathbf{p}_i$ by $\mathbf{p}_i^{op}$ one obtains an operator, which is however not unique because of the question of operator ordering. Although there is an ordering which is consistent with quantum electrodynamics (QED), the point is that lack of uniqueness in potentials is the norm in RQFT.

In the following I will describe an approach [3] which makes the source of such ambiguities clear and, more importantly, has a number of desirable features: i) it retains the enormous simplification achieved by the use of Feynman graphs and techniques in the computation of field-theoretic effects, ii) it avoids any a *priori* nonrelativistic approximations, iii) it maintains Lorentz and gauge invariance at any stage of calculation, and iv) it avoids the introduction of cutoffs in the computation of long-range forces.

2 S-MATRIX APPROACH TO EFFECTIVE POTENTIALS

The basic idea is quite simple: we first consider the scattering problem and the associated two-body transition amplitude T. We then ask to what extent T can be regarded as arising from an effective two-body potential, to be used in a Schrödinger type of equation. To be more explicit, we define an interaction operator V, acting directly in configuration space, as a Fourier transform of an on-shell amplitude, obtained from gauge-invariant subsets of Feynman diagrams, modified by appropriate subtractions to avoid double counting; V is constrained by the requirement that when used in an *a priori* specified type of relativistic Schrödinger equation it reproduces $T_{c.m.}$, the value of T in the center-of-momentum system (c.m.s.). For spin-0 particles $A$and B this equation is taken to have the natural form

$$h\phi = W\phi, \quad h = h_0 + V, \tag{3}$$

where h_0 is defined by

$$h_0 = E_A^{op} + E_B^{op}, \tag{4}$$

with

$$E_i^{op} \equiv (m_i^2 + \mathbf{p}_{op}^2)^{1/2}, \quad \mathbf{p}_{op} \equiv -i\partial/\partial\mathbf{r}. \tag{5}$$

The associated potential-theory transition amplitude $T_{pot}$is given by

$$T_{pot} = \langle \mathbf{p}'|V + V(W - h_0 - V + i\epsilon)^{-1}V|\mathbf{p}\rangle. \tag{6}$$

The field-theory transition amplitude T is given in the c.m.s. by

$$T_{c.m.} = M(s,t)/4E_AE_B, \tag{7}$$

where $M(s,t)$ denotes the invariant Feynman amplitude and s and t are the invariant squares of energy and momentum transfer, respectively:

$$s \equiv (p_A + p_B)^2, \quad t \equiv (p_A - p'_A)^2. \tag{8}$$

Here the p's denote initial and final four-momenta. The potential V, which in general will be nonlocal and/or depend parametrically on $s = W^2$, is then required to generate $T_{c.m.}$ from (6),

$$T_{pot} = T_{c.m.}, \tag{9}$$

a condition which is to be satisfied order-by-order in perturbation theory. (To apply (3) to bound states, one looks for normalizable solutions of (3). The associated eigenvalues will correspond to poles of M at values of s

below the threshold $s_0 = (m_A + m_B)^2$ and so can be interpreted as the masses of bound states.)

3 USE OF ANALYTICITY AND UNITARITY

In the computation of potentials from scattering amplitudes it is both convenient and physically appealing to utilize the fact that such amplitudes are analytic functions of the variables on which they depend. In particular, the contribution $M_S(s,t)$ to M from a set S of Feynman diagrams is, for fixed s, usually found to be an analytic function of t, now regarded as a complex variable, with singularities only on the real t-axis. If M_S vanishes as $t \to \infty$ one can use Cauchy's theorem to write $M_S(s,t)$ in the form

$$M_S(s,t) = \pi^{-1} \int dt' \rho_S(s,t')/(t'-t), \tag{10}$$

where $\rho_S(s,t)$, the so-called spectral function, is proportional to the discontinuity of $M_S(s,t)$ across the real t-axis:

$$\rho_S(s,t) = (2i)^{-1}[M_S(s,t+i0) - M_S(s,t-i0)]. \tag{11}$$

A practical advantage of this relationship is that the spectral function is often relatively easy to calculate and/or expressible in terms the amplitudes associated with other physical processes, by use of the ideas of unitarity or generalized unitarity [4]. A conceptual advantage is that ambiguities in the potential V_S corresponding to M_S may be limited by the sensible requirement that it be defined in such a way that is reproduces $M_S(s,t)$ not only in the physical region of the scattering, *i.e.* for

$$s \geq s_0 \equiv (m_A + m_B)^2, \quad -4p^2 \leq t \leq 0, \tag{12}$$

with p the magnitude of the 3-momentum of A or B in the c.m.s., but also for t outside this region, where $M_S(s,t)$ is uniquely determined by analytic continuation. After all, the analyticity properties of $M_S(s,t)$ arise from one of the deepest properties of RQFT, namely locality, and reproducing the properties of field theory is the *leitmotiv* of our approach. For spin-0 particles it is useful to eliminate the kinematic energy factors in (7) by defining a modified potential U via

$$V = y_{op} U y_{op}, \quad y_{op} \equiv (m_A m_B / E_A^{op} E_B^{op})^{1/2} \tag{13}$$

and to require that U be local, depending only parametrically on s or equivalently on p^2. We then find, by inversion of the Fourier transform and use of the spectral representation (10), that $U_S = U_S(r;\ p^2)$ may be expressed directly in terms of ρ_S:

$$U_S(r;p^2) = (16\pi^2 m_A m_B)^{-1} \int dt \rho_S(s,t) exp(-t^{1/2} r). \qquad (14)$$

In general, there is a nearest right-hand branch point at a value $t_0 \geq 0$, equal to the minimum mass of the particle systems being exchanged by A and B in the graphs included in the set S, and a nearest left-hand branch point at a value $\bar{t}_0 < 0$; the function ρ_S vanishes in the interval $(\bar{t}_0, t_0)$. It can be shown that the contribution to U_S from the region $t \leq 0$ always gives rise to a short-range potential, *i.e.* one which vanishes exponentially as r becomes large. However, if $t_0 = 0$, as is always the case when only zero-mass quanta such as photons or neutrinos are exchanged, the integral from 0 to infinity yields a long-range (LR) potential U_S^{LR}, *i.e.* one which falls off as an inverse power of r for large r:

$$U_S^{LR}(r;p^2) = (16\pi^2 m_A m_B r)^{-1} \int_0^\infty dt \rho_S(s,t) exp(-t^{1/2} r), \qquad (15)$$

The relation (15) is especially convenient for the analysis of a long-range force, associated with the exchange of zero-mass quanta. For it is clear from the Laplace transform character of (15) that to determine the asymptotic form of U_S at large r it suffices to know the behavior of ρ_S near $t = 0$.

For the physical examples which have been studied for the $t_0 = 0$ case, the spectral function ρ can be represented in the neighborhood of $t = 0$ by a Laurent expansion in $z = t^{1/2}$, with a simple pole in z. Dropping the subscript S, we have

$$\rho(s,t) = a_2(s) t^{-1/2} + a_3(s) + a_4(s) t^{1/2} + \cdots. \qquad (16)$$

Substitution into (15) then yields an expansion for U^{LR} in inverse powers of r, with coefficients which depend on s or, equivalently, on the square of the c.m. momentum p:

$$U^{LR}(r;p^2) = c_2(p^2) r^{-2} + c_3(p^2) r^{-3} + c_4(p^2) r^{-4} + \cdots, \qquad (17)$$

where

$$c_n(p^2) = [(n-2)!/8\pi^2 m_A m_B] a_n(s). \qquad (18)$$

With this as background, we are ready to turn to some specific applications.

4 POTENTIALS FROM ONE- AND TWO-QUANTUM EXCHANGE

Let us consider some examples of forces arising from the exchange of one or two quanta between spin-0 particles A and B, first for massive quanta and then for massless quanta.

4.1 Exchange of massive spin-0 quanta

As perhaps the simplest example of the techniques sketched above, consider a theory of two complex scalar fields ϕ_A and ϕ_B, both interacting with a neutral scalar field ϕ of mass μ, with an interaction Lagrangian $L_I = -G_A\phi^*{}_A\phi_A\phi + (A \to B)$. The one-quantum exchange amplitude is $M^{(2)} = -G_AG_B/(t-\mu^2)$ and the discontinuity of $(t-\mu^2)^{-1}$ is just $-i\pi\delta(t-\mu^2)$. Thus a simple calculation yields,

$$U^{(2)} = U_Y \equiv -(g_Ag_B/4\pi r)exp(-\mu r)(g_X \equiv G_X/2m_X). \qquad (19)$$

This is of course just the potential first obtained by Yukawa, for infinitely massive nucleons. The corresponding $V^{(2} = y_{op}U_Yy_{op}$ takes into account recoil corrections to the static potential U_Y, to all orders in v/c. Note that $U^{(2)}$ is independent of s only because $M^{(2)}$ is. In higher orders this feature disappears.

A more complicated example in the same theory is provided by considering the fourth-order potential arising from the exchange of two quanta. There are now two Feynman diagrams, a two-rung ladder or "box" diagram with amplitude $M_a^{(4)}$ and a "crossed box" diagram with amplitude $M_b^{(4)}$. Here a key tool is generalized unitarity [4], which states that in order to find discontinuities of Feynman integrals one need only replace selected propagator factors such as $(k^2-m^2+i\epsilon)^{-1}$ by their discontinuity:

$$(k^2-m^2+i\epsilon)^{-1} \to -i\pi\delta(k^2-m^2). \qquad (20)$$

This puts the four-momentum k on the mass shell, $k^2 = m^2$; in addition one must fix the energy to be positive, by including a factor $\theta(k^0)$, so that the effective rule is

$$(k^2 - m^2 + i\epsilon)^{-1} \to -i\pi\theta(k^0)\delta(k^2 - m^2). \quad (21)$$

To avoid double counting, one must subtract the discontinuity of the iteration amplitude $M_I^{(4)}$, generated by $V^{(2)}$ in second-order perturbation theory. One obtains in this way a "net" fourth-order spectral function: $\rho_{net}^{(4)} = \rho^{(4)} - \rho_I^{(4)}$, which now depends not only on t but also on p^2. For $p^2 = 0$, one finds, using (14), with $\eta = 4m/\mu\pi^{1/2}$, that [5]

$$U_{net}^{(4)} = (m/\mu)^2(g_A g_B/4\pi r)^2 exp(-2\mu r)[1-\eta(\mu r)^{-1/2}+O(r^{-1})]. \quad (22)$$

4.2 Exchange of photons

Now let us consider the LRF arising from exchange of photons between spinless systems A and B. If at least one of A and B is neutral, the one-photon exchange potential $V_{1\gamma}$ is short-range. However, the two-photon exchange potential $V_{2\gamma}$ can be long-range. Further, in studying the large-r behavior of $V_{2\gamma}$, we need not worry about the effects of iteration of $V_{1\gamma}$, since this can only contribute to short-range effects. So in some ways this case is actually simpler than that of two charged particles, even if these are pointlike! In the study of $V_{2\gamma}$ a key role is played by the (Compton) amplitude $M_{1\gamma}$ for photon scattering by either particle. For a spinless particle $M_{1\gamma}$ may be written in the form

$$M_{1\gamma}(\sigma, t) = M^{\mu\nu}(p', k'; p, k)\epsilon_\mu \epsilon'^*_\nu \quad (23)$$

where σ and t are the invariant squared energy and momentum transfer,

$$\sigma = (p+k)^2, \quad t = (p-p')^2, \quad \bar{\sigma} = (p-k')^2 = -\sigma - t + 2m^2. \quad (24)$$

The quantity $\bar{\sigma}$ the cross-momentum transfer, is defined for use below. Using Lorentz and gauge invariance one can show that on the photon mass-shell the tensor amplitude $M^{\mu\nu}$ may be written in the form [6]

$$M^{\mu\nu}(p', k'; p, k) = F_E(\sigma, t)T_E^{\mu\nu} + F_M(\sigma, t)T_M^{\mu\nu}. \quad (25)$$

The notation corresponds to a special choice of gauge-invariant tensors which may be regarded as "electric" and "magnetic", but which I need not reproduce here.

Their main feature is that if the particle is neutral, the accompanying coefficients may be shown to have the property

$$F_X(m^2, 0) = 4\pi\alpha_X, \quad (X = E, M), \quad (26)$$

where α_E and α_M denote the static electric and magnetic polarizabilities of the particle. This explains the nomenclature. Moreover, the F_X admit spectral representations of the form

$$F_X(\sigma,t) = \pi^{-1}\int d\sigma' \rho_X(\sigma',t)[(\sigma'-\sigma)^{-1} + (\sigma'-\bar{\sigma})^{-1}]. \qquad (27)$$

where $\bar{\sigma}$ is defined by (24). With $\omega \equiv (\sigma - m^2)/2m$, one finds from (26) and (27) the sum rule

$$2\pi^2\alpha_X = \int_0^\infty d\omega \rho_X(\sigma,0)/\omega. \qquad (28)$$

In terms of such tensors, the amplitude $M_{2\gamma}$ for two-photon exchange is given by

$$M_{2\gamma}^{(4)} = (i/2)\int d^4k d^4k' \, \delta(Q-k-k') M_A(p'_A,k';p_A,-k) : \\ M_B(p'_B,-k';p_B,k)(k^2+i\epsilon)^{-1}(k'^2+i\epsilon)^{-1}. \qquad (29)$$

Here $M_A^{\mu\nu}(p'_A,k';p_A,-k)$ is the tensor amplitude for the emission of two virtual photons by A and $M_B^{\mu\nu}(p'_B,-k';p_B,k)$ that for absorption of two virtual photons by B, both of them off-shell extensions of the on-shell tensors; the colon in (29) denotes a summation over tensor indices.

4.2.1 Both A and B neutral

When one takes the t-discontinuity of $M_{2\gamma}$ by using (21), the photons go on the mass shell and, with both A and B neutral, one can use the form (25) for both M_A and M_B in (29) to carry out the indicated contraction. This already shows that $U_{2\gamma}$ is a quadratic functional of the invariants F_X^A and F_X^B. To obtain the asymptotic form of $U_{2\gamma}$ one needs only the value of the spectral function $\rho_{2\gamma}$ near $t = 0$ and this can be expressed in terms of integrals involving the spectral functions $\rho_X^A(\sigma,0)$ and $\rho_X^B(\sigma,0)$. On use of (26) and the sum rule (28) one finds that for large r and low energies $V_{2\gamma}$ falls of as r^{-7}, with a coefficient which is a quadratic function of the static polarizabilities [6],

$$V_{2\gamma} = -D(\hbar c)/r^7, \\ D \equiv (23/4\pi)(\alpha_E^A\alpha_E^B + \alpha_M^A\alpha_M^B) - (7/4\pi)(\alpha_E^A\alpha_M^B + \alpha_M^A\alpha_E^B). \qquad (30)$$

The purely electric terms coincide with (1), the result of CP. In the case of two atoms, (30) is a good approximation only for separations which are large compared to the maximum wavelengths for dipole emission, of order

$\alpha^{-1}a$ with a the Bohr radius. The potential between two atoms at smaller distances, but still large compared to a, can also be studied by these methods [6]. The form of the integrals involved led to the conjecture that for r large compared to a, a very good approximation to $V_{2\gamma}$ is given by an arctangent function,[7]

$$V_{2\gamma} = -(C/r^6)(2/\pi)arctan(d/r), \tag{31}$$

where $d = (23/8C)\alpha_A\alpha_B$ and C is the factor in the Wang-London potential $V_{WL} = -C/r^6$. Comparison with available numerical calculations shows that (31) interpolates to 2% accuracy between V_{WL} (which neglects retardation) and the asymptotic formula (1).

4.2.2 A neutral, B charged

The same techniques can be applied to the case of a neutral composite A and a charged particle B, since $V_{1\gamma}$ is still short-range [8]. However, the invariant amplitudes F_X^B then contain pole terms and the identification (26) fails. For an elementary B, with charge e_B, these pole terms can be calculated explicitly by using scalar QED; they correspond to contributions to the spectral functions ρ_i^B which are proportional to $\delta(\sigma - m_B^2)$. The result is

$$\begin{aligned} V_{2\gamma}(r) = &-(e_B^2/4\pi)[(1/2)\alpha_E r^{-4} - (11/4\pi)\alpha_E r^{-4}(\lambda_B/r) \\ &- (5/4\pi)\alpha_M r^{-4}(\lambda_B/r) + \cdots] \end{aligned} \tag{32}$$

where $\lambda_B = m_B^{-1}$, the α's refer to the polarizabilities of A, and the dots denote terms which fall off as $1/r^7$ or faster. The $11/4\pi$ term was first found by J. Bernabeu and R. Tarrach [9], using the present methods, and by E. Kelsey and L. Spruch [10] using hybrid QED. The latter authors also suggested that its presence could be tested by study of the fine structure of Rydberg states of helium. Measurements of these, with $n = 10$ for the outer electron, were carried out by S. Lundeen and co-workers over a period of years[11]. It turns out that the asymptotic formula (32) is not accurate enough at $n = 10$, but a general theory based on the present method can be worked out which gives the potential at any separation large compared to the Bohr radius[5]. The whole subject is discussed at length in "Casimir", which includes relevant articles by Lundeen, Spruch, G.W.F. Drake, R. Drachman, and G. Feinberg and me, with references to the literature [3].

5 LRF BETWEEN CHARGED PARTICLES: BEYOND THE COULOMB POTENTIAL

The extension of these methods to the case of two charged particles runs at once into a serious difficulty: Some of the integrals associated with two-photon exchange are infrared (IR) divergent. Thus it appears at first sight that a two-photon exchange potential does not exist! Some reflection leads to the realization that these IR divergences are the counterpart in quantum field theory of the well-known fact that in NRQM the Coulomb interaction cannot be treated in perturbation theory; the second Born approximation diverges, not just for zero momentum transfer but for any value of t. The cure for this problem turns out to be precisely the subtractions which are necessary anyhow to avoid double counting [12]. For concreteness, let us study the case of two point-like spin-0 particles, with charges $e_A$and e_B, and confine our attention to the so-called generalized ladder approximation to $M(s,t)$, *i.e.* to graphs which only involve photon exchange between the particles.

5.1 One-photon exchange potential

Before considering two-photon exchange we must define a one-photon exchange potential $V_{1\gamma}$. Note that however $V_{1\gamma}$ is defined, it must reduce to the Coulomb potential in the static limit. Since this is long-range, the associated iteration amplitude M_I is likely to be equivalent to a long-range potential. Thus, even if one were unaware of the IR divergence problem one would have to compute M_I to find just the long-range part of $V_{2\gamma}$. If one uses Feynman gauge in writing down the (gauge invariant) one-photon exchange amplitude $M_{1\gamma}$, one gets a numerator factor $(p_A + p'_A)\cdot(p_B + p'_B) = s - u$, where $u = (p_A - p'_B)^2$ is the cross momentum transfer. Since $u = 2m_A^2 + 2m_B^2 - s - t$ we have

$$M_{1\gamma} = e_A e_B (2a + t)/t \qquad (a \equiv s - m_A^2 - m_B^2). \tag{33}$$

Simple Fourier transformation of (33) yields a term proportional to U_C, with an energy-dependent coefficient, plus a contact term proportional to $\delta(\mathbf{r})$. Such a potential is not suitable for use in a Schrödinger type of equation. In second-order perturbation theory it would lead to an ultraviolet (UV) divergence. A potential which is iterable can be obtained by first writing $M_{1\gamma}$ in a different form (which does not change its value on the

mass shell) and then finding an equivalent operator in **r**-space which involves derivative operators. One is thereby led to what can be termed a Feynman-gauge inspired (FGI) potential V_1^{FGI},

$$V_1^{FGI} = z'_{op} U_C z'_{op} + y_{op}(\mathbf{p}_{op} \cdot U_C \mathbf{p}_{op}/2m_A m_B) y_{op}, \qquad (34)$$

where $z'_{op} \equiv (1 + \mathbf{p}_{op}^2/2E_A^{op}E_B^{op})^{1/2}$ and $y_{op} = (m_A m_B/E_A^{op}E_B^{op})^{1/2}$ The corresponding Coulomb-gauge inspired (CGI) one-photon exchange potential V_1^{CGI} is given by [13]

$$V_{1\gamma}^{CGI} \equiv \qquad (35)$$
$$y_{op}(\{E_A^{op},\{E_B^{op}, U_C\}\}+(1/2)\{p_i^{op},\{p_j^{op}, (\delta_{ij}+\hat{r}_i\hat{r}_j)U_C]\}\}y_{op})/4m_A m_B.$$

The t-discontinuity of the iteration amplitude M_I obtained from either choice is IR finite but behaves as $1/t$ for small t; this behavior leads to a logarithmic divergence in the spectral integral, consistent with the nature of the IR divergence of M_I itself.

5.2 Two-photon exchange potential

To compute the field theory amplitude $M^{(4)}$ in scalar QED one must study the integrals associated with the five fourth-order Feynman diagrams which enter the game: (a) the two-rung ladder graph, (b) the two-rung crossed ladder graph, (c) the two single-seagull graphs and (e): the double-seagull graph. Both (a) and (b) are UV convergent but IR divergent, whereas (c) and (d) are UV divergent but IR convergent. But the t-discontinuity of each of these is divergence free. The net spectral function behaves again as $1/t$ for small t, corresponding to the IR divergence but the coefficient of t^{-1} is equal and opposite to that appearing in M_I. The difference spectral function ρ_{diff} then goes like $t^{-1/2}$; this is integrable at $t = 0$ and therefore yields a finite $V_{2\gamma}$. On using $V_{1\gamma}^{FGI}$ to compute M_I one finds that [12]

$$V_{2\gamma}^{FGI} = c_2^{FGI} r^{-2} + c_3^{FGI} r^{-3} + \cdots \qquad (36)$$

where, with $k \equiv e_A e_B/4\pi$,

$$c_2^{FGI} = k^2/2(m_A + m_B), \quad c_3^{FGI} = -7k^2/6\pi mAmB. \qquad (37)$$

In contrast, use of V_1^{CGI} yields [13],

$$V_{2\gamma}^{CGI} = c_2^{CGI} r^{-2} + c_3^{CGI} r^{-3} + \cdots \qquad (38)$$

where

$$c_2^{CGI} = 0, \quad c_3^{CGI} = -7k^2/6\pi m_A m_B. \tag{39}$$

Thus we see that in the case of two charged particles the leading asymptotic behavior of $V_{2\gamma}$ depends on the precise definition of $V_{1\gamma}$. This observation resolves a longstanding puzzle in the literature and shows that in the case of two charged particles not just the concept of potential but even that of its asymptotic form is not without ambiguity. Furthermore, one can show that the difference between the two choices of $V_{1\gamma}$ is connected with the form of the so-called orbit-orbit interaction, $U_{o\text{-}o}$. To see this, note that in the n.r. limit (34) yields as the leading correction to U_C an interaction $U_{o\text{-}o}$ of the form

$$U_{o-o}^{FGI} = \{p_i^{op}, \{p_j^{op}, \delta_{ij}, U_C\}\}/4m_A m_B, \tag{40}$$

whereas (35) yields

$$U_{o-o}^{CGI} = (1/2)\{p_i^{op}, \{p_j^{op}, (\delta_{ij} + \hat{r}_i\hat{r}_j), U_C\}\}/4m_A m_B \tag{41}$$

The latter is a manifestly hermitian form of the orbit-orbit interaction $U_{o\text{-}o}$ familiar from atomic physics, usually obtained by reduction of the Breit operator to n.r. form. It can be shown that in the computation of atomic level shifts to order $\alpha^2 Ry$, the difference between these two forms of $U_{o\text{-}o}$ is precisely accounted for by the $1/r^2$ term in (36). One can also understand the difference between the two V's in the framework of classical electrodynamics [13, 14].

6 INCLUSION OF SPIN-1/2

6.1 Continuum dissolution

The inclusion of spin-1/2 particles is straightforward, once one recognizes the main pitfall encountered when dealing with relativistic Dirac-like equations. As is by now well known, if one wishes to describe the interactions of electrons using such equations, *e.g.* in the context of the helium atom, even the Coulomb potential U_C must be accompanied by positive-energy Casimir-type projection operators. Otherwise, there are no normalizable solutions which correspond to the discrete spectrum of the atom or ion [15, 16]. The cure for this disease, which I like to call "contin-

uum dissolution", is to go back to first principles. Using field theory, one finds that putative interaction potentials such as U_C always come accompanied by positive-energy projection operators which keep the (product) bound states from mixing with the negative-energy states responsible for a continuous degeneracy. In the absence of an external field, i.e. in the pure two-body case, the problem is ameliorated because of momentum conservation. However, starting from field theory one again finds that the effective interaction operators come equipped with projection operators for the Dirac particles. In particular, for two spin-1/2 particles the counterpart of Eq. (3) is

$$[h_A^{op} + h_B^{op} + \Lambda_{++}{}^{op} U \Lambda_{++}{}^{op}]\psi = E\psi, \tag{42}$$

where

$$h_A^{op} = \alpha_A \cdot \mathbf{p}^{op} + \beta_A m_A, \quad h_B^{op} = -\alpha_B \cdot \mathbf{p}^{op} + \beta_B m_B, \tag{43}$$

and Λ_{++}^{op} is the product of free (Casimir-type) positive-energy projection operators:

$$\Lambda_{++}^{op} = \Lambda_{+;A}^{op} \Lambda_{+;B}^{op}, \quad \Lambda_{+;i}^{op} = (E_i^{op} + h_i^{op})/2E_i^{op}. \tag{44}$$

6.2 One- and two-photon exchange potentials

With this understanding, the one-photon exchange potential takes the form

$$V_{1\gamma} = \Lambda_{++}^{op} U_{1\gamma} \Lambda_{++}^{op}, \tag{45}$$

where $U_{1\gamma}$ is required to reproduce $M_{1\gamma}$ when sandwiched between on-shell Dirac spinor plane waves. This leads to two natural choices for $U_{1\gamma}$ which are local in Dirac-spinor space,

$$U_{1\gamma}^{FGI} = U_C + U_G, \quad U_{1\gamma}^{CGI} = U_C + U_B, \tag{46}$$

where U_G and U_B are the Gaunt and Breit potentials defined by

$$\begin{aligned} U_G &= (e_A e_B/4\pi r)(1 - \alpha_A \cdot \alpha_B), \\ U_B &= (e_A e_B/4\pi r)(1 - (1/2)[\alpha_A \cdot \alpha_B + \alpha_A \cdot \hat{\mathbf{r}} \alpha_A \cdot \hat{\mathbf{r}}]) \end{aligned} \tag{47}$$

The computation of $V_{2\gamma}$ for the case when either one or both particles have spin-1/2 is a major undertaking, especially in the latter case. If one uses $V_{1\gamma}^{FGI}$ to compute M_I, the spin-independent part (more precisely, the Dirac-matrix independent part) of $U_{2\gamma}$ is the same as that found for two spin-0 particles. For the mixed case of, say, A with spin-0 and B with

spin-1/2, the computation of the spin- dependent part has already been carried out [17]. One finds a correction $V_{2\gamma}^{s-o}$ to the spin-orbit potential $V_{1\gamma}^{s-o}$ coming from $V_{1\gamma}$, which for large r is proportional to $\sigma\cdot\ell/r^4$. If A has structure, $V_{2\gamma}^{s-o}$ also contains a spin-orbit polarizability potential falling of as $\sigma\cdot\ell/r^6$ for large r. These terms will also be present in the case of two spin-1/2 particles but in addition there will be a correction $V_{2\gamma}^{s-s}$ to the spin-spin potential $V_{1\gamma}^{s-s}$ coming from $V_{1\gamma}$, which remains to be calculated. There are a number of physical situations in which it may be possible to detect the effects of $V_{2\gamma}^{s-o}$. Typically these involve measurements of bound state energies in exotic atoms, where one particle has spin 1/2 and another has spin 0. Examples include anti-protonic atoms with a spin-0 nucleus, such as $\bar{p} - He^4$, pionic atoms with a spin-1/2 nucleus, such as pionic hydrogen, and the pi-muon bound state known as pi-muonium. Certain aspects of $V_{2\gamma}^{s-o}$ may be observable in Rydberg states of helium-like ions whose nuclei have spin-1/2 [8].

7 CONCLUDING REMARKS

We have seen that the concept of a potential in RQFT is rather subtle. Once one departs from the static approximation the demands of relativity inevitably lead to velocity dependence in the classical limit and, correspondingly, to energy dependence and/or nonlocality of any potential designed to reproduce or facilitate the calculations of the predictions of QFT. Such dependence may be handled in a variety of ways. Some of the ambiguity in the potential is reduced by the requirement that it reproduce the field-theoretic scattering amplitude not only in the physical region of the scattering, but outside this region where the amplitude is defined by analytic continuation. The use of analyticity, when combined with generalized unitarity, turns out to be also a powerful tool in computing the potential, especially for large separations. We have seen how this method can be used to obtain and generalize old results, such as that of Casimir and Polder on the retarded van der Waals potential between atoms, in a way which makes it clear why the result is universal, depending only on general features of field theory such as locality and gauge invariance.

Extension to the case of a neutral particle and a charged particle leads to an effective potential which has proved to be useful in the analysis of Ryd-

berg levels of the helium atom. Extension to the case of two charged particles reveals a new feature in the concept of potential. Different choices of the one-photon exchange potential $V_{1\gamma}$ may lead to different results for the leading term in expansion in powers of r^{-1} of the two-photon exchange potential $V_{2\gamma}$; this feature has a counterpart in classical electrodynamics [13]. Extension of the method to spin-1/2 particles, either composite or elementary, has led to formulas for the e^4 corrections to the spin-orbit interaction; when work on similar corrections to the spin-spin interaction is completed one will be able to reanalyze the spin-dependent level structure of a number of two-body physical systems and perhaps gain new insight into some aspects of bound-state QED, especially for states of large orbital angular momentum and relatively large separation between the constituents.

In conclusion, let me return to the quote from CP. With regard to the physical background of (1), we have seen that in the S-matrix approach the (generalized) form of this result emerges as naturally from two-photon exchange as, say, the Yukawa potential arises from one-meson exchange. One sees at once that the potential is a quadratic functional of the tensor amplitudes for two-photon emission for each of the particles. Gauge invariance forces these amplitudes to be quadratic in the four-momenta k and k' of the photons and unitarity forces their components to be proportional to $t^{1/2}$. Thus the spectral function $\rho_{2\gamma}$ is proportional to $(t^{1/2})^4 = t^2$ for small momentum transfer t. Since $\rho \propto t^N$ implies $V \propto 1/r^{2N+3}$, the $1/r^7$ behavior is manifest. Because only the low-energy values of the invariant amplitudes F_X enter for large r and these are proportional to the static polarizabilities, the coefficient of $1/r^7$ must be a quadratic polynomial in these polarizabilities. It remains an elusive goal to obtain a simple physical undestanding of the appearance of the primes 23 and 7 in the final result.

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