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Galilean conformal algebras and AdS/CFT

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Abstract: Non-relativistic versions of the AdS/CFT conjecture have recently been investigated in some detail. These have primarily been in the context of the Schrodinger symmetry group. Here we initiate a study based on a *different* non-relativistic conformal symmetry: one obtained by a parametric contraction of the relativistic conformal group. The resulting Galilean conformal symmetry has the same number of generators as the relativistic symmetry group and thus is different from the Schrodinger group (which has fewer). One of the interesting features of the Galilean Conformal Algebra is that it admits an extension to an *infinite* dimensional symmetry algebra (which can potentially be dynamically realised). The latter contains a Virasoro-Kac-Moody subalgebra. We comment on realisations of this extended symmetry in a boundary field theory. We also propose a somewhat unusual geometric structure for the bulk gravity dual to any realisation of this symmetry. This involves taking a Newton-Cartan like limit of Einstein's equations in anti de Sitter space which singles out an AdS_2 comprising of the time and radial direction. The infinite dimensional Virasoro extension is identified with the asymptotic isometries of this AdS_2.

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JHEP07(2009)037

Contents

1 Introduction **1**

2 Non-relativistic conformal symmetries **4**
- 2.1 Schrodinger symmetry 4
- 2.2 Contraction of the relativistic conformal group 5

3 The infinite dimensional extended GCA **7**

4 The bulk dual **9**
- 4.1 Newton-Cartan theory of gravity 10
- 4.2 Newtonian limit of gravity on AdS_{d+2} 11

5 GCA in the bulk **12**
- 5.1 Contraction of the bulk isometries 13

6 Concluding remarks **15**

A Galilean isometries **16**

B Killing vectors of AdS_{d+2} and bulk contraction **17**

JHEP07(2009)037

1 Introduction

Even after more than a decade, the AdS-CFT conjecture [1] continues to throw up rich, new avenues of investigation. One such recent direction has been to consider extensions of the conjecture from its original relativistic setting to a non-relativistic context. This opens the door to potential applications of the spirit of gauge-gravity duality to a variety of real-life strongly interacting systems. It was pointed out in [2] that the Schrodinger symmetry group [3, 4], a non-relativistic version of conformal symmetry, is relevant to the study of cold atoms. A gravity dual possessing these symmetries was then proposed in [7, 8] (see also [9, 10] for a somewhat different bulk realisation). Further developments along this line can be found in [11]–[38].

Instead of the Schrodinger group, in this paper, we will consider an *alternative* non-relativistic realisation of conformal symmetry and begin a study of its consequences and realisations in the context of the AdS/CFT conjecture. This symmetry will be obtained by considering the nonrelativistic group contraction of the relativistic conformal group

SO($d+1,2$) in $d+1$ space-time dimensions.[1] The process of group contraction leads, in $d=3$, for instance, to a fifteen parameter group (like the parent SO(4, 2) group) which contains the ten parameter Galilean subgroup. This Galilean conformal group is to be contrasted with the twelve parameter Schrodinger group (plus central extension) with which it has in common only the Galilean subgroup. The Galilean conformal group is, in fact, different from the Schrodinger group in some crucial respects, which we will describe in more detail later. For instance, the dilatation generator $\tilde{D}$ in the Schrodinger group scales space and time differently $x_i \to \lambda x_i, t \to \lambda^2 t$. Whereas the corresponding generator D in the Galilean Conformal Algebra (GCA) scales space and time in the *same* way $x_i \to \lambda x_i, t \to \lambda t$. Relatedly, the GCA does *not* admit a mass term as a central extension. Thus, in some sense, this symmetry describes "massless" or "gapless" non-relativistic theories, like the parent relativistic group but unlike the Schrodinger group.

However, the most interesting feature of the GCA seems to be its natural extension to an infinite dimensional symmetry algebra (which we will also often denote as GCA when there is no risk of confusion). This is somewhat analogous, as we will see, to the way in which the finite conformal algebra of SL$(2,C)$ in two dimensions extends to two copies of the Virasoro algebra. We will see that it is natural to expect this extended symmetry to be dynamically realised (perhaps partially) in actual systems possesing the finite dimensional Galilean conformal symmetry. Indeed, it has been known (see [47] and references therein) that there is a notion of a "Galilean isometry" which encompasses the so-called Coriolis group of arbitrary time dependent (but spatially homogeneous) rotations and translations. In this language, our infinite dimensional algebra is that of "Galilean conformal isometries". As we will see, it contains one copy of a Virasoro together with an SO(d) current algebra (on adding the appropriate central extension).

In addition to possible applications to non-relativistic systems, one of the motivations for studying the contracted SO($d+1,2$) conformal algebra is to examine the possibility of a new tractable limit of the parent AdS/CFT conjecture. In fact, the BMN limit [46] of the AdS/CFT conjecture is an instance where, as result of taking a particular scaling limit, one obtains a contraction of the original SO$(4,2) \times$ SO(6) (bosonic) global symmetry.[2] In our case, the non-relativistic contraction is obtained by taking a similar scaling limit on the parent theory. Like in the BMN case, taking this limit would isolate a closed subsector of the full theory. The presence of an enhanced symmetry in our scaling limit raises interesting possibilities about the solvability of this subsector. We defer the detailed study of this aspect for later.

There are, however, some important differences here from a BMN type limit which have to do with the nature of taking the scaling. Normally the BMN type scaling leads to

[1] The process of group contraction is, of course, standard and may have been applied by many people to the relativistic conformal group. To the best of our knowledge, [39] is a recent reference with the explicit results of this contraction of the relativistic conformal group. This reference goes on to study a realisation of the $2+1$ dimensional case, which has some special features.

[2] An algebraically equivalent contraction to ours, of the isometries of $AdS_5 \times S^5$, was studied in [53] as an example of a non-relativistic string theory. However, the embedding of this contraction in AdS_5 is not manifestly such that it corresponds to a non-relativistic CFT on the boundary. We will comment further on this at a later stage.

JHEP07(2009)037

a Penrose limit of the geometry in the vicinity of some null geodesic. These are typically pp-wave like geometries whose isometry is the same as that of the contracted symmetry group on the boundary. The non-relativistic scaling limit that leads to the GCA on the boundary is at first sight more puzzling to implement in the bulk. This is because, under the corresponding scaling, the bulk metric degenerates in the spatial directions x_i. Thus one might think one has some kind of singular limit in the bulk description. However, this degeneration is a feature common to all non-relativistic limits. It arises, for instance, in taking the Newtonian gravity limit of Einstein's equations in asymptotically flat space. However, in this case, there is a well defined geometric description of the limit despite the degeneration of the metric. This description, originally due to Cartan, and studied fairly thoroughly by geometers describes Newtonian gravity in terms of a non-dynamical metric but with a dynamical *non-metric* connection.[3] The Einstein's equations reduce to equations determining the curvature of this connection in terms of the matter density. These are nothing but the Poisson equations for the Newtonian gravitational potential. In geometric terms the spacetime takes the form of a vector bundle with fibres as the spatial R^d over a base R which is time, together with an affine connection related to the gradient of the Newtonian potential.

We propose a similar limiting description for the bulk geometry in our case. The main difference is that the time and the radial direction together constitute an AdS_2 with a non-degenerate metric. Thus one has a geometry with an AdS_2 base and the spatial R^d fibred over it. Once again there is no overall spacetime metric. The dynamical variables are affine connections determined by the limiting form of Einstein's equations. As a check of this proposal, we will see that the infinite dimensional GCA symmetries are realised in this bulk geometry as asymptotic isometries. In fact the Virasoro generators of the GCA are precisely the familiar generators of asymptotic global isometries of AdS_2. These generators will also be seen to reduce to the generators of the GCA on the boundary.

The paper is organised as follows. In the next section we first review the Schrodinger symmetry algebra in order to set notation and contrast it with the Galilean Conformal Algebra, which we obtain through group contraction on $\mathrm{SO}(d+1,2)$. In section 3 we describe the infinite dimensional extension of this algebra and its physical significance. Section 4 moves onto the bulk realisation of the non-relativistic contraction. We propose a geometric description of the bulk physics analogous to the Newton-Cartan theory. In section 5 we lend support to this proposal by finding the vector fields corresponding to the GCA and its infinite dimensional extension as well as their action on the bulk AdS. In section 6 we close with a laundry list of things left undone. In two appendices we elaborate on a couple of points of the main text.

[3] See the textbook [59] Chap.12 for a basic discussion. We thank T. Padmanabhan for bringing this to our attention.

JHEP07(2009)037

2 Non-relativistic conformal symmetries

2.1 Schrodinger symmetry

The Schrodinger symmetry group in $(d+1)$ dimensional spacetime (which we will denote as $Sch(d,1)$) has been studied as a non-relativistic analogue of conformal symmetry. It's name arises from being the group of symmetries of the free Schrodinger wave operator in $(d+1)$ dimensions. In other words, it is generated by those transformations that commute with the operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$. However, this symmetry is also believed to be realised in interacting systems, most recently in cold atoms at criticality.

The symmetry group contains the usual Galilean group (denoted as $G(d,1)$) with its central extension.

$$\begin{aligned}
&[J_{ij}, J_{rs}] = so(d) \\
&[J_{ij}, B_r] = -(B_i\delta_{jr} - B_j\delta_{ir}) \\
&[J_{ij},\, P_r] = -(P_i\delta_{jr} - P_j\delta_{ir}), \qquad [J_{ij},\, H] = 0 \\
&[B_i, B_j] = 0, \qquad [P_i, P_j] = 0, \qquad [B_i, P_j] = m\delta_{ij} \\
&[H, P_i] = 0, \qquad [H, B_i] = -P_i.
\end{aligned} \tag{2.1}$$

Here J_{ij} $(i,j = 1 \ldots d)$ are the usual $\mathrm{SO}(d)$ generators of spatial rotations. P_r are the d generators of spatial translations and B_j those of boosts in these directions. Finally H is the generator of time translations. The parameter m is the central extension and has the interpretation as the non-relativistic mass (which also appears in the Schrodinger operator S).

As vector fields on the Galilean spacetime $R^{d,1}$, they have the realisation (in the absence of the central term)

$$\begin{aligned}
&J_{ij} = -(x_i\partial_j - x_j\partial_i) \qquad && H = -\partial_t \\
&P_i = \partial_i && B_i = t\partial_i
\end{aligned} \tag{2.2}$$

In addition to these Galilean generators there are *two* more generators which we will denote by $\tilde{K}, \tilde{D}$. $\tilde{D}$ is a dilatation operator, which unlike the relativistic case, scales time and space differently. As a vector field $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ so that

$$x_i \to \lambda x_i, \qquad t \to \lambda^2 t. \tag{2.3}$$

$\tilde{K}$ acts something like the time component of special conformal transformations. It has the form $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$ and generates the finite transformations (parametrised by μ)

$$x_i \to \frac{x_i}{(1+\mu t)}, \qquad t \to \frac{t}{(1+\mu t)}. \tag{2.4}$$

These two additional generators have non-zero commutators

$$\begin{aligned}
&[\tilde{K}, P_i] = B_i, \qquad && [\tilde{K}, B_i] = 0, \qquad && [\tilde{D}, B_i] = -B_i \\
&[\tilde{D}, \tilde{K}] = -2\tilde{K}, && [\tilde{K}, H] = -\tilde{D}, && [\tilde{D}, H] = 2H.
\end{aligned} \tag{2.5}$$

JHEP07(2009)037

The generators $\tilde{K}, \tilde{D}$ are invariant under the spatial rotations J_{ij}. We also see from the last line that $H, \tilde{K}, \tilde{D}$ together form an SL$(2, R)$ algebra. The central extension term of the Galilean algebra is compatible with all the extra commutation relations.

Note that there is no analogue in the Schrodinger algebra of the spatial components K_i of special conformal transformations. Thus we have a smaller group compared to the relativistic conformal group. In $(3+1)$ dimensions the Schrodinger algebra has twelve generators (ten being those of the Galilean algebra) and the additional central term. Whereas the relativistic conformal group has fifteen generators. In the next subsection we will discuss how to get a nonrelativistic conformal group through group contraction. In the process of group contraction one does not lose any generators and hence the Galilean Conformal Algebra we find will have the same number of generators as the group SO$(d+1, 2)$.

2.2 Contraction of the relativistic conformal group

We know that the Galilean algebra $G(d, 1)$ arises as a contraction of the Poincare algebra ISO$(d, 1)$. Physically this comes from taking the non-relativistic scaling

$$t \to \epsilon^r t \qquad x_i \to \epsilon^{r+1} x_i \tag{2.6}$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v_i \sim \epsilon$ to zero (in units where $c = 1$). We have allowed for a certain freedom of scaling through the parameter r, since we might have other scales in the theory with respect to which we would have to take the above nonrelativistic limit. We will later consider the example of nonrelativistic fluid mechanics, in which we have a scale set by the temperature. In this case the natural scaling corresponds to $r = -2$. However, for the process of group contraction the parameter r will play no role apart from modifying an over all factor which is unimportant. Hence we will mostly take $r = 0$.

Starting with the expressions for the Poincare generators $(\mu, \nu = 0, 1 \ldots d)$

$$J_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad P_\mu = \partial_\mu, \tag{2.7}$$

the above scaling gives us the Galilean vector field generators of (2.2)

$$\begin{aligned} J_{ij} &= -(x_i \partial_j - x_j \partial_i) & P_0 = H &= -\partial_t \\ P_i &= \partial_i & J_{0i} = B_i &= t\partial_i. \end{aligned} \tag{2.8}$$

They obey the commutation relations (without central extension) of (2.1). This should be contrasted with the Poincare commutators

$$\begin{aligned} &[J_{ij}, J_{rs}] = so(d) \\ &[J_{ij}, B_r] = -(B_i \delta_{jr} - B_j \delta_{ir}) \\ &[J_{ij},\ P_r] = -(P_i \delta_{jr} - P_j \delta_{ir}), \qquad [J_{ij}, H] = 0 \\ &[B_i, B_j] = -J_{ij}, \qquad [P_i, P_j] = 0, \qquad [B_i, P_j] = \delta_{ij} H \\ &[H, P_i] = 0, \qquad [H, B_i] = -P_i \end{aligned} \tag{2.9}$$

JHEP07(2009)037

To obtain the Galilean Conformal Algebra, we simply extend the scaling (2.6) to the rest of the generators of the conformal group $\mathrm{SO}(d+1,2)$. Namely to

$$D = -(x\cdot\partial) \qquad K_\mu = -(2x_\mu(x\cdot\partial) - (x\cdot x)\partial_\mu) \tag{2.10}$$

where D is the relativistic dilatation generator and K_μ are those of special conformal transformations. The non-relativistic scaling in (2.6) now gives (see also [39])

$$\begin{aligned} D &= -(x_i\partial_i + t\partial_t) \\ K &= K_0 = -(2tx_i\partial_i + t^2\partial_t) \\ K_i &= t^2\partial_i. \end{aligned} \tag{2.11}$$

Note that the dilatation generator $D = -(x_i\partial_i + t\partial_t)$ is the *same* as in the relativistic theory. It scales space and time in the same way $x_i \to \lambda x_i, t \to \lambda t$. Therefore it is different from the dilatation generator $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ of the Schrodinger group. Similarly, the temporal special conformal generator K in (2.11) is different from $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$. Finally, we now have spatial special conformal transformations K_i which were not present in the Schrodinger algebra. Thus the generators of the Galilean Conformal Algebra are $(J_{ij}, P_i, H, B_i, D, K, K_i)$.

Since the usual Galilean algebra $G(d,1)$ for the generators (J_{ij}, P_i, H, B_i) is a subalgebra of the GCA, we will not write down their commutators. The other non-trivial commutators of the GCA are [39]

$$\begin{aligned} &[K, K_i] = 0, && [K, B_i] = K_i, && [K, P_i] = 2B_i \\ &[J_{ij}, K_r] = -(K_i\delta_{jr} - K_j\delta_{ir}), && [J_{ij}, K] = 0, && [J_{ij}, D] = 0 \\ &[K_i, K_j] = 0, && [K_i, B_j] = 0, && [K_i, P_j] = 0, \qquad [H, K_i] = -2B_i, \\ &[D, K_i] = -K_i, && [D, B_i] = 0, && [D, P_i] = P_i, \\ &[D, H] = H, && [H, K] = -2D, && [D, K] = -K. \end{aligned} \tag{2.12}$$

This can again be contrasted with commutators of the corresponding relativistic generators

$$\begin{aligned} &[K, K_i] = 0, && [K, B_i] = K_i, && [K, P_i] = 2B_i \\ &[J_{ij}, K_r] = -(K_i\delta_{jr} - K_j\delta_{ir}), && [J_{ij}, K] = 0, && [J_{ij}, D] = 0 \\ &[K_i, K_j] = 0, && [K_i, B_j] = \delta_{ij}K, && [K_i, P_j] = 2J_{ij} + 2\delta_{ij}D \\ &[H, K_i] = -2B_i, && [D, K_i] = -K_i, && [D, B_i] = 0, \qquad [D, P_i] = P_i, \\ &[D, H] = H, && [H, K] = -2D, && [D, K] = -K. \end{aligned} \tag{2.13}$$

We can also compare the relevant commutators in (2.12) with those of (2.5) and we notice that they too are different. Thus the Schrodinger algebra and the GCA only share a common Galilean subgroup and are otherwise different. In fact, one can verify using the Jacobi identities for (D, B_i, P_j) that the Galilean central extension in $[B_i, P_j]$ is *not* admissible in the GCA. This is another difference from the Schrodinger algebra, which as mentioned above, does allow for the central extension. Thus in some sense, the GCA is the symmetry of a "massless" (or gapless) nonrelativistic system. We will discuss some possible realisations in the next section.

JHEP07(2009)037

3 The infinite dimensional extended GCA

The most interesting feature of the GCA is that it admits a very natural extension to an infinite dimensional algebra of the Virasoro-Kac-Moody type.[4] To see this we denote

$$\begin{aligned} L^{(-1)} &= H, & L^{(0)} &= D, & L^{(+1)} &= K, \\ M_i^{(-1)} &= P_i, & M_i^{(0)} &= B_i, & M_i^{(+1)} &= K_i. \end{aligned} \tag{3.1}$$

The finite dimensional GCA which we had in the previous section can now be recast as

$$\begin{aligned} &[J_{ij}, L^{(n)}] = 0, && [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\ &[J_{ij}, M_k^{(m)}] = -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), && [M_i^{(m)}, M_j^{(n)}] = 0, \\ &[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}. \end{aligned} \tag{3.2}$$

The indices $m, n = 0, \pm 1$ We have made manifest the $\mathrm{SL}(2,R)$ subalgebra with the generators $L^{(0)}, L^{(\pm 1)}$. In fact, we can define the vector fields

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \\ M_i^{(n)} &= t^{n+1}\partial_i \end{aligned} \tag{3.3}$$

with $n = 0, \pm 1$. These (together with J_{ij}) are then exactly the vector fields in (2.2) and (2.11) which generate the GCA (without central extension).

If we now consider the vector fields of (3.3) for *arbitrary* integer n, and also define

$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \tag{3.4}$$

then we find that this collection obeys the current algebra

$$\begin{aligned} &[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)} && [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ &[J_a^{(n)}, J_b^{(m)}] = f_{abc}J_c^{(n+m)} && [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{3.5}$$

The index a labels the generators of the spatial rotation group $\mathrm{SO}(d)$ and f_{abc} are the corresponding structure constants. We see that the vector fields generate a $\mathrm{SO}(d)$ Kac-Moody algebra without any central terms. In addition to the Virasoro and current generators we also have the commuting generators $M_i^{(n)}$ which function like generators of a global symmetry. We can, for instance, consistently set these generators to zero. The presence of these generators therefore do not spoil the ability of the Virasoro-Kac-Moody generators to admit the usual central terms in their commutators.

What is the meaning of this infinite dimensional extension? Do these additional vector fields generate symmetries?

There is a relatively simple interpretation for the generators $M_i^{(n)}, L^{(n)}, J_a^{(n)}$. We know that $P_i = M_i^{(-1)}, B_i = M_i^{(0)}, K_i = M_i^{(1)}$ generate uniform spatial translations, velocity

[4] After obtaining these results we came to learn of a similar Virasoro extension of the Schrodinger group [5]. The actual algebra is different from the one described here.

JHEP07(2009)037

boosts and accelerations respectively. In fact, it is simple to see from (3.3) that the $M_i^{(n)}$ generate arbitrary time dependent (but spatially independent) accelerations.

$$x_i \to x_i + b_i(t). \tag{3.6}$$

Similarly the $J_{ij}^{(n)}$ in (3.4) generate arbitrary time dependent rotations (once again space independent)

$$x_i \to R_{ij}(t)x_j \tag{3.7}$$

These two set of generators together generate what is sometimes called the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime [47].

Recall that in the absence of gravity Galilean spacetime is characterised by a degenerate metric. The time intervals are much larger than any space-like intervals in the nonrelativistic scaling limit (2.6). We thus have an absolute time t and spatial sections with a flat Euclidean metric. We can, in a precise sense, describe the analogue of the isometries in this Galilean spacetime. The Coriolis group by virtue of preserving the spatial slices (at any given time) are the maximal set of isometries. See appendix A for more details. This realisation of the current algebra in our context is a bit like the occurence of a loop group.

The generators $L^{(n)}$ have a more interesting action in acting both on time as well as space. We can read this off from (3.3)

$$\to f(t), \qquad x_i \to \frac{df}{dt}x_i. \tag{3.8}$$

Thus it amounts to a reparametrisation of the absolute time t. Under this reparametrisation the spatial coordinates x_i act as vectors (on the worldline t). It seems as if this is some kind of "conformal isometry" of the Galilean spacetime, rescaling coordinates by the arbitrary time dependent factor $\frac{df}{dt}$.

With this interpretation of the infinite extension of the GCA, one might expect that it ought to be partially or fully dynamically realised in physical systems where the finite GCA is (partially or fully) realised. We will see below an example which lends support to this idea. We will also see in section 5 that the bulk geometry which we propose as the dual has the extended GCA among its *asymptotic* isometries. An analogy might be two dimensional conformal invariance where the Virasoro algebra is often a symmetry when the finite conformal symmetry of $\mathrm{SL}(2,C)$ is realised. And the (two copies of the) Virasoro generators are reflected in the bulk AdS_3 as asymptotic isometries.

Given that the Galilean limit can be obtained by taking a definite scaling limit within a relativistic theory, we expect to see the GCA (and perhaps its extension) as a symmetry of some subsector within every relativistic conformal field theory. For instance, in the best studied case of $\mathcal{N} = 4$ Yang-Mills theory, we ought to be able to isolate a sector with this symmetry. One clue is the presence of the $\mathrm{SL}(2,R)$ symmetry together with the preservation of spatial rotational invariance. One might naively think this should be via some kind of conformal quantum mechanics obtained by considering only the spatially independent modes of the field theory. But this is probably not totally correct for the indirect reasons explained in the next paragraph.

JHEP07(2009)037

Recently, the nonrelativistic limit of the relativistic conformal hydrodynamics, which describes the small fluctuations from thermal equilibrium, have been studied [40–42]. One recovers the non-relativistic incompressible Navier-Stokes equation in this limit. The symmetries of this equation were then studied by [41] (see also [42]). One finds that all the generators of the finite GCA are indeed symmetries[5] except for the dilatation operator D.[6] In particular it has the K_i as symmetries. It is not surprising that the choice of a temperature should break the scaling symmetry of D.[7] The interesting point is that the arbitrary accelerations $M_i^{(n)}$ are also actually a symmetry [43] (generating what is sometimes called the Milne group [47]). Thus we have a part of the extended GCA as a symmetry of the non-relativistic Navier-stokes equation which should presumably describe the hydrodynamics in every nonrelativistic field theory. In particular, the closed non-relativistic subsector within every relativistic conformal field theory should have a hydrodynamic description governed by the Navier-Stokes equation. This might seem to suggest that this sector ought to have more than just the degrees of freedom of a conformal quantum mechanics.

Coming back to the Navier-Stokes equation, if the viscosity is set to zero, one gets the incompressible Euler equations

$$\partial_t v_i(x,t) + v_j \partial_j v_i(x,t) = -\partial_i p(x,t) \tag{3.9}$$

In this case one has the entire finite dimensional GCA being a symmetry since D is now also a symmetry. It is the viscous term which breaks the symmetry under equal scaling of space and time. This shows that one can readily realise "gapless" non-relativistic systems in which space and time scale in the same way![8]

4 The bulk dual

Given a particular instance of an AdS/CFT duality, we should be able parametrically to take the non-relativistic scaling limit on both sides of the duality. On the field theory side, as we have discussed, the relativistic conformal invariance reduces to the GCA with a possible infinite dimensional dynamical extension. On the string theory side it should be possible to take a similar scaling limit along the lines of the non-relativistic limit studied in [51–53]. Below we will only consider features of this scaling limit when the parent bulk theory is well described by gravity. This will already involve some novel features. This has to with the fact that the usual pseudo-riemannian metric degenerates when one takes a non-relativistic limit. Nevertheless, there is a well defined, albeit somewhat unfamiliar, geometric description of gravity in such a limit [59]. In the (asymptotically) flat space case this is known as the Newton-Cartan theory of gravity which captures Newtonian gravity in a geometric

[5] For a realisation of the Schrodinger symmetry in the context of the Navier-Stokes equation see [44, 45].

[6] The generator K acts trivially.

[7] However, one can define an action of the $\tilde{D}$ as in (2.3) to be a symmetry.

[8] Inonu and Wigner [49] have considered representations of the Galilean group without the mass extension and concluded that a particle interpretation of states of the irreducible representations is subtle. In particular such states are not localisable. Just as in the case of relativistic conformal group it is likely that observables such as the S-matrix are ill-defined. We thank Sean Hartnoll for bringing this reference to our attention.

JHEP07(2009)037

setting. This is a *non-metric* gravitational theory. The dynamical variables are affine connections. Einstein's equations reduce to equations for the curvature of these non-metric connections. One can generalise this to the case of a negative cosmological constant as well. A variant of this is what we propose below as the right framework for the gravity dual of systems with the GCA. In the next subsection we will briefly review features of the Newton-Cartan theory and then go onto describe the case with a negative cosmological constant.

4.1 Newton-Cartan theory of gravity

In the Newton-Cartan description of gravity, the $(d+1)$ dimensional spacetime M has a time function t on it which foliates the spacetime into d dimensional spatial slices. Stated more precisely (see for example [60]): one defines a contravariant tensor $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \ldots d)$ such there is a time 1-form $\tau = \tau_\mu dx^\mu$ which is orthogonal to γ in the sense that $\gamma^{\mu\nu}\tau_\mu = 0$. The metric γ, which has three positive eigenvalues and one zero eigenvalue, will be the non-dynamical spatial metric on slices orthogonal to the worldlines defined by τ. There is no metric on the spacetime as a whole. In fact, its geometric structure is that of a fibre bundle with a one dimensional base (time) and the d dimensional spatial slices as fibres.

The dynamics is encoded in a torsion free affine connection $\Gamma^\mu_{\nu\lambda}$ on M. We will demand that this connection is compatible with both γ and τ i.e.

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho \tau_\nu = 0. \tag{4.1}$$

This enables one to define a time function t ("absolute time") since we have $\tau_\mu = \nabla_\mu t$. Unlike the Christoffel connections which are determined by the spacetime metric in Einstein's theory, this Newton-Cartan connection is not fixed by just these conditions. One has to impose some additional relations. Defining $R^{\mu\nu}_{\lambda\sigma} = \gamma^{\nu\alpha}R^\mu_{\lambda\alpha\sigma}$, one can define a Newtonian connection as one which obeys the additional condition $R^{\mu\nu}_{\lambda\sigma} = R^{\nu\mu}_{\sigma\lambda}$.[9]

In the presence of matter sources specified by a contravariant second rank stress tensor $T^{\mu\nu}$, which is additionally required to be covariantly conserved $\nabla_\mu T^{\mu\nu} = 0$, we can write down the field equations which determine the connection in terms of the sources. This is best done by introducing a "time" like metric $g_{\mu\nu} = \tau_\mu\tau_\nu$ which is orthogonal to the spatial metric $\gamma_{\mu\nu}$. The field equations are then familiar in form

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{4.2}$$

where $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$ and $T = g_{\alpha\beta}T^{\alpha\beta}$. Note that to define the Ricci tensor $R_{\mu\nu}$ (unlike the Ricci scalar R) one does not need a metric, only the affine connection.

When one chooses coordinates such that $\gamma = \delta^{ij}\partial_i \otimes \partial_j$ $(i, j = 1 \ldots d)$, $\tau = dt$, the non-zero components of the Newtonian connection take the form (imposing appropriate boundary conditions at infinity) $\Gamma^i_{00} = \partial_i\Phi$. The field equations then reduce to Poisson's equations with Φ being the Newtonian gravitational potential and the source being the matter density $\rho = T^{00}$.

[9] In Einstein gravity with the Christoffel connection and γ being the *nondegenerate* spacetime metric, this relation is identically satisfied but here it has to be imposed additionally.

JHEP07(2009)037

This is, of course, an intrinsic characterisation of Newtonian gravity. Not unsurprisingly, this geometric structure has also been shown to arise in the degenerating limit of a usual Einsteinian geometry [54]. Namely, one can study a one parameter (ϵ) family of usual Lorentzian signature metrics, with the non-relativistic limit $\epsilon \to 0$ leading to a degenerate metric. The condition that this limiting geometry be a Newtonian spacetime is satisfied under fairly mild conditions on the ϵ dependence of the Lorentzian metric (and therefore the associated geometric objects such as the covariant derivative, curvature tensor etc.). This shows that the nonrelativistic scaling limit is a sensible one to take of a generic Einsteinian geometry.

4.2 Newtonian limit of gravity on AdS_{d+2}

We would like to parametrically carry out the non-relativistic scaling on the bulk AdS_{d+2} which would capture the physics of the nonrelativistic limit in the $(d+1)$ dimensional boundary theory. In the next section we will describe the bulk scaling in more detail. Here we will simply motivate its qualitative features and give the resulting Newton-Cartan like description of the bulk geometry.

We know that the boundary metric degenerates in the nonrelativistic limit with the d spatial directions scaling as $x_i \propto \epsilon$ while $t \propto \epsilon^0$. We expect this feature to be shared by the bulk metric. One expects that the geometry on constant radial sections to have such a scaling. Since the radial direction of the AdS_{d+2} is an additional dimension, we have to fix its scaling. The radial direction is a measure of the energy scales in the boundary theory via the holographic correspondence. We therefore expect it to also scale like time i.e. as ϵ^0. This means that in the bulk the time and radial directions of the metric *both* survive when performing the scaling. Together these constitute an AdS_2 sitting inside the original AdS_{d+2}.

What this implies for the dynamics is that we should have a Newton-Cartan like description but with the special role of time being replaced by an AdS_2. The geometric structure, in analogy with that of the previous section, is that of a fibre bundle with AdS_2 base and the d dimensional spatial slices as fibres.

Accordingly, we will consider a ("spatial") metric $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \dots d+1)$ which now has *two* zero eigenvalues corresponding to the time and radial directions. (In a canonical choice of coordinates these directions will correspond to $\mu = 0, d+1$). Mathematically the two null eigenvectors will be taken to span the space of left invariant 1-forms of AdS_2. These will also define the AdS_2 metric $g_{\alpha\beta}$ in the usual way (This is the analogue of the time metric defined in the previous subsection).

We will once again have dynamical, torsion free affine connections $\Gamma^\mu_{\nu\lambda}$ which are compatible with both the spatial and AdS_2 metrics

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho g_{\alpha\beta} = 0. \tag{4.3}$$

There will also be Christoffel connections from the AdS_2 and spatial metrics which will not be dynamical if we do not allow these metrics, specifically $g_{\mu\nu}$, to fluctuate. We will also impose the condition below (4.1) on the Riemann tensor.

In standard Poincare coordinates where $\gamma = z^2 \delta^{ij}\partial_i \otimes \partial_j$ $(i, j = 1 \dots d)$ and $g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{z^2}(dt^2 - dz^2)$, the non-zero components of the dynamical connection can

JHEP07(2009)037

be taken to be $\Gamma^i_{ab}(t,z,x_i) = \partial_i \Phi_{ab}(t,z,x_i)$ (with $a,b = 0, d+1$). There will be Christoffel components from γ and g as mentioned above.

The field equations are the expected modification of (4.2)

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{4.4}$$

where $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$ and $T = g_{\alpha\beta}T^{\alpha\beta}$ and Λ is the cosmological constant. These are dynamical equations for the fields $\Phi_{ab}(t,z,x_i)$ once the stress tensor $T^{ab}(t,z,x_i)$ in the AdS_2 directions is specified.

5 GCA in the bulk

In this section we will carry out the non-relativistic scaling limit on the AdS_5 piece of the bulk. We will also do this for the SO(4, 2) isometries of AdS_5 and obtain the same contracted algebra as in section3.[10] We will then see how the infinite dimensional extension of this algebra is realised in the bulk. They will have the interpretation as being the generators of asymptotic isometries of the bulk Newton-Cartan like geometry described in the previous section. Since asymptotic isometries, under appropriate circumstances, act on the physical hilbert space of the theory, one finds support for the assertion that the infinite extension can be dynamically realised.

Consider the metric of AdS_5 in Poincare coordinates

$$ds^2 = \frac{1}{z'^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz'^2) = \frac{1}{z'^2}(dt'^2 - dz'^2 - dx_i^2) \tag{5.1}$$

The nonrelativistic scaling limit that we will be considering is, as motivated in the previous section

$$t', z' \to \epsilon^0 t', \epsilon^0 z' \qquad x_i \to \epsilon^1 x_i. \tag{5.2}$$

In this limit we see that only the components of the metric in the (t', z') directions survive to give the metric on an AdS_2. The d dimensional spatial slices parametrised by the x_i are fibred over this AdS_2. The Poincare patch has a horizon at $z' = \infty$ and to extend the coordinates beyond this we will choose to follow an infalling null geodesic, in an analogue of the Eddington-Finkelstein coordinates. Therefore define $z = z'$ and $t = t' + z'$. In these coordinates

$$ds^2 = \frac{1}{z^2}(-2dtdz + dt^2) = \frac{dt}{z^2}(dt - 2dz). \tag{5.3}$$

[10] As mentioned earlier, an algebraically equivalent contraction of the bulk isometries was carried out in [53](see also [50]). However the actual embedding of this contracted algebra in AdS is not manifestly that of a nonrelativistic CFT on the boundary. In particular, their foliation of the bulk corresponds to a boundary geometry which is a time dependent $AdS_2 \times S^2$. This is natural from the point of view of considering the worldvolume of a half BPS string. By considering Poincare coordinates and foliating the bulk in terms of $R^{3,1}$ slices, it is plausible that their limit will be related directly to ours. We thank Jaume Gomis for helpful communication in this regard.

JHEP07(2009)037

5.1 Contraction of the bulk isometries

In the infalling Eddington-Finkelstein coordinates, the Killing vectors of AdS_5 read as

$$\begin{aligned} P_i &= \partial_i,\ B_i = (t-z)\partial_i - x_i\partial_t \\ K_i &= (t^2 - 2tz - x_j^2)\partial_i + 2tx_i\partial_t + 2zx_i\partial_z + 2x_ix_j\partial_j \\ J_{ij} &= -(x_i\partial_j - x_j\partial_i) \\ H &= -\partial_t,\ D = -t\partial_t - z\partial_z - x_i\partial_i \\ K &= -(t^2 + x_i^2)\partial_t - 2z(t-z)\partial_z - 2(t-z)x_i\partial_i \end{aligned} \tag{5.4}$$

Here we have used the same labelling for the bulk generators as on the boundary to facilitate easy comparison. Some additional details are given in appendix B.

Carrying out the scaling (5.2) we obtain the contracted Killing vectors

$$\begin{aligned} &P_i = \partial_i,\ B_i = (t-z)\partial_i, \quad K_i = (t^2 - 2tz)\partial_i, \qquad J_{ij} = -(x_i\partial_j - x_j\partial_i) \\ &H = -\partial_t, \qquad\qquad D = -t\partial_t - z\partial_z - x_i\partial_i, \quad K = -t^2\partial_t - 2(t-z)(z\partial_z + x_i\partial_i) \end{aligned} \tag{5.5}$$

We see that at the boundary $z = 0$ these reduce to the contracted Killing vectors of the relativistic conformal algebra. It can also be checked that these obey the same algebra as (2.1) and (2.12) or equivalently (3.2) after the relabelling of (3.1).

The interpretation of most of the generators is straightforward. We note that the H, K, D are scalars under the spatial $\mathrm{SO}(d-1)$ and generate, as before, an $\mathrm{SL}(2, R)$. We identify this as the isometry group of the AdS_2 base of our Newton-Cartan theory.

We can again define an infinite family of vector fields in the bulk

$$\begin{aligned} M_i^{(m)} &= (t^{m+1} - (m+1)zt^m)\partial_i \\ J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i) \\ L^{(n)} &= -t^{n+1}\partial_t - (n+1)(t^n - nzt^{n-1})(x_i\partial_i + z\partial_z) \end{aligned} \tag{5.6}$$

These vector fields reduce on the boundary to (3.3) and (3.4).

It is rather remarkable that these vector fields also obey the commutation relations of the Virasoro-Kac-Moody algebra, the same as in the boundary theory

$$\begin{aligned} &[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)} \qquad &&[L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ &[J_a^{(n)}, J_b^{(m)}] = f_{abc}J_c^{(n+m)} \qquad &&[L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{5.7}$$

How do we interpret all these additional vector fields from the point of view of the bulk? Firstly, notice that the vector fields $M_i^{(n)}$ and $J_a^{(n)}$ only act on the spatial coordinates x_i (with t, z dependent coefficents). From the viewpoint of the fibre bundle structure, these are simply rotations and translations on the spatial slices which happen to be dependent on time as well as z. These are the isometries of the spatial metric γ of the previous section. They are also trivially isometries of the AdS_2 metric since they do not act on those coordinates. In general, these transformations will have a non-trivial effect on the dynamical connection coefficient (though trivial action on the non-dynamical christoffel

JHEP07(2009)037

coefficients). This is not unusual since it is only the vacuum configuration of the bulk theory (in which the dynamical connections vanish) which preserves the full symmetry.

Now we come to the action of the Virasoro generators, $L^{(n)}$ which act non-trivially on all coordinates. We have under its action (with infinitesimal parameter a_n)

$$\begin{aligned} z \to \tilde{z} &= z[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ t \to \tilde{t} &= t[1 + a_n t^n] \\ x_i \to \tilde{x_i} &= x_i[1 + a_n(n+1)(t^n - nzt^{n-1})]. \end{aligned} \tag{5.8}$$

In other words,

$$\begin{aligned} dz \to \tilde{dz} &= dz[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ &\quad + z a_n n(n+1) t^{n-2}[(t - (n-1)z)dt - t dz] \\ dt \to \tilde{dt} &= dt[1 + (n+1)a_n t^n] \\ dx_i \to \tilde{dx_i} &= dx_i[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ &\quad + n(n+1)a_n x_i t^{n-2}[(t - (n-1)z)dt - t dz]. \end{aligned} \tag{5.9}$$

To see how this acts on the Newton-Cartan structure, consider first the above action on the original Poincare metric on AdS_5 but transformed to the Eddington-Finkelstein coordinates (5.3). Only after that do we take the scaling limit (5.2). We find

$$\begin{aligned} ds^2 &= \frac{1}{z^2}(-2dtdz + dt^2 + dx_i^2) \to \frac{1}{z^2}(-2dtdz + dt^2 + dx_i^2) \\ &\quad + 2n(n^2-1)a_n t^{n-2} dt^2 - 2\frac{a_n n(n+1)}{z^2} x_i dx_i[(t - (n-1)z)dt - t dz]. \end{aligned} \tag{5.10}$$

We now see that on taking the scaling limit (5.2) we have

$$ds^2 = \frac{1}{z^2}(-2dtdz + dt^2) \to \frac{1}{z^2}(-2dtdz + dt^2 + 2n(n^2-1)a_n z^2 t^{n-2} dt^2). \tag{5.11}$$

As expected the $\mathrm{SL}(2,R)$ subgroup $L^{(0)}, L^{(\pm 1)}$ are exact isometries. The other $L^{(n)}$ are not exact isometries. However, they are asymptotic isometries (See [55] and references therein). Near the boundary $z = 0$ the diffeomorphisms generated by these vector fields leave the metric unchanged upto a factor which has a falloff like z^2. Thus these do not affect the non-normalizable mode of the metric.

One expects that when the charges for these asymptotic isometries are constructed, then just as in the Brown-Henneaux construction for AdS_3 [55](and recent generalisations to AdS_2 [56]), there will actually be a central term due to boundary contributions. Thus the Virasoro algebra will presumably act in a faithful way on the physical Hilbert space.

We also notice from the action of the $L^{(n)}$ (5.10) on the spatial metric that on the slices of constant t, z, the action is again an isometry. Thus the $L^{(n)}, J_a^{(n)}, M_i^{(N)}$ together generate (asymptotic) isometries of the spatial and AdS_2 metrics γ^{ij} and g_{ab}. Therefore it seems natural to consider the action of these generators on the Newton-Cartan like geometry

JHEP07(2009)037

6 Concluding remarks

We have seen that the nonrelativistic conformal symmetry obtained as a scaling limit of the relativistic conformal symmetry has several novel features which make it a potentially interesting case for further study. The GCA, we have argued, is different from the Schrodinger group which has been studied recently. It also has the advantage of being embedded within the relativistic theory. Hence we ought to have realisations of the GCA in every interacting relativistic conformal field theory. The obvious question is to understand this sector in a particular case such as $\mathcal{N} = 4$ Super Yang-Mills theory. And to see whether the infinite dimensional extension can be dynamically realised (and its central charge computed). We have provided indications why this might be the case generically.

Relatedly, in the bulk gravity dual to such a system one ought to be able to independently compute the central term in the Virasoro algebra *a la* Brown-Henneaux. In such cases one should be able to use the more general Kac-Moody algebra to constrain the theory and its correlation functions much more. A straightforward generalisation of our results would be to a supersymmetric extension of the Kac-Moody algebra. These and related questions are currently under investigation.

The bulk description in terms of a Newton-Cartan like geometry is somewhat unfamiliar and it would be good to understand it better. In particular, one needs a precise bulk-boundary dictionary to characterise the duality. At least implicitly this is determined by taking the parametric limit of the relativistic duality.

Then there is the question of how such non-metric theories lift to string theories. This is something we have not touched upon at all in this note. One might hope to get some guidance from previous studies of nonrelativistic string theories, though in all these cases one had additional fields like the two form $B_{\mu\nu}$ turned on which made the sigma model well defined. It is therefore not completely clear how to define a string theory on these Newton-Cartan like geometries.[11]

In the case of the Schrodinger symmetry the dual gravity theory is proposed to live in *two* higher dimensions than the field theory. This also provided the route for embedding the dual geometry in string theory. It is interesting to ask if there is something analogous in our case, whereby the GCA is realised as a standard isometry of a higher dimensional geometry (e.g. $(d+3)$ dimensional for a $(d+1)$ dimensional field theory).

Coming back to the boundary theory, it is interesting to ask whether there are intrinsically non-relativistic realisations of the GCA, perhaps in a real life system. It is encouraging in this context that the incompressible Euler equations concretely realise the GCA, providing an example of a gapless non-relativistic system.

Note added. It has been brought to our attention that closely related infinite dimensional algebras have been studied in the context of statistical mechanical systems in [61].

[11] There have been works attempting to quantise nonrelativistic theories of gravity directly in a canonical framework [57, 58]. It would be interesting to see if these can be generalised to the case with negative cosmological constant. This could have interesting implications for gauge-gravity dualities in the non-relativistic setting.

JHEP07(2009)037

It would be interesting to study the precise connection as well as the potential realisations in statistical mechanics further.

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A Galilean isometries

In the Newton-Cartan spacetime described in section 4, we do not have a spacetime metric. Therefore the usual notion of an isometry as generated by a Killing vector field of the metric is not applicable. There is consequently some ambiguity in the definition of an isometry. We will paraphrase here some of the different possibilities as outlined in [47].

1. **Galilei Algebra**: This consists of all vector fields X satisfying

$$L_X\gamma^{\mu\nu} = 0 \qquad L_X\tau = 0 \qquad L_X\Gamma^{\alpha}_{\mu\nu} = 0. \tag{A.1}$$

   This gives rise to the usual set of vector fields which generate the finite dimensional Galilean algebra of uniform translations (in space and time), uniform velocity boosts and spatial rotations.

2. **Milne Algebra**: This consists of all vector fields X satisfying

$$L_X\gamma^{\mu\nu} = 0 \qquad L_X\tau = 0 \qquad L_X\Gamma^{\nu\alpha}_{\mu} = 0 \tag{A.2}$$

   where $\Gamma^{\nu\alpha}_{\mu} = \gamma^{\beta\nu}\Gamma^{\alpha}_{\mu\beta}$. The set of vector fields X satisfying this condition is an infinite dimensional extension of the Galilean algebra, now involving arbitrary time dependent boosts/accelerations.

3. **Coriolis Algebra**: This consists of all vector fields X satisfying

$$L_X\gamma^{\mu\nu} = 0 \qquad L_X\tau = 0 \qquad L_X\Gamma^{\mu\nu\alpha} = 0 \tag{A.3}$$

   where $\Gamma^{\mu\nu\alpha} = \gamma^{\rho\mu}\gamma^{\beta\nu}\Gamma^{\alpha}_{\rho\beta}$. The set of vector fields X satisfying this condition is a further infinite dimensional extension of the Milne algebra, now involving in addition to the arbitrary boosts or accelerations, arbitrary time dependent rotations as well.

JHEP07(2009)037

B Killing vectors of AdS_{d+2} and bulk contraction

Here we list the killing vectors of AdS_{d+2} in the $d+3$ dimensional Minkowskian embedding space and then rewrite them in intrinsic AdS coordinates. For taking the contraction, we will work in Poincare coordinates.

We denote flat $d+1$-dimensional space with co-ordinates y_a with $a = 1, .., d+1$. Embedding equation:

$$uv + \eta_{ab} y^a y^b = 1 \tag{B.1}$$

$$ds^2 = dudv + \eta_{ab} dy^a dy^b \tag{B.2}$$

where η_{ab} has a form $diag(1, -1, -1, \ldots -1)$. Also note that we have rescaled the AdS_{d+2} radius to 1.

We wish to write in explicit SO$(d+1, 2)$ notation. So we choose $u = y_0 + y_{d+2}$ and $v = y_0 - y_{d+2}$. Now we would be interested in the co-ordinates on the Poincare patch.

$$z = v^{-1} = (y_0 - y_{d+2})^{-1} \tag{B.3}$$

$$t = v^{-1} y_1 = (y_0 - y_{d+2})^{-1} y_1 \tag{B.4}$$

$$x_i = v^{-1} y_i = (y_0 - y_{d+2})^{-1} y_i \tag{B.5}$$

where we label $i = 2, 3 \ldots d+1$.

The constraint equation becomes:

$$u = z - (z)^{-1}(t^2 - x_i^2) \tag{B.6}$$

which gives the metric on the Poincare patch (5.1).

We can take the inverse transformations and express the derivatives of the Poincare co-ordinates in terms of derivatives of y's and take various linear combinations to obtain

$$\begin{aligned}
M_{01} &= -(y_0\partial_1 - y_1\partial_0) \\
&= -\frac{1}{2}(z^2 + 1 + t^2 + x_i^2)\frac{\partial}{\partial t} - zt\frac{\partial}{\partial z} - tx_i\frac{\partial}{\partial x_i} \\
M_{0i} &= -(y_0\partial_i + y_i\partial_0) \\
&= -\frac{1}{2}(z^2 + 1 - t^2 + +x_i^2 + x_j^2)\frac{\partial}{\partial x_i} + zx_i\frac{\partial}{\partial z} + tx_i\frac{\partial}{\partial t} + x_i^2\frac{\partial}{\partial x_i} + x_ix_j\frac{\partial}{\partial x_j} \quad (j \neq i) \\
M_{1i} &= -(y_1\partial_i + y_i\partial_1) = -t\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial t} \\
M_{ij} &= -(y_i\partial_j - y_j\partial_i) = -\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right) \\
M_{0,d+2} &= -(y_0\partial_{d+2} + y_{d+2}\partial_0) = -z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i} \\
M_{1,d+2} &= -(y_1\partial_{d+2} + y_{d+2}\partial_1) \\
&= -\frac{1}{2}(z^2 + t^2 + x_i^2 - 1)\frac{\partial}{\partial t} - zt\frac{\partial}{\partial z} - tx_i\frac{\partial}{\partial x_i}
\end{aligned}$$

JHEP07(2009)037

$$
\begin{aligned}
M_{i,d+2} &= -(y_{d+2}\partial_i + y_i\partial_{d+2}) \\
&= -\frac{1}{2}(z^2 - t^2 + x_i^2 + x_j^2 - 1)\frac{\partial}{\partial x_i} + z x_i \frac{\partial}{\partial z} + t x_i \frac{\partial}{\partial t} + x_i^2 \frac{\partial}{\partial x_i} + x_i x_j \frac{\partial}{\partial x_j} \qquad (j \neq i)
\end{aligned}
$$

In the above equations, the repeated indices j are summed over in M_{0i} and $M_{i,d+2}$.

To connect with our notation for the boundary generators, we define:

$$
\begin{aligned}
H &= M_{01} - M_{1,d+2}; & K &= M_{01} + M_{1,d+2}; & D &= M_{0,d+2} \\
P_i &= -M_{0i} + M_{i,d+2}; & K_i &= M_{0i} + M_{i,d+2}; & B_i &= -M_{1i} \\
M_{ij} &= M_{ij}
\end{aligned} \tag{B.7}
$$

After transforming to infalling Eddington-Finkelstein coordinates the generators of (B.7) become then ones given in (5.4) We then carry out the contraction on these Killing vectors using the scaling: $z = \epsilon^0 \tilde{z}$, $t = \epsilon^0 \tilde{t}$ and $x_i = \epsilon^1 \tilde{x}_i$ and obtain (5.5).

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JHEP07(2009)037