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PATH INTEGRATION AND THE FUNCTIONAL MEASURE*

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ABSTRACT

The functional measure for the Feynman path integral is investigated, and it is argued that non-trivial measure factors should not be automatically discarded as is often done. The fundamental hypothesis of path integration is stated in its Hamiltonian formulation and is used, together with the Faddeev-Popov ansatz, to derive the general form of the canonical functional measure for all gauged or ungauged theories of integer spin fields in any number of spacetime dimensions. This general result is then used to calculate the effective functional measures for scalar, vector, and gravitational fields in more than two dimensions at energies low compared to the Planck Mass. It is shown that these results indicate the self-consistency and plausibility of the canonical functional measure over other functional measures and suggest an important relationship between bosonic and fermionic degrees of freedom. The canonical functional measure factors associated with fields of half-integer spin and with auxiliary fields are also derived.

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1. Introduction

The path integral formulation of quantum field theory, developed by Richard Feynman, has been responsible for many of the most important developments in theoretical physics over the past several decades. However, one aspect of the Feynman path integral which has received relatively little attention is the functional measure. A recent series of elegant papers by Fujikawa [1] have shown that anomalies can be best understood as arising from the transformation properties of the functional measure; a number of papers stretching back over the past decade-and-a-half have disputed the correct form of the functional measure for gravitation [2-7]; and a few other papers have touched on the subject of the functional measure in one way or another. Virtually all of the remaining multitude of papers on the subject of quantum field theory and path integration have ignored the issue entirely.

This state of affairs has been due to two very simple reasons. First, as we shall see below, the canonical functional measure for most ordinary theories is trivial, with the measure factor being equal to unity. Second, and more importantly, any non-trivial measure factors present would be formally set equal to zero under several very popular regularization schemes.

In Sec. 2, I argue that in a wide range of situations, these regularizations—dimensional regularization and zeta function regularization—are not necessarily legitimate, and the results which they produce should be viewed with caution. Any measure factors which are discarded under these schemes but which survive under other, more intuitively simple regularization schemes should be retained. Under this line of reasoning, the functional measure factors derived in the remaining sections of this paper may be meaningful.

In Sec. 3, I state the fundamental hypothesis of the path integral formulation of quantum field theory, and use it to derive the canonical functional measures for theories of bosonic or fermionic integer spin fields in which all degrees of freedom are physical. I then utilize the Faddeev-Popov ansatz to generalize this result to gauged theories of integer spin fields as well.

In Sec. 4, I apply these results to calculate the canonical functional measures for various scalar and vector theories in an arbitrary number of dimensions. In curved spacetime, the measures of these fields often involve mixings with the measure for gravitation, but I argue that these terms should be neglected at energies low compared to the Planck Mass. Hence, some of the results obtained are valid only in more than two dimensions, where a dimensionful Planck Mass exists.

In Sec. 5, I consider the case of the canonical functional measure for gravitation in an arbitrary number of dimensions. The derivation of this measure is identical in form to those preceding, but is slightly more difficult to actually carry out, and yields the canonical functional measure for gravitation in more than two dimensions at energies low compared to the Planck Mass.

In Sec. 6, the results of the previous sections are analyzed and several interesting conclusions drawn. First, it is suggested that fermions should best be understood as possessing bosonic physical anti-degrees of freedom (and *vice versa*). Second, it is noted that although the canonical functional measure for quantum fields in curved spacetime is usually not manifestly covariant under general coordinate transformations, its non-covariance may be required to cancel the possibly non-covariant point permutation Jacobian produced by a general coordinate transformation; a similar argument may indicate the flaw in the previous derivation of the gravitation functional measure by Fujikawa [7]. Finally, it is noted that only the canonical functional measure possesses the correct form to allow Kaluza-Klein theories to be automatically self-consistent on the quantum level. This appears to strengthen the likelihood that the canonical functional measure is indeed the correct functional measure for a quantum field theory.

In Sec. 7, I derive the canonical functional measure for theories of half-integer spin fields, whose kinetic terms are linear in derivatives. I apply this result to

Dirac, Majorana, and Weyl spinor fields in an arbitrary number of dimensions.

In Sec. 8, I show the proper means of dealing with auxiliary fields, which possess no dynamics, and use the result to determine the canonical functional measures for massive vector field theories in an arbitrary number of dimensions.

Throughout this paper, I shall use units in which $\hbar = c = k = 1$ and all quantities are measured in GeV. My metric convention will be timelike, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, \ldots, -1)$, and Greek letters will range over spacetime coordinates, while capital Latin letters will be completely general field indices. I will adopt usage of the rationalized Newton's Constant, $\bar{G} = 8\pi G$, with the n dimensional (rationalized) Planck Mass being $M_{Planck} = (\bar{G})^{\frac{1}{n-2}}$.

Under my terminology, a propagating degree of freedom will be called "gauge" if the Lagrangian is invariant under a change in its value, and "physical" otherwise. Non-propagating degrees of freedom will be called "constrained" if they correspond to the first-class constraints always paired with gauge conditions [11], and "auxiliary" otherwise. Except for this last distinction (which is more semantic than substantive), this nomenclature accords with supersymmetry usage [12]. These definitions differ slightly from the more common ones in which a field must have on-shell states in order to be truly "physical;" but such a definition breaks down under the Faddeev-Popov procedure, becoming either ambiguous or leading to the non-conservation of physical degrees of freedom.

2. Why the Functional Measure Matters

Consider a quantum field theory defined on some arbitrary space and based on canonical fields Q_A and a Lagrangian $\mathcal{L}[Q_A]$. In its Lagrangian formulation, the Feynman path integral defines a generating functional for the Green's functions of the theory,

$$Z = \int [dQ_A] exp^{i \int d^n x \mathcal{L}[Q_A]}, \tag{2.1}$$

with

$$[dQ_A] = \prod_x M[Q_A]dQ_A \tag{2.2}$$

being the functional measure which should be used. From this generating functional one may calculate the Green's functions either by adding a source term to the action or by using the relationship

$$\langle F[Q_A]\rangle = \frac{\int [dQ_A]F[Q_A]exp^{i\int d^n x\mathcal{L}[Q_A]}}{\int [dQ_A]exp^{i\int d^n x\mathcal{L}[Q_A]}} \tag{2.3}$$

In this procedure, there is no reason for us to assume that the functional measure factor $M[Q_A]$ is trivial, *i.e.* that $[dQ_A] = \prod_x dQ_A$. A non-trivial functional measure factor corresponds to the presence of additional terms in the effective action of our field theory, and these in principle could have a significant impact on the behavior of our theory. Yet this very important aspect of the path integral—the issue of the correct form of the functional measure and the possible existence of non-trivial measure factors—has, with some notable exceptions, received very little attention.

This lack of attention is due to two very simple reasons. First, as we shall see below in Sec. 5, the canonical functional measures for most ordinary field theories are trivial "flat" measures, with measure factors equal to unity. Second, and more importantly, all non-trivial measure factors in the functional measure may be transferred to the effective Lagrangian by means of fictitious "measure ghost" fields similar to the better known Faddeev-Popov ghosts of the gauge-fixing technique [8]. However, all loops involving these measure ghosts will carry factors of the form

$$\int \frac{d^n k}{(2\pi)^n}(k^2)^N, \tag{2.4}$$

with n being our spacetime dimensionality. And these, despite being highly divergent, are formally set equal to zero under dimensional regularization (or zeta-function regularization). The widespread dominance of dimensional regularization over the past decade has nearly eliminated functional measure factors from the thoughts of most theoreticians*

However, dimensional regularization may be argued to be a valid regularization technique only so long as it is unambiguous. This criterion seems roughly satisfied in situations in which the underlying topology of background spacetime is trivial. For example, flat Minkowski space in four dimensions, M^4, can be extended in a natural manner to flat Minkowski space in ω dimensions, M^ω, resulting in a unique regularization of all divergent quantities.

Since ordinary particle physics is usually done on a flat Minkowski background, dimensional regularization is more or less reasonable, and is widely employed. But in situations in which the background space has non-trivial topology (as is often the case with gravity and always the case with Kaluza-Klein theories), this approach breaks down completely.

For example, suppose that our background space is $M^4 \times S^1$. Extending the dimensionality of this space can be done in a number of different ways, namely

$$M^4 \times S^1 \longmapsto M^\omega \times S^1 \;\text{ or }\; M^4 \times S^\omega \;\text{ or }\; M^4 \times (S^1)^\omega \tag{2.5}$$

or any combination of these, and as Hawking has pointed out, the regularized values of divergent loops are dependent upon which dimensional continuation one chooses [10]. Dimensional regularization makes little sense under these circumstances, even apart from its well-known inapplicability to theories with chiral or conformal symmetries.

Zeta function regularization is only slightly more respectable in this context. The regularization technique is unambiguous for a non-trivial spacetime background topology, hence its widespread use in gravity and Kaluza-Klein theory.

⋆ This view that dimensional regularization legitimately eliminates all non-trivial functional measure factors is presented most forcefully by 't Hooft [9].

However, as Hawking himself pointed out [10] in the paper which introduced it to the physics community, the results it produces are identical (up to an unimportant constant normalization factor) to those obtained from dimensional regularization if we choose to append only flat extra dimensions to spacetime; hence in some sense, it is dimensional regularization by the back door. It should be realized that dimensional regularization would also produce unambiguous results if we adopt the arbitrary rule that all continued dimensions shall be flat regardless of the background topology of the original space. Furthermore, zeta function regularization is self-consistent to only one loop anyway.

For these and similar reasons, one should hesitate to ignore divergent terms such as (2.4) which are formally equated to zero under certain regularization schemes (such as dimensional or zeta function regularization), but which retain their full divergent character under other, somewhat more intuitively simple regularizations schemes (such as working on a lattice or using a naive cut-off). Given such a cautious approach, non-trivial factors in the functional measure of a quantum field theory should be retained.

3. The Canonical Functional Measure and How to Derive It

Let us now consider the Hamiltonian formulation of the Feynman path integral for quantum fields Q_A. We have

$$Z = \int [d\Pi^A dQ_A] exp^{i\int d^n x \Pi^A \partial_0 Q_A - \mathcal{H}[\Pi^A, Q_A]} \tag{3.1}$$

with our canonical momenta being defined by

$$\Pi^A = \frac{\delta \mathcal{L}}{\delta(\partial_0 Q_A)} \tag{3.2}$$

and with

$$\mathcal{H} = \Pi^A \partial_0 Q_A - \mathcal{L}[Q_A, \partial_0 Q_A]. \tag{3.3}$$

According to the fundamental hypothesis of path integration, our Green's functions may be calculated from our generating functional Z by the Hamiltonian version of (2.3)

$$\left\langle F[\Pi^A, Q_A]\right\rangle = \frac{\int [d\Pi^A dQ_A] F[\Pi^A, Q_A] exp^{i\int d^n x \Pi^A \partial_0 Q_A - \mathcal{H}[\Pi^A, Q_A]}}{\int [d\Pi^A dQ_A] exp^{i\int d^n x \Pi^A \partial_0 Q_A - \mathcal{H}[\Pi^A, Q_A]}}. \tag{3.4}$$

Furthermore, the hypothesis holds that the correct functional measure in this Hamiltonian formulation is the canonical functional measure, in which the integration extends over all physically distinct field configurations, and weights each by the same trivial factor of unity [3, 4, 8].

$$[d\Pi^A dQ_A] = \prod_x \left(d\Pi^A(x) dQ_A(x)\right)_{physical}. \tag{3.5}$$

Under this elegant hypothesis, our generating functional Z represents a quantum partition functional in which each physically distinct unit of classical phase space is weighted by the exponential of its quantum action. It has been argued that only this choice of the functional measure ensures the overall unitarity of our quantum field theory [4-5]. The canonical functional measure for the more commonly seen Lagrangian formulation of the path integral is obtained by formally performing the functional integration over the canonical momenta in (3.1).

The above formulation—in which the canonical coordinates and momenta are independent, quantum mechanically conjugate variables—applies only in the case that our Lagrangian is quadratic in (time) derivatives. This is because the second order differential field equations obtained from such a Lagrangian requires the values of the fields and their first time derivatives to be specified on each spacelike hypersurface in order to determine the subsequent evolution of the fields in the path integral expansion. On the other hand, Lagrangians which contain terms cubic or higher in derivatives tend to lead to violations of unitarity and can be ignored for our purposes. Therefore, let us restrict our attention to Lagrangians which are quadratic in derivatives, and hence are based on fields of integer spin. Furthermore, let us temporarily impose the simplifying assumptions that (A) our Lagrangian is non-degenerate (*i.e.* has no gauge symmetry or constraints on its field variables Q_A, implying that all our canonical variables represent physical degrees of freedom) and (B) all our fields are bosonic in nature.

Under these conditions, we can rewrite our Lagrangian as

$$\mathcal{L} = \frac{1}{2} D^{AB}(\partial_0 Q_A)(\partial_0 Q_B) + E^A(\partial_0 Q_A) + F, \tag{3.6}$$

with D^{AB}, E^A, F being functionals of fields Q_A and their spatial derivatives, and with $det D^{AB} \neq 0$ (our non-degeneracy condition). This implies that our canonical momenta are equal to

$$\Pi^A = \frac{\delta \mathcal{L}}{\delta(\partial_0 Q_A)} = D^{AB}\partial_0 Q_B + E^A. \tag{3.7}$$

We can use this result to solve for the Hamiltonian which corresponds to our Lagrangian

$$\mathcal{H}[\Pi^A, Q_A] = \frac{1}{2}\Pi^A (D^{-1})_{AB}\Pi^B - E^A (D^{-1})_{AB}\Pi^A - F. \tag{3.8}$$

With our Hamiltonian now known, we can directly perform the functional integration over canonical momenta in our Hamiltonian path integral (3.1), and find that our generating functional becomes [8]

$$Z = \int \prod_x dQ_A \left[det D^{AB}\right]^{\frac{1}{2}} exp^{i\int d^n x \mathcal{L}[Q_A, \partial_0 Q_A]}. \tag{3.9}$$

Our path integral has now been put into its Lagrangian form, and we have also determined the form of the canonical functional measure

$$[dQ_A] = \prod_x \Big[det\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\Big)\Big]^{\frac{1}{2}} dQ_A. \tag{3.10}$$

This functional measure is potentially non-trivial.* It should be noted that this measure is independent of the particular ordering we choose for our canonical fields and momenta in the Hamiltonian; all resulting commutators would be at most linear in canonical momenta, and hence would not contribute to the measure factor.

If we relax one of our simplifying assumptions, and allow our canonical fields to be fermionic, it is easy to see that the resulting canonical functional measure would be

$$[dQ_A] = \prod_x \Big[det\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\Big)\Big]^{-\frac{1}{2}} dQ_A, \tag{3.11}$$

with the inverse power of the measure factor being due to the special nature of fermionic integration. We can even consider the situation in which some of our fields are bosonic, some are fermionic, and terms in our Hamiltonian possibly contain mixtures of fermionic and bosonic conjugate momenta. The functional measure factor in this case is simply

$$[dQ_A] = \prod_x \Big[sdet\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\Big)\Big]^{\frac{1}{2}} dQ_A, \tag{3.12}$$

with *sdet*, the superdeterminant, being defined by [12]

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* This same result may be obtained [14] by rotating to Euclidean space, introducing a new "pseudo-time" coordinate, and treating the theory as being one in classical statistical mechanics.

$$sdet\begin{pmatrix} M_{bb} & M_{fb} \\ M_{bf} & M_{ff} \end{pmatrix} = \frac{detM_{bb}}{det(M_{ff} - M_{fb}M_{bb}^{-1}M_{bf})} = \frac{det(M_{bb} - M_{bf}M_{ff}^{-1}M_{fb})}{detM_{ff}}, \tag{3.13}$$

with M_{bb} being the bosonic-bosonic submatrix, M_{ff} the fermionic-fermionic submatrix, and M_{bf} and M_{fb} the mixed bosonic-fermionic submatrices of the supermatrix.

If we relax the second of our simplifying assumptions, and attempt to calculate the functional measure for a theory based on a degenerate Lagrangian, we face a somewhat more difficult task. One approach is to determine the physical portion of the Hamiltonian phase space measure in terms of the total phase space measure by using the restrictions imposed by the constraints on our canonical variables [3-5]; equivalently, we can demand that our measure be chosen so as exactly to cancel all divergences of the form $\delta^{(n)}(0)$ in our Lagrangian effective action [4-6]. However, the easiest and most intuitive approach relies on applying the Faddeev-Popov ansatz to our theory based on a degenerate (*i.e.* gauge invariant) Lagrangian. Under this well-known technique, the Lagrangian for our theory is replaced by

$$\mathcal{L}_0 \mapsto \mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost}. \tag{3.14}$$

The gauge-fixing piece of our new Lagrangian is chosen so as to remove the degeneracy of our original Lagrangian, while the ghost piece compensates for this gauge-fixing piece, and contains additional ghost fields which have commutation relations opposite to those of our original fields. The extra physical degrees of freedom produced by our gauge-fixing procedure are exactly compensated for (and cancelled) by these ghost fields, which (as will be discussed in Sec. 6) should be understood as possessing physical anti-degrees of freedom.

Now the central hypothesis of the Faddeev-Popov procedure is that the quantum field theory based on our modified Lagrangian $\mathcal{L}'$ and fields Q_A, η_B yields identical physical results (*i.e.* Green's functions) to those produced by our original Lagrangian $\mathcal{L}_0$ and fields Q_A. But if this is true, then we can make the modified form of the Lagrangian the starting point of our analysis, and consider the corresponding Hamiltonian formulation in order to determine the canonical functional measure for the Lagrangian formulation of the theory.*

This determination then becomes quite easy. Our modified Lagrangian is non-degenerate, with all its field configurations being physical, and we can directly apply the results previously obtained for non-degenerate Lagrangians. *However, we must consider the measure factors being contributed both from the ordinary fields and from the ghost fields of our modified Lagrangian; our full functional measure factor includes both of these contributions.* The results obtained by this procedure are formally identical to those derived from the more cumbersome constraint procedure [4].

4. Simple Cases

Let us apply this powerful formal machinery of the Hamiltonian path integral analysis to derive the canonical functional measures for various commonly encountered theories.

Consider a scalar field theory in a flat spacetime. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \mathcal{L}_{int}[\phi]. \tag{4.1}$$

Therefore, the canonical functional measure is given by

$$[d\phi] = \prod_x \left[\frac{\delta^2 \mathcal{L}}{\delta(\partial_0\phi)\delta(\partial_0\phi)}\right]^{\frac{1}{2}} d\phi = \prod_x d\phi, \tag{4.2}$$

* This technique was previously noted by Fradkin and Vilkovisky [4] for the case of gravitation, but applied incorrectly.

and is trivial. On the other hand, for a non-linear sigma model with Lagrangian

$$\mathcal{L} = \frac{1}{2(1+\phi_a\phi_a)^2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) + \mathcal{L}_{int}[\phi_a] \quad \text{with} \quad a = 1,2,3, \tag{4.3}$$

the non-trivial functional measure is

$$[d\phi_a] = \prod_x \frac{1}{(1+\phi_a\phi_a)^3} d\phi_a. \tag{4.4}$$

It should be noted that the canonical formalism thus automatically reproduces the correct group-invariant measure which should be used for the non-linear sigma model [14].

Now let us consider the slightly more complicated case of an abelian gauge theory in flat spacetime. Our initial Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2. \tag{4.5}$$

Adding the appropriate gauge-fixing and ghost terms of the Faddeev-Popov ansatz, chosen for Feynman gauge, we obtain the modified effective Lagrangian

$$\mathcal{L}' = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu \bar{\eta})(\partial^\mu \eta). \tag{4.6}$$

The functional measure factor due to our vector field is given by

$$\left[det\left(\frac{\delta^2\mathcal{L}}{\delta(\partial_0 A_\mu)\delta(\partial_0 A_\nu)}\right)\right]^{\frac{1}{2}} = \text{constant}, \tag{4.7}$$

while the functional measure contribution from our ghost fields is

$$\left[det\left(\frac{\delta^2\mathcal{L}}{\delta(\partial_0 \tilde{\eta}_a)\delta(\partial_0 \tilde{\eta}_b)}\right)\right]^{-\frac{1}{2}} = \text{constant}, \tag{4.8}$$

where we have defined $\tilde{\eta}_a = (\eta, \bar{\eta})$. Both of these contributions are trivial constants which can be absorbed into our overall normalization factor, so their product, the total functional measure factor for an abelian gauge theory in flat spacetime is also trivial. Equivalently, the total functional measure could have been written more compactly as the superdeterminant

$$\left[sdet\left(\frac{\delta^2\mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\right)\right]^{\frac{1}{2}} = \text{ constant,} \tag{4.9}$$

with $Q_A = (A_\mu, \eta, \bar{\eta})$. Since there are no Hamiltonian terms which contain both vector field and ghost field canonical momenta, our supermatrix is block diagonal, and its superdeterminant is indeed simply the product of (4.7) and (4.8).

The case of a non-abelian gauge theory in flat spacetime is just as simple. After adding gauge-fixing and Faddeev-Popov ghost terms (again chosen in Feynman gauge) our effective Lagrangian is

$$\begin{aligned}\mathcal{L} = \sum_{a=1}^{K} -\frac{1}{2}(\partial_\mu A_\nu^a)(\partial^\mu A^{\nu a}) + (\partial_\mu \bar{\eta}^a)(\partial^\mu \eta^a)& \\ + \text{ terms linear or lower in derivatives.}&\end{aligned} \tag{4.10}$$

The only portion of the Lagrangian which contributes to the functional measure is the portion quadratic in derivatives, and this portion is exactly the same as for an abelian gauge theory. The total measure factor is once again given by (4.9), where this time $Q_A = (A_\mu^a, \eta^a, \bar{\eta}^a)$. The resulting expression is the result for the abelian case raised to the power of K, the number of gauge fields in the theory, and is once again trivial.

We have shown that the canonical functional measure factors for free scalar fields and free vector gauge fields are completely trivial in flat spacetime. It will be shown below (in Secs. 7 and 8) that the canonical functional measure factors for free spinor and free massive vector fields in flat spacetime are also trivial. Now any interactions involving these various scalar, vector, and spinor fields will be at most linear in derivatives; therefore such interactions cannot contribute to our functional measure factors. This implies that virtually all familiar quantum field theories formulated in flat spacetime have trivial functional measures (the non-linear sigma models, mentioned above, are about the only significant exceptions). This is the only reason that these familiar quantum field theories—formulated by naively ignoring functional measure factors—are nonetheless correct.

If we extend our analysis to curved spacetime (*i.e.* theories in which gravitation is quantized), the functional measure factors for all of these theories become highly non-trivial. For a minimally-coupled scalar field, our Lagrangian is

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sqrt{-g}m^2\phi^2 + \mathcal{L}[\phi, g_{\mu\nu}]. \tag{4.11}$$

The functional measure factor for this scalar field theory is given by

$$\left[det\Big(\frac{\delta^2\mathcal{L}}{\delta(\partial_0\phi)\delta(\partial_0\phi)}\Big)\right]^{\frac{1}{2}}, \tag{4.12}$$

and the canonical functional measure for ϕ is thus

$$[d\phi] = \prod_x (g^{00})^{1/2} g^{1/4} d\phi. \tag{4.13}$$

Moving to the case of an abelian gauge theory in curved spacetime, the situation is slightly more complex. Our initial Lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\lambda\sigma}F_{\mu\lambda}F_{\nu\sigma} \quad \text{with} \\ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu.\end{aligned} \tag{4.14}$$

Applying the curved space form of the Faddeev-Popov ansatz [4,20] and choosing Feynman gauge, we obtain the modified Lagrangian

$$\begin{aligned}\mathcal{L}' = &-\frac{1}{2}\sqrt{-g}g^{\mu\nu}g^{\lambda\sigma}(\nabla_\mu A_\lambda - \nabla_\lambda A_\mu)\nabla_\nu A_\sigma - \frac{1}{2}\sqrt{-g}g^{\mu\nu}g^{\lambda\sigma}\nabla_\mu A_\nu \nabla_\lambda A_\sigma \\ &+ \partial_\mu(\bar\eta(-g)^{-1/4})g^{\mu\nu}\sqrt{-g}\partial_\nu\eta.\end{aligned} \tag{4.15}$$

Obtaining the functional measure factor for such a Lagrangian is slightly more difficult than it might seem. This is because the theory actually contains terms involving the time derivatives of our metric field as well as the time derivatives of our vector and ghost fields, most obviously since

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu} A_\sigma = \partial_\mu A_\nu - \frac{1}{2} A^\sigma(-g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu}). \qquad (4.16)$$

Therefore, the Hamiltonian which we obtain from our Lagrangian will contain terms which mix the canonical momenta of our vector and ghost fields with those of the gravitational field, and the supermatrix whose superdeterminant produces our functional measure factor

$$\left(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\right) \text{ with } Q_A = (g_{\mu\nu}, A_\mu, \eta, \bar{\eta}, \ldots) \qquad (4.17)$$

will *not* be block diagonal in its gravitational and vector/ghost sectors, and, in general, will be quite difficult to evaluate.

Fortunately, there is a means of resolving this serious difficulty. If we are working in more than two dimensions the graviton-graviton diagonal block element of our measure supermatrix which derives from the purely gravitational portion of our Lagrangian

$$\mathcal{L}_{grav} = -\frac{1}{2G}\sqrt{-g}R, \qquad (4.18)$$

is proportional to a power of the Planck Mass (the square of the Planck Mass in four dimensions), while the graviton-graviton and graviton-vector/ghost block elements obtained from (4.15) are merely proportional to the appropriate powers of the values of the vector/ghost fields. Therefore, at ordinary energies, these latter entries contribute negligibly to the superdeterminant, and may be ignored in a computation of the total functional measure. Our supermatrix becomes effectively block diagonal in its vector/ghost and gravitational sectors, and the low energy effective functional measure for the vector/ghost sector is given by

$$\left[sdet\Big(\frac{\delta^2\mathcal{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\Big)\right]^{\frac{1}{2}} = \frac{\left[det(-g)^{1/2}g^{00}g^{\lambda\sigma}\right]^{1/2}}{\left[\left[(-g)^{1/4}g^{00}\right]^2\right]^{1/2}} = (g^{00})^{\frac{n-2}{2}}g^{\frac{n-3}{2}} \qquad (4.19)$$

with $Q_A = (A_\mu, \bar{\eta}, \eta)$ and with n being the dimensionality of our spacetime.

This approximation based on neglecting terms involving the ratios of our various quantum fields to the Planck Mass is completely justified if we are regulating our theory by using a naive cut-off (or inverse lattice spacing) which is small compared to the Planck Mass. Such a procedure may seem questionable, but is our only option given the lack of an acceptable full theory of quantum gravitation. In fact, the terms being neglected would have to be neglected under any circumstances. This is because the low energy effective Lagrangian for the true theory of quantum gravitation presumably contains higher mass-dimension interactions which are suppressed by appropriate inverse powers of the Planck Mass, and these (completely unknown) terms would produce contributions to our functional measure factors of exactly the same magnitude as the above mixing terms. Since we have no choice but to neglect the contributions of the unknown residue terms of quantum gravity, we must neglect the mixing terms as well.

Thus, the canonical functional measure for an abelian vector field theory in n dimensional curved spacetime is given by

$$[dA_\mu]^{(n)} = \prod_x (g^{00})^{\frac{n-2}{2}}g^{\frac{n-3}{4}}dA_\mu d\bar{\eta}d\eta. \qquad (4.20)$$

This analysis is repeated exactly for the case of a non-abelian gauge theory: since the non-abelian character of the theory only manifests itself in Lagrangian terms containing fewer than two derivatives, it has no effect on the form of the functional measure (just as it did not in the flat spacetime case). If our theory has K vector fields (*i.e.* if the adjoint representation of our symmetry group is K dimensional, our canonical functional measure is

$$[dA^a_\mu]^{(n)} = \prod_x \left((g^{00})^{\frac{n-2}{2}} g^{\frac{n-3}{4}}\right)^K dA^a_\mu d\bar{\eta}^a d\eta^a. \tag{4.21}$$

5. The Canonical Functional Measure for Quantum Gravitation

Now that these simple *hors d'oeuvres* have been served and eaten (and perhaps even digested), we are properly prepared to begin the main course: deriving the canonical functional measure for gravitation itself. The techniques to be used are no different; but the calculation is much more involved.

The Einstein-Hilbert Lagrangian for pure gravitation may be written as [15]

$$\begin{aligned}\mathcal{L}_{EH} =& -\frac{1}{2\bar{G}}\sqrt{-g}R \\ =& -\frac{\sqrt{-g}}{8\bar{G}}(g^{\nu\lambda}g^{\sigma\tau}g^{\mu\rho} - g^{\nu\mu}g^{\lambda\rho}g^{\sigma\tau} + 2g^{\nu\mu}g^{\lambda\tau}g^{\sigma\rho} - 2g^{\tau\mu}g^{\rho\sigma}g^{\nu\lambda})g_{\nu\lambda,\sigma}g_{\mu\rho,\tau}. \\ &+ \text{ total divergence.}\end{aligned} \tag{5.1}$$

Let us choose our gauge-fixing interaction to be

$$\mathcal{L}_{GF} = \frac{1}{2\alpha}\eta_{\alpha\beta}F^\alpha F^\beta = \frac{1}{2\alpha}\eta_{\alpha\beta}\left(\frac{1}{\sqrt{\bar{G}}}h_\mu{}^\beta(-g)^{-1/4}\partial_\sigma(\sqrt{-g}g^{\sigma\mu})\right)^2. \tag{5.2}$$

Note that this term in the action has the proper negative-definite form only after we have Wick-rotated to Euclidean space. The Faddeev-Popov ghost interactions designed to compensate for this new gauge-fixing interaction are given by the formal expression

$$\mathcal{L}_{ghost} = \bar{\eta}_\beta \frac{\delta F^\beta}{\delta \xi^\nu}\eta^\nu \equiv \bar{\eta}_\beta \delta F^\beta|_{\xi^\nu=\eta^\nu}, \tag{5.3}$$

where $\bar{\eta}, \eta$ are our Faddeev-Popov ghost fields and ξ^ν represents our infinitesimal gauge transformation parameter, corresponding in this case to an infinitesimal general coordinate transformation. The Lagrangian in (5.3) is evaluated by noting the behavior of our various metric objects under such a transformation, namely

$$\begin{aligned} \delta x^\mu &= \xi^\mu \\ \delta g_{\mu\nu} &= g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda \\ \delta g^{\mu\nu} &= g^{\mu\lambda}\partial_\lambda \xi^\nu + g^{\lambda\nu}\partial_\lambda \xi^\mu \\ \delta(-g)^K &= 2K(-g)^K \partial_\nu \xi^\nu \\ \delta h_\nu{}^\beta &= h_\nu{}^\beta \partial_\mu \xi^\nu . \end{aligned} \tag{5.4}$$

Using these relations and relabelling a few of our indices for purposes of convenience, we obtain

$$\begin{aligned} \delta F^\beta = \frac{1}{\sqrt{G}}(-g)^{-1/4}\Big[& h_\nu{}^\beta \partial_\mu \xi^\nu \partial_\rho(\sqrt{-g} g^{\rho\mu}) - \frac{1}{2}\partial_\nu \xi^\nu h_\mu{}^\beta \partial_\rho(\sqrt{-g} g^{\rho\mu}) \\ & + h_\mu{}^\beta \partial_\rho(\sqrt{-g}\partial_\nu \xi^\nu g^{\rho\mu}) + h_\mu{}^\beta \partial_\nu(\sqrt{-g}\partial_\rho \xi^\nu g^{\rho\mu}) \\ & + h_\nu{}^\beta \partial_\mu(\sqrt{-g}\partial_\rho \xi^\nu g^{\rho\mu})\Big]. \end{aligned} \tag{5.5}$$

From this expression, we can obtain the ghost interactions of our Lagrangian by applying (5.3).

Half of our ghost fields transform like the components of a world vector, η^ν, while the other half transform like world scalars, $\bar{\eta}_\beta$. This is inconvenient, so we should use a change of variables and redefine $\bar{\eta}_\mu \equiv \bar{\eta}_\beta h_\mu{}^\beta$.* Also, we should use integration by parts to put our Lagrangian into a form in which no field has more than one time derivative acting upon it (the proper form for making the transition to the Hamiltonian formalism). After these two modifications, our

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* Actually, such a naive change of variables in our path integral is not quite correct (contrary to the claims of Coleman [8] and 't Hooft [16]). But in a paper which has received insufficient notice, Gervais and Jervicki [17] have worked out the correct procedure for changing functional variables of integration, and described the additional terms in the effective action which must be added at two loops and higher. For our purposes, the important point is that these additional terms do not involve the canonical momenta, hence do not contribute to the functional measure.

ghost Lagrangian assumes its final form

$$\mathcal{L}_{ghost} = \frac{1}{\sqrt{\bar{G}}}\Big[(-g)^{-1/4}\bar{\eta}_\nu\partial_\mu\eta^\nu\partial_\rho(\sqrt{-g}g^{\rho\mu}) - \frac{1}{2}(-g)^{-1/4}\bar{\eta}_\mu\partial_\nu\eta^\nu\partial_\rho(\sqrt{-g}g^{\rho\mu})$$
$$- \partial_\rho\big((-g)^{-1/4}\bar{\eta}_\mu\big)\sqrt{-g}g^{\rho\mu}\partial_\nu\eta^\nu - \partial_\nu(-g)^{-1/4}\bar{\eta}_\mu\big)\sqrt{-g}g^{\mu\rho}\partial_\rho\eta^\nu$$
$$- \partial_\tau\big((-g)^{-1/4}\bar{\eta}_\nu\big)\sqrt{-g}g^{\tau\rho}\partial_\rho\eta^\nu\Big]. \tag{5.6}$$

Therefore, our complete modified Lagrangian for gravitation is

$$\mathcal{L}' = \mathcal{L}_{EH} + \mathcal{L}_{GF} + \mathcal{L}_{ghost}, \tag{5.7}$$

and our gravitational functional measure factor is given by

$$\Big[sdet\Big(\frac{\delta^2\mathcal{L}'}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\Big)\Big]^{\frac{1}{2}}, \tag{5.8}$$

with $Q_A = (g_{\mu\nu}, \bar{\eta}_\mu, \eta^\nu)$.

Evaluating the superdeterminant of such a supermatrix, containing graviton-graviton, graviton-ghost, and ghost-ghost sectors, would be a very formidable computation. However, just as in the case of vector fields, we may fortunately use energy scaling arguments to simplify our task considerably. First, note that the nature of the Faddeev-Popov ansatz has ensured that the dimensional constant in front of our ghost action is $\frac{1}{\sqrt{\bar{G}}}$, while the constant in front of our Einstein-Hilbert and gauge-fixing action terms is $\frac{1}{G}$. Now in n dimensions, $\bar{G}$ has units of $(mass)^{-(n-2)}$. Therefore, our ghost fields have units of $[\bar{\eta}\eta] = (mass)^{\frac{n-2}{2}}$. Choosing how to distribute these units between a ghost and its conjugate ghost is arbitrary since they enter our Lagrangian only in pairs (and the choice will not affect the argument which follows); therefore, let us choose a symmetric distribution, with $[\bar{\eta}] = [\eta] = (mass)^{\frac{n-2}{4}}$.

Let us rewrite the matrix appearing in (5.8) in slightly more detailed form as

$$\begin{pmatrix} \frac{\delta^2 \mathcal{L}_{EH}}{\delta \dot{g}_A \delta \dot{g}_B} + \frac{\delta^2 \mathcal{L}_{GF}}{\delta \dot{g}_A \delta \dot{g}_B} + \frac{\delta^2 \mathcal{L}_{ghost}}{\delta \dot{g}_A \delta \dot{g}_B} & \frac{\delta^2 \mathcal{L}_{ghost}}{\delta \dot{\eta}_A \delta \dot{g}_B} \\ \frac{\delta^2 \mathcal{L}_{ghost}}{\delta \dot{g}_A \delta \dot{\eta}_B} & \frac{\delta^2 \mathcal{L}_{ghost}}{\delta \dot{\eta}_A \delta \dot{\eta}_B} \end{pmatrix}, \tag{5.9}$$

with $\eta_A = (\bar{\eta}_\mu, \eta^\nu)$ and $g_A = g_{\mu\nu}, \mu \geq \nu$. These different blocks represent the bosonic-bosonic, fermionic-fermionic, and mixed bosonic-fermionic sectors of our supermatrix. Note that the bare and gauge-fixing portions of our total Lagrangian contribute only to the bosonic sector since they contain no fermionic (ghost) fields.

Now the bare and gauge-fixing contributions to our bosonic-bosonic submatrix have the approximate magnitude

$$\frac{\delta^2 \mathcal{L}_{EH}}{\delta \dot{g}_A \delta \dot{g}_B} + \frac{\delta^2 \mathcal{L}_{GF}}{\delta \dot{g}_A \delta \dot{g}_B} \approx (M_{Planck})^{n-2}, \tag{5.10}$$

while the ghost Lagrangian contribution to this submatrix has the form

$$\frac{\delta^2 \mathcal{L}_{ghost}}{\delta \dot{g}_A \delta \dot{g}_B} \approx (M_{Planck})^{\frac{n-2}{2}} (\bar{\eta}\eta). \tag{5.11}$$

Therefore, just as in the earlier vector field case, the contribution from (5.11) may be neglected compared with the contribution from (5.10) when evaluating the superdeterminant of (5.9) (so long as the submatrix in (5.10) is non-singular, which it is, since that requirement determined our choice of gauge-fixing interaction).

Next, let us consider the scaling behavior of the remaining contributions to the supermatrix (5.9). These scale as

$$\mathbf{M} = \begin{pmatrix} (M_{Planck})^{n-2} & (M_{Planck})^{\frac{n-2}{2}} \eta \\ (M_{Planck})^{\frac{n-2}{2}} \eta & (M_{Planck})^{n-2} \end{pmatrix}. \tag{5.12}$$

(The fact that the different blocks of our supermatrix possess different mass-dimensions is not at all alarming, and indeed should be expected since our various fields have different mass-dimensions; we are only interested in the superdeterminant of this supermatrix, which *is* homogenous in mass-dimensions.) If we are working at energies low compared to the Planck Mass, the dominant contribution to this superdeterminant comes from the block-diagonal determinants (so long as these are non-singular). All off-block-diagonal contributions are suppressed by at least

$$\frac{\bar{\eta}\eta}{(M_{Planck})^{\frac{n-2}{2}}}. \tag{5.13}$$

If we neglect these Planck Mass suppressed off-block-diagonal contributions (using the same reasoning as in the vector field case), our gravitational canonical functional measure assumes the simple form

$$\left[det\Big(\frac{\delta^2(\mathcal{L}_{EH}+\mathcal{L}_{GF})}{\delta(\partial_0 g_A)\delta(\partial_0 g_B)}\Big)\right]^{1/2}\left[det\Big(\frac{\delta^2\mathcal{L}_{ghost}}{\delta(\partial_0\eta_A)\delta(\partial_0\eta_B)}\Big)\right]^{-1/2}. \tag{5.14}$$

We can begin to evaluate the first of these two determinants by putting our gauge-fixing Lagrangian into a more convenient form. First, let us use the relation

$$\partial_\sigma(\sqrt{-g}g^{\sigma\mu}) = -(-g)^{1/2}g^{\lambda\sigma}\Gamma^{\mu}_{\lambda\sigma} = -\frac{1}{2}(-g)^{1/2}g^{\lambda\sigma}g^{\mu\rho}(-g_{\lambda\sigma,\rho}+2g_{\lambda\rho,\sigma}) \tag{5.15}$$

to rewrite our gauge-fixing interaction (5.2) as

$$\mathcal{L}_{GF} = \frac{1}{8\alpha\bar{G}}\sqrt{-g}g^{\lambda\sigma}g^{\rho\tau}g^{\mu\nu}(-g_{\lambda\sigma,\mu}+2g_{\mu\lambda,\sigma})(-g_{\rho\tau,\nu}+2g_{\nu\rho,\tau}). \tag{5.16}$$

Using the symmetries of the indices, two of the terms can be combined, and the entire expression put into its penultimate form

$$\mathcal{L}_{GF} = \frac{1}{8\alpha\bar{G}}\sqrt{-g}\big(g^{\lambda\sigma}g^{\rho\tau}g^{\mu\nu} - 4g^{\rho\tau}g^{\lambda\mu}g^{\sigma\nu} + 4g^{\lambda\mu}g^{\rho\nu}g^{\sigma\tau}\big)g_{\lambda\sigma,\mu}g_{\rho\tau,\nu}. \tag{5.17}$$

Relabelling our indices, and combining this expression with (5.1), we obtain the complete graviton self-interaction (in Feynman gauge, $\alpha = 1$)

$$\mathcal{L}_{EH} + \mathcal{L}_{GF} = -\frac{\sqrt{-g}}{8\bar{G}}\Big(4g^{\lambda\rho}g^{\mu\sigma}g^{\nu\tau} - 4g^{\mu\sigma}g^{\rho\tau}g^{\nu\lambda} + 2g^{\nu\lambda}g^{\mu\tau}g^{\sigma\rho} - 2g^{\tau\lambda}g^{\rho\sigma}g^{\nu\mu} - g^{\nu\lambda}g^{\mu\rho}g^{\sigma\tau}\Big)g_{\nu\mu,\sigma}g_{\lambda\rho,\tau}. \tag{5.18}$$

We are interested in computing the determinant of the matrix

$$\frac{\delta}{\delta(\partial_0 g_{\alpha\beta})}\frac{\delta}{\delta(\partial_0 g_{\gamma\delta})}(\mathcal{L}_{EH} + \mathcal{L}_{GF}), \tag{5.19}$$

with $\beta \geq \alpha, \delta \geq \gamma$. Proceeding carefully, we should note that our variational derivatives $\frac{\delta}{\delta(\partial_0 g_{\mu\nu})}$ act only on the terms which appear at the extreme right of (5.18), and that

$$\frac{\delta}{\delta(\partial_0 g_{\alpha\beta})}\frac{\delta}{\delta(\partial_0 g_{\gamma\delta})}g_{\nu\mu,\sigma}g_{\lambda\rho,\tau} = \delta^0_\sigma\delta^0_\tau\Big[(\delta^\alpha_\nu\delta^\beta_\mu + \delta^\alpha_\mu\delta^\beta_\nu)(\delta^\gamma_\lambda\delta^\delta_\rho + \delta^\gamma_\rho\delta^\delta_\lambda) + (\delta^\alpha_\lambda\delta^\beta_\rho + \delta^\alpha_\rho\delta^\beta_\lambda)(\delta^\gamma_\nu\delta^\delta_\mu + \delta^\gamma_\mu\gamma^\delta_\nu)\Big]. \tag{5.20}$$

Therefore, (5.19) can be computed to be

$$-\frac{\sqrt{-g}}{2\bar{G}}\big[2g^{\alpha 0}g^{\beta 0}g^{\gamma\delta} + 2g^{\gamma 0}g^{\delta 0}g^{\alpha\beta} - g^{\alpha 0}g^{\gamma 0}g^{\beta\delta} - g^{\alpha 0}g^{\delta 0}g^{\beta\gamma} - g^{\beta 0}g^{\gamma 0}g^{\alpha\delta} - g^{\beta 0}g^{\delta 0}g^{\alpha\gamma} - g^{00}g^{\alpha\gamma}g^{\beta\delta} - g^{00}g^{\alpha\delta}g^{\beta\gamma}\big]. \tag{5.21}$$

This $\frac{n(n+1)}{2}$ by $\frac{n(n+1)}{2}$matrix determinant (with n being the dimensionality of our spacetime) is extremely difficult to evaluate in general. However, the overall functional factor of $\sqrt{-g}$ contributes a determinantal factor of $g^{\frac{n(n+1)}{4}}$; and, in keeping with our previous approximations, if we assume that we are working at energies low compared with the Planck Mass, our metric field can be expanded out in the linearized tensor field approximation to gravitation, namely

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \quad \text{with} \quad \eta^{\mu\nu} = diag(+1,-1,\ldots,-1) \quad \text{and} \quad h^{\mu\nu} \ll 1. \tag{5.22}$$

Inserting this into (5.21), a lengthy computation shows that the square-root of the determinant of our graviton-graviton measure matrix is (to lowest order in $h^{\mu\nu}$ and dropping an unimportant constant factor) given by

$$g^{\frac{n(n+1)}{8}}\Big(1 - \frac{(n+1)(n+2)}{4}h^{00} - \frac{n+1}{2}\sum_a h^{aa}\Big) = (g^{00})^{\frac{n(n+1)}{4}} g^{\frac{n^2-3n-4}{8}} \tag{5.23}$$

as might have been guessed from naive index-counting arguments. (Actually, the determinant is singular in two spacetime dimensions, but this need not concern us since most of our previous approximations were invalid in two dimensions as well.)

Finally, there remains only the computation of the determinant of our ghost-ghost measure matrix from (5.6). This is given by

$$\begin{aligned} det\Big[\frac{\delta^2 \mathcal{L}_{ghost}}{\delta(\partial_0\eta_A)\delta(\partial_0\eta_B)}\Big] &= det\Big[\Big(\frac{\delta^2 \mathcal{L}_{ghost}}{\delta(\partial_0\bar{\eta}_\mu)\delta(\partial_0\eta^\nu)}\Big)^2\Big] \\ &= det\Big[\Big((-g)^{1/4}(2g^{\sigma 0}\delta^\mu_\sigma\delta^0_\nu + g^{00}\delta^\mu_\sigma\delta^\sigma_\nu)\Big)^2\Big], \end{aligned} \tag{5.24}$$

and the measure factor contributed by the inverse of the square-root of this determinant is

$$g^{\frac{n}{4}}(g^{00})^{-n}. \tag{5.25}$$

Thus, the total canonical functional measure for gravitation in n spacetime dimensions is

$$[dg_{\mu\nu}]^{(n)} = \prod_x (g^{00})^{\frac{n(n-3)}{4}} g^{\frac{n^2-5n-4}{8}} dg_{\mu\nu} d\bar{\eta}_\mu d\eta^\nu. \tag{5.26}$$

This expression differs by a factor of $g^{1/2}$ from the four dimensional special case worked out by Leutwyler [2], Fradkin and Vilkovisky [4], and others [6]. This is because under my choice of canonical variables, both of the ghost fields transform as world vectors, while under theirs, one would transform as a world vector and the other as a tangent space vector and set of world scalars. The extra factor of $g^{1/2}$ merely represents the determinant of the Jacobian between these two arbitrary choices of functional integration variables. Although all of these authors wrongly ignore the effects which correspond to the graviton-graviton and graviton-ghost blocks in our measure supermatrix deriving from the ghost Lagrangian, they are fortunate enough to obtain the correct answer (for four dimensions) anyway: as we have seen, these additional contributions are negligible at energies far below the Planck Mass.

The particular functional measure factor in (5.26) was first suggested (for the four dimensional case) by Leutwyler [2], and first derived in detail by Fradkin and Vilkovisky [4], who used the constraint-elimination procedure previously suggested by Faddeev and Popov [3]. Faddeev and Popov's own derivation had contained an error and resulted in a different answer, which they have since retracted.*

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* I am grateful to Richard Woodard for informing me of this last fact.

6. What These Results Mean

To summarize our results, the canonical functional measures for scalar, vector, and gravitational fields in n dimensional curved spacetime and at low energies compared to the Planck Mass are given by

$$\begin{aligned}
[d\phi]^{(n)} &= \prod_x (g^{00})^{1/2} g^{1/4} d\phi \equiv \prod_x M_\phi^{(n)} d\phi \\
[dA_\mu]^{(n)} &= \prod_x (g^{00})^{\frac{n-2}{2}} g^{\frac{n-3}{4}} dA_\mu d\bar{\eta} d\eta \equiv \prod_x M_{A_\mu}^{(n)} dA_\mu d\bar{\eta} d\eta \\
[dg_{\mu\nu}]^{(n)} &= \prod_x (g^{00})^{\frac{n(n-3)}{4}} g^{\frac{n^2-5n-4}{8}} dg_{\mu\nu} d\bar{\eta}_\mu d\eta^\nu \equiv \prod_x M_{g_{\mu\nu}}^{(n)} dg_{\mu\nu} d\bar{\eta}_\mu d\eta^\nu .
\end{aligned} \tag{6.1}$$

These results contain several interesting features. First, the exponent of the g^{00} piece of the measure is always equal to the number of physical degrees of freedom divided by two. It is easy to see why this result is true in general for theories of the form which we have been discussing. In theories in which all degrees of freedom are physical, bosonic degrees of freedom each contribute a factor of $(g^{00})^{1/2}$ and fermionic degrees of freedom each contribute a factor of $(g^{00})^{-1/2}$. Now if we consider a theory with some non-physical degrees of freedom, *i.e.* some gauged and constrained variables, the number of first class constrained variables will always be equal to the number of gauge degrees of freedom [11]. After applying the Faddeev-Popov ansatz, all of the previously unphysical degrees of freedom become physical, but we have added an additional set of ghost fields having opposite commutation relations to those of our original fields. The number of ghost fields added is twice the number of gauge degrees of freedom, hence equal to the sum of the number of gauge degrees of freedom and the number of constrained variables. Therefore, the extra factors of $(g^{00})^{1/2}$ coming from our newly added physical field degrees of freedom is exactly cancelled by the extra factors of $(g^{00})^{-1/2}$ coming from our newly added ghosts, leaving a factor of $(g^{00})^{1/2}$ to the power of the number of original physical degrees of freedom.

As an example, let us consider the case of gravitation in four dimensions. Before applying the Faddeev-Popov ansatz, there are two physical degrees of freedom (ten total degrees of freedom minus four gauge degrees of freedom minus four constrained degrees of freedom). After applying the Faddeev-Popov ansatz, there are ten bosonic physical degrees of freedom and eight fermionic physical degrees of freedom; cancelling measure factors, there remains the same factor as that for a theory of two (bosonic) physical degrees of freedom.[†]

As we shall see below in Sec. 8, this result relating the number of physical degrees of freedom in the theory to the form of the canonical functional measure holds equally well in the case of massive theories, and appears to be true in general for all field theories of integer spin (a similar relationship holds for theories of half-integer spin as well, as we shall see below in Sec. 7). This conjecture is strongly supported by a naive analysis of the transformation properties of the functional measure in its Hamiltonian formulation (*i.e.* counting the number of physical canonical momenta in the theory, each of which transforms like $(g^{00})^{1/2}$).

Besides providing an excellent means of specifying the number of physical degrees of freedom in any field theory, this result emphasizes a very important point. *Bosons and fermions are best thought of as having "oppositely signed" physical degrees of freedom; bosonic fields have positive physical degrees of freedom and fermionic fields have negative physical degrees of freedom (or vice versa).* There are a number of other ways of seeing why this is the natural way of counting physical degrees of freedom, but the above is one of the clearest.

† If we choose to follow the slightly more conventional but less elegant nomenclature in which degrees of freedom must exist on shell to be considered truly "physical," we are faced with difficulties. First, Faddeev-Popov ghost fields violate the spin-statistics theorem; hence they cannot exist as states and would not be considered "physical." But now we must either arbitrarily define certain of our A_μ (or $g_{\mu\nu}$) components to be "unphysical" as well (which makes little sense, since after applying the Faddeev-Popov procedure all the components are on an equal footing) or allow the Faddeev-Popov procedure to lead to the non-conservation of "physical" degrees of freedom. Furthermore, we must severely modify our statement of the fundamental hypothesis of path integration for it to produce the correct canonical functional measure. For both of these reasons, the nomenclature used in this paper seems much preferable.

This notion of "oppositely signed" physical degrees of freedom for bosons and fermions is not merely a useless abstraction. Consider the case of gravitation in two dimensions. It is well-known that this theory has negative one physical degrees of freedom [18] (this is because there are more gauge conditions and constraints in our theory than there are independent variables). A theory with negative one physical degrees of freedom sounds like a peculiar and pathological curiosity, until we realize that it simply means that there is one more fermionic field than bosonic field in the theory. And this is indeed the correct interpretation: after applying the Faddeev-Popov ansatz, our theory would contain three bosonic fields g_{00}, g_{01}, g_{11} and four fermionic ghost fields $\bar{\eta}_0, \bar{\eta}_1, \eta^0, \eta^1$. Nothing in the least mysterious or pathological is involved.

A second interesting point is simply the mere presence of the factor $(g^{00})^K$ in the canonical functional measure. The measure is not at all manifestly covariant under general coordinate transformations (the factors of $det g_{\mu\nu}$, which naively transform like tensor densities rather than as scalars, can be shown by a trivial calculation [3] to actually be invariant under an infinitesimal general coordinate transformation). However, such a naive covariance analysis assumes that the point permutation Jacobian for a general coordinate transformation is unity, *i.e.* that

$$\prod_{x} d\phi(x) = \prod_{x'} d\phi(x') \quad \text{where} \quad x' = x + \xi(x), \tag{6.2}$$

and this is not at all certain. In fact, Fradkin and others [4-5] have repeatedly argued that this Jacobian should not be unity, and that the non-covariance of the g^{00} factor in the canonical measure is required in order to cancel this other non-covariance, and render the entire measure invariant. Their argument demonstrates the self-consistency of these assumptions.

Fujikawa has also derived an expression for the functional measure for gravitation, and has generalized it to n dimensional spacetime [7]. His procedure assumes that the BRST extension of general covariance is unbroken by anomalies arising from the non-invariance of the functional measure. The measure factors which he derives agree with those above, except for the absence of all g^{00} factors. However, Fujikawa's analysis tacitly assumes that the point permutation Jacobian for a BRST transformation is unity; this is not at all clear, and perhaps accounts for the discrepancy between Fujikawa's results and those of this paper, which are based on the canonical formalism.

Finally, de Witt's brief sketches presenting a plausible functional measure for gravitation [19] neglect those terms corresponding to the ghost-ghost and graviton-ghost blocks of the measure supermatrix, as well as the factors of g^{00}. In fact, his result is identical to (5.2), except for the absence of the g^{00} factors.

A final interesting feature of the canonical functional measures derived above and summarized in (6.1) is that they factorize in an enlightening manner. Specifically, the measure factor for a vector field theory in $n+1$ dimensions is equal to the product of the measure factor for a vector field theory in n dimensions and the measure factor for a scalar field theory in n dimensions, namely

$$M_{A_\mu}^{(n+1)} = M_{A_\mu}^{(n)} M_{\phi}^{(n)}. \tag{6.3}$$

Similarly, the measure for gravitation in $n+1$ dimensions factorizes into the product of the measures for gravitation, vector field, and scalar field in n dimensions

$$M_{g_{\mu\nu}}^{(n+1)} = M_{g_{\mu\nu}}^{(n)} M_{A_\mu}^{(n)} M_{\phi}^{(n)}. \tag{6.4}$$

This is not merely a curiosity; it is an absolute requirement needed for a toroidally compactified Kaluza-Klein theory to make sense on the quantum level, and if it were not satisfied, such Kaluza-Klein theories would be quantum-mechanically inconsistent. Therefore, it is indeed fortunate that the canonical functional measure satisfies this condition. Furthermore, although this factorization is necessary, it is not sufficient, and as I have shown elsewhere [21], the somewhat stronger true consistency condition is also (and *automatically*) satisfied by the canonical functional measure . *None of the other functional measures suggested in the literature (including Fujikawa's) satisfy this consistency condition, and each would lead to the complete inconsistency of Kaluza-Klein theory.* This fact significantly strengthens the likelihood that the canonical functional measure, besides being the simplest and most elegant, is also the correct functional measure for a quantum field theory.

7. Theories Linear in Derivatives

Now that we have determined the canonical functional measures for theories quadratic in time derivatives, let us turn to the case of theories which are linear in time derivatives. The former involved particles of integer spin; these will involve particles of half-integer spin.

The central feature of theories linear in time derivatives is that in their usual form their Feynman path integral representation cannot be put into Hamiltonian formulation. This is because the field equations of these theories are first order in time derivatives, implying that only the values of the fields themselves (and not also the first time derivatives of the fields) need be specified on each spacelike hypersurface. In fact, any conjugate momenta which we might care to define in the usual manner (such as $\Pi^A \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 Q_A)}$) are merely proportional to combinations of the fields themselves and carry no independent information. For this reason, the formal machinery developed in Sec. 2 is initially inappropriate for this situation. The canonical functional measure must be obtained through a similar but slightly different procedure.

The basic idea of the approach which we shall use is intuitively simple. According to the fundamental hypothesis of path integration, the functional measure factor should be unity in the Hamiltonian formulation, in which our functional integration variables are independent, canonically conjugate fields and momenta. Therefore, we shall redefine the variables of our theory so that half of our original fields retain the properties of fields *and the other half assume the properties* *of canonical momenta, conjugate to those fields.*, becoming what one might call "canonical pseudo-momenta." Since the nature of such a theory of canonically conjugate fields and pseudo-momenta is formally distinguishable from the usual case of a theory based on fields and momenta, one may invoke the fundamental hypothesis of path integration to argue that the canonical functional measure factor should be unity in the "pseudo-Hamiltonian formulation" of the theory. From this assumption, it is easy to derive the non-trivial functional measure factors which would be present in other, more commonplace formulations of the theory. A non-trivial functional measure factor may be easily understood as being the product of the various factors by which we must multiply some of our fields to give them the characteristics of canonical momenta.

Let us consider then a Lagrangian based on fields Q_A. which is linear in the derivatives of these fields, and with a kinetic term at most quadratic in the fields themselves (this last restriction follows automatically by counting mass-dimensions). For the moment, let us also assume that these fields Q_A are all bosonic. Under these conditions, our Lagrangian can be written as

$$\mathcal{L} = E^A \partial_0 Q_A - F, \tag{7.1}$$

with E^A and F being functions of the Q_A (as well as possibly some other fields in our complete theory).

Now this expression can be rewritten as

$$\mathcal{L} = \frac{\delta \mathcal{L}}{\delta(\partial_0 Q_A)} \partial_0 Q_A - F = \frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta Q_B} Q_B \partial_0 Q_A - F, \tag{7.2}$$

where we have used the fact that the kinetic term of $\mathcal{L}$ is at most quadratic in Q_A. Next, if we use integration by parts to shift the time derivative which appears in the first portion of (7.2), we obtain

$$\mathcal{L} = -\frac{d}{dt}\Big(\frac{\delta\mathcal{L}}{\delta(\partial_0 Q_A)}\Big)Q_A - F = -\frac{\delta^2\mathcal{L}}{\delta(\partial_0 Q_A)\delta Q_B}(\partial_0 Q_B)Q_A - F. \tag{7.3}$$

Therefore, if we require our theory to be consistently defined under integration by parts (any theory linear in derivatives can be put into a form which meets this requirement), we obtain the condition

$$\frac{\delta^2\mathcal{L}}{\delta(\partial_0 Q_A)\delta Q_B} = -\frac{\delta^2\mathcal{L}}{\delta(\partial_0 Q_B)\delta Q_A} \equiv N^{AB} \qquad (7.4).$$

Note that N^{AB} is *not* a function of the Q_A (though it might be a function of other fields appearing in our complete theory) and that

$$\mathcal{L} = N^{AB}Q_B\partial_0 Q_A - F. \tag{7.5}$$

We can now place a useful restriction on the form of N^{AB} by considering the nature of our fields. Since the Q_A have half-integer spin, we can choose to work in a basis of chiral eigenstates, *i.e.* a basis in which our fields are labelled by a chirality index. We can separate our fields into two categories, $Q_A = (Q_L, Q_R)$, with L being a new index ranging over the "left-handed" fields and R a new index ranging over the "right-handed" fields. For our purposes, we will define a field to be "left-handed" if it either creates left-handed particles or annihilates right-handed particles; "right-handed" fields satisfy the opposite requirement. (Whether or not particles of different chirality are connected by the presence of a mass term in our Lagrangian is immaterial, and the left-right symmetry or lack thereof in our entire theory is equally irrelevant.)

Since the kinetic term of our Lagrangian should preserve chirality, each kinetic piece must contain one right-handed and one left-handed field, implying that the matrix N^{AB} has the block-off-diagonal form

$$N^{AB} = \begin{pmatrix} 0 & N^{RL} \\ N^{LR} & 0 \end{pmatrix}. \qquad (7.6)$$

This allows us to rewrite our Lagrangian as

$$\mathcal{L} = N^{RL} Q_L \partial_0 Q_R + N^{LR} Q_R \partial_0 Q_L - F = 2N^{RL} Q_L \partial_0 Q_R - F, \qquad (7.7)$$

where we have used integration by parts to shift the time derivative, and absorbed any extra terms produced into a redefinition of F. Finally, let us redefine our left-handed field variables by

$$P^R = 2N^{RL} Q_L. \qquad (7.8)$$

Our Lagrangian now assumes its final form

$$\mathcal{L} = P^R \partial_0 Q_R - F[P^R, Q_R]. \qquad (7.9)$$

But now our newly defined left-handed fields P^R appear in this theory exactly as if they were the canonical momenta conjugate to our right-handed fields Q_R; they are in fact our canonical pseudo-momenta. Therefore, by our fundamental hypothesis, the functional measure for this theory is given by

$$[dP^R dQ_R] = \prod_x dP^R dQ_R. \qquad (7.10)$$

(Note that in deriving this result we have tacitly assumed that our Lagrangian is non-degenerate, *i.e.* that $detN^{AB} \neq 0$; in fact, if there had been gauged and constrained variables in our theory, it would have been necessary to apply a form of the Faddeev-Popov ansatz before following the above procedure.)

This form of the functional measure differs by a Jacobian factor from the more usual form in which all the functional variables of integration are our original fields. The canonical functional measure in the more common form is therefore

$$[dQ_A] = [dP^R dQ_R] = \prod_x det \frac{\delta P^R}{\delta Q_L} dQ_L dQ_R = \prod_x det \frac{\delta P^R}{\delta Q_L} dQ_A, \qquad (7.11)$$

with the canonical functional measure factor being*

$$det \frac{\delta P^R}{\delta Q_L} = det N^{RL} = det\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_R)\delta Q_L}\Big) = \Big[det\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta Q_B}\Big)\Big]^{1/2}, \qquad (7.12)$$

This result has been derived under the assumption that all of our fields Q_A are bosonic. However, we have carefully chosen our derivation in such a way that all the steps leading to (7.12) are equally correct for fermionic Q_A. Hence, the canonical functional measure factor for the case in which some of our Q_A are bosonic and others are fermionic is simply the appropriate generalization of the Jacobian determinant appearing in (7.12), namely

$$[dQ_A] = \prod_x \Big[sdet\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 Q_A)\delta Q_B}\Big)\Big]^{1/2} dQ_A. \qquad (7.13)$$

Actually, in practice, all the half-integer spin fields with which we will concern ourselves shall be fermionic.

Let us now apply these general results to compute the canonical functional measure factors for spinor fields. The form of the measure matrix in (7.13) ensures that only the kinetic portion of the spinor interaction is relevant to this

* It is also possible to derive this same result by adding a pseudo-time coordinate and utilizing a statistical mechanics approach [14].

computation. For a Dirac spinor field in n dimensional flat spacetime, the kinetic Lagrangian term is

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi), \tag{7.14}$$

with each of our n dimensional Dirac spinor fields $\bar{\psi}, \psi$ having $2^{[n/2]-1}$ components and $Q_A = (\bar{\psi}, \psi)$ having $2^{[n/2]}$ components. For this choice of functional integration variables, our functional measure factor is given by $(sdet\mathrm{M})^{1/2}$, with M being a $2^{[n/2]} \times 2^{[n/2]}$ block-diagonal matrix, each of whose 2×2 blocks is given by

$$\begin{pmatrix} 0 & \frac{i}{2}\gamma^0 \\ -\frac{i}{2}\gamma^0 & 0 \end{pmatrix}. \tag{7.15}$$

The determinant is a constant, hence the canonical functional measure is trivial. The functional measure factors for Majorana or Weyl spinor fields would be given by the square-root of this Dirac field measure factor, and would be equally trivial.

Next, let us consider the more interesting case of a Dirac spinor field acting in curved spacetime (*i.e.* coupled to a quantized gravitational field). The kinetic portion of the Lagrangian is [20]

$$\mathcal{L} = \frac{i}{2}\sqrt{-g}\left[h_{\alpha}^{\ \mu}\bar{\psi}\gamma^{\alpha}(\nabla_{\mu}\psi) - h_{\alpha}^{\ \mu}(\nabla_{\mu}\bar{\psi})\gamma^{\alpha}\psi\right]. \tag{7.16}$$

As in earlier cases, our covariant derivatives contain terms in which time derivatives act on the metric field. However, these terms do not contribute to the graviton measure matrix since all such contributing terms must be quadratic in time derivatives. On the other hand, the spinor measure matrix derived from (7.16) is non-trivial, and has a block-diagonal form, with each of the $2^{[n/2]-1}$ 2×2 blocks being given by

$$\begin{pmatrix} 0 & \frac{i}{2}\sqrt{-g}\gamma^{\alpha}h_{\alpha}^{\ 0} \\ -\frac{i}{2}\sqrt{-g}\gamma^{\beta}h_{\beta}^{\ 0} & 0 \end{pmatrix}. \tag{7.17}$$

Dropping all unimportant constant factors, the determinant of the block in (7.17) is

$$g\gamma^\alpha\gamma^\beta h_\alpha^{\ 0} h_\beta^{\ 0} = \frac{1}{2} g\{\gamma^\alpha,\gamma^\beta\} h_\alpha^{\ 0} h_\beta^{\ 0} = \frac{1}{2} g 2\eta^{\alpha\beta} h_\alpha^{\ 0} h_\beta^{\ 0} = g g^{00}. \tag{7.18}$$

Therefore, the canonical functional measure—containing the square-root of the superdeterminant of the entire measure matrix—is

$$[d\bar{\psi} d\psi]^{(n)} = \prod_x (g g^{00})^{-2^{[n/2]-2}} d\bar{\psi} d\psi. \tag{7.19}$$

Once again, the functional measure factors for Weyl or Majorana spinors in n dimensional curved spacetime would simply be the square-root of this quantity.

It should be noted that the exponent of the g^{00} piece appearing in the measure is equal to one-fourth the number of physical degrees of freedom, with fermionic degrees of freedom being once again counted with a minus sign. This appears to be true in general for theories of half-integer spin.

8. Measure Factors for Auxiliary Fields and Massive Vectors

Now that we have determined the canonical functional measures for fields whose kinetic term contains two derivatives (integer spin fields) and fields whose kinetic term contains one derivative (half-integer spin fields), we should also consider the proper means of treating fields whose "kinetic" (*i.e.* quadratic) terms contain no derivatives. Such fields are usually called "auxiliary," and are most often discussed in the context of supersymmetric theories, in which they play a crucial role by closing the symmetry algebra [12,13]. However, they enter into even as simple a theory as massive electro-magnetism.

First, it should be noted that since they are non-propagating, auxiliary fields are by definition non-physical and so *should not* be integrated over under the fundamental path integral hypothesis (see Sec. 3). This may be best understood by realizing that auxiliary fields do not have canonical commutation relations, and are purely classical objects. Being classical fields, they are not true functional variables of integration, do not enter into the functional measure, and should be treated as classical constant factors. However, if for reasons of convenience or elegance we wish to treat our auxiliary fields as functional variables of integration appearing in the path integral, we must multiply our functional measure by a compensating factor to cancel the result of this additional integration.

For example, suppose that the functional measure for a theory based on physical fields F_A is given by

$$[dF_A] = \prod_x M[F_A] dF_A. \tag{8.1}$$

Now further suppose that the Lagrangian for this theory also contains a non-physical auxiliary field Q, which appears at most quadratically in $\mathcal{L}$ and whose quadratic term has the form

$$\mathcal{L} = -\frac{1}{2} D[F_A] Q^2. \tag{8.2}$$

By our fundamental hypothesis, the canonical functional measure for our theory containing both F_A and Q is still just given by (8.1), since Q is not a physical field. Functionally integrating over Q would correspond to inserting the value of Q determined by our classical field equations into the Lagrangian, and would thus remove Q from our theory while possibly adding new terms to the effective Lagrangian produced; such a procedure is perfectly legitimate for a non-propagating field. However, this Gaussian functional integration over Q would also produce an additional functional measure factor, namely

$$\int \prod_x dQ\ exp^{iS} = \prod_x \left(D[F_A]\right)^{-1/2} exp^{iS_{eff}}, \tag{8.3}$$

if we assume that Q is bosonic. Generalizing this result to the case of functionally integrating over an arbitrary number of bosonic and fermionic auxiliary fields is simple and yields the result

$$\int \prod_x dQ_A \, exp^{iS} = \prod_x \Big[sdet\Big(\frac{\delta^2 \mathcal{L}}{\delta Q_A \delta Q_B}\Big)\Big]^{-1/2} exp^{iS_{eff}}. \tag{8.4}$$

Therefore, the canonical functional measure for our physical fields F_A and auxiliary fields Q_B, although still just given by (8.1), can also be written as

$$[dF_A dQ_A] = \prod_x M[F_A] dF_A = \prod_x \Big[sdet\Big(\frac{\delta^2 \mathcal{L}}{\delta Q_A \delta Q_B}\Big)\Big]^{1/2} M[F_A] dQ_A dF_A. \tag{8.5}$$

Often this latter form is preferable.

Let us apply these results concerning auxiliary fields to the specific case of a massive vector field in n dimensional curved spacetime. Our Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g} g^{\mu\nu} g^{\lambda\sigma} (\partial_\mu A_\lambda - \partial_\lambda A_\mu)(\partial_\nu A_\sigma - \partial_\sigma A_\nu) - \frac{1}{2} m^2 \sqrt{-g} g^{\mu\nu} A_\mu A_\nu. \tag{8.6}$$

Of the polarizations of A_μ which appear in this Lagrangian, A_0 is auxiliary and the remainder are physical. No gauge degrees of freedom are present. In the Hamiltonian formulation, our canonical functional measure is given by

$$[dA_\mu] = \prod_{x,m} \Pi^m A_m. \tag{8.7}$$

Integrating over the canonical momenta (see Sec. 4) produces the functional measure factor for our physical fields A_m in their Lagrangian formulation

$$\begin{aligned} \Big[det\Big(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 A_m)\delta(\partial_0 A_n)}\Big)\Big]^{1/2} &= \big[det(\sqrt{-g} g^{m0} g^{n0} - \sqrt{-g} g^{00} g^{mn})\big]^{1/2} \\ &= (g^{00})^{\frac{n-2}{2}} g^{\frac{n-3}{4}}. \end{aligned} \tag{8.8}$$

This result applies for energies low compared to the Planck Mass.

Meanwhile, if we choose to functionally integrate over our auxiliary field A_0 as well, we must include the compensating functional measure factor of $(g^{00})^{1/2}g^{1/4}$ (the mass term factor of m is simply a constant and can be neglected). Therefore, the canonical functional measure for a massive vector field in n dimensional curved spacetime is given by

$$[dA_\mu] = \prod_x (g^{00})^{\frac{n-1}{2}} g^{\frac{n-2}{4}} dA_\mu. \tag{8.9}$$

It should be noted that the exponent of the g^{00} piece in the measure once again equals one-half the number of physical degrees of freedom in this integer spin theory. It should also be noted that the functional measure for this n dimensional massive vector field is equal to the product of the measures for an n dimensional massless vector field and an n dimensional scalar field. This equality is necessary for the conventional analysis of the Higgs mechanism to be correct on the quantum level. As I show elsewhere [21], this consistency as well as the quantum consistency of Kaluza-Klein theories mentioned previously are both special cases of the automatic consistency of the canonical functional measure under field redefinitions.

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