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Einstein-Maxwell-dilaton theory: Black holes, wormholes, and applications to AdS/CMT

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Orientador

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to Maria, Pedro, and Sônia Goulart, my greatest supporters

"There is nothing so annoying as to be fairly rich, of a fairly good family, pleasing presence, average education, to be "not stupid," kindhearted, and yet to have no talent at all, no originality, not a single idea of one's own-to be, in fact, "just like everyone else." Of such people there are countless numbers in this world-far more even than appear. They can be divided into two classes as all men can-that is, those of limited intellect, and those who are much cleverer. The former of these classes is the happier. To a commonplace man of limited intellect, for instance, nothing is simpler than to imagine himself an original character, and to revel in that belief without the slightest misgiving. Many of our young women have thought fit to cut their hair short, put on blue spectacles, and call themselves Nihilists. By doing this they have been able to persuade themselves, without further trouble, that they have acquired new convictions of their own. Some men have but felt some little qualm of kindness towards their fellow-men, and the fact has been quite enough to persuade them that they stand alone in the van of enlightenment and that no one has such humanitarian feelings as they. Others have but to read an idea of somebody else's, and they can immediately assimilate it and believe that it was a child of their own brain. The "impudence of ignorance," if I may use the expression, is developed to a wonderful extent in such cases;-unlikely as it appears, it is met with at every turn. ... those belonged to the other class-to the "much cleverer" persons, though from head to foot permeated and saturated with the longing to be original. This class, as I have said above, is far less happy. For the "clever commonplace" person, though he may possibly imagine himself a man of genius and originality, nonetheless has within his heart the deathless worm of suspicion and doubt; and this doubt sometimes brings a clever man to despair. (As a rule, however, nothing tragic happens;-his liver becomes a little damaged in the course of time, nothing more serious. Such men do not give up their aspirations after originality without a severe struggle,-and there have been men who, though good fellows in themselves, and even benefactors to humanity, have sunk to the level of base criminals for the sake of originality)."

-Fyodor Dostoyevsky, *The Idiot*

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Resumo

No contexto de teorias de Einstein-Maxwell-dilaton, estudamos buracos negros, buracos de minhoca e aplicações à correspondência anti-de Sitter/Teoria de Matéria Condensada. Apresentamos a solução de buracos negro dyonica para a teoria de Einstein-Maxwell-dilaton escrita completamente em termos de constantes de integração, e então investigamos como definir parâmetros físicos dependentes e independentes. Escolhendo condições de contorno apropriadas para o dilaton no infinito, construímos buracos negros sem massa e uma ponte de Einstein-Rosen que satisfaz a condição de energia nula. Construímos uma solução carregada analítica de buraco de minhoca atravessável para a teoria de Einstein-Maxwell-phantom-dilaton que é livre de singularidades e conecta dois espaços de Minkowski. Usando o teorema de Gauss-Bonnet calculamos o ângulo de deflexão de um raio de luz que passa próximo à este buraco de minhoca. Apresentamos o formalismo da função entropia de Sen e o aplicamos para o cálculo analítico da entropia do buraco negro extremo de uma teoria de supergravidade com $\mathcal{N} = 8$ em quatro dimensões. No contexto de holografia, calculamos coeficientes de transporte na presença de campos magnéticos para teorias com um termo topológico na ação. Definimos quantidades radialmente independentes subtraindo as correntes de magnetização, e então estudamos perturbações lineares em torno do horizonte a fim de expressar as condutividades elétrica, termoelétrica e térmica em termos de somente propriedades do horizonte. Combinamos as fórmulas para as condutividades com os dados do horizonte calculados usando o formalismo de Sen, e expressamos analiticamente as condutividades à temperatura zero para várias teorias cujas soluções de buraco negro não são conhecidas analiticamente.

Palavras Chaves: Buracos negros; buracos de minhoca; AdS/CMT.

Áreas do conhecimento: Relatividade Geral; Buracos negros; Holografia.

Abstract

In the context of Einstein-Maxwell-dilaton theory, we study black holes, wormholes and applications to the anti-de Sitter/Condensed Matter Theory correspondence. We present the dyonic black hole solution to the Einstein-Maxwell-dilaton theory written fully in terms of integration constants, and then investigate how to define dependent and independent physical parameters. Choosing appropriate boundary conditions for the dilaton at infinity, we construct massless black holes and an Einstein-Rosen bridge that satisfies the null energy condition. We construct an analytical charged traversable wormhole solution to the Einstein-Maxwell-phantom-dilaton theory which is free of singularities and connects two Minkowski spacetimes. Using the Gauss-Bonnet theorem we compute the deflection angle of a light ray passing close to this wormhole. We present the Sen's entropy function method and apply it to compute analytically the entropy of the extremal black hole of a gauged $\mathcal{N} = 8$ supergravity theory in four dimensions. In the holographic context, we compute the transport coefficients in the presence of magnetic fields for theories with a topological term in the action. We define radially independent quantities by subtracting off the magnetization currents, and then study linear perturbations around the horizon in order to express the electric, thermoelectric and heat conductivities in terms of horizon properties only. We combine the formulae for the conductivities with the horizon data computed using Sen's entropy function method, and express analytically the conductivities at zero temperature for several theories whose the full black hole solutions are not known analytically.

Keywords: Black holes; Wormholes; AdS/CMT.

Areas: General Relativity; Black Holes; Holography.

List of publications

[1] A. de Souza Dutra, Prieslei E. D. Goulart, "Nonlinear two-field models from orbit equation deformations", **Phys. Rev. D 84**, 105001.

[2] Prieslei Goulart, Horatiu Nastase, "Massive ABJM and black hole entropy in the presence of field strength coupling to curvature", arXiv: 1502.05949 [hep-th]

[3] Prieslei Goulart, "Dyonic AdS4 black hole entropy and attractors via entropy function", **JHEP 09 (2016) 003**, arXiv: 1512.05399 [hep-th]

[4] Prieslei Goulart, "Dyonic black holes and dilaton charge in string theory", arXiv: 1611.03093 [hep-th]

[5] Prieslei Goulart, "Massless black holes and charged wormholes in string theory", arXiv: 1611.03164 [hep-th]

[6] Johanna Erdmenger, Daniel Fernandéz, Prieslei Goulart and Piotr Witkowski, "Conductivities from attractors", **JHEP 03 (2017) 147**, arXiv: 1611.09381 [hep-th]

[7] Prieslei Goulart, "Phantom wormholes in Einstein-Maxwell-dilaton theory", arXiv: 1708.00935 [gr-qc]

Chapter 3 is based on papers number [3] and [4]. Chapter 4 is based on paper number [7]. Chapter 6 is based on paper number [3]. Chapter 8 is based on paper number [6]. The figures used in each chapter are also referenced to the respective papers.

Contents

Chapter 1

Introduction

It is undeniable that a new era in Physics has just started. At the moment of the writing of this thesis, both the Advanced Laser Interferometer Gravitational-Wave Observatory, abbreviated as LIGO, and the Advanced Virgo Interferometer have anounced a joint detection of gravitational waves emmitted by the merger of two black holes [1]. This was the fourth detection done by LIGO, and the first done by Virgo. The detection of gravitational waves has just rendered the 2017 Nobel Prize in Physics to Barry Barish, Kip Thorne and Rainer Weiss, the idealizers of the LIGO experiment. This certainly left no space for questioning the existence of gravitational waves and black holes, both predicted theoretically as solutions to field equations of General Relativity.

Black holes are objects that arise from the collapse of stars with large masses compared to the mass of the Sun. Their mass is concentrated entirely on a pointlike region of space, called singularity, and they are covered by a surface from which nothing can escape classically, called event horizon. This is why we use the term "black hole" to designate these objects that do not even emit light. When two black holes are in a binary system that is about to merger, they emit a strong signal of gravitational waves. The mass of the big black hole formed after the merger is smaller than the sum of the masses of the two black holes before the merger: gravitational waves carry the rest of the energy. The hope is that the experiments become precise enough so that we can extract more information about the sources using this "new light". Perhaps fundamental fields inside the black holes, like scalars or vector fields, leave imprints on these gravitational waves. This means that we would be able to probe their constituents and learn whether or not they are charged under such scalars and vector fields. If this is the case, then we would be able to say, for instance, if a black hole has the features of a string theory black hole: it would be a very strong indication that string theory is a fundamental theory of nature if a charged black hole contains scalar fields coupling in specific ways to the vector fields!

In this thesis we will deal with the so-called Einstein-Maxwell-dilaton theory. This is the simplest low-energy effective theory that arises in string theory. The presence of a scalar background field called dilaton, which couples to the gauge fields, is a feature these theories. This generalizes the Einstein-Maxwell theory by adding the dilaton with its respective kinetic term and also by coupling it to the Maxwell term in the action. In general, this coupling is an exponential function of the dilaton. The presence of the dilaton enriches a lot the physics of the solutions to the field equations compared to the Reissner-Nordström solution, which is the charged black hole solution of the Einstein-Maxwell theory. In the case of black holes with electric and magnetic charges, called dyonic black holes, the dilaton field is attracted to a fixed point on the horizon. This phenomenon is called "attractor mechanism", and we will also deal with this topic here.

The Einstein-Maxwell-dilaton theory has its origins in the so-called Kaluza-Klein compactification scheme [2], [3]. At that time, the electromagnetic and gravitational fields were thought to be the components of a five dimensional gravitational field. The four dimensional gauge invariance arose as a consequence of the invariance of the five dimensional theory under coordinate transformations. The fifth dimension must be compact, and with very small radius of compactification. When the five dimensional gravity theory has one spatial dimension compactified on a torus, the resulting theory is the four dimensional Einstein gravity coupled to a $U(1)$ gauge theory and to a single massless scalar. In other words, the Maxwell's theory of electromagnetism arises naturally from a higher dimensional theory. The extra scalar field is called dilaton. One of the problems of this scheme is that the dilaton is associated to the radius of compactification, but the four dimensional theory has no dilaton potential. This means that the radius of the compact dimension could not be fixed. Due to the problems of the scheme, the subject remained dormant for quite a long time.

The dream of unification of the forces was revived with the advent of string theory. In order to avoid anomalies, string theory requires the existence of several spacetime dimensions. The bosonic string theory, for instance, requires that there exists twenty six spacetime dimensions, whereas supersymmetric string theory, called superstring theory, requires ten. Since we live in four spacetime dimensions, the other dimensions of the theory must be compact. The problem with the dilaton can be fixed in this context, since in superstring theory there is a mechanism that generates a dilaton potential. Consequently, these compactification schemes can be used in string theory, and consequently, the Einstein-Maxwell-dilaton became interesting again. The natural way to follow would then be to find the solutions to the theory, and study their physics. The investigation of the solutions to the Einstein-Maxwell-dilaton theory and their applications are the main topics of this thesis. In order to be more specific, we will study the black hole and wormhole solutions to the theory, the computation of the entropy of such black holes, and how black holes can be applied in AdS/CFT.

The history of the black hole solutions of the Einstein-Maxwell-dilaton theory (6.1) is actually very interesting. The solution was first presented in 1982 by Gary Gibbons, as can be seen in reference [4], and it was clarified in reference [5] by Gibbons himself together with Kengo Maeda. The solution they presented had electric and magnetic charges, which is called dyonic, and of course one can easily obtain the magnetically charged one just by setting the electric charge to zero. The magnetically charged solution was "rediscovered" by Garfinkle, Horowitz and Strominger in [6]. It is fair to mention that the solution they presented had already been given by Gibbons [4], so, at that time, it was not new anymore. But they discussed important points in their paper, such as the problems related to the dilaton field on the horizon of the black hole in the string and Einstein frame, and also that the electrically charged solution could be obtained via S-duality rotation of the solution. The dyonic solution of reference [4] was found for the case when the dilaton field at infinity, refered to as ϕ_0 in this thesis, was set to zero. This was included in the paper by Kallosh, Linde, Ortín, Peet and van Proeyen, in reference [7], and some authors call it the most general dyonic black hole solution of the Einstein-Maxwell-dilaton theory, although the only difference of the solution in [7] compared to the one in [4] is the inclusion of such parameter. The original solutions of reference [7] are actually the multicenter ones.

There are many motivations for considering black hole solutions besides the ones related to astrophysics. In theoretical physics, black holes are believed to be the perfect object to address the quantum aspects of gravity. The quantum processes taking place near the horizon lead the black hole to evaporate [8]. We also know that the horizon area surrounding the singularity has a large entropy. Although we will not present this in this thesis, it was shown by Strominger and Vafa that this matches the counting of their inner microstates in the five dimensional string theory context [9]. The idea that the entropy counts the microstates of black holes in string theory motivated Sen to develop a whole formalism to compute the entropy of extremal black holes [10]. This is the Sen's entropy function: an efficient method to compute the entropy of extremal black holes even without knowing the full black hole solution. Once there are more efficient methods to count the microstates of a black hole, we can compare to its entropy in order to check if they match.

There are also other kinds of solutions that we will consider in this thesis. They are the Einstein-Rosen bridges and the traversable wormholes. Both are referred to as just wormholes nowadays. The traversable wormholes are tunnels in spacetime that would allow rapid insterstellar travel, and the Einstein-Rosen bridges also has the same features of such tunnels, but they are constructed out from deforming black hole solutions. The problems with both solutions is that they require the existence of exotic kinds of matter to exist, and this will be explained later. But still, they are very interesting objects and we will investigate them here as well.

The fact that the physics of black holes is encoded in the properties of the horizon is one example of holography: the properties of a (d+1)-dimensional theory is encoded in its d-dimensional boundary surface. Another example of holography is the anti-de Sitter/Conformal Field Theory correspondence, or just AdS/CFT for short. Maldacena proposed in 1997 that there exists a duality between a (d+1)-dimensional gravity theory in anti-de Sitter space and a d-dimensional conformal field theory [11]. So, difficult problems to be handled in conformal field theories might become simpler to be handled using its gravitational dual. Nowadays, the AdS/CFT correspondence is used as an attempt to handle strongly coupled systems in Condensed Matter Theory, and such approach is called AdS/CMT for short. As black holes have temperature, they are the thermodynamical objects used in gravity in order to describe, or at least to predict something, in condensed matter theory. We will also apply the black hole solutions of the Einstein-Maxwell-dilaton theory in this context, and compute transport coefficients holographically.

By no means we intend to be mathematically formal. As this thesis is about solutions to the Einstein-Maxwell-dilaton theory and also about their applications in holography, one might say that it would be necessary to have an introduction to both topics. Of course this would turn the reading more tedious and would occupy more space. We judged more important to introduce more carefully only the topic of solutions, which is done in the next chapter. This will be of great relevance, since we will present the language used in the context of black holes and wormholes. The topics related to the AdS/CFT applied to condensed matter theory will be done in chapter 7. In order to have a more detailed discussion about string theory we refer to the books by Polchinski [12] and [13]. For AdS/CFT we refer to the book by Nastase [14].

In chapter 2 we review the simplest black holes and wormhole solutions, and discuss the properties that will be relevant for the thesis.

In chapter 3 we present the dyonic black hole solution to the Einstein-Maxwell-dilaton theory. We will explain its features, how to define physical parameters, how to recover other known solutions, and how this theory presents the phenomenon called attractor mechanism. Then, we discuss that it is possible to obtain massless black holes under certain boundary conditions, and from it, to construct Einstein-Rosen bridges that satisfy the null energy condition. This whole chapter is based in references [15] and [16].

In chapter 4, we construct traversable wormholes for the Einstein-Maxwell-phantom-dilaton theory. A phantom dilaton is the same as the dilaton field, but its kinetic term in the action appears with flipped sign, so that it has negative kinetic energy. The wormhole is electrically charged and contains no singularities. We plot the embedding diagram for this wormhole, as well as the fields of the theory. The wormhole also deviates light, since it is a gravitational object just like a black hole. We then use the Gauss-Bonnet theorem and compute the deflection angle for a light ray passing close to it. All the results of this chapter are based in reference [17].

In chapter 5 we present the entropy function formalism developed by Sen. This is a review based on references [10] and [18]. We show how to construct the entropy function from first principles and also how to obtain the attractor equations.

In chapter 6 we apply the entropy function method to compute the entropy for a very complicated theory, the $U(1)^4$ supergravity theory, which is the bosonic part of a gauged $\mathcal{N} = 8$ supergravity theory. First, we obtain the solutions to simpler Einstein-Maxwell-dialton theories in the presence of a potential. Then, we show how we can infer the solution to the attractor equations for the $U(1)^4$ supergravity theory. The entropy of the black hole is computed analytically, and expressed in two different ways. This chapter is based on reference [19].

In chapter 7, we review the AdS/CFT and AdS/CMT correspondences, and show how to calculate holographic conductivities in the presence of magnetic fields. The main references for this chapter are [20] and [21]. The computation we present is more general than the one in [21], since it includes a topological term in the action. This generalization is the topic of ongoing research that will be published soon, in collaboration with Luis Alejo. Basically, we show how to define radially independent currents in the presence of perturbations around the black hole. This requires to subtract off the contributions from the magnetization currents to the total currents. The resulting current is then independent of the radial coordinate in the presence of perturbations, which allows us to compute the conductivities analytically, and express them in terms on the properties of the horizon only.

Finally, in chapter 8, we apply the Sen's entropy function formalism in the context of AdS/CMT correspondence. We adapt the entropy function method to the case of black holes with planar horizons, and obtain the horizon data for Einstein-Maxwell-dilaton theories with specific scalar potentials. We insert the horizon data into the expressions for the electric, thermoelectric and heat conductivities derived in chapter 7. The results are expressed in terms of the charges of the black hole and coupling constants of the theory. We show that all conductivities scale as $\sim N^{3/2}$ for a constant potential. We also show that the ratio between the heat conductivity and the temperature is finite in our approach. We study how the attractor equations and the conductivities for the constant potential case change under S-duality transformations. All the results of this chapter are based on reference [22].

In chapter 9 we conclude.

Chapter 2

Basic concepts about black holes and wormholes

This is an introductory chapter about black holes and wormholes. It is our intention to introduce both subjects and establish the language that will be used in the whole thesis. Along the thesis, we will use words like horizon, singularity, throat, and all of them are defined here by reviewing the simplest known black holes and wormhole solutions.

2.1 Weak-field limit

The weak-field limit in General Relativity corresponds to considering the metric of the spacetime as being the Minkowski metric $\eta_{\mu\nu}$ corrected with a linear perturbation $h_{\mu\nu}$, i.e.

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.1}$$

The Einstein's field equations reduce to the following system

$$\nabla^2 h_{00} = 8\pi G_N \rho, \;\; \nabla^2 h_{ij} = 8\pi G_N \rho \delta_{ij}, \tag{2.2}$$

and

$$\nabla^2 h_{0i} = 0, \tag{2.3}$$

where the operator ∇^2 is the Laplacian operator. In this derivation, one has to assume that the metric is independent of time, and that energy-momentum tensor is dominated by the energy density $T^{00} = \rho(\vec{r})$. The solutions can be written in terms of the Newton's gravitational potential $U_N = -\frac{G_N M}{r}$ using

$$h_{00} = -2U_N, \ \ h_{ij} = -2\delta_{ij}U_N, \ \ h_{oi} = 0. \tag{2.4}$$

In the weak field limit, the metric is given by

$$ds^2 = -(1+2U_N)dt^2 + (1-2U_N)(dr^2 + r^2 d\Omega_2^2). \tag{2.5}$$

The weak-field limit has many applications in Celestial Mechanics, but here, we will only use it to associate the integration constants of a black hole solution to the mass parameter M, which, of course, will be interpreted as the mass of the black hole.

2.2 Schwarzschild solution

In this subsection we will study the so-called Schwarzschild black hole. This is a solution to Einstein's field equations in vacuum, i.e. $T_{\mu\nu} = 0$. As this is one of the simplest solutions in General Relativity in four dimensions, we will give a special treatment to it, since this allows us to introduce important concepts that will be used in the entire thesis, such as the definition of event horizon, singularity, and the relation between physical quantities and boundary conditions. These concepts will be used in following subsection in the study of charged black holes, and also in chapter 3 in the context of black holes with dilaton fields.

The Einstein's equations can be derived from the Einstein-Hilbert action, which is just

$$I = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R, \tag{2.6}$$

where R is the Ricci scalar, sometimes called curvature scalar. Varying the action with respect to $g^{\mu\nu}$ gives the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.7}$$

Now, we introduce the concept of static and spherically symmetric solutions to the equations of motion in curved spacetimes. For the purpose of this section, we write a spherically symmetric metric as

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega_2^2, \tag{2.8}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. A spacetime is said to be *stationary* if the metric elements are time-independent, i.e. $\partial_t g_{\mu\nu} = 0$. We can perform a coordinate transformation in the time direction and obtain a metric that is now time dependent, containing crossed terms of the kind $g_{rt} dr dt$. A spacetime is said to be *static* if it has a time-like Killing vector field orthogonal to the family of surfaces defined by $t =$ constant. If we adapt a metric to be stationary and impose that it contains a time-like Killing vectror field orthogonal to the family of surfaces defined by $t =$ constant, then the metric will not contain crossed terms of the kind g_{0i}. In other words, a spacetime is said to be static if the metric elements are time-independent and $g_{0i} = 0$.

The Schwarzschild solution is a static and spherically symmetric solution to the Einstein's equations in vacuum. After taking the trace of (7.30) and eliminating the Ricci scalar, we have

$$R_{\mu\nu} = 0. \tag{2.9}$$

The Einstein's equations in vacuum for the ansatz (2.8) reduce to the system of two equations

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0, \tag{2.10}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0. \tag{2.11}$$

Adding (2.10) and (2.11) we have

$$\lambda' + \nu' = 0, \tag{2.12}$$

and after integrating it

$$\lambda + \nu = \text{const.} \tag{2.13}$$

One can rescale the time coordinate to choose a coordinate system in which const $=$ 0, so that $\nu = -\lambda$. Then, integrating the remaining equation we obtain

$$e^{-\lambda} = 1 - \frac{r_0}{r}, \tag{2.14}$$

where r_0 is another integration constant. We have to impose boundary conditions in order to give an interpretation to this constant. Notice that we can define a new radial coordinate defined by the equation

$$r = \left(1 + \frac{r_0}{4\bar{r}}\right)^2 \bar{r}. \tag{2.15}$$

This gives

$$1 - \frac{r_0}{r} = \left(\frac{1 - r_0/(4\bar{r})}{1 + r_0/(4\bar{r})}\right)^2, \quad dr = d\bar{r}\left[1 - \left(\frac{r_0}{4\bar{r}}\right)\right], \tag{2.16}$$

and the line element will be

$$ds^2 = -\left(\frac{1 - r_0/(4\bar{r})}{1 + r_0/(4\bar{r})}\right)^2 dt^2 + \left(1 + \frac{r_0}{4\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega_2^2). \tag{2.17}$$

Expanding the metric until first order in $(\bar{r}/r)$ we obtain

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 + \frac{r_0}{r}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega_2^2). \tag{2.18}$$

Comparing (2.18) to (4.24), we see that

$$r_0 = 2G_N M. \tag{2.19}$$

The vacuum solution to the Einstein's equations is then written as

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{2.20}$$

This is the **Schwarzschild black hole solution**. There are two special "hypersurfaces" that seem to make the metric divergent, i.e. $r = 0$ and $r = 2G_N M$. The time spent for the light emitted by an infalling object to reach an observer far away from the the hypersurface $r = 2G_N M$ is defined by the equation

$$ds^2 = 0. \tag{2.21}$$

Assuming for simplicity that the motion occurs in the radial direction, the time is given by

$$\Delta t = \int_{r_1}^{r_2} \sqrt{\frac{g_{rr}}{g_{tt}}} dr, \tag{2.22}$$

where r_1 is the position of the infalling object and r_2 is the position of the observer. For the Schwarzschild solution (2.20) this gives

$$\Delta t = [r + 2G_N M \ln(r - 2G_N M)]_{r_1}^{r_2}. \tag{2.23}$$

When the infalling object achieves the hypersurface $r = 2G_N M$, this variation of time diverges. In other words, after crossing the hypersurface $r = 2G_N M$ the infalling object will not be seen anymore since the light it emits takes an infinite amount of time to achieve an observer at position r_2. Everything that is inside the sphere defined by the hypersurface $r = 2G_N M$ is unnaccessible for an observer outside of it. The hypersurface $r = 2G_N M$ defines the **event horizon** of the black hole, and we will denote it by r_H. In principle, one is led to think that the infalling object may feel something special while crossing the surface. We will show that this is not exactly the case. In fact, the Schwarzschild black hole (2.20) was written in a coordinate system that is valid only in the region $r > r_H$. In order to make the solution be valid in the region $r < r_H$ we need to change the coordinate system in such a way that our spacetime includes the internal region of the black hole. In order to do that, we first introduce the tortoise coordinates r_*

$$\frac{dr}{1 - \frac{2G_N M}{r}} = dr_* \to r_* = r + 2G_N M \ln(\frac{r}{2G_N M} - 1), \tag{2.24}$$

which gives the metric

$$ds^2 = \left(1 - \frac{2G_N M}{r}\right)(-dt^2 + dr_*^2) + r^2 d\Omega_2^2. \tag{2.25}$$

Then, we introduce the null coordinates

$$u = t - r_*, \quad v = t + r_*, \tag{2.26}$$

in such a way that light rays travel with $u =$ constant or $v =$ constant. This will give the metric written in Kruskal coordinates,

$$ds^2 = \left(1 - \frac{2G_N M}{r}\right) du dv + r^2 d\Omega_2^2. \tag{2.27}$$

This is what we call the Kruskal spacetime. The extension to the region inside the horizon can be done by introducing the Kruskal-Szekeres coordinates,

$$U = -4G_N M e^{-\frac{u}{4G_N M}}, \;\; V = 4G_N M e^{\frac{v}{4G_N M}}. \tag{2.28}$$

The metric in this coordinate system is given by

$$ds^2 = -\frac{2G_N M}{r} e^{-\frac{r}{2G_N M}} dU dV + r^2 d\Omega_2^2, \tag{2.29}$$

where $r = r(U, V)$. This is the Schwarzschild solution in null-Kruskal coordinates. This coordinate sysem is valid everywhere, except at the point $r = 0$. We say that the hypersurface $r = 2G_N M$ is a singularity of the metric, and this, as we saw, can be removed by introducing appropriate coordinate systems. The singularity r_S can not be removed by changing the coordinate system, being the **singularity of the spacetime**. All the mass of the black hole is concentrated in this small region of the spacetime. If any scalar constructed out of curvature tensors is singular in one point, this is the point where the singularity of the spacetime is located.

2.3 Reissner-Nordström solution

Suppose the spacetime now contains a non-trivial massless gauge field A_μ. This theory is described by the Einstein-Maxwell action, written as

$$I = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{2.30}$$

where the field strength $F_{\mu\nu}$ has the usual definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.31}$$

The equations of motion with respect to the metric are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2G_N\left(F_{\mu\gamma}F_{\nu}{}^{\gamma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \qquad (2.32)$$

Varying the action with respect to the gauge field gives the Maxwell's equations in curved spacetimes, written as

$$\nabla_\mu F^{\mu\nu} = 0. \qquad (2.33)$$

The field strength must satisfy the Bianchi identities

$$\nabla_{[\mu}F_{\nu\,\rho]} = 0. \qquad (2.34)$$

The Bianchi identities can also be written as

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \qquad (2.35)$$

where the dual field strength $\tilde{F}^{\mu\nu}$ is given by

$$\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}F_{\rho\sigma}. \qquad (2.36)$$

In this expression, $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol, for which $\tilde{\epsilon}^{tr\theta\phi} = 1$. The solution to the gauge field equations, consistent with the Bianchi identities, are

$$F_{rt} = \frac{Q}{r^2}, \quad F_{\theta\phi} = P\sin\theta, \qquad (2.37)$$

where Q and P have the interpretation of electric and magnetic charges respectively. The Einstein's equations are the same as in the Schwarzschild case, supplemented by the terms dependent on the charges. After integrating the resulting equation we obtain

$$e^{-\lambda} = 1 + \frac{r_0}{r} + \frac{G_N(Q^2+P^2)}{r^2}. \qquad (2.38)$$

We can follow the same procedure done in the previous section and use the weak-field limit in order to define the integration constant r_0 in terms of the mass of the black hole M. After this, the charged black hole solution can be written as

$$ds^2 = -e^{-\lambda}dt^2 + e^{\lambda}dr^2 + r^2 d\Omega_2^2,$$

$$e^{-\lambda} = \left(1 - \frac{2G_N M}{r} + \frac{G_N(Q^2 + P^2)}{r^2}\right), \tag{2.39}$$

This is the so-called **Reissner-Nordström solution**.

For the discussion that follows we will take $G_N = 1$. In order to find the event horizons, we solve $g_{tt} = 0$, and obtain

$$r_\pm = M \pm \sqrt{M^2 - (Q^2 + P^2)}. \tag{2.40}$$

There are three situations:

- $M > \sqrt{Q^2 + P^2}$

  This corresponds to the situation in which the black hole has one inner and one outer horizon, respectively given by r_- and r_+;

- $M < \sqrt{Q^2 + P^2}$

  In this case, the black hole has no horizon. The singularity is still present, and we call this solution a **naked singularity**. By analyzing the motion of an infalling particle following a timelike geodesics, one concludes that the gravitational force produced by the naked singularity is repulsive. Then, one can argue that this situation is not physically acceptable, since the naked singularity will destroy itself with its own gravitational field. There also exists the so-called **cosmic censorship conjecture**, which states that the gravitational collapse of matter satisfying physical energy conditions forbids the formation of naked singularities. In other words, if the gravitational collapse forms singularities, it will always be covered by a horizon;

- $M = \sqrt{Q^2 + P^2}$

  In this case the two horizons coincide, and we have a solution with one degenerate horizon. The metric becomes

  $$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2. \tag{2.41}$$

  We will see that the temperature of this solution is zero, and this is called **extremal** solution. Because this solution saturates the Bogolmony, Prasad, and Sommerfield bound, which states that the mass is "equal" to the charge, we sometimes call it a **BPS** black hole.

By computing scalars formed witht the curvature tensor for the Reissner-Nordström solution, we can check that the singularity is located at $r_S = 0$. If we set the charges to zero, we recover the Schwarzschild solution (2.20).

2.4 Black hole thermodynamics

In 1973, Bardeen, Carter and Hawking published a seminal work [23] entitled "*The four laws of black hole mechanics*". Once the concepts of event horizon and singularity of black holes were well defined, the natural direction would be to study how the black hole evolve when a particle of very small mass falls into it. In fact, this is what they did in that paper. It was known at the time that Killing vectors could be used to create conserved currents, which could be integrated to give rise to conserved quantities. The integrals of such currents are all compressed in the so-called "Smarr formula", times called "Smarr integrals". The conserved quantities used in the original paper were the mass of the black hole and the angular momentum. At that time, the authors did not really give the correct interpretation to the laws they had just derived, and did not relate the laws to usual laws of thermodynamics. Jacob Bekenstein then suggested [24] that there is a similarity between black holes and thermodynamics, arguing that the entropy of a black hole must be proportional to its area. At first, Hawking refused to accept this idea based on the fact that black holes do not emit any radiation, so they must not have an entropy. But then, in 1974, Hawking itself published a paper corroborating Bekenstein's idea [8]. The fact that black holes do not emit radiation is correct, but only classically. Taking into account the quantum process of creation of pairs happening very near the horizon of a black hole, Hawking proved that black holes do emmit particles quantum mechanically. This changed the interpretation of the laws of black hole mechanics, which since then started being called "the laws of black hole thermodynamics". In other words, Hawking proved that black holes emit particles, and consequently are thermodynamical objects with an associated entropy S_H and temperature T_H, given by

$$S_H = \frac{A_H}{4G_N}, \;\; T_H = \frac{\kappa}{4\pi}, \tag{2.42}$$

where A_H is the area of the horizon, and κ is the surface gravity of the black hole, which, for the non-rotating solutions of this thesis, is just given by

$$\kappa = \frac{(g_{tt})'}{2}. \tag{2.43}$$

The original derivation of the laws is very complicated, and we will not present it here. Instead, we will only list them. They are:

- **Zeroth law**
  If the energy-momentum tensor satisfies the dominant energy condition, then the surface gravity is constant on the horizon of the black hole.

- **First law**
  If a stationary black hole of mass M, charge Q and angular momentum J, is taken into a new black hole with parameters $M + dM$, $Q + dQ$ and $J + dJ$ by a quasi-static process, then

  $$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ, \tag{2.44}$$

  with surface gravity κ, angular velocity Ω_H and electric potential Φ_H on the horizon.

- **Second law**
  If the energy momentum tensor satisfies the weak energy condition, then the area of the event horizon is a non-decreasing function of time

  $$dA \geq 0. \tag{2.45}$$

Some comments are in order. First, the first law of thermodynamics gets modified in the presence of a scalar field called dilaton, which couples to the electromagnetic field strength. The dilaton has a charge, and we will see that this charge also gives a contribution to the first law. Second, the area law is not true when the black hole emits Hawking particles. This means that black holes can evaporate and disappear completely. This does not contradict the laws of thermodynamics though, since it can be shown that the sum of the entropy of the black hole with the entropy of the emitted particles are always greater or equal than zero. Third and last, unlike usual thermodynamical systems that has entropy proportional to its volume, the entropy of black holes is proportional to the area of its event horizon. This is a remarkable observation, and it is a strong indication of the holographic principle: all the information of the black holes are encoded on its event horizon surface! In conclusion, black holes really behave as thermodynamical objects, and respect a set of laws which are analogues to the laws of thermodynamics.

2.5 Einstein-Rosen bridges

In the previous sections we discussed the difference between a singularity of the spacetime and the event horizon of a black hole. This notion was not so well-understood even during Einstein's time. As we said, a singularity of the spacetime can not be removed by coordinate transformations, whereas the singularities that can be removed in general define the event horizon of a black hole. In this section we will introduce and discuss in a simple way the concept of a "bridge", which is a region that connects two slices of the spacetime.

The concept of a bridge was made mathematically consistent by Einstein and Rosen [25], although they did not have a complete understanding of their mathematical construction at the time. An idea that always intrigued scientists of the old and new generation is that neutral and charged black holes resemble very much elementary particles, at least classically speaking. Black holes are pointlike objects, and they can be described completely by a small set of parameters. They are the mass, the angular momentum, and the charges associated to the elementary fields that characterize the black hole, which can be the electric and magnetic charges associated to a gauge field, and also the dilaton charge which is the charge associated to the a scalar field called dilaton. Particles are also characterized by a very small set of parameters, i.e. the mass, spin, and charge. The angular momentum is a classical quantity, and it differs from the spin in the sense that the spin is quantized. But still, the notion of spin is that it is a sort of intrinsic angular momentum, which carries some resemblance with the classical angular momentum. The most remarkable difference is perhaps that black holes have an event horizon, and elementrary particles do not. The lack of a precise definition of event horizon led the scientists of the old generation to consider it as if it were real singularities of the spacetime. We will see that this confusion was the reason why Einstein and Rosen introduced the concept of a bridge [25].

In order to describe elementary particles as solutions to General Relativity, one of the first requirements is that the space must be smooth outside the region where the particle mass is concentrated. In other words, the event horizon must not be present. The current techniques to remove the apparent singularity causing the event horizon surface in the metric are based on choosing an appropriate coordinate system that covers the spacetime outside and inside the black hole. In the original work [25], Einstein and Rosen called the event horizon surface a "special kind of singularity", and showed how to remove this kind of singularity for the Schwarzschild and Reissner-Nordström solution. We will show how they did this for both cases.

Consider the Schwarzschild solution (2.20), with $G_N = 1$. The horizon (or "special kind of singularity" in Einstein and Rosen sense) is at $r = 2M$. Let us introduce the new coordinate

$$u^2 = r - 2M, \tag{2.46}$$

so that the metric is written as

$$ds^2 = -\left(\frac{u^2}{u^2 + 2M}\right) dt^2 + 4(u^2 + 2M)du^2 + (u^2 + 2M)^2 d\Omega_2^2. \tag{2.47}$$

In this coordinate system, the surface $r = 2M$ is located at $u = 0$, and so the space is free of singularities. Notice that this coordinate system excludes the region inside the black hole, which contains the real singularity of the metric at $r = 0$. Notice also that the new coordinate system is valid for positive and negative values of u, i.e. it is valid from $-\infty < u < +\infty$. The hypersurface $u = 0$ has area $A = 16\pi M^2$, which is the minimum surface area allowed in this spacetime. This hypersurface connects the "positive slice" with the "negative slice" of the spacetime, and it is called **throat**.

This is just one example of a general method to construct spacetimes containing two slices connected by a throat from black hole solutions (see reference [26]). This kind of spacetime is called an **Einstein-Rosen bridge**. Because (2.47) was constructed out of a black hole without charge, some authors refer to it as neutral bridge. This construction fails for $M < 0$ because it requires the existence of a horizon.

Consider now the Reissner-Nordström solution (2.39). In order to have the same bridge construction, Einstein and Rosen first set the mass M to zero, which turned the solution into a naked singularity, i.e. it has no horizon. But the existence of a horizon is a necessary condition for the bridge construction, so, Einstein and Rosen turned the charges charges to imaginary values, i.e. $(Q^2 + P^2) \equiv -\epsilon^2$, for a constant ϵ. The resulting solution now has a horizon, so the bridge construction is possible. In order to see this, they introduced the coordinate

$$u^2 = r^2 - \epsilon^2, \tag{2.48}$$

and this brings the metric into the bridge form

$$ds^2 = -\frac{u^2}{u^2 + \epsilon^2}dt^2 + du^2 + (u^2 + \epsilon^2)d\Omega_2^2. \tag{2.49}$$

The throat is located at $u = 0$, and the minimum area is $A = 4\pi\epsilon^2$. In reference [26] the solution (2.49) is called quasi-charged bridge. Although the object described by (2.49) is massless, Einstein and Rosen wanted to interpret it as the electron, of course for $P = 0$. This attempt failed badly because the bridge construction was only possible for $M = 0$ and for imaginary charges. But the electron has a non-zero mass, and real charges. If we allow for non-zero mass and real charges, and adjust them so that the mass and charge of the Reissner-Nordström solution are exactly the mass and the charge of the electron, then this would be a naked singularity because $m_e \approx 10^{-22} Q_e$ in units for which $c = G_N = 1$. In conclusion, the quasi-charged bridge can not describe the electron because the electric charge is much greater than its mass in such units. Of course we could also try to use the Kerr-Newmann solution but this does not work either, as was discussed in [26].

The natural question at this point is whether some object coming from the asymptotic regions at $u = -\infty$ could cross the throat and arrive at the other asymptotic region at $u = +\infty$. It was shown that the neutral bridge is dynamical [27], and it pinches off so fast that not even light can cross the throat separating the two sheets. The quasi-charged bridge clearly violates the null energy condition everywhere in the spacetime, since we had to consider imaginary charges. So, by its own, it represents a rather unphysical situation. The neutral and quasi-charged bridges are just black holes disguised in different coordinate systems, and this does not exclude their true singularities, as we can change back to the old coordinate system. An observer crossing and Einstein-Rosen bridge will actually fall into a black hole!

2.6 Traversable wormholes

The Einstein-Rosen bridges discussed in the previous sections are thought as tunnels in spacetime. They can connect two different regions, which can be of the same spacetime, or which can be of different spacetimes. In this section we will discuss a different kind of solution to General Relativity. This corresponds to what we call nowadays as **wormholes**, which allow an object to cross it without having any problem such as those we encountered for Einstein-Rosen bridges. The term wormhole was introduced for the first time by John Archibald Wheeler in 1957, and it corresponds to solutions connected by a throat, but that were not constructed from black hole solutions. This implies that the wormhole spacetime is totally free of true singularities (singularities of the spacetime). It is worth pointing out that the terms Einstein-Rosen bridges and wormholes have been constantly used in the literature as defining the same object. Some authors make a differentiation in tems of what we call the traversability of a wormhole. So, the neutral bridge of the previous section, for instance, is what they call a **non-traversable wormhole**, since nothing crosses from one slice of the spacetime to the other. Of course, a **traversable wormhole** is an object of the same kind but that would allow objects to cross its throat. From now on, we will use this classification in terms of the traversability of the solutions, which we will call indistinguably as wormhole. So, this section is dedicated to the discussion of traversable wormholes.

The interest in wormhole solutions in General Relativity was revived after the work of Morris and Thorne [28], where they defined the basic properties of a traversable wormhole. The paper was written for pedagogical reasons, but it was the first time somebody defined the properties of a basic traversable wormhole. As an example, they studied the properties of one solution, known as Bronikov-Ellis wormhole, which will be reviewed here.

Consider the following metric

$$ds^2 = -dt^2 + dr^2 + (r^2 + l^2)d\Omega_2^2, \tag{2.50}$$

where l is a real non-zero constant. This metric is free of singularities, and it is defined for positive and negative values of the radial coordinate r. Notice that the area of the two-dimensional surface is $A = 4\pi(r^2 + l^2)$, which is obtained by integrating the angular part of the metric, has a minimum at $r = 0$. In other words, the position $r = 0$ is the location of the **throat**. This structure, as we saw before, defines a wormhole. The two asymptotic regions at $r = \pm\infty$ are two Minkowski spacetimes. The metric (2.50) defines the Bronikov-Ellis wormhole [29, 30]. Any scalar quantity computed out of the curvature tensor is finite everywhere, so the spacetime has no singularities. This is the simplest example of a traversable wormhole. This is a spherically symmetric solution of the following theory

$$S = \int \sqrt{-g}\left(R + 2\partial_\mu\phi\partial^\mu\phi\right), \tag{2.51}$$

where the scalar field that solves the equations of motion is also free of singularities, and is written as

$$\phi(r) = \phi_0 + \arctan\left(\frac{r}{l}\right), \tag{2.52}$$

for a constant ϕ_0. The factor of 2 in front of the kinetic term was inserted only for future convenience. Notice that the kinetic term of the scalar field has positive sign, and for our metric convention $(-, +, +, +)$, this implies that the kinetic energy of the scalar field is negative. This type of scalar field will be called in this thesis as **phantom scalar***. The most problematic aspect of this kind of matter is that its

*Some authors refer to it as **ghost scalars** as well.

energy-momentum tensor does not satisfy the null energy condition, i.e. $T_{\mu\nu}k^\mu k^\nu < 0$ for a null vector k^μ. Matter of this kind is called **exotic matter**. This is the main reason why traversable wormholes are so underappreciated in theoretical physics.

Now, we construct the so-called **embedding diagram** for the wormhole. Without loss of generality, we set $t = \text{const}$ and $\theta = \pi/2$. The metric for this slice is then written as

$$ds^2 = dr^2 + (r^2 + l^2)d\phi^2. \tag{2.53}$$

We redefine our radial coordinate such that

$$(r^*)^2 \equiv r^2 + l^2. \tag{2.54}$$

We can take derivatives to show that

$$dr = \frac{r^*}{\sqrt{(r^*)^2 - l^2}}dr^*. \tag{2.55}$$

Then, (2.53) is written as

$$ds^2 = \frac{(r^*)^2}{(r^*)^2 - l^2}dr^{*2} + r^{*2}d\phi^2. \tag{2.56}$$

The Euclidean metric of the embedding space is the same as the one used in [28]

$$ds^2 = dz^2 + dr^{*2} + r^{*2}d\phi^2 = \left[1 + \left(\frac{dz}{dr*}\right)^2\right]dr^{*2} + r^{*2}d\phi^2. \tag{2.57}$$

Comparing (2.56) with (2.57) we obtain

$$\frac{dz}{dr*} = \pm\left(\frac{(r^*)^2}{(r^*)^2 - l^2} - 1\right)^{1/2}. \tag{2.58}$$

This equation can be easily integrated and gives

$$z(r^*) = \pm l \ln\left[\frac{r^*}{l} + \sqrt{\frac{(r^*)^2}{l^2} - 1}\right]. \tag{2.59}$$

This defines the embedded surface, and it is shown in figure 2.1. Notice that the throat of the Bronikov-Ellis wormhole is located at $r^* = l$, which implies that

$z(r^*) = 0$. As we mentioned, this is the simplest example of a traversable wormhole, supported by exotic matter. In chapter 4, we will present a novel wormhole solution of this kind, but in the presence also of electric charges. In order to do so, we will use the Einstein-Maxwell-dilaton theory, but we will make the dilaton field to be the phantom scalar. We end this section mentioning that there are other interesting issues related to traversable wormholes. Perhaps the most intriguing one is the fact that wormhole spacetimes present closed timelike curves, which raises several paradoxes, but this is definitely out of the scope of this thesis and we will not discuss this topic here. We again refer to [26] for a complete discussion.

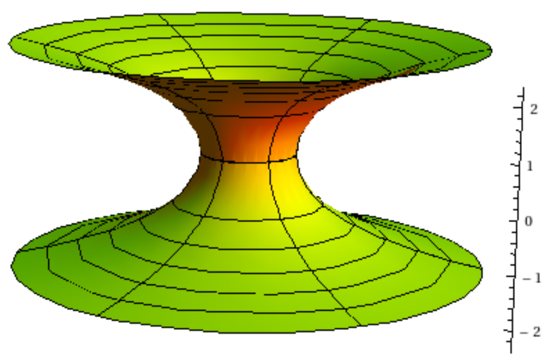


Figure 2.1: Embedding diagram for the Bronikov-Ellis wormhole: $l = 1$

Chapter 3

Dyonic black holes in string theory

In this chapter we will present the most important theory for this thesis: The Einstein-Maxwell-dilaton theory. We will first see how the theory arises in string theory, and also the duality transformation of the equations of motion. Then, we will study the black holes of theory. We will also show how to construct Einstein-Rosen bridges, which we will call just wormholes for convenience.

3.1 The Einstein-Maxwell-dilaton theory

In this section we review the low-energy effective action of four-dimensional heterotic string theory, i.e. the Einstein-Maxwell-axion-dilaton theory. We follow the definitions and normalizations of reference [31]. Since we will be interested in the gravitating solutions and duality transformations of the theory, we will restrict our attention only to the bosonic fields, composed by the metric $g_{S\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, the dilaton ϕ, and a $U(1)$ gauge field A_μ. The action is*

$$S_S = \int d^4x\sqrt{-g_S}e^{-2\phi}(R_S + 4g_S^{\mu\mu'}\partial_\mu\phi\partial_{\mu'}\phi - \frac{1}{12}g_S^{\mu\mu'}g_S^{\nu\nu'}g_S^{\tau\tau'}H_{\mu\nu\tau}H_{\mu'\nu'\tau'} \\ -\frac{1}{8}g_S^{\mu\mu'}g_S^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}). \qquad (3.1)$$

*The dilaton field ϕ in this action is related to the dilaton field Φ of reference [31] as $\Phi = 2\phi$.

The field strength and the three-form field have the following definitions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.2}$$

$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}) - \frac{1}{4}(A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu} + A_\rho F_{\mu\nu}). \tag{3.3}$$

The index S indicates that we are in the string frame, where $g_{S\mu\nu}$ is the σ-model metric. The conformal transformation of the metric can be used in order to bring the theory to the Einstein frame. In fact, this transformation is written as

$$g_{S\mu\nu} = e^{2\phi} g_{\mu\nu}. \tag{3.4}$$

This induces a transformation on the D-dimensional Ricci scalar, and the relation between the Ricci scalar in the string frame and in the Einstein frame is given by

$$R_S = e^{-2\phi}\left[R - 2(D-1)\nabla^2\phi - (D-2)(D-1)\partial_\mu\phi\partial^\mu\phi\right]. \tag{3.5}$$

Replacing (3.4) and (3.5) in the action (3.1), we can write it in the Einstein frame, i.e.

$$S = \int d^4x\sqrt{-g}(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-4\phi}H_{\mu\nu\tau}H^{\mu\nu\tau} - \frac{1}{8}e^{-2\phi}F_{\mu\nu}F^{\mu\nu}). \tag{3.6}$$

Notice that the conformal transformation (3.4) flips the sign of the kinetic term for the dilaton field. The equations of motion for (3.6) are[†]

$$R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{12}g_{\mu\nu}e^{-4\phi}H^2 - \frac{1}{16}g_{\mu\nu}F^2 + \frac{1}{4}e^{-4\phi}H_{\mu\rho\tau}H_\nu{}^{\rho\tau} + \frac{1}{4}e^{-2\phi}F_{\mu\rho}F_\nu{}^\rho, \tag{3.7}$$

$$\nabla_\rho(e^{-4\phi}H^{\mu\nu\rho}) = 0, \tag{3.8}$$

$$\nabla_\mu(e^{-2\phi}F^{\mu\nu}) + \frac{1}{2}e^{-4\phi}H^{\rho\mu\nu}F_{\rho\mu} = 0, \tag{3.9}$$

$$\nabla^\mu\nabla_\mu\phi + \frac{1}{6}e^{-4\phi}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{8}e^{-2\phi}F_{\mu\nu}F^{\mu\nu} = 0. \tag{3.10}$$

[†]Notice that the equation for the metric in reference [31] is combined with the equation for the dilaton.

The Bianchi identity for $H_{\mu\nu\rho}$ is

$$\frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = -\frac{3}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{3.11}$$

with

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{3.12}$$

where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol, with $\tilde{\epsilon}^{0123} = 1$. Using the equation of motion for $B_{\mu\nu}$, given by (3.8), we can define a scalar field a, called **axion**, such that

$$H^{\mu\nu\rho} = -\frac{1}{\sqrt{-g}}e^{4\phi}\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\sigma a. \tag{3.13}$$

This is consistent with the equation of motion (3.8) because

$$\begin{aligned}\nabla_\rho\left(-\frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\sigma a\right) &= -\frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}(\partial_\rho\partial_\sigma a - \Gamma^\alpha_{\rho\sigma}\partial_\alpha a)\\ &= 0.\end{aligned} \tag{3.14}$$

With lower indices the equation (3.13) is written as

$$H_{\mu\nu\rho} = -\sqrt{-g}e^{4\phi}\tilde{\epsilon}_{\mu\nu\rho\sigma}\partial^\sigma a. \tag{3.15}$$

Notice that, using $\tilde{\epsilon}^{\mu\nu\rho\sigma}\tilde{\epsilon}_{\nu\rho\sigma\delta} = -3!\delta^\mu_\delta$, the Bianchi identity (3.11) becomes

$$\nabla^\mu(e^{4\phi}\partial_\mu a) = \frac{1}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.16}$$

For the discussion that follows, it is convenient to define a complex field λ and the field strengths F_+ and F_- as

$$\lambda = a + ie^{-2\phi} \equiv \lambda_1 + i\lambda_2, \tag{3.17}$$

$$F_\pm = F \pm i\tilde{F}. \tag{3.18}$$

We aim to rewrite all the equations of motion in terms of these new complex fields. We rewrite the following terms as

$$2\nabla_\mu\phi\nabla_\nu\phi = -\frac{1}{2(\lambda_2)^2}\partial_\mu a\partial_\nu a + \frac{1}{4(\lambda_2)^2}(\partial_\mu\lambda\partial_\nu\overline{\lambda} + \partial_\nu\lambda\partial_\mu\overline{\lambda}), \tag{3.19}$$

$$\frac{1}{4}g_{\nu\gamma}e^{-4\phi}H_{\mu\rho\tau}H^{\gamma\rho\tau} = -\frac{1}{2(\lambda_2)^2}g_{\mu\nu}\partial_\delta a\partial^\delta a + \frac{1}{2(\lambda_2)^2}\partial_\mu a\partial_\nu a, \tag{3.20}$$

$$\frac{1}{12}g_{\mu\nu}e^{-4\phi}H_{\alpha\beta\rho}H^{\alpha\beta\rho} = -\frac{1}{2(\lambda_2)^2}g_{\mu\nu}\partial_\delta a\partial^\delta a. \tag{3.21}$$

Using these expressions, equation (3.7) becomes

$$R_{\mu\nu} = \frac{\partial_\mu\overline{\lambda}\partial_\nu\lambda + \partial_\nu\overline{\lambda}\partial_\mu\lambda}{4(\lambda_2)^2} + \frac{1}{4}\lambda_2 F_{\mu\rho}F_\nu{}^\rho - \frac{1}{16}\lambda_2 g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{3.22}$$

Also, computing the terms of (3.10) separately, we have

$$\nabla_\mu\nabla^\mu\phi = -\frac{1}{2i\lambda_2}(\nabla_\mu\nabla^\mu\lambda - \nabla_\mu\nabla^\mu\overline{\lambda}) - \frac{1}{4(\lambda_2)^2}(\nabla_\mu\lambda\nabla^\mu\lambda + \nabla_\mu\overline{\lambda}\nabla^\mu\overline{\lambda} - 2\nabla_\mu\lambda\nabla^\mu\overline{\lambda}), \tag{3.23}$$

$$\frac{e^{-2\phi}}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} = -\frac{1}{4(\lambda_2)^2}(\nabla_\mu\lambda\nabla^\mu\lambda + \nabla_\mu\overline{\lambda}\nabla^\mu\overline{\lambda} + 2\nabla_\mu\lambda\nabla^\mu\overline{\lambda}). \tag{3.24}$$

Then equation (3.10) is rewritten as

$$-\frac{1}{2i(\lambda_2)^2}(\nabla_\mu\nabla^\mu\lambda - \nabla_\mu\nabla^\mu\overline{\lambda}) - \frac{1}{2(\lambda_2)^3}(\nabla_\mu\lambda\nabla^\mu\lambda + \nabla_\mu\overline{\lambda}\nabla^\mu\overline{\lambda}) + \frac{1}{8}F_{\mu\nu}F^{\mu\nu} = 0. \tag{3.25}$$

Using the definition (3.13), we can also rewrite the equation (3.16) as

$$-\frac{1}{2i(\lambda_2)^2}(\nabla_\mu\nabla^\mu\lambda + \nabla_\mu\nabla^\mu\overline{\lambda}) - \frac{1}{2(\lambda_2)^3}(\nabla_\mu\lambda\nabla^\mu\lambda - \nabla_\mu\overline{\lambda}\nabla^\mu\overline{\lambda}) + \frac{1}{8i}F_{\mu\nu}\tilde{F}^{\mu\nu} = 0. \tag{3.26}$$

Adding equations (3.25) e (3.26), and using $F_{+\mu\nu}F_-^{\mu\nu} = 0$, the equations of motion take a more elegant form, i.e.

$$\frac{\nabla^\mu\nabla_\mu\lambda}{(\lambda_2)^2} + i\frac{\nabla_\mu\lambda\nabla^\mu\lambda}{(\lambda_2)^3} - \frac{i}{16}F_{-\mu\nu}F_-{}^{\mu\nu} = 0. \tag{3.27}$$

Using the definition (3.18), the gauge field equation is easily rewritten as

$$\nabla_\mu(\lambda F_+{}^{\mu\nu} - \overline{\lambda}F_-{}^{\mu\nu}) = 0. \tag{3.28}$$

Finally, we also rewrite the Bianchi identity in terms of the complex fields as

$$\nabla_\mu(F_+{}^{\mu\nu} - F_-{}^{\mu\nu}) = 0. \tag{3.29}$$

The equations of motion written in terms of the complex fields (3.17) and (3.18) allow us to see explicitly one very special kind of invariance. First, notice we can shift the field λ by a constant c, i.e.

$$\lambda \to \lambda + c, \tag{3.30}$$

and the equations of motion and the Bianchi identities are invariant. Under the transformations

$$\lambda \to -\frac{1}{\lambda}, \quad F_+ \to -\lambda F_+, \quad F_- \to -\overline{\lambda} F_-, \tag{3.31}$$

the field λ_2 transforms as $\lambda_2/\lambda\overline{\lambda}$. We can verify that equation (3.27) is invariant, and that the equations (3.28) and (3.29) gets rotated. Equation (3.22) transforms into itself plus an extra term, given by

$$-\frac{\lambda_1(\lambda_2)^2}{|\lambda|^2}(2F_{\mu\rho}\tilde{F}^{\rho}_{\nu} + 2F_{\nu\rho}\tilde{F}^{\rho}_{\mu} - g_{\mu\nu}F_{\rho\tau}\tilde{F}^{\rho\tau}). \tag{3.32}$$

This term is identically zero in four dimensions. What we have just shown is that transformations (3.30) and (3.31) are a symmetry transformation of the equations of motion. The two transformations together generate the group $\mathrm{SL}(2,\mathbb{R})$ under which

$$\lambda \to \frac{\tilde{a}\lambda + b}{c\lambda + d}, \quad F_+ \to -(c\lambda + d)F_+, \tag{3.33}$$

where $\tilde{a}d - bc = 1$, and we used a tilde on the parameter $\tilde{a}$ to differentiate it from the axion field. It is important to stress that $\mathrm{SL}(2,\mathbb{R})$ is the invariance group of the equations of motion, but not of the action. This can be seen explicitly if we rewrite the action (3.1) in terms of the complexified fields, and then, after applying (3.32), the Maxwell's term transforms into itself with a minus sign, i.e. this term does not allow the action to be invariant.

It is interesting to notice that we can obtain the same equations of motion derived above from the following action

$$I = \int d^4x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{8}e^{4\phi}\partial_\mu a\partial^\mu a - \frac{1}{8}e^{-2\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}aF_{\mu\nu}\tilde{F}^{\mu\nu}\right). \tag{3.34}$$

The action needs to be written necessarilly in this way, including these numerical factors, so that the equations of motion be invariant under the duality transformations (3.33). In order to be consistent with the definitions used by most authors in the literature, we will rescale the fields in this theory as[‡]

$$a \to \frac{a}{2}, \quad A^{\mu} \to \frac{A^{\mu}}{\sqrt{8}}, \tag{3.35}$$

so that the action (3.34) is rewritten as

$$I = \int d^4x\sqrt{-g}\left(R - 2\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}e^{4\phi}\partial_{\mu}a\partial^{\mu}a - e^{-2\phi}F_{\mu\nu}F^{\mu\nu} - aF_{\mu\nu}\tilde{F}^{\mu\nu}\right). \tag{3.36}$$

This action describes the **Einstein-Maxwell-axion-dilaton theory**. Of course, this is a simplified version of a low-energy effective action in string theory in four dimensions. In this thesis we will be mostly concerned about the simplified version of this theory, in which the axion field is consistently set to zero, i.e. we will be concerned about the following theory

$$S = \int d^4x\sqrt{-g}\left(R - 2\partial_{\mu}\phi\partial^{\mu}\phi - W(\phi)F_{\mu\nu}F^{\mu\nu}\right). \tag{3.37}$$

We remind that $(16\pi G_N) \equiv 1$. From now on, we will refer to this action as the **Einstein-Maxwell-dilaton theory**. In the next chapters we will also have potential term for the dilaton in the action. The function $W(\phi)$ accounts for all the kinds of coupling to the gauge fields, and will be fixed later. In order to fix the notation, we rewrite the equations of motion for the metric, dilaton and gauge field, and Bianchi identities, which are respectively

$$R_{\mu\nu} = 2\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}W(\phi)F_{\rho\sigma}F^{\rho\sigma} + 2W(\phi)F_{\mu\rho}F_{\nu}{}^{\rho}, \tag{3.38}$$

$$\nabla_{\mu}(\partial^{\mu}\phi) - \frac{1}{4}\frac{\partial W(\phi)}{\partial\phi}F_{\mu\nu}F^{\mu\nu} = 0, \tag{3.39}$$

$$\nabla_{\mu}\left(W(\phi)F^{\mu\nu}\right) = 0, \tag{3.40}$$

$$\nabla_{[\mu}F_{\rho\sigma]} = 0. \tag{3.41}$$

[‡]Rescaling fields does not break invariance under $\mathrm{SL}(2,\mathbb{R})$ invariance.

In supergravity theories it is common to have more than one dilaton field, and also more gauge fields, but all the equations of motion obtained here are easily generalized for these cases. When $W(\phi) = e^{-2\phi}$, (3.37) is the bosonic sector of $SU(4)$ version of $\mathcal{N} = 4$ supergravity theory [32], for zero axion field, which is the low-energy effective theory for heterotic strings, as we just saw.

3.2 General metric and electric-magnetic duality

The most general form for a static and spherically symmetric solution is written as

$$ds^2 = -e^{-\lambda} dt^2 + e^{\lambda} dr^2 + C^2(r) d\Omega_2^2, \tag{3.42}$$

where all the metric elements depend on the radial coordinate r. In this chapter we will study only the case for which

$$W(\phi) = e^{-2\phi}. \tag{3.43}$$

Equation (3.38) are non-trivial for the R_{00}, R_{11}, and R_{22} components (the R_{33} component is the same as the R_{22} component). Adding R_{00} with R_{11}

$$-\frac{2C''}{C} = 2(\phi')^2, \tag{3.44}$$

where the primes denote derivative with respect to the radial coordinate. Now, computing R_{11} minus R_{22} we have

$$(e^{-\lambda} C^2)'' = 2. \tag{3.45}$$

We can rewrite R_{00} as

$$\frac{d}{dr}\left(C^2 \frac{d}{dr}(e^{-\lambda})\right) = -2C^2 e^{-2\phi}(F_{rt}F^{rt} - F_{\theta\phi}F^{\theta\phi}). \tag{3.46}$$

The dilaton equation of motion can be put in the form

$$\frac{d}{dr}\left(e^{-\lambda} C^2 \phi'\right) = -C^2 e^{-2\phi}\left(F_{rt}F^{rt} + F_{\theta\phi}F^{\theta\phi}\right). \tag{3.47}$$

By adding and subtracting (3.46) and (3.47) we can form the following equations

$$\frac{d}{dr}\left(\frac{C^2}{2}\frac{d}{dr}(e^{-\lambda}) + e^{-\lambda}C^2\phi'\right) = -2C^2e^{-2\phi}F_{rt}F^{rt}, \tag{3.48}$$

$$\frac{d}{dr}\left(\frac{C^2}{2}\frac{d}{dr}(e^{-\lambda}) - e^{-\lambda}C^2\phi'\right) = 2C^2e^{-2\phi}F_{\theta\phi}F^{\theta\phi}. \tag{3.49}$$

In section 3.1, we saw that the equations of motion of the Einstein-Maxwell-axion-dilaton theory are invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations, given by (3.33). In the present case, the axion field was set to zero, so the set of transformations (3.33) gets simplified. This smaller invariance of the equations of motion is called **S-duality**, and it relates the old and new fields through

$$F^{\mu\nu} \to \frac{1}{2\sqrt{-g}}e^{-2\phi}\tilde{\epsilon}^{\mu\nu\rho\sigma}F_{\rho\sigma}, \;\; \phi \to -\phi, \tag{3.50}$$

where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol. The importance of having the equations (3.48) and (3.49) written in such a way resides in the fact that we can see explicitly the invariance under S-duality duality transformation (3.50). The analysis of the duality transformations of section (3.1) was for a general background. Here, after fixing the background to be given by (3.42), we see explicitly that the Maxwell's equations and Bianchi identities rotate into each other. In the same way, the same happens to equations (3.48) and (3.49), showing explicitly that the duality transformation is not only an invariance of the gauge field equations but also of the Einstein's and dilaton equations combined.

3.3 Full dyonic black hole solution

In order to find the dyonic black hole solution, we must solve the system of equations given by (4.5), (3.44), (3.45), (3.46) and (3.47). Of course, there will be some integration constants that will appear in the solution and that must be fixed in terms of the physical charges of the black hole. The solutions presented in [4], [5], [6] and [7] are all written in terms of the physical charges. We first present a solution written in terms of only the integration constants, and will show how they are related to the physical charges. The dyonic black hole solution to the Einstein-Maxwell-dilaton theory is given by

$$e^{-\lambda} = \frac{(r-r_1)(r-r_2)}{(r+d_0)(r+d_1)}, \quad C^2(r) = (r+d_0)(r+d_1), \tag{3.51}$$

$$e^{2\phi} = e^{2\phi_0}\frac{r+d_1}{r+d_0}, \tag{3.52}$$

$$F_{rt} = \frac{e^{2\phi_0}Q}{(r+d_0)^2}, \quad F_{\theta\phi} = P\sin\theta. \tag{3.53}$$

Here, Q is the electric charge, P is the magnetic charge, ϕ_0 is the value of the dilaton at infinity, and r_1, r_2, d_0, and d_1 are integration constants. The dilaton equation of motion (3.39) implies that all the parameters of the solution must satisfy the following relations

$$(d_0-d_1)[(r_1+r_2)+(d_0+d_1)] = 2(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2), \tag{3.54}$$

$$(d_0-d_1)(d_0d_1 - r_1r_2) = 2(d_1e^{2\phi_0}Q^2 - d_0e^{-2\phi_0}P^2), \tag{3.55}$$

$$(d_0-d_1)[-(r_1+r_2)d_0d_1 - (d_0+d_1)r_1r_2] = 2(d_1^2e^{2\phi_0}Q^2 - d_0^2e^{-2\phi_0}P^2). \tag{3.56}$$

Notice that we can isolate (r_1+r_2) in (3.54) and r_1r_2 in (3.55) to obtain

$$(r_1+r_2) = 2\frac{(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2)}{(d_0-d_1)} - (d_0+d_1), \tag{3.57}$$

$$r_1r_2 = d_0d_1 - 2\frac{(d_1e^{2\phi_0}Q^2 - d_0e^{-2\phi_0}P^2)}{(d_0-d_1)}. \tag{3.58}$$

Replacing (3.57) and (3.58) in (3.56), we see that this equation is satisfied trivially. In other words, this system of three equations reduces to a system of just two linearly independent equations. Notice that we can combine the equations (3.54) and (3.55) to obtain

$$d_0^2 + (r_1+r_2)d_0 + r_1r_2 = 2e^{2\phi_0}Q^2, \tag{3.59}$$

$$d_1^2 + (r_1+r_2)d_1 + r_1r_2 = 2e^{-2\phi_0}P^2. \tag{3.60}$$

For future convenience, we solve these equations for d_0 and d_1 in terms of the other parameters, and obtain

$$d_0 = \frac{-(r_1+r_2) \pm \sqrt{(r_1-r_2)^2 + 8e^{2\phi_0}Q^2}}{2}, \tag{3.61}$$

$$d_1 = \frac{-(r_1 + r_2) \pm \sqrt{(r_1 - r_2)^2 + 8e^{-2\phi_0} P^2}}{2}. \tag{3.62}$$

The same procedure can be repeated for equation (3.46), and the same system of two linearly independente equations can be obtained.

In order to have general expressions, we compute two quantities of interest in terms of the integration constants, which are the temperature of the black hole, given by the last equation of (2.42),

$$T = \frac{1}{4\pi} \frac{(r_2 - r_1)}{(r_2 + d_0)(r_2 + d_1)}, \tag{3.63}$$

and the dilaton charge, defined as

$$\Sigma = \frac{1}{4\pi} \int d\Sigma^\mu \nabla_\mu \phi = \frac{(d_0 - d_1)}{2}. \tag{3.64}$$

Of course, depending on the values of d_0 and d_1 the dilaton charge can be positive or negative. The constants r_2 and r_1 will be identified with the outer and inner horizon respectively. As we mentioned, no boundary condition was imposed on the solution. In order to define the mass of the black hole, we expand g_{tt} for large values of r. So, in the asymptotic region, g_{tt} is written as

$$g_{tt} = -\left(1 - \frac{(d_0 + d_1 + r_1 + r_2)}{r}\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{3.65}$$

In the same way as we did for the solutions of chapter 2, we identify the parameter M with the mass of the black hole, consistent with the weak-field limit metric (4.24), i.e.

$$2M = (d_0 + d_1 + r_1 + r_2). \tag{3.66}$$

The Ricci scalar written in terms of the integration constants is given by

$$R = \frac{(d_0 - d_1)^2 (r - r_1)(r - r_2)}{2(r + d_0)^3 (r + d_1)^3}. \tag{3.67}$$

In order to avoid problems with causality, the solution must be restricted to the domain $C^2(r) \geq 0$. Therefore, the singularity is at $r_S = -d_0$ when $d_0 > d_1$, or at $r_S = -d_1$ when $d_1 > d_0$.

3.4 Relation to known solutions

In this section we show how to recover the known non-extremal and extremal black hole solutions discussed before. In order to do so, we must write the solution in terms of the mass, electric charge, magnetic charge and dilaton charge. We make this comparison using the notation of reference [7].

- Non-extremal dyonic black hole:

In order to obtain this solution, we solve (3.57) and (3.58) for r_1 and r_2, by making the redefinitions $r_1 \equiv r_-$ and $r_2 \equiv r_+$. The result is

$$r_\pm = -\frac{(d_0+d_1)}{2} + \frac{(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2)}{(d_0-d_1)} \pm \left(\frac{\Delta}{4}\right)^{1/2}, \tag{3.68}$$

with

$$\frac{\Delta}{4} = \left(\frac{d_0-d_1}{2}\right)^2 + \frac{(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2)^2}{(d_0-d_1)^2} - (e^{2\phi_0}Q^2 + e^{-2\phi_0}P^2). \tag{3.69}$$

The definition of mass (3.66) gives

$$M = \frac{(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2)}{(d_0-d_1)}. \tag{3.70}$$

Using also the definition of the dilaton charge, we write this expression as

$$M \cdot \Sigma = \frac{(e^{\phi_0}Q + e^{-\phi_0}P)}{\sqrt{2}} \frac{(e^{\phi_0}Q - e^{-\phi_0}P)}{\sqrt{2}}. \tag{3.71}$$

We will comment on this expression later. Notice that the parameter $\frac{(d_0+d_1)}{2}$ can be removed by shifting the coordinates, i.e. $r \equiv \rho - \frac{(d_0+d_1)}{2}$. Then the full solution will be written as

$$e^{-\lambda} = \frac{(\rho-\rho_+)(\rho-\rho_-)}{(\rho+\Sigma)(\rho-\Sigma)}, \quad C^2(\rho) = (\rho+\Sigma)(\rho-\Sigma), \tag{3.72}$$

$$e^{2\phi} = e^{2\phi_0}\frac{\rho-\Sigma}{\rho+\Sigma}, \tag{3.73}$$

$$F_{\rho t} = \frac{e^{2\phi_0}Q}{(\rho+\Sigma)^2}, \quad F_{\theta\phi} = P\sin\theta, \tag{3.74}$$

with the outer and inner horizons given by

$$\rho_{\pm} = M \pm \sqrt{\Sigma^2 + M^2 - (e^{2\phi_0}Q^2 + e^{-2\phi_0}P^2)}. \tag{3.75}$$

This is exactly the solution given in reference [7]. Notice that the dilaton charge given in the same reference is the negative of the one used here. This solution contains four independent parameters: the mass M, the electric charge Q, the magnetic charge P, and the dilaton at infinity ϕ_0. The magnetically charged solutions of [5] and [6] are obtained from it by setting $Q = 0$. By setting further that $P = 0$ we recover the Schwarzschild solution (2.20).

- Extremal black holes:

Just like in the case of the Reissner-Nordström solution, when the inner and outer horizon in (3.68) coincide, i.e. $r_+ = r_- \equiv r_H$, we have an extremal ($T = 0$) black hole. The horizon is now located at r_H, and, from (3.61) and (3.62) for instance, it is related to the other parameters as

$$r_H + d_0 = \pm\sqrt{2}e^{\phi_0}Q, \;\; r_H + d_1 = \pm\sqrt{2}e^{-\phi_0}P. \tag{3.76}$$

The fact that there is more than one possibility of choosing signs will be of relevance for the analysis of next section. In order to identify this case with the one found in the literature, we make the change of coordinates $\rho = r - r_H$ and then the horizon of the extremal black hole is located at $\rho = 0$. The extremal black hole solution is therefore

$$\begin{aligned} ds^2 &= -e^{-2U}dt^2 + e^{2U}(d\rho^2 + \rho^2 d\Omega_2^2), \\ e^{2U} &= \left(1 \pm \frac{\sqrt{2}e^{\phi_0}Q}{\rho}\right)\left(1 \pm \frac{\sqrt{2}e^{-\phi_0}P}{\rho}\right), \end{aligned} \tag{3.77}$$

$$e^{2\phi} = e^{2\phi_0}\frac{(\rho \pm \sqrt{2}e^{-\phi_0}P)}{(\rho \pm \sqrt{2}e^{\phi_0}Q)}, \tag{3.78}$$

$$F_{rt} = \frac{e^{2\phi_0}Q}{(\rho \pm \sqrt{2}e^{\phi_0}Q)^2}, \quad F_{\theta\phi} = P\sin\theta. \tag{3.79}$$

Notice that we have used isotropic coordinates, i.e. $\rho^2 = x_1^2 + x_2^2 + x_3^2$, and consequently $d\rho^2 + \rho^2 d\Omega_2^2 = d\vec{x}^2$. The parameter M given in (3.66) can be positive, zero, or negative, depending on the choice of signs and charges in (3.61) and (3.62). We keep dependency on this arbitrary choice of signs. Using (3.64) and (3.66) we have the following possible configurations

$$M = \frac{e^{\phi_0}Q \pm e^{-\phi_0}P}{\sqrt{2}}, \quad \Sigma = \frac{e^{\phi_0}Q \mp e^{-\phi_0}P}{\sqrt{2}}, \tag{3.80}$$

$$M = \frac{e^{-\phi_0}P \pm e^{\phi_0}Q}{\sqrt{2}}, \quad \Sigma = \frac{e^{-\phi_0}P \mp e^{\phi_0}Q}{\sqrt{2}}. \tag{3.81}$$

When we take the upper sign in (3.80) we recover exactly the extremal dyonic solution found in [7]. But this is just one possibility, since (3.80) and (3.81) show that there are three more. Notice that any configuration of signs respects the product between the mass and the dilaton charge given by equation (3.71). It is important to notice that we wrote the explicit dependence of the dilaton field (3.78) in terms of the electric charge Q, magnetic charge P and dilaton at infinity ϕ_0.

3.5 Dependent and independent parameters

In order to recover the solution found by Kallosh et al. [7], given by (3.72), (3.73), and (3.74), we had to define the mass M using (3.70). One could argue that this identification makes it clear that the mass M is a dependent parameter, of the kind $M(Q, P, \phi_0, \Sigma)$, and the dilaton charge Σ is an independent parameter. This argument could be supported by the fact that the horizons (3.68) blow up in the limit when $d_0 \to d_1$, i.e. when $\Sigma \to 0$, resulting in a divergent metric. Moreover, it seems that the zero mass limit is well-defined in (3.70). But one could for instance, take the dilaton charge as an independent parameter, and say that (3.70) is instead written as

$$\Sigma = \frac{(d_0 - d_1)}{2} = \frac{(e^{2\phi_0}Q^2 - e^{-2\phi_0}P^2)}{2M}. \tag{3.82}$$

In this picture, the mass now becomes an independent parameter, and the dilaton charge has a dependency on the other parameters of the form $\Sigma(Q, P, \phi_0, M)$. Moreover, in this picture, the limit when the dilaton charge is zero, i.e. $d_0 \to d_1$, is well defined, although the mass M now can not be zero, since this will lead to a divergent metric. This is exactly the case presented by the authors of [7]. In fact, we see clearly that claiming that the dilaton charge is the dependent parameter and the mass is independent is just one possibility, since the equations of motion allow us to choose also the other case. In other words, the only restriction we find is that the product between the mass and the dilaton charge must be given by (3.71), which is respected also in the extremal case, as we can see from (3.80) and (3.81). Notice also that the extremal solution of [7] has mass and dilaton charge written as (3.80) with only upper sign. Another possibility would be to take both the mass and dilaton charge as dependent parameters in the non-extremal solution. For this case, one is forced by (3.71) to choose one of the possibilities represented by equations (3.80) and (3.81). But for this picture, when both are dependent parameters, one can not escape from the extremal limit, which can be easily seen by inserting M and Σ in (3.68). So, in order to write a non-extremal solution in terms of the physical charges of the black hole, one is forced to choose whether the dilaton charge or the mass is a dependent parameter.

The advantage of writing the solution in terms of integration constants, instead of physical charges, is beyond the discussion on which parameters are dependent or independent. We will see that without defining the integration constants in terms of the mass or dilaton charge, we solve some puzzles related to the dilaton field in the extremal limit, as was stated in the introduction. Moreover, in the picture adopted by Kallosh et al. [7], we have a description of black holes whose dilaton charge is the dependent parameter, with a well-defined limit when the dilaton charge is zero. In the other picture we take the mass of the black hole as a dependent parameter, with a well-defined zero mass limit. The zero mass limit can not be taken directly from (3.82), which raises the question of whether this limit really exists or is ill-defined. In fact, it is easy to satisfy equations (3.54), (3.55), and (3.56) at the same time in such a way to construct a massless solution. This massless black hole solution and its physical significance will be discussed in section 3.8. Moreover, this solution can be used to construct charged Einstein-Rosen bridges satisfying the null energy condition.

3.6 The attractor mechanism

The **no hair theorem** stablishes that black holes are entirely described by a very little set of parameters. This means that if a black hole has an a scalar field as one of its fields, then the scalar field charge is written in terms of the other black hole parameters, which are the mass, the electric and magnetic charges, and the angular momentum. The most important feature concerning a black hole with scalar fields is that this scalar radiated away from the black hole. In the end of the process the black hole will no longer have this field, which justifies why we say "a black hole has no hair". In this case, it is assumed that the scalars have no coupling to the gauge fields. This assumption is not valid for the theories we are interested in, since, as we saw, the dilaton field couples to the gauge fields. We also saw in the previous section that we have the freedom to choose the dilaton charge or the mass of the black hole as independent parameters. In the literature it is always assumed without justification that the dilaton charge is the dependent parameter, so that for the black holes of the Einstein-Maxwell-dilaton theory the dilaton charge is written in terms of the mass, electric and magnetic charge, and the value of the dilaton at infinity. In this sense, we say that the scalar field that is radiated away has a **primary hair**, and in the case of the dilaton, which is not radiated away, we say that it has **secondary hair**: its charge is still a dependent parameter but the dilaton is not radiated from the black hole.

In the case of an extremal black hole, the mass and the dilaton charge depends on the value of the dilaton at infinity and on the electric and magnetic charges. A very important phenomenon happens for the extremal black holes given by (3.77), (3.78) and (3.79). Notice that the dilaton field written in isotropic coordinates, given by equation (3.78), is well defined on the horizon $\rho = 0$. In fact, if we evaluate it on the horizon we obtain

$$e^{2\phi_H} = \frac{P}{Q}. \tag{3.83}$$

Notice that the value of the dilaton at infinity, ϕ_0 does not appear explicitly in this result. This is what defines the **attractor mechanism for extremal black holes**: no matter what the conditions imposed on ϕ_0 are, the value of the dilaton on the horizon is always the same, and it depends only on the electric and magnetic charges of the black hole.

The attractor mechanism was discovered in the context of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supergravity [33], [34]. Later, it was discovered that supersymmetry does not play an important role in the attractor phenomenon [35]: this is a consequence of the near-horizon geometry of extremal black holes which is always $\mathrm{AdS}_2 \times \mathcal{M}^2$, where $\mathcal{M}^2$ is the space that defines the geometry of the horizon[§]. This is understood as the following. Consider any supergravity theory. One then sets the fermions to zero and find the solution for the metric and bosonic fields. Then, one replaces the solution into the supersymmetric variation, δ_ξ, of the gravitiny ψ^α_μ. If the solution satisfies

$$\delta_\xi \psi^\alpha_\mu = 0, \tag{3.84}$$

then the solution represents a **supersymmetric black hole**, or equivalently, a **BPS black hole**. If the black hole preserves none of the supersymmetries, then we have a **non-supersymmetric black hole**, or equivalently, a **non-BPS black hole**. All non-extremal black holes are non-supersymmetric. The attractor mechanism can happen for supersymmetric or non-supersymmetric extremal black holes, since all extremal black holes have $\mathrm{AdS}_2 \times \mathcal{M}^2$ near-horizon geometry.

3.7 Thermal properties

The thermodynamical properties follow easily. The general black hole temperature and entropy are given by equation (2.42), and written explicitly as

$$T = \frac{1}{4\pi} \frac{(r_+ - r_-)}{(r_+ + d_0)(r_+ + d_1)}, \quad S = \pi(r_+ + d_0)(r_+ + d_1). \tag{3.85}$$

Notice that the temperature and entropy are written in terms of the integration constants. In the extremal limit, $r_+ = r_- \equiv r_H$, the temperature is naturally zero and the entropy is completely independent on the choice of sign configurations in (3.80) and (3.81), i.e.

$$T = 0, \quad S = 2\pi QP. \tag{3.86}$$

[§]In this thesis we will consider the cases in which $\mathcal{M}^2$ is S^2 or $\mathbb{R}^2$.

This also shows that the entropy for extremal black holes is insensitive to boundary conditions i.e. it does not depend on ϕ_0. This is in agreement with the attractor mechanism [33], [34]. The entropy was first computed in [7], and confirmed in several papers dealing also with the attractor mechanism. The value of the dilaton field on the horizon is given by (3.83).

The first law of black hole thermodynamics (2.44) needs a modification to include the dependence upon the moduli ϕ_0. This was done in [36] and we write it here. Although we dealt with static solutions, we also include angular momentum for completeness. For fixed mass, $(r_1 + r_2) = 2M$, this is

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \psi^\Lambda dq_\Lambda + \chi_\Lambda dp^\Lambda - \Sigma^a d\phi_a, \tag{3.87}$$

where M is the mass, κ is the surface gravity, A is the area of the horizon, Ω is the angular velocity, J is the angular momentum, ψ^Λ and χ_Λ are the electrostatic and magnetostatic potentials, q_Λ and p^Λ are the electric and magnetic charges, $\Lambda = 1, ..., n$ is an index labelling all the charges, Σ^a is the dilaton charge, ϕ_a is the dilaton at infinity, and $a = 1, ..., m$ is an index labelling the dilatons evaluated at infinity.

Also in [36], the authors raised two questions concerning the value of the dilaton on the horizon for extremal black holes:

(i) Why is $\phi_{H,\text{extreme}}$ independent of ϕ_0?

(ii) Why is $\phi_{H,\text{extreme}}$ given by

$$\left(\frac{\partial M_{\text{extreme}}}{\partial \phi}\right)_{((p,q),\phi=\phi_{\text{extreme}})} = 0? \tag{3.88}$$

The second question was answered by the authors in the same refence. It turns out that equation (3.88) is equivalent to

$$\Sigma(\phi_{\text{fix}}, (p,q)) = 0. \tag{3.89}$$

But black holes with vanishing scalar charge must have spatially constant moduli fields: $\phi(r) = \phi_{H,\text{extreme}} = \phi_0$. So, for equation (3.89) to be satisfied, we must choose ϕ_0 to be $\phi_{H,\text{extreme}}$. Here, surprisingly, the answers to both questions arise naturally from the solution describing the dilaton field (3.52). Because the extremal solution is independent of boundary conditions or any definition of dilaton charge and mass, we can compute $\phi_{H,\text{extreme}}$ from (3.78) without really worrying about how to define mass and dilaton charge. This is achieved just by choosing $\rho = 0$, which is the position of the horizon, and this gives

$$e^{2\phi_{H,\text{extreme}}} = \frac{P}{Q}. \tag{3.90}$$

We see that the factors of ϕ_0 cancel out of (3.78), which gives an answer to the first question. This miraculous cancellation of ϕ_0 in the solution for the dilaton (3.78) is the root of the attractor mechanism for extremal black holes, and it could only be seen here because we wrote d_0 and d_1 in terms of the other parameters. For the second question, we have

$$\Sigma(\phi_{\text{fix}}, (p, q)) = 0 \Rightarrow d_0 = d_1. \tag{3.91}$$

Using the expressions for d_0 and d_1, given by (3.61) and (3.62), this is achieved if the value of the dilaton at infinity is given by

$$e^{2\phi_0} = \frac{P}{Q}. \tag{3.92}$$

This is the same as (3.90), which implies $\phi_{H,\text{extreme}} = \phi_0$. We see that it is mathematically trivial to answer these questions once the dilaton field is written as (3.78). In other words, we have just written the constants d_0 and d_1 appearing in (3.52) using (3.61) and (3.62) in the extremal limit.

3.8 Massless pointlike dyonic solutions

In this section we describe the results of reference [16]. Basically, we show that it is possible to construct massless black holes for the Einstein-Maxwell-dilaton theory, whose solution is described by equations (3.51), (3.52) and (3.53). The important point is the observation that this solution is totally free of boundary conditions, which must be imposed on r_1 and r_2 for the present discussion. In the asymptotic region, the g_{tt} component of the metric has an expansion given by equation (3.65). Notice that (3.54) has the definition of mass (3.65) in it. In order to obtain $M = 0$, we set $d_1 = -d_0 = -\Sigma$ and $r_1 = -r_2 \equiv r_H$ at the same time, and this implies directly that

$$e^{2\phi_0} = \pm\frac{P}{Q}. \tag{3.93}$$

We will discuss the situations corresponding to each sign. Equation (3.55) fixes the event horizons as

$$r_H^2 = \Sigma^2 \mp 2QP. \tag{3.94}$$

Notice that the positive sign in (3.93) implies that we must take the minus sign in (3.94). This also respects (3.56), which shows consistency with all the equations motion.

- $e^{2\phi_0} = -\frac{P}{Q}$

We choose the minus sign in (3.93), and discuss the physical relevance of this choice later. The non-extremal solution is written as

$$ds^2 = -e^{-\lambda}dt^2 + e^{\lambda}dr^2 + C^2(r)d\Omega_2^2,$$

$$e^{-\lambda} = \frac{(r-r_+)(r-r_-)}{(r^2-\Sigma^2)}, \quad C^2(r) = (r^2 - \Sigma^2), \tag{3.95}$$

$$e^{2\phi} = -\frac{P}{Q}\frac{(r-\Sigma)}{(r+\Sigma)}, \tag{3.96}$$

$$F_{rt} = -\frac{P}{(r+\Sigma)^2}, \quad F_{\theta\phi} = P\sin\theta. \tag{3.97}$$

The horizon and singularity are located at

$$r_+ = +\sqrt{\Sigma^2 + 2QP}, \quad r_S = |\Sigma|. \tag{3.98}$$

Notice that the area of the two-sphere shrinks to zero at r_S. This excludes $r_- = -\sqrt{\Sigma^2 + 2QP}$ as an inner horizon, since the angular part of the metric will flip sign when an observer approaches this region, leading to problems with causality. The temperature T and entropy S associated to this object are given by

$$T = \frac{1}{4\pi}\frac{\sqrt{\Sigma^2 + 2QP}}{2QP}, \quad S = 2\pi QP. \tag{3.99}$$

The temperature and the entropy are positive quantities. Notice also that the dyonic massless black hole can not become extremal, since this implies that we have an imaginary dilaton charge. The entropy depends only on the electric and magnetic charges, and has the same value as the entropy of extremal black holes of Einstein-Maxwell-dilaton theory with arbitrary dilaton charge and non-zero mass, given by (3.86). When the dilaton charge is zero, we still have a non-extremal black hole, with horizon at $r_+ = +\sqrt{2QP}$. This shows that, at the critical point of the moduli space, i.e. $\Sigma = 0$, we indeed have a massless black hole solution, which is non-extremal and have temperature and entropy given by (3.99).

We see that it is indeed possible to construct a massless dyonic black hole solution for the Einstein-Maxwell-dilaton theory. In order to do that, we had to fix the dilaton field at infinity to be imaginary. The minus sign in (3.93) seems to spoil this solution, although all the physical quantities are real. Zero mass electrically charged solutions of Einstein-Maxwell-dilaton theory were discussed by Gibbons and Rasheed in [37]. These authors obtained massless solutions for such a theory by flipping the sign of the kinetic term of the dilaton or the gauge field, or of both terms at the same time, introducing the term "anti" to express which kinetic term has a flipped sign. The Einstein-Maxwell-anti-dilaton theory for instance, has a positive kinetic term in the action for the dilaton, and so on. The motivation for doing so was based on Dyson's argument [37]: The properties of the theory

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \tag{3.100}$$

do not depend analytically on the coupling constant e^2. If this was not the case, then perturbation theory around the origin would be convergent in powers of e^2, and also in powers of $-e^2$. For the negative sign, the particles would attract, destabilizing the vacuum. As extremal charged black holes behave as charged particles, the authors of [37] used Dyson's argument [38] to study what happens to black holes in theories with positive kinetic terms. In fact, they construct massless electrically charged black holes and wormholes for such theories. Here, studying the dyonic case, we did not flip the sign of any kinetic term, but the massless solution introduced, as stated above, an imaginary dilaton field at infinity. One can check that it is possible to obtain the same massless solution with a real dilaton field if we flip the sign of the kinetic term for the gauge field, which was the same situation studied in [37] for the electrically charged case. So, by trying to avoid problems with the dilaton field we end up transfering the problem to the gauge field, making it have negative kinetic energy. This does not come as a surprise. As we will use this massless solution to build wormholes in the next section, it is worth pointing out that there are other examples in the literature in which wormholes exist when the fields have negative kinetic energy, or imaginary electromagnetic charges. Euclidean wormholes were shown to exist as solutions to low-energy effective actions in string theory, whether when the charges are imaginary [39], or when one of the fields have negative kinetic energy [40]. Notice also that non-Abelian gauge fields with negative kinetic energies have zero mass monopoles, as was pointed out in reference [41]. Our analysis shows that, even with an imaginary dilaton field at infinity, this masless solution seems physically acceptable, since the charges, which are the physical observables, are real. Our intention is not to prove whether or not this is the case, but instead, to show that this solution can be used to construct Einstein-Rosen bridges. As the scalar field does not seem to be physical, one would expect that the null energy condition be violated. We will also show that this is not the case: the null energy condition is satisfied.

- $e^{2\phi_0} = +\frac{P}{Q}$

This other massless solution is written as

$$ds^2 = -e^{-\lambda}dt^2 + e^{\lambda}dr^2 + C^2(r)d\Omega_2^2,$$

$$e^{-\lambda} = \frac{(r-r_+)(r-r_-)}{(r^2-\Sigma^2)}, \quad C^2(r) = (r^2 - \Sigma^2), \tag{3.101}$$

$$e^{2\phi} = \frac{P}{Q}\frac{(r-\Sigma)}{(r+\Sigma)}, \tag{3.102}$$

$$F_{rt} = \frac{P}{(r+\Sigma)^2}, \quad F_{\theta\phi} = P\sin\theta. \tag{3.103}$$

The singularity is located at

$$r_S = |\Sigma|. \tag{3.104}$$

The quantities $r_\pm = \pm\sqrt{\Sigma^2 - 2QP}$ are always smaller than r_S, so this solution represents a naked singularity. The problems related to the previous case are absent here, since the value of the dilaton field at infinity is a real quantity. But still, this massless solution does not have a horizon. As we saw in section 2.5, the existence of a horizon is a requirement for the construction of Einstein-Rosen bridges, so this solution can not be used for this purpose.

3.9 Bridge construction

In this section we construct Einstein-Rosen bridges using the massless solution (3.95), (3.96) and (3.97). The procedure is the same as was done for the Schwarzschild and Reissner-Nordström solutions in section 2.5. Again, we will use the term "wormhole" to name this Einstein-Rosen bridge until the end of the chapter. We first consider the more general case and fix $r_1 = -r_2 \equiv r_0$ in the solution (3.51), (3.52) and (3.53). Notice that, by switching to the coordinates $u^2 = r^2 - r_0^2$, the metric (3.51) is written as

$$\begin{aligned} ds^2 &= -\frac{u^2}{(u^2+r_0^2)}\frac{dt^2}{f(u)} + f(u)du^2 + (u^2+r_0^2)f(u)d\Omega_2^2, \\ f(u) &= \left(1 + \frac{d_0}{\sqrt{u^2+r_0^2}}\right)\left(1 + \frac{d_1}{\sqrt{u^2+r_0^2}}\right). \end{aligned} \tag{3.105}$$

This is a genuine charged wormhole solution: It connects one Minkowski space at $u = -\infty$ to another at $u = +\infty$. The throat of the wormhole is located at $u = 0$, and it has radius

$$R_{\text{throat}} = \sqrt{(r_0 + d_0)(r_0 + d_1)}, \tag{3.106}$$

where d_0 and d_1 will be determined by equations (3.54), (3.55), and (3.56). The term inside the square root is positive when we take the minus sign in (3.93), and the throat of the wormhole will always be greater than zero for $|Q|, |P| > 0$. Notice that this solution is valid only outside the horizon, $r > +r_0$. So, for the full massless non-extremal solution (3.95), we must take the minus sign in (3.93), and $d_1 = -d_0 \equiv -\Sigma$.

The solution is

$$ds^2 = -\frac{u^2}{(u^2+2QP)}dt^2 + \frac{u^2+2QP}{u^2+\Sigma^2+2QP}du^2 + (u^2+2QP)d\Omega_2^2. \tag{3.107}$$

At the critical point of the moduli space, $\Sigma = 0$, this bridge is exactly the Einstein-Rosen bridge (2.49), with $\epsilon^2 = 2QP$. This is due to the fact that $Q \propto -P$, which fulfills $\epsilon^2 \propto -Q^2$. The radius of the throat is

$$R_{\text{throat}} = \sqrt{2QP}. \tag{3.108}$$

We see that the charged wormholes in the Einstein-Maxwell-dilaton theory may come from the non-extremal massless dyonic solutions.

3.10 Null Energy condition

In the construction of the charged wormhole arising from the Reissner-Nordström solution, given by (2.49), we needed to consider imaginary charges. Notice that this is not the situation with the charged wormhole of the Einstein-Maxwell-dilaton theory, given by (3.105). As stated before, exotic matter violates the null energy condition, and we now check whether this is the case for the present solution or not. In order to simplify the analysis, we will use the coordinate system for which the radial coordinate is r, but the same analysis can be done using the coordinate system for which the radial coordinate is u. This does not change the conclusions, since the null energy condition does not depend on the choice of coordinate system. Everything we do in this section is to follow the same analysis as in [28], but, of course, using the non-extremal massless solution (3.95). Again, taking $r_1 = -r_2 \equiv r_0$, the Ricci tensors for the metric (3.51) are given by

$$\begin{aligned} R_{tt} = &\frac{(r^2-r_0^2)}{2(d_0+r)^4(d_1+r)^4}\left[d_0^2\left(2d_1^2+2d_1r+r^2-r_0^2\right)\right.\\ &\left.+\,2d_0r\left(d_1^2-r_0^2\right)+d_1^2\left(r^2-r_0^2\right)-2d_1rr_0^2-2r^2r_0^2\right], \end{aligned} \tag{3.109}$$

$$R_{rr} = \frac{r_0^2-d_0d_1}{(d_0+r)(d_1+r)\left(r^2-r_0^2\right)}, \tag{3.110}$$

$$
\begin{aligned}
R_{\theta\theta} = {} & \frac{1}{2(d_0+r)^2(d_1+r)^2}\left[d_0^2\left(2d_1^2+2d_1 r+r^2-r_0^2\right)\right. \\
& \left. + 2d_0 r\left(d_1^2-r_0^2\right)+d_1^2\left(r^2-r_0^2\right) -2d_1 r r_0^2-2r^2 r_0^2\right], \qquad (3.111) \\
R_{\phi\phi} = {} & R_{\theta\theta}\sin^2\theta. \qquad (3.112)
\end{aligned}
$$

The curvature tensor is

$$
R=\frac{(d_0-d_1)^2\left(r^2-r_0^2\right)}{2(d_0+r)^3(d_1+r)^3}. \qquad (3.113)
$$

We choose orthonormal basis vectors [28]:

$$
\mathbf{e}_{\hat{t}}=\left(\frac{(r+d_0)(r+d_1)}{(r^2-r_0^2)}\right)^{1/2}\mathbf{e}_t, \qquad (3.114)
$$

$$
\mathbf{e}_{\hat{r}}=\left(\frac{(r^2-r_0^2)}{(r+d_0)(r+d_1)}\right)^{1/2}\mathbf{e}_r, \qquad (3.115)
$$

$$
\mathbf{e}_{\hat{\theta}}=\left(\frac{1}{(r+d_0)(r+d_1)}\right)^{1/2}\mathbf{e}_\theta, \qquad (3.116)
$$

$$
\mathbf{e}_{\hat{\phi}}=\left(\frac{1}{(r+d_0)(r+d_1)}\right)^{1/2}\frac{1}{\sin\theta}\mathbf{e}_\phi. \qquad (3.117)
$$

In this basis the metric coefficients take the form $\mathbf{g}_{\hat{\alpha}\hat{\beta}}=\mathbf{e}_{\hat{\alpha}}\cdot\mathbf{e}_{\hat{\beta}}=\eta_{\hat{\alpha}\hat{\beta}}=\mathrm{diag}(-1,1,1,1)$. Einstein's equations take the form

$$
G_{\hat{\mu}\hat{\nu}}=8\pi G_N T_{\hat{\mu}\hat{\nu}}. \qquad (3.118)
$$

We remind the reader that in our units $(16\pi G_N)=1$. The components of the energy momentum tensor are $T_{\hat{t}\hat{t}}=\rho(r)$, $T_{\hat{r}\hat{r}}=-\tau(r)$, $T_{\hat{\theta}\hat{\theta}}=T_{\hat{\phi}\hat{\phi}}=p(r)$, where $\rho(r)$ is the energy density measured by the static observer, $\tau(r)$ is the tension per unit area measured in the radial direction, and $p(r)$ is the pressure that is measured in the directions orthogonal to the radial direction. They are given by

$$
\begin{aligned}
\rho(r) = {} & \frac{1}{2(d_0+r)^3(d_1+r)^3}\left[4d_0 r\left(d_1^2-r_0^2\right)2d_0^2\left(2d_1^2+2d_1 r+r^2-r_0^2\right)\right. \\
& \left. +2d_1^2\left(r^2-r_0^2\right)-4d_1 r r_0^2-4r^2 r_0^2+(d_0-d_1)^2\left(r^2-r_0^2\right)\right], \qquad (3.119) \\
-\tau(r) = {} & \frac{1}{2(d_0+r)^3(d_1+r)^3}\left[(4r_0^2-4d_0 d_1)(d_0+r)(d_1+r)\right.
\end{aligned}
$$

$$-2(d_0-d_1)^2(r^2-r_0^2)\big]\,. \tag{3.120}$$

The null energy condition in the hatted coordinate system states that

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} \geq 0. \tag{3.121}$$

In the same coordinate system, the null vector is given by $k^{\hat{\mu}} = (1,1,0,0)$, and the null energy condition results in

$$\rho(r) - \tau(r) = \frac{(d_0-d_1)^2(r^2-r_0^2)}{2(d_0+r)^3(d_1+r)^3}. \tag{3.122}$$

Evaluating this for the metric (3.95), in which $e^{2\phi_0} = -P/Q$, and $d_1 = -d_0 \equiv -\Sigma$, we have

$$\rho(r_0) - \tau(r_0) = \frac{2\Sigma^2(r^2-(\Sigma^2+2QP))}{(r^2-\Sigma^2)^3} \geq 0. \tag{3.123}$$

This is a proof that the null energy condition is satisfied, and the massless charged wormhole solution of the Einstein-Maxwell-dilaton theory in equation (3.95) does not require exotic matter to exist. The coordinate system is valid only outside the horizon, and the only way to saturate the bound is at the critical point of the moduli space, i.e.

$$\Sigma = 0 \Rightarrow \rho(r_0) - \tau(r_0) = 0. \tag{3.124}$$

This is a very intriguing fact. It states that, as long as the dilaton exists, the energy density of the spacetime is always greater than the tension per unit area. The massless pointlike objects presented here are entirely new, and, of course, the charged wormholes may be understood as a generalization of the charged Einstein-Rosen bridge, for the case when we include the dilaton in the theory.

There are some topics concerning the massless solution which might be the subject of future research. It is necessary to check the stability of the massless solution (3.95), (3.96) and (3.97). It is known that negative mass is in general a sign of instability, but a full analysis is necessary to prove that our massless solutions are stable under small perturbations of the metric. We hope to analyze whether or not the same argument concerning the traversability of the neutral bridges applies here. This does not necessarily mean that one can cross this wormhole, unless there is a mechanism that avoids the objects to hit the singularity. It is intriguing though that the massless black holes solution, with an imaginary dilaton field at infinity, allowed us to construct a wormhole that satisfies the null energy condition. As stated in the text, an imaginary dilaton field at infinity can be avoided by allowing the gauge fields to have negative kinetic energy. Quantum mechanically, energy is allowed to admit negative values (at least locally), but we are dealing with a classical theory. In general, a classical theory admiting fields whose kinetic energy is negative violates the null energy condition, but here we just saw a counterexample of such claim. The physical observables computed here are the temperature, entropy, mass, electric charge, and magnetic charge, and all of them are real quantities. The dilaton charge does not depend on ϕ_0, and this is also a real quantity. The electric charge appears in F_{rt} as $e^{2\phi_0}Q$, and this is real, although ϕ_0 is imaginary. These facts are strong indications that the solution (3.95), (3.96), and (3.97) is indeed physically acceptable, but a more careful analysis is necessary in order to make such a claim.

Chapter 4

Traversable wormholes of Einstein-Maxwell-phantom-dilaton theory

In the last chapter we investigated the bridge construction for the Einstein-Maxwell-dilaton theory. We first solved the field equations and then imposed appropriate boundary conditions on the solution in order to obtain the desired profile of a wormhole. In other words, these wormholes were constructed from a black hole solution with "unusual" boundary conditions. As we saw in section 2.6, it is possible to construct a genuine traversable wormhole in general relativity if we start with a theory containing one phantom scalar field. The price we pay for such a construction is that the null energy condition is violated since the beginning, and this makes the theory become classically unacceptable, although there is some hope that it can be acceptable quantum mechanically. In this chapter we will follow the same philosophy, and start with a theory that contains one phantom field: we will turn only the dilaton field into a phantom field in the Einstein-Maxwell-dilaton theory. We will present an analytical traversable wormhole solution for the Einstein-Maxwell-phantom-dilaton theory*. The solution exists in the presence of electric charge, and

*For a numerical study about the role of phantom scalar fields in the gravitational collapse, see reference [42].

can be understood as a charged generalization of the Bronikov-Ellis wormhole, since we can recover it by setting the electric charge to zero. We will present the theory, the solution, and then present the embedding diagram. But we will also do something else in this chapter: We will compute the deflection of light passing close to the wormhole using the Gauss-Bonnet theorem. We also discuss that the magnetically charged solution can be obtained via S-duality, as the equations of motion are invariant under such transformation.

4.1 Motivation for considering phantom fields

As we saw in section 2.6, phantom fields are defined as the fields whose kinetic term appears in the action with flipped sign[†], resulting in a negative kinetic energy for such fields. There are cosmological observations [43][44] suggesting that the existence of a fluid with negative pressure could be a phantom field, which serves as models for dark energy. From a theoretical point of view, for a nontrivial kinetic term $P(-\frac{1}{2}(\partial\phi)^2)$, ghost fields give a ghost condensate in a consistent infrared modification of gravity [45]. We will not investigate the stability of the solution, but, although negative energies are in general a sign of instabilities, there are arguments that if there are instabilities, they can be cured [46]. In string theory, phantom fields appear in the study of "negative branes" [47][48] (called also as "topological anti-branes" or "ghost branes"). In the same way as ordinary branes, negative branes are extended objects which give rise to a gauge group $SU(M)$, for a stack of M negative branes on top of each other, where M is a negative Chan-Pathon factor associated to the endpoint of the string. They cancel the effects of ordinary branes. Then, $SU(N|M)$ symmetry can be realized for a stack of N ordinary D-branes, and M negative D-branes. It was argued in [49][50] that, if $\mathcal{N} = 4$ $SU(N|M)$ gauge theories exist, they must be holographically dual to $\mathrm{AdS}_5 \times S^5$ because they are indistinguishable from $SU(N-M)$ theory to all order in $1/(N-M)$ (for $N > M$). Also the relation among string dualities, the signature of spacetime, and phantom fields, was carefully studied in [51].

[†]Remember that the metric signature used in this thesis is $(-,+,+,+)$

An interesting direction of research is to find and classify the charged solutions in the presence of phantom fields. Some results were obtained previously for the Einstein-Maxwell-dilaton theory. We can, for instance, flip the sign of the kinetic term of the gauge field or the dilaton field for this theory. In fact, Gibbons and Rasheed showed [37] that it is possible to construct massless black holes and wormholes solutions for the new theories with flipped signs. As we saw in section 3.8, it is possible to set the value of the dilaton at infinity to a specific imaginary value, and construct massless black holes for the Einstein-Maxwell-dilaton theory, whose observables are all real. Moreover, the same massless solutions were used to construct Einstein-Rosen bridges which satisfy the Null Energy Condition. There are also black hole solutions found for the case when only the dilaton field of the phantom type [52]. In this chapter we consider only phantom dilaton field, so that the gauge fields have positive kinetic energy. For convenience, we follow [37] and call such a theory as Einstein-Maxwell-anti-dilaton, due to the fact that the only field whose kinetic term has flipped sign is the dilaton.

Of course the wormhole solution we present does not satisfy the null energy condition. This means that the phantom dilaton serves as the matter with negative energy necessary to keep the throat of the wormhole open, which is classically unacceptable. We know nowadays that violations of the null energy condition happen in quantum mechanics[‡], which, together with the cosmological observations discussed in the previous paragraphs, is another motivation for the study of wormholes in the presence of phantom fields.

4.2 Einstein-Maxwell-anti-dilaton theory

The fields of the Einstein-Maxwell-anti-dilaton theory are the metric $g_{\mu\nu}$, a gauge field A_μ, and a phantom scalar ϕ. This is just the Einstein-Maxwell-dilaton theory

[‡]The violations happen only locally. The integrated version of the null energy condition is still valid.

with a positive kinetic term for the dilaton, whose action is written as

$$S = \int d^4x\sqrt{-g}\left(R + 2\partial_\mu\phi\partial^\mu\phi - e^{-2\phi}F_{\mu\nu}F^{\mu\nu}\right). \tag{4.1}$$

In this chapter, we take units in which $(16\pi G_N) \equiv 1$. The field strength has the usual form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.2}$$

The equations of motion for the metric, anti-dilaton, gauge field, and the Bianchi identities are respectively:

$$R_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}e^{-2\phi}F_{\rho\sigma}F^{\rho\sigma} - 2e^{-2\phi}F_{\mu\rho}F_\nu{}^\rho = 0, \tag{4.3}$$

$$\nabla_\mu(\partial^\mu\phi) - \frac{1}{2}e^{-2\phi}F_{\mu\nu}F^{\mu\nu} = 0, \tag{4.4}$$

$$\nabla_\mu\left(e^{-2\phi}F^{\mu\nu}\right) = 0, \tag{4.5}$$

$$\nabla_{[\mu}F_{\rho\sigma]} = 0. \tag{4.6}$$

4.3 Electrically charged wormholes

We want to study electrically charged wormholes in a space with spherical symmetry, so the ansatz for the metric is the same as the one in equation (3.42). The equations of motion are slightly modified compared to the ones of chapter 3. They are written as

$$\frac{C''}{C} = (\phi')^2, \tag{4.7}$$

$$(e^{-\lambda}C^2)'' = 2, \tag{4.8}$$

$$\frac{d}{dr}\left(C^2\frac{d}{dr}(e^{-\lambda})\right) = -2C^2e^{-2\phi}F_{rt}F^{rt}, \tag{4.9}$$

$$\frac{d}{dr}\left(e^{-\lambda}C^2\phi'\right) = C^2e^{-2\phi}F_{rt}F^{rt}. \tag{4.10}$$

We solve firt the gauge field equation (4.5), whose solution is

$$F_{rt} = \frac{Q}{e^{-2\phi}C^2}. \tag{4.11}$$

Replacing this in the previous equations, we find an analytical wormhole solution given by

$$e^{-\lambda} = \exp\left[\frac{Q^2}{2c_1^2}e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} - \frac{2(b_1-c_1)}{l}\arctan\left(\frac{r}{l}\right) - (2b_2-c_2)\right], \tag{4.12}$$

$$C^2 = (r^2+l^2)\exp\left[-\frac{Q^2}{2c_1^2}e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} + \frac{2(b_1-c_1)}{l}\arctan\left(\frac{r}{l}\right) + (2b_2-c_2)\right], \tag{4.13}$$

$$\phi = -\frac{Q^2}{4c_1^2}e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} + \frac{b_1}{l}\arctan\left(\frac{r}{l}\right) + b_2, \tag{4.14}$$

$$F_{rt} = \frac{Q}{(r^2+l^2)}e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)}, \tag{4.15}$$

with the condition

$$c_1 = b_1 \pm \sqrt{b_1^2 - l^2}. \tag{4.16}$$

The process for obtaining this solution is based on guess work, and is very tedious and definitely not very enlightening. That is why we do not describe it here. This is a real solution valid in the whole spacetime. Apart from the term (r^2+l^2) in (4.13) and (4.15), the solution depends only on exponentials of the inverse of tangent function of the radial coordinate. Clearly, it contains no singularity, as a traversable wormhole must be. The wormhole connects one Minkowski spacetime located at $r = +\infty$ with another one at $r = -\infty$. As we mentioned in the introduction of the chapter, for $Q = 0$, we recover the anti-Fisher solution [53][54], which is a generalization of the Bronikov-Ellis solution presented in section 2.6. The anti-Fisher solution is written as

$$ds^2 = -e^{-\lambda}dt^2 + e^{\lambda}dr^2 + C^2(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$e^{-\lambda} = \exp\left[-\frac{2(b_1-c_1)}{l}\arctan\left(\frac{r}{l}\right) - (2b_2-c_2)\right], \tag{4.17}$$

$$C^2(r) = (r^2+l^2)\exp\left[\frac{2(b_1-c_1)}{l}\arctan\left(\frac{r}{l}\right) + (2b_2-c_2)\right], \tag{4.18}$$

$$\phi(r) = \frac{b_1}{l}\arctan\left(\frac{r}{l}\right) + b_2, \tag{4.19}$$

also with the condition (4.16). We recover the Bronikov-Ellis wormhole [29][30], given by equations (2.50) and (2.52), if we set $b_1 = l$ (which implies $b_1 = c_1$), and also $2b_2 = c_2$. For other solutions involving different interacting theories with Lagrangian $\mathcal{L} \sim R - 2(\nabla\phi)^2 - Z(\phi)F^2$, where $Z(\phi)$ are different functions of the scalar field, see references [55], [56].

For reasons that will be explained below, it is more convenient to express our solution (4.12), (4.13), (4.14) and (4.15) in terms of the constants Q, l, b_1, b_2, c_1 and c_2. In general, static black hole and wormhole solutions are expressed in terms of the asymptotic charges such as the mass M, the electric charge q, and the dilaton Σ. For a traversable wormhole we have two asymptotic regions, so we can compute the charges for each asymptotic region and write them in terms of these constants. Using equation (B.8) derived in appendix B, the wormhole metric can be expressed in the positive asymptotic region, i.e. $r \to +\infty$, as

$$ds^2 \approx -e^{-m_1}\left(1 - \frac{m_2}{r}\right)dt^2 + e^{m_1}\left(1 - \frac{m_2}{r}\right)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (4.20)$$

where m_1 and m_2 are given by equation (B.6) also in appendix A. Making the following scale redefinitions

$$t \to e^{m_1/2}\tau, \;\; r \to e^{-m_1/2}u, \quad (4.21)$$

the metric becomes

$$ds^2 \approx -\left(1 - \frac{m_2 e^{m_1/2}}{u}\right)d\tau^2 + \left(1 - \frac{m_2 e^{m_1/2}}{u}\right)^{-1}[du^2 + u^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (4.22)$$

The term multiplying the spatial part of the metric can be expanded, and we finally obtain

$$ds^2 \approx -\left(1 - \frac{m_2 e^{m_1/2}}{u}\right)d\tau^2 + \left(1 + \frac{m_2 e^{m_1/2}}{u}\right)[du^2 + u^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (4.23)$$

In the weak-field limit the static metric is expressed as

$$ds^2 = -(1 + 2U_N)d\tau^2 + (1 - 2U_N)[du^2 + u^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (4.24)$$

where $U_N = -\frac{M}{r}$ is the Newtonian potential, and M is the mass parameter of the gravitational object. Comparing equations (4.23) and (4.24) we identify the mass parameter in the positive asymptotic region as

$$M_+ = \frac{m_2 e^{m_1/2}}{2}. \tag{4.25}$$

Writing this explicitly we obtain

$$M_+ = \left(-b_1 + c_1 + \frac{Q^2 e^{c_2 + \frac{c_1 \pi}{l}}}{2c_1}\right) \exp\left[-\frac{Q^2 e^{c_2 + \frac{c_1\pi}{l}}}{4c_1^2} + \frac{(b_1 - c_1)\pi}{2l} + \frac{(2b_2 - c_2)}{2}\right]. \tag{4.26}$$

This approximation is valid only for the positive asymptotic region, i.e. $r \to +\infty$. The result for the negative asymptotic region, i.e. $r \to -\infty$, is obtained by flipping the signs of the terms containing factors of π, since $\lim_{r\to-\infty} \arctan(r/l) = -\pi/2$. So, the mass parameter for the negative asymptotic region is written as

$$M_- = \left(-b_1 + c_1 + \frac{Q^2 e^{c_2 - \frac{c_1 \pi}{l}}}{2c_1}\right) \exp\left[-\frac{Q^2 e^{c_2 - \frac{c_1\pi}{l}}}{4c_1^2} - \frac{(b_1 - c_1)\pi}{2l} + \frac{(2b_2 - c_2)}{2}\right]. \tag{4.27}$$

The electric charge in the positive asymptotic region q_+ and in the negative asymptotic region q_- are defined through the integral

$$q_\pm = \frac{1}{4\pi} \int_{r\to\pm\infty} F_{\mu\nu} n^\mu m^\nu \sqrt{g_{\theta\theta} g_{\phi\phi}}\, d\theta d\phi, \tag{4.28}$$

where $m^\mu = (1, 0, 0, 0)$ and $n^\mu = (0, 1, 0, 0)$. This gives

$$\begin{aligned} q_\pm &= \frac{1}{4\pi} \int_{r\to\pm\infty} d\theta d\phi \frac{Q}{(r^2 + l^2)} e^{c_2 + \frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} (r^2 + l^2) e^\lambda \\ &= Q e^{c_2 + \frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} e^\lambda \Big|_{r\to\pm\infty}. \end{aligned} \tag{4.29}$$

This implies that

$$q_\pm = Q \exp\left[-\frac{Q^2}{2c_1^2} e^{c_2 \pm \frac{c_1\pi}{l}} \pm \frac{b_1 \pi}{l} + 2b_2\right]. \tag{4.30}$$

The dilaton charges at each region, $\Sigma_\pm$, are defined through

$$\phi \approx \phi_\pm - \frac{\Sigma_\pm}{r} + \dots, \tag{4.31}$$

where $\phi_\pm$ is the value of the dilaton at the positive region for plus sign and negative asymptotic region for minus sign. Using the approximations given in appendix A, this gives

$$\phi \approx -\frac{Q^2 e^{c_2 \pm \frac{c_1 \pi}{l}}}{4c_1^2} \pm \frac{b_1 \pi}{2l} + b_2 - \frac{1}{r}\left[\frac{Q^2 e^{c_2 \pm \frac{c_1 \pi}{l}}}{2c_1} + b_1\right]. \tag{4.32}$$

Then we can extract

$$\phi_+ = -\frac{Q^2}{4c_1^2} e^{c_2 + \frac{c_1 \pi}{l}} + b_2 + \frac{b_1 \pi}{2l}, \tag{4.33}$$

$$\phi_- = -\frac{Q^2}{4c_1^2} e^{c_2 - \frac{c_1 \pi}{l}} + b_2 - \frac{b_1 \pi}{2l}, \tag{4.34}$$

$$\Sigma_+ = \frac{Q^2 e^{c_2 + \frac{c_1 \pi}{l}}}{2c_1} + b_1, \tag{4.35}$$

$$\Sigma_- = \frac{Q^2 e^{c_2 - \frac{c_1 \pi}{l}}}{2c_1} + b_1. \tag{4.36}$$

This allows us to rewrite the mass parameters $M_\pm$ and the charges $q_\pm$ as

$$M_+ = (\Sigma_+ - 2b_1 + c_1) e^{\phi_+ - \frac{c_1 \pi}{2l} - \frac{c_2}{2}}, \tag{4.37}$$

$$M_- = (\Sigma_- - 2b_1 + c_1) e^{\phi_- + \frac{c_1 \pi}{2l} + \frac{c_2}{2}}, \tag{4.38}$$

$$q_+ = Q e^{2\phi_+}, \tag{4.39}$$

$$q_- = Q e^{2\phi_-}. \tag{4.40}$$

Notice that we can not express the constants Q, b_1, b_2 and c_2 solely in terms of the charges at each asymptotic region[§]. This is the reason why it is more convenient to express all our results in terms of these integration constants instead of the asymptotic charges. We will use these results in the end section 7.

For the analysis that follows, we will need the Ricci tensors and the Ricci scalar for our wormhole solution (4.12), (4.13), (4.14) and (4.15). Using equations (4.12) and (4.13), we compute the Ricci tensors, which are given by

$$R_{tt} = \frac{Q^2}{(r^2 + l^2)^2} \exp\left[\frac{Q^2}{c_1^2} e^{\frac{2c_1}{l} \arctan\left(\frac{r}{l}\right) + c_2} - \frac{2(2b_1 - 3c_1)}{l} \arctan\left(\frac{r}{l}\right)\right.$$

[§]Remind that c_1 depends on b_1 and l by (4.16)

$$-4b_2 + 3c_2\big], \tag{4.41}$$

$$R_{rr} = \frac{-1}{2c_1^2\left(l^2+r^2\right)^2}\Big[4c_1^2\left((b_1-c_1)^2+l^2\right) + Q^4 e^{\frac{4c_1}{l}\arctan\left(\frac{r}{l}\right)+2c_2}$$

$$+2c_1Q^2(c_1-2b_1)e^{\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)+c_2}\Big], \tag{4.42}$$

$$R_{\theta\theta} = \frac{Q^2}{(r^2+l^2)}e^{\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)+c_2}, \tag{4.43}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{4.44}$$

From these results we can easily obtain the Ricci scalar, an it is written as

$$R = \frac{-1}{2c_1^2\left(l^2+r^2\right)^2}\left(Q^2 e^{\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)+c_2} - 2b_1c_1\right)^2$$

$$\times\exp\left(\frac{2(c_1-b_1)}{l}\arctan\left(\frac{r}{l}\right) - 2b_2 + \frac{Q^2}{2c_1^2}e^{\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)+c_2} + c_2\right). \tag{4.45}$$

Notice that the Ricci scalar (8.4) is finite everywhere in the spacetime, i.e. it does not contain any singularity in the range $-\infty < r < +\infty$. This means that the solution is neither a black hole nor a naked singularity. One can check that other scalar invariants constructed out from the Riemann tensors are also finite everywhere. Also, due to the smoothness of the spacetime, all geodesics are complete. In order to check the energy conditions, we choose orthonormal basis vectors [28]:

$$\mathbf{e}_{\hat{t}} = e^{\lambda/2}\frac{\partial}{\partial t}, \mathbf{e}_{\hat{r}} = e^{-\lambda/2}\frac{\partial}{\partial r}, \mathbf{e}_{\hat{\theta}} = \frac{1}{C}\frac{\partial}{\partial\theta}, \mathbf{e}_{\hat{\phi}} = \frac{1}{C\sin\theta}\frac{\partial}{\partial\phi}. \tag{4.46}$$

In the hatted coordinated system, the components of the energy momentum tensor are $T_{\hat{t}\hat{t}} = \rho(r)$, $T_{\hat{r}\hat{r}} = -\tau(r)$, $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r)$, where $\rho(r)$ is the energy density measured by the static observer, $\tau(r)$ is the tension per unit area measured in the radial direction, and $p(r)$ is the pressure that is measured in the directions orthogonal to the radial direction. The null energy condition is written as

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} \geq 0, \tag{4.47}$$

where the null vector is given by $k^{\hat{\mu}} = (1,1,0,0)$. As we are using units in which $(16\pi G_N) = 1$, we can use Einstein's equations and the fact that $k_{\hat{\mu}}k^{\hat{\mu}} = 0$ in order to show that $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = 2R_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}}$. The term $R_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}}$ is just twice the curvature scalar given by (8.4). So, for the wormhole solution (4.12), (4.13), (4.14) and (4.15), the null energy condition (4.47) is not satisfied, since (8.4) is a stricly negative function of the radial coordinate. It is important to emphasize that this fact does not depend on any choice of integration constants. The curvature is negative everywhere in the spacetime, and it is finite at the throat.

An interesting fact about the theory (4.1), is that, although the dilaton now is a phantom field, the equations of motion for the theory (4.1) have the same S-duality invariance of the Einstein-Maxwell-dilaton theory, i.e. the equations of motion are invariant under

$$\phi \to -\phi, \;\; F^{\mu\nu} \to \frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{2\sqrt{-g}} F_{\rho\sigma}. \tag{4.48}$$

We can transform our electrically charged solution to a magnetically charged solution simply by applying the S-duality transformation (4.48). Unfortunately, we could not obtain a traversable wormhole solution for the Einstein-Maxwell-anti-dilaton theory with both electric and magnetic charges, i.e. a dyonic traversable wormhole solution.

4.4 The throat of the wormhole

As was saw in sections 2.5 and 2.6, the throat of the wormhole corresponds to the surface of minimal area. Notice that the wormhole metric can be cast in the form

$$ds^2 = -e^{-\lambda}dt^2 + e^{\lambda}\left[dr^2 + (r^2 + l^2)d\Omega_2^2\right]. \tag{4.49}$$

The spatial part of the metric would be the same as the Bronikov-Ellis wormhole case, given by (2.1), if the factor $e^{-\lambda}$ were set to unity. In (2.1) the minimal surface happens when the radial coordinate is $r = 0$, but this is not the case here. The function $C^2(r) = e^{\lambda}(r^2 + l^2)$ has a minimum when

$$(C^2)'(r_{\min}) = 0, \;\; (C^2)''(r_{\min}) > 0. \tag{4.50}$$

These two conditions imply that, at the minimum radius

$$2CC' = 0, \;\; 2(C')^2 + 2CC'' > 0. \tag{4.51}$$

The function C is non-zero everywhere, so the position of the minimum, r_{min}, can be found solving

$$C'(r_{\text{min}}) = 0. \tag{4.52}$$

The second condition implies that the signs of $C(r_{\text{min}})$ and $C''(r_{\text{min}})$ must be the same. Of course, if $2(C')^2 + 2CC'' = 0$, then we must analyse the behavior of the first derivative around the inflection point to check that it is indeed a minimum. The first condition implies that the position of the throat r_{min} is found solving the equation

$$r_{\text{min}} - \frac{Q^2}{2c_1} e^{c_2 + \frac{2c_1}{l} \arctan\left(\frac{r_{\text{min}}}{l}\right)} + b_1 - c_1 = 0. \tag{4.53}$$

The constant r_{min} that solves this equation can have different values because the inverse tangent is a multi-valued function, i.e. it has different branches. In our analysis, we consider only the principal value of the inverse tangent, i.e.

$$\frac{r}{l} = \tan y \to -\frac{\pi}{2} < y < \frac{\pi}{2}. \tag{4.54}$$

As we are interested only in principal values, equation (4.53) gives one single value for r_{min} for any integration constants. An easy way to see this is to plot the function $C^2(r)$, and then check that this value is unique and is indeed a minimum.

4.5 Topological charge and plots

A topological charge is a conserved quantity, also obtained by integrating a current, that is not associated to any Noether symmetry. The phantom dilaton ϕ, given by equation (4.14), is of topological nature, so it must have a topological charge associated to it. In fact, we can define the following current

$$j^{\mu} \sim \tilde{\epsilon}^{\mu\nu} \partial_{\nu} \phi, \tag{4.55}$$

such that

$$N = \beta \int_{-\infty}^{+\infty} dr \tilde{\epsilon}^{tr} \partial_r \phi = \beta[\phi(+\infty) - \phi(-\infty)], \tag{4.56}$$

for a constant β, which we will fix as $\beta = 1$. We have assumed that the field ϕ depends only on the radial coordinate. Here, $\tilde{\epsilon}^{\mu\nu}$ is the antisymmetric Levi-Civita symbol, and $\tilde{\epsilon}^{tr} = 1$. For $l > 0$ we have

$$\lim_{r\to\pm\infty} \arctan\left(\frac{r}{l}\right) = \pm\frac{\pi}{2}(1+4n), \quad n = 0, 1, 2, ... \tag{4.57}$$

We choose the principal branch in order to be consistent with the choice of interval in equation (4.54), i.e. $n = 0$. At the two asymptotic regions, (4.14) gives

$$\phi(\pm\infty) = -\frac{Q^2}{4c_1^2} e^{c_2 \pm \frac{c_1 \pi}{l}} + b_2 \pm \frac{b_1 \pi}{2l}. \tag{4.58}$$

So, for our choices, the topological charge is

$$N = \phi(+\infty) - \phi(-\infty) = -\frac{Q^2}{2c_1^2} e^{c_2} \sinh\left(\frac{c_1\pi}{l}\right) + \frac{b_1\pi}{l}. \tag{4.59}$$

The factor c_2 is very important in our discussion: it shapes the profile of the scalar field. Depending on the choice of integration constants, the dilaton can be connected to two different vacua, i.e. it is a **kink**, or it can be connected to the same vacuum, i.e. it is a **lump**. The topological charge must be zero in order to obtain a lump. This implies

$$\phi(\infty) = \phi(-\infty) \Rightarrow e^{c_2} = \frac{2c_1^2 b_1 \pi}{Q^2 l} \frac{1}{\sinh\left(\frac{c_1\pi}{l}\right)}. \tag{4.60}$$

Now, we show plots of the fields of the theory. The dilaton field shown in figure 4.1 represents the case when it is a lump. We also plot the coupling $e^{2\phi}$ for the same values of constants in figure 4.2. Notice that it also has a lump profile. The electric field for the same values of constants is also plotted in figure 4.3. For values of c_2 other than those in (4.60), the dilaton has a kink or anti-kink profile. We plot only the case when $c_2 = 0$ for the dilaton, shown in figure 4.4, for the exponential coupling, shown in figure 4.5. The electric field for $c_2 = 0$ is also plotted in figure 4.6.

4.6 Embedding diagram

In this section we follow the same procedure as was done in section 2.6, and construct the embedding diagram for the wormhole. Again, we set $t =$ const and

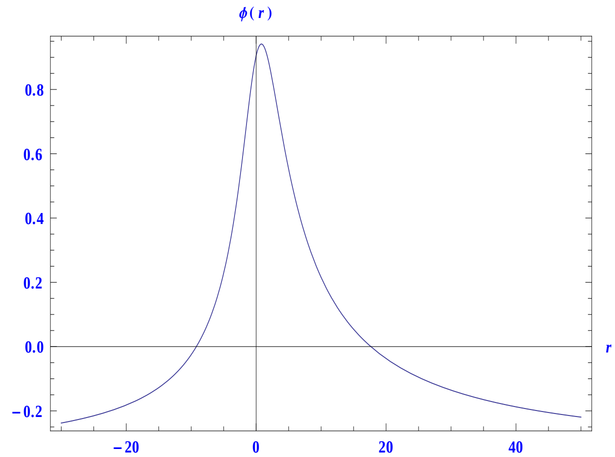


Figure 4.1: Dilaton field. $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

$\theta = \pi/2$. The metric is then written as

$$ds^2 = e^{\lambda}(dr^2 + (r^2 + l^2)d\phi^2). \tag{4.61}$$

Comparing (2.53) and (4.61), we see that the present case differs from the Bronikov-Ellis case only by a "conformal factor" e^{λ}. We define a new radial coordinate such that

$$(r^*)^2 \equiv e^{\lambda}(r^2 + l^2). \tag{4.62}$$

The technical problem now is that, unlike the Bronikov-Ellis wormhole, we can not invert this equation in order to write $r(r^{\star})$. But we will still be able to draw the embedding diagram. Notice that

$$dr^* = \left(r - \frac{Q^2}{2c_1} e^{c_2 + \frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} + b_1 - c_1 \right) \frac{e^{\lambda/2}}{\sqrt{r^2 + l^2}} dr. \tag{4.63}$$

We redefine the term inside the parenthesis as

$$g(r^*) \equiv r - \frac{Q^2}{2c_1} e^{c_2 + \frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} + b_1 - c_1, \tag{4.64}$$

where it is implicit the relation $r(r^{\star})$. Then, (4.61) is rewritten as

$$ds^2 = \frac{r^2 + l^2}{g^2} dr^{*2} + r^{*2} d\phi^2. \tag{4.65}$$

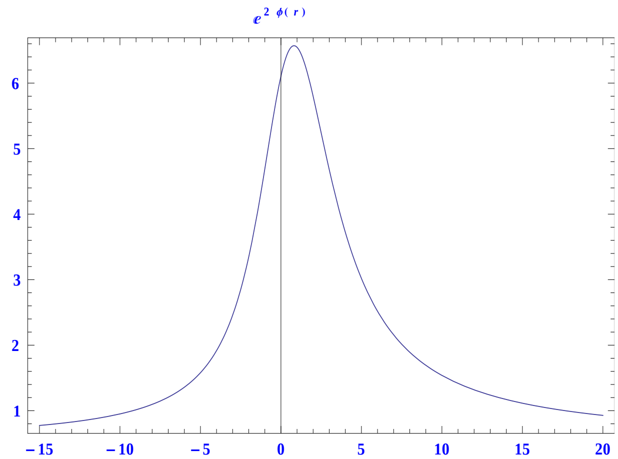


Figure 4.2: Exponential coupling $e^{2\phi}$*.* $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

The Euclidean metric of the embedding space is the same as the one used in section 2.6, and it is given by equation (2.57). This implies that

$$\frac{dz}{dr*} = \pm \left(\frac{r^2 + l^2}{g^2} - 1 \right)^{1/2} . \tag{4.66}$$

The function $z(r^\star)$ has two parts, one for the positive sign, $z_+(r^\star)$, and another one for the negative sign, $z_-(r^\star)$. So we have two differential equations written as

$$\frac{dz_+}{dr*} = + \left(\frac{r^2 + l^2}{g^2} - 1 \right)^{1/2}, \quad \frac{dz_-}{dr*} = - \left(\frac{r^2 + l^2}{g^2} - 1 \right)^{1/2} . \tag{4.67}$$

We integrate these equations numerically. The embedding diagram is shown in figure 4.7. The boundary condition we imposed in order to plot the embedding diagram is just $z_+(r_{\text{min}}) = z_-(r_{\text{min}}) = 0$. Just like the embedding diagram of figure 2.1 represents the Bronikov-Ellis wormhole, the embedding diagram of figure 4.7 represents the embedding diagram for the wormhole of the Einstein-Maxwell-anti-dilaton theory. It connects the upper region $z_+(r^\star)$ with the lower region $z_-(r^\star)$ by a minimal surface, the throat, with area greater than zero.

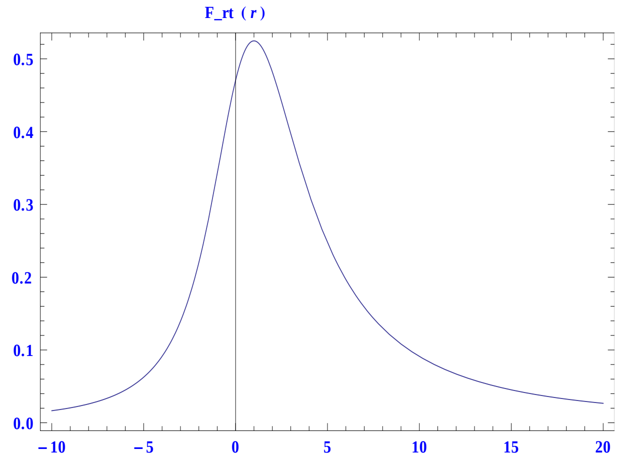


Figure 4.3: Electric field. $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

4.7 Deflection angle via Gauss-Bonnet theorem

A wormhole is a gravitational object just like black holes are. So, its gravitational field also curves the spacetime around it. A light ray can pass close to a wormhole and have its trajectory changed, or it can cross the wormhole from one side to the other and have also its trajectory changed. In this section, we will study only the first situation for our wormhole spacetime. When the bending of the trajectory is small, the observer sees the source in the space at small angle compared to the source's original position. We call this effect **lensing**, or **gravitational lense**, and this section is devoted to the computation of the angle of deviation as seen by the observer, called **deflection angle**.

In a novel geometrical approach to gravitational lensing theory, Gibbons and Werner showed how the Gauss-Bonnet theorem can be applied to the computation of the light deflection angle in the weak deflection limit for static and spherically symmetric spacetimes [57]. This can be applied for black holes, topological defects, and also for wormholes. The application of this method for wormhole cases was done in references [58][59].

Consider the following oriented surface domain $\mathcal{D}$, with boundary $C_R = \gamma_R \cup \gamma_{\bar{g}}$,

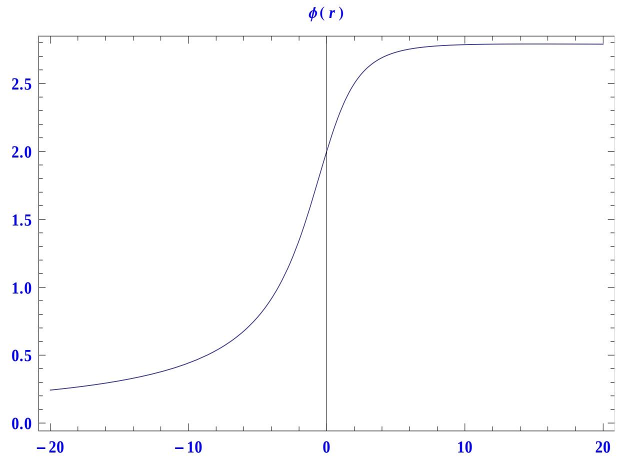


Figure 4.4: Dilaton field. $c_2 = 0$, $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

as described by figure 4.8. The Gauss-Bonnet theorem states that

$$\int\int_{\mathcal{D}} K dS + \int_{\partial\mathcal{D}} \kappa dt + \sum_i \alpha_i = 2\pi\chi(\mathcal{D}), \tag{4.68}$$

where K is the Gaussian curvature associated to the Riemannian metric $\bar{g}$, κ is the geodesic curvature for $C_R : \{t\} \to \mathcal{D}$, and α_i is the extrerior angle with i^{th} vertex. $\chi(\mathcal{D})$ is the Euler characteristic, which is $\chi(\mathcal{D}) = 1$ for a non-singular domain, and $\chi(\mathcal{D}) = 0$ for a singular domain [57]. As we know, the wormhole spacetime is non-singular, so we must use $\chi(\mathcal{D}) = 1$. In order to find the Gaussian curvature, we consider without loss of generality only null geodesics $ds^2 = 0$ on the equatorial plane $\theta = \pi/2$. Then, the wormhole metric reduces to

$$dt^2 = \bar{g}_{ij} dx^i dx^j = e^{2\lambda} dr^2 + e^{2\lambda}(r^2 + l^2) d\varphi^2. \tag{4.69}$$

This is called **optical metric**. The Christoffel connections for this metric are given by

$$\bar{\Gamma}^r_{\varphi\varphi} = -r + \frac{Q^2}{c_1} e^{c_2 + \frac{2c_1}{l} \arctan\left(\frac{r}{l}\right)} - 2(b_1 - c_1), \tag{4.70}$$

$$\bar{\Gamma}^r_{r\varphi} = \frac{1}{(r^2 + l^2)} \left[r - \frac{Q^2}{c_1} e^{c_2 + \frac{2c_1}{l} \arctan\left(\frac{r}{l}\right)} + 2(b_1 - c_1) \right]. \tag{4.71}$$

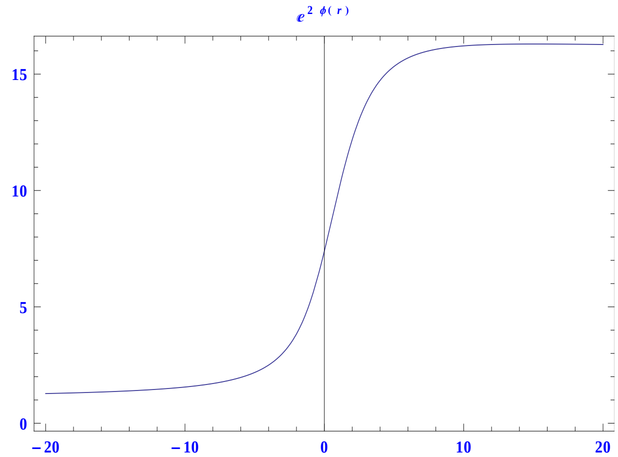


Figure 4.5: Exponential coupling $e^{2\phi}$. $c_2 = 0$, $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

We will use these connections only later. We introduce Regge-Wheeler tortoise coordinate $r^\star$, such that

$$dr^\star = e^\lambda dr, \quad f^2(r^\star) = e^{2\lambda}(r^2 + l^2), \tag{4.72}$$

where $r = r(r^\star)$. The optical metric (4.69) then takes the form

$$dt^2 = dr^{\star 2} + f^2(r^\star) d\varphi^2. \tag{4.73}$$

The Gaussian curvature K is related to the Riemann tensor through the relation

$$R_{r\varphi r\varphi} = K(\bar{g}_{r\varphi}\bar{g}_{\varphi r} - \bar{g}_{rr}\bar{g}_{\varphi\varphi}) = -K \det \bar{g}. \tag{4.74}$$

The Gaussian curvature is expressed as [57]

$$K = -\frac{R_{r\varphi r\varphi}}{\det \bar{g}} = -\frac{1}{f(r^\star)} \frac{d^2 f(r^\star)}{dr^{\star 2}}. \tag{4.75}$$

Notice that we can return to the original radial coordinate r writing this expression as

$$K = -\frac{1}{f(r^\star)} \left[\frac{dr}{dr^\star} \frac{d}{dr} \left(\frac{dr}{dr^\star} \right) \frac{df}{dr} + \left(\frac{dr}{dr^\star} \right)^2 \frac{d^2 f}{dr^2} \right]. \tag{4.76}$$

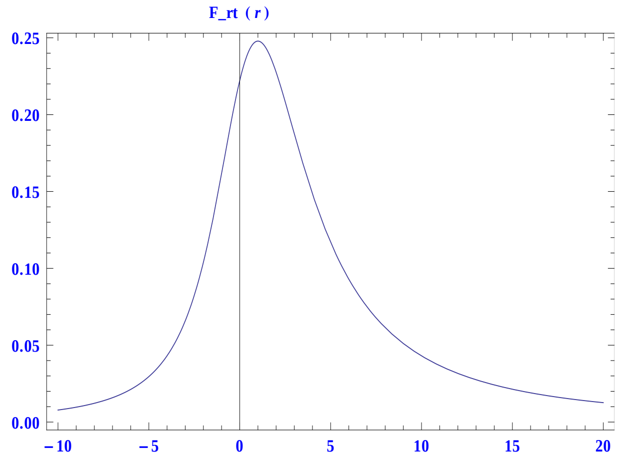


Figure 4.6: Electric field. $c_2 = 0$, $Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$.

Using the definitions (4.72) we obtain

$$\begin{aligned} K &= -\frac{1}{e^{2\lambda}(r^2+l^2)}\left[\lambda' r + \frac{l^2}{(r^2+l^2)} + (r^2+l^2)\lambda''\right] \\ &= -\frac{1}{e^{2\lambda}(r^2+l^2)^2}\left[\frac{Q^2}{c_1} r e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} - 2Q^2 e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)} - 2r(b_1-c_1) + l^2\right]. \end{aligned} \tag{4.77}$$

For the metric (4.69), we have $\sqrt{\bar{g}} = e^{2\lambda}\sqrt{r^2+l^2}$. We will compute the first integral in (4.68) only to leading order for large values of r, i.e.

$$\begin{aligned} \int\int_D K dS &= \int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty dr d\varphi \sqrt{\det\bar{g}} K \\ &= \int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty dr d\varphi e^{2\lambda}\sqrt{r^2+l^2} K \\ &\approx -\int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty dr d\varphi \frac{1}{r^2}\left[\frac{Q^2}{c_1} e^{c_2+\frac{c_1\pi}{l}} - 2(b_1-c_1)\right] \\ &\approx -\frac{2}{b}\left[\frac{Q^2}{c_1} e^{c_2+\frac{c_1\pi}{l}} - 2(b_1-c_1)\right]. \end{aligned} \tag{4.78}$$

The second integral is a little more involved. In the original paper [57], the examples presented always had for the second integral the result $\pi + \delta$, but this is not always

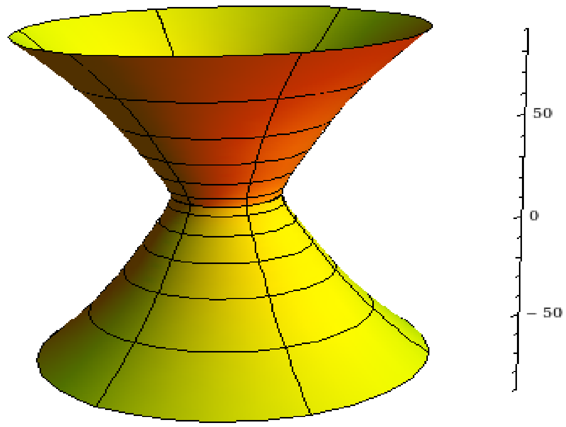


Figure 4.7: Embedding diagram:
$Q = 2$, $b_1 = 5$, $l = 3$, $c_1 = 1$, $b_2 = 3$, $e^{c_2} = (3\sqrt{3}+4)/(2e^{2\pi/9})$.

the case. In fact, the presence of topological defects gives a different contribution [60], and this is the situation here. In order to compute the second integral, we must first find κ. Let us define the velocity and acceleration vectors along the curve γ respectively as $\dot{\gamma}$ and $\ddot{\gamma}$. The velocity vector is normalized such that $\bar{g}(\dot{\gamma}, \dot{\gamma}) = 1$. For very large values of R the sum of the external angles for the source and the observer is approximatelly π, i.e. $\alpha_S + \alpha_O \to \pi$. The geodesic curvature is computed using the relation

$$\kappa = \bar{g}(\nabla_{\dot{\gamma}}\dot{\gamma}, \ddot{\gamma}). \tag{4.79}$$

Along $\gamma_{\bar{g}}$, $\kappa(\gamma_{\bar{g}}) = 0$ because $\gamma_{\bar{g}}$ is a geodesic. We must then compute

$$\kappa(\gamma_R) = |\nabla_{\dot{\gamma}_R}\dot{\gamma}_R|, \tag{4.80}$$

where $\dot{\gamma}_R$ is the velocity vector along the curve γ_R. The radial component of this expression is given by

$$(\nabla_{\dot{\gamma}_R}\dot{\gamma}_R)^r = \dot{\gamma}_R^\mu \partial_\mu \dot{\gamma}_R^r + \dot{\gamma}_R^\mu \bar{\Gamma}^r_{\mu\nu}\dot{\gamma}_R^\nu. \tag{4.81}$$

Because we confine our attention on the countour $C_R := r(\phi) = R = \text{const}$ for large R, there is no change in the radial distance, so $\dot{\gamma}_R^r = 0$, and $\bar{g}_{\varphi\varphi}(\dot{\gamma}_R^\varphi)^2 = 1$. So, (4.81)

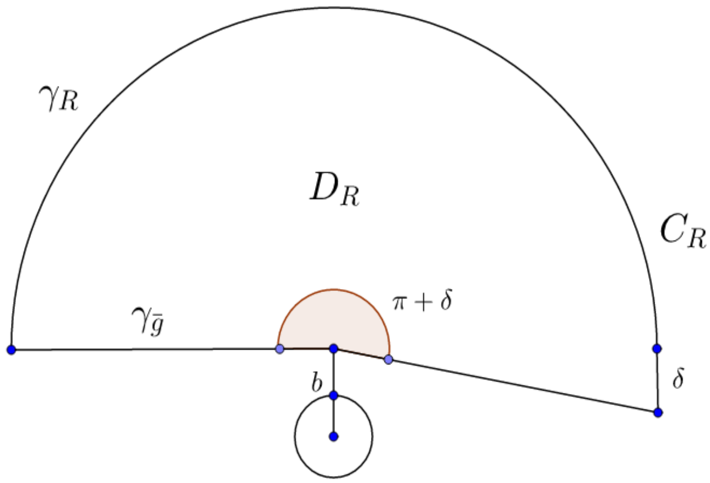


Figure 4.8: Gravitational lens: D_R is the domain enclosed by the boundary C_R, b is the impact parameter, and δ is the deflection angle.

is written as

$$(\nabla_{\dot{\gamma}_R}\dot{\gamma}_R)^r = \bar{\Gamma}^r_{\varphi\varphi}(\dot{\gamma}^\phi_R)^2. \tag{4.82}$$

Using (4.70) we have

$$(\nabla_{\dot{\gamma}_R}\dot{\gamma}_R)^r = \frac{1}{e^{2\lambda}(r^2+l^2)}\left(-r+\frac{Q^2}{c_1}e^{c_2+\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)}-2(b_1-c_1)\right). \tag{4.83}$$

Using the optical metric (4.69), we also obtain

$$dt = e^\lambda\sqrt{r^2+l^2}d\varphi. \tag{4.84}$$

Combining the results (4.83) and (4.84), and also using Taylor expansions, we evaluate the integral as

$$\int_{\partial D}\kappa dt = \int_0^{\pi+\delta}(\nabla_{\dot{\gamma}_R}\dot{\gamma}_R)^r d\varphi \approx -e^{-m_1}(\pi+\delta). \tag{4.85}$$

This is valid for large values of R and for small values of δ. Inserting the results (4.78) and (4.85) in (4.68), we can finally evaluate the light deflection angle, given by

$$\delta = \frac{2e^{m_1}}{b}\left[2(b_1 - c_1) - \frac{Q^2}{c_1}e^{c_2 + \frac{c_1\pi}{l}}\right] - \pi(1 + e^{m_1}). \tag{4.86}$$

Of course this must be a small value for the deflection angle, since we used a lot of approximations valid in the weak deflection angle limit to obtain this result. If we wish to study the bending of light away from this limit, the Gauss-Bonnet theorem method is not valid anymore, and we must solve analytically the geodesic equation.

This result was derived only for the positive region, but we can easily generalize it for the negative region as well. In fact, using the asymptotic values of the dilaton (4.33) and (4.34), and the definition of the asymptotic charges (4.35), (4.36), (4.37), (4.38), (4.39), and (4.40), the deflection angle in both regions can also be written as

$$\delta_+ = -\frac{4M_+}{b}e^{\phi_+ - \frac{c_1\pi}{2l} - \frac{c_2}{2}} - \pi(1 + e^{2\phi_+ - \frac{c_1\pi}{l} - c_2}), \tag{4.87}$$

$$\delta_- = -\frac{4M_-}{b}e^{\phi_- + \frac{c_1\pi}{2l} + \frac{c_2}{2}} - \pi(1 + e^{2\phi_- + \frac{c_1\pi}{l} + c_2}). \tag{4.88}$$

Chapter 5

Sen's entropy function

In this chapter we introduce an efficient method to compute the entropy and the value of fields on the horizon of an extremal black hole. This is the Sen's entropy function method. The analysis of this chapter is done for black holes with spherical symmetry, but it can be generalized for black holes with planar horizons as well. In fact, we will do this in section 8.1, where black holes with planar horizon are used to compute conductivities for strongly coupled systems at zero temperature.

5.1 $\mathrm{AdS}_2\times\mathrm{S}^2$ near horizon geometry

A general metric describing a charged and non-rotating black hole solution with spherical horizon can always be cast in the following form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + C^2(r)d\Omega_2^2, \tag{5.1}$$

where $f(r)$ and $C(r)$ are functions of the radial coordinate only. The horizon is located at the position $r = r_+$, so that in the near horizon region the function $f(r)$ is Taylor expanded as

$$f(r) \approx f(r_+) + (r - r_+)f'(r_+) + \frac{(r - r_+)^2}{2!}f''(r_+) + \dots\ , \tag{5.2}$$

where the primes denote derivatives with respect to the radial coordinate. The first term in the expansion, $f(r_+)$, is zero due to the definition of the horizon. The temperature of the black hole is defined as $T \sim f'(r_+)$, so the second term is also zero due to the fact that an extremal black hole has $T = 0$. These facts imply that the third term is the dominant one, and the metric can be approximated as

$$ds^2 \approx -\frac{(r-r_+)^2}{2} f''(r_+) dt^2 + \frac{2}{(r-r_+)^2} \frac{dr^2}{f''(r_+)} + C^2(r_+) d\Omega_2^2. \tag{5.3}$$

We change coordinates to

$$r - r_+ \equiv \rho, \quad t \equiv \frac{2}{f''(r_+)} \tau. \tag{5.4}$$

The metric in this coordinate system is written as

$$ds^2 \approx \frac{2}{f''(r_+)} \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + C^2(\rho + r_+) d\Omega_2^2. \tag{5.5}$$

The horizon now is located at $\rho = 0$. The metric is factored out in two parts. The term inside the parenthesis is the metric of an AdS_2 space. The angular part is the metric of the S^2 space. So, we say that the metric of an extremal black hole has $\mathrm{AdS}_2 \times \mathrm{S}^2$ geometry. The functions $f''(r)$ and $C^2(r)$ are shown to give positive constants on the horizon all extremal black holes considered in this thesis. So, this space is indeed the product of the two dimensional anti de-Sitter space with the two-sphere S^2. The term multiplying the AdS_2 part is called the AdS_2 radius, and the term multiplying the S^2 part is called the S^2 radius.

5.2 Entropy function and attractor equations

In this section we will derive the Sen's entropy function for extremal black holes. This was introduced for the first time in [10], but in this thesis we follow the more detailed derivation given in the review [18].

Consider a four-dimensional gravity theory coupled to Abelian massless fields $A_\mu^{(i)}$ and neutral scalar fields ϕ^s. The Lagrangian of the theory is generically written as $\sqrt{-g}\mathcal{L}$, where $\mathcal{L}$ depends on the scalars ϕ^s, on the field strengths $F_{\mu\nu}^{(i)} \equiv \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}$, on the inverse metric $g^{\mu\nu}$, and on the Riemann tensor $R_{\mu\nu\rho\sigma}$. The Lagrangian may also depend on the covariant derivatives of all these fields. As we saw in the previous section, the near-horizon metric of an extremal black hole can be conveniently written as

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{5.6}$$

$$\phi_s = u_s, \;\; F^{(i)}_{rt} = e_i, \;\; F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta, \tag{5.7}$$

where v_1, v_2, u_s, e_i and p_i are constants. The Riemann tensor for the metric (5.6) is written as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \;\; \alpha,\beta,\gamma,\delta = r,t \\ R_{mnpq} &= v_2^{-1}(g_{mp}g_{nq} - g_{mq}g_{np}), \;\; m,n,p,q = \theta,\phi. \end{aligned} \tag{5.8}$$

Due to the fact that the fields are constants near the horizon, all the covariant derivatives of the scalars ϕ_s, of the field strength $F^{(i)}_{\mu\nu}$, and of the Riemann tensor $R_{\mu\nu\rho\sigma}$ appearing in the action vanish. One result that will be used later and that can be obtained easily from the above is the Ricci scalar, which is given by

$$R = -\frac{2}{v_1} + \frac{2}{v_2}. \tag{5.9}$$

Let us denote by $f(\vec{u},\vec{v},\vec{e},\vec{p})$ the Lagrangian density integrated over the angular variables θ and ϕ in the near horizon region, i.e.

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) = \int d\theta d\phi \sqrt{-g}\mathcal{L}. \tag{5.10}$$

In general supergravity theories we have several gauge fields, so we have written for short $\vec{p}$ and $\vec{q}$ to represent the electric and magnetic charges associated to all gauge fields. The equations of motion for the scalar and the metric correspond to extremizing f with respect to $\vec{u_s}$ and $\vec{v}$. They are given by

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_i} = 0. \tag{5.11}$$

The equation of motion and Bianchi identities for the gauge fields are respectivelly given by

$$\partial_r \left(\frac{\delta \mathcal{S}}{\delta F^{(i)}_{rt}} \right) = 0, \;\; \partial_r F^{(i)}_{\theta\phi} = 0, \tag{5.12}$$

where $\mathcal{S}$ is the action. These equations are satisfied for the near horizon metric (5.6). Integrating over the angular variables we have

$$\int d\theta d\phi \frac{\delta \mathcal{S}}{\delta F_{rt}^{(i)}} = a_i, \quad \int d\theta d\phi F_{\theta\phi}^{(i)} = b_i \tag{5.13}$$

where a_i, b_i are radially independent constants. Using (5.7), we have

$$\begin{aligned}
\frac{\delta \mathcal{S}}{\delta F_{rt}^{(i)}} = \sqrt{-g} F^{(i)rt} = \frac{\partial}{\partial F_{rt}^{(i)}} \left[\sqrt{-g}\mathcal{L}\right] \\
a_i = \int d\theta d\phi \frac{\delta \mathcal{S}}{\delta F_{rt}^{(i)}} = \int d\theta d\phi \frac{\partial}{\partial F_{rt}^{(i)}} \left[\sqrt{-g}\mathcal{L}\right] = \frac{\partial}{\partial e_i}\left[\int d\theta d\phi \sqrt{-g}\mathcal{L}\right] = \frac{\partial f}{\partial e_i}, \\
b_i = \int d\theta d\phi \frac{p_i}{4\pi} \sin\theta = p_i.
\end{aligned}$$

This implies that a_i and b_i are

$$a_i = \frac{\partial f}{\partial e_i}, \quad b_i = p_i. \tag{5.14}$$

The constants a_i and b_i are integrals of the electric and magnetic fluxes, written as

$$q_i = \frac{\partial f}{\partial e_i}, \quad b_i = p_i. \tag{5.15}$$

Then, we conclude that q_i is associated to the electric charge and p_i is associated to the magnetic charge of the extremal black hole. Then, we end up with a system of three equations

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_i} = 0, \quad \frac{\partial f}{\partial e_i} = q_i, \tag{5.16}$$

and three variables $\vec{u}$, $\vec{v}$ and $\vec{e}$. In other words, we end up with a completely determined system. Let us define now a new function as

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\pi[e_i q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})]. \tag{5.17}$$

The system of equations (5.16) is reproduced in terms of this new function as

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0. \tag{5.18}$$

This means that the near horizon metric and fields can be determined by extremizing one single function $\mathcal{E}$.

There is a general formula for computing the entropy of a black hole when the theory contains terms with higher derivatives in the Riemann tensor. This is called Wald formula [61]. Its derivation is out of the scope of this thesis, and we will only present the result here. We refer to [61][62][63][64] for a detailed derivation of this result. The Wald formula is written as

$$S_{BH} = -8\pi \int_H d\theta d\phi \sqrt{-g_{tt}g_{rr}} \frac{\delta \mathcal{S}}{\delta R_{rtrt}}. \tag{5.19}$$

It is our intention to show that the function $\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})$ reduces to the Wald formula after extremization, following the same procedure presented in reference [18]. In order to compute the quantity $\frac{\delta \mathcal{S}}{\delta R_{rtrt}}$ we must do the following:

1. We express $\mathcal{S}$ in terms of covariant derivatives of symmetrized fields replacing the antisymmetrized covariant derivatives of such fields in terms of the Riemann tensor;
2. then, we treat $R_{\mu\nu\rho\sigma}$ as an ideependent variable.

In our case, the covariant derivatives of all tensors vanish, so

$$\frac{\delta \mathcal{S}}{\delta R_{rtrt}} = \sqrt{-g} \frac{\partial \mathcal{L}}{\partial R_{rtrt}}, \tag{5.20}$$

and we keep only terms that do not involve covariant derivatives in $\mathcal{L}$. Then $\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}$ can be computed from

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \delta R_{\mu\nu\rho\sigma}. \tag{5.21}$$

We replace (5.20) in (5.19) and obtain

$$\begin{aligned} S_{BH} &= -8\pi \int_H d\theta d\phi \sqrt{-g_{tt}g_{rr}} \sqrt{-g_{tt}g_{rr}g_{\theta\theta}g_{\phi\phi}} \frac{\partial \mathcal{L}}{\partial R_{rtrt}} \\ &= 8\pi g_{tt}g_{rr} \frac{\partial \mathcal{L}}{\partial R_{rtrt}} A_H \\ &= 8\pi(-v_1 r^2) \left(\frac{v_1}{r^2}\right) \frac{\partial \mathcal{L}}{\partial R_{rtrt}} A_H \\ &= -8\pi v_1^2 \frac{\partial \mathcal{L}}{\partial R_{rtrt}} A_H. \end{aligned} \tag{5.22}$$

In order to express this entropy in terms of the function f previouly defined we write $f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})$, which is similar to the right hand side of equation (5.10), except by the fact that we multiply each factor of R_{rtrt} em $\mathcal{L}$ by a factor of λ. This implies the following equality

$$\left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1} = \int d\theta d\phi\sqrt{-g}R_{\alpha\beta\gamma\delta}\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}, \tag{5.23}$$

where $\alpha,\beta,\gamma,\delta$ runs over r and t only. Notice that $\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}$ is proportional to $(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})$, and, with the correct normalization factors, it is written as

$$\frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}} = -v_1^2(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})\frac{\partial\mathcal{L}}{\partial R_{rtrt}}. \tag{5.24}$$

Using (5.24) and (5.8) we can rewrite (5.23) as

$$\begin{aligned}\left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1} &= \int d\theta d\phi\sqrt{-g}[-v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})][-v_1^2(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})]\frac{\partial\mathcal{L}}{\partial R_{rtrt}}\\ &= \int d\theta d\phi\sqrt{-g}[v_1^2(g_{\alpha\gamma}g^{\alpha\gamma}g_{\beta\delta}g^{\beta\delta}-\delta^\gamma_\delta\delta^\delta_\gamma-\delta^\delta_\gamma\delta^\gamma_\delta+g_{\alpha\delta}g^{\alpha\delta}g_{\beta\gamma}g^{\beta\gamma})]\frac{\partial\mathcal{L}}{\partial R_{rtrt}}.\end{aligned} \tag{5.25}$$

We use $\sqrt{-g}=v_1\sqrt{g_{\theta\theta}g_{\phi\phi}}$, and write

$$\left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1} = 4v_1^2\int d\theta d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}}\frac{\partial\mathcal{L}}{\partial R_{rtrt}}. \tag{5.26}$$

As $\frac{\partial\mathcal{L}}{\partial R_{rtrt}}$ is independent of θ and ϕ, we have

$$\left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1} = 4v_1^2\frac{\partial\mathcal{L}}{\partial R_{rtrt}}A_H. \tag{5.27}$$

Replacing (5.27) em (5.22), The entropy of the black hole can be rewritten as

$$S_{BH} = -2\pi\left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1}. \tag{5.28}$$

We must express the right hand side of this expression in terms of derivatives of f with respect to $\vec{u}$, $\vec{v}$, $\vec{e}$, and $\vec{p}$. As $\mathcal{L}$ is invariant under reparametrization of the coordinates r and t, each factor of R_{rtrt} in f_λ must appear in the combination $\lambda g^{rr} g^{tt} R_{rtrt} = \lambda \left(-\frac{1}{v_1 r^2}\right)(v_1) = -\lambda v_1^{-1}$, each factor of $F^{(i)}_{rt}$ must appear in the combination $\sqrt{-g^{tt} g^{rr}} F^{(i)}_{rt} = e_i v_1^{-1}$, each factor of $F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}$ and $\phi_s = u_s$ must appear without multiplying v_1 or powers of it. As we mentioned, the terms with covariant derivatives of $F^{(i)}_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, and ϕ_s vanish. Notice that v_1 also appears in $f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ due to the term $\sqrt{-g} \propto v_1$. So, the expression

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = v_1 g(\vec{u}, \vec{v_2}, \vec{p}, \lambda v_1^{-1}, \vec{e} v_1^{-1}) \tag{5.29}$$

is valid for some function g. With this definition, the following identity is true:

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} + v_1 \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial v_1} + e_i \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial e_i} - f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = 0. \tag{5.30}$$

Now, we impose that $\lambda = 1$ in this expression, and use the equation for v_1 given in (5.16), so that the above identity gives as the result

$$\left.\frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda}\right|_{\lambda=1} = -e_i \frac{\partial f(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial e_i} + f(\vec{u}, \vec{v}, \vec{e}, \vec{p}). \tag{5.31}$$

This result allows us to rewrite (5.28) as

$$S_{BH} = 2\pi \left(e_i \frac{\partial f(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial e_i} - f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \right). \tag{5.32}$$

We use (5.16) and replace the term containing the derivative of f with respect to e_i by q_i, so that we can interpret $\frac{S_{BH}}{2\pi}$ as a Legendre transform of $f(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ with respect to e_i after $\vec{u}$ and $\vec{v}$ are eliminated using the equations of motion (5.16). Finally, using (5.17), we express the quantity S_{BH} as

$$S_{BH} = \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}), \tag{5.33}$$

which is the entropy of the extremal black hole at the extremum (5.18). The function defined in (5.17) is the **Sen's entropy function**, and the set of equations that extremize this function, given by (5.18), are called **attractor equations**. The Sen's entropy function method has become a very powerful tool for the computation of the entropy of extremal black holes for a huge class of theories arising in string theory as their low-energy limit (supergravity). Due to the fact that the formalism is valid also in the presence of higher derivative terms in the Riemann tensor, the Sen's entropy function gives also the corrections to the entropy coming from these higher derivative terms, i.e.

$$S_{BH} = \mathcal{E} = \frac{A_H}{4G_N} + \text{corrections}. \tag{5.34}$$

These corrections are called "classical corrections" in the worldsheet string theory sense, since they do not come with powers of the string coupling α', which are called "worldsheet quantum corrections". Another reason why Sen's entropy function is so powerful is that it allows us to investigate the attractor mechanism for extremal black holes without the need of knowing explicitly the black hole solution. In other words, the solution to the attractor equations (5.18) gives not only the attractor values of the scalar fields on the horizon but also the near horizon metric and gauge fields. We will refer to this as **horizon data**, and we will show that this data can be used to compute conductivities for extremal black holes in the context of AdS/CMT in chapter 8. All the analysis made in this section was based in equations of motion, and did not make use of specific forms of the higher derivative terms.

Although we emphasized the power of Sen's entropy function method, it is also to stress out its limitations, which are:

- The Lagrangian is not allowed to have a mass term for the gauge fields $A_\mu^{(i)}$, since we the entropy function is constructed in order to agree with the Wald formula for the entropy of a black hole, which is not valid when the gauge field is massive. So, the formalism is valid only when the gauge fields appears in the action through their field strengths;

- The fields are only allowed to depend on the radial coordinate for theories of the EMD type. This implies that the scalars of the theory must not depend on the time or the horizon coordinates. This happens because the attractor mechanism is not present when one of the scalars depends on a coordinate other than the radial one.

Of course one can test that the method gives in fact the entropy of the black hole and near horizon data for the cases when the black hole solution is known explicitly. This can be asily done for the Reissner-Nordström black hole, for the dyonic black hole of Einstein-Maxwell-dilaton theory in absence of a scalar potential, and for some other explicit solutions.

Chapter 6

AdS$_4$ dyonic black hole entropy and attractors via entropy function

In this chapter we will present the computation for the dyonic extremal black hole entropy for the bosonic part of a $\mathcal{N} = 8$ supergravity theory in AdS$_4$ space, which is sometimes called $U(1)^4$ theory due to the fact that it contains four Abelian gauge fields. This result was obtained by the student and can be found in reference [19]. In essence, we will study Einstein-Maxwell-dilaton theories in the presence of a potential for the scalar field. In the first part, we will just show how to apply the entropy function formalism for these cases, and compute the black hole entropy for some models which are not necessarily embedded in string theory. This will allow us to learn important lessons about how to solve the attractor equations for the more general theory that we just mentioned, i.e. the $U(1)^4$ theory in AdS$_4$ space.

6.1 The general Einstein-Maxwell-dilaton theory

In order to be as general as possible, we write the Einstein-Maxwell-dilaton action in the presence of a scalar potential as

$$S = \int d^4x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - W(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)\right), \tag{6.1}$$

for some function of the dilaton $W(\phi)$, which is in general an exponential function of the dilaton, and for some general scalar potential $V(\phi)$. In the next section we will choose specific forms for these functions in order to solve the attractor equations. Here, we define the field strength as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{6.2}$$

and we take units in which $(16\pi G) = 1$. The $U(1)^4$ gauged supergravity contains more scalars and gauge fields, and will be discussed in details later. For now, let us focus attention only to this simple case. The equations of motion are:

- for the metric:

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}W(\phi)F_{\rho\sigma}F^{\rho\sigma} + 2W(\phi)F_{\mu\rho}F_\nu{}^\rho + \frac{1}{2}g_{\mu\nu}V(\phi); \tag{6.3}$$

- for the dilaton:

$$\nabla_\mu(\partial^\mu\phi) - \frac{1}{4}\frac{\partial W(\phi)}{\partial\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\frac{\partial V}{\partial\phi} = 0; \tag{6.4}$$

- for the gauge field:

$$\nabla_\mu\left(W(\phi)F^{\mu\nu}\right) = 0. \tag{6.5}$$

We also have the Bianchi identities:

$$\nabla_{[\mu}F_{\rho\sigma]} = 0. \tag{6.6}$$

6.2 Entropy function for the Einstein-Maxwell-dilaton theory with a potential

In this section we compute explicitly the entropy of the extremal black holes considered in the text. Instead of choosing specific models, we try to keep some sort of generality by not making use of any explicit functional form for the coupling of the dilaton field with the field strength, i.e. $W(\phi)$. This means that, apart from the constant potential, the potentials $V(\phi)$ we consider are written in terms of $W(\phi)$ in a specific way. Then, we will show that we can make generic assumptions about the solutions to the attractor equations for the case of $U(1)^4$ theory.

We define the near horizon field strength components as $F_{rt} = e$ and $F_{\theta\phi} = P\sin\theta$. Also, as mentioned in chapter 5, v_1 is the AdS_2 radius, v_2 is the S^2 radius, and here we define the value of the scalar on the horizon as u_D, where the subscript D stands for dilaton. We follow the procedure of chapter 5 for obtaining the entropy function. We assume that the near horizon metric is given by (5.6), which implies that the Ricci scalar is written as (5.9), and then, we assume that the near horizon fields are written as above. We integrate the Lagrangian over the angular variables to obtain the function f, written in equation (5.10), for the theory (6.1). This gives

$$f = 4\pi v_1 v_2 \left[-\frac{2}{v_1} + \frac{2}{v_2} + W(u_D)\left(-\frac{e^2}{v_1^2} + \frac{P^2}{v_2^2}\right) - V(u_D)\right], \tag{6.7}$$

and the entropy function is just $\mathcal{E} = 2\pi[Qe - f]$, which gives

$$\mathcal{E} = 2\pi\left[Qe - 8\pi(v_1 - v_2) - 4\pi v_1 v_2 W(u_D)\left(\frac{e^2}{v_1^2} - \frac{P^2}{v_2^2}\right) + 4\pi v_1 v_2 V(u_D)\right]. \tag{6.8}$$

The attractor equations (5.18) are

$$Q - 8\pi v_1 v_2 W(u_D)\frac{e}{v_1^2} = 0, \tag{6.9}$$

$$-2 + v_2 W(u_D)\left(\frac{e^2}{v_1^2} + \frac{P^2}{v_2^2}\right) + v_2 V(u_D) = 0, \tag{6.10}$$

$$2 - v_1 W(u_D)\left(\frac{e^2}{v_1^2} + \frac{P^2}{v_2^2}\right) + v_1 V(u_D) = 0, \tag{6.11}$$

$$\frac{\partial W(u_D)}{\partial u_D}\left(\frac{e^2}{v_1^2} - \frac{P^2}{v_2^2}\right) - \frac{\partial V(u_D)}{\partial u_D} = 0. \tag{6.12}$$

We now show that the attractor equations allow us to show that the function $\mathcal{E}$ is indeed the entropy of the extremal black hole. Using (6.9) we eliminate Q in (6.8), which gives

$$\mathcal{E} = 2\pi\left[-8\pi(v_1 - v_2) + 4\pi v_1 v_2 W(u_D)\left(\frac{e^2}{v_1^2} + \frac{P^2}{v_2^2}\right) + 4\pi v_1 v_2 V(u_D)\right]. \tag{6.13}$$

Multiplying (6.10) by v_1 and adding to (6.11) multiplied by v_2 we obtain the following expression for the potential

$$v_1 v_2 V(u_D) = (v_2 - v_1). \tag{6.14}$$

This expression holds for general scalar fields ϕ^I and gauge fields A^I_μ, and we will also use it in the next section. We also eliminate the potential in (6.13) and then obtain

$$\mathcal{E} = 2\pi \left[-4\pi(v_1 - v_2) + 4\pi v_1 v_2 W(u_D) \left(\frac{e^2}{v_1^2} + \frac{1}{v_2^2} \frac{p^2}{(4\pi)^2} \right) \right]. \tag{6.15}$$

Now, multiplying (6.10) by v_1 and subtracting (6.11) multiplied by v_2 we have

$$v_1 v_2 W(u_D) \left(\frac{e^2}{v_1^2} + \frac{P^2}{v_2^2} \right) = (v_1 + v_2), \tag{6.16}$$

and we also eliminate this term from (6.15). This implies that the entropy can be written only in terms of the S^2 radius, i.e.

$$\mathcal{E} = 16\pi^2 v_2. \tag{6.17}$$

By dimensional analysis, we recover the term $16\pi G$, and conclude that this corresponds to the usual $A/4$ term of the black hole entropy. Notice that we can solve (6.9) directly, giving

$$\frac{e}{v_1} = \frac{Q}{8\pi v_2 W(u_D)}. \tag{6.18}$$

We will analyze the solutions of the attractor equations for some specific potentials. We make a list for the cases we investigated.

- $V(\phi) = 0$.

This is the simplest case. The solution to the attractor equations and the entropy are given by

$$W(u_D) = \frac{Q}{8\pi P},\ v_1 = v_2 = \frac{QP}{8\pi},\ e = P, \tag{6.19}$$

$$\mathcal{E} = 2\pi QP. \tag{6.20}$$

Notice that this result is independent of the functional form of $W(u_D)$. For the case when $W(\phi) = e^{-2\phi}$, we can use the extremal analytical solution for the Einstein-Maxwell-dilaton theory without potential, given in reference [7], and see that the entropy and near horizon data obtained here are the same as there.

- $V(\phi) = 2\Lambda$.

The solution to the attractor equations for this case is

$$W(u_D) = \frac{Q}{8\pi P}, \;\; e = P\frac{v_1}{v_2}, \tag{6.21}$$

$$v_1 = \frac{1}{\Lambda}\frac{\frac{1}{2\Lambda}\left(1 \pm \sqrt{1 + \frac{\Lambda QP}{2\pi}}\right) - \frac{QP}{8\pi}}{\frac{QP}{4\pi} - \frac{1}{2\Lambda}\left(1 \pm \sqrt{1 + \frac{\Lambda QP}{2\pi}}\right)}, \;\; v_2 = \frac{1}{2\Lambda}\left(1 \pm \sqrt{1 - \frac{\Lambda QP}{2\pi}}\right), \tag{6.22}$$

$$\mathcal{E} = \frac{8\pi^2}{\Lambda}\left(1 - \sqrt{1 - \frac{\Lambda QP}{2\pi}}\right). \tag{6.23}$$

For convenience, we left the value of the electric field in an implicit form. We also took the minus sign in v_2 in order to write the entropy. The reason is that, after Taylor expanding for small values of the cosmological constant, we recover the previous result for zero potential when $\Lambda \to 0$ only with the negative sign, which is a consistency check of the correctness of the method.

- $V(\phi) = \beta W(\phi)$.

For this case, (6.9) and (6.12) give directly

$$W^2(u_D) = \frac{Q^2}{(8\pi)^2}\frac{1}{(P^2 + \beta v_2^2)}. \tag{6.24}$$

Replacing this in (6.11) we find v_2 and all the rest can be found after some algebraic manipulation

$$W(u_D) = \frac{Q}{8\pi P}\left(1 - \frac{\beta Q^2}{(8\pi)^2}\right)^{1/2} \quad e = \frac{P}{\left(1 - \frac{\beta Q^2}{(8\pi)^2}\right)^{3/2}} \tag{6.25}$$

$$v_1 = \frac{QP}{8\pi} \frac{1}{\left(1 - \frac{\beta Q^2}{(8\pi)^2}\right)^{3/2}}, \; v_2 = \frac{QP}{8\pi} \frac{1}{\left(1 - \frac{\beta Q^2}{(8\pi)^2}\right)^{1/2}}, \tag{6.26}$$

$$\mathcal{E} = \frac{2\pi QP}{\left(1 - \frac{\beta Q^2}{(8\pi)^2}\right)^{1/2}}. \tag{6.27}$$

- $V(\phi) = \frac{\beta}{W(\phi)}$.

This case is a bit simpler to be obtained. The solutions are

$$W(u_D) = \frac{Q}{8\pi P} \frac{1}{(1 - \beta P^2)^{1/2}} \; e = \frac{P}{(1 - \beta P^2)^{1/2}} \tag{6.28}$$

$$v_1 = \frac{QP}{8\pi} \frac{1}{(1 - \beta P^2)^{3/2}}, \; v_2 = \frac{QP}{8\pi} \frac{1}{(1 - \beta P^2)^{1/2}}, \tag{6.29}$$

$$\mathcal{E} = \frac{2\pi QP}{(1 - \beta P^2)^{1/2}}. \tag{6.30}$$

Notice that we can achieve the zero potential case by setting the constant β to zero. Notice also that the AdS_2 and S^2 radii, v_1 and v_2, are related in equations (6.26) and (6.29), by the exchange $Q/(8\pi) \leftrightarrow P$. Of course, these are simplified cases, and generalizations can be done, for instance, by choosing other potentials that could be combinations of these two examples.

In these last two examples, we chose the potential to be proportinal of inversely proportional to the coupling $W(\phi)$. Notice that adding a potential of these kinds change the entropy and near-horizon data of the theory without scalar potential, given by (6.19) and (6.20), just by a function of the charges. We will see in the next section that this will allow us to gain insights about the solution to the attractor equations for a specific supergravity theory, whose potential is a combination of the gauge coulplings of these kinds.

6.3 $U(1)^4$ gauged supergravity and AdS$_4$ dyonic black hole entropy

In this section we present a brief review of an AdS$_4$ gauged supergravity theory, as we mentioned. The theory is the $U(1)^4$ gauged supergravity in four dimensions, which follows from a truncation of the maximal $\mathcal{N}=8$ $SO(8)$ supergravity down to the Cartan subgroup of $SO(8)$. The bosonic action with the same field definition and coefficients given in [65] is

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[& R - \frac{1}{2}\left((\partial\phi^{(12)})^2 + (\partial\phi^{(13)})^2 + (\partial\phi^{(14)})^2\right) - V \\ & -2\left(e^{-\lambda_1}(F^{(1)}_{\mu\nu})^2 + e^{-\lambda_2}(F^{(2)}_{\mu\nu})^2 + e^{-\lambda_3}(F^{(3)}_{\mu\nu})^2 + e^{-\lambda_4}(F^{(4)}_{\mu\nu})^2\right)\Big], \end{aligned} \tag{6.31}$$

and

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu, \tag{6.32}$$

and the scalar combinations λ are given by

$$\begin{aligned} \lambda_1 &= -\phi^{(12)} - \phi^{(13)} - \phi^{(14)}, \\ \lambda_2 &= -\phi^{(12)} + \phi^{(13)} + \phi^{(14)}, \\ \lambda_3 &= \phi^{(12)} - \phi^{(13)} + \phi^{(14)}, \\ \lambda_4 &= \phi^{(12)} + \phi^{(13)} - \phi^{(14)}. \end{aligned} \tag{6.33}$$

The scalar potential is

$$V = -4g^2\left(\cosh\phi^{(12)} + \cosh\phi^{(13)} + \cosh\phi^{(14)}\right). \tag{6.34}$$

The scalars are not all independent, and satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0. \tag{6.35}$$

We can write the scalar $\phi^{(ij)}$ in terms of the fields λ as

$$\phi^{(12)} = \frac{1}{4}(-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4), \tag{6.36}$$

$$\phi^{(13)} = \frac{1}{4}(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4), \tag{6.37}$$

$$\phi^{(14)} = \frac{1}{4}(-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4). \tag{6.38}$$

We can rewrite parts of the kinetic terms as

$$\begin{aligned}\partial_\mu \phi^{12} \partial^\mu \phi^{12} = \frac{1}{16} \big[& (\partial_\mu \lambda_1)^2 + (\partial_\mu \lambda_2)^2 + (\partial_\mu \lambda_3)^2 + (\partial_\mu \lambda_4)^2 \\ & + 2\partial_\mu \lambda_1 \partial^\mu \lambda_2 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_3 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_4 \\ & - 2\partial_\mu \lambda_2 \partial^\mu \lambda_3 - 2\partial_\mu \lambda_2 \partial^\mu \lambda_4 + 2\partial_\mu \lambda_3 \partial^\mu \lambda_4 \big], \end{aligned} \tag{6.39}$$

$$\begin{aligned}\partial_\mu \phi^{13} \partial^\mu \phi^{13} = \frac{1}{16} \big[& (\partial_\mu \lambda_1)^2 + (\partial_\mu \lambda_2)^2 + (\partial_\mu \lambda_3)^2 + (\partial_\mu \lambda_4)^2 \\ & - 2\partial_\mu \lambda_1 \partial^\mu \lambda_2 + 2\partial_\mu \lambda_1 \partial^\mu \lambda_3 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_4 \\ & - 2\partial_\mu \lambda_2 \partial^\mu \lambda_3 + 2\partial_\mu \lambda_2 \partial^\mu \lambda_4 - 2\partial_\mu \lambda_3 \partial^\mu \lambda_4 \big], \end{aligned} \tag{6.40}$$

$$\begin{aligned}\partial_\mu \phi^{14} \partial^\mu \phi^{14} = \frac{1}{16} \big[& (\partial_\mu \lambda_1)^2 + (\partial_\mu \lambda_2)^2 + (\partial_\mu \lambda_3)^2 + (\partial_\mu \lambda_4)^2 \\ & - 2\partial_\mu \lambda_1 \partial^\mu \lambda_2 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_3 + 2\partial_\mu \lambda_1 \partial^\mu \lambda_4 \\ & + 2\partial_\mu \lambda_2 \partial^\mu \lambda_3 - 2\partial_\mu \lambda_2 \partial^\mu \lambda_4 - 2\partial_\mu \lambda_3 \partial^\mu \lambda_4 \big]. \end{aligned} \tag{6.41}$$

The full kinetic term is

$$\begin{aligned}(\partial_\mu \phi^{12})^2 + (\partial_\mu \phi^{13})^2 + (\partial_\mu \phi^{14})^2 = \frac{1}{16} \big[& 3(\partial_\mu \lambda_1)^2 + 3(\partial_\mu \lambda_2)^2 + 3(\partial_\mu \lambda_3)^2 \\ + 3(\partial_\mu \lambda_4)^2 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_2 - 2\partial_\mu \lambda_1 \partial^\mu \lambda_3 & - 2\partial_\mu \lambda_1 \partial^\mu \lambda_4 - 2\partial_\mu \lambda_2 \partial^\mu \lambda_3 \\ & - 2\partial_\mu \lambda_2 \partial^\mu \lambda_4 - 2\partial_\mu \lambda_3 \partial^\mu \lambda_4 \big]. \end{aligned} \tag{6.42}$$

The potential term is written as

$$V = -2g^2 \left(e^{\phi^{12}} + e^{\phi^{13}} + e^{\phi^{14}} + \frac{1}{e^{\phi^{12}}} + \frac{1}{e^{\phi^{13}}} + \frac{1}{e^{\phi^{14}}} \right), \tag{6.43}$$

and in terms of the fields λ we have

$$V = -2g^2 \left[e^{\frac{1}{2}(\lambda_1+\lambda_2)} + e^{\frac{1}{2}(\lambda_1+\lambda_3)} + e^{\frac{1}{2}(\lambda_1+\lambda_4)} + e^{\frac{1}{2}(\lambda_2+\lambda_3)} + e^{\frac{1}{2}(\lambda_2+\lambda_4)} + e^{\frac{1}{2}(\lambda_3+\lambda_4)} \right]. \tag{6.44}$$

In order to have the Maxwell term written with a factor 1/4, we redefine the exponential of the fields as

$$\frac{X_I}{\sqrt{8}} \equiv e^{-\frac{\lambda_I}{2}} \Rightarrow \partial_\mu \lambda_I = -2 \frac{\partial_\mu X_I}{X_I}. \tag{6.45}$$

The potential is then

$$V = -\frac{g^2}{4}\sum_{I<J}\frac{1}{X_I X_J}. \tag{6.46}$$

The action is rewritten as

$$S = \int d^4x\sqrt{-g}\left[R - \frac{1}{32}\left(3\sum_{I=1}^{4}(\partial_\mu\lambda_I)^2 - 2\sum_{I<J}\partial_\mu\lambda_I\partial^\mu\lambda_J\right) - \frac{1}{4}\sum_{I=1}^{4}X_I^2(F^I_{\mu\nu})^2 - V\right]. \tag{6.47}$$

But notice that, due to (6.35), we can write

$$X_1X_2X_3X_4 = 1, \tag{6.48}$$

$$\left(\sum_{I=1}^{4}\partial_\mu\lambda_I\right)^2 = 0 = \sum_{I=1}^{4}(\partial_\mu\lambda_I)^2 + 2\sum_{I<J}\partial_\mu\lambda_I\partial^\mu\lambda_J, \tag{6.49}$$

and this allows us to rewrite the kinetic term in the action, so that we have

$$S = \int d^4x\sqrt{-g}\left[R - \frac{1}{8}\sum_{I=1}^{4}(\partial_\mu\lambda_I)^2 - \frac{1}{4}\sum_{I=1}^{4}X_I^2(F^I_{\mu\nu})^2 - V(X)\right]. \tag{6.50}$$

We now apply the the entropy function formalism to the four-dimensional theory (6.50). This was done for the first time in [66] by Morales and Samtleben. They considered the case when the black hole solution has only electric charges. The solution found by the authors is written in an implicit form, depending on the values of a set of parameters rather than the electric charges*. Here, we consider both electric and magnetic charges. Notice that we can set the potential to zero by choosing $g = 0$, so, after having solved the attractor equations for $g \neq 0$ we must be able to recover the result obtained for zero potential by taking the limit $g \to 0$. The near horizon scalars and gauge fields are

$$X_I = u_I,\ \ F^I_{rt} = e^I,\ \ F^I_{\theta\phi} = p^I\sin\theta. \tag{6.51}$$

The kinetic terms for the scalars give zero contribution to the entropy function. One can easily do some identifications and add a summation in the entropy function

*Notice that the authors of [66] did not really solve the attractor equations.

(6.15) in order to obtain

$$\mathcal{E} = 2\pi \left[e_I q^I - 4\pi v_1 v_2 \left(-\frac{2}{v_1} + \frac{2}{v_2} + \frac{1}{2} \sum_{I=1}^{4} u_I^2 \left(\frac{e_I^2}{v_1^2} - \frac{p_I^2}{v_2^2} \right) + 4g^2 \sum_{I<J}^{4} u_I u_J \right) \right]. \tag{6.52}$$

The attractor equations (5.18) are

$$\frac{e_I}{v_1} - \frac{q^I}{(4\pi) u_I^2 v_2} = 0, \tag{6.53}$$

$$2 - \frac{v_2}{2} \sum_{I=1}^{4} u_I^2 \left(\frac{e_I^2}{v_1^2} + \frac{p_I^2}{v_2^2} \right) + v_2 V(u) = 0, \tag{6.54}$$

$$-2 + \frac{v_1}{2} \sum_{I=1}^{4} u_I^2 \left(\frac{e_I^2}{v_1^2} + \frac{p_I^2}{v_2^2} \right) + v_1 V(u) = 0, \tag{6.55}$$

$$u_I \left(\frac{e_I^2}{v_1^2} - \frac{p_I^2}{v_2^2} \right) - \frac{\partial V(u)}{\partial u_I} = 0. \tag{6.56}$$

- V(X)=0.

We first solve for null potential in order to recover the results presented here for the case when the potential is non-trivial. Notice that, if we also set the magnetic charges equal to zero, equation (6.56) will give zero dilaton coupling or zero gauge field on the horizon. The gauge field coupling is an exponential of the dilaton fields, so it can not be zero. But we also know that the electric field is also non-zero on the horizon of a black hole. This means that for zero potential and zero magnetic charges the attractor equations do not make sense. For the dyonic case, one can easily find the solutions:

$$e_I = p_I, \;\; u_I^2 = \frac{q^I}{4\pi p_I}, \;\; v_1 = v_2 = \frac{1}{2(4\pi)} \sum_{I=1}^{4} q^I p_I, \tag{6.57}$$

$$\mathcal{E} = 2\pi \sum_{I=1}^{4} q^I p_I. \tag{6.58}$$

From the results of the previous section we see that this is expected, as we have done nothing but add indices to our fields.

- $V(X) = -\frac{g^2}{4}\sum_{I<J}^{4}\frac{1}{X_I X_J}$.

We will show how to solve the attractor equations step by step for this case. Equation (6.56) gives

$$\sum_{I=1}^{4} u_I^2 \frac{e_I^2}{v_1^2} = \sum_{I=1}^{4} u_I^2 \frac{p_I^2}{v_2^2} - 2V(u), \tag{6.59}$$

and replacing this directly in (6.54) we obtain a quadratic equation

$$v_2^2 V(u) + v_2 - \frac{1}{2}\sum_{I=1}^{4} u_I^2 p_I^2 = 0, \tag{6.60}$$

whose solution is

$$v_2 = \frac{-1 \pm \sqrt{1 + 2V(u)\sum_{I=1}^{4} u_I^2 p_I^2}}{V(u)}. \tag{6.61}$$

In order to check if this result has the correct limit we expand it for $V(u) \to 0$, giving

$$v_2 = \frac{1}{V(u)}\left(-1 \pm \left(1 + V(u)\sum_{I=1}^{4} u_I^2 p_I^2 + \mathcal{O}(V(u)^2)\right)\right). \tag{6.62}$$

Taking the plus sign we can recover the previous case setting the potential to zero. This is again an indication that the case for zero potential should be recovered from this case by setting the potential to zero, i.e. $g^2 \to 0$.

Inserting also (6.59) in (6.55) we find directly

$$v_1 = \frac{2v_2^2}{\sum_{I=1}^{4} u_I^2 p_I^2}. \tag{6.63}$$

Also, from (6.53) we can obtain

$$\sum_{I=1}^{4} u_I^2 \frac{e_I^2}{v_1^2} = \sum_{I=1}^{4}\left(\frac{q^I}{4\pi}\right)^2 \frac{1}{u_I^2 v_2^2}. \tag{6.64}$$

Inserting this in (6.59) we have the following equation

$$\sum_{I=1}^{4}\left(\frac{q^I}{4\pi}\right)^2 \frac{1}{u_I^2} = \sum_{I=1}^{4} u_I^2 p_I^2 - 2v_2^2 V(u). \tag{6.65}$$

We can eliminate the last term containing v_2^2 using (6.60), and obtain

$$v_2 = \frac{1}{2}\sum_{I=1}^{4}\left(\frac{q^I}{4\pi}\right)^2\frac{1}{u_I^2}. \tag{6.66}$$

In order to have a solution for the attractor equations one should use (6.66) and replace v_2 in (6.60) or in (6.65), and then solve it for u_I, i.e. write u_I in terms of the electric and magnetic charges. It turns out that finding solutions for the resulting equations is a non-trivial task due to the amount of scalar fields and all summations involving them. Here is how we proceed to solve the problem. In the previous analysis, we saw that the black hole entropy for zero potential is recovered from the black hole entropy in the presence of a potential, always by taking the limit when the potential is zero. So, we must recover (6.58) when the potential is set to zero for the present case. In other words, all other parameters obtained here should also match the ones obtained before in (6.57) in this limit. By direct observation, the value of the general coupling $W(u_D)$ on the horizon of the black hole obtained in the cases for non-trivial potential appears as the product of the result for zero potential case and a function of the charges. We are then led to consider a solution for the dilaton, given by

$$u_I^2 = \frac{q^I}{4\pi p_I}F(q,p)^{1/2}, \tag{6.67}$$

where $F(q,p)$ is a generic function of the charges that will be fixed by the attractor equations. With this solution we obtain for the other constant fields

$$v_2 = \frac{1}{2(4\pi)}\frac{1}{F(q,p)^{1/2}}\sum_{I=1}^{4}q^I p_I, \tag{6.68}$$

$$v_1 = \frac{1}{2(4\pi)}\frac{1}{F(q,p)^{3/2}}\sum_{I=1}^{4}q^I p_I, \tag{6.69}$$

$$e_I = \frac{p_I}{F(q,p)^{3/2}}. \tag{6.70}$$

Of course, for now this is just a guess and we need to determine $F(q,p)$ and check that the solution is consistent. Notice that everything will have the correct limit if, at zero potential, the function $F(q,p) \to 1$. In order to obtain this function, we insert these possible solutions into the attractor equations. The first thing to notice is that that (6.53) gives an identity. Equations (6.54), (6.55), (6.56), all lead to the same equation. This in fact shows the correctness of our solution (6.67), and is the consistency check. The resulting equation is quadratic in $F(q,p)$, given by

$$F(q,p)^2 - F(q,p) + \frac{g^2}{8}\left(\sum_{I=1}^{4} q^I p_I\right)\left(\sum_{J<K}\sqrt{\frac{p_J p_K}{q^J q^K}}\right) = 0. \tag{6.71}$$

The solution is found easily and it is given by

$$F(q,p) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{g^2}{2}\left(\sum_{I=1}^{4} q^I p_I\right)\left(\sum_{J<K}\sqrt{\frac{p_J p_K}{q^J q^K}}\right)}\right). \tag{6.72}$$

In the limit $g \to 0$ we must have $F(q,p) \to 1$, so the positive sign is the correct one. The dyonic extremal black hole entropy for the $U(1)^4$ theory is then

$$\mathcal{E} = 2\pi\left(\sum_{I=1}^{4} q^I p_I\right)\left[\frac{1}{2}\left(1 + \sqrt{1 - \frac{g^2}{2}\left(\sum_{I=1}^{4} q^I p_I\right)\left(\sum_{J<K}\sqrt{\frac{p_J p_K}{q^J q^K}}\right)}\right)\right]^{-1/2}. \tag{6.73}$$

As the dilaton fields are not all independent we can combine the constraint for the scalar fields (6.48), which is written as $u_1 u_2 u_3 u_4 = 1$ near the horizon, with (6.67). This allows us to compute the function $F(q,p)$ in a simpler way, which can also be written as

$$F(q,p) = (4\pi)^2\sqrt{\frac{p^1 p^2 p^3 p^4}{q^1 q^2 q^3 q^4}}. \tag{6.74}$$

Now, the entropy will be

$$\mathcal{E} = \frac{1}{2}\left(\sum_{I=1}^{4} q^I p_I\right)\left(\frac{q^1 q^2 q^3 q^4}{p^1 p^2 p^3 p^4}\right)^{1/4}. \tag{6.75}$$

Notice that, when written in this form, the entropy does not show explicitly the dependence on the coupling constant g^2. One can obtain g^2 as a function of the charges of the black hole by using (6.72) and (6.74), i.e.

$$g^2 = \frac{8(4\pi)^4\left(\sqrt{\frac{p^1 p^2 p^3 p^4}{q^1 q^2 q^3 q^4}} - \frac{p^1 p^2 p^3 p^4}{q^1 q^2 q^3 q^4}\right)}{\left(\sum_{I=1}^{4} q^I p_I\right)\left(\sum_{J<K}\sqrt{\frac{p_J p_K}{q^J q^K}}\right)}. \tag{6.76}$$

This means that, for this geometry, the charges must be chosen in such a way to satisfy (6.76) on the horizon, since g^2 is a parameter defining the theory that one can freely vary. This condition on the charges and couplings of the potential arose because of the constraint (6.48). Such constraints are present for gauged supergravities in order to obtain consistent truncations. In other words, (6.76) is a condition on the charges of the black hole, and not on the coupling constant g^2. In the absence of such constraint the electric and magnetic charges can also vary freely.

Equations (6.73) and (6.75) are equivalent, and they are the most important results of this section. They represent the entropy of the extremal dyonic black hole, which is a solution of the $U(1)^4$ supergravity theory (6.50). As we said before, the advantage of using Sen's entropy function to determine the black hole entropy is that we do not need to know the full black hole solution of the theory. In fact, the full dyonic black hole solution of the theory (6.50) is unknown. All we had to do was to find the solution to the attractor equations. Once the full solution is found, one can check that it must have the near-horizon data and entropy found here. More importantly is the fact that, once there is a method to count the microstates of a black hole, one must be able to compute the entropy of the black hole using $S \sim \ln(\text{microstates})$, and this entropy must match (6.73), or its equivalent (6.75). This was not done yet and will be the topic of a future investigation.

If the dyonic black hole solution is invariant under duality transformation, $u_I \to u_I^{-1}$ and $(q_I/4\pi) \leftrightarrow p^I$, then the dilaton field is invariant, since $F \to F^{-1}$[†]. But notice that the entropy changes to

$$\mathcal{E} = 8\pi^2 \left(\sum_{I=1}^{4} q^I p_I \right) \left(\frac{p^1 p^2 p^3 p^4}{q^1 q^2 q^3 q^4} \right)^{1/4} . \tag{6.77}$$

If one makes the rescaling $q^I \equiv \epsilon$, then the electric charge disappears from this

[†]As was pointed out in [65], for the case of purely electric or purely magnetic black holes, there is a direct correspondence between the supersymmetry properties of the electric and magnetic solutions in the absence of gauging ($g = 0$), which apparently does not happen for the gauged theory. However it is not clear yet if the dyonic solution of this theory preserves the supersymmetry properties under electric-magnetic duality. Here we just take the dual in order to show that it is possible to recover the entropy computed for purely magnetic black holes.

formula, and the result is just

$$\mathcal{E} = 8\pi^2 \left(\sum_{I=1}^{4} p_I \right) \left(p^1 p^2 p^3 p^4 \right)^{1/4} . \tag{6.78}$$

In order to obtain the same result from equation (6.73), one should take the limit of magnetic black holes, i.e. $\epsilon \to 0$. A formula for purely magnetic black hole was found for an $\mathcal{N} = 2$ supersymmetric theory in reference [67], which is given by

$$S = 2\pi^2 \left(\prod_{I=1}^{4} \frac{p^I}{g_I} \right)^{1/4} , \tag{6.79}$$

where $g_I = g\xi_I$, for some constant ξ_I. As pointed out in the same reference, such a model can be embedded in $\mathcal{N} = 8$ gauged supergravity. The gravitino is charged in gauged supergravity, and so, for topological consistency in a magnetic backgroung field, the charges should satisfy the Dirac quantization condition: for BPS configurations this is just $\sum g_I p^I \in \kappa$, where κ is the curvature of the horizon geometry. In the case of an S^2 horizon we have $\kappa = 1$. For the $U(1)^4$ theory the Dirac quantization is written as $-8\pi \sum g_I p^I = \mathbb{Z}$. Using this condition, it is easy to see that equations (6.78) and (6.79) agree for some choice of constant ξ_I, which shows that one can recover the purely electric case from the dyonic one, if the dyonic solution is invariant under electric-magnetic duality in the presence of gaugings. It should be emphasized that the Sen's formalism allows us to obtain information about the fields in the near horizon for general extremal black holes: they do not necessarily need to be BPS, and this is the reason why we should use $-8\pi \sum g_I p^I = \mathbb{Z}$, instead of the Dirac quantization for BPS black holes.

The question that naturally arises in this work is whether solutions of the kind (6.67) represent a general feature of the scalar fields for any theory of the kind (6.1). All the results obtained here depended strongly on the functional form of the dilaton potential. Some complications in the equations might arise when the potential is a complicated function (like logarithmic or exponential)of the couplings to the field strength ($W(\phi)$ in (6.1) and u_I^2 in (6.50)). Potentials having polynomial functional forms of these couplings are the most common ones in gauged supergravities, and, at least in 4 spacetime dimensions, it seems that (6.67) will allow one to solve the attractor equations for any of these theories. Of course, one should check it case by case, since the results presented here are only an illustration of why this should work, but not a complete proof. Also, for more general theories in D-dimensions containing Chern-Simons terms and higher order derivatives, like RF^2 and R^2 for instance, the attractor equations might present higher powers of v_1 and v_2, and such solutions are much more difficult to be obtained, but of course, in the limit when the coupling constants of such terms go to zero, we must recover the case for zero potential. How our results change in such cases may be a subject of future work.

Chapter 7

Holographic conductivities

In this chapter we describe how to compute the electric, electrothermal and heat conductivities just using the horizon data. The derivation of the main results of this chapter is tedious, but even so we will show all the steps of the computation, since this was not done in details in any of the papers we cite in the text.

7.1 The AdS/CFT correspondence

As we mentioned in the introduction, the AdS/CFT correspondence is an example of holography: a (d+1)-dimensional theory is equally described by a d-dimensional theory. The main idea is related to the concept of duality: if two different theories describe the same physical system, then we say that they are dual to each other, and therefore there must exist a map between them.

In string theory we have objects with p spatial dimensions called "Dp branes. A D0 brane is a point particle, a D1 brane is a string, a D2 brane is a membrane, etc. They can live on the top of each other, so we can have for instance a stack of N coincident Dp branes with certain low energy excitations. Dp branes gravitate, and how strongly they gravitate depends on their tension and the strength of the gravitational force. In the original example, Maldacena considered a stack of N coincident D3 branes. The gravitational backreaction is determined by the number N of coincident D3 branes and the dimensionless string coupling constant g_s. The tension of the N D3 branes is proportional to N/g_s and the strength of gravity is proportional to g_s^2, so the gravitational effects of the D3 branes can be neglected when

$$\lambda \equiv 4\pi g_s N \ll 1. \tag{7.1}$$

The surviving fields in this limit are all massless, and they are described a field theory called $\mathcal{N} = 4$ Super Yang-Mills theory. The Lagrangian density schematically written as

$$\mathcal{L} \sim \frac{N}{\lambda} tr \left(F^2 + (\nabla \Phi)^2 + i\bar{\Psi} \not{D} \Psi + i\bar{\Psi}[\Phi, \Psi] - [\Phi, \Phi]^2 \right), \tag{7.2}$$

where F is the $SU(N)$ field strength, Φ is a scalar field and Ψ is a fermionic field. There are six scalars and four fermions in the theory. All these fields are $N \times N$ matrices. λ is called 't Hooft coupling. When λ is small, the theory (7.2) remains wealy coupled in the large N limit, and consequently can be treated perturbatively.

Now, we consider the opposite regime, i.e.

$$\lambda \equiv 4\pi g_s N \gg 1. \tag{7.3}$$

The effects of gravity are strong, and the D3 branes collapse and form an object called black brane. In the near horizon region of the black brane, there is a very strong gravitational redshift seen by an observer far from the horizon. The low energy excitations must then live near the horizon. The geometry for this case is $\mathrm{AdS}_5 \times S^5$, described by the metric

$$ds^2 = L^2 \left(\frac{-dt^2 + d\vec{x}^2 + dr^2}{r^2} + d\Omega_5^2 \right), \tag{7.4}$$

where t and $\vec{x}$ are coordinates along the D3 brane worldvolume, r is orthogonal to the D3 brane called radial coordinate along the bulk, and $d\Omega_5^2$ is the metric of the five-sphere. In this coordinate system, the horizon is located at $r \to \infty$. The relation between the AdS radius L, the sting length L_s and the Planck length L_p is given by

$$L = \lambda^{1/4} L_s = (4\pi N)^{1/4} L_p. \tag{7.5}$$

In the strong 't Hooft coupling regime and large N, the radius of curvature L is very large compared to both the string length L_s, which controls the effects of highly excited string states, and the Planck length L_p, which controls the quantum gravity effects. Such effects can be neglected, and the excitations of the near horizon region will be described by classical gravitational perturbations of the background (7.4).

So, the low energy excitations of N D3 branes are described by a weakly interacting large N field theory, the $\mathcal{N}=4$ Super Yang-Mills, at weak 't Hooft coupling, whereas at strong 't Hooft coupling they are described gravitational perturbations about $\mathrm{AdS}_5 \times S^5$. The AdS/CFT correspondence [11] is the conjecture that this set of decoupled degrees of freedom interpolates between these weak and strong coupling descriptions.

Another example of the correspondence deals with a field theory in $(2+1)$-dimensions, called ABJM model due to its discoverers Aharony, Bergman, Jafferis and Maldacena [68]. This theory is dual to 11-dimensional supergravity on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. There are other examples of the duality for different dimensions as well (see [14]).

In the original formulation, there was an AdS space, dual to a conformal field theory. Moreover, the duality was derived from a system of branes. But later, a more general formulation was put forward, gauge/gravity duality, where these assumptions were relaxed.

In quantum field theory, the basic observables are the multi-point functions. A special kind of operators $\mathcal{O}_i$, called single-trace operators, factorize in quantum field theory becoming classical in the large N limit. Multi-point functions can be obtained from the generating functional

$$Z_{\mathrm{QFT}}[h_i(x)] \equiv \langle e^{i \sum_i \int dx h_i(x) \mathcal{O}_i} \rangle_{\mathrm{QFT}}. \tag{7.6}$$

In AdS space, observables are dynamical fields in the bulk, but we can consider a Dirichlet problem in which the fields are fixed on the boundary. In this case, it is possible to construct a partition function of the thoory as a function of boundary values $h_i(x)$ of all the bulk fields ϕ_i

$$Z_{\mathrm{Gravity}}[h_i(x)] \equiv \int^{\phi_i \to h_i} \left(\prod_i \mathcal{D}\phi_i \right) e^{iS[\phi_i]}. \tag{7.7}$$

For a quantum field theory with a gravity dual there must be a one-to-one correspondence between single trace operators $\mathcal{O}_i$ on the field theory side and the bulk dynamical fields ϕ_i on the gravity side. This observation was formulated by Gubser, Klebanov, Polyakov and Witten [69] [70], and it is written as

$$Z_{\text{QFT}}[h_i(x)] = Z_{\text{Gravity}}[h_i(x)]. \tag{7.8}$$

This defines the AdS/CFT dictionary, which is the statement that the boundary value h_i for a bulk field ϕ_i is a source for an operator $\mathcal{O}_i$ in the dual quantum field theory.

7.2 The AdS/CMT correspondence

The difficult problems in condensed matter physics are at strong coupling. As we saw in the previous section, the AdS/CFT correspondence provides a strong/weak coupling description of the same physical system. In the last few years, the AdS/CFT duality has been used to handle phenomenologically such problems in condensed matter theory. The philosophy behind it is that we can extract the desired properties of some condensed matter systems by using some gravity theory.

The canonical examples of the AdS/CFT duality involve known gravity solutions of string theory. This is the so-called **top-down approach**. We can assume that AdS/CFT is a universal duality between quantum field theories and gravity theories. In this more general scenario, the AdS/CFT is just an example of a **gauge/gravity duality**. So, we can, for instance, study the properties of a gravity theory which does not necessarily come from string theory and assume that this has a quantum field theory dual. In other words, we would be dealing with a **bottom-up approach**. The bottom-up approach is widely used in order to handle problems in condensed matter systems. As the physics of both sides of the duality must match, one can in principle use gravity to make predictions or explain phenomena for these condensed matter systems in certain limits when the field theory description is untractable or even unknown. This idea goes by the name of AdS/CMT correspondence.

It turns out that condensed matter systems are at finite temperature, which requires the gravity theory to have also a well-defined notion of temperature. This is achieved by the introduction of a black hole on the gravity side. In this thesis, we apply the black hole solutions to the Einstein-Maxwell-dilaton theory in the context of AdS/CMT correspondence. The physical observables we compute are the transport coefficients, or just conductivities for short.

In order to introduce a chemical potential, consider a gauge field a_μ of a conformal field theory, associated to a global symmetry. This field is the source of a current J^μ associated to this symmetry. Consider now a gauge field in the bulk, A_μ, associated to the same, but local, symmetry in AdS space. When A_μ is evaluated at the boundary of the AdS, it is associated to a_μ, which is the source of the current J^μ of the conformal field theory, whose coupling is written as $\int d^d x J^\mu a_\mu$. In other words, introducing charge J^0 in the conformal field theory corresponds to having

$$A = A_0(r)dt + ... \to a_0 dt, \tag{7.9}$$

as we approach the boundary at $z \to 0$. In the gravity picture, a_0 is associated to the electric charge Q of a black hole in AdS space. The integral over J^0, the charge density of the conformal field theory, gives rise to the electric charge q, so that $\int J^0 a_0$ is identified with $q\mu$. Here, μ is the chemical potential, so the gauge field in AdS must satisfy the boundary condition

$$A \to \mu dt \quad \text{as} \quad r \to 0. \tag{7.10}$$

The charge density in the conformal field theory case is found by

$$\rho = \langle J^0 \rangle = \left. \frac{\delta S_{\text{Sugra}}}{\delta a_0} \right|_{a_0 = 0}. \tag{7.11}$$

One of the transport coefficients we are interested in is the electric conductivity, σ. Applying an electric field $\vec{E}(\omega)$ to a system induces a current $\vec{J}(\omega)$. The electric conductivity $\sigma(\omega)$ is computed using Ohm's law, written as

$$\vec{J}(\omega) = \sigma(\omega)\vec{E}(\omega). \tag{7.12}$$

Here, ω is the frequency. Suppose we shake the electric field at frequency ω. If the system responds at the same frequency, this means we are in the domain of linear response. The quantities that depend on the frequency are called AC. So, $\sigma(\omega)$ is the electric AC conductivity. In the limit when $\omega \to 0$, we have what we call a DC quantity.

In order to compute AC conductivities in the domain of linear response, one must use spectral functions, which are based on retarded Green's function. From them, one can derive Kubo formulae for the transport coefficients, at finite frequency. The derivation of such formulae is out of the scope of the thesis, and we refer to reference [20] for more details. This will not be important for the present work, since we are interested in computing only DC conductivities.

The computation of DC transport coefficients for strongly interacting systems with a net charge density ρ is a challenging task. In order to obtain finite results for the conductivities, a mechanism for breaking translation invariance must be introduced. Otherwise, momentum will not dissipate, and conductivities will blow up. In AdS/CMT, the breaking of translation invariance is achieved by the introduction of fields with specific profiles that do the the job. This was the case, for instance, of massive gravity theories [71, 72, 73], lattice models [74, 75, 76], and linear axions [77]. We will not discuss the details here, but these references are a good guide to understand such mechanism.

7.3 Magnetic fields and thermoelectric transport

We will show how to compute transport coefficients in the presence of magnetic fields. There are some subtleties that must be taken into account, as will be explained. The idea is to consider some specific theories and see how they respond when we apply an external electric field, $\vec{E}$, and a thermal gradient, $\vec{\nabla}T$. The electric and heat currents mix, so the transport coefficients will be associated to the electric and heat transport. They will be given below.

It was explained in the condensed matter set up [78] that the electric and heat currents receive additional contributions in the presence of a magnetic field. This was also discussed in the context of holography and dyonic black holes in reference

[79]. Basically the total electric and heat currents are decomposed as

$$\vec{J}^{\text{total}} = \vec{J}^{\text{phys}} + \vec{J}^{\text{mag}},$$
$$\vec{Q}^{\text{total}} = \vec{Q}^{\text{phys}} + \vec{Q}^{\text{mag}}. \tag{7.13}$$

We will drop the label "phys" and refer to the physical currents only as $\vec{J}$ and $\vec{Q}$. So, after subtracting off the magnetization currents from the total currents, we obtain the physical currents. These physical currents are related to the transport coefficients through

$$J^i = \sigma_{ij} E^j - \alpha_{ij} (\nabla T)_j, \tag{7.14}$$

$$Q^i = \alpha_{ij} T E^j - \bar{\kappa}_{ij} (\nabla T)_j. \tag{7.15}$$

From these expressions we can obtain directly the electrical conductivity, σ_{ij}, and the electrothermal conductivity α_{ij}. We can also obtain $\bar{\kappa}_{ij}$ which is related with the thermal conductivity through

$$\hat{\kappa} = \hat{\bar{\kappa}} - T \hat{\alpha} \cdot \hat{\sigma}^{-1} \cdot \hat{\alpha} \quad . \tag{7.16}$$

Then, κ_{ij} is the true heat conductivity, which is the same as $\bar{\kappa}_{ij}$ in the absence of electric field. For simplicity, we will also refer to $\bar{\kappa}_{ij}$ as heat conductivity in the text. The off-diagonal conductivities are antisymmetric, i.e. $\sigma_{xy} = -\sigma_{yx}$, $\alpha_{xy} = -\alpha_{yx}$ and $\bar{\kappa}_{xy} = -\bar{\kappa}_{yx}$. We do not get into details of how to derive the magnetization currents. This can be found in [78] and [79]. The most important point for us is the fact that we can define physical currents in such a way that they do not depend on the radial coordinate along the bulk. This means that we can evaluate our expression in any hypersurface along the bulk, and not necessarily only on the AdS boundary. Moreover, we will obtain analytical expressions for the conductivities that depend only on the fields and metric of the theory evaluated on the chosen hypersurface. It is convenient though to choose this hypersurface as being the horizon of the black hole, i.e. the conductivities will depend only on the horizon data.

7.4 The model

The theory we will consider is given by*

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_N}\left(R - \frac{1}{2}[(\partial\phi)^2 + \Phi(\phi)\left((\partial\chi_1)^2 + (\partial\chi_2)^2\right)] - V(\phi)\right)\right.$$
$$\left. - \frac{Z(\phi)}{4g_4^2}F^2 - W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \quad (7.17)$$

where ϕ is the dilaton, a real scalar field. Notice that the dilaton field was rescaled by a factor of 2 compared to the one used in the theory (6.1). The couplings $Z(\phi)$, $\Phi(\phi)$, $W(\phi)$ and the potential $V(\phi)$ depend only on the dilaton, and $F_{\mu\nu}$ is an Abelian field strength. Under specific choices of the couplings and $V(\phi)$, the theory will present S-duality. The field strength and dual field strength are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{\sqrt{-g}}F_{\rho\sigma}. \quad (7.18)$$

In the literature, the fields χ_i are called "linear axions" because of the classical solutions chosen for them. They are not the string theory axion fields related to the $\mathrm{SL}(2,\mathbb{R})$ invariance of the equations of motion of the Einstein-Maxwell-axion-dilaton theory (3.36). The name was just inspired in the fact that their kinetic term have a similar coupling to stringy axion kinetic term. But notice that the stringy axion also couples to the field strength, which is not the case with the linear axions. The solutions one considers for them are

$$\chi_1 = k_1 x, \quad \chi_2 = k_2 y. \quad (7.19)$$

As we are interested in holographic applications of the model, the potential must satisfy the conditions

$$V(0) = -\frac{6}{L^2}, \quad V'(0) = 0, \quad (7.20)$$

which guarantees the existence of and AdS solution, where the prime denotes derivative with respect to the dilaton. This theory is then dual to a three-dimensional

*We take the potential in the action with a minus sign, and write the gauge coupling g_4^2 explicitly for future convenience.

conformal field theory. The background metric takes the form

$$ds^2 = -\mathcal{U}\,dt^2 + \mathcal{U}^{-1}dr^2 + e^{2\mathcal{V}}(dx^2 + dy^2), \tag{7.21}$$

where $\mathcal{U}$ and $\mathcal{V}$ are functions of r. We consider the metric in the near-horizon region. Notice that the function $\mathcal{U}(r)$ appearing in (7.21) can be Taylor expanded around the horizon of the black hole as

$$\mathcal{U}(r) \approx \mathcal{U}(r_H) + (r - r_H)\mathcal{U}'(r_H) + \mathcal{O}(r^2). \tag{7.22}$$

The first term in this expansion vanishes at the horizon by definition. Since we want to consider black holes at finite temperature, we stop the expansion at the linear term. Notice that such term vanishes for extremal black holes. So in the near-horizon region the metric is written as

$$ds^2 \approx -(r - r_H)\mathcal{U}'(r_H)dt^2 + \frac{1}{(r - r_H)\mathcal{U}'(r_H)}dr^2 + e^{2\mathcal{V}(r_H)}(dx^2 + dy^2). \tag{7.23}$$

The two-dimensional Rindler metric is written as

$$ds^2 = -(\kappa z)^2 dt^2 + dz^2. \tag{7.24}$$

Notice that this metric can be written as a product between a Rindler spacetime and $\mathbb{R}^2$ by making the redefinition

$$(r - r_H) \equiv \frac{\mathcal{U}'(r_H)}{4}z^2, \;\; \kappa \equiv \frac{\mathcal{U}'(r_H)}{2}, \tag{7.25}$$

where κ is the surface gravity. We will choose the appropriate sign for it later. The relation between the surface gravity and temperature is given by (2.42). Notice that we can find directly the following scaling symmetries for the metric :

$$t \to \lambda t, \;\; \kappa \to \lambda^{-1}\kappa, \tag{7.26}$$

$$t \to \chi^{-1}t, \;\; (r - r_H) \to \chi(r - r_H), \;\; \mathcal{U}'(r_H) \to \chi\,\mathcal{U}'(r_H), \tag{7.27}$$

$$e^{\mathcal{V}(r_H)} \to \xi e^{2\mathcal{V}(r_H)}, \;\; x \to \xi^{-1}x, \;\; y \to \xi^{-1}y. \tag{7.28}$$

We can expand the radially dependent background fields near the horizon r_+ as

$$\begin{aligned}
\mathcal{U} &\sim 4\pi T(r-r_+)+...,\\
a &\sim a_+(r-r_+)+...,\\
V &\sim V_+ + ...,\\
\phi &\sim \phi_+ + ...,
\end{aligned} \tag{7.29}$$

where T is the temperature of the black hole, which is identified with the temperature of the dual field theory. For this chapter, the important equations of motion are the equations for the metric and gauge field, given respectively by

$$R_{\mu\nu}=\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}g_{\mu\nu}V(\phi)+\frac{(16\pi G_N)}{4g_4^2}Z(\phi)\left(2F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{7.30}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left(\frac{Z(\phi)}{g_4^2}F^{\mu\nu}+4W(\phi)\tilde{F}^{\mu\nu}\right)\right]=0. \tag{7.31}$$

Another quantity of interest is the gauge current, computed as

$$\langle J^\mu\rangle=\frac{\delta S_{\text{on-shell}}}{\delta A_\mu}\bigg|_{\text{boundary}}=\sqrt{-g}\left(\frac{Z(\phi)}{g_4^2}F^{\mu\nu}+4W(\phi)\tilde{F}^{\mu\nu}\right). \tag{7.32}$$

When μ is the time index, the gauge current defines the charge density of the boundary theory

$$\rho=\langle J^t\rangle. \tag{7.33}$$

7.5 Magnetization and energy magnetization densities

In this section we review the computation of the energy magnetization, as was done in appendix A of [21], and apply this for our case, which contains the topological term $W(\phi)F\tilde{F}$. Applying a magnetic field to the boundary via a source $A_x^{(0)}=-By$, the magnetization density is given by

$$M=-\frac{1}{V}\frac{\partial S_E}{\partial B}, \tag{7.34}$$

where V is the volume of the boundary, and S_E is the Euclidean action. Applying a source $\delta g_{tx}^{(0)} = -B_1 y$, allows us to define an analogous quantity for the metric,

$$M_E = -\left.\frac{1}{V}\frac{\partial S_E}{\partial B_1}\right|_{B_1=0}. \tag{7.35}$$

The Euclidean action is obtained from the Lorentzian action by using Wick rotation, $t \to -i\tau$, and this gives

$$S_E = -\int d^4x\sqrt{g}\left[\frac{1}{16\pi G_N}\left(R - \frac{1}{2}(\partial\phi)^2 + V(\phi)\right) - \frac{Z(\phi)}{4g_4^2}F^2 + W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{7.36}$$

The ansatz for the background fields in the presence of such source is

$$A_t = a(r),\ \ A_x = -By + (a(r)-\mu)B_1 y, \tag{7.37}$$

$$ds^2 = -\mathcal{U}(r)(dt + B_1 y dx)^2 + \frac{dr^2}{\mathcal{U}(r)} + e^{2\mathcal{V}(r)}(dx^2+dy^2). \tag{7.38}$$

Notice that the inverse metric is given by

$$g^{\mu\nu} = \begin{bmatrix} B_1^2 e^{-2\mathcal{V}}y^2 - \frac{1}{\mathcal{U}} & 0 & -B_1 e^{-2\mathcal{V}}y & 0 \\ 0 & \mathcal{U} & 0 & 0 \\ -B_1 e^{-2\mathcal{V}}y & 0 & e^{-2\mathcal{V}} & 0 \\ 0 & 0 & 0 & e^{-2\mathcal{V}} \end{bmatrix} \tag{7.39}$$

The non-trivial field strengths are written as

$$F_{rt} = a'(r),\ F_{rx} = a'(r)B_1 y,\ F_{xy} = -B + (a(r)-\mu)B_1. \tag{7.40}$$

We compute each term appearing in S_E^{Maxwell} separately, i.e.

$$F_{\mu\nu}F^{\mu\nu} = 2g^{rr}g^{tt}(F_{rt})^2 + 4g^{rr}g^{tx}F_{rt}F_{rx} + 2g^{yy}g^{xx}(F_{xy})^2 + 2g^{rr}g^{xx}(F_{rx})^2 \tag{7.41}$$

$$= 2\mathcal{U}\left(\frac{-B_1^2y^2\mathcal{U} + e^{2\mathcal{V}}}{-\mathcal{U}e^{2\mathcal{V}}}\right)(a')^2 + 4\mathcal{U}(-B_1 y e^{-2\mathcal{V}})(a'B_1 y) \tag{7.42}$$

$$+ e^{-2\mathcal{V}}e^{-2\mathcal{V}}[-B + (a(r)-\mu)B_1]^2 + 2\mathcal{U}e^{-2\mathcal{V}}(a'(r)B_1 y)^2 \tag{7.43}$$

$$= -2(a')^2 + 2e^{-4\nu}[-B + (a(r) - \mu)B_1]^2, \quad (7.44)$$

$$\tilde{\epsilon}^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 8\tilde{\epsilon}^{rtxy} F_{rt} F_{xy} = -8a'(r)(-B + (a(r) - \mu)B_1). \quad (7.45)$$

This gives

$$S_E^{\text{Maxwell}} = \int d^4x \sqrt{g} \left[\frac{Z(\phi)}{4g_4^2} \left(2(a')^2 + 2e^{-4\nu}[-B + (a(r) - \mu)B_1]^2\right) \right.$$
$$\left. -4W(\phi)a'(r)(-B + (a(r) - \mu)B_1) \right]. \quad (7.46)$$

Notice that this expression is written in Euclidean metric, so the signs of the terms containing time components of the metric are reversed. The magnetization density (7.34) is obtained by making $B_1 = 0$ and then computing

$$M = -\frac{1}{V} \frac{\partial S_E}{\partial B} = -\int_{r_+}^{\infty} dr \left(\frac{e^{-2\nu} Z(\phi) B}{g_4^2} - 4W(\phi)a'(r) \right). \quad (7.47)$$

The magnetization density (7.35) is obtained by differentiating with respect to B_1 and then making $B_1 = 0$. This gives

$$M_E = -\int_{r_+}^{\infty} dr \left(\frac{e^{-2\nu} Z(\phi) B}{g_4^2} - 4W(\phi)a'(r) \right) (\mu - a(r)). \quad (7.48)$$

The heat magnetization is given by

$$M_Q = M_E - \mu M = \int_{r_+}^{\infty} dr \left(\frac{e^{-2\nu} Z(\phi) B}{g_4^2} - 4W(\phi)a'(r) \right) a(r). \quad (7.49)$$

7.6 Electric currents

In order to compute the conductivities we study the perturbations of the background solution. The sources applied to the boundary fields are linear. The perturbation ansatz used here is the same as in [21][76], given by

$$A_x = -By + (-E + \xi a(r))t + \delta A_x(r)$$
$$A_y = \delta A_y(r)$$
$$g_{tx} = -\xi t \mathcal{U} + e^{2\nu} \delta h_{tx}(r)$$

$$
\begin{aligned}
g_{ty} &= e^{2\mathcal{V}}\delta h_{ty}(r) \\
g_{rx} &= e^{2\mathcal{V}}\delta h_{rx}(r) \\
g_{ry} &= e^{2\mathcal{V}}\delta h_{ry}(r) \\
\chi_1 &= kx + \delta\chi_1(r) \\
\chi_2 &= ky + \delta\chi_2(r),
\end{aligned} \tag{7.50}
$$

where $E_i = E\delta_{ix}$ is the external electric field and $(\nabla T)_i = \xi\delta_{ix}T$ is the temperature gradient. The the infinitesimal parameters are $\delta h_{\mu\nu}$, δA_i, $\delta\chi_i$, E and ξ. In matrix form, the metric is written as

$$
g = \begin{pmatrix} -\mathcal{U} & 0 & e^{2\mathcal{V}}\delta h_{tx} - t\mathcal{U}\xi & e^{2\mathcal{V}}\delta h_{ty} \\ 0 & \frac{1}{\mathcal{U}} & e^{2\mathcal{V}}\delta h_{rx} & e^{2\mathcal{V}}\delta h_{ry} \\ e^{2\mathcal{V}}\delta h_{tx} - t\mathcal{U}\xi & e^{2\mathcal{V}}\delta h_{rx} & e^{2\mathcal{V}} & 0 \\ e^{2\mathcal{V}}\delta h_{ty} & e^{2\mathcal{V}}\delta h_{ry} & 0 & e^{2\mathcal{V}} \end{pmatrix}. \tag{7.51}
$$

Notice that we also have $\sqrt{-g} \approx e^{2\mathcal{V}} + \mathcal{O}(2)$, where $\mathcal{O}(2)$ represents terms which are of second order in perturbations. The inverse metric at linear order is given by

$$
g^{-1} = \begin{pmatrix} -\frac{1}{\mathcal{U}} & 0 & \left(\frac{\delta h_{tx}}{\mathcal{U}} - e^{-2\mathcal{V}}t\xi\right) & \frac{\delta h_{ty}}{\mathcal{U}} \\ 0 & \mathcal{U} & -\mathcal{U}\delta h_{rx} & -\mathcal{U}\delta h_{ry} \\ \left(\frac{\delta h_{tx}}{\mathcal{U}} - e^{-2\mathcal{V}}t\xi\right) & -\mathcal{U}\delta h_{rx} & e^{-2\mathcal{V}} & 0 \\ \frac{\delta h_{ty}}{\mathcal{U}} & -\mathcal{U}\delta h_{ry} & 0 & e^{-2\mathcal{V}} \end{pmatrix}. \tag{7.52}
$$

The field strengths with lower indices have the following components

$$
\begin{aligned}
F_{rt} &= a', \\
F_{tx} &= -E + \xi a, \\
F_{xy} &= B, \\
F_{rx} &= \xi a' t + \delta A'_x, \\
F_{ry} &= \delta A'_y.
\end{aligned} \tag{7.53}
$$

The gauge field equations imply for the x component

$$
0 \;=\; \partial_t\left(\frac{\sqrt{-g}Z(\phi)}{g_4^2}F^{tx} + 4\sqrt{-g}W(\phi)\tilde{F}^{tx}\right) + \partial_r\left(\frac{\sqrt{-g}Z(\phi)}{g_4^2}F^{rx} + 4\sqrt{-g}W(\phi)\tilde{F}^{rx}\right)
$$

$$
+ \quad \partial_y \left(\frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{yx} + 4\sqrt{-g}W(\phi)\tilde{F}^{yx} \right), \tag{7.54}
$$

and similarly for the y component

$$
\begin{aligned}
0 \;&=\; \partial_t \left(\frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{ty} + 4\sqrt{-g}W(\phi)\tilde{F}^{ty} \right) + \partial_r \left(\frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{ry} + 4\sqrt{-g}W(\phi)\tilde{F}^{ry} \right) \\
&+\; \partial_x \left(\frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{xy} + 4\sqrt{-g}W(\phi)\tilde{F}^{xy} \right).
\end{aligned} \tag{7.55}
$$

In order to compute correctly (7.54) and (7.55), we need to compute the various components of field strength with upper indices, which are given by

$$
\begin{aligned}
F^{tx} &= g^{t\alpha}g^{x\beta}F_{\alpha\beta}, \\
&= g^{tt}g^{xr}F_{tr} + g^{tt}g^{xx}F_{tx} + g^{tx}g^{xt}F_{xt} + g^{tx}g^{xr}F_{xr} + g^{ty}g^{xr}F_{yr} + g^{ty}g^{xx}F_{yx}. \\
&= (-\frac{1}{\mathcal{U}})(-\mathcal{U}\delta h_{rx})(-a') + (-\frac{1}{\mathcal{U}})e^{-2\mathcal{V}}(\xi a - E) \\
&\quad +(\frac{\delta h_{tx}}{\mathcal{U}} - e^{-2\mathcal{V}}t\xi)^2(E - \xi a) + (\frac{\delta h_{tx}}{\mathcal{U}} - e^{-2\mathcal{V}}t\xi)(-\mathcal{U}\delta h_{rx})(-\xi a't - \delta A'_x) \\
&\quad +\frac{\delta h_{ty}}{\mathcal{U}}(-\mathcal{U}\delta h_{rx})(-\delta A'_y) + (\frac{\delta h_{ty}}{\mathcal{U}})e^{-2\mathcal{V}}(-B) \\
&= -\left(a'\delta h_{rx} + \frac{B}{\mathcal{U}}e^{-2\mathcal{V}}\delta h_{ty} + \frac{\xi a}{\mathcal{U}}e^{-2\mathcal{V}} - \frac{E}{\mathcal{U}}e^{-2\mathcal{V}} \right).
\end{aligned} \tag{7.56}
$$

A similar analysis follows for the other components, and we obtain

$$
\begin{aligned}
F^{xr} &= g^{x\alpha}g^{r\beta}F_{\alpha\beta} \\
&= g^{rr}g^{xt}F_{tr} + g^{rx}g^{xt}F_{tx} + g^{rx}g^{rx}F_{xr} + g^{ry}g^{xr}F_{ry} + g^{rr}g^{xx}F_{xr} + g^{ry}g^{xx}F_{xy} \\
&= -(a'\delta h_{tx} + \mathcal{U}e^{-2\mathcal{V}}\delta A'_x + \mathcal{U}e^{-2\mathcal{V}}B\delta h_{ry}), \qquad (7.57) \\
F^{xy} &= g^{xm}g^{yn}F_{mn}, \\
&= g^{xm}(g^{yt}F_{mt} + g^{yr}F_{mr} + g^{yy}F_{xy}), \\
&= g^{xr}g^{yt}F_{rt} + g^{xx}g^{yt}F_{xt} + g^{xt}g^{yr}F_{tr} \\
&\quad +g^{xx}g^{yr}F_{xr} + g^{xr}g^{yy}F_{ry} + g^{xx}g^{yy}F_{xy}. \\
&= Ze^{-4\mathcal{V}}B \qquad (7.58) \\
F^{ty} &= g^{t\alpha}g^{y\beta}F_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
&= g^{tt}g^{yr}F_{tr} + g^{tx}g^{yt}F_{xt} + g^{tx}g^{yr}F_{xr} + g^{tx}g^{xr}F_{yr} + g^{tx}g^{yy}F_{xy} \\
&= -(a'\delta h_{ry} - \frac{e^{-2\mathcal{V}}}{\mathcal{U}}B\delta h_{tx} + e^{-4\mathcal{V}}Bt\xi), \qquad (7.59)\\
F^{ry} &= g^{r\alpha}g^{y\beta}F_{\alpha\beta} \\
&= g^{rr}g^{yt}F_{rt} + g^{rr}g^{yy}F_{ry} + g^{rx}g^{yt}F_{xt} + g^{rx}g^{yr}F_{xr} + g^{rx}g^{yy}F_{xy} + g^{ry}g^{yr}F_{yr} \\
&= (a'\delta h_{ty} + \mathcal{U}e^{-2\mathcal{V}}\delta A'_y - \mathcal{U}e^{-2\mathcal{V}}B\delta h_{rx}). \qquad (7.60)
\end{aligned}
$$

Similarly, we can compute the components of the dual field strengths with upper indices, and this gives

$$
\begin{aligned}
\sqrt{-g}\tilde{F}^{tx} &= F_{yr} = -\delta A'_y, \qquad (7.61)\\
\sqrt{-g}\tilde{F}^{ty} &= F_{rx} = (\xi a't + \delta A'_x), \qquad (7.62)\\
\sqrt{-g}\tilde{F}^{rx} &= F_{ty} = 0, \qquad (7.63)\\
\sqrt{-g}\tilde{F}^{ry} &= F_{tx} = -(-E + \xi a), \qquad (7.64)\\
\sqrt{-g}\tilde{F}^{xy} &= F_{tr} = -a'. \qquad (7.65)
\end{aligned}
$$

We now replace all field strength components on (7.54) and (7.55). First, notice that ∂_x and ∂_y terms vanish, because no component is dependent on the x or y coordinate. For (7.54) we have

$$
\begin{aligned}
-\partial_t \left[\frac{1}{g_4^2}\left(a'Ze^{2\mathcal{V}}\delta h_{rx} + \frac{\xi a Z}{\mathcal{U}} + \frac{ZB}{\mathcal{U}}\delta h_{ty} - \frac{EZ}{\mathcal{U}}\right) + 4W\delta A'_y\right] = 0 \\
= -\partial_r\left(\frac{\sqrt{-g}Z}{g_4^2}F^{rx} + 4\sqrt{-g}W\tilde{F}^{rx}\right). \qquad (7.66)
\end{aligned}
$$

The left hand side is clearly time independent, and that is why it is zero. No field strength depends on the coordinate x and y, and that is why Maxwell's equations for this case imply that the term appearing inside the parenthesis the last line is radially independent. Similarly, for (7.55) we have

$$
\begin{aligned}
&\partial_t\left[-\frac{1}{g_4^2}\left(a'e^{2\mathcal{V}}Z\delta h_{ry} - \frac{BZ}{\mathcal{U}}\delta h_{tx} + ZBe^{-2\mathcal{V}}t\xi\right) + 4W(\xi a't + \delta A'_x)\right] \\
&= -\frac{Z}{g_4^2}\xi e^{-2\mathcal{V}}B + 4\xi W a'
\end{aligned}
$$

$$= \partial_r \left(\frac{Z}{g_4^2} \sqrt{-g} F^{yr} + 4\sqrt{-g} W \tilde{F}^{yr} \right). \tag{7.67}$$

The term on the second line is the result of the action of the partial derivative with respect to time on the first line. This implies that in the presence of the perturbations (7.50) the term inside the parenthesis on the last line is not radially independent. The appropriate radially independent currents are constructed by subtracting off the magnetization current contributions, i.e.

$$\begin{aligned} J^x &= \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{xr} + 4\sqrt{-g} W(\phi) \tilde{F}^{xr}, \\ J^y &= \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{yr} + 4\sqrt{-g} W(\phi) \tilde{F}^{yr} - \xi M(r), \end{aligned} \tag{7.68}$$

where $M(r)$ is given by (7.47). Using (7.57), the x-component is written explicitly as

$$J^x = -\frac{Z}{g_4^2} a' e^{2\mathcal{V}} \delta h_{tx} - \frac{Z}{g_4^2} \mathcal{U} \delta A'_x - \frac{Z}{g_4^2} \mathcal{U} B \delta h_{ry}. \tag{7.69}$$

There is no contribution from the topological term containing $\tilde{F}$, since $\tilde{F}^{xr} = 0$. Similarly, for y-component

$$J^y = -\frac{Z}{g_4^2} \mathcal{U} \delta A'_y - \frac{Z}{g_4^2} e^{2\mathcal{V}} a' \delta h_{ty} + \frac{Z}{g_4^2} B \mathcal{U} \delta h_{rx} + 4W(-E + \xi a) - \xi M(r). \tag{7.70}$$

The most important observation at this point is that **these currents are independent of the radial coordinate, so we can evaluate them at any hypersurface in the bulk**. For the purposes of this thesis, it is convenient to choose this hypersurface as being the horizon of the black hole. In order to do so, we have to impose the regularity conditions [21]

$$\begin{aligned} \delta A_i &= -\frac{E_i}{4\pi T} \ln(r - r_+) + O(r - r_+), \\ \delta \chi_i &= O((r - r_+)^0), \\ \delta h_{ti} &= \mathcal{U} \delta h_{ri} - \frac{\xi_i \mathcal{U}}{4\pi e^{2\mathcal{V}} T} \ln(r - r_+) + O(r - r_+) \end{aligned} \tag{7.71}$$

in the the near horizon region, where r_+ is the position of the horizon. We remind that, for our ansatz, we have $E_i = E\delta_{ix}$ and $\xi_i = \delta_{ix}\xi$. Notice that $M(r)$, given by

(7.47), vanishes on the horizon. Then, we can compute the conductivities explicitly in terms of the horizon data. Using the regularity conditions (7.71), we evaluate the currents of the horizon, which gives

$$
\begin{aligned}
J^x &= \frac{Z}{g_4^2}E_x - \frac{Z}{g_4^2}e^{2\mathcal{V}}a'\delta h_{tx} - \frac{Z}{g_4^2}B\delta h_{ty}\Big|_{r_+}, \\
J^y &= \frac{Z}{g_4^2}E_y - e^{2\mathcal{V}}Za'\delta h_{ty} + \frac{Z}{g_4^2}B\delta h_{tx} - 4WE\Big|_{r_+}.
\end{aligned} \tag{7.72}
$$

Now we need to determine the graviton fluctuations δh_{ti} in order to obtain the conductivities. This is done using the linearized Einstein's equations

$$
\begin{aligned}
\mathcal{U}(e^{4\mathcal{V}}\delta h_{tx}')' &- \left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2\mathcal{V}}k^2\Phi\right)\delta h_{tx} + \frac{2\kappa^2}{g_4^2}ZB\mathcal{U}e^{2\mathcal{V}}a'\delta h_{ty} = -\frac{2\kappa^2}{g_4^2}Ze^{2\mathcal{V}}a'\delta a_x', \\
\mathcal{U}(e^{4\mathcal{V}}\delta h_{ty}')' &- \left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2\mathcal{V}}k^2\Phi\right)\delta h_{ty} - \frac{2\kappa^2}{g_4^2}ZB\mathcal{U}e^{2\mathcal{V}}a'\delta h_{tx} = -\frac{2\kappa^2}{g_4^2}Ze^{2\mathcal{V}}a'\delta a_y' \\
&+\frac{2\kappa^2}{g_4^2}ZB(-E+\xi a),
\end{aligned} \tag{7.73}
$$

where we used the notation[†] $2\kappa^2 = 16\pi G_N$. Notice that the there is no contribution from the $F\tilde{F}$ term, since this is topological, i.e. it does not depend on the metric of the spacetime. Using the regularity conditions (7.71) we can show that these equations reduce to

$$
\begin{aligned}
\left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2\mathcal{V}}k^2\Phi\right)\delta h_{tx} - \frac{2\kappa^2}{g_4^2}ZBe^{2\mathcal{V}}a'\delta h_{ty} &= -\frac{2\kappa^2}{g_4^2}Ze^{2\mathcal{V}}a'E + e^{2\mathcal{V}}\mathcal{U}\xi, \\
\left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2\mathcal{V}}k^2\Phi\right)\delta h_{ty} - \frac{2\kappa^2}{g_4^2}ZBe^{2\mathcal{V}}a'\delta h_{tx} &= \frac{2\kappa^2}{g_4^2}ZBE,
\end{aligned} \tag{7.74}
$$

evaluated at the horizon r_+. This is a system of two equations and two variables, so it is solvable. The result is

$$
\delta h_{tx} = \frac{\frac{2\kappa^2}{g_4^2}Ze^{4\mathcal{V}}a'k^2\Phi}{(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2\mathcal{V}}k^2\Phi)^2 + (\frac{2\kappa^2}{g_4^2}Z)^2B^2e^{4\mathcal{V}}a'^2}E
$$

[†]We apologize for using the same letter κ as was used in the definition of the surface gravity. This is just to keep the same notation of reference [22]. Notice that in this whole chapter the only quantity that appears explicitly in our formulae is the temperature T, and not the surface gravity.

$$+\frac{(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}a'k^2\Phi)e^{2\mathcal{V}}\mathcal{U}'\xi}{(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}k^2\Phi)^2+(\frac{2\kappa^2}{g_4^2}Z)^2B^2e^{4\mathcal{V}}a'^2},$$

$$\begin{aligned}\delta h_{ty} =& \frac{\frac{2\kappa^2}{g_4^2}ZB}{(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}k^2\Phi)}E \\ &-\frac{(\frac{2\kappa^2}{g_4^2}ZBe^{2\mathcal{V}}a')e^{2\mathcal{V}}\mathcal{U}'\xi}{[(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}k^2\Phi)^2+(\frac{2\kappa^2}{g_4^2}Z)^2B^2e^{4\mathcal{V}}a'^2]} \\ &-\frac{\frac{2\kappa^2}{g_4^2}ZBe^{2\mathcal{V}}a'(e^{4\mathcal{V}}\frac{2\kappa^2}{g_4^2}Za'k^2\Phi E)}{(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}k^2\Phi)[(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2\mathcal{V}}k^2\Phi)^2+(\frac{2\kappa^2}{g_4^2}Z)^2B^2e^{4\mathcal{V}}a'^2]}.\end{aligned} \tag{7.75}$$

Now, we can replace the graviton fluctuations (7.75) in the currents (7.72) and write them in terms of the horizon data. We compare the resulting expressions with equation (7.14) so that we can extract the conductivities

$$\sigma_{xx} = \left.\frac{e^{2\mathcal{V}}k^2\Phi(2\kappa_4^2g_4^4\rho^2+2\kappa_4^2B^2Z^2+g_4^2Ze^{2\mathcal{V}}k^2\Phi)}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2\mathcal{V}}k^2\Phi)^2}\right|_{r_+}, \tag{7.76}$$

$$\sigma_{xy} = \left.4\kappa_4^2B\rho\frac{\kappa_4^2g_4^4\rho^2+\kappa_4^2B^2Z^2+g_4^2Ze^{2\mathcal{V}}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2\mathcal{V}}k^2\Phi)^2}-4W\right|_{r_+}, \tag{7.77}$$

$$\alpha_{xx} = \left.\frac{2\kappa_4^2g_4^4s\rho e^{2\mathcal{V}}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2\mathcal{V}}k^2\Phi)^2}\right|_{r_+}, \tag{7.78}$$

$$\alpha_{xy} = \left.2\kappa_4^2sB\frac{2\kappa_4^2g_4^4\rho^2+2\kappa_4^2B^2Z^2+g_4^2Ze^{2\mathcal{V}}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2\mathcal{V}}k^2\Phi)^2}\right|_{r_+}. \tag{7.79}$$

We expressed our results in terms of the charge density ρ. This can be computed using (7.32) and (7.33), which gives

$$\rho = \left.\frac{Ze^{2\mathcal{V}}a'}{g_4^2}+4WB\right|_{r_+}. \tag{7.80}$$

And we also expressed the results in terms of the entropy density, given by

$$s = \left.\frac{4\pi e^{2\mathcal{V}}}{2\kappa_4^2}\right|_{r_+}. \tag{7.81}$$

Our results differ from the ones obtained in [21] only by the inclusion of the term $-4W$ in the off-diagonal electric conductivity σ_{xy}. This is expected, since we are adding a topological term to the theory. The thermoelectric conductivities computed here are the same.

7.7 Heat currents

In the previous section, we confined our attention to the electrical currents. Now, we focus on the heat currents, and show also how to compute heat conductivities in terms of horizon data. This will confirm our computation of the electrothermal conductivities as well. Again, we follow the same procedure as was done in [21], but including the topological term that appears in the action. The computation of the conductivities in the presence of topological terms was done in a different way in [80]. There, the authors claim that the conductivities can be computed analytically by solving a set of equations, called by them as Stokes equations. We are interested in the case when the conductivities are deduced completely in terms of horizon data, and that is why we adopted the approach of reference [21], and we explain out computation now.

Consider an arbitrary vector that satisfies

$$\nabla_\mu k^\mu = 0. \tag{7.82}$$

It is straightforward to construct the following set of relations:

$$\begin{aligned}
\nabla^{[\mu}k^{\nu]} - \nabla^{(\mu}k^{\nu)} &= -\nabla^\nu k^\mu \\
\nabla_\mu(\nabla^{[\mu}k^{\nu]}) &= \nabla_\mu(\nabla^{(\mu}k^{\nu)}) - \nabla_\mu\nabla^\nu k^\mu \\
&= \nabla_\mu(\nabla^{(\mu}k^{\nu)}) - [\nabla_\mu, \nabla^\nu]k^\mu \\
&= \nabla_\mu(\nabla^{(\mu}k^{\nu)}) - R^\nu{}_\mu k^\mu,
\end{aligned} \tag{7.83}$$

where $R^\nu{}_\mu$ is the Ricci tensor. Now, we define the **bulk two-form tensor** $G^{\mu\nu}$ as

$$G^{\mu\nu} = -2\nabla^{[\mu}k^{\nu]} - Zk^{[\mu}F^{\nu]\rho}A_\rho - \frac{1}{2}(\psi - 2\theta)H^{\mu\nu}, \tag{7.84}$$

where the functions ψ and θ are also defined through

$$\nabla_\rho\psi = (\mathcal{L}_k A)_\rho = k^\mu\partial_\mu A_\rho + A_\mu\partial_\rho k_\mu, \tag{7.85}$$

$$\nabla_\rho \theta = k^\mu F_{\mu\rho} - \frac{1}{2}\xi_\rho k^\mu A_\mu. \tag{7.86}$$

Of course these are just definitions, but we intend to relate the bulk two-form (7.84) with the heat currents, and then define our radially independent currents. This is a construction, and we show that it works for the perturbation (7.50). The idea in defining the bulk two-form (7.84) is to make the following identification at the linearized level

$$\partial_\mu(\sqrt{-g}G^{\mu i}) = 0, \qquad \text{as } r \to \infty. \tag{7.87}$$

So, we can identify the following radially independent fluxes as

$$\langle \vec{Q}^{(tot)i} \rangle = \sqrt{-g}G^{ri}. \tag{7.88}$$

In absence of perturbations, this would be identified with the heat current, and it would be a conserved quantity. But similarly to what happened to the electrical currents, equation (7.87) is no longer satisfied in the presence of the perturbations (7.50). We will see this explicitly now.

Taking the covariant derivative of (7.84), and using the equation of motion (7.31), i.e. $\nabla_\mu H^{\mu\nu} \equiv \nabla_\mu(Z(\phi)F^{\mu\nu} + 4g_4^2 W(\phi)\tilde{F}^{\mu\nu}) = 0$, and (7.82), we obtain

$$\begin{aligned}
\nabla_\mu G^{\mu\nu} &= -2\nabla_\mu(\nabla^{[\mu}k^{\nu]}) - \nabla_\mu(Zk^{[\mu}F^{\nu]\rho}A_\rho) - \frac{1}{2}\nabla_\mu(\psi - 2\theta)H^{\mu\nu} - \frac{1}{2}(\psi - 2\theta)\nabla_\mu H^{\mu\nu} \\
&= -2\nabla_\mu(\nabla^{[\mu}k^{\nu]}) - \frac{1}{2}[k^\mu \nabla_\mu Z F^{\nu\rho}A_\rho + Z\nabla_\mu k^\mu F^{\nu\rho}A_\rho + Zk^\mu \nabla_\mu F^{\nu\rho}A_\rho \\
&\quad + Zk^\mu F^{\nu\rho}\nabla_\mu A_\rho - \nabla_\mu Z k^\nu F^{\mu\rho}A_\rho - Z\nabla_\mu k^\nu F^{\mu\rho}A_\rho - Zk^\nu \nabla_\mu F^{\mu\rho}A_\rho - Zk^\nu F^{\mu\rho}\nabla_\mu A_\rho] \\
&\quad - \frac{1}{2}\nabla_\mu(\psi - 2\theta)ZF^{\mu\nu} - \frac{1}{2}\nabla_\mu(\psi - 2\theta)4g_4^2 W\tilde{F}^{\mu\nu}.
\end{aligned} \tag{7.89}$$

The Lie derivative of F is written as[‡]

$$(\mathcal{L}_k F)^{\nu\rho} = k^\mu \nabla_\mu F^{\nu\rho} - \nabla_\mu k^\nu F^{\mu\rho} - \nabla_\mu k^\rho F^{\nu\mu}. \tag{7.90}$$

For future purposes k^μ will only have time component. Notice that $Z(\phi)$ depends only on the radial coordinate, which implies that $k^\mu \nabla_\mu Z = 0$. Then, we can rewrite

[‡]We emphasize that k^μ is not a Killing vector in the presence of perturbations (7.50).

(7.89) as

$$\nabla_\mu G^{\mu\nu} = -2\nabla_\mu(\nabla^{[\mu}k^{\nu]}) - \frac{1}{2}[Zk^\mu F^{\nu\rho}\nabla_\mu A_\rho - \nabla_\mu Z k^\nu F^{\mu\rho}A_\rho - Zk^\nu \nabla_\mu F^{\mu\rho}A_\rho$$
$$- Zk^\nu F^{\mu\rho}\nabla_\mu A_\rho + ZA_\rho(\mathcal{L}_k F)^{\nu\rho} + Z\nabla_\mu k^\rho F^{\nu\mu}A_\rho]$$
$$- \frac{1}{2}\nabla_\mu(\psi - 2\theta)ZF^{\mu\nu} - \frac{1}{2}\nabla_\mu(\psi - 2\theta)4g_4^2 W\tilde{F}^{\mu\nu}. \tag{7.91}$$

Using (7.85) and (7.86), we replace $\nabla_\mu(\psi - 2\theta)$ in the term containing Z in the last line, and obtain the expression

$$\nabla_\mu G^{\mu\nu} = -2\nabla_\mu(\nabla^{[\mu}k^{\nu]}) - \frac{1}{2}[Zk^\mu F^{\nu\rho}\nabla_\mu A_\rho - \nabla_\mu Z k^\nu F^{\mu\rho}A_\rho - Zk^\nu \nabla_\mu F^{\mu\rho}A_\rho$$
$$- Zk^\nu F^{\mu\rho}\nabla_\mu A_\rho + 2Zk^\rho\nabla_\mu A_\rho F^{\mu\nu} - Zk^\rho \nabla_\rho A_\mu F^{\mu\nu}] - \frac{ZA_\rho(\mathcal{L}_k F)^{\nu\rho}}{2}$$
$$+ \frac{1}{2}ZF^{\nu\mu}s_\mu - \frac{1}{2}\nabla_\mu(\psi - 2\theta)4{g_4}^2 W\tilde{F}^{\mu\nu}, \tag{7.92}$$

where $s_\mu \equiv= k^\nu F_{\nu\mu} - \nabla_\mu\theta$. The terms with covariant derivatives in A_μ can be combined in such a way to obtain

$$Zk^\mu F^{\nu\rho}\nabla_\mu A_\rho + 2Zk^\rho \nabla_\mu A_\rho F^{\mu\nu} - Zk^\rho\nabla_\rho A_\mu F^{\mu\nu} = 2k^\rho F_{\mu\rho}F^{\mu\nu}. \tag{7.93}$$

Notice also that

$$Zk^\nu F^{\mu\rho}\nabla_\mu A_\rho = \frac{Z}{2}k^\nu F^{\mu\rho}F_{\mu\rho}. \tag{7.94}$$

The expression (7.92) can be simplified using these two last equations, and also using (7.83), i.e.

$$\nabla_\mu G^{\mu\nu} = 2R^\mu{}_\nu k^\nu - 2\nabla_\mu(\nabla^{(\mu}k^{\nu)}) - \frac{1}{2}[-\nabla_\mu Z k^\nu F^{\mu\rho}A_\rho - Zk^\nu\nabla_\mu F^{\mu\rho}A_\rho]$$
$$+ \frac{1}{2}ZF^{\nu\mu}s_\mu - \frac{ZA_\rho(\mathcal{L}_k F)^{\nu\rho}}{2} + Zk^\rho F_{\mu\rho}F^{\nu\mu} + \frac{Z}{4}k^\nu F_{\mu\rho}F^{\mu\rho}$$
$$- \frac{1}{2}\nabla_\mu(\psi - 2\theta)4{g_4}^2 W\tilde{F}^{\mu\nu}. \tag{7.95}$$

The curvature term can be replaced in this equation using (7.30). This cancels out the terms $Zk^\nu F_{\mu\rho}F^{\mu\rho}$ and $Zk^\rho F_{\rho\mu}F^{\mu\nu}$ appearing in (7.95), and the resulting expression is just

$$\nabla_\mu G^{\mu\nu} = Vk^\nu - 2\nabla_\mu(\nabla^{(\mu}k^{\nu)}) - \frac{ZA_\rho(\mathcal{L}_k F)^{\nu\rho}}{2} - \frac{1}{2}ZF^{\mu\nu}s_\mu - \frac{1}{2}\nabla_\mu(\psi - 2\theta)4g_4^2 W\tilde{F}^{\mu\nu}$$

$$+\frac{1}{2}[\nabla_\mu Z k^\nu F^{\mu\rho} A_\rho + Z k^\nu \nabla_\mu F^{\mu\rho} A_\rho]. \tag{7.96}$$

The term inside the square brackets can be expressed as $\frac{1}{2}k^\nu A_\rho \nabla_\mu (Z F^{\mu\rho})$. Using again the equation of motion (7.31), and the Bianchi identity $\nabla_\mu \tilde{F}^{\mu\nu} = 0$, we have

$$\frac{1}{2}\nabla_\mu Z F^{\mu\rho} A_\rho k^\nu + \frac{1}{2} Z \nabla_\mu F^{\mu\rho} A_\rho k^\nu = \frac{1}{2}\nabla_\mu (Z F^{\mu\rho}) A_\rho k^\nu = -2 g_4^2 \nabla_\lambda W \tilde{F}^{\lambda\rho} A_\rho k^\nu. \tag{7.97}$$

Finally, the expression takes the form

$$\begin{aligned}\nabla_\mu G^{\mu\nu} = {} & V k^\nu - 2\nabla_\mu (\nabla^{(\mu} k^{\nu)}) + \frac{1}{2} Z F^{\nu\mu} s_\mu - \frac{Z}{2} A_\rho (\mathcal{L}_k F)^{\nu\rho} \\ & - 2 g_4^2 (\partial_\mu W) \tilde{F}^{\mu\rho} A_\rho k^\nu - 2 g_4^2 W \tilde{F}^{\mu\nu} \nabla_\mu (\psi - 2\theta).\end{aligned} \tag{7.98}$$

Notice that, using (7.85) and (7.86), we can compute the following integral

$$\begin{aligned}\int dx^\rho \nabla_\rho (\psi - 2\theta) = {} & \int dx^\rho k^\mu \partial_\mu A_\rho + \int dx^\rho A_\mu \nabla_\rho k^\mu \\ & - 2\int dx^\rho k^\mu F_{\mu\rho} + \int dx^\rho \xi_\rho i_k A.\end{aligned} \tag{7.99}$$

As we mentioned before, we will consider the case for which $k^\mu = (\partial_t)^\mu = \delta_t^\mu$, so the integral reduces to

$$\begin{aligned}\int dx^\rho \nabla_\mu (\psi - 2\theta) &= \int dx^\rho k^\mu \partial_\mu A_\rho + \int dx^\rho A_\mu \partial_\rho k^\mu - 2\int dx^\rho k^\mu F_{\mu\rho} + \int dx^\rho \xi_\rho k^\mu A_\mu \\ &= \int dx^\rho \partial_t A_\rho + \int dx^\rho A_\mu \partial_\rho \delta_t^\mu - 2\int dx^\rho F_{t\rho} + \int dx \xi A_t \\ &= \int dx^\rho \partial_t A_x - 2\int dx F_{tx} - 2\int dr F_{tr} + \int dx \xi a(r) \\ &= -\int dx(-E + a\xi) - 2\int dr(-a'(r)) + \int dx \xi a \\ &= Ex + 2a\end{aligned} \tag{7.100}$$

This result will allow us to evaluate the respective term $(\psi - 2\theta)$ when we calculate the components of the bulk two-form (7.84) in the presence of the perturbations (7.50). So, using $k^\mu = (\partial_t)^\mu = \delta_t^\mu$, the component G^{ri} is written as

$$G^{ri} = -\nabla^r k^i + \nabla^i k^r - Z k^{[r} F^{i]\sigma} A_\sigma - \frac{1}{2}(2a + Ex) H^{ri}$$

$$
\begin{aligned}
&= -g^{r\alpha}(\partial_\alpha k^i + \Gamma^i_{\alpha\beta}k^\beta) + g^{i\alpha}(\partial_\alpha k^r + \Gamma^r_{\alpha\beta}k^\beta) - Z(k^r F^{i\sigma} - k^i F^{r\sigma})A_\sigma \\
&\quad -\frac{1}{2}(2a + Ex)H^{ri} \\
&= -g^{r\alpha}(\partial_\alpha \delta^i_t + \Gamma^i_{\alpha t}) + g^{i\alpha}(\partial_\alpha \delta^r_t + \Gamma^r_{\alpha\beta}k^\beta) - \frac{1}{2}(2a + Ex)H^{ri} \\
&= -g^{r\alpha}\Gamma^i_{\alpha t} + g^{i\alpha}\Gamma^r_{\alpha t} - \frac{1}{2}(2a + Ex)H^{ri}. \qquad (7.101)
\end{aligned}
$$

We wrote explicitly all the connexions in the presence of pertubations (7.50) in appendix A. The linear order contributions to (7.101) come only for the case when $\alpha = r$ and $\alpha = i$ because all connections are at least of linear order in perturbation. So, the components of the metric $g^{r\alpha}$ and $g^{i\alpha}$ must contribute only with their zero order term in perturbation. Writing this explicitly we obtain

$$
g^{r\alpha}\Gamma^x_{\alpha t} = g^{rr}\Gamma^x_{rt} = \frac{\mathcal{U}}{2}\left(\delta h'_{tx} - \frac{\delta h_{tx}\mathcal{U}'}{\mathcal{U}} + 2\delta h_{tx}\mathcal{V}'\right) = \frac{\mathcal{U}e^{-2\mathcal{V}}}{2}\left(\frac{e^{2\mathcal{V}}\delta h_{tx}}{\mathcal{U}}\right)', \qquad (7.102)
$$

$$
g^{r\alpha}\Gamma^y_{\alpha t} = g^{rr}\Gamma^y_{rt} = \frac{\mathcal{U}}{2}\left(\delta h'_{ty} - \frac{\delta h_{ty}\mathcal{U}'}{\mathcal{U}} + 2\delta h_{ty}\mathcal{V}'\right) = \frac{\mathcal{U}e^{-2\mathcal{V}}}{2}\left(\frac{e^{2\mathcal{V}}\delta h_{ty}}{\mathcal{U}}\right)', \qquad (7.103)
$$

$$
g^{xx}\Gamma^r_{xt} = \frac{e^{-2\mathcal{V}}}{2}\mathcal{U}\left(\xi t\mathcal{U}' - e^{2\mathcal{V}}\left(\delta h'_{tx} + 2\delta h_{tx}\mathcal{V}'\right)\right), \qquad (7.104)
$$

$$
g^{yy}\Gamma^r_{yt} = -\frac{\mathcal{U}}{2}\left(\delta h'_{ty} + 2\delta h_{ty}\mathcal{V}'\right), \qquad (7.105)
$$

$$
g^{xt}\Gamma^r_{tt} = \left(\frac{\delta h_{tx}}{\mathcal{U}} - e^{-2\mathcal{V}}t\xi\right)\frac{1}{2}\mathcal{U}(r)\mathcal{U}'(r), \qquad (7.106)
$$

$$
g^{yt}\Gamma^r_{tt} = \delta h_{ty}\frac{\mathcal{U}'}{2}. \qquad (7.107)
$$

Notice that our quantities are defined in the limit when $r \to \infty$, so the term $a(r)$ dominates over Ex, and the heat current is

$$
\langle Q^i \rangle = U^2\left(\frac{e^{2\mathcal{V}}\delta h_{ti}}{\mathcal{U}}\right)' + a(r)\sqrt{-g}(Z(\phi)F^{ri} + 4g_4^2 W(\phi)\tilde{F}^{ri}). \qquad (7.108)
$$

Now we can check that the flux $\langle Q^i \rangle$ is not conserved in the presence of perturbations, just like we did for the electromagnetic current. Notice that we have to compute

$$
\begin{aligned}
\partial_r(\sqrt{-g}G^{rx}) &= -\partial_t(\sqrt{-g}G^{tx}) - \partial_y(\sqrt{-g}G^{yx}), \\
\partial_r(\sqrt{-g}G^{ry}) &= -\partial_t(\sqrt{-g}G^{ty}) - \partial_x(\sqrt{-g}G^{xy}). \qquad (7.109)
\end{aligned}
$$

In the definition of the bulk two-form (7.84), the term containing derivatives of k and the term $k^{[y}F^{x]\sigma}A_\sigma$ in G^{yx} are zero because $k^y = k^x = 0$. So, we obtain

$$G^{yx} = -g^{y\alpha}\Gamma^x_{\alpha t} + g^{x\alpha}\Gamma^y_{\alpha t} - \frac{1}{2}(2a+Ex)H^{yx} \tag{7.110}$$

In this expression, the terms involving the connections cancel out because they are of second order in perturbation. Since H^{yx} is independent of y, this implies that $\partial_y(\sqrt{-g}G^{yx}) = 0$.

The next term we analyze is G^{tx}, which is written as

$$\begin{aligned} G^{tx} &= -\nabla^t k^x + \nabla^x k^t - Z(k^t F^{x\sigma} - k^x F^{t\sigma})A_\sigma - \frac{1}{2}(2a+Ex)H^{tx} \\ &= -g^{t\beta}(\partial_\beta \delta^x_t + \Gamma^x_{\beta t}) + g^{x\beta}(\partial_\beta \delta^t_t + \Gamma^t_{\beta t}) - Zk^t F^{x\sigma} - \frac{1}{2}(2a+Ex)H^{tx} \\ &= -g^{t\beta}\Gamma^x_{\beta t} + g^{x\beta}\Gamma^t_{\beta t} - ZF^{x\sigma}A_\sigma + \frac{1}{2}(2a+Ex)H^{tx} \\ &= -g^{tt}\Gamma^x_{tt} - g^{tr}\Gamma^x_{rt} - g^{tx}\Gamma^x_{xt} - g^{ty}\Gamma^x_{yt} + g^{xt}\Gamma^t_{tt} + g^{xr}\Gamma^t_{rt} + g^{xx}\Gamma^t_{xt} + g^{xy}\Gamma^t_{yt} \\ &\quad - ZF^{xt}A_t - ZF^{xr}A_r - ZF^{xy}A_y + \frac{1}{2}(2a+Ex)(ZF^{tx} + 4g_4^2 W\tilde{F}^{tx}). \end{aligned} \tag{7.111}$$

Using the fact that $g^{xy} = 0$, $g^{rt} = 0$, $A_r = 0$ and that Γ^t_{tt}, Γ^t_{xt}, Γ^x_{xt} and Γ^x_{yt} are of second order in perturbation, then (7.111) is reduced to

$$G^{tx} = -g^{tt}\Gamma^x_{tt} + g^{xr}\Gamma^t_{rt} - ZF^{xt}A_t - ZF^{xy}A_y + \frac{1}{2}(2a+Ex)(ZF^{tx} + 4g_4^2 WF_{yr}). \tag{7.112}$$

This term does not depend on the t coordinate, so $\partial_t(\sqrt{-g}G^{tx}) = 0$.

The next component is

$$G^{xy} = -g^{x\alpha}\Gamma^y_{\alpha t} + g^{y\alpha}\Gamma^x_{\alpha t} + \frac{1}{2}(2a+Ex)(ZF^{xy} + 4g_4^2 WF_{tr}). \tag{7.113}$$

Using the same arguments as before, we are left with

$$\begin{aligned} G^{xy} &= \frac{1}{2}(2a+Ex)(ZF^{xy} + 4g_4^2 WF_{tr}) \\ &= \frac{1}{2}(2a+Ex)(Ze^{-4\nu}B - 4g_4^2 e^{-2\nu}a'). \end{aligned} \tag{7.114}$$

Now, it is easy to show that

$$\partial_x(\sqrt{-g}G^{xy}) = \frac{E}{2}(Ze^{-2\mathcal{V}}B - 4g_4^2 W a'). \tag{7.115}$$

Finally, the component G^{ty} is written explicitly as

$$\begin{aligned}
G^{ty} &= -g^{t\beta}\Gamma^y_{\beta t} + g^{y\beta}\Gamma^t_{\beta t} - Z(k^t F^{y\sigma} - k^y F^{t\sigma})A_\sigma - \frac{1}{2}(2a + Ex)(ZF^{ty} + 4WF_{xr}) \\
&= -g^{tt}\Gamma^y_{tt} + g^{yr}\Gamma^t_{rt} - \frac{1}{2}ZF^{yt}A_t - ZF^{yx}A_x - \frac{1}{2}(2a + Ex)(ZF^{ty} + 4WF_{xr}) \\
&= \frac{\mathcal{U}'}{\mathcal{U}}\delta h_{ry} - \frac{1}{2}Z(e^{-4\mathcal{V}}B\xi t + \cdots)a - \frac{1}{2}Z(-e^{-4\mathcal{V}}B(-E + \xi a)t) \\
&\quad - \frac{1}{2}(2a + Ex)[Z(e^{-4\mathcal{V}}B\xi t + \cdots) - 4g_4^2 W(\xi a' t + \delta A'_x)],
\end{aligned} \tag{7.116}$$

where "$\cdots$" represents terms that do not depend on the t coordinate. So,

$$\begin{aligned}
\partial_t(\sqrt{-g}G^{ty}) = &-\frac{1}{2}Ze^{-2\mathcal{V}}B\xi a - \frac{1}{2}Ze^{-2\mathcal{V}}B(-E + \xi a) \\
&-\frac{1}{2}(2a + Ex)(Ze^{-2\mathcal{V}}B\xi + 4g_4^2 W a'\xi).
\end{aligned} \tag{7.117}$$

Collecting our results we obtain the following

$$\begin{aligned}
\partial_r(\sqrt{-g}G^{rx}) &= -\partial_t(\sqrt{-g}G^{tx}) - \partial_y(\sqrt{-g}G^{yx}) \\
&= 0, \\
\partial_r(\sqrt{-g}G^{ry}) &= -\partial_t(\sqrt{-g}G^{ty}) - \partial_x(\sqrt{-g}G^{xy}) \\
&= (2g_4^2 W a' - e^{-2\mathcal{V}}ZB)(E - 2\xi a(r)).
\end{aligned} \tag{7.118}$$

We just showed that the fluxes Q^i are no longer conserved in the presence of perturbations. So, we redefine Q^i as

$$\begin{aligned}
Q^x &= \mathcal{U}^2(\frac{e^{2\mathcal{V}}\delta h_{tx}}{\mathcal{U}})' - a(r)\sqrt{-g}H^{rx}, \\
Q^y &= \mathcal{U}^2(\frac{e^{2\mathcal{V}}\delta h_{ty}}{\mathcal{U}})' - a(r)\sqrt{-g}H^{ry} - ME - 2M_Q\xi,
\end{aligned} \tag{7.119}$$

where M and M_Q are given by equations (7.47) and (7.49). In these expressions, we subtracted off the magnetization currents and now the fluxes Q^i are conserved quantities in the presence of perturbations, i.e.

$$\partial_r Q^i = 0, \quad r \to \infty. \tag{7.120}$$

As these are radially independent quantities, we do the same as we did for the electric fluxes and evaluate them on the horizon. Notice that $M(r)$ and $M_Q(r)$ vanish on the horizon. Using the near horizon expansions (7.29) it is easy to see that the term containing $a(r)$ also vanishes because $a(r) \sim a_+(r - r_+)$. Finally, using $\mathcal{U} \sim 4\pi T(r - r_+)$, we obtain

$$\mathcal{U}^2(\frac{e^{2\mathcal{V}}\delta h_{ti}}{\mathcal{U}})' = \mathcal{U}(e^{2\mathcal{V}}\delta h_{ti})' - \mathcal{U}'e^{2\mathcal{V}}\delta h_{ti}. \tag{7.121}$$

The first term is zero on the horizon, and we remain with

$$Q^i = -\mathcal{U}'e^{2\mathcal{V}}\delta h_{ti}\big|_{r_+}. \tag{7.122}$$

We already computed the graviton perturbations δh_{ti}, which are given by (7.75). Finally, we extract the conductivities using (7.15). The thermoelectric conductivity α confirms the results of the previous section, and are written as

$$\alpha_{xx} = \frac{2\kappa_4^2 g_4^4 s\rho e^{2\mathcal{V}}k^2\Phi}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2\mathcal{V}}k^2\Phi)^2}\bigg|_{r_+}, \tag{7.123}$$

$$\alpha_{xy} = 2\kappa_4^2 sB\frac{2\kappa_4^2 g_4^4\rho^2 + 2\kappa_4^2 B^2 Z^2 + g_4^2 Z e^{2\mathcal{V}}k^2\Phi}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2\mathcal{V}}k^2\Phi)^2}\bigg|_{r_+}. \tag{7.124}$$

We also extract the heat conductivity $\bar{\kappa}$

$$\bar{\kappa}_{xx} = \frac{2\kappa_4^4 g_4^2 s^2 T(2\kappa_4^2 B^2 Z + g_4^2 e^{2\mathcal{V}}k^2\Phi)}{4\kappa_4^6 g_4^4 B^2\rho^2 + \kappa_4^2(2\kappa_4^2 B^2 Z + g_4^2 e^{2\mathcal{V}}k^2\Phi)^2}\bigg|_{r_+}, \tag{7.125}$$

$$\bar{\kappa}_{xy} = \frac{4\kappa_4^4 g_4^4 s^2 T\rho B}{4\kappa_4^4 g_4^4 B^2\rho^2 + (2\kappa_4^2 B^2 Z + g_4^2 e^{2\mathcal{V}}k^2\Phi)^2}\bigg|_{r_+}. \tag{7.126}$$

The formulae (7.76), (7.77), (7.78), (7.79), (7.125) and (7.126) are the main results of this chapter. We would like to stress that, in the context of holography, the physical observables are always computed at the boundary of the AdS spacetime. Due to the fact that we are obtaining physical observables from radially independent quantities, we can evaluate our expressions on any hypersurface in the bulk, and we chose it to be the horizon of the black hole. So, the conductivities can be computed analytically by knowing the fields and the metric evaluated on the horizon of the black hole, i.e. by knowing the horizon data.

Chapter 8

Conductivities from attractors

As we saw in the previous chapter, holography provides a simple way of computing conductivities in models for condensed matter systems. Applying electric fields and thermal gradients induces linear perturbations about the black hole, and the matrix of thermoelectric conductivities is obtained by solving the linearized perturbation equations. Although the formalism of the previous chapter involves a large set of theories one can consider, the most commonly studied case in the literature deals with electrically charged anti-de Sitter-Reissner-Nordström black holes [81], where the electric charge is the dual of the chemical potential of the field theory. In the presence of chemical potential and magnetic field on the CFT side, the gravity dual contains a dyonic charged black hole, i.e. a black hole with both electric and magnetic charges. The simplest example is the dyonic anti-de Sitter-Reissner-Nordström planar black hole, for which the electric (σ), thermoelectric ($\alpha, \bar{\alpha}$) and heat conductivities ($\bar{\kappa}$) are given by

$$\begin{aligned}
\sigma^{ij} &= \frac{\rho}{B}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\alpha^{ij} &= \overline{\alpha}^{ij} = \frac{s}{B}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\overline{\kappa}^{ij} &= \frac{s^2 T g_4^2}{B(\rho^2 g_4^4 + B^2)}\begin{pmatrix} B & \rho g_4^2 \\ -\rho g_4^2 & B \end{pmatrix}.
\end{aligned} \tag{8.1}$$

Here, ρ is the charge density, B the magnetic field, g_4 the constant coupling to the field strength and s the entropy density. These conductivities were computed in [82, 83]. One very important point for this chapter is that these conductivities are valid for all temperatures.

As the full dyonic anti-de Sitter-Reissner-Nordström solution is known analytically, the conductivities (8.1) can be written in terms of the charges of the black hole. In general, the full analytical gravity solutions for more elaborate supergravity theories are not known, and writing these conductivities explicitly in terms of the charges of the black hole at finite temperature is not possible.

We saw that it is possible to obtain the horizon data for extremal black holes using Sen's entropy function method, discussed in chapter (5). The advantage of using this method relies on the fact that we do not need to obtain the full extremal black hole solution in order to obtain such data: All we need to do is to solve the attractor equations. In this chapter, we give a prescription to compute these conductivities explicitly at zero temperature via Sen's entropy function method even for theories whose full dyonic black hole solution is not known. In order to do so, we combine the horizon data, which is obtained for Einstein-Maxwell-dilaton theories, with the formulae for the thermoelectric conductivities (7.76), (7.77), (7.78), (7.79), (7.125) and (7.126). We first present the results for which the topological term is absent, i.e. we make $W(\phi) = 0$ in (7.17). The results for this case were all presented in reference [22].

The results that we obtain for the horizon data are written not only in terms of the charges of the black holes but also in terms of the coupling constants of the theory. These explicit results allow us to analyze the behavior of the conductivities in field theory dual in terms of the rank of the gauge group N, of the magnetic field applied and of the charge density. The Hall conductivity is universal, and is the same as the case of AdS-Reissner-Nordström, i.e. it reads $\sigma_{xy} = \rho/B$. When the potential is just a cosmological constant, we show that the thermoelectric conductivity at zero temperature is given by

$$\alpha_{xy} = \frac{1}{3\gamma}\sqrt{\frac{\mathcal{Q}}{3B}}\, N^{3/2}\,, \tag{8.2}$$

with $Q \equiv g_4^2\rho$ the normalized charge density of the black hole and γ a dimensionless constant which fixes the relation between metric and gauge field from supergravity. g_4 is the gauge theory coupling in the gravity action, which is chosen such that the Einstein-Hilbert and gauge kinetic terms have the same scaling with N. The constant potential model is used as a prototype for more involved dilaton theories. In particular, we consider theories with exponential coupling to the field strengths and exponential scalar potentials, and a theory with quadratic dilaton expansion in both the coupling to the field strengths and the potential. We calculate the electric and thermoelectric conductivities for these theories as well. For all these theories, we also compute the ratio between the heat conductivities and the temperature under the assumption that this is a finite quantity at $T = 0$. Surprisingly, $\kappa_{xx}/T = \kappa_{xy}/T$ for the constant scalar potential model. Moreover, the scaling with N is $\kappa_{xx}/T = \kappa_{xy}/T \sim N^{3/2}$, similarly to the other conductivities.

One physical motivation to compute holographic conductivities at $T = 0$ is related to the topic of **quantum phase transitions** in condensed matter. At temperatures close to zero, some condensed matter systems may present different phases which are accessed by varying parameters that are not related to the temperature, for instance the pressure, chemical potential, or magnetic fields. On the gravity side, by varying parameters such as chemical potential, magnetic fields, or the value of the constant coupling to the scalar potential, one might have different black hole phases, or phases for which the black hole does not even exist, and these are the different phases that might have a dual conformal field theory interpretation. Since we are dealing with bottom-up models, we do not have a map between the gravity theory we study and the conformal field theory describing a condensed matter system. So, how the different phases appearing on the gravity side are related to the different phases of some materials at temperatures close to zero will be left for future investigation. The explicit computation of the conductivities done in this section is just the first step towards this direction.

8.1 Entropy function for black holes with planar horizons

In this section we adapt the Sen's entropy function method, discussed in chapter 5, for black holes with planar horizons.

The near-horizon geometry of an extremal dyonic black hole in four dimensions is $\mathrm{AdS}_2 \times S^2$ [84], and in the planar limit this becomes $\mathrm{AdS}_2 \times \mathbb{R}^2$. So, the starting point of the formalism is to consider that the near-horizon geometry of the planar black hole is $\mathrm{AdS}_2 \times \mathbb{R}^2$, whose general form is written as

$$ds^2 = v\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + wd\vec{x}^2, \tag{8.3}$$

where the constants* v and w are the AdS_2 radius and the $\mathbb{R}^2$ radius†, respectively. The curvature associated to the near-horizon metric (8.3) is

$$R = -\frac{2}{v}. \tag{8.4}$$

The scalar and vector fields are constants for this geometry and are written as

$$\phi_s = u_s, \quad F_{rt}^{(A)} = e_A, \quad F_{\theta\phi}^{(A)} = B_A, \tag{8.5}$$

where e_A and B_A are related to the integrals of the magnetic and electric fluxes, which are in turn related to the electric and magnetic charges, respectively. The attractor mechanism states that the value of the scalars on the horizon of the extremal black hole is independent of any asymptotic condition at infinity. This value is completely determined by the electric and magnetic charges of the black hole. The function $f(u_s, v, w, e_A, p_A)$ is defined as the Lagrangian density $\sqrt{-\det g}\mathcal{L}$ evaluated for the near-horizon geometry (8.3) and integrated over the planar horizon variables [10], [18],

$$f(u_s, v_i, e_A, p_A) = \int dxdy\sqrt{-\det g}\mathcal{L}. \tag{8.6}$$

*In this section we use constants v and w instead of v_1 and v_2.

†The volume of $\mathbb{R}^2$ is infinite, but we will only deal with finite densities in the whole paper.

We extremize this function with respect to u_s, v, w and e_A by

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v} = 0, \quad \frac{\partial f}{\partial w} = 0, \quad \frac{\partial f}{\partial e_A} = Q^A, \tag{8.7}$$

where the first equation is the equation of motion for the scalar, and the second and third are the equations of motion for the metric. Next, one defines the entropy function

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\pi[e_A Q^A - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})]. \tag{8.8}$$

The equations that extremize the entropy function are

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v} = 0, \quad \frac{\partial \mathcal{E}}{\partial w} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_A} = 0 \ , \tag{8.9}$$

and are called the attractor equations. At the extremum, this new function equals the entropy of the black hole

$$S_{BH} = \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}). \tag{8.10}$$

In the context of planar black holes the horizon has infinite area, so we consider the entropy density, since this is a finite quantity. The solutions of the equations (8.9) are the near-horizon data that will be used later to compute the conductivities.

The attractor mechanism is independent of supersymmetry, and relies only on the near horizon geometry [10], which is $\mathrm{AdS}_2 \times \mathbb{R}^2$ in our case. In the study of spherical black holes with $AdS_2 \times S^2$ near-horizon geometry, the attractor mechanism is present even after the inclusion of α' corrections [10][85][86][87][88][89]. It is now understood that the long throat of AdS_2 is the basis of the attractor phenomenon [10][90][91]. Since the $\mathrm{AdS}_2 \times \mathbb{R}^2$ near-horizon geometry is the starting point of the construction of Sen's entropy function, finding solutions to the attractor equations guarantees that the attractor mechanism exist for the theories analyzed.

8.2 Dyonic anti-de Sitter-Reissner-Nordström planar black hole

In this section we give one example of application of the entropy function. We will compute the near horizon data for one known solution, the dyonic anti-de Sitter-

Reissner-Nordström black hole. Then, we will show that we can obtain the same results using Sen's method. This is just an illustrative example that the method works also for planar black holes. We consider the Einstein-Maxwell theory, which contains the metric and a gauge field. Since we want to study solutions in AdS, the theory must contain a constant potential. The action in the notation of reference [81] is written as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa_4^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4g_4^2}F_{\mu\nu}F^{\mu\nu}\right]. \tag{8.11}$$

Here, L is the AdS_4 radius, and $2\kappa_4^2 = 16\pi G_N$. The gauge coupling g_4 is assumed to scale as $\sim N^{-3/4}$. Although we will not give an explicit derivation, we point out that the relation between the gravity coupling κ^2, the AdS_4 radius, and the rank N of the gauge group of the field theory is given by [82]

$$\frac{2L^2}{\kappa_4^2} = \frac{\sqrt{2}N^{3/2}}{6\pi}. \tag{8.12}$$

The dyonic anti-de Sitter-Reissner-Nordström black hole is the solution to this theory, and it is written as

$$ds^2 = \frac{L^2}{u^2}\left(-f(u)dt^2 + \frac{du^2}{f(u)}\right) + \frac{L^2}{u^2}(dx^2 + dy^2), \tag{8.13}$$

$$f(u) = 1 - \left(1 + \frac{u_+^2\mu^2 + u_+^4 B^2}{\gamma^2}\right)\left(\frac{u}{u_+}\right)^3 + \left(\frac{u_+^2\mu^2 + u_+^4 B^2}{\gamma^2}\right)\left(\frac{u}{u_+}\right)^4, \tag{8.14}$$

$$F = \frac{\mu}{u_+}du \wedge dt + B dx \wedge dy. \tag{8.15}$$

Notice that the horizon of the black hole is located at u_+, and the asymptotic region is achieved when $u \to \infty$. B is the magnetic field and μ is the chemical potential. The temperature and the constant γ^2 are given by

$$T = \frac{1}{4\pi u_+}\left(3 - \frac{u_+^2\mu^2 + u_+^4 B^2}{\gamma^2}\right), \quad \gamma^2 = \frac{2g_4^2 L^2}{\kappa_4^2}. \tag{8.16}$$

Notice that, as $g_4 \sim N^{-3/4}$, γ is independent of N. The black hole becomes extremal when it achieves zero temperature. This happens when

$$\frac{u_+^2\mu^2 + u_+^4 B^2}{\gamma^2} = 3. \tag{8.17}$$

The charge density ρ can be computed using (7.32) and (7.33), which gives

$$\rho = \frac{\sqrt{-g}}{g_4^2} F^{tr}\Big|_{\text{bdry}}, \tag{8.18}$$

where the "bdry" subscript denotes that the expression should evaluated at the boundary. Substituting the field strength, we obtain (see [81])

$$\rho = \frac{2L^2}{\kappa_4^2} \frac{\mu}{u_+ \gamma^2}. \tag{8.19}$$

As we intend to make a comparison between the quantities computed from the full extremal solution with the ones obtained via the entropy function, we write all of them in terms of the charge density, since this is the quantity that appears in the entropy function. So, using (8.17), the value of u_+ for the extremal black hole, is given by the relation

$$\frac{1}{u_+^2} = \sqrt{\frac{1}{3\gamma^2} (B^2 + g_4^4 \rho^2)}. \tag{8.20}$$

In order to see how the $\mathrm{AdS}_2 \times \mathbb{R}^2$ near-horizon geometry arises from the full extremal solution, we Taylor expand the function $f(u)$ around the horizon:

$$f(u) \approx f(u_+) + (u - u_+) f'(u_+) + \frac{(u - u_+)^2}{2} f''(u_+), \tag{8.21}$$

where the primes define derivatives with respect to u. The first term in the expansion is zero due to the definition of the horizon of the black hole, and the second one is zero due to the fact that we consider extremal black holes, as can be seen from the definition of temperature (2.42). So, for the extremal black hole

$$f''(u_+) = \frac{12}{u_+^2}. \tag{8.22}$$

The metric (8.13) becomes

$$ds^2 \approx \frac{L^2}{u_+^2} \left(-\frac{6}{u_+^2} (u - u_+)^2 dt^2 + \frac{u_+^2}{6} \frac{du^2}{(u - u_+)^2} \right) + \frac{L^2}{u_+^2} (dx^2 + dy^2). \tag{8.23}$$

Defining new coordinates as

$$r \equiv u - u_+, \;\; \tau \equiv \frac{6}{u_+^2} t, \tag{8.24}$$

and the metric and gauge field will be

$$ds^2 \approx \frac{L^2}{6}\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \frac{L^2}{u_+^2}(dx^2 + dy^2), \tag{8.25}$$

$$F = \frac{\rho g_4^2 u_+^2}{6} dr \wedge d\tau + B dx \wedge dy. \tag{8.26}$$

This just shows that the anti-de Sitter-Reissner-Nordström planar black hole has $\mathrm{AdS}_2 \times \mathbb{R}^2$ near-horizon geometry.

We now compute the near-horizon data using Sen's entropy function formalism. We define the near-horizon metric and the gauge field as

$$ds^2 = v\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + w(dx^2 + dy^2), \tag{8.27}$$

$$F = e\, dr \wedge d\tau + B dx \wedge dy. \tag{8.28}$$

The Lagrangian for this background reads

$$\sqrt{-g}\mathcal{L} = \frac{1}{\kappa_4^2}\left(-w + \frac{3}{L^2} v w\right) + \frac{w}{2 g_4^2 v} e^2 - \frac{v}{2 g_4^2 w} B^2, \tag{8.29}$$

and the entropy function is just

$$\mathcal{E} = 2\pi[eQ - \int dx dy \sqrt{-g}\mathcal{L}]. \tag{8.30}$$

Computing derivatives with respect to the fields we obtain the attractor equations

$$e = g_4^2 \frac{v}{w} \tilde{Q}, \tag{8.31}$$

$$\frac{w}{v^2} e^2 + \frac{1}{w} B^2 - \frac{6 g_4^2}{\kappa_4^2 L^2} w = 0, \tag{8.32}$$

$$\frac{2 g_4^2}{\kappa_4^2} - \frac{e^2}{v} - \frac{v}{w^2} B^2 - \frac{6 g_4^2}{\kappa_4^2 L^2} v = 0, \tag{8.33}$$

where $\tilde{Q} = Q/\mathrm{Vol}\,\mathbb{R}^2$. We can easily solve this system and obtain the solution

$$e = \frac{g_4^2 \tilde{Q}}{2}\sqrt{\frac{\gamma^2}{3(g_4^4 \tilde{Q}^2 + B^2)}},\ v = \frac{L^2}{6},\ w = L^2 \sqrt{\frac{1}{3\gamma^2}(g_4^4 \tilde{Q}^2 + B^2)}, \tag{8.34}$$

where the definition of γ^2 is given in (8.16). The constant $\tilde{Q}$ is a parameter in Sen's entropy function method that is proportional to the charge density ρ. In order to obtain the same near-horizon we make the identification

$$\rho \equiv \tilde{Q}. \tag{8.35}$$

Comparing the above results with (8.25) and (8.26), we see that we have obtained exactly the same near-horizon metric and gauge fields via the entropy function, as we wanted to do.

8.3 DC conductivities at $T = 0$ for Einstein-Maxwell-Dilaton theories

Now, we apply the Sen's entropy function for the Einstein-Maxwell-dilaton theory with the topological term and general couplings $Z(\phi)$ and $W(\phi)$, and potential $V(\phi)$. We must take $\Phi(\phi) = 0$ in (7.17), since the scalars χ_i depend on the horizon coordinates and are not attracted to a fixed point on the horizon. The four-dimensional Einstein-Maxwell-dilaton theory we consider is then

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_N}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) - \frac{Z(\phi)}{4g_4^2}F_{\mu\nu}F^{\mu\nu} - W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{8.36}$$

Using (8.3), (8.4) and (8.5), we compute the Lagrangian in the near-horizon region, i.e.

$$\sqrt{-g}\mathcal{L} = \frac{1}{16\pi G_N}\left(-2w - wvV(u_D)\right) + \frac{Z(u_D)}{2g_4^2}\left(\frac{w}{v}e^2 - \frac{v}{w}B^2\right) + 4W(u_D)eB, \tag{8.37}$$

where u_D is the value of the dilaton field on the horizon. The entropy function (8.8) is then

$$\mathcal{E} = 2\pi[e_A Q^A - \text{Vol }\mathbb{R}^2\sqrt{-g}\mathcal{L}]. \tag{8.38}$$

The attractor equations for this system are

$$\frac{Q}{\text{Vol }\mathbb{R}^2} - \frac{Z(u_D)}{g_4^2}\frac{w}{v}e - 4W(u_D)B = 0, \tag{8.39}$$

$$\frac{Z(u_D)}{2g_4^2}\left(\frac{w}{v^2}e^2+\frac{B^2}{w}\right)+\frac{w}{16\pi G_N}V(u_D)=0, \tag{8.40}$$

$$\frac{2}{16\pi G_N}-\frac{Z(u_D)}{2g_4^2}\left(\frac{1}{v}e^2+\frac{v}{w^2}B^2\right)+\frac{v}{16\pi G_N}V(u_D)=0, \tag{8.41}$$

$$-\frac{1}{2g_4^2}\frac{\partial Z(u_D)}{\partial u_D}\left(\frac{w}{v}e^2-\frac{v}{w}B^2\right)-4\frac{\partial W(u_D)}{\partial u_D}eB+\frac{wv}{16\pi G_N}\frac{\partial V(u_D)}{\partial u_D}=0. \tag{8.42}$$

Using (8.39) we eliminate Q from (8.38), and obtain

$$\mathcal{E}=2\pi\mathrm{Vol}\,\mathbb{R}^2\left[\frac{1}{(16\pi G_N)}\left(2w+wvV(u_D)+\frac{Z(u_D)}{2g_4^2}\left(\frac{w}{v}e^2+\frac{v}{w}B^2\right)\right]\right.. \tag{8.43}$$

We combine equations (8.40) and (8.41) and obtain

$$V(u_D)=-\frac{1}{v}, \tag{8.44}$$

$$\frac{Z(u_D)}{2g_4^2}\left(\frac{e^2}{v^2}+\frac{B^2}{w^2}\right)=\frac{1}{(16\pi G_N)}\frac{1}{v}, \tag{8.45}$$

and replacing these in (8.38), we obtain

$$\mathcal{E}=\frac{4\pi w\mathrm{Vol}\,\mathbb{R}^2}{16\pi G_N}=\frac{w\mathrm{Vol}\,\mathbb{R}^2}{4G_N}=\frac{A}{4G_N}. \tag{8.46}$$

This is the expected Hawking formula for the entropy of the black hole. The attractor equations (8.39), (8.40), (8.41) and (8.42) are general in the sense that we did not fix the model yet by choosing specific functional forms for the functions $Z(\phi)$, $W(\phi)$ and $V(\phi)$. After fixing these functions, we solve the system and obtain the near-horizon data. We wish to insert the near-horizon into the expressions for the conductivities, given by equations (7.76), (7.77), (7.78), (7.79), (7.125) and (7.126), and then write the conductivities at zero temperature explicitly in terms of the parameters of the black hole. But, in order to do so, we must first establish a map between the metric elements and fields appearing in Sen's formalism with the metric and fields appearing in the conductivity formulae, since they appear with different notation in each case. Take for instance the ratio e/v. Both the electric field e and the AdS_2 radius v appears in Sen's formalism. We will show that this ratio is related to the quantity $a'(r)$ appearing in the expressions for the conductivity through equation (8.57).

The field strength for the theory (8.36) is

$$F = a'(r)dr \wedge dt + Bdx \wedge dy, \tag{8.47}$$

which is valid anywhere in the bulk. Consider now the field strength and the metric in the near-horizon region. The function $\mathcal{U}(r)$ appearing in (7.21) can be Taylor expanded around the horizon of the black hole as

$$\mathcal{U}(r) \approx \mathcal{U}(r_H) + (r - r_H)\mathcal{U}'(r_H) + \frac{(r - r_H)^2}{2}\mathcal{U}''(r_H) + \mathcal{O}(r^3). \tag{8.48}$$

Again, the first term in this expansion vanishes at the horizon by definition, and the linear term vanishes for extremal black holes. So in the near-horizon region the metric is written as

$$ds^2 = -\frac{(r - r_H)^2}{2}\mathcal{U}''(r_H)dt^2 + \frac{2}{(r - r_H)^2\mathcal{U}''(r_H)}dr^2 + e^{2\mathcal{V}(r_H)}(dx^2 + dy^2). \tag{8.49}$$

In order to see how the $\mathrm{AdS}_2 \times \mathbb{R}^2$ geometry emerges, we need to choose an appropriate coordinate system. For our case this is

$$r - r_H \to \tilde{\rho}, \;\; t \to \frac{2\tau}{\mathcal{U}''(r_H)}, \tag{8.50}$$

so that

$$ds^2 = \frac{2}{\mathcal{U}''(r_H)}\left(-\tilde{\rho}^2 d\tau^2 + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2}\right) + e^{2\mathcal{V}(r_H)}(dx^2 + dy^2), \tag{8.51}$$

The metric has $\mathrm{AdS}_2 \times \mathbb{R}^2$ as its near horizon geometry, as expected. A direct comparison with (8.3) shows that the term multiplying the AdS_2 part of this metric is identified with v, and the term multiplying the $\mathbb{R}^2$ part is identified with w. Also, under the change of coordinates (8.50), the field strength changes to

$$F = \frac{2a'(r_H)}{\mathcal{U}''(r_H)}d\tilde{\rho} \wedge d\tau + Bdx \wedge dy. \tag{8.52}$$

We compare with (8.5) and conclude that the $(\tilde{\rho}\tau)$ component of the field strength is identified with e, and the angular part containing the magnetic field is the same. This provides us with the quantities that map the horizon data obtained via Sen's entropy function to the quantities appearing in the expressions (7.76), (7.77), (7.78), (7.79), (7.125), (7.126). They are written as

$$v = \frac{2}{\mathcal{U}''(r_H)}, \quad w = e^{2\mathcal{V}(r_H)}, \quad e = \frac{2a'(r_H)}{\mathcal{U}''(r_H)} = va'(r_H). \tag{8.53}$$

Notice that the gauge current is computed as

$$\langle J^\mu \rangle = \frac{\delta S_{\text{on-shell}}}{\delta A_\mu}\bigg|_{\text{boundary}} = \sqrt{-g}\left(\frac{Z(\phi)}{g_4^2}F^{\mu\nu} + 4W(\phi)\tilde{F}^{\mu\nu}\right). \tag{8.54}$$

This means that the boundary charge density for the theory (8.36) is given by the horizon expression[‡]

$$\rho = \frac{Z(u_D)wa'(r_H)}{g_4^2} + 4W(u_D)B\bigg|_{r_+}. \tag{8.55}$$

The entropy density is given by the Hawking formula

$$s = \frac{4\pi w}{16\pi G_N}\bigg|_{r_+}. \tag{8.56}$$

Notice that v and w in these expressions come from the metric elements. Using the identification (8.53), we see that

$$\frac{e}{v} = a'(r_H), \tag{8.57}$$

so, by replacing this in the charge density and using the attractor equation (8.39), the charge density is related to the quantity $\tilde{Q}$ in Sen's entropy function as

$$\rho = \tilde{Q}. \tag{8.58}$$

Now that we have a map between the quantities related to the entropy function and those appearing in the formulae for the conductivities, we can now fix our theory and compute analytically the conductivities at $T = 0$ in terms of the charges of the black hole. In order to do so, we first write the formulae for the conductivities (7.76), (7.77), (7.78), (7.79), (7.125) and (7.126) for the case when $\Phi(\phi) = 0$, i.e.

$$\sigma_{xx} = 0, \tag{8.59}$$

[‡]Notice that $A_t = a(r)$ vanishes at the horizon but $a'(r)$ does not.

$$\sigma_{xy} = \frac{\rho}{B}, \tag{8.60}$$

$$\alpha_{xx} = 0, \tag{8.61}$$

$$\alpha_{xy} = \frac{s}{B}, \tag{8.62}$$

$$\frac{\bar{\kappa}_{xx}}{T} = \frac{s^2 Z}{g_4^2 \left(\rho^2 + \frac{B^2 Z^2}{g_4^4}\right)}, \tag{8.63}$$

$$\frac{\bar{\kappa}_{xy}}{T} = \frac{\rho}{B} \frac{s^2}{\left(\rho^2 + \frac{B^2 Z^2}{g_4^4}\right)}. \tag{8.64}$$

Notice that we have written the ratio $\bar{\kappa}/T$, which is shown to be a finite quantity for our set up. In reference [92], a bound on κ/T was derived[§]. In that paper, it was argued that the heat conductivity is always non-zero at finite temperature, so long as the dilaton potential is bounded from below. Our approach shows that this is also the case even for $T \to 0$.

With these results, we are now able to compute conductivities explicitly for different theories. We will analyse several models separately. First we will write the solution to the attractor equations and then we combine with the expressions for the conductivities (8.59), (8.60), (8.61), (8.62), (8.63) and (8.64). At the moment of the writing of this thesis, we have obtained the results for the attractor equations only for the cases when the topological term in the action (7.17) is absent, i.e.

$$W(\phi) = 0. \tag{8.65}$$

The analysis of models in which this term is present is the topic of future work.

8.4 Massless scalar

In this model, we choose the scalar potential to be[¶]

$$V(\phi) = -\frac{6}{L^2}. \tag{8.66}$$

[§] Their result was derived in absence of topological terms in the action.

[¶] We wrote $V(\phi)$ only to keep notation. The potential is a constant and does not really depend on the dilaton field ϕ.

Here, L is the AdS_4 scale, and we aim to use equation (8.12) in order to have a map between the number of colors in the gauge theory N and the constants in the gravity theory. Notice that the Einstein-Maxwell-dilaton theory with this constant potential satisfies the requirement given by equations (7.20), so this is dual to a three-dimensional conformal field theory. In this section, we solve the attractor equations for this theory and then write the conductivities explicitly in terms of the black hole parameters for the extremal case. This is the simplest potential one can consider, but, to the best of our knowledge, a full dyonic black hole solution for this theory is not known, so writing explicitly the conductivities in terms of the black hole parameters at finite temperature is not possible. We use equation (8.12) and write the conductivities in terms of the rank of the gauge group N.

The solution to the attractor equations is

$$Z(u_D) = g_4^2 \frac{\tilde{Q}}{B}, \quad e = \sqrt{\frac{L^2 B}{6(16\pi G_N)\tilde{Q}}}, \quad v = \frac{L^2}{6}, \quad w = \sqrt{\frac{L^2(16\pi G_N)\tilde{Q}B}{6}}. \tag{8.67}$$

The solution is independent of the functional form of the coupling $Z(u_D)$. The entropy density is just

$$s = 4\pi\sqrt{\frac{L^2}{6}\frac{\tilde{Q}B}{(16\pi G_N)}} = \frac{1}{3\gamma}\sqrt{\frac{\mathcal{Q}B}{3}}\, N^{3/2}\,. \tag{8.68}$$

Here, we defined

$$\mathcal{Q} \equiv g_4^2 \rho \tag{8.69}$$

as the normalized charge density of the black hole. As we mentioned, we used equation (8.12) to rewrite s explicitly in terms of the rank N of the gauge group. Notice that $\mathcal{Q}$ is independent of N, indeed we have

$$\rho = \tilde{Q} = \frac{\mathcal{Q}}{g_4^2} = \mathcal{Q}\frac{2L^2}{\gamma^2\kappa_4^2} = \frac{\sqrt{2}\mathcal{Q}}{6\pi\gamma^2}\, N^{3/2}\,. \tag{8.70}$$

Replacing these results in (8.59), (8.60), (8.61), (8.62), (8.63) and (8.64) we write the non-zero conductivities as

$$\sigma_{xy} = \frac{\tilde{Q}}{B} = \frac{\sqrt{2}\mathcal{Q}}{6\pi\gamma^2 B}\, N^{3/2}, \tag{8.71}$$

$$\alpha_{xy} = 4\pi\sqrt{\frac{\tilde{Q}}{6B}\frac{L^2}{16\pi G_N}} = \frac{1}{3\gamma}\sqrt{\frac{\mathcal{Q}}{3B}}\,N^{3/2}, \tag{8.72}$$

$$\frac{\bar{\kappa}_{xx}}{T} = \frac{\bar{\kappa}_{xy}}{T} = \frac{(4\pi)^2}{12}\frac{L^2}{(16\pi G)} = \frac{\pi}{9\sqrt{2}}N^{3/2}. \tag{8.73}$$

Notice that the values for the ratios $\bar{\kappa}_{xx}/T$ and $\bar{\kappa}_{xy}/T$ coincide at $T = 0$.

The result (8.71) is a general result for Einstein-Maxwell-dilaton theories. The thermoelectric conductivity (8.72) and the ratios (8.73) for constant potential and at zero temperature were obtained for the first time in reference [22]. We note that all conductivities scale as $N^{3/2}$, as is generically expected for theories in 2+1 dimensions [82]. Unlike the electric and thermoelectric conductivities (8.71) and (8.72), the ratios (8.73) do not depend on the electric and magnetic charges of the black hole.

If we set the dilaton to zero in the Einstein-Maxwell-dilaton theory, then these conductivities should reduce to the ones computed for the AdS-RN black hole (8.1), since the potential used here is the same potential as in section 8.2. In supergravity theories, $Z(\phi)$ is generally an exponential of the type $Z(\phi) = e^{\tilde{\gamma}\phi}$ for an arbitrary constant $\tilde{\gamma}$, so it reduces to $Z(0) = 1$ if the dilaton is set to zero. We make the comparison with section 8.2 in order to have a consistency check of our new results. Notice that in (8.1) the entropy density is written in terms of the charges, so we first need to express explicitly the conductivities of the extremal AdS-RN case in terms of the charges too. For $W(\phi) = 0$, (8.55) and (8.56) give the corresponding expressions for s and ρ, which can be evaluated for the horizon data of (8.34), leading to the results

$$\sigma_{xy} = \frac{\tilde{Q}}{B} = \frac{\sqrt{2}\mathcal{Q}}{6\pi\gamma^2 B}\,N^{3/2}, \tag{8.74}$$

$$\alpha_{xy} = \frac{s}{B} = \frac{4\pi}{B}\frac{L^2}{2\kappa^2}\sqrt{\frac{1}{3\gamma^2}\,(B^2 + g_4^4\rho^2)} = \frac{N^{3/2}}{3B\sqrt{2}\gamma}\sqrt{\frac{1}{3}\,(B^2 + \mathcal{Q}^2)}. \tag{8.75}$$

On the other hand, if we set $\phi = 0$, then the first equation in (8.67) implies the constraint

$$\mathcal{Q} = B, \tag{8.76}$$

which can be inserted into (8.74) and (8.75) to give

$$\sigma_{xy} = \frac{\sqrt{2}}{6\pi\gamma^2} N^{3/2}, \;\; \alpha_{xy} = \frac{N^{3/2}}{3\sqrt{3}\gamma}. \tag{8.77}$$

Indeed, the same results are obtained by inserting (8.76) into the results of this section, i.e. (8.71) and (8.72). In general terms, all the results obtained for the Einstein-Maxwell-dilaton theory reduces to Einstein-Maxwell theory, whose black hole solution is the anti-de Sitter-Reissner-Nordström solution, with an extra constraint on the charges given by (8.76). Notice that this contraint results from the dilaton equation of motion in the $\phi = 0$ case. From this analysis, it is natural to assume that the conductivities of the Einstein-Maxwell-dilaton theory scales with the same powers of N as in the case of the conductivities of Einstein-Maxwell theory.

8.5 Exponential couplings

This model is defined by[‖]

$$Z(\phi) = e^{\gamma\phi}, \;\; V(\phi) = 2\beta e^{-\delta\phi}. \tag{8.78}$$

A potential written as an exponential of the dilaton is called Liouville potential. Again, we deal with a model whose analytical dyonic black hole solutions are unknown. There are black hole solutions with cylindrical symmetry for this theory presented in [93]. This theory was discussed numerically on page 23 in [94] for electrically charged black holes. But notice that in the same reference, the analysis done so far for black holes with spherical horizons were all numerical. In other words, there are no analytical computation for the conductivities of these theories. In the same way as we did in the previous section, we will express the conductivities analytically for the zero temperature case as well.

As $V'(0) \neq 0$, equations (7.20) are not satisfied, showing that this system does not admit a conformal field theory dual. The motivation for considering these

[‖]The parameter γ here is an arbitrary constant. This must not be confused with the parameter γ of equation (8.16), which will not appear in the remaining part of this chapter.

models comes from interesting examples of top-down theories that are very difficult to treat analytically, even with the present approach. For instance, $V(\phi) = -\frac{6}{L^2}\cosh(\phi/\sqrt{3})$, $Z(\phi) = 1/\cosh(\phi\sqrt{3})$ is a string-theory inspired model whose attractor equations cannot be solved analytically due to the high powers of the variables in the algebraic equations. The potential (8.78) is used as an approximation of this potential, provided that the value of scalar field on the horizon is large. So the conductivities we compute here are good approximations once this condition is satisfied.

Notice that γ and δ are parameters defining the theory, and the constant β must be related to the AdS_4 radius**. As discussed in [93, 94], the case $\gamma\delta = 1$ is of special interest since the associated models arise within string theory. We first write the formulae which are valid for general values of γ and δ. Using (8.39) and (8.42) we have

$$\frac{e}{v} = \frac{g_4^2\tilde{Q}}{we^{\gamma u_D}}, \tag{8.79}$$

$$\frac{e^2}{v^2} = \frac{B^2}{w^2} - \frac{4\beta\delta}{\gamma}\frac{g_4^2}{(16\pi G_N)}e^{-(\delta+\gamma)u_D}. \tag{8.80}$$

Replacing the last equation in (8.40) we have

$$\frac{B^2}{w^2} = -2\beta\frac{g_4^2}{(16\pi G_N)}\left(1-\frac{\delta}{\gamma}\right)e^{-(\delta+\gamma)u_D}. \tag{8.81}$$

Combining these three equations, we obtain as solution to the attractor equations

$$e^{u_D} = \left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma-\delta)}{(\gamma+\delta)}\right]^{\frac{1}{2\gamma}}, \tag{8.82}$$

$$e = \frac{g_4^3\tilde{Q}}{B}\sqrt{\frac{(\gamma-\delta)}{-2\beta(16\pi G)\gamma}}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma-\delta)}{(\gamma+\delta)}\right]^{\frac{\delta-3\gamma}{4\gamma}}, \tag{8.83}$$

$$v = -\frac{1}{2\beta}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma-\delta)}{(\gamma+\delta)}\right]^{\frac{\delta}{2\gamma}}, \tag{8.84}$$

**This was argued in reference [22]

$$w = B\sqrt{\frac{(16\pi G)\gamma}{-2\beta g_4^2\,(\gamma-\delta)}}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma-\delta)}{(\gamma+\delta)}\right]^{\frac{\delta+\gamma}{4\gamma}}. \tag{8.85}$$

The entropy density is given by

$$s = 4\pi B\sqrt{\frac{\gamma}{-2\beta g_4^2(16\pi G)\,(\gamma-\delta)}}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma-\delta)}{(\gamma+\delta)}\right]^{\frac{\delta+\gamma}{4\gamma}}. \tag{8.86}$$

As stated before, we are interested in the cases for which $\delta = 1/\gamma$. So, the non-zero conductivities are given by

$$\sigma_{xy} = \frac{\tilde{Q}}{B}, \tag{8.87}$$

$$\alpha_{xy} = \frac{4\pi\gamma}{\sqrt{-2\beta g_4^2(16\pi G)\,(\gamma^2-1)}}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma^2-1)}{(\gamma^2+1)}\right]^{\frac{\gamma^2+1}{4\gamma^2}} \tag{8.88}$$

$$\frac{\bar{\kappa}_{xx}}{T} = \frac{s^2}{2\rho B}\frac{(\gamma^2+1)}{\gamma^2}, \tag{8.89}$$

$$\frac{\bar{\kappa}_{xy}}{T} = \frac{s^2}{2\rho B}\frac{\sqrt{(\gamma^2+1)(\gamma^2-1)}}{\gamma^2}. \tag{8.90}$$

Notice that we expressed the results in terms of the entropy density squared, which is given by

$$s^2 = \frac{(4\pi)^2B^2\gamma^2}{-2\beta g_4^2(16\pi G)\,(\gamma^2-1)}\left[\frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\gamma^2-1)}{(\gamma^2+1)}\right]^{\frac{\gamma^2+1}{2\gamma^2}}, \tag{8.91}$$

and it is non-zero for $\gamma = 1,\ \sqrt{3}$. Unlike the massless scalar of the previous section, the ratios $\bar{\kappa}_{xx}/T$ and $\bar{\kappa}_{xy}/T$ for this model are not the same at $T = 0$.

The cases $\gamma = \sqrt{3}$ and $\gamma = 1$ are of special interest [94],: $\gamma = 1$ arises from string theory in four dimensions with a vector arising from the Neveu-Schwarz sector, while $\gamma = \sqrt{3}$ arises from a Kaluza-Klein reductions. The case for which $\gamma = \sqrt{3}$ gives the following conductivities

$$\sigma_{xy} = \frac{\tilde{Q}}{B}, \tag{8.92}$$

$$\alpha_{xy} = \frac{2\pi}{\sqrt{(16\pi G)}}\sqrt{-\frac{3}{\beta}}\frac{1}{g_4^{2/3}2^{1/3}}\left(\frac{\tilde{Q}}{B}\right)^{\frac{2}{3}}, \tag{8.93}$$

$$\frac{\bar{\kappa}_{xx}}{T} = \frac{(4\pi)^2}{(-2\beta)(16\pi G)}\left(\frac{g_4^2\rho}{4B}\right)^{1/3}, \tag{8.94}$$

$$\frac{\bar{\kappa}_{xy}}{T} = \frac{\sqrt{2}(4\pi)^2}{(-2\beta)(16\pi G)}\left(\frac{g_4^2\rho}{4B}\right)^{1/3}. \tag{8.95}$$

For $\gamma = 1$ we have

$$\sigma_{xy} = \frac{\tilde{Q}}{B}, \tag{8.96}$$

$$\alpha_{xy} = 2\pi\frac{g_4}{\sqrt{(16\pi G_N)}}\frac{\tilde{Q}}{B}\sqrt{-\frac{1}{\beta}}, \tag{8.97}$$

$$\frac{\bar{\kappa}_{xx}}{T} = \frac{(4\pi)^2}{(-2\beta)(16\pi G)}\frac{g_4^2\rho}{2B}, \tag{8.98}$$

$$\frac{\bar{\kappa}_{xy}}{T} = 0. \tag{8.99}$$

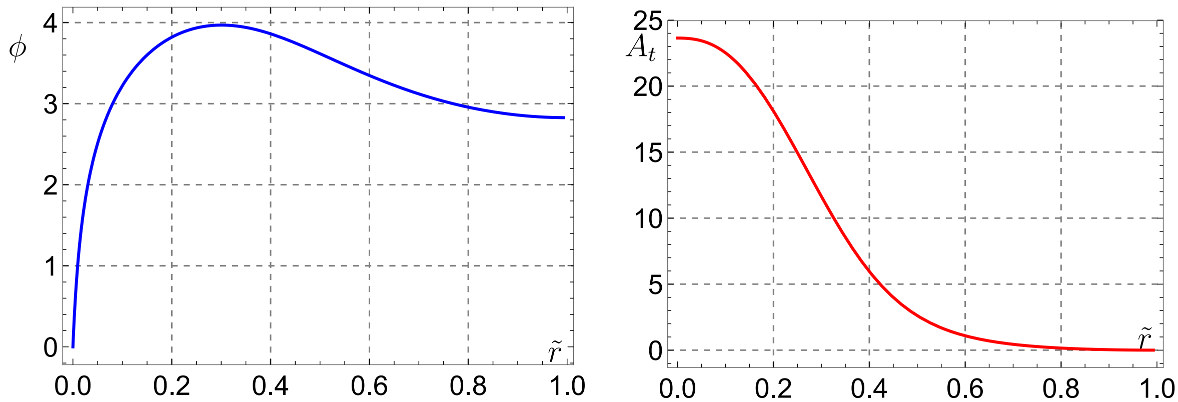
In figures 8.1 and 8.2 we computed numerically the bulk solution that connects this solution in the IR to AdS_4 in the UV, in order to check that the attractor mechanism provides the appropriate solution for these special values of γ (and δ). For more details on the numerical UV completion of the solutions, see Appendix B of reference [22].

8.6 Quadratic couplings

In this bottom-up model both potential and gauge coupling are second order polynomials given by

$$Z(\phi) = 1 + \frac{\alpha}{2}\phi^2, \;\; V(\phi) = -\frac{6}{L^2} + \frac{\beta}{2L^2}\phi^2. \tag{8.100}$$

This theory was investigated in [95] and, by the same arguments of the model in the previous section, it may be viewed as an approximation of more complicated top-down models in which the scalar is small near the horizon. The constants are



(a) Dilaton ϕ profile along the bulk.

(b) Gauge field A_t profile along the bulk.

Figure 8.1: Result of RG flow for the case $\gamma = -\sqrt{3}$, $\delta = -1/\sqrt{3}$. In this case, the magnetic field needs to be $B \gg Q$ in order for the coupling $Z(\phi)$ to be well-approximated by an exponential near the horizon. We chose $B/Q = 100$ for this plot. The horizon is at $\tilde{r} = 1$, the boundary at $\tilde{r} = 0$.

arbitrary, so we expect this model to capture universal features. First notice that (8.42) reduces to

$$-\frac{\alpha u_D\left(\frac{2B^2}{w^2} - \frac{2e^2}{v^2}\right)}{g_4^2} - \frac{8\beta u_D}{16\pi G L^2} = 0\,. \tag{8.101}$$

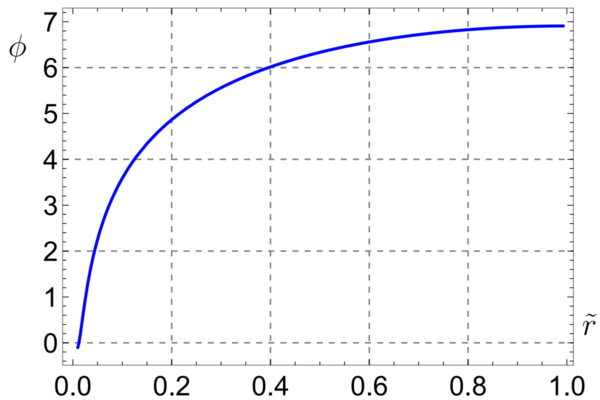
Therefore there are two possible solutions. The first is $u_D = 0$, in which case the other three attractor equations are solved by the dyonic anti-de Sitter-Reissner-Nordström geometry without scalar, as in the example of section 8.2. For the next solution, we assume $u_D \neq 0$. Equations (8.39) and 8.42 give

$$\frac{e}{v} = \frac{g_4^2 \tilde{Q}}{w Z(u_D)}, \tag{8.102}$$

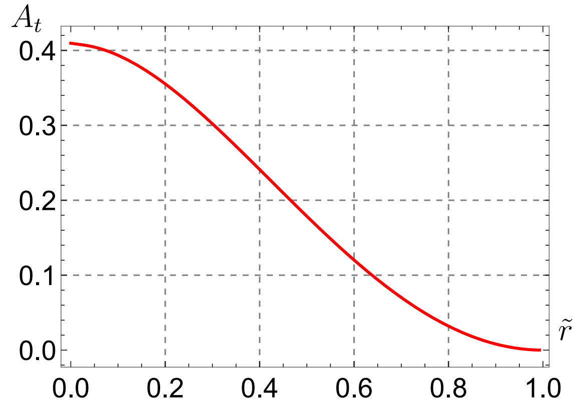
$$\frac{1}{2g_4^2}\frac{e^2}{v^2} = \frac{1}{2g_4^2}\frac{B^2}{w^2} + \frac{\beta}{\alpha}\frac{1}{(16\pi G)L^2}. \tag{8.103}$$

We use (8.103) to eliminate the term containing e from (8.45) and write

$$\frac{B^2}{g_4^2 w^2} Z(u_D) = \frac{1}{L^2(16\pi G)}\left(6 - \frac{\beta}{\alpha}\right) - \frac{\beta}{L^2(16\pi G)} u_D^2. \tag{8.104}$$



(a) Dilaton ϕ profile along the bulk.



(b) Gauge field A_t profile along the bulk.

Figure 8.2: Result of RG flow for the case $\gamma \simeq -1$, $\delta \simeq -1$ and $B/Q = 1$. In this case, the magnetic field is not required to be large. The horizon is at $\tilde{r} = 1$, the boundary at $\tilde{r} = 0$.

We also use (8.102) to eliminate e in (8.103) and write

$$\frac{1}{g_4^2 w^2}\left(\frac{g_4^4 \tilde{Q}^2}{Z(u_D)^2} - B^2\right) = \frac{2\beta}{\alpha}\frac{1}{L^2(16\pi G)}. \tag{8.105}$$

Combining equations (8.104) and (8.105) we can write the quadratic equation for u_D^2,

$$u_D^4 + \frac{u_D^2}{\alpha}\left(1 + \frac{\beta}{(\beta - 6\alpha)}\frac{g_4^4\tilde{Q}^2}{B^2}\right) + \frac{1}{\alpha^2}\left(1 + \frac{g_4^4\tilde{Q}^2}{B^2}\frac{(\beta - 6\alpha)}{(\beta + 6\alpha)}\right) = 0. \tag{8.106}$$

The solution for this equation is

$$1+\frac{\alpha}{2}u_D^2 = -\frac{g_4^4\tilde{Q}^2}{B^2}\frac{\beta}{(3\alpha - 2\beta)} \pm \sqrt{-3 + \frac{g_4^2\tilde{Q}^2}{B^2(\beta + 6\alpha)}\left((24\alpha - 6\beta) + \frac{g_4^2\tilde{Q}^2}{B^2}\frac{\beta^2}{(6\alpha + \beta)}\right)}. \tag{8.107}$$

This is the solution to the attractor equations for u_D written in terms of the charges. The expressions for w, v and e written in terms of the charges as provided by the attractor equations are rather lengthy, and we don't write them explicitly here.

Instead, we write them in terms of u_D^2, i.e.

$$e = \frac{v}{w}\frac{g_4^2\tilde{Q}}{(1+\frac{\alpha}{2}u_D^2)}, \tag{8.108}$$

$$v = \frac{L^2}{(6-\frac{\beta}{2}u_D^2)}, \tag{8.109}$$

$$w = \sqrt{\frac{L^2(16\pi G)B^2\alpha}{g_4^2}\frac{(1+\frac{\alpha}{2}u_D^2)}{6\alpha-\beta(1+\alpha u_D^2)}}. \tag{8.110}$$

The entropy density is found directly and is given by

$$s = 4\pi B\sqrt{\frac{L^2\alpha}{g_4^2(16\pi G)}\frac{(1+\frac{\alpha}{2}u_D^2)}{6\alpha-\beta(1+\alpha u_D^2)}}. \tag{8.111}$$

From this result we express the conductivities in terms of the black hole parameters as

$$\sigma_{xy} = \frac{\tilde{Q}}{B}, \tag{8.112}$$

$$\alpha_{xy} = 4\pi\sqrt{\frac{L^2\alpha}{g_4^2(16\pi G)}\frac{(1+\frac{\alpha}{2}u_D^2)}{6\alpha-\beta(1+\alpha u_D^2)}}, \tag{8.113}$$

where again u_D^2 is obtained from equation (8.107). The ratios involving the heat conductivities are rather lengthy and we did not write them explicitly here.

It is natural to ask about the stability of these solutions. In order to tackle this question we perform a simple analysis: It is likely that possible instabilities are triggered by the scalar sector. We therefore calculate the equation for scalar fluctuations on the dyonic anti-de Sitter-Reissner-Nordström background, and then compare the effective mass to the Breitenlohner-Freedman (BF) bound. Since the near-horizon geometry is translationally invariant in $t,\ x,\ y$ directions, we investigate an Ansatz for the scalar of the form

$$\phi(r,t,x,y) = R(r)e^{-i\omega t+k_1x+k_2y} \tag{8.114}$$

subject to

$$\Box_{AdS_2\times\mathrm{R}^2}\phi - \alpha\phi - \frac{2\kappa^2}{4g_4^2}\frac{\beta}{L^2}F_{\mu\nu}F^{\mu\nu}\phi = 0. \tag{8.115}$$

From this equation, following the logic of [96], we extract the effective mass generated by the scalar potential and interaction with the background electromagnetic field,

$$m_{\text{eff}}^2 = \frac{B^2(6\alpha+\beta) + g_4^4\tilde{Q}^2(\beta-6\alpha)}{6\left(B^2+g_4^4\tilde{Q}^2\right)}. \tag{8.116}$$

The BF bound states that geometry is unstable against scalar perturbations if their mass satisfies:

$$m_{\text{eff}}^2 \leq -\frac{1}{4}. \tag{8.117}$$

Two consequences follow from this: First, the possible phase transition is driven by the electric field in the bulk, which is dual to the chemical potential of the boundary theory, while the magnetic field tends to stabilize the uncondensed phase by increasing the effective mass. Second, this instability does not occur for all models (i.e not for all values of $\alpha,\ \beta$): In the limit

$$\lim_{\tilde{Q}\to\infty} m_{\text{eff}}^2 = \frac{1}{6}(\beta-6\alpha), \tag{8.118}$$

a violation of the BF bound (8.117) requires that

$$6\alpha-\beta > \frac{3}{2}. \tag{8.119}$$

If this is satisfied, the instability occurs at

$$\tilde{Q}^2 = \frac{B^2(6\alpha+\beta+\frac{3}{2})}{g_4^2(6\alpha-\beta-\frac{3}{2})}. \tag{8.120}$$

This superficial analysis indicates that the model may exhibit a possible quantum critical phase transition (i.e. a phase transition at $T=0$). Instabilities due to fluctuations in the non-scalar sector are also conceivable. It is also not clear what happens if the condition (8.119) is not satisfied. This question would require a more sophisticated analysis which we leave for future work.

8.7 S-duality of the attractor equations and conductivities

In this section we make a rough analysis of the invariance of the attractor equations and conductivities under S-duality transformations. In [83], the authors derive conductivities from a set of generalized Stokes equations at the horizon. These Stokes equations are shown to be invariant under S-duality for a 3+1-dimensional gravity dual. Here, in order to examine the S-duality transformation properties of the attractor equations considered in the sections above, we restric our attention to theories whose potential is even and which satisfy $Z(-\phi) = Z(\phi)^{-1}$.

The S-duality transformations are given by

$$F^{\mu\nu} \to Z(\phi)\frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{2\sqrt{-g}}F_{\rho\sigma}, \ \ \phi \to -\phi, \tag{8.121}$$

where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol, with $\tilde{\epsilon}^{trxy} = 1$. If we focus on horizon of the extremal black hole, then the S-duality transformation results in

$$-\frac{e}{v^2} \to -\frac{1}{vw}Z(u_D)B, \ \ \frac{B}{w^2} \to -\frac{1}{vw}Z(u_D)e, \ \ Z(u_D) \to \frac{1}{Z(u_D)}. \tag{8.122}$$

The S-duality transformation of the charge density ρ is obtained from its definition within Einstein-Maxwell-dilaton theory, from which we obtain

$$\rho = \sqrt{-g}\frac{Z(\phi)}{g_4^2}F^{tr} \to \sqrt{-g}\frac{1}{Z(\phi)g_4^2}Z(\phi)\frac{\tilde{\epsilon}^{trxy}}{\sqrt{-g}}F_{xy} = \frac{B}{g_4^2}. \tag{8.123}$$

Using $\tilde{Q} = \rho$ we see directly that the transformation for the magnetic field in (8.122) is given by $B \to -g_4^2\rho$. Applying the transformations (8.122) and (8.123) to the attractor equations, we see that (8.40), (8.41) and (8.42) are invariant, and (8.39) gives a trivial identity. This shows that the attractor equations are invariant under S-duality transformations. However note that S-duality is an invariance of the equations of motion, but not of the action, so we may expect that the entropy function (8.8) is not invariant under S-duality either. In fact, as was pointed out in reference [18], applying an S-duality transformation gives rise to a new entropy function in the attractor formalism, which might generate new attractor equations. Nevertheless, the extremization of both the initial and the S-dual entropy functions yields the same black hole entropy, which shows that the black hole entropy is invariant under S-duality.

The result (8.68) for the entropy density contains a square root of the product of the charges. This comes from the fact that the attractor equations involve squares. For investigating the S-duality properties of our explicit expressions for the entropy and for the conductivities, we have to take care of the signs carefully when taking square roots of quadratic expressions. As an example, the result for the thermoelectric conductivity of (8.72) should be written as

$$\alpha_{xy} = \mathrm{sgn}(B)\, 4\pi \sqrt{\frac{L^2|\tilde{Q}|}{6|B|} 16\pi G}\,. \tag{8.124}$$

Then, the transformed conductivities are given by

$$\sigma_{xy} \to -\frac{B}{g_4^4 \tilde{Q}}, \quad \alpha_{xy} \to -\frac{4\pi}{g_4^2 \tilde{Q}} \sqrt{\frac{\tilde{Q} B}{6} \frac{L^2}{(16\pi G)}}, \tag{8.125}$$

with $\tilde{Q} \to \rho$. Performing the transformation once more, we see that the conductivities transform as

$$\sigma_{xy} \to \sigma_{xy}, \quad \alpha_{xy} \to -\alpha_{xy}\,, \tag{8.126}$$

which is precisely what authors of [83] find.
The result for κ/T is independent of the absolute values of charges for the massless scalar model, but again its sign depends on signs of charges. This implies that after transforming the charges once we have

$$\bar{\kappa}_{xx} \to \bar{\kappa}_{xx}, \quad \bar{\kappa}_{xy} \to -\bar{\kappa}_{xy}\,. \tag{8.127}$$

Transforming the charges again results in

$$\bar{\kappa}_{xx} \to \bar{\kappa}_{xx}, \quad \bar{\kappa}_{xy} \to \bar{\kappa}_{xy}\,, \tag{8.128}$$

which again is consistent with general transformation laws of [83].

8.8 Generalizations and partial conclusions

In this chapter we computed conductivities at zero temperature for Einstein-Maxwell-dilaton theories. The expressions we obtained are written in terms of the extremal black hole parameters, and we showed that the off-diagonal components of the electric and thermoelectric conductivities scale as

$$\sigma_{xy} \sim N^{3/2}, \quad \alpha_{xy} \sim N^{3/2} \tag{8.129}$$

for a constant potential, where N is the rank of the gauge group of the conformal field theory dual to the Einstein-Maxwell-dilaton theory. We argued that this should also be the case for theories with other kinds of potential. We briefly discussed that in the $T = 0$ limit the Einstein-Maxwell-dilaton presents different phases, which may be related to quantum phase transitions. All the results were obtained by applying Sen's entropy function method in the AdS/CMT context. We also computed κ_{xx}/T and κ_{xy}/T for Einstein-Maxwell-dilaton theories assuming that these ratios are finite at $T = 0$. For a constant potential, κ_{xx}/T is equal to κ_{xy}/T, and they also scale as

$$\frac{\kappa_{xx}}{T} = \frac{\kappa_{xy}}{T} \sim N^{3/2}. \tag{8.130}$$

Explicit analytical zero-temperature expressions are expected to be very useful in particular for universality arguments in AdS/CMT, see for instance [97, 98] and references therein. We expect that the results of this paper may be generalized to more involved geometries relevant in that context.

Although we investigated the simplest Einstein-Maxwell-dilaton theories, generalizations of this approach are indeed possible. The two most immediate ones are the following: One direction is to generalize the equations for conductivities (7.76), (7.77), (7.78), (7.79), (7.125), (7.126) to take into account multiple scalar fields and gauge fields coupled in the non-minimal way. In the context of AdS/CFT it is useful to consider maximally supersymmetric supergravities, and including more scalars and field strengths will guarantee that we can handle the four-dimensional $\mathcal{N} = 8$ gauged supergravity. Solving the attractor equations for the planar case, the conductivities of the three-dimensional CFT can be expressed explicitly in terms of the black hole parameters and of N, just as we did in the paper for simpler cases.

Another possibility is to generalize the computation of the conductivities to higher-dimensional supergravity theories. This means that the gravity theory will have a Chern-Simons term in odd dimensions, which may complicate the computation. The entropy function method is valid only for even-dimensional gravity theories, but it is possible to perform dimensional reduction down to even dimensions, treat the even dimensional theory as an effective theory, and then compute the horizon data, in the same spirit as [99]. That would allow to use the entropy function method to obtain the conductivities at zero temperature also for these cases, and in particular for the case most studied in the literature, which is the supergravity dual of four-dimensional $\mathcal{N} = 4$ Super Yang-Mills theory.

Yet another possibility is to extend the superficial analysis of the phase transition considered in subsection 8.6. It would be interesting to look not only at the scalar instabilities but also at the full linearized Einstein-Maxwell-dilaton system on both backgrounds. Moreover, as we saw in 8.6, the scalar instability occurs only for particular values of parameters α, β, but the coexistence of phases with and without scalar fields seems to happen also for other values of those parameters. Exploring this phase diagram would probably require knowledge of the full solution, and therefore one would probably be forced to invoke some numerical methods. Apart from this scalar condensation it seems possible that there may be another phase transition in the regime of vanishing magnetic field, which may lead to a scaling geometry in the far IR.

Our results imply that the attractor mechanism is a very important ingredient in the computation of zero temperature conductivities, as well as in the study of quantum phase transitions.

Chapter 9

Conclusions

In this thesis we studied several aspects of the black hole and wormhole solutions to the Einstein-Maxwell-dilaton theory. We studied Sen's entropy function formalism applied to the computation of the entropy of extremal black holes of theories of interest as well as to the holographic context.

In chapter 3 we presented the analytical dyonic black hole solution of the Einstein-Maxwell-dilaton theory. This solution was presented in its bare form, i.e. without imposing boundary conditions to write it in terms of the physical observables. In this sense, the physical observables of the dyonic black hole solution are the mass M, the electric charge Q, the magnetic charge P and the dilaton charge Σ. Although it is widely mentioned in the black hole literature that the dilaton charge is a parameter that depends on the others (see for instance [7]), we showed that the only requirement imposed by the equations of motion is that the product of the dilaton charge Σ and the mass M must be written in terms of the other parameters. We discussed how to define dependent and independent parameters. We constructed massless black hole solutions by imposing that the value of the dilaton at infinity be an imaginary number. The observables in this case are all physical in the sense that they are real quantities: they depend only on the exponential of the dilaton at infinity and this gives a real number. We used such massless solutions to construct Einstein-Rosen bridges, and proved that these bridges satisf the null energy condition.

In chapter 4 we gave an analytical traversable wormhole solution to the Einstein-Maxwell-phantom-dilaton theory. This wormhole is electrically charged, and the magnetically charged one can be easily found by applying S-duality transformations, since this is an invariance of the equations of motion. As a traversable wormhole must be, this solution contains no singularities, and develops a throat, which is the minimum surface that connects the two asymptotic spaces. In the same way as was done for the dyonic black holes of chapter 3, we showed how to recover other known wormhole solutions by setting some parameters to zero. We computed the topological charges and constructed numerically the embedding diagram, just like was done for the Bronikov-Ellis wormhole. As a physical application, we computed the deflection angle of light in the weak field limit using the Gauss-Bonnet theorem.

In chapter 5 we reviewed the Sen's entropy function formalism. We constructed the entropy function using the fact that the near horizon geometry of an extremal black hole is always $\mathrm{AdS}_2 \times S^2$. From it, we derived the so-called called attractor equations, whose solutions extremize the entropy function, resulting in the entropy of the black hole. The formalism can also be used to extract the near horizon data of extremal black holes, i.e. the near horizon metric and fields.

In chapter 6 we used the Sen's entropy function formalism to compute the entropy of Einstein-Maxwell-dilaton theories in the presence of a scalar potential. Specifically, we computed the horizon data and entropy for models whose potential was proportional to the couplings to the field strength. This investigation was of special interest, since we wanted to solve the attractor equations for a more complicated theory, which is the $U(1)^4$ gauged supergravity. This theory has four gauge fields, all coupled to exponentials of combinations of the three dilaton fields. By analogy with the simplest models, we inferred how the solution to the dilatons on the horizon must be, and checked that our guess solved analytically all the atrractor equations. We then expressed the black hole entropy for the $U(1)^4$ gauged supergravity in terms of all the electric and magnetic charges, and also in terms of the coupling to the potential. The scalar fields of this theory must satisfy a constraint, and this was used to express the entropy only in terms of the four electric and four magnetic charges of the black hole. We also discussed that is possible to recover the entropy for magnetically charged black holes calculated in the context of $\mathcal{N} = 2$ supergravity.

In chapter 7 we computed transport coefficients in the presence of a magnetic field in holography. We considered Einstein-Maxwell-dilaton theories in the presence of linear axions, which breaks translation symmetry, and also in the presence of a topological term of the kind $W(\phi)F\tilde{F}$. In order to extract the conductivities by applying linear perturbations around the black hole, we needed to define the radially independent currents, which are the physical currents. This required to subtract off the magnetization currents from the total currents, and showed in details that the new currents were indeed radially independent. By solving linearized Einstein's equations for the perturbations, we extracted the electric conductivity, σ_{ij}, the thermoelectric conductivity, α_{ij}, and the heat conductivity, $\bar{\kappa}_{ij}$. There are two important points in this computation. First, the conductivities are all analytical. Second, they depend only on the value of the metric and fields on the horizon of the black hole, which whe call horizon data. This feature arises from the fact that they were computed from radially independent currents, so, instead of evaluating them at the boundary of the AdS spacetime as is usual in holography, we chose to evaluate them on the horizon of the black hole. This means that, once we know the black hole horizon data, we can obtain the conductivities analytically and express them in terms of the parameters of the black hole.

In chapter 8 we introduced a novel approach to compute holographic conductivities at zero temperature. We combined the horizon data, computed from Sen's formalism, with the expressions for the conductivities derived in chapter 7. We first adapted the Sen's formalism to the case when the black hole has planar horizon, and then wrote the attractor equations for the Einstein-Maxwell-dilaton theories. Three models were considered in that chapter. The massless scalar is the simplest model, with a constant potential. We solved the attractor equations, e inserted the horizon data in the formulae for the conductivities. We expressed the conductivities in terms of the electric and magnetic charges, and also in terms of the Newton constant G_N and AdS scale L. The combination of G_N and L allowed us to use the map given by equation (8.12) and express the results in terms of the rank N of the gauge group of the conformal field theory. All the conductivities for this model scale as $\sim N^{3/2}$. Our computation gave a finite result for the ratio $\bar{\kappa}_{ij}/T$, and also showed that $\bar{\kappa}_{xx}/T = \bar{\kappa}_{xy}/T \sim N^{3/2}$. The other two cases we analyzed were the exponential coupling and the quadratic coupling models. The conductivities were also expressed analytically, although there is no known map that would allow us to express them in terms of the rank N of the gauge group for these cases. We also studied the invariance of the attractor equations under S-duality transformations, and showed how the conductivities change for massless case, which is consistent with the literature.

Appendix A

Christoffel symbols

Using (7.50), we can obtain non-trivial the Christoffel symbols

$$
\begin{aligned}
\Gamma^t_{rt} &= \frac{\mathcal{U}'(r)}{2\mathcal{U}(r)} \\
\Gamma^t_{xr} &= \frac{\epsilon\left(\xi t\,\mathcal{U}'(r) - 2\xi t\,\mathcal{U}(r)\mathcal{V}'(r) - e^{2\mathcal{V}(r)}\delta h'_{tx}(r)\right)}{2\mathcal{U}(r)} \\
\Gamma^t_{ry} &= -\frac{\epsilon e^{2V(r)}\delta h'_{ty}(r)}{2U(r)} \\
\Gamma^r_{tt} &= \frac{1}{2}\mathcal{U}(r)\mathcal{U}'(r) \\
\Gamma^r_{rr} &= -\frac{\mathcal{U}'(r)}{2\mathcal{U}(r)} \\
\Gamma^r_{tx} &= \frac{1}{2}\epsilon U(r)\left(\xi t U'(r) - e^{2V(r)}\left(\delta h'_{tx}(r) + 2\delta h_{tx}(r)V'(r)\right)\right) \\
\Gamma^r_{rx} &= -\epsilon e^{2\mathcal{V}(r)}\delta h_{rx}(r)\mathcal{U}(r)\mathcal{V}'(r) \\
\Gamma^r_{xx} &= -e^{2\mathcal{V}(r)}\mathcal{U}(r)\mathcal{V}'(r) \\
\Gamma^r_{ty} &= -\frac{1}{2}\epsilon\,\mathcal{U}(r)e^{2\mathcal{V}(r)}\left(\delta h'_{ty}(r) + 2\delta h_{ty}(r)\mathcal{V}'(r)\right) \\
\Gamma^r_{yy} &= -e^{2\mathcal{V}(r)}\mathcal{U}(r)\mathcal{V}'(r) \\
\Gamma^x_{tt} &= \epsilon\,\mathcal{U}(r)\left(-\xi e^{-2\mathcal{V}(r)} - \frac{1}{2}\delta h_{rx}(r)\mathcal{U}'(r)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^x_{tr} &= \frac{1}{2}\epsilon\left(\delta h'_{tx}(r) - \frac{\delta h_{tx}(r)\,\mathcal{U}'(r)}{\mathcal{U}(r)} + 2\delta h_{tx}(r)\mathcal{V}'(r)\right)\\
\Gamma^x_{rr} &= \epsilon\left(\delta h'_{rx}(r) + \frac{\delta h_{rx}(r)\,\mathcal{U}'(r)}{2\,\mathcal{U}(r)} + 2\delta h_{rx}(r)\mathcal{V}'(r)\right)\\
\Gamma^x_{xr} &= \mathcal{V}'(r)\\
\Gamma^x_{xx} &= \epsilon\delta h_{rx}(r)\,\mathcal{U}(r)e^{2\mathcal{V}(r)}\mathcal{V}'(r)\\
\Gamma^x_{yy} &= \epsilon\delta h_{rx}(r)\,\mathcal{U}(r)e^{2\mathcal{V}(r)}\mathcal{V}'(r)\\
\Gamma^y_{tt} &= -\frac{1}{2}\epsilon\delta h_{ry}(r)\,\mathcal{U}(r)\,\mathcal{U}'(r)\\
\Gamma^y_{rt} &= \frac{1}{2}\epsilon\left(\delta h'_{ty}(r) - \frac{\delta h_{ty}(r)\,\mathcal{U}'(r)}{\mathcal{U}(r)} + 2\delta h_{ty}(r)\mathcal{V}'(r)\right)\\
\Gamma^y_{rr} &= \epsilon\left(\delta h'_{ry}(r) + \frac{\delta h_{ry}(r)\,\mathcal{U}'(r)}{2\,\mathcal{U}(r)} + 2\delta h_{ry}(r)\mathcal{V}'(r)\right)\\
\Gamma^y_{xx} &= \epsilon\delta h_{ry}(r)\,\mathcal{U}(r)e^{2\mathcal{V}(r)}\mathcal{V}'(r)\\
\Gamma^y_{yr} &= \mathcal{V}'(r)\\
\Gamma^y_{yy} &= \epsilon\delta h_{ry}(r)\,\mathcal{U}(r)e^{2\mathcal{V}(r)}\mathcal{V}'(r). \qquad \text{(A.1)}
\end{aligned}
$$

Appendix B

Details of the approximation

In this appendix we present explicitly the detail of the approximation used to obtain the result (4.85). The series expansion for the inverse tangent is written as

$$\arctan\left(\frac{x}{l}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{x}{l}\right)^{2n+1}, \quad \left|\frac{x}{l}\right| < 1. \tag{B.1}$$

Notice that the limit for when the expansion is valid corresponds to $|x| < l$. We are interested in the limit when $|r| > l$, so we must use the following identity and corresponding expansion

$$\arctan\left(\frac{r}{l}\right) = \frac{\pi}{2} - \arctan\left(\frac{l}{r}\right) \approx \frac{\pi}{2} - \frac{l}{r} + \frac{l^3}{3r^3} - \frac{l^5}{5r^5} + \frac{l^7}{7r^7} + ... \tag{B.2}$$

The exponential of the inverse tangent has then the following expansion

$$\begin{aligned} \exp\left[\frac{2c_1}{l}\arctan\left(\frac{r}{l}\right)\right] &\approx \exp\left[\frac{2c_1}{l}\left(\frac{\pi}{2} - \frac{l}{r}\right)\right] \\ &\approx e^{\frac{c_1\pi}{l}}\left(1 - \frac{2c_1}{r}\right), \end{aligned} \tag{B.3}$$

$$\frac{2(b_1 - c_1)}{l}\arctan\left(\frac{r}{l}\right) \approx \frac{(b_1 - c_1)\pi}{l} - \frac{2(b_1 - c_1)}{r}. \tag{B.4}$$

These will be enough to expand the function λ for large r, which results in

$$\lambda \approx -\frac{Q^2 e^{c_2 + \frac{c_1\pi}{l}}}{2c_1^2} + \frac{(b_1 - c_1)\pi}{l} + 2b_2 - c_2 + \left(-2b_1 + 2c_1 + \frac{Q^2 e^{c_2 + \frac{c_1\pi}{l}}}{c_1}\right)\frac{1}{r}. \tag{B.5}$$

Defining

$$m_1 \equiv -\frac{Q^2 e^{c_2+\frac{c_1\pi}{l}}}{2c_1^2} + \frac{(b_1-c_1)\pi}{l} + 2b_2 - c_2, \quad m_2 \equiv -2b_1 + 2c_1 + \frac{Q^2 e^{c_2+\frac{c_1\pi}{l}}}{c_1}, \tag{B.6}$$

we have

$$\lambda \approx m_1 + \frac{m_2}{r}, \tag{B.7}$$

$$e^{-\lambda} \approx e^{-m_1}\left(1 - \frac{m_2}{r}\right). \tag{B.8}$$

In the text, we used only the leading term in this expansion.

Bibliography

[1] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, "GW170814: A three-detector observation of gravitational waves from a binary black hole coalescence," *Submitted to: Phys. Rev. Lett.* (2017) , `arXiv:1709.09660 [gr-qc]`.

[2] T. Kaluza, "On the Problem of Unity in Physics," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1921** (1921) 966–972.

[3] O. Klein, "Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English)," *Z. Phys.* **37** (1926) 895–906. [Surveys High Energ. Phys.5,241(1986)].

[4] G. W. Gibbons, "Antigravitating Black Hole Solitons with Scalar Hair in N=4 Supergravity," *Nucl. Phys.* **B207** (1982) 337–349.

[5] G. W. Gibbons and K.-i. Maeda, "Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields," *Nucl. Phys.* **B298** (1988) 741.

[6] D. Garfinkle, G. T. Horowitz, and A. Strominger, "Charged black holes in string theory," *Phys. Rev.* **D43** (1991) 3140. [Erratum: Phys. Rev.D45,3888(1992)].

[7] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet, and A. Van Proeyen, "Supersymmetry as a cosmic censor," *Phys. Rev.* **D46** (1992) 5278–5302, `arXiv:hep-th/9205027 [hep-th]`.

[8] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220. [,167(1975)].

[9] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys. Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029 [hep-th].

[10] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," *JHEP* **09** (2005) 038, arXiv:hep-th/0506177 [hep-th].

[11] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[12] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 2007.

[13] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge University Press, 2007.

[14] H. Nastase, *Introduction to the ADS/CFT Correspondence.* Cambridge University Press, 2015. http://www.cambridge.org/academic/subjects/physics/theoretical-physics-and-mathematical-physics/introduction-adscft-correspondence?format=HB&isbn=9781107085855.

[15] P. Goulart, "Dyonic black holes and dilaton charge in string theory," arXiv:1611.03093 [hep-th].

[16] P. Goulart, "Massless black holes and charged wormholes in string theory," arXiv:1611.03164 [hep-th].

[17] P. Goulart, "Phantom wormholes in Einstein-Maxwell-dilaton theory," arXiv:1708.00935 [gr-qc].

[18] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," *Gen. Rel. Grav.* **40** (2008) 2249–2431, arXiv:0708.1270 [hep-th].

[19] P. Goulart, "Dyonic AdS_4 black hole entropy and attractors via entropy function," *JHEP* **09** (2016) 003, arXiv:1512.05399 [hep-th].

[20] S. A. Hartnoll, A. Lucas, and S. Sachdev, "Holographic quantum matter," arXiv:1612.07324 [hep-th].

[21] M. Blake, A. Donos, and N. Lohitsiri, "Magnetothermoelectric Response from Holography," *JHEP* **08** (2015) 124, arXiv:1502.03789 [hep-th].

[22] J. Erdmenger, D. Fernandez, P. Goulart, and P. Witkowski, "Conductivities from attractors," *JHEP* **03** (2017) 147, arXiv:1611.09381 [hep-th].

[23] J. M. Bardeen, B. Carter, and S. W. Hawking, "The Four laws of black hole mechanics," *Commun. Math. Phys.* **31** (1973) 161–170.

[24] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[25] A. Einstein and N. Rosen, "The Particle Problem in the General Theory of Relativity," *Phys. Rev.* **48** (1935) 73–77.

[26] M. Visser, *Lorentzian wormholes: From Einstein to Hawking*. 1995.

[27] R. W. Fuller and J. A. Wheeler, "Causality and Multiply Connected Space-Time," *Phys. Rev.* **128** (1962) 919–929.

[28] M. S. Morris and K. S. Thorne, "Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity," *Am. J. Phys.* **56** (1988) 395–412.

[29] K. A. Bronnikov, "Scalar-tensor theory and scalar charge," *Acta Phys. Polon.* **B4** (1973) 251–266.

[30] H. G. Ellis, "Ether flow through a drainhole - a particle model in general relativity," *J. Math. Phys.* **14** (1973) 104–118.

[31] A. Sen, "Electric magnetic duality in string theory," *Nucl. Phys.* **B404** (1993) 109–126, arXiv:hep-th/9207053 [hep-th].

[32] E. Cremmer, J. Scherk, and S. Ferrara, "SU(4) Invariant Supergravity Theory," *Phys. Lett.* **B74** (1978) 61–64.

[33] S. Ferrara, R. Kallosh, and A. Strominger, "N=2 extremal black holes," *Phys. Rev.* **D52** (1995) 5412–5416, `arXiv:hep-th/9508072 [hep-th]`.

[34] S. Ferrara and R. Kallosh, "Supersymmetry and attractors," *Phys. Rev.* **D54** (1996) 1514–1524, `arXiv:hep-th/9602136 [hep-th]`.

[35] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, "Non-supersymmetric attractors," *Phys. Rev.* **D72** (2005) 124021, `arXiv:hep-th/0507096 [hep-th]`.

[36] G. W. Gibbons, R. Kallosh, and B. Kol, "Moduli, scalar charges, and the first law of black hole thermodynamics," *Phys. Rev. Lett.* **77** (1996) 4992–4995, `arXiv:hep-th/9607108 [hep-th]`.

[37] G. W. Gibbons and D. A. Rasheed, "Dyson pairs and zero mass black holes," *Nucl. Phys.* **B476** (1996) 515–547, `arXiv:hep-th/9604177 [hep-th]`.

[38] F. J. Dyson, "Divergence of perturbation theory in quantum electrodynamics," *Phys. Rev.* **85** (1952) 631–632.

[39] E. Bergshoeff, A. Collinucci, U. Gran, D. Roest, and S. Vandoren, "Non-extremal instantons and wormholes in string theory," *Fortsch. Phys.* **53** (2005) 990–996, `arXiv:hep-th/0412183 [hep-th]`.

[40] N. Arkani-Hamed, J. Orgera, and J. Polchinski, "Euclidean wormholes in string theory," *JHEP* **12** (2007) 018, `arXiv:0705.2768 [hep-th]`.

[41] U. Lindstrom and M. Rocek, "A Note on the Seiberg-Witten solution of N=2 superYang-Mills theory," *Phys. Lett.* **B355** (1995) 492–493, `arXiv:hep-th/9503012 [hep-th]`.

[42] A. Nakonieczna, M. Rogatko, and Å. Nakonieczny, "Dark sector impact on gravitational collapse of an electrically charged scalar field," *JHEP* **11** (2015) 012, `arXiv:1508.02657 [hep-th]`.

[43] S. Hannestad, "Dark energy and dark matter from cosmological observations," *Int. J. Mod. Phys.* **A21** (2006) 1938–1949, arXiv:astro-ph/0509320 [astro-ph]. [,364(2005)].

[44] **WMAP** Collaboration, J. Dunkley *et al.*, "Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Likelihoods and Parameters from the WMAP data," *Astrophys. J. Suppl.* **180** (2009) 306–329, arXiv:0803.0586 [astro-ph].

[45] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty, and S. Mukohyama, "Ghost condensation and a consistent infrared modification of gravity," *JHEP* **05** (2004) 074, arXiv:hep-th/0312099 [hep-th].

[46] F. Piazza and S. Tsujikawa, "Dilatonic ghost condensate as dark energy," *JCAP* **0407** (2004) 004, arXiv:hep-th/0405054 [hep-th].

[47] C. Vafa, "Brane / anti-brane systems and U(N|M) supergroup," arXiv:hep-th/0101218 [hep-th].

[48] T. Okuda and T. Takayanagi, "Ghost D-branes," *JHEP* **03** (2006) 062, arXiv:hep-th/0601024 [hep-th].

[49] C. Vafa, "Non-Unitary Holography," arXiv:1409.1603 [hep-th].

[50] R. Dijkgraaf, B. Heidenreich, P. Jefferson, and C. Vafa, "Negative Branes, Supergroups and the Signature of Spacetime," arXiv:1603.05665 [hep-th].

[51] C. M. Hull, "Duality and the signature of space-time," *JHEP* **11** (1998) 017, arXiv:hep-th/9807127 [hep-th].

[52] G. Clement, J. C. Fabris, and M. E. Rodrigues, "Phantom Black Holes in Einstein-Maxwell-Dilaton Theory," *Phys. Rev.* **D79** (2009) 064021, arXiv:0901.4543 [hep-th].

[53] O. Bergmann and R. Leipnik, "Space-Time Structure of a Static Spherically Symmetric Scalar Field," *Phys. Rev.* **107** (1957) 1157–1161.

[54] I. Z. Fisher, "Scalar mesostatic field with regard for gravitational effects," *Zh. Eksp. Teor. Fiz.* **18** (1948) 636–640, arXiv:gr-qc/9911008 [gr-qc].

[55] K. A. Bronnikov and G. N. Shikin, "Interacting Fields in General Relativity," *Izv. Vuz. Fiz.* **1977N9** (1977) 25–30.

[56] K. A. Bronnikov, V. N. Melnikov, G. N. Shikin, and K. P. Staniukowicz, "SCALAR, ELECTROMAGNETIC, AND GRAVITATIONAL FIELDS INTERACTION: PARTICLE - LIKE SOLUTIONS," *Annals Phys.* **118** (1979) 84–107.

[57] G. W. Gibbons and M. C. Werner, "Applications of the Gauss-Bonnet theorem to gravitational lensing," *Class. Quant. Grav.* **25** (2008) 235009, arXiv:0807.0854 [gr-qc].

[58] K. Jusufi, "Deflection Angle of Light by Wormholes using the Gauss-Bonnet Theorem," arXiv:1706.01244 [gr-qc].

[59] K. Jusufi, A. Övgün, and A. Banerjee, "Light Deflection by Charged Wormholes in Einstein-Maxwell-Dilaton Theory," arXiv:1707.01416 [gr-qc].

[60] K. Jusufi, "Gravitational lensing by Reissner-Nordström black holes with topological defects," *Astrophys. Space Sci.* **361** (2016) no. 1, 24, arXiv:1510.08526 [gr-qc].

[61] R. M. Wald, "Black hole entropy is the Noether charge," *Phys. Rev.* **D48** (1993) no. 8, R3427–R3431, arXiv:gr-qc/9307038 [gr-qc].

[62] T. Jacobson, G. Kang, and R. C. Myers, "On black hole entropy," *Phys. Rev.* **D49** (1994) 6587–6598, arXiv:gr-qc/9312023 [gr-qc].

[63] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," *Phys. Rev.* **D50** (1994) 846–864, arXiv:gr-qc/9403028 [gr-qc].

[64] T. Jacobson, G. Kang, and R. C. Myers, "Black hole entropy in higher curvature gravity," in *Heat Kernels and Quantum Gravity Winnipeg, Canada, August 2-6, 1994*. 1994. arXiv:gr-qc/9502009 [gr-qc]. http://inspirehep.net/record/382515/files/arXiv:gr-qc_9502009.pdf.

[65] M. J. Duff and J. T. Liu, "Anti-de Sitter black holes in gauged N = 8 supergravity," *Nucl. Phys.* **B554** (1999) 237–253, arXiv:hep-th/9901149 [hep-th].

[66] J. F. Morales and H. Samtleben, "Entropy function and attractors for AdS black holes," *JHEP* **10** (2006) 074, arXiv:hep-th/0608044 [hep-th].

[67] S. L. Cacciatori and D. Klemm, "Supersymmetric AdS(4) black holes and attractors," *JHEP* **01** (2010) 085, arXiv:0911.4926 [hep-th].

[68] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, arXiv:0806.1218 [hep-th].

[69] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[70] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[71] D. Vegh, "Holography without translational symmetry," arXiv:1301.0537 [hep-th].

[72] M. Blake and D. Tong, "Universal Resistivity from Holographic Massive Gravity," *Phys. Rev.* **D88** (2013) no. 10, 106004, arXiv:1308.4970 [hep-th].

[73] R. A. Davison, "Momentum relaxation in holographic massive gravity," *Phys. Rev.* **D88** (2013) 086003, arXiv:1306.5792 [hep-th].

[74] M. Blake, D. Tong, and D. Vegh, "Holographic Lattices Give the Graviton an Effective Mass," *Phys. Rev. Lett.* **112** (2014) no. 7, 071602, arXiv:1310.3832 [hep-th].

[75] A. Donos and J. P. Gauntlett, "Holographic Q-lattices," *JHEP* **04** (2014) 040, arXiv:1311.3292 [hep-th].

[76] A. Donos and J. P. Gauntlett, "Novel metals and insulators from holography," *JHEP* **06** (2014) 007, arXiv:1401.5077 [hep-th].

[77] T. Andrade and B. Withers, "A simple holographic model of momentum relaxation," *JHEP* **05** (2014) 101, arXiv:1311.5157 [hep-th].

[78] N. R. Cooper, B. I. Halperin, and I. M. Ruzin, " Thermoelectric response of an interacting two-dimensional electron gas in a quantizing magnetic field," *Phys. Rev.* **B55** (1997) 2344, arXiv:cond-mat/9607001 [cond-mat].

[79] S. A. Hartnoll, P. K. Kovtun, M. Muller, and S. Sachdev, "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes," *Phys. Rev.* **B76** (2007) 144502, arXiv:0706.3215 [cond-mat.str-el].

[80] A. Donos, J. P. Gauntlett, T. Griffin, N. Lohitsiri, and L. Melgar, "Holographic DC conductivity and Onsager relations," *JHEP* **07** (2017) 006, arXiv:1704.05141 [hep-th].

[81] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class. Quant. Grav.* **26** (2009) 224002, arXiv:0903.3246 [hep-th].

[82] S. A. Hartnoll and P. Kovtun, "Hall conductivity from dyonic black holes," *Phys. Rev.* **D76** (2007) 066001, arXiv:0704.1160 [hep-th].

[83] A. Donos, J. P. Gauntlett, T. Griffin, and L. Melgar, "DC Conductivity of Magnetised Holographic Matter," *JHEP* **01** (2016) 113, arXiv:1511.00713 [hep-th].

[84] H. K. Kunduri, J. Lucietti, and H. S. Reall, "Near-horizon symmetries of extremal black holes," *Class. Quant. Grav.* **24** (2007) 4169–4190, arXiv:0705.4214 [hep-th].

[85] A. Sen, "Entropy function for heterotic black holes," *JHEP* **03** (2006) 008, arXiv:hep-th/0508042 [hep-th].

[86] B. Sahoo and A. Sen, "BTZ black hole with Chern-Simons and higher derivative terms," *JHEP* **07** (2006) 008, arXiv:hep-th/0601228 [hep-th].

[87] B. Sahoo and A. Sen, "Higher derivative corrections to non-supersymmetric extremal black holes in N=2 supergravity," *JHEP* **09** (2006) 029, arXiv:hep-th/0603149 [hep-th].

[88] M. Alishahiha and H. Ebrahim, "Non-supersymmetric attractors and entropy function," *JHEP* **03** (2006) 003, arXiv:hep-th/0601016 [hep-th].

[89] B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, "Non-supersymmetric attractors in R**2 gravities," *JHEP* **08** (2006) 004, arXiv:hep-th/0602022 [hep-th].

[90] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, "Rotating attractors," *JHEP* **10** (2006) 058, arXiv:hep-th/0606244 [hep-th].

[91] R. Kallosh, N. Sivanandam, and M. Soroush, "The Non-BPS black hole attractor equation," *JHEP* **03** (2006) 060, arXiv:hep-th/0602005 [hep-th].

[92] S. Grozdanov, A. Lucas, and K. Schalm, "Incoherent thermal transport from dirty black holes," *Phys. Rev.* **D93** (2016) no. 6, 061901, arXiv:1511.05970 [hep-th].

[93] C. Charmousis, B. Gouteraux, and J. Soda, "Einstein-Maxwell-Dilaton theories with a Liouville potential," *Phys. Rev.* **D80** (2009) 024028, arXiv:0905.3337 [gr-qc].

[94] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," *JHEP* **11** (2010) 151, arXiv:1005.4690 [hep-th].

[95] M. Cadoni and P. Pani, "Holography of charged dilatonic black branes at finite temperature," *JHEP* **04** (2011) 049, arXiv:1102.3820 [hep-th].

[96] S. Moroz, "Below the Breitenlohner-Freedman bound in the nonrelativistic AdS/CFT correspondence," *Phys. Rev.* **D81** (2010) 066002, arXiv:0911.4060 [hep-th].

[97] J. Erdmenger, B. Herwerth, S. Klug, R. Meyer, and K. Schalm, "S-Wave Superconductivity in Anisotropic Holographic Insulators," *JHEP* **05** (2015) 094, arXiv:1501.07615 [hep-th].

[98] K.-Y. Kim and C. Niu, "Homes' law in Holographic Superconductor with Q-lattices," *JHEP* **10** (2016) 144, arXiv:1608.04653 [hep-th].

[99] D. Astefanesei, N. Banerjee, and S. Dutta, "Near horizon data and physical charges of extremal AdS black holes," *Nucl. Phys.* **B853** (2011) 63–79, arXiv:1104.4121 [hep-th].