TWISTOR THEORY AND GEOMETRIC QUANTIZATION

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Introduction

In this talk, I shall show how Penrose's twistor formalism [4] arises through the application, with the aid of a few geometrical tricks, of the Kostant-Souriau geometric quantization theory [3,10,11] to massless spinning particles in Minkowski space. This is not the usual way of introducing twistors, still less does it reflect the historical development of the subject, but it does have the advantage of showing up some of the similarities in the fundamental ideas of the two theories.

However, in spite of these similarities, it is important to realize that their motivations are very different. The ultimate aim of geometric quantization is the construction of a unified theory of the irreducible unitary representations of connected Lie groups by first geometrizing and then generalizing the physicist's concept of quantization. When applied to simple physical systems, such as those invariant under the Poincaré group, geometric quantization results in the synthesis within a geometric framework of various well understood techniques from conventional quantum mechanics; it does not incorporate any new physical ideas.

In his twistor theory, on the other hand, Penrose is trying to develop a new formalism for relativistic quantum field theory and, eventually, to lay the foundations for a quantum theory of gravity. Moreover, Penrose has often stressed that he is looking for a formalism which only works in four dimensional space-time. In a sense, the existence of such a formalism would explain the dimension and signature of the real world.

But, even allowing for these differences in outlook, there are a number of practical benefits which can be derived from a comparison of the two theories:

1) If one tries to construct a manifestly conformally invariant theory of massless particles by applying geometric quantization to the conformal

group $O(1,3)$ - or, rather, to its fourfold cover, $SU(2,2)$ - one runs into a number of difficulties (which I shall describe later). Penrose's twistor contour integration techniques provide a way of circumventing these difficulties and it is possible that, when translated into a suitable form, these techniques will lead to a friutful generalization of the Kostant-Souriau theory.

2) Kostant and Souriau's geometric formulation of standard quantum mechanics is ideally suited to answering the question: How much of twistor theory is an elegant restatement of old ideas, and how much is new physics?

3) Geometric quantization is often difficult to work with in practice: one is frequently forced to rely on the introduction of special coordinate systems. However, by using some of the tricks suggested by twistor theory, it is possible to quantize massless and massive particles in a covariant way and thus to obtain an example of a completely geometric application of the theory.

Notation

The notation used here for the $SL(2,\mathbb{C})$ spinor calculus is essentially the same as that described by Penrose [6] and Pirani [8] . Capital Roman letters are used for spinor indices (which run over 0,1) and lower case Roman letters for space-time vector and tensor indices (which run over 0,1,2,3); primed indices are used to denote conjugate spinors. The Einstein range and summation conventions are used throughout.

The correspondence between a vector Y^a and its spinor equivalent $Y^{AA'}$ is given explicitly, in <u>any</u> proper orthochronous Lorentz frame, by

$$Y^a \leftrightarrow Y^{AA'} = \begin{bmatrix} Y^{00'} & Y^{01'} \\ Y^{10'} & Y^{11'} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} Y^0 + Y^1 & Y^2 + iY^3 \\ Y^2 - iY^3 & Y^0 - Y^1 \end{bmatrix} \qquad (1)$$

Spinor indices are raised and lowered with the Levi-Civita symbols

$$\epsilon^{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \epsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad (2)$$

and their complex conjugates $\epsilon^{A'B'}$ and $\epsilon_{A'B'}$ (these are all $SL(2,\mathbb{C})$ invariant). Thus, for example,

$$\chi^A = \epsilon^{AB}\chi_B, \ \chi_B = \epsilon_{AB}\chi^A \ \text{ and } \ \bar{\chi}_{B'} = \epsilon_{A'B'}\bar{\chi}^{A'}. \tag{3}$$

This is consistent with the usual convention for raising and lowering space-time indices since $\epsilon_{AB}\epsilon_{A'B'}$ is the spinor equivalent of the space-time metric g_{ab}. Finally, the flat spinor connection $\nabla_{AA'}$ is given in Lorentz coordinates $\{x^a\}$ by

$$\sqrt{2}\,\nabla_{AA'} = \begin{bmatrix} \partial_0 + \partial_1 & \partial_2 - i\partial_3 \\ \partial_2 + i\partial_3 & \partial_0 - \partial_1 \end{bmatrix} \qquad \partial_a = \frac{\partial}{\partial x^a} \tag{4}$$

(In the Battelle convention [6], spinor and tensor indices are regarded as abstract labels indicating the type of the geometric object to which they are attached. Thus, for example, Y^a is actually a vector, rather than the components of a vector, and (1) can be rewritten: $Y^a = Y^{AA'}$. Though this convention will not be used explicitly, it can be used to reinterpret all the equations below as relations between geometric objects, rather than the components of geometric objects.)

Massless Particles: Canonical Formalism

In classical relativistic mechanics, the kinematical variables of a massless particle with helicity $s \geqslant 0$ can be represented by a position vector X^a (relative to some origin O) and two future pointing null vectors I^a and J^a (normalized so that $I_a J^a = 1$). In terms of these, the momentum and angular momentum are given by

$$p_a = I_a \quad \text{and} \quad M^{ab} = -s\,\epsilon^{abcd} I_c J_d + X^a I^b - X^b I^a, \tag{5}$$

the form of M^{ab} being fixed by the condition that the spin vector

$$s_a = \tfrac{1}{2}\epsilon_{abcd}\,p^b M^{cd} \tag{6}$$

should be parallel to the momentum (ϵ_{abcd} is the alternating tensor).

Before quantizing this system, it is necessary to construct the classical phase space: in practical terms, this means finding a suitable expression for the symplectic 2-form of the system in terms of the variables X^a, I^a and J^a. In this search, there are two guiding principals:

1) The system is to be an elementary relativistic system: this means that the Poincare group P must act transitively on the phase space as a group of canonical transformations. By the Kostant-Kirillov-Souriau theorem [3], therefore, the phase space must be locally isomorphic (as a P-symplectic space) with an orbit in the dual of the Poincare Lie algebra.

2) The physical variables p_a and M^{ab} are to have their usual interpretation as generators of P.

The first implication of these is that each point in the phase space is determined by the values of p_a and M^{ab} alone, or, after a little calculation, that one must identify $(\tilde{X}^a, \tilde{I}^a, \tilde{J}^a)$ and (X^a, I^a, J^a) whenever

$$\tilde{X}^a = X^a + Z^a, \quad \tilde{I}^a = I^a \quad \text{and} \quad \tilde{J}^a = J^a + \epsilon^{abcd} I_b J_c Z_d - \tfrac{1}{2} Z^b Z_b I^a \qquad (7)$$

for some Z^a such that $Z^a I_a = 0$. The resulting manifold is six dimensional and has topology $\mathbb{R}^4 \times S^2$. (The need for this identification reflects the fact that even classically a massless spinning particle is not localizable: it occupies an entire null hyperplane.)

It is then not hard to show that one, and hence the only, symplectic form on M_s which gives the correct Poisson brackets for p_a and M^{ab} is

$$\sigma = s\,\epsilon_{abcd} I^a J^b dI^c \wedge dJ^d - dX^a \wedge dI_a \qquad (8)$$

(see Souriau [10], p.190). (As a 2-form on the nine dimensional (X^a, I^a, J^a)-space, σ is degenerate. However, the vectors in this space which annihilate σ are precisely those which generate the identification (7). Thus σ projects into a nondegenerate 2-form on M_s.)

Prequantization

The first stage in the prequantization of (M_s, σ) is to replace I^a and J^a by two spinors o^A and ι^A chosen so that

$$I^a \leftrightarrow o^A \bar{o}^{A'}, \quad J^a \leftrightarrow \iota^A \bar{\iota}^{A'} \quad \text{and} \quad o_A \iota^A = 1. \qquad (9)$$

(This is possible since I^a and J^a are null, so that $I^{AA'}$ and $J^{AA'}$ are singular.) If then

$$\pi_{A'} = \bar{o}_{A'} \quad \text{and} \quad \omega^A = s\iota^A + i X^{AA'} \bar{o}_{A'} \tag{10}$$

the identification (7) becomes $(\tilde{X}^a, \tilde{\omega}^A, \tilde{\pi}_{A'}) \simeq (X^a, \omega^A, \pi_{A'})$ whenever

$$\tilde{X}^a = X^a + Z^a, \quad \tilde{\pi}_{A'} = e^{it}\pi_{A'} \quad \text{and} \quad \tilde{\omega}^A = e^{it}\omega^A \tag{11}$$

for some $t \in \mathbb{R}$ and for some real Z^a such that $Z^{AA'}\bar{\pi}_A \pi_{A'} = 0$.

Thus a point of M_s can be fixed by specifying the pair $(\omega^A, \pi_{A'})$: the corresponding values of X^a are then given as the solutions of the linear equation

$$(\omega^A - i X^{AA'} \pi_{A'})\bar{\pi}_A = s \tag{12}$$

and the values of the momenta are given explicitly by

$$p_a \leftrightarrow \bar{\pi}_A \pi_{A'} \quad \text{and} \quad M^{ab} \leftrightarrow i(\omega^{(A} \bar{\pi}^{B)} \epsilon^{A'B'} - \epsilon^{AB} \bar{\omega}^{(A'} \pi^{B')}). \tag{13}$$

The four complex (eight real) dimensional vector space in which ω^A and $\pi_{A'}$ are independent variables is called <u>twistor space</u> (denoted $\mathbb{T}$). A twistor (that is, an element of $\mathbb{T}$) can be represented either as a pair $(\omega^A, \pi_{A'})$ of spinors or as a quadruple $Z^\alpha = (Z^1, Z^2, Z^3, Z^4)$ where

$$(Z^1, Z^2) = (\omega^0, \omega^1) \quad \text{and} \quad (Z^3, Z^4) = (\pi_{0'}, \pi_{1'}). \tag{14}$$

(Again, the index α can be interpreted as an "abstract" index.)

The twistors which correspond to points in M_s are those which lie in the surface $G_s \subset \mathbb{T}$ given by

$$g(Z^\alpha) = Z^\alpha \bar{Z}_\alpha = \omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'} = 2s \tag{15}$$

where $\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$ is the <u>Hermitian conjugate</u> of Z^α. (The map

$$g : \mathbb{T} \longrightarrow \mathbb{R} : Z^\alpha \longmapsto Z^\alpha \bar{Z}_\alpha = \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'} \tag{16}$$

defines a pseudo-Hermitian metric on $\mathbb{T}$ of signature $(+,+,-,-)$.) Because the freedom available in the choice of the phases of o^A and ι^A, the projection*

$$pr : G_s \longrightarrow M_s : Z^\alpha = (\omega^A, \pi_{A'}) \longmapsto (p_a, M^{ab}) \tag{17}$$

defined by (13) is not one-to-one and, in fact, $pr(Z^\alpha) = pr(Y^\alpha)$ whenever $Z^\alpha = e^{it} Y^\alpha$ for some $t \in \mathbb{R}$.

Now if, in T, one introduces the symplectic 2-form

$$\sigma = i\,(dZ^\alpha \wedge d\overline{Z}_\alpha) \tag{18}$$

(which gives T the structure of a pseudo-Kähler manifold) then the Hamiltonian vector field generated by $g : T \longrightarrow \mathbb{R}$ is

$$X_g = i\,(Z^\alpha \frac{\partial}{\partial Z^\alpha} - \overline{Z}_\alpha \frac{\partial}{\partial \overline{Z}_\alpha}) \tag{19}$$

which has closed integral curves of the form $t \longmapsto e^{it} Z_0^\alpha$. In other words,

$$M_s = G_s / X_g . \tag{20}$$

Moreover, the restiction of σ to G_s is degenerate and is annihilated by X_g (which is tangent to G_s), so that σ projects into a closed 2-form in M_s. A short calculation shows that this is precisely the symplectic form introduced above.

The point of this is that, while the symplectic structure of (M_s, σ) is not exact, that of (T, σ) is, since, in T,

$$\sigma = d\theta \quad \text{where} \quad \theta = \tfrac{1}{2} i(Z^\alpha d\overline{Z}_\alpha - \overline{Z}_\alpha dZ^\alpha). \tag{21}$$

This can be exploited in the prequantization of (M_s, σ), as follows:

If, in the bundle space of the trivial line bundle $G_s \times \mathbb{C} \longrightarrow G_s$, one puts:

$$\alpha = \theta + \frac{1}{2\pi i} \frac{dz}{z} \quad \text{and} \quad Y = X_g - 2\pi i\,(X_g \lrcorner\, \theta) \frac{\partial}{\partial z} = X_g - \frac{2is}{\hbar} \cdot \frac{\partial}{\partial z}$$

*When $s = 0$, the twistors of the form $Z = (\omega^A, 0)$ must be omitted from G_s: the reason for this will be made clear later.

(where z is the coordinate in $\mathbb{C}$, and, in the units used here, $\hbar = (2\pi)^{-1}$) then α is a connection form on $G_s \times \mathbb{C}$ and the integral curves of Y are parallel and are of the form

$$t \longmapsto \left(e^{it} Z_0^{\alpha}, \; e^{-2is\hbar^{-1}t} \cdot z_0 \right) \; ; \quad Z_0^{\alpha} \in T, \quad z_0 \in \mathbb{C} \, . \tag{22}$$

These are closed whenever $2s = \oint \frac{1}{2\pi}\theta$ (the integral being taken around an orbit of X_g in G_s) is an integral multiple of $\hbar$, in which case $L = (G_s \times \mathbb{C})/_Y$ is a Hausdorff manifold, and, in fact, a line bundle over M_s (with the projection $L \longrightarrow M_s$ making the diagram

$$\begin{array}{ccc} G_s \times \mathbb{C} & \longrightarrow & L \\ \downarrow & & \downarrow \\ G_s & \longrightarrow & M_s \end{array}$$

commute). Moreover, $Y \lrcorner \alpha = 0$ and $\mathcal{L}_Y \alpha = 0$, so that α projects onto a connection form (also denoted α) on L; it follows from the definition of α that the curvature of this connection is σ. Thus, when $2s\hbar^{-1}$ is integral, this construction is an explicit prequantization for (M_s, σ).

Quantization

The next stage is to find a polarization for T which is invariant under the action of the Poincaré group. This action is easily found from equation (10). Under translation through A^a,

$$(\omega^A, \pi_{A'}) \longmapsto (\omega^A + i A^{AA'}\pi_{A'}, \pi_{A'}) \tag{23}$$

and under the Lorentz rotation defined by $L^A{}_B \in SL(2,\mathbb{C})$,

$$(\omega^A, \pi_{A'}) \longmapsto (L^A{}_B\, \omega^B, \; \overline{M}^{B'}{}_{A'}\, \pi_{B'}) \quad \text{where} \quad M^C{}_A L^A{}_B = \delta^C_B \tag{24}$$

(Note that both transformations preserve the form $\epsilon_{\alpha\beta\gamma\delta}\, dZ^{\alpha}\wedge dZ^{\beta}\wedge dZ^{\gamma}\wedge dZ^{\delta}$ and the Kähler structure of T, and thus define elements of $SU(2,2)$.)

It follows that the real polarization F of T, spanned at each point by the vectors $\partial/\partial\omega^A$ and $\partial/\partial\overline{\omega}^{A'}$ is Poincaré invariant. It is also Lie propagated by X_g.

This polarization induces a polarization $\tilde{F}$ of M_s: explicitly, $\tilde{F}$

is the projection into M_s of the distribution H on G_s defined by

$$H : Z \in G_s \longmapsto F_Z \cap T_Z(G_s) \subset T_Z(G_s)$$

where $T_Z(G_s)$ is the tangent space to G_s at Z (H is also Lie propagated by X_g); $\tilde{F}$ is, in fact, the 'natural' polarization of M_s, in the sense that it is spanned by the generating vector fields of the translation subgroup of P.

The integral manifolds of $\tilde{F}$ are the surfaces in M_s given by $\bar{\pi}_A$ = const., $\pi_{A'}$ = const. so that the points of the factor space $M_s/\tilde{F}$ are parameterized by $\pi_{A'}$ and $\bar{\pi}_A$ (modulo phase); in other words, $M_s/\tilde{F}$ is simply the future half of the light cone in momentum space (denoted N_+).

Now, up to normalization, there is a unique Lorentz invariant volume element ν on N_+, given in coordinates by

$$\nu = \frac{1}{p_0}(dp_1 \wedge dp_2 \wedge dp_3) = i(d\bar{\pi}^A \wedge d\bar{\pi}_A \wedge (\pi^{B'} d\pi_{B'}) - d\pi^{A'} \wedge d\pi_{A'} \wedge (\bar{\pi}^B d\bar{\pi}_B)) \qquad (25)$$

Thus the wave functions of the $\tilde{F}$ polarization can be written in the form

$$\psi = \tau \nu^{\frac{1}{2}} \qquad (26)$$

where $\tau : M_s \longrightarrow L$ is a section of L which is covariantly constant on the leaves of F (the precise meaning of the square root $\nu^{\frac{1}{2}}$ - which is not important here - is discussed in detail by Blattner [1]).

The prequantization of (M_s, σ) allows the sections of L to be realized in a particularly simple way: to be precise, any smooth function $f : G_s \longrightarrow \mathbb{C}$ which is homogeneous of degree $-2s\hbar^{-1}$ in Z^α defines a section τ_f of L, which makes this diagram commute:

$$\begin{array}{ccc} G_s \times \mathbb{C} & \longrightarrow & L \\ \downarrow\uparrow f & & \downarrow\uparrow \tau_f \\ G_s & \longrightarrow & M_s \end{array}$$

Conversely, any section $\tau : M_s \longrightarrow L$ can be obtained in this way from a homogeneous function $f_\tau : G_s \longrightarrow \mathbb{C}$. Furthermore, τ will be covariantly constant on the leaves of F if, and only if, f_τ is covariantly constant in the directions in H (as a section of the trivial bundle $G_s \times \mathbb{C}$), that is, if, and only if, f_τ is of the form

$$f_{\tau} : Z^{\alpha} = (\omega^{A}, \pi_{A'}) \longmapsto k_{\tau}(\bar{\pi}_{A}, \pi_{A'}) \exp(\tfrac{1}{2}\hbar^{-1}(\omega^{A}\bar{\pi}_{A} - \bar{\omega}^{A'}\pi_{A'})$$
$$= k\ \ (\bar{\pi}_{A}, \pi_{A'}) \exp(i\hbar^{-1} p_{a}X^{a}) \qquad (27)$$

where k_{τ} is homogeneous of degree $-2s\hbar^{-1}$ in $\pi_{A'}$:

$$\hbar(\bar{\pi}_{A}\frac{\partial k_{\tau}}{\partial \bar{\pi}_{A}} - \pi_{A'}\frac{\partial k_{\tau}}{\partial \pi_{A'}}) = 2s\, k_{\tau} \ . \qquad (28)$$

In this realization, the inner product of the two wave functions

$$\psi = \tau\, \nu^{\frac{1}{2}} \quad \text{and} \quad \psi' = \tau'\, \nu^{\frac{1}{2}}$$

is given by

$$\langle \psi, \psi' \rangle = \int_{N_{+}} k_{\tau}\bar{k}_{\tau'} \cdot \nu \qquad (29)$$

(since it is invariant under phase transformations of $\pi_{A'}$, $k_{\tau}\bar{k}_{\tau'}$ is a well defined function on N_{+}).

Finally, the relationship of this to the conventional quantum description of massless particles can be seen by introducing the spinor field

$$\chi_{A'B'C'\ldots} = \int_{N_{+}} [\pi_{A'}\pi_{B'}\pi_{C'} \ldots f_{\tau}]\nu = \int_{N_{+}} [\pi_{A'}\pi_{B'} \ldots k_{\tau}e^{i\hbar^{-1}p_{a}X^{a}}]\,\nu \qquad (30)$$

(with $2s\hbar^{-1}$ indices). The integrand is again independent of the phase of $\pi_{A'}$ and $\chi_{A'B'C'\ldots}$ satisfies the zero-rest-mass field equation

$$\nabla^{AA'}\chi_{A'B'C'\ldots} = 0 \qquad (31)$$

Twistor Theory

So far, nothing exceptional has been achieved. It was already known that geometric quantization leads in a straightforward way to the conventional quantum description of a free massless particle [9,10]. The twistor formalism has done little except to make the calculations simpler.

The parting of the ways between conventional physics and twistor theory proper come when one tries to repeat this analysis in a way which respects the invariance of massless under the full conformal group of Minkowski space (denoted $C(1,3)$). Except for one minor subtlety, which I shall deal with presently, the construction of the classical phase space and its prequantization are unchanged. The trouble is that the polarization introabove is not invariant under the action of $C(1,3)$ on T. This action is found by introducing a new geometrical realization of a twistor:

The definition of the ω^A part of a twistor Z^α, corresponding to some fixed point of M_s, depends on the choice of an origin O in Minkowski space $\mathcal{M}$. If O is translated through Y^a then the $\pi_{A'}$ part of Z^α is unaltered, while ω^A transforms according to

$$\omega^A \longmapsto \omega^A - i\,Y^{AA'}\pi_{A'} \tag{32}$$

The spinor ω^A can, therefore, be regarded as a field on $\mathcal{M}$ rather than as a fixed (but origin dependent) quantity. From (32), this field satisfies the twistor equation,

$$\nabla_{A'}{}^{(A}\omega^{B)} = 0 \tag{34}$$

and, in fact, the general solution of this is

$$(\omega^A)_x = \omega_0^A - i\,X^{AA'}\pi_{A'}, \quad \text{where } x \in \mathcal{M} \text{ and } \pi_{A'} \text{ is fixed.} \tag{35}$$

Here, ω_0^A is the value of ω^A at $0 \in \mathcal{M}$ and X^a is the position vector of x relative to 0. Thus the map $\omega^A \mapsto (\omega_0^A, \pi_{A'})$ defines an isomorphism between T and the solution space of the twistor equation.

The point of this is that (34) is conformally invariant in the sense that if $\rho : \mathcal{M} \to \mathcal{M}$ is a conformal isometry, and if ω^A satisfies (34), then so does $\rho(\omega)^A$, but with $\nabla_{AA'}$ replaced by the connection $\tilde{\nabla}_{AA'}$ of the conformally transformed metric $\tilde{g}_{ab} = \rho(g)_{ab} = \Omega^2 g_{ab}$, which is related to $\nabla_{AA'}$ by

$$\tilde{\nabla}_{AA'}\chi^B = \nabla_{AA'}\chi^B + \epsilon_A{}^B\Upsilon_{A'C}\chi^C \quad \text{where } \Upsilon_a = \nabla_a \ln\Omega. \tag{36}$$

Thus if ρ fixes the origin 0 then its action on T is

$$\rho : Z^\alpha = (\omega_0^A, \pi_{A'}) \longmapsto (L^A{}_B\omega_0^B, \bar{M}^{B'}{}_{A'}(\pi_{B'} + i\,U_{BB'}\omega_0^B)) \tag{37}$$

where $U_a = (\Upsilon_a)_0$ and $L^A{}_B$ is the $SL(2,\mathbb{C})$ transformation of the spin space at 0 induced by ρ and $M^C{}_A L^A{}_B = \delta^C_B$. Again, this transformation defines an element of $SU(2,2)$.

The subtlety mentioned above arises because, strictly speaking, the only conformal isometries which fix 0 are the Lorentz rotations and the dilatations. The full 15-dimensional group $C(1,3)$ acts not on $\mathfrak{M}$ but on its compactification $\overline{\mathfrak{M}}$ which is obtained from $\mathfrak{M}$ by attaching the 'null cone at infinity' [5] . The only role that this plays here is that, when $s = 0$, the phase space of a conformally invariant particle includes points at infinity corresponding to twistors of the form $(\omega^A, 0)$, and has topology $S^3 \times S^2 \times \mathbb{R}$ rather than $\mathbb{R}^4 \times S^2$.

One now sees that there is an obvious conformally invariant polarization for T: this is the Kähler polarization K, spanned at each point by the antiholomorphic vectors $\{\partial/\partial\bar{Z}_\alpha\}$.

As before, K induces a polarization $\tilde{K}$ on each M_s (when $s \neq 0$, this is Kähler, but when $s = 0$, $\tilde{K} \cap \bar{\tilde{K}}$ is one dimensional). The $\tilde{K}$ wave functions are of the form $\psi = \tau \mu^{\frac{1}{2}}$ where τ is a section of L, covariantly constant in the directions in $\tilde{K}$ and μ is a 3-form, orthogonal to $\tilde{K}$ and Lie propagated by the directions in $\tilde{K}$.

Again, τ can be represented by a function $f_\tau : G_s \longrightarrow \mathbb{C}$, only now f_τ must be of the form

$$f_\tau : Z^\alpha \longmapsto k_\tau(Z^\alpha)\, e^{-\hbar^{-1} Z^\alpha \bar{Z}_\alpha} \tag{38}$$

where $k_\tau : T \longrightarrow \mathbb{C}$ is holomorphic and homogeneous of degree $-2s\hbar^{-1}$ in Z^α.

This time, there is no natural choice for μ. However, if $u : T \longrightarrow \mathbb{C}$ is holomorphic and homogeneous of degree -2 then

$$\mu = u^2\, \epsilon_{\alpha\beta\gamma\delta}\, Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \tag{39}$$

projects from G_s to a 3-form in M_s which has the desired properties. Thus the $\tilde{K}$ wave functions can be represented symbolically in the form

$$\psi = k_\tau (u^2\, \epsilon_{\alpha\beta\gamma\delta}\, Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta)^{\frac{1}{2}} \tag{40}$$

where $h = k_\tau u : T \longrightarrow \mathbb{C}$ is holomorphic and homogeneous of degree $-2s\hbar^{-1} - 2$. In short, there is a one-to-one correspondence between holomorphic homogeneous twistor functions of degree $-2s\hbar^{-1}-2$ and $\tilde{K}$ wave functions on M_s.

At this point, the Kostant-Souriau procedure breaks down: such functions are necessarily singular, and, even allowing singularities, there are no square integrable wave functions.

Penrose's solution to this problem begins with the observation that if $h : T \to \mathbb{C}$ is holomorphic and homogeneous of degree $-2s\hbar^{-1}-2$ then the space-time field

$$(\varphi_{A'B'C'\ldots}))_x = \oint \pi_{A'}\pi_{B'}\pi_{C'} \ldots h(-iX^{EE'}\pi_{E'}, \pi_{E'})\pi^{F'} d\pi_{F'} \quad (41)$$

(where X^a is the position vector of x) satisfies the zero-rest-mass field equation (31). The integral here is around any closed one dimensional contour in the $\pi_{A'}$ space which avoids the singularities of h. Because of the homogeneity of h, the integrand is a closed 1-form, so that the integral depends only on x and the cohomology class of the contour (in the complement of the singularity set of h). Provided the contour varies continuously as x varies, the result will be a smooth, and, indeed, analytic, field in $\mathcal{M}$.

When s is negative, the fields have unprimed indices and are given by homogeneous holomorphic functions on the dual space T^*, so that, in place of (41) one has

$$(\varphi_{ABC\ldots})_x = \oint \pi_A \pi_B \pi_C \ldots g(\pi_E, iX^{EE'}\pi_E)\pi^F d\pi_F; \quad g : T^* \to \mathbb{C} \quad (42)$$

The idea is that, eventually, all the momentum space integrals of conventional quantum theory - such as inner products, charge integrals and Feynman diagrams - should be replaced by integrals over compact contours in twistor space (or in the product of several twistor spaces). These are necessarily finite, so there would be no need for renormalization.

For example, when $s = \frac{1}{2}\hbar$, the inner product of two fields $\varphi_{A'}$ and $\psi_{B'}$ can be given by an integral over an 8-dimensional contour in $T \times T^*$

$$\langle\varphi, \psi\rangle = (2\pi i)^{-3} \oint (Z^\alpha W_\alpha)^{-1} h(Z^\alpha)\, g(W_\alpha) \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\pi\rho\sigma\tau} dZ^\alpha \wedge \cdots \wedge dZ^\delta \wedge dW_\pi \wedge \cdots \wedge dW_\tau \quad (43)$$

Here h generates $\varphi_{A'}$, and g generates $\bar{\psi}_B$ and the contour is fixed by the choice of contours for $\varphi_{A'}$ and $\bar{\psi}_B$. Similar expressions exist for other values of the helicity, but when $s \geq \hbar$, the contour is compact-with-boundary rather than closed.

Massive Particles

I shall end by making a few remarks about massive particles. Again, the kinematical variables can be represented by a position vector X^a and two future pointing null vectors $I^a \leftrightarrow o^A \bar{o}^{A'}$ and $J^a \leftrightarrow \iota^A \bar{\iota}^{A'}$, only now (5), (6) and (8) must be replaced by

$$p_a = 2^{-\frac{1}{2}} m(I_a + J_a) , \quad M^{ab} = -s\, \epsilon^{abcd} I_c J_d + X^a p^b - X^b p^a \tag{44}$$

$$\tilde{X}^a = X^a + r p^a , \quad \tilde{I}^a = I^a \text{ and } \tilde{J}^a = J^a, \quad \text{where } r \in \mathbb{R} \tag{45}$$

$$\sigma = s\, \epsilon_{abcd} I^a J^b dI^c \wedge dJ^d + m\, dI_a \wedge dX^a . \tag{46}$$

One way to quantize this system is to first form the 12-dimensional exact symplectic space (W, σ) in which X^a, p_a and $\chi^A = s^{\frac{1}{2}}(2m)^{-\frac{1}{2}} o^A$ are independent variables and in which

$$\sigma = d(p_a dX^a + i\, p_{AA'} (\chi^A d\bar{\chi}^{A'} - \bar{\chi}^{A'} d\chi^A)). \tag{47}$$

The phase space $(M_{m,s}, \sigma)$ of a particle with rest mass m and spin $s > 0$ is recovered from (W, σ) by taking the 10-surface $G_{m,s} \subset W$ on which $p_a p^a = m^2$ and $p_{AA'} \chi^A \bar{\chi}^{A'} = s$ and factoring out the two commuting Hamiltonian vector fields generated by $p_a p^a$ and $p_{AA'} \chi^A \bar{\chi}^{A'}$. As before, when $2s$ is an integral multiple of $\hbar$, the prequantization line bundle can be expressed as a factor space of $G_{m,s} \times \mathbb{C}$.

In this notation, the polarization $\tilde{F}$ used by Renouard [9] is induced from the polarization of (W, σ) spanned by the vector fields $\partial/\partial \bar{\chi}^{A'}$ and $\partial/\partial X^a$. The factor space $M_{m,s}/\tilde{F}$ is $N_{m,+} \times S^2$ where $N_{m,+}$ is the future m-mass shell in momentum space and the $\tilde{F}$-wave functions are of the form

$$\psi : (X^a, p_a, \chi^A) \mapsto \left[\varphi_{ABC\ldots} \chi^A \chi^B \chi^C \ldots e^{i\hbar^{-1} p_a X^a} \right] (\nu \wedge \chi^A d\chi_A)^{\frac{1}{2}} \tag{48}$$

where ν is the invariant volume element on $N_{m,+}$ and $\varphi_{ABC\ldots}$ depends only on p_a. The conventional quantum mechanical description is recovered by introducing the space-time field

$$\tilde{\varphi}_{ABC\ldots} = \int_{N_{m,+}} \left[\varphi_{ABC\ldots}\, e^{i\hbar^{-1} p_a X^a} \right] \nu \tag{49}$$

Alternatively, one can recover $(M_{m,s}, \sigma)$ from the product $T \times T = \{(Z^\alpha, Y^\alpha)\}$ of two twistor spaces with the exact symplectic form

$$\sigma = i\, (dZ^\alpha \wedge d\bar{Z}_\alpha + dY^\alpha \wedge d\bar{Y}_\alpha) \tag{50}$$

To be precise, $(M_{m,s}, \sigma)$ is the symplectic manifold obtained from the

11-surface in $T \times T$ given by

$$Z^\alpha \bar{Z}_\alpha = s, \quad Y^\alpha \bar{Y}_\alpha = -s, \quad Z^\alpha \bar{Y}_\alpha = 0 \quad \text{and} \quad M = \tfrac{1}{2} m^2 \tag{52}$$

where

$$M = (I^{\alpha\beta} \bar{Z}_\alpha \bar{Y}_\beta)(I_{\alpha\beta} Z^\alpha Y^\beta) \quad \text{and} \quad I^{\alpha\beta} = \begin{bmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{bmatrix} \tag{53}$$

by factoring out the commuting Hamiltonian vector fields generated by $Z^\alpha \bar{Z}_\alpha$, $Y^\alpha \bar{Y}_\alpha$ and M (in Dirac's terminology, $Z^\alpha \bar{Y}_\alpha = 0$ is a first class constraint, that is, σ restricts to a nondegenerate 2-form on the 14-surface $Z^\alpha \bar{Y}_\alpha = 0$). Explicitly,

$$Z^\alpha = e^{ir} \left(\tfrac{1}{2} s \lambda^{-1} \iota^A + i x^{AA'} (\lambda \bar{o}_{A'}), \lambda \bar{o}_{A'} \right) \tag{54}$$

$$Y^\alpha = e^{it} \left(\tfrac{1}{2} s \lambda^{-1} o^A + i x^{AA'} (\lambda \bar{\iota}_{A'}), \lambda \bar{\iota}_{A'} \right) \tag{55}$$

where $\lambda = m^{\frac{1}{2}} 2^{-\frac{1}{4}}$ and $r, t \in \mathbb{R}$.

This representation of $(M_{m,s}, \sigma)$ is not unique: the choice made for the right hand sides in (52) can be changed, subject to the constraint that the momentum and angular momentum defined by Z^α and Y^α should add to give the momentum and angular momentum of the massive particle. The transformations of Z^α and Y^α which preserve this constraint form a classical 'internal symmetry' group isomorphic with $SU(2) \times \tilde{E}(2)$ [4, 7] ($\tilde{E}(2)$ is the double covering of the Euclidean group $E(2)$).

This time, the natural choice for the polarization in $T \times T$ is the Kähler polarization spanned by the vector fields $\partial / \partial \bar{Z}_\alpha$ and $\partial / \partial \bar{Y}_\alpha$. This choice results in a representation of quantized massive particles by holomorphic functions on $T \times T$ of fixed homogeneities in their two arguments. Again, the space-time fields are obtained by contour integration.

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Most of the ideas presented here were formed during the dialogue which has been taking place in Oxford between the twistor theorists and those working on geometric quantization, and it would be futile to attempt to trace specific ideas to particular individuals. However, I should mention the names of Keith Hannabuss and Alan Carey (who will be publishing a paper on

this subject shortly), George Sparling, Paul Tod and, of course, Roger Penrose himself.

The correspondence between the symplectic structure of twistor space and that of the phase space of a zero spin zero mass particle was first observed by Crampin and Pirani [2].

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Footnote: After this talk, Professor Kostant suggested that, in the conformally invariant case, the Kostant-Souriau theory could be saved by constructing the quantum Hilbert space from certain cohomology groups associated with the prequantization line bundle and the antiholomorphic polarization. Unfortunately, this does not work since the wave functions would then be represented by products of $\frac{1}{2}$-forms with holomorphic forms on T satisfying $\mathcal{L}_{X_g}\beta = -2is\hbar^{-1}\beta$, where $\beta = \beta_{\alpha\beta\ldots}dZ^{\alpha}\wedge dZ^{\beta}\ldots$, and the singularities would still be present.