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# Nambu Geometry in Quantum Hall Effect and Topological Insulator

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**Abstract.** This short review is a contribution to the conference proceeding of IF-YITP Symposium VI, 2016. We discuss how Nambu geometry emerges in the context of higher dimensional quantum Hall effect or A-class topological insulators [1].

## 1. Introduction

Non-commutative geometry (NCG) is a promising framework of quantum geometry to extend concepts of classical geometry. While the classical geometry is formulated on infinitely divisible spaces, it is considered that quantum geometry consists of indivisible finite volume elements. A natural scale of quantum space-time will be the Planck scale,

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}[\mathrm{m}]. \tag{1}$$

As the Einstein's theory of general relativity is based on Riemann geometry, the quantum geometry needs a new mathematical framework of quantum space-time. NCG is expected to play the role of such mathematical formulation. Interestingly, NCG also appears effectively in the context of quantum Hall effect (QHE). In QHE, the electrons undergo cyclotron motion with the cyclotron radius,

$$\ell_B = \sqrt{\frac{\hbar}{eB}} \approx 8.1 \times 10^{-9}[\mathrm{m}] \quad \text{for } B = 10[\mathrm{T}]. \tag{2}$$

Though the cyclotron scale is typically $10^{27}$ times much larger than the Planck scale, QHE realizes the NCG framework due to the level projection [2, 3, 4]. Concretely, on 2D plane in magnetic field, the electron coordinates of cyclotron motion realizes the Heisenberg algebra of NC plane

$$[X, Y] = i{\ell_B}^2, \tag{3}$$

and similarly on a two-sphere in monopole background, electron coordinates satisfy the the $SU(2)$ algebra of fuzzy two-sphere [5, 6, 7]

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k. \tag{4}$$

Here both $\ell_B$ and $\alpha = 2r/I$ ($r$ is the radius of sphere, and $I/2$ is the magnetic charge of the monopole) denote constants of dimension of length related to the indivisible volume element.

It is expected that knowledge obtained from the analysis of the QHE leads us to a better understanding of quantum geometry. We will discuss higher D generalizations of the spherical case (4) in the following.

## 2. Two Ways to Higher D Non-commutative Geometry

Since the fuzzy two-sphere is based on the $SU(2)$ Lie algebra, one natural way to generalize (4) is to adopt higher D Lie algebras [8, 9, 10, 11, 12, 13]. For realization of $2k$-D fuzzy sphere, the $SO(2k+1)$ gamma matrices are adopted as the coordinates $X_a$ $(a = 1, 2, \cdots, 2k+1)$:

$$[X_a, X_b] = i\alpha X_{ab}, \tag{5}$$

where $X_{ab}$ denote the $SO(2k+1)$ operators. Alternatively by using the quantum Nambu bracket, *i.e.*, a generalized commutator for $n$ entities [14, 15]

$$[X_1, X_2, \cdots, X_n] = \mathrm{sgn}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots, X_{\sigma(n)}, \tag{6}$$

we can generalize (4) to represent the fuzzy $2k$-sphere as [16, 17, 18]

$$[X_{a_1}, X_{a_2}, \cdots, X_{a_{2k}}] = i^k \alpha^{2k-1} \epsilon_{a_1 a_2 \cdots a_{2k+1}} X_{a_{2k+1}}. \tag{7}$$

Hereafter, we will consider the quantum Nambu bracket with even entities. (For odd entities, simple use of the quantum Nambu bracket contains critical difficulties [15].) Though the above two algebras, (5) and (7), take superficially different forms, in both cases $X_a$ are given by the $SO(2n+1)$ gamma matrices, and hence (5) and (7) represent the same fuzzy $2k$-sphere.

The fuzzy two-sphere appears physically in the context of the Landau problem on two-sphere [19]:

$$H = -\frac{1}{2M} \sum_{i=1,2,3} {D_i}^2|_{r=1} \tag{8}$$

where $D_i$ are the covariant derivatives whose commutator gives the monopole field strength,

$$[D_i, D_j] = i\epsilon_{ijk} \frac{I}{2r^2} x_k. \tag{9}$$

Here $I/2$ ($I$: integer) represents the charge of the monopole. The energy eigenvalues of the Hamiltonian are readily obtained as

$$E_n(I) = \frac{1}{2M}(n(n+1) + I(n+\frac{1}{2})), \tag{10}$$

and the degeneracy of the lowest Landau level ($n = 0$) is

$$d_{LLL}(I) = I + 1. \tag{11}$$

The total $SU(2)$ angular momentum operators are constructed as

$$L_i = -i\epsilon_{ijk} x_j D_k + \frac{1}{\alpha} x_i, \tag{12}$$

and in the lowest Landau level, $L_i$ are reduced to

$$L_i \rightarrow \frac{1}{\alpha} x_i. \tag{13}$$

Therefore in the lowest Landau level, $x_i$ can be identified with the operator

$$X_i = \alpha L_i \tag{14}$$

and satisfy the fuzzy two-sphere algebra (4). This is the mechanism of the emergent NCG in the context of the lowest Landau level. Heuristically, the NC algebra (4) can be obtained from (9) by the replacement

$$D_i \;\rightarrow\; iX_i. \tag{15}$$

For relativistic case, the Dirac operator acts as the relativistic Hamiltonian. In the presence of non-trivial bundle topology of magnetic field, the Dirac Hamiltonian necessarily accommodates the zero-modes with the number

$$\mathrm{ind}(-i\not{D}) = d_{LLL}(I-1) = I, \tag{16}$$

which coincides with the 1st Chern number on the two-sphere

$$c_1 = \frac{1}{2\pi}\int_{S^2} F = I. \tag{17}$$

This demonstrates the index theorem, $\mathrm{ind}(-i\not{D}) = c_1$.

## 3. Higher D Quantum Hall Effect and Non-Abelian Gauge Field

The higher D study of the QHE was initiated by the work of Zhang and Hu [20]. They utilized the quaternions to generalize the QHE in four-dimensional space. After their proposal, many works have been devoted to the developments of higher D QHE [21, 22] (see Refs.[23, 24] as reviews and references therein). In particular, we explored higher D QHE on arbitrary even D spheres [25] and on supersphere [26, 27]. The constructions are summarized in Fig.1.

| 2D | 4D | 2k D | 2\|2 D |
|---|---|---|---|
| Complex # | Quaternions | Clifford algebra | Grassmann # |
| U(1) | SU(2) | SO(2k) | *U*(1) |
| Dirac monopole | Yang monopole | Non-abelian monopole | Landi super-monopole |
| 1st Chern # | 2nd Chern # | kth Chern # | 1st Chern # |
| Fuzzy two-sphere | Fuzzy four-sphere | Fuzzy 2k-sphere | Fuzzy super-sphere |

**Figure 1.** Higher D and supersymmetric generalizations of QHE.

In the setup, we adopted the non-Abelian monopoles introduced in Refs.[28, 29, 30, 31, 32]. For QHE on $S^6$, we used the $SO(6)$ monopole, and in general, for QHE on $S^{2k}$ $SO(2k)$. The $SO(2k)$ monopole gauge fields are explicitly given by

$$A_{\mu=1,2,\cdots,2k} = -\frac{1}{r(r+x_{2k+1})}\sigma_{\mu\nu}x_\nu, \qquad A_{2k+1} = 0, \quad (\sigma_{\mu\nu} : SO(2k) \text{ Weyl generators}) \tag{18}$$

and the covariant derivatives are constructed as $D_a = \partial_a + iA_a$. Their commutators yield the field strength,

$$[D_a, D_b] = iF_{ab} \tag{19}$$

where

$$F_{\mu\nu} = \frac{1}{r(r+x_{2k+1})}\sigma_{\mu\nu}x_\nu - \frac{1}{r^2}x_{[\mu}A_{\nu]}, \quad F_{\mu,2k+1} = \frac{1}{r^2}(r+x_{2k+1})A_\mu. \tag{20}$$

Using the covariant derivative, we introduce the Landau Hamiltonian on $S^{2k}$ as

$$H = -\frac{1}{2Mr^2}\sum_{a=1}^{2k+1} {D_a}^2|_{r=1}. \tag{21}$$

The Landau levels are explicitly derived as

$$E_n = \frac{1}{2M}(n(n+2k-1) + I(n+\frac{k}{2})) \quad (n = 0, 1, 2, \cdots). \tag{22}$$

For $2k = 4$ and 6, the lowest Landau level ($n = 0$) degeneracies are given by

$$d_{LLL}^{(2k=4)}(I) = \frac{1}{6}(I+1)(I+2)(I+3), \quad d_{LLL}^{(2k=6)}(I) = \frac{1}{360}(I+1)(I+2)(I+3)^2(I+4)(I+5). \tag{23}$$

The non-Abelian monopoles have the non-trivial Chern numbers guaranteed by the homotopy theorem:

$$\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z}. \tag{24}$$

For $k = 2$ and 3 of (20), the Chern numbers

$$c_k = \frac{1}{k!(2\pi)^k}\int_{S^{2k}} \mathrm{tr}(F^k) \tag{25}$$

are calculated as

$$c_2(I) = \frac{1}{6}I(I+1)(I+2), \quad c_3(I) = \frac{1}{360}I(I+1)(I+2)^2(I+3)(I+4). \tag{26}$$

From (23) and (26), we can confirm that the validity of the index theorem in higher D:

$$\mathrm{Ind}(-i\not{D})_I = d_{LLL}(I-1) = c_k(I). \tag{27}$$

More general discussion can be found in Ref.[33].

Recalling the replacement (15), we can find that the commutation relation (19) implies the NCG

$$[X_a, X_b] = i\alpha X_{ab}, \tag{28}$$

which exactly reproduces (5).

## 4. Topological Insulator and Tensor Gauge Field

We have constructed a higher D QHE by introducing the non-Abelian monopole. While the 2D restriction was relaxed, magnetic field is still necessary for the set-up. However, the condition about the existence of magnetic field can also be relaxed. The strong spin-orbit coupling takes the place of external magnetic field, and the time reversal counterpart of the QHE, called the quantum spin Hall system, was proposed theoretically [34, 35, 36] and subsequently observed experimentally [37]. Since the spin-orbit coupling is not specific to 2D, the spin-orbit coupling can give rise to the 3D analogue of the quantum spin Hall system referred to as the topological insulator (in a narrow sense) [38, 39, 40]. Thus there are 3D and 2D analogues of the QHE, and there may be no wonder if other cousins of QHE exist. Indeed, a comprehensive list of such cousins, *i.e.* topological insulators in a broad sense, was proposed in Refs.[41, 42, 43] according to the random matrix theory classification based on discrete symmetries. Among them, the original 2D QHE belongs to the so-called A-class, and so the higher D QHE can be identified with higher D entity of the A-class. In Refs.[44, 45], the quantum Nambu geometry was shown to appear as the geometry of the A-class topological insulators by level projection. Meanwhile, as discussed above, the higher D QHE is realized in the non-Abelian monopole background. Therefore, there must be underlying relations between the quantum Nambu geometry and the non-Abelian monopole background. We clarify the relations in the remaining sections.

While the non-Abelian generalization is concerned with the internal gauge space of monopole, there is another way to generalize monopole based on external space extension: The internal gauge group is still $U(1)$, but the external indices are added to the gauge field and the gauge field become a tensor-type [46, 47] (such tensor gauge field is called the Kalb-Ramond field [48]). For 5D and 7D, the tensor monopole gauge fields are given by

$$\begin{aligned} C_{abc} &= -\frac{1}{6r^3} I(I+1)(I+2) \left( \frac{1}{r+x_5} + \frac{r}{(r+x_5)^2} \right) \epsilon_{abcd5} x_d, \\ C_{abcdef} &= -\frac{1}{40r^5} I(I+1)(I+2)^2(I+3)(I+4) \left( \frac{1}{r+x_7} + \frac{r}{(r+x_7)^2} + \frac{2}{3} \frac{r^2}{(r+x_7)^3} \right) \epsilon_{abcdef7} x_f \end{aligned} \tag{29}$$

and the corresponding field strengths, $G = dC$, are obtained as

$$G_{abcd} = \frac{1}{2r^5} I(I+1)(I+2) \epsilon_{abcde} x_e, \quad G_{abcdef} = \frac{1}{8r^7} I(I+1)(I+2)^2(I+3)(I+4) \epsilon_{abcdefg} x_g. \tag{30}$$

The integrations of $G$ on $S^{2k}$ yield

$$\int_{S^4} G = \frac{1}{6} I(I+1)(I+2), \quad \int_{S^6} G = \frac{1}{360} I(I+1)(I+2)^2(I+3)(I+4). \tag{31}$$

These results coincide with the Chern numbers (26). Even before the integration, we can make an exact map from the non-Abelian monopole gauge field to the tensor monopole gauge field though the Chern-Simons term

$$C^{(2k-1)}[A] = k \int_0^1 dt \ \mathrm{tr}(A(dA + it^2 A^2)^{k-1}) \tag{32}$$

and then

$$G^{2k} = \mathrm{tr} \ F^k. \tag{33}$$

One may convince himself by substituting $SU(2)$ and $SO(6)$ monopole gauge fields (18) to the 4D and 6D Chern-Simons terms

$$C^{(3)}[A] = \mathrm{tr}(AdA + \frac{2}{3} iA^3), \quad C^{(5)}[A] = \mathrm{tr}(A(dA)^2 + \frac{3}{2} iA^3 dA - \frac{3}{5} A^5), \tag{34}$$

to derive (29). From (33), we have

$$\frac{1}{r^{2k+1}} x_a = \frac{2}{(2k)! c_k} \epsilon_{a a_1 a_2 \cdots a_{2k-1} a_{2k}} \mathrm{tr}(F_{a_1 a_2} \cdots F_{a_{2k-1} a_{2k}}). \tag{35}$$

## 5. Quantum Nambu Geometry and Index Theorem

Armed with the observation of the correspondence between the non-Abelian monopole and tensor monopole, we revisit the non-Abelian Landau problem on $S^{2k}$:

$$H = -\frac{1}{2Mr^2}\sum_{a=1}^{2k+1} D_a D_a|_{r=1} = \frac{1}{2M}\sum_{a>b} {\Lambda_{ab}}^2, \tag{36}$$

where $\Lambda_{ab} = -x_a D_b + i x_b D_a$ denote the covariant angular momentum. The total angular momentum operators are constructed as

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab}. \tag{37}$$

In the lowest Landau level, the covariant derivatives are quenched and from (37) $F_{ab} \sim L_{ab}$. Consequently, (35) gives the relation

$$X_a = \alpha \frac{I}{(2k)!c_k} \epsilon_{a a_1 a_2 \cdots a_{2k-1} a_{2k}} L_{a_1 a_2} \cdots L_{a_{2k-1} a_{2k}}. \tag{38}$$

The $S^{2k}$ coordinates now become the operator given by (38), and $X_a$ satisfy the higher D algebra of the quantum Nambu bracket (7). Thus, through the correspondence between the non-Abelian and tensor monopoles, we have shown that the quantum Nambu geometry realizes as the geometry of the lowest Landau level [Fig.2].

| | Quantum Nambu Geometry | Monopole |
|---|---|---|
| 2D | 2-bracket $[X_i, X_j] = i\epsilon_{ijk} X_k$ | U(1) monopole |
| 4D | 4-bracket $[X_a, X_b, X_c, X_d] = -\epsilon_{abcde} X_e$ | SU(2) monopole ⇕ 3-tensor monopole |
| 6D | 6-bracket $[X_{a_1}, X_{a_2}, \cdots, X_{a_6}] = -i\epsilon_{a_1 \cdots a_7} X_{a_7}$ | SO(6) monopole ⇕ 5-tensor monopole |
| 2k-D | 2k-bracket $[X_{a_1}, X_{a_2}, \cdots, X_{a_{2k}}] = i^k \epsilon_{a_1 \cdots a_7} X_{a_{2k+1}}$ | SO(2k) monopole ⇕ (2k-1)-tensor monopole |

**Figure 2.** Correspondence between the quantum Nambu geometries and the non-Abelian or tensor monopole background.

The higher D fuzzy sphere coordinates are essentially the higher D gamma matrices which can be constructed from the low D ones. Similarly, the higher D quantum Nambu algebra incorporates the low D quantum Nambu algebras, for instance,

$$[X_1, X_2, X_3, X_4] = [X_1, X_2][X_3, X_4] - [X_1, X_3][X_2, X_4] + \cdots \tag{39}$$

and

$$[X_1, X_2, X_3, X_4, X_5, X_6] = [X_1, X_2, X_3, X_4][X_5, X_6] - [X_1, X_2, X_3, X_5][X_4, X_6] + \cdots . \tag{40}$$

These algebraic structures manifest a dimensional hierarchy of the higher D QHE: The low D QH liquids condense to form higher D QH liquid [25]. The index theorem also enforces the observation. The index of the Dirac operator (27) counts the number of the quantum states in the lowest Landau level, and the index theorem tells that such quantum states correspond to the number of the finite volume elements on the quantum geometry given by the Chern number. The index of the non-Abelian Dirac operator on the $2k$D sphere behaves as

$$\text{Ind}(-i\not{D}) = d_{LLL}(I-1) \sim I^{\frac{1}{2}I(I+1)} = I \cdot I^2 \cdot I^3 \cdots I^k, \tag{41}$$

and we see the dimensional hierarchy: The right-hand side of (41) implies that the higher D quantum geometry consists of lower D quantum geometries. Such hierarchy is quite similar to the D-brane structure in string theory [49, 50].

## 6. Summary

We gave a short review about the Nambu geometry in the context of the monopole magnetic field background along with the development of the higher D QHE and topological insulator. The higher D QHE rediscovered in the recent development of the topological insulator naturally realizes the quantum Nambu geometry. As the Nambu bracket formation is equivalent to the Lie bracket formulation for fuzzy spheres, two superficially different non-Abelian and tensor monopoles are shown to be related by the Chern-Simons term. The index theorem manifests itself in the context of higher D QHE and guarantees the dimensional hierarchy of the quantum geometry. While the present review focused on the even D Nambu geometry, related works about odd D Nambu geometry can be found in Refs.[51, 45, 52].

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