Moduli Space of (0,2) Conformal Field Theories

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Physics
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Abstract

In this thesis we study aspects of (0,2) superconformal field theories (SCFTs), which are suitable for compactification of the heterotic string. In the first part, we study a class of (2,2) SCFTs obtained by fibering a Landau-Ginzburg (LG) orbifold CFT over a compact Kähler base manifold. While such models are naturally obtained as phases in a gauged linear sigma model (GLSM), our construction is independent of such an embedding. We discuss the general properties of such theories and present a technique to study the massless spectrum of the associated heterotic compactification. We test the validity of our method by applying it to hybrid phases of GLSMs and comparing spectra among the phases. In the second part, we turn to the study of the role of accidental symmetries in two-dimensional (0,2) SCFTs obtained by RG flow from (0,2) LG theories. These accidental symmetries are ubiquitous, and, unlike in the case of (2,2) theories, their identification is key to correctly identifying the IR fixed point and its properties. We develop a number of tools that help to identify such accidental symmetries in the context of (0,2) LG models and provide a conjecture for a toric structure of the SCFT moduli space in a large class of models. In the final part, we study the stability of heterotic compactifications described by (0,2) GLSMs with respect to worldsheet instanton corrections to the space-time superpotential following the work of Beasley and Witten. We show that generic models elude the vanishing theorem proved there, and may not determine supersymmetric heterotic vacua. We then construct a subclass of GLSMs for which a vanishing theorem holds.

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1

Introduction

Perhaps the most striking example of the fruitful interaction between physics and geometry is general relativity, where classical geometry is key and guide in predicting and understanding phenomena like singularities and more generally the structure of spacetime. One would expect that string theory, as a quantum theory of gravity, should be equipped with its own geometric tools and intuitions which would constitute the mathematical framework for developing the theory. As it stands, the definition of this mathematical framework is a formidable challenge even at the classical level. In order to better understand this statement, we should recall that in string theory there are two different quantities, g_s and α', which can be thought as expansion parameters. The former, known as the *string coupling*, is the vacuum expectation value of the dilaton field and it can be considered the stringy generalization of $\hbar$ in quantum field theory (QFT). The loop expansion of Feynman diagrams in QFT is replaced in string theory by a sum over worldsheet topologies, where g_s is a weight for these different topologies. The classical limit, which we referred to above, is defined by taking $g_s \to 0$, and it corresponds to string theory formulated on a spherical worldsheet. The second quantity, α', is merely a scale that controls the auxiliary quantum field theory defined on a worldsheet of fixed genus.

The full formulation of string theory should describe the theory for general values of both these "parameters". In fact, even in the classical limit $g_s \to 0$, the geometry that characterizes string theory is far from the classical Riemannian geometry underlying general relativity or Yang-Mills theory. We refer to this new set of geometric tools as *stringy* geometry, and we reserve the term *quantum* geometry for the full quantum formulation of string theory.

In this thesis we will focus on compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string preserving $N = 1$ supersymmetry in four dimensions in the classical limit $g_s \to 0$. The main reason for us to focus on this subset in the space of superstring theories is that the choice of gauge bundle gives rise to phenomenologically intriguing models, with gauge groups in the four-dimensional spacetime that embed the Standard Model's. Moreover, these degrees of freedom are well-described by the worldsheet approach we wish to pursue. Other superstring theories, upon compactification to lower dimensions, can give rise to phenomenologically attractive gauge groups. However, these are obtained by D-branes wrapping circles in the non-compact dimensions, in general together with orbifold planes. These features do not have clear worldsheet corresponding degrees of freedom, thus this set of compactifications is hard to analyze via the methods of this thesis.

In the rest of this chapter we will start from the field theory limit of string theory ($g_s \to 0$, $\alpha' \to 0$) and work our way up to the subject of interest for this thesis, namely (0,2) superconformal field theories in two dimensions.

1.1 From supergravity onto the worldsheet

A natural place to start the analysis of the compactification of the heterotic string is in some large radius limit, if available. The goal of this section is to show that this subset of compactifications, while certainly interesting, merely constitutes a corner of a larger landscape. In fact, we will be mostly interested in the worldsheet approach to heterotic compactifications, which will lead us to the study of conformal field theories as a tool to explore the stringy geometry of the moduli space.

However, we find it instructive to first give a swift review of the supergravity approach to hetorotic compactifications as well as some spacetime aspects.

1.1.1 Heterotic supergravity

The bosonic massless string spectrum of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory in ten dimensions consists of the following:

- The metric G_{MN};
- The heterotic B-field B_2. Its field strength is given by H_3;
- The $\mathrm{E}_8 \times \mathrm{E}_8$ gauge-field A_M. Its field strength is given by F_2;
- The dilaton Φ.

Here the indices $M, N = 1, \ldots, 10$, parametrize the ten dimensional target space. The fermion content of the theory is given by the gravitino Ψ_M, the dilatino λ and the gaugino χ. While the main body of this thesis will consider compactifications to $10-d$ dimensions, for now we want to study the low-energy behavior of the heterotic string in ten dimensions. This theory has $N = 1$ supersymmetry, that is, it is invariant under the action of 16 supercharges [1]. This high degree of supersymmetry completely determines the low-energy action, whose bosonic part is given by[2]

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[R + 4|\partial_\mu \Phi|^2 - \frac{1}{2}|\tilde{H}_3|^2 - \frac{\alpha'}{4} \left(\mathrm{tr}|F_2|^2 - \mathrm{tr}|R_2|^2 \right) \right] , \quad (1.1)$$

[1] This is also true for type I, and indeed the low-energy actions are the same, except that the heterotic theory does not admit R-R fields.

[2] For notation and more details we refer to [19].

where R is the scalar curvature, R_2 is the Riemann tensor 2-form, and the field strengh H_3 is twisted by

$$\tilde{H}_3 = dB_2 + \frac{\alpha'}{4}(\omega_{\rm L}^{\rm CS} - \omega_{\rm YM}^{\rm CS}) \ . \tag{1.2}$$

The two $\omega^{\rm CS}$ terms are the Lorentz and Yang-Mills Chern-Simons terms respectively

$$\omega_{\rm L}^{\rm CS} = {\rm tr}\left(\omega\wedge\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right) \ , \quad \omega_{\rm YM}^{\rm CS} = {\rm tr}\left(A\wedge A + \frac{2}{3}A\wedge A\wedge A\right) \ , \tag{1.3}$$

where ω is the spin connection. The field $\tilde{H}_3$ satisfies a Bianchi identity

$$d\tilde{H}_3 = \frac{\alpha'}{4}\left({\rm tr}|R_2|^2 - {\rm tr}|F_2|^2\right) \ . \tag{1.4}$$

The supersymmetry variations of the fermions of the theory are

$$\begin{aligned}
\delta\Psi_M &= \left(\partial_M + \frac{1}{4}\omega_M^{NP}\Gamma_{NP}\right)\epsilon - \frac{1}{8}\tilde{H}_{NPM}\Gamma^{NP}\epsilon \ , \\
\delta\lambda &= -\frac{1}{2\sqrt{2}}\left(\Gamma^M\partial_M\Phi - \frac{1}{12}\tilde{H}_{MNP}\Gamma^{MNP}\right)\epsilon \ , \\
\delta\chi &= -\frac{1}{8}F_{MN}\Gamma^{MN}\epsilon \ .
\end{aligned} \tag{1.5}$$

A solution is supersymmetric if the variations above vanish. The appearance of $\tilde{H}_3$ in the first equation in (1.5) gives it the interpretation of torsion.

Now we are ready to start considering possible supersymmetric solutions obeying (1.4). We restrict our attention to the case of interest, that is when the target space factors as $\mathbb{R}^{1,3}\times X$, where X is a six-dimensional internal manifold. In this case, solving (1.5) implies that X is complex with Hermitian form $J_{\mu\overline{\nu}}$ such that

$$\begin{aligned}
\tilde{H}_3 &= \frac{i}{2}(\bar{\partial} - \partial)J \ , \\
J^{\mu\overline{\nu}}F_{\mu\overline{\nu}} &= F_{\mu\nu} = F_{\overline{\mu\nu}} = 0 \ .
\end{aligned} \tag{1.6}$$

The indices μ,ν and their barred counterparts parametrize the complex coordinates of the internal manifold X. The first equation is the statement that if $\tilde{H}_3 \neq 0$, X is non-Kähler and the Bianchi identity now reads

$$i\partial\bar{\partial}J = \frac{\alpha'}{4}\left(\mathrm{tr}|R_2|^2 - \mathrm{tr}|F_2|^2\right) \ . \tag{1.7}$$

Throughout the rest of this thesis, we will only consider the case of torsion-free models, so the space X will be Kähler. We therefore restrict ourselves to the case $\tilde{H}_3 = 0$ from here on. With this assumption, the second equation of (1.5) implies that the dilaton is constant, while the vanishing of the variation of Ψ_μ forces ϵ to be covariantly constant on X. This amounts to the fact that X has SU(3) holonomy, which is equivalent to the space being Kähler and Ricci flat, which in turn means that X has vanishing first Chern class. That is, X is a Calabi-Yau manifold. Now we need to analyze the second line of (1.5), which constrains the bundle $\mathcal{F}$ over X. The equations $F_{\mu\nu} = F_{\overline{\mu\nu}} = 0$ imply that the bundle is homolorphic, while the first equation is referred to as Hermitian-Yang-Mills. When the manifold is Kähler, there is a powerful result known as the Donaldson-Uhlenbeck-Yau theorem. This theorem states that given a holomorphic vector bundle over a Kähler space the second line of (1.5) admits solutions when the bundle is poly-stable. Stability is a topological condition which in rough terms can be stated as follows: a vector bundle is said to be stable if it is more ample than any proper sub-bundle[3]. This definition of stability is not rigorous, but we will not need a more precise definition of stability for the purpose of this thesis. In Chapter 4, when studying the linear sigma model approach to (0,2) models, we will assume the existence of such a stable bundle over the Calabi-Yau threefold. Moreover, a bundle is said to be poly-stable if it splits as a direct sum of stable bundles. Finally, the Bianchi identity can be recast as the topological condition $\mathrm{ch}_2(\mathcal{F}) = \mathrm{ch}_2(T_X)$, where T_X is the tangent bundle of X.

[3] In the case of bundles over a Riemann surface, we say that W is a stable bundle if and only if $\deg(W)/\mathrm{rank}(W) > \deg(V)/\mathrm{rank}(V)$ for each proper sub-bundle V of W.

1.1.2 The non-linear sigma model

A different approach, which exploits the underlying worldsheet theory, is the non-linear sigma model. This is a two dimensional theory of maps $\phi^i : \Sigma \to X$, where X is a Riemannian manifold equipped with a metric g and a closed two-form B, and Σ is a Riemann surface. In addition, we have the superpartners of ϕ^i, which we denote as ψ^i and which transform as Grassman-valued sections of the pullback of the tangent bundle of X, $K^{\frac{1}{2}} \otimes \phi^*(T_X)$, and a set of left-moving fermions γ^I, which instead transform as sections of $K^{-\frac{1}{2}} \otimes \phi^*(\mathcal{E})$, where $\mathcal{E}$ is a holomorphic vector bundle over X. Here, K is the canonical line bundle of Σ. The action for this theory is given by

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \left[\frac{1}{2} g_{i\bar{\jmath}} (\partial\phi^i \bar{\partial}\overline{\phi}^{\bar{\jmath}} + \partial\overline{\phi}^{\bar{\jmath}} \bar{\partial}\phi^i) + B_{i\bar{\jmath}} (\partial\phi^i \bar{\partial}\overline{\phi}^{\bar{\jmath}} - \partial\overline{\phi}^{\bar{\jmath}} \bar{\partial}\phi^i) + g_{i\bar{\jmath}} \overline{\psi}^{\bar{\jmath}} D_{\bar{z}} \phi^i \right.$$
$$\left. + \mathcal{H}_{I\bar{J}} \overline{\gamma}^{\bar{J}} D_z \gamma^I + R_{\bar{J} I \bar{\jmath} i} \overline{\gamma}^{\bar{J}} \gamma^I \psi^i \overline{\psi}^{\bar{\jmath}} \right] , \quad (1.8)$$

where the covariant derivative $D_{\bar{z}}$ is constructed by pulling back the Christoffel connection on T_X, and the covariant derivative D_z is constructed by pulling back the Hermitian connection constructed from the metric $\mathcal{H}$ on X. See also [81] for notation and a generalization of (1.8) to comprise H-fluxes.

Let us have a look at the symmetries of this theory. If X is complex Kähler, then the theory posseses (0,2) supersymmetry, which is enhanced to (2,2) supersymmetry when $\mathcal{E} = T_X$. One way to see this is that in this case the action can be explicitly written in (0,2) (or (2,2)) superspace, and (1.8) becomes the component action obtained by evaluating the superderivatives. Conformal invariance, on the other hand, is a much tricker business, as in general (1.8) does not possess this symmetry. A way to check this is to consider the metric g as a coupling constant and compute the β function with respect to it. As a result, at lowest order in α' the β function is proportional to the Ricci tensor of X. Hence, a necessary condition for conformality is X to be a Calabi-Yau manifold. We have thus recovered, from the worldsheet perspective, one of the conditions that characterized the heterotic supergravity solutions. In fact, one can push this further. If one computes the β function with respect to the two-form B and the metric g at the appropriate order in α' one recovers (1.5) (we are ignoring the dilaton dependent part of the action here, but this extends to it as well). We then see the relation between the non-linear sigma model and the supergravity approaches: constraints from conformal invariance translate into spacetime equations of motion.

1.1.3 Stringy geometry

The discussion up to this point is valid in the field theory regime, where the curvature of the background is small compared to the string scale. When the curvature becomes large, this approximation is not valid anymore, and we need to substitute our geometric interpretation by the properties of abstract conformal field theory. In other words, instead of considering the target space to be $\mathbb{R}^{1,3} \times X$, we will consider more in general $\mathbb{R}^{1,3} \times \mathrm{CFT}$, where the "internal" conformal field theory must satisfy some constraints, for example a fixed central charge. These conformal field theories naturally come with a moduli space, that is there exists a collection of operators that we can use to deform the theory to reach a nearby conformal field theory. We refer to these operators as exactly marginal. The study of different aspects of this moduli space is the subject of this thesis.

We can start characterizing the moduli space by restricting ourselves to theories that posses (2,2) superconformal symmetry. These theories come with a moduli space that it is locally a product of two factors, which have a geometrical interpretation when the conformal field theory is realized as a non-linear sigma model with a Calabi-Yau target space. In fact, the two different types of marginal operators correspond to the two different kinds of deformations that preserve the Calabi-Yau condition, namely Kähler and complex structure deformations.

We can picture the Kähler moduli space for a given Calabi-Yau X as a cone, given by elements of $J \in H^2(X, \mathbb{R})$ satisfying certain positivity properties. Moreover, by adding the contribution of the two-form B, the relevant object becomes the so-called complexified Kähler form $J+iB$. Its corresponding complexified Kähler moduli space is also described by a cone, whose walls, in real codimension one, are parametrized by metrics that fail to satisfy the positivity properties and therefore should lead to a singular theory of some sort.

In order to briefly describe the complex structure moduli space, let us consider the simple case of a Calabi-Yau given as a hypersurface in a (weighted) projective space. The equation that defines the hypersurface depends on some coefficients, and different choices for these correspond to different choices for the complex structure, modulo the field redefinitions that act on the coordinates. The locus where the Calabi-Yau is singular is referred to as the discriminant locus, and it is a complex codimension one sublocus.

This picture is unsatisfactory for different reasons. For example, what happens when we shrink the area of a given curve, or in other words when we move towards the wall of the complexified Kähler cone? The problem in probing this regime is that perturbation theory is not reliable anymore, as one of the scales in the theory, namely the size of the shrinking curve, is now comparable to the string scale.

Mirror symmetry [53] provides an elegant answer to this, as under mirror symmetry the complex structure moduli space and the complexified Kähler moduli space get exchanged in the mirror Calabi-Yau $\widetilde{X}$. Thus, the region near to the wall in X is mapped to a particular complex structure in $\widetilde{X}$, which comes with the advantage that now we are free to choose any Kähler form. In particular, we can pick a Kähler form deep in the Kähler cone, where the perturbative analysis (in $\widetilde{X}$) is valid. The resolution of the puzzle is provided by the previous paragraph: the singular theories in the complex structure of $\widetilde{X}$ are a complex codimension one sublocus of the complex structure moduli space of $\widetilde{X}$. Translating back to X, this means that the locus of singular theories in the complex Kähler moduli space is complex codimension one as well. In particular, the theories near and on the wall of the Kähler cone are generally well-behaved, and we can ask what happens if we cross the wall. What we discover is that, by a mathematical procedure called flop, we enter into the Kähler cone for a topologically different, but birationally equivalent, Calabi-Yau [10]. By choosing a path that does not intersect the singular locus, this process is smooth in the sense that the conformal field theory makes sense for *any* point along the path. Thus we uncovered two properties of stringy geometry: topology chance is a smooth process and perfectly well defined physics can arise from singular geometries (as in orbifolds).

If one wants to study compactifications of type II string theory, one might be quite satisfied with this picture, at least from a theoretical point of view in the context of perturbative string theory. However, if we want to study compactifications of the heterotic strings, this does not suffice even at the perturbative level. In fact, while a theory with (2,2) superconformal symmetry can be completed to a consistent heterotic vacuum, the moduli space $\mathcal{M}_{(2,2)}$ of (2,2) theories is only a sub-locus of the moduli space of a more generic heterotic compactification. In fact, as we will show in the next section, $N = 1$ supersymmetry in spacetime requires at least (0,2) superconformal symmetry on the worldsheet, and we will denote this general moduli space as $\mathcal{M}_{(0,2)}$.

The natural first step in this exploration is towards how $\mathcal{M}_{(2,2)}$ sits inside $\mathcal{M}_{(0,2)}$, and which properties extend off the (2,2) locus. One can then imagine starting from a theory exhibiting (2,2) symmetry and considering small deformations that preserve only (0,2) symmetry. Again referring to the case of a Calabi-Yau manifold described by a hypersurface in some toric variety, these moduli correspond to deforming the bundle for the left-moving fermions away from the tangent bundle. The study of various aspects of these deformations has been carried out by various groups in the past decade. There are two main lessons we can extrapolate from this body of work. The first one is that, despite some technical challenges, many properties that hold on the (2,2) locus keep holding in the deformed theory [11, 76, 17]. For this reason we are especially interested in exploring theories which do not exhibit a (2,2) locus, as it is for these theories that we expect many more exotic things to happen. The second lesson is that even in this confined area there are many issues that are not yet resolved. For example, at the moment there is not a complete generalization of the mirror map to the bundle moduli, even though some steps have been made towards it [80, 78].

The author hopes to have provided convincing evidence that the study of these matters, despite being a classic topic in string theory, is both important and in need of more profound understanding. Except in some special cases, it is very hard to give a global description of this moduli space. For this reason we will follow a different approach: we will study limiting points/corners in the moduli space, and develop techniques to compute physically interesting quantities. By comparing these features between different corners it is then possible to learn more about the structure and the global properties of the moduli space.

1.2 Superconformal algebras

In this section we are going to provide some basics about supersymmetric CFTs in two dimensions. The conformal group in two dimensions is somehow special, as the *local* conformal group is infinite dimensional. The word local here means that not all of these transformations are well-defined on the Riemann sphere $\mathbb{P}^1$. The transformations that are well defined comprise the *global* conformal group, which is isomorphic to $\mathrm{SO}(3,1) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. These transformations are translations, dilations and special conformal transformations, and they are realized on the states of the CFT by the operators L_{-1}, L_0 and L_1 respectively. These satisfy the algebra

$$[L_1, L_{-1}] = 2L_0 \ , \qquad\qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1} \ . \tag{1.9}$$

An central notion is the one of a primary operator. An operator $\Phi(z, \overline{z})$ is said to be primary if it transforms as

$$\Phi(z, \overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^{h_\Phi} \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}_\Phi} \Phi(f(z), \overline{f}(\overline{z})) \tag{1.10}$$

under the transformation

$$z \to f(z) \ , \qquad\qquad \overline{z} \to \overline{f}(\overline{z}) \ . \tag{1.11}$$

If (1.10) holds only when (1.11) are restricted to global conformal transformations, we say that Φ is quasi-primary.

The most important object in any CFT is the energy-momentum tensor, which has the following OPE with itself

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ . \tag{1.12}$$

Each of the terms on the RHS of the above OPE has a specific meaning. The first term indicates the fact that $T(w)$ is not a primary operator unless the central charge vanishes, $c = 0$. However, it is quasi-primary, *i.e.*, $\mathrm{SL}(2, \mathbb{C})$ primary, for any value of c. The second term means that $T(z)$ is an operator of weight (2,0). It is then possible to expand the energy-momentum tensor in modes as

$$T(z) = \sum_n L_n z^{-n-2} \ , \tag{1.13}$$

where the summand -2 in the exponent is appropriate for operators of weight $h = 2$. The modes $L_{0,\pm 1}$ are precisely the generators of the global conformal transformations in (1.9). An equivalent way of phrasing the information contained in the OPE (1.12) is in terms of commutators of the modes

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \ . \tag{1.14}$$

Since OPEs and (anti-)commutators provide the same content about the algebra, henceforth we will stick to the former. Now we want to extend the above non-supersymmetric algebra to posses $N = 1$ supersymmetry. This is possible if there is an operator $G(z)$ of weight (3/2,0), the worldsheet superpartner of $T(z)$, with the following OPEs

$$T(z)G(w) \sim \frac{3/2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \ ,$$

$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} \ , \tag{1.15}$$

and (1.12) still holds. This algebra is further enhanced to $N = 2$ SUSY when it is possible to write [4]

$$G(z) = \frac{1}{\sqrt{2}}G^+(z) + \frac{1}{\sqrt{2}}G^-(z) \ , \tag{1.16}$$

with OPE

$$G^+(z)G^-(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} \ ,$$

$$G^\pm(z)G^\pm(w) \sim 0 \ ,$$

$$T(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{z-w} \ . \tag{1.17}$$

We see the appearance of a operator $J(z)$ which is a current of weight (1,0), therefore we need to complete the algebra

$$T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \ ,$$

$$J(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{z-w} \ ,$$

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2} \ . \tag{1.18}$$

[4] We are ignoring a possible phase between the two terms.

1.2.1 The Sugawara decomposition

We have seen the appearance of a $\mathfrak{u}(1)$ current in the $N = 2$ superconformal algebra above. This has some very peculiar consequences that we will use abundantly in the work presented in this thesis. In fact, even when the supersymmetry structure of the algebra does not provide us with a natural candidate for such a $\mathfrak{u}(1)$ symmetry, as happens on the left-moving side in a (0,2) SCFT, we will restrict our attention to models for which the existence of such a current is guaranteed. In this section, we are going to sketch some of its properties.

The $J(z)$ current of weight (1,0) that appeared above is the simplest example of a Kac-Moody (KM) algebra, corresponding to a $\mathfrak{u}(1)$ algebra. In general, the OPE is given by

$$J(z)J(w) = \frac{r}{z-w} \ , \tag{1.19}$$

where $r > 0$ is known as the level of the algebra, and we assume $r \in \mathbb{Z}$. In the SCFTs we will consider in this work, r will be related to the rank of the gauge bundle associated to the corresponding heterotic compactification. A field is KM primary if and only if

$$J(z)\Phi(w) \sim q\frac{\Phi(w)}{z-w} \ . \tag{1.20}$$

We will now describe how it is possible to "factorize" this KM dependence. We will use this fact in the next section to prove a crucial fact for a SCFT to describe a $N = 1$ SUSY heterotic compactification in $d = 4$ dimensions. Let us represent the KM current as $J = i\sqrt{r}\partial H$, where H is a free chiral boson. Now, a KM primary field Φ with charge q under the U(1) and weights $(h, \overline{h})$ can be decomposed as

$$\Phi(z) = \exp(iq/\sqrt{r}H)(z)\widehat{\Phi}(z) \ , \tag{1.21}$$

where $\widehat{\Phi}(z)$ is KM neutral and has weights $(h - q^2/2r, \overline{h})$. This is the Sugawara decomposition.

1.3 Spacetime and worldsheet supersymmetry

The bosonic string theory has a number of features that historically made the theory not suitable for realizing a conceivable model of particle physics [5]. The two main issues are the following:

1. The closed string spectrum has a tachyon. This means that the vacuum of the theory is unstable, or in other words that we are considering the theory around a local maximum of the potential. Also the open string spectrum has its own tachyons, but these are somehow more benign. In fact, they have been interpreted as the decay of D-branes into closed string-radiation (see for example [62]).

2. The spacetime spectrum does not contain fermions. This is a substantial problem if we want string theory to generate the particle content of the Standard Model.

This leads us into the study of superstring theories. There are two approaches to this, namely supersymmetry on the worldsheet (RNS formalism) or supersymmetry in spacetime (GS formalism). It is easy to show that they are equivalent in ten dimensional Minkowski spacetime, but since the main point of this thesis is to further the understanding of string compactifications from the point of view of the worldsheet, we are naturally going to implement the former approach.

This thesis aims at studying properties of (0,2) SCFTs relevant for heterotic compactifications. It is perhaps useful to review the classic result [43] that relates (0,2) SCFTs on the worldsheet and $N = 1$ SUSY in $d = 4$ spacetime. For ease of exposition we will switch our convention and consider the left-moving part of the algebra to be supersymmetric. We now assume (1,0) SUSY on the worldsheet,

[5] This statement is not entirely fair, because half of the worldsheet theory in the heterotic string is indeed purely bosonic.

as it is necessary for a consistent string background [57], and show that $N = 1$ spacetime SUSY corresponds to (2,0) SUSY on the worldsheet. We start by writing the spacetime supercurrents

$$V^{\alpha}_{-\frac{1}{2}}(z) = e^{-\phi/2}\mathcal{S}_{\alpha}\Sigma(z) \ , \qquad \overline{V}^{\overline{\alpha}}_{-\frac{1}{2}}(z) = e^{-\phi/2}\overline{\mathcal{S}}_{\overline{\alpha}}\overline{\Sigma}(z) \ , \tag{1.22}$$

where $\alpha = (\pm\frac{1}{2}, \pm\frac{1}{2})$ and $\overline{\alpha} = (\pm\frac{1}{2}, \mp\frac{1}{2})$, and the spacetime supercharges are

$$Q_{\alpha} = \oint dz V^{\alpha}_{-\frac{1}{2}}(z) \ , \qquad \overline{Q}_{\overline{\alpha}} = \oint dz \overline{V}^{\overline{\alpha}}_{-\frac{1}{2}}(z) \ . \tag{1.23}$$

Let us explain what these quantities are. Q_{α} and $\overline{Q}_{\overline{\alpha}}$ are Weyl spinors representing the four supercharges of $N = 1$ SUSY in $d = 4$, which satisfy the algebra

$$\{Q_{\alpha}, \overline{Q}_{\overline{\beta}}\} = \sigma^{\mu}_{\alpha\overline{\beta}}P_{\mu} \ , \qquad \{Q_{\alpha}, Q_{\beta}\} = \{\overline{Q}_{\overline{\alpha}}, \overline{Q}_{\overline{\beta}}\} = 0 \ , \tag{1.24}$$

where the index μ parametrizes the Minkowski spacetime. The currents (1.22) are built out of three different pieces: $\exp(-\frac{1}{2}\phi)$ is a spin field for the (β, γ) Faddeev-Popov ghost SCFT that one introduces when gauge-fixing the Polyakov path-integral for the superstring; the terms $\mathcal{S}_{\alpha}$ and $\overline{\mathcal{S}}_{\overline{\alpha}}$ are Weyl spin fields for the fermions corresponding to the flat four dimensional Minkowski directions; finally, Σ and $\overline{\Sigma}$ are fields in the internal SCFT. Although our main focus will be on these internal fields, we need to determine first the OPEs for the spin fields. This is easily done by considering first the bosonization of the spin fields (this is already explicit for the ghosts), *i.e.*, by writing the spin fields as the exponential of free bosons. Then, the dimensions of the fields and their OPEs just follow from the ones for the bosonic fields (see for example (2.2.14) and (2.2.17) of [88] with $\alpha' = 1$). We can then determine the dimension h of the operators Σ and $\overline{\Sigma}$, and we present the result in the following table

	$e^{-\phi/2}$	$\mathcal{S}_{\alpha} = e^{i\alpha\cdot H}$	$\overline{\mathcal{S}}_{\overline{\alpha}} = e^{i\overline{\alpha}\cdot H}$	$V^{\alpha}_{-\frac{1}{2}}$	$\overline{V}^{\overline{\alpha}}_{-\frac{1}{2}}$	Σ	$\overline{\Sigma}$
h	3/8	1/4	1/4	1	1	3/8	3/8

(1.25)

The OPEs are

$$\exp(-\frac{1}{2}\phi)(z)\exp(-\frac{1}{2}\phi)(w) \sim \frac{\exp(-\phi)}{(z-w)^{1/4}} ,$$

$$\mathcal{S}_\alpha \overline{\mathcal{S}}_{\overline{\alpha}} \sim \sigma^\mu_{\alpha\overline{\beta}}\psi_\mu(w) ,$$

$$\mathcal{S}_\alpha \mathcal{S}_\beta \sim \frac{\eta_{\alpha\beta} I}{(z-w)^{1/2}} , \tag{1.26}$$

where I is the identity operator and similarly for $\overline{\mathcal{S}}_{\overline{\alpha}}$. Now, (1.24) determines the Σ and $\overline{\Sigma}$ OPEs

$$\Sigma(z)\Sigma(w) \sim (z-w)^{3/4}\mathcal{O}(w) + \cdots ,$$

$$\overline{\Sigma}(z)\overline{\Sigma}(w) \sim (z-w)^{3/4}\overline{\mathcal{O}}(w) + \cdots ,$$

$$\Sigma(z)\overline{\Sigma}(w) \sim \frac{I}{(z-w)^{-3/4}} , \tag{1.27}$$

where $\mathcal{O}$ and $\overline{\mathcal{O}}$ are some dimension 3/2 operators. For example, the first equation of (1.24) implies that

$$\left(e^{-\phi/2}\mathcal{S}_\alpha\Sigma\right)(z)\left(e^{-\phi/2}\overline{\mathcal{S}}_{\overline{\alpha}}\overline{\Sigma}\right)(w) \sim \frac{\exp(-\phi)}{(z-w)^{1/4}}\sigma^\mu_{\alpha\overline{\beta}}\psi_\mu(w)\Sigma(z)\Sigma(w) \tag{1.28}$$

has a simple pole proportional to $\sigma^\mu_{\alpha\overline{\beta}}$. The other two equations follow similarly by imposing that the second set of equalities in (1.24) are satisfied.

At this point, we implement our assumption that we have already a $N = 1$ SCFT on the worldsheet, that is, we have operators $T(z)$ and $G(z)$ satisfying (1.12) and (1.15) for $c = 9$. A first result that will be useful later on comes from the supersymmetry invariance of the gravitino vertex operator $\exp(\phi)G(z)$. In other words

$$\left(e^{-\phi/2}\mathcal{S}_\alpha\Sigma\right)(z)\left(e^{\phi}G\right)(w) \sim (z-w)^{1/2}\exp(1/2\phi)\mathcal{S}_\alpha(w)\Sigma(z)G(w) \tag{1.29}$$

must have no single pole. This means that the most singular term in the OPEs $\Sigma(z)G(w)$ and $\overline{\Sigma}(z)G(w)$ must be proportional to $(z-w)^{-1/2}$.

In order to study the next order in the $\Sigma\overline{\Sigma}$ OPE we consider the four-point function

$$f(z_1, z_2, z_3, z_4) = \langle \Sigma(z_1)\overline{\Sigma}(z_2)\Sigma(z_3)\overline{\Sigma}(z_4)\rangle \ . \tag{1.30}$$

The OPEs (1.27) and SL$(2,\mathbb{C})$ invariance constrain the above function to be

$$f(z_1, z_2, z_3, z_4) = \left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}}\right)^{3/4} \ . \tag{1.31}$$

We can expand this expression as $z_{12} \to 0$ and we obtain

$$f(z_1, z_2, z_3, z_4) = z_{12}^{-3/4} z_{34}^{-3/4} \left[1 + \frac{3}{4}\frac{z_{12}z_{43}}{z_{23}z_{24}} + \cdots\right] \ . \tag{1.32}$$

The second order in the expansion signals the presence of an operator of dimension $-1/4 + 2(3/8) = 1$ in the $\Sigma\overline{\Sigma}$ OPE

$$\Sigma(z)\overline{\Sigma}(w) \sim \frac{I}{(z-w)^{-3/4}} + (z-w)^{1/4}\frac{1}{2}J(w) \ , \tag{1.33}$$

thus J is a $\mathfrak{u}(1)$ Kac-Moody current. In particular, we can see that Σ and $\overline{\Sigma}$ have charge $\pm 3/2$ under this symmetry

$$J(z)\Sigma(w) \sim \frac{3/2\Sigma(w)}{z-w} \ ,$$

$$J(z)\overline{\Sigma}(w) \sim \frac{-3/2\overline{\Sigma}(w)}{z-w} \ . \tag{1.34}$$

This is particularly nice, since we are starting to gain the structure of a more general algebra. It is possible to implement the Sugawara decomposition described in the previous section to all the operators we considered so far in the theory. In practice, let us write $J(z) = i\sqrt{3}\partial H(z)$, where $H(z)$ is a free scalar. From (1.34) we see that $\Sigma = \exp(i\sqrt{3}/2H)$ and similarly $\overline{\Sigma} = \exp(-i\sqrt{3}/2H)$. For these there is no need to multiply the exponential by a J-neutral operator because the dimensions already work out. Instead, G does not have a definite charge under J, and we decompose it as follows

$$G = \sum_q \exp(iq/\sqrt{3}H) G_0^q \ . \tag{1.35}$$

We can use these representations to determine for which values of q we are able to reproduce the OPE between Σ and $\overline{\Sigma}$ with G. For example, in the case of $\Sigma(z)G(w)$ we find

$$\left(\exp(i\sqrt{3}/2H)\Sigma_0\right)(z)\left(\sum_q \exp(iq/\sqrt{3}H)G_0^q\right)(w) \sim \sum_q (z-w)^{q/2}\cdots \ , \tag{1.36}$$

from which we conclude that $q=-1$. The same analysis with $\overline{\Sigma}$ gives $q=1$, therefore we have found a decomposition $G=\frac{1}{\sqrt{2}}(G^+ + G^-)$ analogous to (1.16), which satisfies

$$J(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{z-w} \ . \tag{1.37}$$

The above OPE can be rewritten as

$$G(w)J(z) \sim -\frac{1}{\sqrt{2}}\frac{G^+(w)-G^-(w)}{z-w} \ , \tag{1.38}$$

that is $(J, -\sqrt{2}G^+ + \sqrt{2}G^-)$ assemble in a primary $N=1$ superfield. It is then convenient to define $\overline{G} \equiv \frac{1}{\sqrt{2}}(G^+(w) - G^-(w))$ which means that

$$G(z)\overline{G}(w) \sim -\frac{\frac{1}{2}J(w)}{(z-w)^2} - \frac{\frac{1}{4}\partial J(w)}{z-w} \ . \tag{1.39}$$

For the last part of the argument it is convenient to expand the operators in modes

$$G(z) = \frac{1}{2}\sum_r G_r z^{-r-3/2} \ , \quad \overline{G}(z) = \frac{1}{2}\sum_r \overline{G}_r z^{-r-3/2} \ , \quad J(z) = \sum_n J_n z^{-n-1} \ . \tag{1.40}$$

These satisfy the relations

$$[J_n, J_m] = \frac{c}{3} m\delta_{m,-n}, \quad [J_n, G_r] = \overline{G}_{n+r}, \quad [J_n, \overline{G}_r] = G_{n+r}, \quad \{G_r, \overline{G}_s\} = (s-r)J_{r+s}, \tag{1.41}$$

which follow from (1.18), (1.38) and (1.39). Now we compute

$$\begin{aligned}\{\overline{G}_r, \overline{G}_s\} &= \{[J_0, G_r], \overline{G}_s\} = J_0 G_r \overline{G}_s - G_r J_0 \overline{G}_s + \overline{G}_s J_0 G_r - \overline{G}_s G_r J_0 \\ &= -\{G_r, [J_0, \overline{G}_s]\} + [J_0, \{G_r, \overline{G}_s\}] = -\{G_r, G_s\} - (r-s)[J_0, J_{r+s}] \\ &- \{G_r, G_s\} \ , \end{aligned} \tag{1.42}$$

which, translated back into the OPE language, yields

$$\overline{G}(z)\overline{G}(w) \sim G(z)G(w) \ . \tag{1.43}$$

This means that the GG and $\overline{GG}$ OPEs have the same singular terms. This fact combined with (1.39) yield the remaining relations in (1.17). We have thus recovered the structure of a $N = 2$ superconformal algebra, proving the claim that $N = 1$ SUSY in spacetime implies (2,0) SUSY on the worldsheet.

1.4 Organization of the thesis

The remainder of this thesis is divided into three parts.

The work described in Chapter 2 is published in [27] and it was carried out in collaboration with Ilarion Melnikov and Ronen Plesser. In this chapter we undertake the study of hybrid theories with (2,2) supersymmetry. Roughly, a hybrid model is a Landau-Ginzburg orbifold fibered non-trivially over a compact Kähler base. Although the existence of such theories was known for more then two decades, their properties have remained largely unexplored until recent years. In this work, we present an intrinsic definition of a hybrid theory, that is independent of a GLSM embedding. In order to do so, we derive several geometric constraints that characterize the flow of the theory in the IR to a non-trivial fixed point. We also present a method to compute the massless spectrum of the theory, which corresponds to first-order deformations of the theory.

The work described in Chapter 3 is published in [26] and it was carried out in collaboration with Ilarion Melnikov and Ronen Plesser. In this chapter we study

RG flows of (0,2) Landau-Ginzburg models. The superpotential of the UV theory does not get renormalized under the RG flow, while the corrections to the kinetic terms are believed to be irrelevant. This fact makes the problem very tractable, as the IR behavior of the theory is encoded only in the superpotential. The UV theory has in general a set of field redefinitions consistent with the symmetries of the action, and the endpoints of RG flows corresponding to theories in the same orbit of field redefinitions must coincide. If the UV superpotential admits a point of enhanced symmetry, the R-symmetry at the conformal point might differ from the naïve one of the UV theory for general values of the superpotential. This rather simple observation has two striking consequences: first, the moduli space of the theory is stratified according to basins of attraction of orbits of enhanced symmetry; second, the conformal manifold describing a theory with the naïve UV properties (central charges, etc.) might be empty. In this work, we study these "accidents" in the context of (0,2) Landau-Ginzburg orbifolds, and, for a large subclass of theories, we propose a geometric conjecture for the conformal manifold.

The work described in Chapter 4 is published in [25] and it was carried out in collaboration with Ronen Plesser. In this chapter we undertake a study of α' non-perturbative corrections in the context of gauged-linear sigma models (GLSMs). It is well known that, while most calculations are done in some appropriate large radius limit (see for example the spectrum computation of Chapter 2), instanton effects are important, as they can lift classically flat directions and they might destabilize the vacuum and ruin conformal invariance. Indeed, at first it was thought that this was the destiny of a general (0,2) model [35]. It was then shown that (0,2) models are non-perturbatively stable when the gauge bundle splits non-trivially over every rational curve in the Calabi-Yau [40]. Furthermore, there can be cases where the individual instantons do contribute but these contributions sum up to zero [11]. It was argued in the context of (0,2) GLSMs, that this is indeed the case [17]. However, the argument, while elegant and powerful, was not fully exploited and the only concrete evidence was based on a very simple example. In this work, we show that the argument does not apply to all linear models, and we provide a counterexample. We then conclude by proving a theorem that defines a class of linear models that are truly conformally invariant.

2

(2,2) hybrid conformal field theories

2.1 Introduction

Just what is a hybrid anyway? In constructing two-dimensional superconformal field theories (SCFTs) relevant for superstring vacua we are used to two sorts of massless fluctuating fields: those corresponding to a non-linear sigma model (NLSM), and those corresponding to a Landau-Ginzburg (LG) theory. The former define a classically conformally invariant system. Under favorable conditions, e.g. a Calabi-Yau target space and world-sheet supersymmetry, the background fields can be chosen to preserve superconformal invariance, and when the background is weakly coupled in a "large radius limit" (i.e. the background fields have small gradients), the theory reduces to a free-field limit. The latter have superpotential interactions that explicitly break scale invariance; however, under favorable conditions, e.g. a quasi-homogeneous superpotential, the IR limit of such a theory defines a non-trivial SCFT.

In each case, the utility of the description is two-fold: at a fundamental level, we can use the weakly coupled UV theory to define a SCFT; as a practical matter, the weakly coupled description, combined with non-renormalization theorems that follow from supersymmetry, allow us to identify and compute certain protected quantities such as chiral rings and massless spectra of the associated string vacua in terms of the UV degrees of freedom.

By now the reader has surely guessed what is meant by a hybrid [97, 10]: it is a two-dimensional theory that includes both types of massless fluctuating fields: ones that have classically conformally invariant NLSM self-interactions, as well as some that self-interact via a superpotential; of course an interesting hybrid also has interactions between the two types of degrees of freedom. A hybrid is a fibered theory, where the fiber is a LG theory with potential whose coefficients depend on the fields of the base NLSM. The potential is chosen so that its critical point set is the base target space. We then have two important questions: what are the criteria for a hybrid theory to flow to a SCFT? how do we generalize NLSM/LG techniques to compute physical quantities?

It is well-known that all of these descriptions— large radius limits of NLSMs, Landau-Ginzburg orbifolds (LGOs), and hybrid loci—arise as phases of (2,2), and more generally (0,2) gauged linear sigma models (GLSMs) [97]. The GLSM philosophy is that each phase should yield a limiting locus where at least protected quantities should be amenable to computation via the UV weakly-coupled field theory description. Such techniques are known for large radius NLSM and LGO phases but not for more general phases. In this work, we take a step in developing techniques for what we will call the "good hybrid" phases of a GLSM.[1]

Although this does not cover a generic GLSM phase, and there are perhaps good reasons [8] that we should not expect a simple description for a generic phase, it does increase the set of special points in the moduli space amenable to exact computations; this can lead to useful insights into stringy moduli space as in [11, 7,

[1] Along the way we obtain a simple and direct description of the massless spectrum for the large radius limit of a (0,2) NLSM — an application to CY NLSMs with non-standard embedding may be found in appendix A.4 .

4, 28]. In addition, our definition of a good hybrid model, although inspired by the GLSM construction, will not explicitly invoke the GLSM. Thus, we are in principle providing a new class of UV theories that can lead to SCFTs without a known GLSM embedding.

In this chapter we will focus on hybrid theories with (2,2) world-sheet supersymmetry that are suitable for supersymmetric string compactification, i.e. ones with integral $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ R-symmetry charges; as in the case of LGO string vacua, this is achieved by taking an appropriate orbifold.

While such models offer a good point of departure, it is clear that a more general (0,2) hybrid framework will be both computationally useful and conceptually illuminating. While we are not going to tackle (0,2) hybrids in this thesis, for now we note that just like (2,2) LG models, the hybrids incorporate a class of Lagrangian deformations away from the (2,2) locus. These are obtained by smoothly deforming the (2,2) superpotential to a more general (0,2) form.

In what follows, we first give a broad geometric description of (2,2) hybrids, construct a Lagrangian for a good hybrid model and study its symmetries. With that basic structure in hand, we turn to a technique, valid in the large base volume limit and generalizing the well-known (2,2) and (0,2) LGO results of [61, 41], to compute the massless heterotic spectrum of a hybrid compactification. We then apply the techniques to a number of examples and conclude with a brief discussion of applications and further directions.

2.2 A geometric perspective

The geometric setting for our theory is a (2,2) NLSM constructed with (2,2) chiral superfields. Consider a Kähler manifold $\boldsymbol{Y}_0$ equipped with a holomorphic function—the superpotential W—chosen so that its critical point set is a compact subset $B \subset \boldsymbol{Y}_0$. More precisely, dW, a holomorphic section of the cotangent bundle $T^*_{\boldsymbol{Y}_0}$, has the property that $dW^{-1}(0) = B \subset \boldsymbol{Y}_0$. We call this *the potential condition.* A LG model, with $\boldsymbol{Y}_0 \simeq \mathbb{C}^n$ and B being the origin, is a familiar example. A compact $\boldsymbol{Y}_0$ necessarily has a trivial superpotential, and the resulting theory is just a standard compact NLSM.

We say a geometry satisfying the potential condition has a *hybrid model* iff the local geometry for $B \subset \boldsymbol{Y}_0$ can be modeled by $\boldsymbol{Y}$ — the total space of a rank n holomorphic vector bundle $X \to B$ over a compact smooth Kähler base B of complex dimension d. The point of this definition is that the superpotential interactions will lead to a suppression of finite fluctuations of fields away from B, so that the low energy physics of the original NLSM will be well-approximated by the restriction to the hybrid model. Our main task will be to describe this low energy physics, and in what follows we will concentrate on the hybrid model geometry $\boldsymbol{Y}$. In many examples (e.g. the LG theories) $\boldsymbol{Y} \simeq \boldsymbol{Y}_0$, but our results apply to the more general situation where $\boldsymbol{Y}$ is simply a local model. A simple example of a hybrid geometry, where $X = \mathcal{O}(-2)$ over $B = \mathbb{P}^1$, is presented in appendix A.1.

In order to be reasonably confident that the low energy limit of a hybrid model is a (2,2) SCFT, we will need the geometry to satisfy several additional conditions intimately related to the existence of chiral symmetries and GSO projections. It will be easiest to discuss these after we introduce the explicit Lagrangian realization of this geometry. In our examples these features will already be present in the "UV" completion of the hybrid model, offered either by $\boldsymbol{Y}_0$ or some other high energy description such as a GLSM.[2]

A final geometric comment, relevant for heterotic applications, concerns (0,2)-preserving deformations of these theories. (2,2) theories often admit a class of smooth (0,2) deformations, where the left-moving fermions couple to a vector bundle $\mathcal{E} \to \boldsymbol{Y}$, a deformation of $T_{\boldsymbol{Y}}$, and the (0,2) superpotential is encoded by a holomorphic section

[2] It would be interesting to find hybrid examples where these features emerge accidentally.

$J \in \Gamma(\mathcal{E}^*)$ with $J^{-1}(0) = B$. In the hybrid case there exist (0,2) deformations where $\mathcal{E} = T_{\boldsymbol{Y}}$ but $dJ \neq 0$; such a (0,2) superpotential cannot be integrated to a (2,2) superpotential W. Turning these on leads to a simple class of (0,2) hybrid models.

2.3 Action and symmetries

In this section we construct the (2,2) SUSY UV action for a hybrid model and analyze its symmetries. We begin with the necessary superspace formalism for a flat Euclidean world-sheet with coordinates $(z, \overline{z})$. Since we are interested in (0,2) deformations of (2,2) theories, it will be convenient for us to work with both (2,2) and (0,2) superspaces.[3] Let's start with the latter. Introducing Grassmann coordinates θ and $\overline{\theta}$, we obtain the supercharges

$$\mathcal{Q} = -\frac{\partial}{\partial \theta} + \overline{\theta}\bar{\partial}_{\overline{z}}, \qquad \overline{\mathcal{Q}} = -\frac{\partial}{\partial \overline{\theta}} + \theta\bar{\partial}_{\overline{z}}, \tag{2.1}$$

where $\bar{\partial}_{\overline{z}} \equiv \partial/\partial\overline{z}$. These form a representation of the (0,2) SUSY algebra: $\mathcal{Q}^2 = \overline{\mathcal{Q}}^2 = 0$ and $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = -2\bar{\partial}_{\overline{z}}$. The supercharges are graded by a $\mathrm{U}(1)_{\mathrm{R}}$ symmetry that assigns charge $\overline{\boldsymbol{q}} = 1$ to θ, and they anticommute with the supercovariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial \theta} + \overline{\theta}\bar{\partial}_{\overline{z}}, \qquad \overline{\mathcal{D}} = \frac{\partial}{\partial \overline{\theta}} + \theta\bar{\partial}_{\overline{z}}, \tag{2.2}$$

that satisfy $\mathcal{D}^2 = \overline{\mathcal{D}}^2 = 0$ and $\{\mathcal{D}, \overline{\mathcal{D}}\} = 2\bar{\partial}_{\overline{z}}$.

To build a (2,2) superspace we introduce additional Grassmann variables $\theta', \overline{\theta}'$ and form $\mathcal{Q}'$, $\overline{\mathcal{Q}}'$, as well as $\mathcal{D}'$ and $\overline{\mathcal{D}}'$, by replacing $(\theta, \overline{\theta}, \bar{\partial}_{\overline{z}}) \to (\theta', \overline{\theta}', \partial_z)$, where $\partial_z = \partial/\partial z$. These supercharges and derivatives are graded by $\mathrm{U}(1)_{\mathrm{L}}$ that assigns charge $\boldsymbol{q} = 1$ to θ'.

2.3.1 Multiplets

We are interested in Kähler hybrid models with target space $\boldsymbol{Y}$, and these can be constructed by using bosonic chiral (2,2) superfields and their conjugate anti-chiral

[3] Our superspace conventions are those of [81]; more details may be found in [39] or [95].

multiplets[4] denoted by $\mathcal{Y}^\alpha$ and $\overline{\mathcal{Y}}^{\overline{\alpha}}$, with $\alpha, \overline{\alpha} = 1, \ldots, \dim \boldsymbol{Y}$. These decompose into (0,2) chiral and anti-chiral multiplets as follows:

$$\begin{aligned}
\mathcal{Y}^\alpha &= Y^\alpha + \sqrt{2}\theta'\mathcal{X}^\alpha + \theta'\overline{\theta}'\partial_z Y^\alpha \ , & \overline{\mathcal{Y}}^{\overline{\alpha}} &= \overline{Y}^{\overline{\alpha}} - \sqrt{2}\overline{\theta}'\overline{\mathcal{X}}^{\overline{\alpha}} - \theta'\overline{\theta}'\partial_z \overline{Y}^{\overline{\alpha}} \ , \\
Y^\alpha &= y^\alpha + \sqrt{2}\theta\eta^\alpha + \theta\overline{\theta}\partial_{\overline{z}} y^\alpha \ , & \overline{Y}^{\overline{\alpha}} &= \overline{y}^{\overline{\alpha}} - \sqrt{2}\overline{\theta}\overline{\eta}^{\overline{\alpha}} - \theta\overline{\theta}\partial_{\overline{z}}\overline{y}^{\overline{\alpha}} \ , \\
\mathcal{X}^\alpha &= \chi^\alpha + \sqrt{2}\theta H^\alpha + \theta\overline{\theta}\partial_{\overline{z}}\chi^\alpha \ , & \overline{\mathcal{X}}^{\overline{\alpha}} &= \overline{\chi}^{\overline{\alpha}} + \sqrt{2}\overline{\theta H}^{\overline{\alpha}} - \theta\overline{\theta}\partial_{\overline{z}}\overline{\chi}^{\overline{\alpha}} \ .
\end{aligned} \tag{2.3}$$

The Y^α are bosonic (0,2) chiral multiplets, while the $\mathcal{X}^\alpha$ are chiral fermi multiplets, with lowest component a left-moving fermion χ^α; the H^α and their conjugates are auxiliary non-propagating fields.[5]

Since $\boldsymbol{Y}$ is the total space of a vector bundle, it will occasionally be useful to split the y^α into base and fiber coordinates, which we will denote by $y^\alpha = (y^I, \phi^i)$, with $I = 1, \ldots, d$ and $i = 1, \ldots, n$. The y^I are then coordinates on the base manifold B, while the ϕ^i parametrize the fiber directions.

2.3.2 The (2,2) hybrid action

The two-derivative (2,2) action is a sum of kinetic and potential terms, with

$$\begin{aligned}
S_{\text{kin}} &= \frac{1}{4\pi}\int d^2z\ \mathcal{D}\overline{\mathcal{D}}\mathcal{L}_{\text{kin}}, & \mathcal{L}_{\text{kin}} &= \frac{1}{2}\overline{\mathcal{D}}'\mathcal{D}'\boldsymbol{K}(\mathcal{Y}, \overline{\mathcal{Y}}), \\
S_{\text{pot}} &= \frac{\sqrt{2}m}{4\pi}\int d^2z\ \mathcal{D}\mathcal{W}(Y, \mathcal{X}) + \text{c.c.}, & \mathcal{W} &= \frac{1}{\sqrt{2}}\mathcal{D}'W(\mathcal{Y}) \ .
\end{aligned} \tag{2.4}$$

As is well-known, the kinetic term leads to a $\boldsymbol{Y}$ NLSM with a Kähler metric g. The superpotential W is a holomorphic function on $\boldsymbol{Y}$ satisfying the potential condition, i.e. $dW(0)^{-1} = B$; m is a parameter with dimensions of mass. If the metric g is

[4] Recall that a chiral superfield A satisfies the constraints $\overline{\mathcal{D}}A = \overline{\mathcal{D}}'A = 0$; more general (2,2) multiplets (twisted chiral and semi-chiral) are reviewed in, for instance, [74].

[5] A comment on Euclidean conventions: the charge conjugation operator $\mathcal{C}$, inherited from Minkowski signature, conjugates the complex bosons and acts as $\mathcal{C}(\chi) = \overline{\chi}$ and $\mathcal{C}(\overline{\chi}) = -\chi$ for every fermion χ.

well-behaved, then the potential condition leads a suppression of field fluctuations away from $B \subset \boldsymbol{Y}$ via the bosonic potential

$$S \supset \frac{|m|^2}{2\pi} \int d^2z \; g^{\alpha\overline{\beta}} \partial_\alpha W \partial_{\overline{\beta}} \overline{W} \;, \tag{2.5}$$

and at low energies (as compared to $|m|$) the kinetic term can be taken to be quadratic in the fiber directions, i.e. the Kähler potential is

$$\boldsymbol{K} = K(y^I, \overline{y}^{\overline{I}}) + \phi h(y^I, \overline{y}^{\overline{I}}) \overline{\phi} + \ldots, \tag{2.6}$$

where K is a Kähler potential for a metric on B, h is a Hermitian metric on $X \to B$, and $\ldots$ denotes neglected terms in the fiber coordinates. Using the base–fiber decomposition the metric $g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} \boldsymbol{K} \equiv \boldsymbol{K}_{\alpha\overline{\beta}}$ then takes the form

$$g = (K_{I\overline{J}} - \phi \mathcal{F}_{I\overline{J}} h \overline{\phi}) dy^I d\overline{y}^{\overline{J}} + D\phi h \overline{D\phi} + \ldots, \tag{2.7}$$

where $\mathcal{A} = \partial h h^{-1}$ is the Chern connection for the metric h, $\mathcal{F} = \overline{\partial} \mathcal{A}$ is its (1,1) curvature, and $D\phi = d\phi + \phi \mathcal{A}$ is the corresponding covariant derivative.

Positivity of the metric and the case $\boldsymbol{Y} \simeq \boldsymbol{Y}_0$

In many cases we need not worry about higher order corrections to g in order to define a sensible theory. As in the simple case of LG models, this would be a situation where we need not consider the distinction between $\boldsymbol{Y}$ and $\boldsymbol{Y}_0$ from above. Examining the form of g, we see that a necessary condition is that $\phi \mathcal{F}_{I\overline{J}} h \overline{\phi}$ is non-positive for all points in $\boldsymbol{Y}$.[6] We say a bundle $X \to B$ is non-positive if it admits a Hermitian metric h that satisfies this non-positivity condition.

Thus, to use (2.6) to define a UV-complete theory, we are led to a geometric question: what are the non-positive bundles over B? This is closely related to classical questions in algebraic geometry regarding positive and/or ample bundles,

[6] Suppose there is a point $p \in B$ and $\phi_0 \in \pi^{-1}(p)$ such that the Hermitian form $\phi_0 \mathcal{F}_{I\overline{J}} h \overline{\phi}_0$ has a positive eigenvalue. Then taking $\phi = t\phi_0$, for sufficiently large t the metric g will cease to be positive.

and using those classical results we can easily find sufficient conditions for non-positivity. Recall that a line bundle $L \to B$ is said to be positive if its (1,1) curvature form is positive; it is said to be negative if the dual bundle L^* is positive [54, 73]. Taking $X = \oplus_i L_i$, a sum of negative and trivial line bundles, leads to many examples of non-positive bundles.

We should stress two points: first, even this set of examples leads to many previously unexplored SCFTs. Second, more generally, we do not need to assume that $\boldsymbol{Y} \simeq \boldsymbol{Y}_0$ or that g has no higher-order terms in the fibers. The low energy limit of a UV theory with a hybrid model will be well-described by our action, and the potential condition will imply that the fiber corrections to the metric will not be important to the low energy physics. We will analyze one such example below, where X is a sum of a positive and a negative bundle.

(0,2) action

Since we are interested in heterotic applications as well as (0,2) deformations, it is useful to have the manifestly (0,2) supersymmetric action obtained by integrating over $\theta', \overline{\theta}'$ in (2.4). Absorbing the superpotential mass scale m into W the result is

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{2}(\boldsymbol{K}_\alpha \partial_z Y^\alpha - \boldsymbol{K}_{\overline{\alpha}} \partial_z \overline{Y}^{\overline{\alpha}}) + g_{\alpha\overline{\beta}} \mathcal{X}^\alpha \overline{\mathcal{X}}^{\overline{\alpha}} \ , \qquad \mathcal{W} = \mathcal{X}^\alpha W_\alpha \ . \tag{2.8}$$

where $\boldsymbol{K}_\alpha \equiv \partial \boldsymbol{K}/\partial Y^\alpha$, $W_\alpha \equiv \partial W/\partial Y^\alpha$, etc. It is a simple matter to obtain the classical equations of motion from the (0,2) action[7]. The result is

$$\overline{\mathcal{D}}\, \overline{\mathcal{X}}_\alpha = \sqrt{2} W_\alpha, \qquad \overline{\mathcal{D}} \left[g_{\alpha\overline{\beta}} \partial \overline{Y}^{\overline{\beta}} + g_{\alpha\overline{\beta},\gamma} \overline{\mathcal{X}}^{\overline{\beta}} \mathcal{X}^\gamma \right] = \sqrt{2} \mathcal{X}^\beta W_{\alpha\beta} \ , \tag{2.9}$$

where we defined the fermi superfield $\overline{\mathcal{X}}_\alpha \equiv g_{\alpha\overline{\beta}}(Y, \overline{Y}) \overline{\mathcal{X}}^{\overline{\beta}}$.

[7] If A and B are (0,2) superfields, then $\mathcal{D}\overline{\mathcal{D}}(AB)|_{\theta,\overline{\theta}=0} = 0 \quad \forall B \implies A = 0$; any chiral (anti-chiral) superfield, say $\delta\mathcal{X}$ ($\delta\overline{\mathcal{X}}$), can be expressed as $\overline{\mathcal{D}}P$ ($\mathcal{D}\overline{P}$) for some superfield P.

Component action

Finally, we can integrate over the remaining (0,2) superspace coordinates θ and $\overline{\theta}$ to obtain the component action. The auxiliary field $\overline{H}^{\overline{\alpha}}$ is determined by the equations of motion (2.9):

$$g_{\alpha\overline{\beta}}\overline{H}^{\overline{\beta}} = g_{\alpha\overline{\beta},\overline{\gamma}}\overline{\eta}^{\overline{\gamma}}\overline{\chi}^{\overline{\beta}} + W_\alpha \ , \tag{2.10}$$

and using this as well as $\overline{\chi}_\alpha \equiv g_{\alpha\overline{\beta}}\overline{\chi}^{\overline{\beta}}$ we obtain

$$\begin{aligned} 2\pi L = g_{\alpha\overline{\beta}}\left(\bar{\partial}_{\bar{z}} y^\alpha \partial_z \overline{y}^{\overline{\beta}} + \overline{\eta}^{\overline{\beta}} D_z \eta^\alpha\right) + \overline{\chi}_\alpha \overline{D}_{\bar{z}}\chi^\alpha - \overline{\eta}^{\overline{\beta}}\eta^\alpha R_{\alpha\overline{\beta}\gamma}{}^{\delta}\overline{\chi}_\delta\chi^\gamma - \chi^\alpha\eta^\beta D_\beta W_\alpha \\ + \overline{\chi}^{\overline{\alpha}}\overline{\eta}^{\overline{\beta}} D_{\overline{\beta}}\overline{W}_{\overline{\alpha}} + g^{\overline{\beta}\alpha}W_\alpha \overline{W}_{\overline{\beta}} \ , \end{aligned} \tag{2.11}$$

where the covariant derivatives are defined with the Kähler connection $\Gamma^\alpha_{\beta\gamma} \equiv g_{\gamma\overline{\beta},\beta}g^{\overline{\beta}\alpha}$, e.g.

$$\overline{D}_{\bar{z}}\chi^\alpha = \bar{\partial}_{\bar{z}}\chi^\alpha + \bar{\partial}_{\bar{z}}y^\beta\Gamma^\alpha_{\beta\gamma}\chi^\gamma \ , \qquad D_\alpha W_\beta = W_{\beta\alpha} - \Gamma^\gamma_{\alpha\beta}W_\gamma \ , \tag{2.12}$$

and the curvature is $R_{\alpha\overline{\beta}\gamma}{}^{\delta} \equiv \Gamma^\delta_{\alpha\gamma,\overline{\beta}}$. This is a complicated interacting theory, and in general it is not clear that one set of fields is preferred to another (say using $\overline{\chi}_\alpha$ instead of $\overline{\chi}^{\overline{\alpha}}$); however, for the purpose of determining the massless spectrum, it turns out to be useful to introduce another field redefinition to keep track of the non-zero left-moving bosonic excitations:

$$\rho_\alpha \equiv g_{\alpha\overline{\alpha}}\partial\overline{y}^{\overline{\alpha}} + \Gamma^\delta_{\alpha\gamma}\overline{\chi}_\delta\chi^\gamma \ , \tag{2.13}$$

in terms of which the left-moving kinetic terms take a strikingly simple form:

$$\begin{aligned} 2\pi L = \rho_\alpha\bar{\partial}_{\bar{z}}y^\alpha + \overline{\chi}_\alpha\bar{\partial}_{\bar{z}}\chi^\alpha + \eta^\alpha\left[g_{\alpha\overline{\beta}}D_z\overline{\eta}^{\overline{\beta}} + \overline{\eta}^{\overline{\beta}}R_{\alpha\overline{\beta}\gamma}{}^{\delta}\overline{\chi}_\delta\chi^\gamma + \chi^\beta D_\alpha W_\beta\right] \\ + \overline{\chi}^{\overline{\alpha}}\overline{\eta}^{\overline{\beta}}D_{\overline{\beta}}\overline{W}_{\overline{\alpha}} + g^{\overline{\beta}\alpha}W_\alpha\overline{W}_{\overline{\beta}} \ . \end{aligned} \tag{2.14}$$

Unlike the other fields ρ does not transform as a section of the pull-back of a bundle on $\boldsymbol{Y}$ under target space diffeomorphisms; this will have important consequences below.

2.3.3 *Symmetries*

We now examine the symmetries of the hybrid Lagrangian.

The $\overline{\boldsymbol{Q}}$ *supercharge*

Our action respects (2,2) SUSY generated by the superspace operators $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, as well as their left-moving images. We define the action of the corresponding operators $\boldsymbol{Q}$ and $\overline{\boldsymbol{Q}}$ by

$$\sqrt{2}[\xi\boldsymbol{Q} + \overline{\xi\boldsymbol{Q}}, A] \equiv -\xi\mathcal{Q}A - \overline{\xi\mathcal{Q}}A, \tag{2.15}$$

where ξ is an anti-commuting parameter and A is any superfield. In order to avoid writing the graded commutator, we will use a condensed notation $\overline{\xi\boldsymbol{Q}} \cdot A \equiv [\overline{\xi\boldsymbol{Q}}, A]$. For our subsequent study of the right-moving Ramond ground states, we will be particularly interested in the action of $\overline{\boldsymbol{Q}}$. Using the superfields in (2.3), we obtain

$$\begin{aligned}
&\overline{\boldsymbol{Q}} \cdot y^{\alpha} = 0, && \overline{\boldsymbol{Q}} \cdot \chi^{\alpha} = 0, && \overline{\boldsymbol{Q}} \cdot \eta^{\alpha} = \bar{\partial}_{\bar{z}} y^{\alpha} \ , && \overline{\boldsymbol{Q}} \cdot H^{\alpha} = \bar{\partial}_{\bar{z}} \chi^{\alpha} \ , \\
&\overline{\boldsymbol{Q}} \cdot \overline{y}^{\overline{\alpha}} = -\overline{\eta}^{\overline{\alpha}} \ , && \overline{\boldsymbol{Q}} \cdot \overline{\chi}^{\overline{\alpha}} = \overline{H}^{\overline{\alpha}} \ , && \overline{\boldsymbol{Q}} \cdot \overline{\eta}^{\overline{\alpha}} = 0 \ , && \overline{\boldsymbol{Q}} \cdot \overline{H}^{\overline{\alpha}} = 0 \ .
\end{aligned} \tag{2.16}$$

The action of the remaining supercharges is easily obtained from this one by conjugation and/or switching left- and right-moving fermions. Eliminating the auxiliary fields by their equations of motion we obtain

$$\begin{aligned}
&\overline{\boldsymbol{Q}} \cdot y^{\alpha} = 0 \ , && \overline{\boldsymbol{Q}} \cdot \chi^{\alpha} = 0 \ , && \overline{\boldsymbol{Q}} \cdot \eta^{\alpha} = \bar{\partial}_{\bar{z}} y^{\alpha} \ , \\
&\overline{\boldsymbol{Q}} \cdot \overline{y}^{\overline{\alpha}} = -\overline{\eta}^{\overline{\alpha}} \ , && \overline{\boldsymbol{Q}} \cdot \overline{\chi}_{\alpha} = W_{\alpha} \ , && \overline{\boldsymbol{Q}} \cdot \overline{\eta}^{\overline{\alpha}} = 0 \ .
\end{aligned} \tag{2.17}$$

From (2.9) it follows that up to the $\overline{\eta}$ equations of motion we also have $\overline{\boldsymbol{Q}} \cdot \rho_{\alpha} = \chi^{\beta} W_{\beta\alpha}$. Hence we can decompose $\overline{\boldsymbol{Q}}$ as $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{Q}}_0 + \overline{\boldsymbol{Q}}_W$, where the non-trivial action is

$$\overline{\boldsymbol{Q}}_0 \cdot \overline{y}^{\overline{\alpha}} = -\overline{\eta}^{\overline{\alpha}} \ , \quad \overline{\boldsymbol{Q}}_0 \cdot \eta^{\alpha} = \bar{\partial}_{\bar{z}} y^{\alpha} \ , \quad \overline{\boldsymbol{Q}}_W \cdot \overline{\chi}_{\alpha} = W_{\alpha} \ , \quad \overline{\boldsymbol{Q}}_W \cdot \rho_{\alpha} = \chi^{\beta} W_{\beta\alpha} \ . \tag{2.18}$$

These satisfy $\overline{\boldsymbol{Q}}_0^2 = \overline{\boldsymbol{Q}}_W^2 = \{\overline{\boldsymbol{Q}}_0, \overline{\boldsymbol{Q}}_W\} = 0$.[8] $\overline{\boldsymbol{Q}}_0$ is the supercharge for the NLSM with $W = 0$, while $\overline{\boldsymbol{Q}}_W$ incorporates the effect of a non-trivial potential.

Chiral U(1) *symmetries*

The $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ symmetries play an important role in relating the UV hybrid model to the IR physics of the corresponding SCFT. In the classical NLSM with $W = 0$ the presence of these symmetries is a consequence of the existence of an integrable, metric-compatible complex structure on $\boldsymbol{Y}$. In terms of component fields, the symmetries leave the bosonic fields invariant, while rotating the fermions as follows:

$$\mathrm{U}(1)^0_{\mathrm{L}} \ : \ \delta^0_L \eta = 0, \qquad \delta^0_L \chi = -i\epsilon\chi \ ; \quad \mathrm{U}(1)^0_{\mathrm{R}} \ : \ \delta^0_R \eta = -i\epsilon\eta, \qquad \delta^0_R \chi = 0 \ , \tag{2.19}$$

where ϵ is an infinitesimal real parameter. These naive symmetries are explicitly broken by the superpotential, but they can be improved if the geometry $(\boldsymbol{Y}, g)$ admits a holomorphic Killing vector V satisfying $\mathcal{L}_V W = W$.[9] V generates a non-chiral symmetry action

$$\delta_V Y^\alpha = i\epsilon V^\alpha(Y), \quad \delta_V \overline{Y}^{\overline{\alpha}} = -i\epsilon \overline{V}^{\overline{\alpha}}(\overline{Y}); \qquad \delta_V \mathcal{X}^\alpha = i\epsilon V^\alpha_{,\beta}\mathcal{X}^\beta \ , \delta_V \overline{\mathcal{X}}^{\overline{\alpha}} = -i\epsilon \overline{V}^{\overline{\alpha}}_{,\overline{\beta}} \overline{\mathcal{X}}^{\overline{\beta}} \ , \tag{2.20}$$

and it is easy to see that $\delta_{L,R} \equiv \delta^0_{L,R} + \delta_V$ are symmetries of the classical action.

While $\mathrm{U}(1)_{\mathrm{diag}} \subset \mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ has a non-chiral action on the fermions and hence is non-anomalous, $\mathrm{U}(1)_{\mathrm{L}}$ is a chiral symmetry that will be anomaly free iff $c_1(T_{\boldsymbol{Y}}) = 0$, a condition satisfied when $\boldsymbol{Y}$ is a non-compact Calabi-Yau manifold, i.e.

[8] If we keep the terms in $\overline{\boldsymbol{Q}} \cdot \rho$ proportional to $\overline{\eta}$ equations of motion and decompose that into a W-independent and W-dependent contributions, we find that the decomposition $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{Q}}_0 + \overline{\boldsymbol{Q}}_W$ into a pair of nilpotent anti-commuting operators holds without use of equations of motion; for us the result of (2.18) will be sufficient.

[9] Holomorphic Killing vectors satisfy $V^\alpha_{,\overline{\beta}} = 0$ and $\mathcal{L}_V g = 0$. They are a familiar topic in supersymmetry—see, e.g., Appendix D of [94]. Note that on a compact Kähler manifold a Killing vector field is holomorphic, but this can fail on a non-compact manifold. Killing vectors on Kähler manifolds are further discussed in [14, 83].

$\boldsymbol{Y}$ has a trivial canonical bundle $K_{\boldsymbol{Y}} \simeq \mathcal{O}_{\boldsymbol{Y}}$. In what follows we assume $K_{\boldsymbol{Y}}$ is indeed trivial (this is stronger than $c_1(T_{\boldsymbol{Y}}) = 0$). When $X = \oplus_i L_i$, a sum of line bundles such that $\otimes_i L_i$ is negative, then since $K_{\boldsymbol{Y}} = K_B \otimes_i L_i^*$ the anti-canonical class of B is very ample and B is Fano.[10]

In what follows we will denote the conserved charge for $\mathrm{U}(1)_{\mathrm{L}}$ ($\mathrm{U}(1)_{\mathrm{R}}$) by J_0 ($\overline{J}_0$) and its eigenvalues on various operators and states by $\boldsymbol{q}$ ($\overline{\boldsymbol{q}}$).

R-symmetries for good hybrid models

We would like to identify the UV $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ symmetries described above with their counterparts in the conjectured IR SCFT. As usual, there is a small subtlety in doing this when V is not unique. In practice this is easily achieved by picking a sufficiently generic superpotential and more generally, one could use c-extremization [23] to fix $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ up to the usual caveats of accidental IR symmetries.

More importantly, in order for the UV R-symmetry of the hybrid model to be a good guide to the IR physics, we need V to be a vertical vector field, i.e. $\mathcal{L}_V \pi^*(\omega) = 0$ for all forms $\omega \in \Omega^\bullet(B)$, and in particular the $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ symmetries fix B pointwise. We denote a model where this is the case a *good hybrid*. As we show in Appendix A.2 this implies

$$V = \textstyle\sum_{i=1}^n q_i \phi^i \frac{\partial}{\partial \phi^i} + \text{c.c.} \qquad (2.21)$$

for some real charges q_i. The q_i have to be compatible with the transition functions defining $X \to B$, and since $\mathcal{L}_V W = W$, and W is polynomial in every patch, $q_i \in \mathbb{Q}_{\geqslant 0}$. In a LG theory, i.e B a point, standard results show that if the potential condition is satisfied then without loss of generality $0 < q_i \leqslant 1/2$ [69, 67]. More generally, the potential condition requires that $W(y^I, \phi)$, thought of locally as a LG potential for the fiber fields ϕ depending on the "parameters" y^I, should be non-singular in a

[10] A variety is Fano iff its anti-canonical class is ample; Fano varieties are quite special: for instance $H^i(B, \mathcal{O}) = 0$ for $i > 0$, $\mathrm{Pic}(B) \simeq H^2(B, \mathbb{Z})$; in addition, they are classified in dimension $d \leqslant 3$ and admit powerful criteria for evaluating positivity of bundles [73].

small neighborhood of any generic point in B. Hence, the range of allowed q_i is the same for a hybrid theory as it is for LG models.

The orbifold action

Our main interest in the hybrid SCFTs is for applications to supersymmetric compactification of type II or heterotic string theories. For left-right symmetric theories this requires the existence of $\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R}$ symmetries with integral $\boldsymbol{q}$, $\overline{\boldsymbol{q}}$ charges of all (NS,NS) sector states [15]. Our hybrid theory, if it flows as expected to a $c = \overline{c} = 9$ SCFT in the IR will not satisfy this condition. Fortunately, the solution is the same as it is for Gepner models [51] or LG orbifolds [92, 58]: we gauge the discrete symmetry Γ generated by $\exp[2\pi i J_0]$, where J_0 denotes the conserved $\mathrm{U}(1)_\mathrm{L}$ charge; since all fields have $\boldsymbol{q} - \overline{\boldsymbol{q}} \in \mathbb{Z}$, the orbifold by Γ is sufficient to obtain integral charges.

In the line bundle case with $q_i = n_i/d_i$ we then see that $\Gamma \simeq \mathbb{Z}_N$, with N the least common multiple of $(d_1, \ldots, d_n)$. Since Γ is embedded in a continuous non-anomalous symmetry we expect the resulting orbifold to be a well-defined quantum field theory, and the resulting orbifold SCFT will be suitable for a string compactification.

In addition to the introduction of twisted sectors and the projection, the orbifold has one important consequence for the physics of hybrid models: it allows us to consider more general "orbi-bundles," where the fiber in $X \to B$ is of the form $\mathbb{C}^n/\Gamma$, and the transition functions are defined up to the orbifold action. For instance, we will examine a theory with $B = \mathbb{P}^3$ and $X = \mathcal{O}(-5/2) \oplus \mathcal{O}(-3/2)$, where the orbifold $\Gamma = \mathbb{Z}_2$ reflects both of the fiber coordinates.[11]

[11] A GLSM embedding of this hybrid model is given in section 2.5 of [1] .

2.3.4 The quantum theory and the hybrid limit

Having defined the classical hybrid model's Lagrangian and discussed its symmetries, we now discuss the quantum theory. To orient ourselves in the issues involved, let's recall the case of (2,2) LG models — the simplest examples of hybrids. These theories have a Lagrangian description at some renormalization scale μ as a free kinetic term for chiral multiplets, and a superpotential interaction with dimensionful couplings m. The theory is weakly coupled when $\mu \gg m$, and we can use the Lagrangian and (approximately) free fields to describe the theory. The low energy limit $\mu \to 0$ is then strongly coupled, and while W is protected by SUSY non-renormalization theorems, the kinetic term receives a complicated but irrelevant set of corrections. There is by now overwhelming evidence that these do flow to the expected SCFTs, in accordance with the original proposals [75, 93], and computations of RG-invariant quantities allow us to use the weakly coupled $\mu \gg m$ description to describe *exactly* the SCFT's (c,c) chiral ring and more generally the $\overline{\boldsymbol{Q}}$-cohomology. Furthermore, the results extend to LGOs suitable for string compactification.[12]

There is a small IR subtlety in using the weakly coupled LG description: the theory at $W = 0$ is non-compact and has all the usual difficulties associated with non-compact bosons. This is of course not very subtle since the theory is free; however, more to the point, in using the weakly coupled description we still keep track of the R-charges and weights that follow from the superpotential and do not consider states supported away from the $W = 0$ locus.

A more general hybrid theory has a similar structure, except that now there are two sorts of couplings: the superpotential couplings m/μ, as well as the choice of Kähler class on the base B. Although the latter coupling is typically encoded in the kinetic D-term, it can also be expressed as a deformation of the twisted chiral

[12] These typically have non-trivial (a,c) rings encoded in the twisted sectors, and that ring structure is not easy to access directly via the LG orbifold description.

superpotential. Hence, the Kähler class and superpotential couplings do not receive quantum corrections. Of course we do expect corrections to the D-terms, but these should be irrelevant just as they are in the LG case. Moreover, there is good evidence, based on GLSM constructions, that the hybrid models with a GLSM UV completion should flow to SCFTs with expected properties (i.e. correct central charges and R-symmetries), and we expect the same to hold for more general hybrid models. As in the LG case, the strict $W = 0$ limit may be subtle, perhaps even more so, since it may require us to specify additional details about the geometry of $\boldsymbol{Y}$. However, we may use the same cure for these IR subtleties as we do in the LG case: use the R-charges and weights encoded by the superpotential and restrict attention to field configurations and states supported on B.

Assuming a hybrid model does flow to an expected SCFT, we would like to have techniques to evaluate RG-invariant quantities such as the $\overline{\boldsymbol{Q}}$-cohomology. It is here that there will be important conceptual and technical differences from the LG case due to the non-trivial base geometry B. For instance, we expect the $\overline{\boldsymbol{Q}}$-cohomology to depend on the choice of Kähler class on B. While there will not be a perturbative dependence, we do in general expect corrections from world-sheet instantons wrapping non-trivial cycles in B. These corrections are suppressed when B is large, which leads us to define the hybrid analogue of the large radius limit of a NLSM: *the hybrid limit*, where the Kähler class of B is taken to be arbitrarily deep in its Kähler cone. In what follows, we will study the $\overline{\boldsymbol{Q}}$-cohomology of a hybrid model in the hybrid limit.

2.4 Massless spectrum of heterotic hybrids

In this section we develop techniques to evaluate the massless spectrum for a compactification of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string based on a $c = \overline{c} = 9$ (2,2) hybrid SCFT.[13]

[13] The SO(32) case can be handled in an entirely analogous fashion.

We first review the standard prescription [51, 92, 61] to obtain a modular invariant theory and identify world-sheet Ramond ground states with massless fermions in spacetime. We then discuss how to enumerate these ground states by studying the $\overline{\boldsymbol{Q}}$ cohomology in the hybrid limit.

2.4.1 Spacetime generalities

In order to describe a heterotic string compactification, we complete our hybrid $c = \overline{c} = 9$ $N = (2,2)$ SCFT internal theory to a critical heterotic theory by adding ten left-moving fermions (with fermion number F_λ) that realize an $\mathfrak{so}(10)$ level 1 current algebra, a left-moving level 1 hidden $\mathfrak{e}_8$ current algebra, and the free $c = 4$, $\overline{c} = 6$ theory of the uncompactified spacetime $\mathbb{R}^{1,3}$.

A modular invariant theory is obtained by performing left- and right- GSO projections. The left-moving GSO projection onto $e^{i\pi J_0}(-)^{F_\lambda} = 1$ is responsible for enhancing the linearly realized $\mathfrak{u}(1)_{\mathrm{L}} \oplus \mathfrak{so}(10)$ gauge symmetry to the full $\mathfrak{e}_6$. The right-moving GSO projection has a similar action, combining $\overline{J}_0$ with the fermion number of the $\mathbb{R}^{1,3}$ theory. Its immediate spacetime consequence is $N = 1$ spacetime supersymmetry, or equivalently, a relation, via spectral flow, between states in right-moving Neveu-Schwarz and Ramond sectors. Spacetime fermions arise in the (NS,R) and (R,R) sectors, and supersymmetry allows us to identify the full spectrum of supermultiplets in the spacetime theory from these states.

The spacetime theory obtained by this procedure will have a model-independent set of massless fermions: the gauginos of the hidden $\mathfrak{e}_8$, the gravitino, and the dilatino. In what follows we focus on the model-dependent massless spectrum. In particular, the hidden $\mathfrak{e}_8$ degrees of freedom are always restricted to their NS ground state and just make a contribution to the left-moving zero-point energy.

On-shell string states have vanishing left- and right-moving energies. For massless states there is no contribution to $\overline{L}_0$ from the $\mathbb{R}^{1,3}$ free fields; massless fermions are thus states in the (R,R) and (NS,R) sectors with vanishing left-moving and right-moving energies. In the (R,R) sector, massless states are associated to the ground states in the internal theory, related by spectral flow to (NS,NS) operators comprising the "chiral rings" [93] of the theory. Massless states in the (NS,R) sector include states related to these by left-moving spectral flow as well as additional states. The main result of [61] is a method for describing these states in LGO theories, which we here extend to hybrids. This relies on the familiar fact that since

$$\{\boldsymbol{Q}, \overline{\boldsymbol{Q}}\} = 2\overline{L}_0 \ ; \quad \boldsymbol{Q}^2 = \overline{\boldsymbol{Q}}^2 = 0 \tag{2.22}$$

the kernel of $\overline{L}_0$ is isomorphic to the cohomology of $\overline{\boldsymbol{Q}}$.

The right-moving GSO projection is onto states with $\overline{\boldsymbol{q}} \in \mathbb{Z} + \frac{1}{2}$; those with $\overline{\boldsymbol{q}} = -1/2$ ($\overline{\boldsymbol{q}} = 1/2$) correspond to chiral (anti-chiral) multiplets, while states with $\overline{\boldsymbol{q}} = \pm 3/2$ are gauginos in vector multiplets. The $\mathrm{U}(1)_{\mathrm{L}}$ charge $\boldsymbol{q}$ determines the $\mathfrak{e}_6$ representation according to the decomposition

$$\begin{aligned}
&\mathfrak{e}_6 \supset \mathfrak{so}(10) \oplus \mathfrak{u}(1) \\
&\mathbf{78} = \mathbf{45}_0 \oplus \mathbf{16}_{-3/2} \oplus \overline{\mathbf{16}}_{3/2} \oplus \mathbf{1}_0 \\
&\mathbf{27} = \mathbf{16}_{1/2} \oplus \mathbf{10}_{-1} \oplus \mathbf{1}_2 \\
&\overline{\mathbf{27}} = \overline{\mathbf{16}}_{-1/2} \oplus \mathbf{10}_1 \oplus \mathbf{1}_{-2} \ .
\end{aligned} \tag{2.23}$$

As in the LG orbifold case [92, 61], the GSO projection can be combined with the hybrid orbifold of $\Gamma = \mathbb{Z}_N$ to an orbifold by $\mathbb{Z}_2 \ltimes \mathbb{Z}_N \cong \mathbb{Z}_{2N}$. Therefore we need to study the $2N$ sectors twisted by $[\exp(i\pi J_0)]^k, k = 0, \ldots, 2N-1$.[14] Spacetime CPT exchanges the k-th and the $(2N-k)$-th sectors, and CPT invariance means we can restrict our analysis to the $k = 0, 1, \ldots, N$. sectors. The states arising in (R,R) (k even) sectors give rise to $\mathfrak{e}_6$-charged matter. This is easy to see since in this case the ground states of the $\mathfrak{so}(10)$ current algebra transform in $\mathbf{16} \oplus \overline{\mathbf{16}}$. Massless $\mathfrak{e}_6$-singlets

[14] That is, schematically, in the k-th twisted sector fields satisfy $\phi(ze^{2\pi i}, \overline{z}e^{-2\pi i}) = [\exp(i\pi J_0)]^k \phi(z, \overline{z})$. We will make these periodicities more precise shortly.

are of particular interest, and they can only arise from (NS,R) sectors, i.e. sectors with odd k.

2.4.2 Left-moving symmetries in cohomology

The action of $\mathrm{U(1)_L}$ commutes with $\overline{\boldsymbol{Q}}$, and following [98, 90], we can find a representative for the corresponding conserved current in $\overline{\boldsymbol{Q}}$-cohomology, denoted by $H_{\overline{\boldsymbol{Q}}}$. Consider the operator

$$\mathcal{J}_L \equiv \mathcal{X}^\beta (D_\beta V^\alpha - \delta^\alpha_\beta)\overline{\mathcal{X}}_\alpha - V^\alpha g_{\alpha\overline{\beta}} \partial_z \overline{Y}^{\overline{\beta}} \ . \tag{2.24}$$

Using (2.9) and $\mathcal{L}_V W = W$ it follows $\overline{\mathcal{D}}\mathcal{J}_L = 0$. Observing that $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are conjugate operators, $\overline{\mathcal{Q}} = -\exp\left[2\overline{\theta}\theta\partial_{\overline{z}}\right]\overline{\mathcal{D}}\exp\left[2\theta\overline{\theta}\partial_{\overline{z}}\right]$, we conclude that

$$J_L \equiv \mathcal{J}_L|_{\theta=0} = \chi^\beta (V^\alpha_{\ ,\beta} - \delta^\alpha_\beta)\overline{\chi}_\alpha - V^\alpha \rho_\alpha \tag{2.25}$$

is $\overline{\boldsymbol{Q}}$-closed and hence has a well-defined action on $H_{\overline{\boldsymbol{Q}}}$. Similarly, we can obtain the remaining generators of the left-moving $N = 2$ algebra in $H_{\overline{\boldsymbol{Q}}}$. To find the energy-momentum generator T we observe that

$$\mathcal{T}_0 = -g_{\alpha\overline{\beta}}\partial_z Y^\alpha \partial_z \overline{Y}^{\overline{\beta}} - \mathcal{X}^\alpha D_z \overline{\mathcal{X}}_\alpha = -\partial_z Y^\alpha \left[g_{\alpha\overline{\beta}}\partial_z \overline{Y}^{\overline{\beta}} - g_{\gamma\overline{\beta},\alpha}\mathcal{X}^\gamma \overline{\mathcal{X}}^{\overline{\beta}}\right] - \mathcal{X}^\alpha \partial_z \overline{\mathcal{X}}_\alpha \tag{2.26}$$

satisfies $\overline{\mathcal{D}}\mathcal{T}_0 = 0$, as does

$$\mathcal{T} \equiv \mathcal{T}_0 - \frac{1}{2}\partial_z \mathcal{J}_L \ . \tag{2.27}$$

The lowest component of $\mathcal{T}$ is $\overline{\boldsymbol{Q}}$-closed and given by

$$T = -\partial y^\alpha \rho_\alpha - \frac{1}{2}\left(\overline{\chi}_\alpha \partial_z \chi^\alpha + \chi^\alpha \partial_z \overline{\chi}_\alpha\right) - \frac{1}{2}\partial_z\left[\chi^\beta \overline{\chi}_\alpha V^\alpha_{\ ,\beta} - V^\alpha \rho_\alpha\right] \ . \tag{2.28}$$

The remaining generators of a left-moving $N = 2$ algebra are obtained from the $\overline{\mathcal{D}}$-closed fields

$$\mathcal{G}^+ \equiv i\sqrt{2}\left[\overline{\mathcal{X}}_\alpha \partial_z Y^\alpha - \partial_z(\overline{\mathcal{X}}_\alpha V^\alpha)\right], \qquad \mathcal{G}^- \equiv i\sqrt{2}\left[\mathcal{X}^\alpha g_{\alpha\overline{\beta}}\partial_z \overline{Y}^{\overline{\beta}}\right] \ , \tag{2.29}$$

yielding the left-moving supercharges $G^\pm$ in $H_{\overline{\boldsymbol{Q}}}$:

$$G^+ = i\sqrt{2}\left[\overline{\chi}_\alpha \partial_z y^\alpha - \partial_z(\overline{\chi}_\alpha V^\alpha)\right] \ , \qquad G^- \equiv i\sqrt{2}\chi^\alpha \rho_\alpha \ . \tag{2.30}$$

Table 2.1: Weights and charges of the fields.

	y^I	ρ_I	χ^I	$\overline{\chi}_I$	ϕ^i	ρ_i	χ^i	$\overline{\chi}_i$
$\boldsymbol{q}$	0	0	-1	1	q_i	$-q_i$	$q_i - 1$	$1 - q_i$
$2h$	0	2	1	1	q_i	$2 - q_i$	$1 + q_i$	$1 - q_i$
$\overline{\boldsymbol{q}}$	0	0	0	0	q_i	$-q_i$	q_i	$-q_i$

2.4.3 Reduction to a curved $bc - \beta\gamma$ system

The action (2.14) determines the OPEs for the left-moving degrees of freedom to be

$$y^\alpha(z)\rho_\beta(w) \sim \frac{1}{z-w}\delta^\alpha_\beta \ , \qquad \chi^\alpha(z)\overline{\chi}_\beta(w) \sim \frac{1}{z-w}\delta^\alpha_\beta \ . \tag{2.31}$$

Using the normal ordering defined by these free-field OPEs we can define T, J, and $G^\pm$ in the quantum theory. This is particularly simple with our choice of fields and Killing vector V: the operators are quadratic in the fields, and it is easy to check that they indeed generate an $N = 2$ algebra with central charge

$$c = 3d + 3\textstyle\sum_{i=1}^n (1 - 2q_i) \ , \tag{2.32}$$

which we recognize as the sum of the fiber LG central charge and the contribution from the base. The $\mathrm{U(1)_L}$ charge J_0 and left-moving Hamiltonian L_0 are obtained in the standard fashion as

$$J_0 = \oint \frac{dz}{2\pi i} J_L(z) \ , \qquad L_0 = \oint \frac{dz}{2\pi i} zT(z) \ , \tag{2.33}$$

and the resulting charge and weight assignments for the fiber fields are given in table 2.1 together with the $\mathrm{U(1)_R}$ charge $\overline{\boldsymbol{q}}$. These currents are trivially annihilated by $\overline{\boldsymbol{Q}}_0$ and commute with $\overline{\boldsymbol{Q}}_W$, whose action is now realized as

$$\overline{\boldsymbol{Q}}_W \equiv \oint \frac{dz}{2\pi i} \ [\chi^\alpha W_\alpha(y)]\,(z) \ . \tag{2.34}$$

It may seem a little bit puzzling that we have been able to reduce the entire problem to a free first order system. What, the reader may ask, encodes the target space geometry, for example? The answer, familiar from [85, 99], is that the free field theory description only applies patch by patch in field-space. That is, we cover $\boldsymbol{Y}$ with open sets U_a and local coordinates x_a^α, and on each $U_{ab} = U_a \cap U_b \neq \varnothing$ $x_b = x_b(x_a)$, and we define the holomorphic transition functions

$$(T_{ba})^\alpha_\beta \equiv \frac{\partial x_b^\alpha}{\partial x_a^\beta}, \qquad (\mathcal{S}_{ba})^\alpha_{\beta\gamma} \equiv (T^{-1}_{ba})^\alpha_\delta (T_{ba})^\delta_{\beta,\gamma} \ . \tag{2.35}$$

The left-moving fields then patch according to

$$\begin{aligned} &y_b^\alpha = x_b^\alpha(y_a)\ , \quad \chi_b^\alpha = (T_{ba})^\alpha_\beta \chi_a^\beta\ , \quad \overline{\chi}_{b\alpha} = (T^{-1}_{ba})^\beta_\alpha \overline{\chi}_{a\beta}\ , \\ &\rho_{b\alpha} =: (T^{-1}_{ba})^\beta_\alpha \left[\rho_{b\beta} - \mathcal{S}^\delta_{ba\beta\gamma} \overline{\chi}_\delta \chi^\gamma\right] :\ , \end{aligned} \tag{2.36}$$

where the transition functions are evaluated at y_a, e.g. $T_{ba} = T_{ba}(y_a)$. Note that the patching of ρ requires a normal-ordering due to singularities in the $y - \rho$ and $\overline{\chi} - \chi$ OPEs. Of course there are similar transformations for the right-moving fields $\overline{y}$ and $\eta, \overline{\eta}$. For instance, the $\overline{\eta}^{\overline{I}}$ transform as sections of $y^*(\overline{T}_B)$.[15]

These transition functions require a careful analysis when we expand about world-sheet instanton configurations, i.e. non-trivial holomorphic maps $\Sigma \to \boldsymbol{Y}$. This, together with non-trivial fermi zero modes in the background of an instanton will lead to world-sheet instanton corrections to $\overline{\boldsymbol{Q}}_0$.[16] These corrections vanish in the hybrid limit where we expand about constant maps $\partial_z y = \bar{\partial}_{\bar{z}} y = 0$, and the only non-trivial $\overline{\boldsymbol{Q}}_0$ action is on the anti-holomorphic zero modes $\overline{\boldsymbol{Q}}_0 \cdot \overline{y}_0^{\overline{\alpha}} = -\overline{\eta}_0^{\overline{\alpha}}$. In fact, since the $\overline{\eta}^{\bar{\imath}}$ are $\overline{\boldsymbol{Q}}$-exact, as far as cohomology is concerned, we can safely ignore the $\overline{\eta}^{\bar{\imath}}$ as well as the anti-holomorphic bosonic fiber zero modes $\overline{\phi}_0^{\bar{\imath}}$. So, the only non-trivial $\overline{\boldsymbol{Q}}_0$ action is on the base anti-holomorphic zero modes: $\overline{\boldsymbol{Q}}_0 \cdot \overline{y}_0^{\overline{I}} = -\overline{\eta}_0^{\overline{I}}$. In what follows we will drop the zero mode subscript on these right-moving fields with the understanding that $\overline{y}$ and $\overline{\eta}$ will denote the base antiholomorphic zero modes.

[15] As we are working on a flat world-sheet throughout this chapter, we do not keep track of the world-sheet spinor properties of the fermionic degrees of freedom.

[16] Since $\overline{\boldsymbol{Q}}_W$ is associated to a chiral superpotential, we do not expect it to be corrected by world-sheet instantons.

2.4.4 Massless states in the hybrid limit

Our task now is to work out, in each twisted sector, the set of GSO-even states that belong to $H_{\overline{\boldsymbol{Q}}}$ and carry left-moving energy $E = 0$. We construct the relevant states (i.e. the only ones with required energy and charges) in the Hilbert space as polynomials in the fermions and non-zero bosonic oscillator modes tensored with wavefunctions of the bosonic zero modes. In a generic twisted sector the bosonic zero modes correspond to the compact base B, while in less generic sectors there can be additional bosonic zero modes. However, since the non-compact bosonic modes will be lifted by the superpotential, in what follows all bosonic wavefunctions will be taken to be polynomial in the fiber fields.

The operators T and J_L can be used to grade the states according to their energy E and left-moving charge $\boldsymbol{q}$, and we can evaluate $\overline{\boldsymbol{Q}}$-cohomology on the states of fixed E and $\boldsymbol{q}$. An important simplification comes from working in the right-moving Ramond ground sector. A look at (2.18) shows that, as far as $\overline{\boldsymbol{Q}}$-cohomology is concerned, we can neglect any states containing oscillators in $\bar{\partial}_{\bar{z}} y^{\alpha}$, as well as any non-zero mode of η^{α}. We choose the Ramond ground state annihilated by the zero modes of η^{α}, so our states will be constructed without η^{α} or right-moving bosonic oscillators. We will call the resulting space of states *the restricted Hilbert space* $\mathcal{H}$. In general this will be infinite-dimensional even at fixed E and $\boldsymbol{q}$.

Twisted modes and ground state quantum numbers

In this section we provide expressions for E, $\boldsymbol{q}$ and $\overline{\boldsymbol{q}}$ of the states in a fixed twisted sector. For simplicity, we work out the case $X = \oplus_i L_i$. The result extends immediately to orbi-bundles of the form $X = \oplus_i L_i^{x_i}$ for $x_i \in \mathbb{Q}$. It should be possible to treat the case of more general X at the price of additional notation.

The first task is to describe the mode expansions of the fields and the quantum numbers of the ground states $|k\rangle$. While we can restrict to right-moving (i.e. anti-holomorphic) zero modes, the left-moving oscillators need to be treated in detail. In each patch of the target space the moding of the left-moving fields in the k-th twisted sector is

$$\begin{aligned} y^\alpha(z) &= \sum_{r\in\mathbb{Z}-\nu_\alpha} y^\alpha_r z^{-r-h_\alpha}, & \chi^\alpha(z) &= \sum_{r\in\mathbb{Z}-\widetilde{\nu}_\alpha} \chi^\alpha_r z^{-r-\widetilde{h}_\alpha}, \\ \rho_\alpha(z) &= \sum_{r\in\mathbb{Z}+\nu_\alpha} \rho_{\alpha r} z^{-r+h_\alpha-1}, & \overline{\chi}_\alpha(z) &= \sum_{r\in\mathbb{Z}+\widetilde{\nu}_\alpha} \overline{\chi}_{\alpha r} z^{-r+\widetilde{h}_\alpha-1}, \end{aligned} \tag{2.37}$$

where

$$\nu_\alpha = \frac{kq_\alpha}{2} \mod 1\ , \quad \widetilde{\nu}_\alpha = \frac{k(q_\alpha-1)}{2} \mod 1\ , \quad \widetilde{h}_\alpha - \frac{1}{2} = h_\alpha = \frac{q_\alpha}{2}\ . \tag{2.38}$$

We choose $0 \leqslant \nu_\alpha < 1$ and $-1 < \widetilde{\nu}_\alpha \leqslant 0$ and recall that the oscillator vacuum $|k\rangle$ is annihilated by all the positive modes. When $\chi, \overline{\chi}$ have zero modes our conventions are that the ground state is annihilated by the χ_0 modes.

The mode (anti)commutators follow from (2.31) and (2.37):

$$[y^\alpha_r, \rho_{\beta s}] = \delta^\alpha_\beta \delta_{r,-s}\ , \qquad \{\chi^\alpha_r, \overline{\chi}_{\beta s}\} = \delta^\alpha_\beta \delta_{r,-s}\ . \tag{2.39}$$

Each oscillator carries the obvious $\boldsymbol{q}, \overline{\boldsymbol{q}}$ charges and contributes minus its mode number to the energy. By using this mode expansion to compute 1-point functions of T and J_L in the oscillator vacuum $|k\rangle$, we determine the quantum numbers of $|k\rangle$. The left- and right-moving charges are given by

$$\begin{aligned} \boldsymbol{q}_{|k\rangle} &= \sum_\alpha \left[(q_\alpha - 1)(\widetilde{\nu}_\alpha + \frac{1}{2}) - q_\alpha(\nu_\alpha - \frac{1}{2})\right], \\ \overline{\boldsymbol{q}}_{|k\rangle} &= \sum_\alpha \left[q_\alpha(\widetilde{\nu}_\alpha + \frac{1}{2}) + (q_\alpha - 1)(-\nu_\alpha + \frac{1}{2})\right], \end{aligned} \tag{2.40}$$

and while the left-moving energy is $E_{|k\rangle} = 0$ for k even, we have

$$E_{|k\rangle} = -\frac{5}{8} + \frac{1}{2}\sum_\alpha \left[\nu_\alpha(1-\nu_\alpha) + \widetilde{\nu}_\alpha(1+\widetilde{\nu}_\alpha)\right], \tag{2.41}$$

for k odd. Note that this includes the usual $-c/24$ shift: $E = L_0 - 1$.

The oscillator vacuum $|k\rangle$ we have constructed is not in general a state in the Hilbert space. To specify a state we need to prescribe a dependence on the bosonic zero modes so as to get a well-defined state, but from above we see that $|k\rangle$ transforms as a section of a holomorphic line bundle $L_{|k\rangle}$ over B. When $X = \oplus_i L_i$ we find (using $K_{\boldsymbol{Y}} = \mathcal{O}_{\boldsymbol{Y}}$)

$$L_{|k\rangle} = \begin{cases} \otimes_i L_i^{(\widetilde{\nu}_i - \nu_i)} & \text{for } k \text{ even,} \\ \otimes_i L_i^{(\widetilde{\nu}_i - \nu_i + \frac{1}{2})} & \text{for } k \text{ odd .} \end{cases} \tag{2.42}$$

From (2.38) we see that if we set $\nu_I = 0$ and $\widetilde{\nu}_I = -k/2 \mod 1$, then $\tau_\alpha = \nu_\alpha - \widetilde{\nu}_\alpha$ is

$$\tau_\alpha = \begin{cases} 0 & \nu_\alpha = 0 \\ 1 & \nu_\alpha \neq 0 \end{cases} \quad \text{for even } k \ ; \quad \tau_\alpha = \begin{cases} 1/2 & \nu_\alpha \leqslant \frac{1}{2} \\ 3/2 & \nu_\alpha > \frac{1}{2} \end{cases} \quad \text{for odd } k \ . \tag{2.43}$$

This shows that $L_{|k\rangle}$ is well-defined because $\tau_\alpha \in \mathbb{Z}$ for k even and $\tau_\alpha \in \mathbb{Z} + \frac{1}{2}$ for k odd. A well-defined ground state can be of the form

$$|\Psi_0^k\rangle = \Psi_0(y', \overline{y})_{\overline{I}_1 \cdots \overline{I}_u} \overline{\eta}^{\overline{I}_1} \cdots \overline{\eta}^{\overline{I}_u} |k\rangle \ , \tag{2.44}$$

where y' denotes bosonic zero modes, the $\overline{\eta}^{\overline{I}}$ are the right-moving superpartners of the base coordinates and Ψ_u^k are (0,u) horizontal forms on $\boldsymbol{Y}$ valued in the holomorphic sheaf $L_{|k\rangle}^*$. In sectors in which there are additional zero modes ($k = 0$ is always an example of this) there are more general ground states, and in (R,R) sectors a subset of these ground states describes the massless spectrum.

This non-trivial vacuum structure is a generalization of familiar limiting cases of the hybrid construction. When $\boldsymbol{Y} = B$ a compact Calabi-Yau manifold, the Ramond ground state is a section of a trivial bundle (the square root of the trivial canonical bundle); in the LGO case each twisted sector has a unique ground state $|k\rangle$.

The double-grading and spectral sequence

Our restricted Hilbert space $\mathcal{H}$ at fixed E and $\boldsymbol{q}$ admits a grading by $\mathrm{U}(1)_\mathrm{R}$ charge, and $\overline{\boldsymbol{Q}}$ acts as a differential, $\overline{\boldsymbol{Q}} : \mathcal{H}_{\overline{\boldsymbol{q}}} \to \mathcal{H}_{\overline{\boldsymbol{q}}+1}$ that preserves the left-moving quantum numbers. A key observation, made in the LG case in [61], that makes the cohomology problem tractable is that in fact $\mathcal{H}$ admits a double-grading compatible with the split $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{Q}}_0 + \overline{\boldsymbol{Q}}_W$ in (2.18). Let U be an operator that assigns charge $+1$ to $\overline{\eta}$, -1 to η, and leaves the other fields invariant. Although U is not a symmetry of the theory when $W \neq 0$, we can still grade our restricted Hilbert space according to the eigenvalues u of U and $p \equiv \boldsymbol{q} - u$, and since $[U, \overline{\boldsymbol{Q}}_0] = \overline{\boldsymbol{Q}}_0$ and $[U, \overline{\boldsymbol{Q}}_W] = 0$ we obtain a double-graded complex with

$$\overline{\boldsymbol{Q}}_0 : \mathcal{H}^{p,u} \to \mathcal{H}^{p,u+1}, \qquad \overline{\boldsymbol{Q}}_W : \mathcal{H}^{p,u} \to \mathcal{H}^{p+1,u} \tag{2.45}$$

acting, respectively, as anticommuting vertical and horizontal differentials. The cohomology of $\overline{\boldsymbol{Q}}$ is thus computed by a spectral sequence with first two stages

$$E_1^{p,u} = H^u_{\overline{\boldsymbol{Q}}_0}(\mathcal{H}^{p,\bullet}), \qquad \text{and} \qquad E_2^{p,u} = H^p_{\overline{\boldsymbol{Q}}_W} H^u_{\overline{\boldsymbol{Q}}_0}(\mathcal{H}^{\bullet,\bullet}) \ . \tag{2.46}$$

In general, E_{r+1} is obtained from E_r as the cohomology of a differential d_r acting as

$$d_r : E_r^{p,u} \to E_r^{p+r,u+1-r} \ . \tag{2.47}$$

We have, for example, $d_0 = \overline{\boldsymbol{Q}}_0$ and $d_1 = \overline{\boldsymbol{Q}}_W$. The differentials at higher stages are produced by a standard zig-zag construction [30]. Since the range of U is $0 \leqslant U \leqslant d$ the differentials vanish for $r > \dim B$, and the sequence converges: $E^{p,u}_{\dim B+1} = E^{p,u}_\infty = H^{p,u}_{\overline{\boldsymbol{Q}}}(\mathcal{H}^{\bullet,\bullet})$.

We now have almost all of the tools to describe the massless spectrum. In each twisted sector there is a geometric structure that organizes the states in the spectral sequence. On $\mathcal{H}$ the $\overline{\boldsymbol{Q}}_0$ action is simply

$$\overline{\boldsymbol{Q}}_0 = -\overline{\eta}^{\bar{I}} \frac{\partial}{\partial \overline{y}^{\bar{I}}}, \tag{2.48}$$

so $\overline{\boldsymbol{Q}}_0$ cohomology amounts to restricting to horizontal[17] Dolbeault cohomology groups, while $\overline{\boldsymbol{Q}}_W$ cohomology imposes further algebraic restrictions.

[17] We mean in the sense of the fiber–base geometry of $\boldsymbol{Y}$.

Since the geometry is typically non-compact the $\overline{\boldsymbol{Q}}_0$ cohomology groups are often infinite-dimensional. Fortunately we can obtain a well-defined counting problem because $\overline{\boldsymbol{Q}}_0$ respects the *fine grading* by a vector $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ that assigns grade $\boldsymbol{r}$ to a monomial $\prod_i \phi_i^{r_i}$.[18] Restricting to a particular grade leads to finite-dimensional vector spaces that, as we show in appendix A.3, are readily computable in terms of sheaf cohomology over B. The fine grading is a refinement of the physically relevant grading by $\boldsymbol{q}$ and E, and therefore it gives an effective method for evaluating the first stage in the spectral sequence $E_1^{p,u}$ at fixed twisted sector, $\boldsymbol{q}$, and E.

The next step is to study the $\overline{\boldsymbol{Q}}_W$ cohomology, i.e. the second stage $E_2^{p,u} = H^p_{\overline{\boldsymbol{Q}}_W}\left(H^u_{\overline{\boldsymbol{Q}}_0}(\mathcal{H}^{\bullet,\bullet})\right)$. Once the first two stages of the spectral sequence are determined, we are able to compute the cohomology of $\overline{\boldsymbol{Q}}$; higher derivatives are then determined by standard zig-zag arguments in terms of the two differentials $\overline{\boldsymbol{Q}}_0$ and $\overline{\boldsymbol{Q}}_W$.

The geometric structure depends on the twisted sector, and rather than presenting a universal framework at the price of opaque notation, we will next consider the relevant geometries in three separate situations:

1. The (R,R) sectors: $k \in 2\mathbb{Z}$. In this case since $E_{|k\rangle} = 0$ we can restrict to zero modes for all the fields, which leads to a very transparent structure.

2. The untwisted (NS,R) sector: $k = 1$. This and its CPT conjugate sector $k = 2N - 1$ are the only states with $E_{|k\rangle} = -1$. In this case the geometry is simply $\boldsymbol{Y}$, and the spectrum involves an interplay between non-trivial base and fiber oscillators.

3. (NS,R) sectors with odd k and $E_{|k\rangle} > -1$. In this case the organizing geometry is a sub-bundle of $\boldsymbol{Y} \to B$, and while the choice of sub-bundle is k-dependent,

[18] This grading has a simple physical interpretation: the $W = 0$ theory has n U(1) symmetries that rotate the fiber fields separately.

the spectrum simplifies since base oscillators have $h = 1$ and do not contribute to the massless states.

We consider these possibilities in turn in the next section.

2.5 Twisted sector geometry

To describe the geometric framework for the various twisted sectors we find it useful to distinguish base and fiber fields, with the latter differentiated according to the values of τ_α. More precisely, we split the coordinates $y^\alpha \to (y^{\alpha'}, \phi^A)$, such that $\tau_{\alpha'} < 1$ and $\tau_A \geqslant 1$. The $y^{\alpha'}$ decompose further into base and fiber directions: $y^{\alpha'} = (y^I, \phi^{i'})$, where $\tau_{i'} < 1$ (since $\nu_I = 0$ for all the base fields $\tau_I < 1$ in all sectors). We decompose the bundle X accordingly as $X = X_k \oplus \oplus_A L_A$ and define

$$\boldsymbol{Y}_k \equiv \text{tot}(X_k \xrightarrow{\pi_k} B). \tag{2.49}$$

The utility of this is that the "light" fields, labeled by α', including the corresponding fermions, are organized by $\boldsymbol{Y}_k$, while the remaining "heavy" fields, labeled by A, are organized by the pull-backs $\pi_k^*(L_A)$. The right-moving sector is considerably simpler: we restrict to zero modes, and as we described at the end of section 2.4.3, the only relevant ones are the zero modes $\overline{y}^{\overline{I}}$ and their $\overline{\boldsymbol{Q}}_0$ superpartners $\overline{\eta}^{\overline{I}}$. We now describe how this works in detail in various twisted sectors.

2.5.1 (R,R) sectors

In this case $E_{|k\rangle} = 0$ as a consequence of the left-moving supersymmetry, and to describe the massless states we can restrict to zero modes for all the fields. A look back at the modes in (2.37) and (2.38) shows that the only fields with zero modes are the light fields. Among these the $\rho_{\alpha'}$ also have no zero modes, while the $\chi_{\alpha'}$ zero modes annihilate the vacuum state. Hence the most general state in the truncated Hilbert space is a linear combination of

$$|\Psi_u^s\rangle = \Psi(y', \overline{x})^{\alpha'_1 \cdots \alpha'_s}_{\overline{I}_1 \cdots \overline{I}_u} \overline{\chi}_{\alpha'_1} \overline{\chi}_{\alpha'_2} \cdots \overline{\chi}_{\alpha'_s} \overline{\eta}^{\overline{I}_1} \cdots \overline{\eta}^{\overline{I}_u} |k\rangle \,. \tag{2.50}$$

The fermions $\overline{\chi}_{\alpha'}$ and $\overline{\eta}^{\overline{I}}$ transform respectively as sections of $T^*_{\boldsymbol{Y}_k}$ and $\pi_k^*(\overline{T}_B)$,[19] while $|k\rangle$ is a section of $L_{|k\rangle} = \pi_k^*(\otimes_A L_A^*)$. Hence to be a well-defined state the wavefunction Ψ_u^s must be a $(0,u)$ horizontal form valued in the holomorphic bundle $\mathcal{E}^s = \wedge^s T_{\boldsymbol{Y}_k} \otimes L^*_{|k\rangle}$.

We can decompose the Ψ according to their eigenvalues under the Lie derivative with respect to the restriction of the holomorphic Killing vector V to $\boldsymbol{Y}_k$: $\mathcal{L}_V\Psi = q_\Psi\Psi$.[20] The resulting $|\Psi\rangle$ has well-defined $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ charges:

$$\boldsymbol{q} = \boldsymbol{q}_{|k\rangle} + q_\Psi + s~, \qquad \overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}_{|k\rangle} + q_\Psi + u~. \tag{2.51}$$

$\overline{\boldsymbol{Q}}_0$ acts by sending $\Psi_u^s \to -\overline{\partial}\Psi_{u+1}^s$, and we can use the fine grading described in appendix A.3 to reduce $\overline{\boldsymbol{Q}}_0$ cohomology to computing the finite-dimensional vector spaces $H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}_k, \mathcal{E}^\bullet)$.

The result is still infinite-dimensional, since these cohomology groups will be non-zero for an infinite set of grades $\boldsymbol{r}$. This is a general feature of any sector with bosonic fiber zero modes. Fortunately, the action of $\overline{\boldsymbol{Q}}_W$, which takes the form

$$\overline{\boldsymbol{Q}}_W = W_{\alpha'}(y')\chi^{\alpha'}~, \tag{2.52}$$

restricts the spectrum further. When W is non-singular we expect a finite-dimensional result, and indeed, this is easy to prove for LG models.[21] It would be useful to give a more general proof for hybrids. At any rate, we see from (2.34) that the $\overline{\boldsymbol{Q}}_W$ action on our state is simply

$$\overline{\boldsymbol{Q}}_W : \Psi_u^s \mapsto (sW_{\alpha'_1}\Psi^{\alpha'_1\alpha'_2\cdots\alpha'_s})_u^{s-1}~. \tag{2.53}$$

The spacetime interpretation of these states is either as $\mathfrak{e}_6$ gauginos ($\boldsymbol{q} = \pm 3/2$) or the $\mathbf{16}_{\pm 1/2}$ components of $\mathbf{27}$s and $\overline{\mathbf{27}}$s.

[19] The pull-back to the world-sheet is irrelevant since in the hybrid limit we consider constant maps.

[20] The Lie derivative has a well-defined action even when $L_{|k\rangle}$ is non-trivial because V is a vertical vector, while the transition functions for $L_{|k\rangle}$ only depend on B.

[21] The result follows from the finite-dimensionality of the Koszul cohomology groups associated to the ideal $\langle W_1, \cdots, W_n\rangle \in \mathbb{C}[\phi_1, \ldots, \phi_n]$ for a non-singular superpotential [65, 79].

$\boldsymbol{Y} = B$

As a simple consistency check we can see that we correctly reproduce the expected spectrum from the unique $k = 0$ (R,R) sector when $\boldsymbol{Y} = B$ a compact Calabi-Yau 3-fold. The non-vanshing $\overline{\boldsymbol{Q}}_0$-cohomology classes, given with multiplicities and $(\boldsymbol{q}, \overline{\boldsymbol{q}})$ charges are

$$\begin{array}{cccc} |0\rangle^{\oplus 1}_{-3/2,-3/2} & |\Psi^3_0\rangle^{\oplus 1}_{3/2,-3/2} & |\Psi^0_3\rangle^{\oplus 1}_{-3/2,3/2} & |\Psi^3_3\rangle^{\oplus 1}_{3/2,3/2}\ , \\ |\Psi^1_1\rangle^{\oplus h^1(T)}_{-1/2,-1/2} & |\Psi^2_2\rangle^{\oplus h^1(T)}_{1/2,1/2} & |\Psi^2_1\rangle^{\oplus h^1(T^*)}_{1/2,-1/2} & |\Psi^1_2\rangle^{\oplus h^1(T^*)}_{-1/2,1/2}\ . \end{array} \tag{2.54}$$

Comparing to (2.23), we see that the first line corresponds to the gauginos, while the second line corresponds to the $\overline{\mathbf{16}}_{-1/2}$ and $\mathbf{16}_{1/2}$ components of $h^1(T)$ chiral $\overline{\mathbf{27}}$ and $h^1(T^*)$ chiral $\mathbf{27}$ multiplets.

2.5.2 The $k = 1$ sector

The $k = 1$ sector is untwisted with respect to the LG orbifold action. It has the richest geometric structure and a number of universal features generalizing those observed for the LGO case [11]. Since $\tau_\alpha = 1/2$ for all the fields, the geometry is simply $\boldsymbol{Y}_1 = \boldsymbol{Y}$, while the vacuum bundle $L_{|k\rangle} = K_{\boldsymbol{Y}}$ is trivial. We also have

$$\boldsymbol{q}_{|1\rangle} = 0\ , \qquad \overline{\boldsymbol{q}}_{|1\rangle} = -3/2\ , \qquad E_{|1\rangle} = -1\ . \tag{2.55}$$

Since $E_{|1\rangle} = -1$ massless states may include non-zero modes of ∂y^I and ρ_I.

We now want to describe the operators that create zero-energy states from $|1\rangle$. It turns out that hybrid theories for which some $q_i = 1/2$ have additional zero-energy states that are not found in more generic theories. We will first describe the zero energy states present generically and then turn to the special states available due to fields with $q_i = 1/2$.

Generic $k = 1$ operators

Ignoring multiplets with $q_i = 1/2$, we list the operators that can carry weight $h \leqslant 1$:[22]

$$\mathcal{O}^{1,s} = \Psi^{1s\alpha_1\cdots\alpha_s}(y)\overline{\chi}_{\alpha_1}\cdots\overline{\chi}_{\alpha_s}\ , \qquad \mathcal{O}^2 = \Psi^2_\alpha(y)\chi^\alpha\ , \qquad \mathcal{O}^3 = \Psi^3_{\alpha\beta}(y)\chi^\alpha\chi^\beta\ ,$$

$$\mathcal{O}^4 = \Psi^4_\alpha(y)\partial y^\alpha\ , \qquad \mathcal{O}^5 = \ :\Psi^{5\alpha}_\beta(y)\overline{\chi}_\alpha\chi^\beta:\ ,$$

$$\mathcal{O}^6 = \ :\Psi^{6\alpha}(y)\rho_\alpha + \Psi^{6\alpha}_{,\beta}(y)\overline{\chi}_\alpha\chi^\beta:\ . \tag{2.56}$$

The index s in $\mathcal{O}^{1s}$ can take values $s = 0, 1, 2, 3$. In each case we only indicated the dependence on the left-moving fields; each Ψ also depends on the $\overline{y}$ and $\overline{\eta}$ zero modes:

$$\Psi^t = \sum_{u=0}^{d}(\Psi^t_u)_{\overline{I}_1\cdots\overline{I}_u}\overline{\eta}^{\overline{I}_1}\cdots\overline{\eta}^{\overline{I}_u}\ , \tag{2.57}$$

and plugging in this expansion, we obtain a set of operators $\mathcal{O}^t_u(z)$. We also used the normal ordering that follows from (2.31) to subtract off the $y\rho$ and $\overline{\chi}\chi$ short-distance singularities. Since our free fields are only defined on open sets covering the target space $\boldsymbol{Y}$, just as in the k even case the wavefunctions Ψ^t_0 have to transform as sections of appropriate holomorphic bundles $\mathcal{E}^t$ over $\boldsymbol{Y}$. For instance, the fermi bilinear term appearing in $\mathcal{O}^8$ is chosen to account for the unusual transition function of ρ_α in (2.36). That is, using (2.36), we find that for two patches U_a and U_b with $U_{ab} \neq \varnothing$ $\mathcal{O}^6_b = \mathcal{O}^6_a$ (i.e. $\mathcal{O}^6$ is well-defined) iff Ψ^6_0 transforms as a section of $T_{\boldsymbol{Y}}$. Similarly, the remaining wavefunctions must transform in the expected way, e.g. Ψ^{1s}_0 as a section of $\wedge^s T_{\boldsymbol{Y}}$ and Ψ^2_0 as a section of $T^*_{\boldsymbol{Y}}$. The wavefunctions for $\Psi^t_{u>0}$ transform as (0,u) horizontal forms valued in $\mathcal{E}^t$, and taking $\overline{\boldsymbol{Q}}_0$ cohomology means the Ψ^t_u taken at a fine grade $\boldsymbol{r}$ define classes in $H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}, \mathcal{E}^\bullet)$. As in the k even case we need to consider all $\boldsymbol{r}$ that contain states with $h = 1$ and non-trivial $\overline{\boldsymbol{Q}}_W$ classes. It

[22] Working with fields, as opposed to modes, avoids complications in patching the non-trivial bosonic oscillators on the base. These complications do not arise in sectors with $E_{|k\rangle} > -1$.

is useful to introduce the following notation for the relevant holomorphic bundles $\mathcal{E}^t$:

$$B_{s,t,q} \equiv \wedge^s T_{\boldsymbol{Y}} \otimes \wedge^t T^*_{\boldsymbol{Y}} \otimes \mathrm{Sym}^q(T_{\boldsymbol{Y}}) \ . \tag{2.58}$$

If we grade the wavefunctions by the eigenvalue of the Lie derivative with respect to the symmetry vector V, i.e. $\mathcal{L}_V \Psi^t_u = q\Psi$, then we obtain the following weights, charges and $\overline{\boldsymbol{Q}}_W$ action for these operators: $\overline{\boldsymbol{q}}_{\mathcal{O}} = q + u$, and

$$\begin{array}{ccccccc}
\text{op.} & \mathcal{O}^{1,s}_u & \mathcal{O}^2_u & \mathcal{O}^3_u & \mathcal{O}^4_u & \mathcal{O}^5_u & \mathcal{O}^6_u \\
\boldsymbol{q}_{\mathcal{O}} & q+s & q-1 & q-2 & q & q & q \\
h_{\mathcal{O}} & \frac{q+s}{2} & \frac{q+1}{2} & \frac{q+2}{2} & \frac{q+2}{2} & \frac{q+2}{2} & \frac{q+2}{2} \\
\overline{\boldsymbol{Q}}_W \cdot & sW_{\alpha_1}\Psi^{1s\alpha_1\cdots\alpha_s}\overline{\chi}_{\alpha_2}\cdots\overline{\chi}_{\alpha_s} & 0 & 0 & 0 & \Psi^{5\beta}_{u\gamma}W_\beta\chi^\gamma & \chi^\alpha\partial_\alpha(\Psi^{6\beta}W_\beta)
\end{array} \tag{2.59}$$

Note that for $s > 0$ the $\mathcal{O}^{1,s}$ can carry negative eigenvalues under $\mathcal{L}_V$, but it is not hard to show that they are bounded by $q > -s/2$. Using these operators we create states in the usual fashion: $|\mathcal{O}^t_u\rangle \equiv \lim_{z\to 0} \mathcal{O}^t_u(z)|1\rangle$. They carry energy $E = h_{\mathcal{O}} - 1$ and charges $\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}_{\mathcal{O}} - 3/2$ and $\boldsymbol{q} = \boldsymbol{q}_{\mathcal{O}}$.

Currents

The $h_{\mathcal{O}} = 1$ $\overline{\boldsymbol{q}}_{\mathcal{O}} = 0$ operators in $\overline{\boldsymbol{Q}}$ cohomology are conserved left-moving currents, and in a generic $k = 1$ sector the corresponding states arise in the bottom row of the spectral sequence:

$$|\mathcal{O}^5_0\rangle \oplus |\mathcal{O}^6_0\rangle \xrightarrow{\overline{\boldsymbol{Q}}_W} |\mathcal{O}^2_0\rangle \ , \tag{2.60}$$

where

$$\Psi^5 \in \oplus_{\boldsymbol{r}} H^0_{\boldsymbol{r}}(\boldsymbol{Y}, B_{1,1,0}) \ , \quad \Psi^6 \in \oplus_{\boldsymbol{r}} H^0_{\boldsymbol{r}}(\boldsymbol{Y}, B_{0,0,1}) \ , \quad \Psi^2 \in \oplus_{\boldsymbol{r}} H^0_{\boldsymbol{r}}(\boldsymbol{Y}, B_{1,0,0}) \ . \tag{2.61}$$

Before taking cohomology, there are a number of states here, including, for example, holomorphic vector fields in $H^0(B, T_B)$ that lift to $\boldsymbol{Y}$ or various enhanced R-symmetries of the $W = 0$ theory. Most of these states are lifted by the superpotential couplings. In fact, for a suitably generic W the only current that survives is J_L, which corresponds to $\Psi^5 = \mathbb{1}$ and $\Psi^6 = -V$; the resulting state is $\overline{\boldsymbol{Q}}_W$ closed as a result of $\mathcal{L}_V W = W$. This gaugino corresponds to the linearly realized $\mathfrak{u}(1)_L \subset \mathfrak{e}_6$. For less generic W additional currents may appear, and of course they are accompanied by additional chiral $\overline{\boldsymbol{q}} = -1/2$ states $|\mathcal{O}_0^2\rangle$ in the cokernel of $\overline{\boldsymbol{Q}}_W$. In spacetime each current corresponds to a gauge boson, and the appearance of extra currents reflects the spacetime Higgs mechanism.

$\boldsymbol{Y} = B$

As in the $k = 0$ case, we examine the case of trivial fiber and a CY target space. Taking $\overline{\boldsymbol{Q}}_0$ cohomology on the space of operators in (2.56), we find the following massless states with $\overline{\boldsymbol{q}} < 0$ (for brevity we omit their conjugates with $\overline{\boldsymbol{q}} > 0$)

$$
\begin{array}{ll}
\mathcal{O}^{1,0}, \mathcal{O}_0^5 \to |1\rangle_{0,-3/2}^{\oplus 1} \oplus \overline{\chi}_\alpha \chi^\alpha |1\rangle_{0,-3/2}^{\oplus 1} & \mathbf{45}_0 \oplus \mathbf{1}_0 \\
\mathcal{O}^{1,1}, \mathcal{O}^2 \to |\mathcal{O}_1^{1,1}\rangle_{1,-1/2}^{\oplus h^1(T)} \oplus |\mathcal{O}_1^2\rangle_{-1,-1/2}^{\oplus h^1(T^*)} & \mathbf{10}_1^{\oplus h^1(T)} \oplus \mathbf{10}_{-1}^{\oplus h^1(T^*)} \\
\mathcal{O}^{1,2}, \mathcal{O}^3 \to |\mathcal{O}_1^{1,2}\rangle_{2,-1/2}^{\oplus h^1(\wedge^2 T)} \oplus |\mathcal{O}_1^3\rangle_{-2,-1/2}^{\oplus h^1(\wedge^2 T^*)} & \mathbf{1}_2^{\oplus h^1(T^*)} \oplus \mathbf{1}_{-2}^{\oplus h^1(T)} \\
\mathcal{O}^4, \mathcal{O}_1^5, \mathcal{O}^6 \to |\mathcal{O}_1^4\rangle_{0,-1/2}^{\oplus h^1(T^*)} \oplus |\mathcal{O}_1^5\rangle_{0,-1/2}^{\oplus h^1(\mathrm{End}\,T)} \oplus |\mathcal{O}_1^6\rangle_{0,-1/2}^{\oplus h^1(T)} & \mathbf{1}_0^{\oplus\{h^1(T)+h^1(T^*)+h^1(\mathrm{End}\,T)\}}
\end{array}
$$

It is not hard to extend this analysis to a more general (0,2) CY NLSM with $\mathfrak{su}(n)$ bundle $\mathcal{V} \neq T_B$. In particular, this offers certainly the most direct world-sheet perspective, in the spirit of [40], on marginal gauge-neutral deformations and agrees with spacetime [44, 3] and world-sheet [81] results on marginal deformations in the large radius limit. This may be found in appendix A.4.

Table 2.2: Quantum numbers for the octic model.

k	$E_{\|k\rangle}$	$\boldsymbol{q}_{\|k\rangle}$	$\overline{\boldsymbol{q}}_{\|k\rangle}$	ℓ_k	ν_i	$\widetilde{\nu}_i$	ν_I	$\widetilde{\nu}_I$
0	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	0	0	0	0
1	-1	0	$-\frac{3}{2}$	0	$\frac{1}{8}$	$-\frac{3}{8}$	0	$-\frac{1}{2}$
2	0	$\frac{1}{2}$	$-\frac{3}{2}$	-2	$\frac{1}{4}$	$-\frac{3}{4}$	0	0
3	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{3}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{2}$
4	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0

	ϕ^i	ρ_i	χ^i	$\overline{\chi}_i$
$\boldsymbol{q}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$
$\overline{\boldsymbol{q}}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

A hybrid example

We will now illustrate how to set up the spectrum computation in a simple but non-trivial hybrid. We consider the "octic model"[23] with $B = \mathbb{P}^1$ and $X = \mathcal{O}(-2) \oplus \mathcal{O}^{\oplus 3}$. The quantum numbers of the ground states of the twisted sectors, as well as charges of the fiber fields are given in table 2.2.

In this example as well as those that follow $\operatorname{Pic} B = H^2(B, \mathbb{Z})$, and the vacuum bundle $L_{|k\rangle}$ is determined by a class in $H^2(B, \mathbb{Z})$. We label the class of the dual bundle $L^*_{|k\rangle}$ by $\ell_k \in H^2(B, \mathbb{Z})$. In this example ℓ_k is simply the degree of the line bundle over $\mathbb{P}^1$.

Let us consider as an example the states at $E = 0$ and $\boldsymbol{q} = 2$ in the $k = 1$ sector. We see from (2.23) that these states belong to $\mathbf{1}_2$ of $\mathfrak{so}(10)$. Energy and charge considerations show that the relevant operators from (2.56) are $\mathcal{O}^{1,s}$, and the states

[23] The name comes from the large radius phase of this much-studied example. Let X_0 be an octic hypersurface in the two-parameter toric resolution of the weighted projective space $\mathbb{P}^4_{\{2,2,2,1,1\}}$. The hybrid model $\mathcal{O}(-2) \oplus \mathcal{O}^{\oplus 3} \to \mathbb{P}^1$ arises as one of the phases of the corresponding GLSM [84].

fit in a double complex

$$\begin{array}{cc|cc}
 & & U & \\
\Psi^{\alpha\beta}_{[2]}\overline{\chi}_\alpha\overline{\chi}_\beta|1\rangle & \Psi^{\alpha}_{[5]}\overline{\chi}_\alpha|1\rangle & & 0 \\
 & & & \\
\Psi^{\alpha\beta}_{[2]}\overline{\chi}_\alpha\overline{\chi}_\beta|1\rangle & \Psi^{\alpha}_{[5]}\overline{\chi}_\alpha|1\rangle & & \Psi_{[8]}|1\rangle \\ \hline
-\frac{3}{2} & -\frac{1}{2} & & \frac{1}{2} \quad p
\end{array} \tag{2.62}$$

The wavefunctions satisfy $\mathcal{L}_V\Psi^{\alpha\beta} = 0$ and $\mathcal{L}_V\Psi^{\alpha} = \Psi^{\alpha}$; in practice this means that each $\Psi_{[d]}(y,\overline{y},\overline{\eta})$ is a quasi-homogeneous polynomial of degree d in the fiber bosons ϕ^i if both indices are vertical, while it is of degree $d-1$ is one of the indices is horizontal. To limit clutter in the notation we suppressed the $\overline{\eta}$s; their number is indicated by the U grading. Recall that the horizontal grading is by $p = \boldsymbol{q} - u$.

Taking $\overline{\boldsymbol{Q}}_0$ cohomology at the relevant $\boldsymbol{q}, \overline{\boldsymbol{q}}, E$ eigenvalues indicated by the subscripts, we obtain

$$\begin{array}{cc|cc}
 & & U & \\
[H^1(\boldsymbol{Y}, B_{2,0,0})]_{2,-1/2,0} & [H^1(\boldsymbol{Y}, B_{1,0,0})]_{2,1/2,0} & & 0 \\
 & & & \\
[H^0(\boldsymbol{Y}, B_{2,0,0})]_{2,-3/2,0} & [H^0(\boldsymbol{Y}, B_{1,0,0})]_{2,-1/2,0} & & [H^0(\boldsymbol{Y}, B_{0,0,0})]_{2,1/2,0} \\ \hline
-\frac{3}{2} & -\frac{1}{2} & & \frac{1}{2} \quad p
\end{array} \tag{2.63}$$

To illustrate the counting, we concentrate on the dimension of

$$[H^0(\boldsymbol{Y}, B_{1,0,0})]_{2,-1/2,0} = [H^0(\boldsymbol{Y}, T_{\boldsymbol{Y}})]_{2,-1/2,0} = \bigoplus_{\sum_i r_i = 4} H^0_{\boldsymbol{r}}(\boldsymbol{Y}, T_{\boldsymbol{Y}}) \ . \tag{2.64}$$

The computation is simple since $\boldsymbol{Y} \simeq \boldsymbol{Y'} \times \mathbb{C}^3$, where $\boldsymbol{Y'}$ is the total space of $\mathcal{O}(-2) \to \mathbb{P}^1$. In this case, as we show in appendix (A.3), the non-trivial graded cohomology groups are

$$H^0_{r_1}(\boldsymbol{Y'}, \mathcal{O}_{\boldsymbol{Y'}}) = \mathbb{C}^{2r_1+1}, \qquad H^0_{r_1}(\boldsymbol{Y'}, T_{\boldsymbol{Y'}}) = \mathbb{C}^{4r_1+4} \ . \tag{2.65}$$

Decomposing $(T_{\boldsymbol{Y}})_{\boldsymbol{r}}$ according to (A.30) we find two types of contributions to $H^0_{\boldsymbol{r}}(\boldsymbol{Y}, T_{\boldsymbol{Y}})$, those with $r_i \geqslant 0$, and those with $r_i = -1$ for $i = 2, 3, 4$:

$$\begin{aligned}
[H^0(\boldsymbol{Y}, T_{\boldsymbol{Y}})]_{2,-1/2,0} &= \bigoplus_{r_1=0}^{4} \left[H^0_{r_1}(\boldsymbol{Y'}, T_{\boldsymbol{Y'}}) \oplus H^0_{r_1}(\boldsymbol{Y'}, \mathcal{O}_{\boldsymbol{Y'}})^{\oplus 3}\right] \otimes \mathbb{C}^{\binom{6-r_1}{4-r_1}} \\
&\quad \oplus \left[\bigoplus_{r_1=0}^{5} H^0_{r_1}(\boldsymbol{Y'}, \mathcal{O}_{\boldsymbol{Y'}}) \otimes \mathbb{C}^{6-r_1}\right]^{\oplus 3} \\
&= \mathbb{C}^{595} \oplus \mathbb{C}^{273} = \mathbb{C}^{868}.
\end{aligned} \tag{2.66}$$

The factors of $\binom{6-r_1}{4-r_1}$ and $(6 - r_1)$ arise from counting monomials, respectively, of degree $4 - r_1$ in three variables and $5 - r_1$ in two variables.

Computing the remaining cohomology groups in a similar fashion we obtain the E_1 stage of the spectral sequence

$$\begin{array}{ccccc}
 & & & U & \\
\mathbb{C}^{18} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & \mathbb{C}^{21} & & 0 \\
 & & & & \\
\mathbb{C}^{126} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & \mathbb{C}^{868} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & \mathbb{C}^{825} \\
-\frac{3}{2} & & -\frac{1}{2} & & \frac{1}{2} \quad p
\end{array} \tag{2.67}$$

Finally, we turn to the computation of the $\overline{\boldsymbol{Q}}_W$ cohomology for these states and for simplicity consider the Fermat superpotential

$$W = S_{[8]}(\phi^1)^4 + (\phi^2)^4 + (\phi^3)^4 + (\phi^4)^4 \ , \tag{2.68}$$

where $S_{[8]} \in H^0(\mathbb{P}^1, \mathcal{O}(8))$. From (2.56) we see that for the states appearing at $p = -\frac{3}{2}$

$$\overline{\boldsymbol{Q}}_W \left(\Psi^{\alpha\beta}_{[2]u} \overline{\chi}_\alpha \overline{\chi}_\beta\right) |1\rangle = 2\Psi^{\alpha\beta}_{[2]} W_\beta \overline{\chi}_\alpha |1\rangle \ , \tag{2.69}$$

and the derivatives of the superpotential that appear are ($a = 2, 3, 4$)

$$W_a = 4(\phi^a)^3 \ , \qquad W_1 = 4S_{[8]}(\phi^1)^3 \ , \qquad W_I = \partial_I S_{[8]}(\phi^1)^4 \ . \tag{2.70}$$

The map (2.69) has vanishing kernel, while the $\overline{\boldsymbol{Q}}_W$ action on the $p = -\frac{1}{2}$ states is

$$\overline{\boldsymbol{Q}}_W \left(\Psi^{\alpha}_{[5]} \overline{\chi}_{\alpha} \right) |1\rangle = \Psi^{\alpha}_{[5]} W_{\alpha} \ . \tag{2.71}$$

Setting this to zero implies $\Psi^{\alpha}_{[5]} = \Phi^{\alpha\beta}_{[2]} W_{\beta}$ for some $\Phi^{\alpha\beta}_{[2]}$ anti-symmetric in its indices. Hence the cohomology in the $(p, u) = (-\frac{1}{2}, 0)$ position is trivial, and the spectral sequence degenerates at

$$\begin{array}{ccc|cc}
 & & U\uparrow & & \\
0 & \mathbb{C}^3 & & 0 & \\
 & & & & \\
0 & 0 & & \mathbb{C}^{83} & \\
\hline
-\frac{3}{2} & -\frac{1}{2} & & \frac{1}{2} & p
\end{array} \tag{2.72}$$

Here we count 86 anti-chiral states in the $\mathbf{1}_2$. These correspond to the 83 polynomial and the 3 non-polynomial deformations of complex structure of the octic hypersurface now determined from the hybrid's point of view.

Extra states in $k = 1$

Multiplets with $q_i = \frac{1}{2}$ can potentially give rise to additional massless states. In a LGO theory these genuinely correspond to massive multiplets that can be integrated out without affecting the IR physics. In general this is not so in the hybrid theory: if a $q_i = \frac{1}{2}$ field is non-trivially fibered then its mass vanishes on the discriminant of W in B, and the field cannot be integrated out globally over B. This leads to a rich structure entirely absent from LGO theories.

To describe the additional operators with $h = 1$ we sadly need a little more notation. Just in this section we use the indices i', j', etc. to denote the multiplets with $q_{i'} = \frac{1}{2}$; the $\alpha, \beta, \ldots$ continue to denote all the fields, while $I, J, \ldots$ denote the fields of the base geometry. Let $X_{\frac{1}{2}} \equiv \oplus_{i'} L_{i'}$ and $\mathcal{A}$ be a holomorphic (in fact diagonal) connection on the bundle $X_{\frac{1}{2}} \to B$. The new operators are then

$$\mathcal{O}^7 = \Psi^{7i'j'k'm'}(y^I)\overline{\chi}_{i'}\overline{\chi}_{j'}\overline{\chi}_{k'}\overline{\chi}_{m'}\ , \qquad \mathcal{O}^8 = \ :\Psi^{8i'j'}_I(y^I)\overline{\chi}_{i'}\overline{\chi}_{j'}\chi^I:\ ,$$

$$\mathcal{O}^9 = \ :\Psi^{9i'j'}(y^I)(\rho_{i'} + \chi^I\mathcal{A}^{k'}_{Ii'}\overline{\chi}_{k'})\overline{\chi}_{j'}:\ . \tag{2.73}$$

The wavefunctions are (0,u) forms valued in the following bundles:

$$\Psi^7:\ \wedge^4 X_{\frac{1}{2}}\ , \qquad \Psi^8:\ \wedge^2 X_{\frac{1}{2}} \otimes T^*_B\ , \qquad \Psi^9:\ X_{\frac{1}{2}} \otimes X_{\frac{1}{2}}\ . \tag{2.74}$$

These operators have weight $h = 1$ and charges

$$\begin{array}{cccc} & \mathcal{O}^7_u & \mathcal{O}^8_u & \mathcal{O}^9_u \\ \boldsymbol{q} & 2 & 0 & 0 \\ \overline{\boldsymbol{q}} & u-2 & u-1 & u-1 \end{array} \tag{2.75}$$

The action of $\overline{\boldsymbol{Q}}_0$ on $\mathcal{O}^7$ is simply to send $\Psi^7_u \to (-\bar\partial\Psi^7)_{u+1}$. Since we used the holomorphic connection $\mathcal{A}$ in $\mathcal{O}^9$ to build a well-defined operator, the $\overline{\boldsymbol{Q}}_0$ action on $\mathcal{O}^8_u + \mathcal{O}^9_u$ is a bit more involved:

$$\begin{aligned} \overline{\boldsymbol{Q}}_0 \cdot (\mathcal{O}^8_u + \mathcal{O}^9_u) = &-(\bar\partial\Psi^{9i'j'})_{\overline{I}_0\cdots\overline{I}_u}\overline{\eta}^{\overline{I}_0}\cdots\overline{\eta}^{\overline{I}_u}(\rho_{i'} + \chi^I\mathcal{A}^{k'}_{Ii'}\overline{\chi}_{k'})\overline{\chi}_{j'} \\ &+ \left[\mathrm{obs}(\Psi^9)^{k'j'}_I - \bar\partial\Psi^{8k'j'}_I\right]_{\overline{I}_0\cdots\overline{I}_u}\overline{\eta}^{\overline{I}_0}\cdots\overline{\eta}^{\overline{I}_u}\overline{\chi}_{k'}\overline{\chi}_{j'}\chi^I\ , \end{aligned} \tag{2.76}$$

where the linear map

$$\mathrm{obs}: \Omega^{0,u}(X_{\frac{1}{2}} \otimes X_{\frac{1}{2}}) \to \Omega^{0,u+1}(\wedge^2 X_{\frac{1}{2}} \otimes T^*_B) \tag{2.77}$$

is given by contracting Ψ^9 with the curvature $\mathcal{F} = \bar\partial\mathcal{A}$:

$$\mathrm{obs}(\Psi^9)^{k'j'}_{I\ \overline{I}_0\cdots\overline{I}_u}d\overline{y}^{\overline{I}_0}\cdots d\overline{y}^{\overline{I}_u} \equiv \tfrac{1}{2}\left(\mathcal{F}_{I\overline{I}_0}{}^{k'}{}_{i'}\Psi^{9i'j'}_{\overline{I}_1\cdots\overline{I}_u} - \mathcal{F}_{I\overline{I}_0}{}^{j'}{}_{i'}\Psi^{9i'k'}_{\overline{I}_1\cdots\overline{I}_u}\right)d\overline{y}^{\overline{I}_0}\cdots d\overline{y}^{\overline{I}_u}\ . \tag{2.78}$$

It is easy to see that $\mathrm{obs}(\Psi^9)$ is $\bar\partial$-closed when Ψ^9 is $\bar\partial$-closed, so that $\mathcal{O}^8_u + \mathcal{O}^9_u$ is $\overline{\boldsymbol{Q}}_0$-closed iff $\bar\partial\Psi^9 = 0$ and $\mathrm{obs}(\Psi^9)$ corresponds to the trivial class in $H^{u+1}(B, \wedge^2 X_{\frac{1}{2}} \otimes T^*_B)$. We will meet examples of such possible "obstruction classes" below, but for now we simply note that obs vanishes in a number of important cases that often arise in particular examples. For instance, $\text{obs}(\Psi^9_d)$ is clearly zero, and $\text{obs} = 0$ for any $\Psi^9 \in H^\bullet(B, L_{j'} \otimes L_{j'})$. A little less trivially, we can also show that obs vanishes for any $\Psi^9 \in H^\bullet(B, \wedge^2 X_{\frac{1}{2}})$.

The $\overline{\boldsymbol{Q}}_W$ action can also be determined;[24] the results are:

$$\begin{aligned} \overline{\boldsymbol{Q}}_W \cdot \mathcal{O}^7 &= 4W_{i'}\Psi^{7i'j'k'm'}\overline{\chi}_{j'}\overline{\chi}_{k'}\overline{\chi}_{m'} \ , \qquad \overline{\boldsymbol{Q}}_W \cdot \mathcal{O}^8 = 2W_{i'}\Psi_I^{8i'j'}\overline{\chi}_{j'}\chi^I \ , \\ \overline{\boldsymbol{Q}}_W \cdot \mathcal{O}^9 &= \Psi^{9i'j'}\Big[(\rho_{i'} - \mathcal{A}_I^{i'}\overline{\chi}_{i'}\chi^I)W_{j'} + (\chi^\alpha W_{i'\alpha} - \chi^I \mathcal{A}_{Ii'}^{k'} W_{k'})\overline{\chi}_{j'}\Big] \ . \end{aligned} \tag{2.79}$$

2.5.3 $k > 1$ (NS,R) sectors

Finally, we turn to (NS,R) sectors with $1 < k < 2N - 1$. These sectors have, in general, two complications relative to the $k = 1$ sector: in general $\boldsymbol{Y}_k \neq \boldsymbol{Y}$, and $|k\rangle$ may transform as a section of a nontrivial bundle over the base B.

The vacuum

Recalling the discussion above (2.49), we split the coordinates as $y^\alpha \to (y^I, \phi^{i'}, \phi^A)$. The quantum numbers of the vacuum are then write the vacuum energy as

$$\begin{aligned} E_{|k\rangle} &= -1 + \frac{1}{2}\left[\sum_{i'}(\nu_{i'} - \frac{q_{i'}}{2}) + \sum_A (1 - \frac{q}{2} - \nu_A)\right] \ , \\ \boldsymbol{q}_{|k\rangle} &= \sum_{i'}(\frac{q_{i'}}{2} - \nu_{i'}) + \sum_A (1 - \frac{q_A}{2} - \nu_A) \ , \\ \overline{\boldsymbol{q}}_{|k\rangle} &= \sum_{i'}(\frac{q_{i'}}{2} - \frac{1}{2} - \nu_{i'}) + \sum_A (\nu_A - \frac{q_A}{2} + \frac{1}{2}) - \frac{d}{2} \ , \end{aligned} \tag{2.80}$$

where d is the dimension of the base B. Note that in the twisted sectors $1 < k < 2N - 1$ we have $E_{|k\rangle} > -1$. The vacuum bundle (2.42) is given by

$$L_{|k\rangle} = \otimes_A L_A^* \ . \tag{2.81}$$

[24] A little care is required in using point-splitting and the free OPE in computing the action on $\mathcal{O}^9$.

Modes and transition functions

Because we have $E_k > -1$ we can restrict attention to the subspace of the Hilbert space generated by the lowest modes of the left-moving fields. That is, we truncate (2.37) to

$$
\begin{aligned}
y^\alpha(z) &= z^{\nu_\alpha - q_\alpha/2}(y^\alpha + z^{-1}\rho^{\dagger\alpha}) , & \rho_\alpha(z) &= z^{q_\alpha/2-\nu_\alpha}(\rho_\alpha + z^{-1}y^\dagger_\alpha) , \\
\chi^\alpha(z) &= z^{\widetilde{\nu}_\alpha - q_\alpha/2-\frac{1}{2}}(\chi^\alpha + z^{-1}\overline{\chi}^{\dagger\alpha}) , & \overline{\chi}_\alpha(z) &= z^{q_\alpha/2+\frac{1}{2}-\widetilde{\nu}_\alpha}(\overline{\chi}_\alpha + z^{-1}\chi^\dagger_\alpha) ,
\end{aligned} \tag{2.82}
$$

where in our restricted Hilbert space $\rho_I = 0$.

The transition functions for these oscillators follow by expanding (2.36). These show that $y^{\alpha'}$ are coordinates on $\mathbf{Y}_k$, while $\chi^{\alpha'}$ ($\overline{\chi}_{\alpha'}$) take values in $T_{\mathbf{Y}_k}$ ($T^*_{\mathbf{Y}_k}$). On the other hand, ϕ^A and λ^A take values in $\hat{Z}_k = \pi_k^*(\oplus L_A)$ and ρ_A and $\overline{\lambda}_A$ in $\hat{Z}_k^*$. As is the case for $k=1$, $\rho_{i'}$ is not a covariant operator due to the fermion bilinear term.

Conserved charges

Inserting (2.82) into our expressions for the conserved charges (2.33) we find in our Hilbert space

$$
\begin{aligned}
L_0 &= \sum_\alpha \left[-\nu_\alpha \phi^\alpha\phi^\dagger_\alpha + (1-\nu_\alpha)\rho_\alpha\rho^{\dagger\alpha} + (1+\widetilde{\nu}_\alpha)\chi^\alpha\chi^\dagger_\alpha - \widetilde{\nu}_\alpha\overline{\chi}_\alpha\overline{\chi}^{\dagger\alpha}\right] , \\
J_0 &= \sum_\alpha \left[(q_\alpha - 1)(\chi^\alpha\chi^\dagger_\alpha - \overline{\chi}_\alpha\overline{\chi}^{\dagger\alpha}) - q_\alpha\left(y^\alpha y^\dagger_\alpha + \rho_\alpha\rho^{\dagger\alpha}\right)\right] , \\
\overline{J}_0 &= \sum_\alpha q_\alpha\left(-y^\alpha y^\dagger_\alpha - \rho_\alpha\rho^{\dagger\alpha} + \chi^\alpha\chi^\dagger_\alpha - \overline{\chi}_\alpha\overline{\chi}^{\dagger\alpha}\right) .
\end{aligned}
$$

States

We again list the operators that can carry weight $h<1$, suppressing the right-moving $\overline{\eta}^{\overline{I}}$ dependence. These can contain at most one operator with $h \geqslant \frac{1}{2}$ and we organize them according to the nature of this operator as

$$
\begin{aligned}
\mathcal{O}^{1,l,m} &= \Xi^{1lm}{}^{\alpha',i_2'\cdots i_l'}_{A_1\cdots A_m}\overline{\chi}_{\alpha'}\overline{\chi}_{i_2'}\cdots\overline{\chi}_{i_l'}\chi^{A_1}\cdots\chi^{A_m}\\
\mathcal{O}^{2,l,m} &= \Xi^{2lm}{}^{i_1'\cdots i_l'}_{\alpha',A_2\cdots A_m}\overline{\chi}_{i_1'}\cdots\overline{\chi}_{i_l'}\chi^{\alpha'}\chi^{A_2}\cdots\chi^{A_m}\\
\mathcal{O}^{3,l,m} &= \Xi^{3lm}{}^{B,i_2'\cdots i_l'}_{A_1\cdots A_m}\overline{\chi}_{B}\overline{\chi}_{i_2'}\cdots\overline{\chi}_{i_l'}\chi^{A_1}\cdots\chi^{A_m}\\
\mathcal{O}^{4,l,m} &= \Xi^{4lm}{}^{i_1'\cdots i_l'}_{B,A_1\cdots A_m}\phi^{B}\overline{\chi}_{i_1'}\cdots\overline{\chi}_{i_l'}\chi^{A_1}\cdots\chi^{A_m}\\
\mathcal{O}^{5,l,m} &= \Xi^{5lm}{}^{j',i_1'\cdots i_l'}_{A_1\cdots A_m}\left[\rho_{j'}-\mathcal{A}_{J}{}^{k'}_{j'}\chi^{J}\overline{\chi}_{k'}\right]\overline{\chi}_{i_1'}\cdots\overline{\chi}_{i_l'}\chi^{A_1}\cdots\chi^{A_m}\ .
\end{aligned}
\tag{2.83}
$$

In constructing $\mathcal{O}^5$ we have introduced a holomorphic (and diagonal) connection on $\oplus_{i'}L_{i'}$. Here the Ξ^t include the dependence on $y^{\alpha'}$ and ρ_A, as well as on the right-moving zero modes of $\overline{y}^{\overline{I}}$. We can make this more explicit by writing, for example,

$$
\Xi^{1lm}{}^{\alpha',i_2'\cdots i_l'}_{A_1\cdots A_m} = \sum_{\mathbf{t}}\Psi^{1lm}_{\mathbf{t}}(y)^{\alpha',i_2'\cdots i_l'}_{A_1\cdots A_m}\prod_{B}\rho_B^{t_B+\sum_{a=1}^m\delta_{B,A_a}}\ , \tag{2.84}
$$

in terms of a vector of integers $t_B \geqslant -1$ such that no negative powers of ρ_B appear. $\mathcal{O}^1$ will now create a well-defined state when acting on $|k\rangle$ provided the wavefunction $\Psi^{1lm}_{\mathbf{t}}$ transforms as a section of a suitable bundle $\mathcal{E}^{1lm}_{\mathbf{t}}$ over $\mathbf{Y}_k$

$$
\mathcal{E}^{1lm}_{\mathbf{t}} = T_{\mathbf{Y}_k}\wedge\left(\wedge^{l-1}\pi_k^*(X_k)\right)\otimes\left(\otimes_B(\pi_k^*L_B^{t_B+1})\right)\ . \tag{2.85}
$$

Note that this takes into account the transformation properties of the vacuum (2.81) and that the odd shift in the power of ρ_B is now seen to be sensible. Incorporating the right-moving fermion zero modes, the wavefunction is in general a (0,u) horizontal form valued in this bundle. These can be fine graded as in A.3 by a vector of integers $\mathbf{r}=(r_{\alpha'})$.

Proceeding in an analogous way with the other operators we find that the wavefunctions take values in the following bundles, organized by $\mathbf{t}$ and the fine grading $\mathbf{r}$

$$\begin{aligned}
\mathcal{E}^{1lm}_{\mathbf{t},\mathbf{r}}(k) &= \left[T_{\mathbf{Y}_k} \wedge \left(\wedge^{l-1}\pi_k^*(X_k)\right) \otimes \left(\otimes_A(\pi_k^* L_A^{t_A+1})\right)\right]_{\mathbf{r}} \\
\mathcal{E}^{2lm}_{\mathbf{t},\mathbf{r}}(k) &= \left[\left(\wedge^{l}\pi_k^*(X_k)\right) \otimes T^*_{\mathbf{Y}_k} \otimes \left(\otimes_A(\pi_k^* L_A^{t_A+1})\right)\right]_{\mathbf{r}} \\
\mathcal{E}^{3lm}_{\mathbf{t},\mathbf{r}}(k) &= \oplus_B \left[\left(\wedge^{l-1}\pi_k^*(X_k)\right) \otimes \left(\otimes_A(\pi_k^* L_A^{t_A+1})\right)\right]_{\mathbf{r}} \\
\mathcal{E}^{4lm}_{\mathbf{t},\mathbf{r}}(k) &= \oplus_B \left[\left(\wedge^{l}\pi_k^*(X_k)\right) \otimes \left(\otimes_A(\pi_k^* L_A^{t_A+1})\right)\right]_{\mathbf{r}} \\
\mathcal{E}^{5lm}_{\mathbf{t},\mathbf{r}}(k) &= \left[\pi_k^*(X_k) \otimes \left(\wedge^{l}\pi_k^*(X_k)\right) \otimes \left(\otimes_A(\pi_k^* L_A^{t_A+1})\right)\right]_{\mathbf{r}}
\end{aligned} \tag{2.86}$$

We need to consider all $\mathbf{t}, \mathbf{r}$ that contain states $\mathcal{O}|k\rangle$ with $E = 0$.

$\overline{\boldsymbol{Q}}$ and cohomology

On states of the form $\mathcal{O}^1_u|k\rangle, \ldots, \mathcal{O}^4_u|k\rangle$ $\overline{\boldsymbol{Q}}_0$ acts as $-\bar\partial$ on horizontal (0,u) forms valued in holomorphic bundles over $\mathbf{Y}_k$, and $\overline{\boldsymbol{Q}}_0$ cohomology is the horizontal Dolbeault cohomology. The action on states of the form $\mathcal{O}^5|k\rangle$ has an added term of the sort already familiar from (2.76,2.77) for the "massive" states in the $k = 1$ sector:

$$\overline{\boldsymbol{Q}}_0 \mathcal{O}^5_u|k\rangle = -\overline{\eta}^{\overline{K}} \left[\bar\partial_{\overline{K}} \mathcal{O}^{5j'}_u + \mathcal{F}_{\overline{K}Jk'}{}^{j'} \chi^J \overline{\chi}_{j'} (\Xi^{5k'}_u)^{i'_1\cdots i'_l}_{A_1\cdots A_m} \overline{\chi}_{i'_1} \cdots \overline{\chi}_{i'_l} \chi^{A_1} \cdots \chi^{A_m}\right] |k\rangle \ , \tag{2.87}$$

where $\mathcal{F}$ is the curvature of $\mathcal{A}$. For $\bar\partial$-closed Ψ^5, the additional "obstruction" term is $\bar\partial$-closed and gives a linear map

$$\text{obs} : \Omega^{0,u}(\mathcal{E}^{5l,m}) \to \Omega^{0,u+1}(\mathcal{E}^{4(l+1),m} \otimes \pi_k^* T^*_B) \ . \tag{2.88}$$

If $\text{obs}(\Psi^5)$ is exact, then we can construct a $\overline{\boldsymbol{Q}}_0$-closed state just as we saw in the $k = 1$ case. We have not encountered a nontrivial obstruction term in any of the examples we considered, and in 2.5.4 we argue that this will be the case in any well-defined model.

The action of $\overline{\boldsymbol{Q}}_W$ is given by the mode expansion of

$$\overline{\boldsymbol{Q}}_W = \oint \frac{dz}{2\pi i} \chi^\alpha W_\alpha = \chi^\alpha \oint \frac{dz}{2\pi i} z^{\widetilde{\nu}_\alpha - q_\alpha/2 - 1/2} W_\alpha + \overline{\chi}^{\dagger\alpha} \oint \frac{dz}{2\pi i} z^{\widetilde{\nu}_\alpha - q_\alpha/2 - 3/2} W_\alpha \ , \tag{2.89}$$

where we write

$$W_\alpha = W_\alpha\left(z^{\nu_\beta - q_\beta/2}(\phi^\beta + z^{-1}\rho^{\dagger\beta})\right) \ . \tag{2.90}$$

We can use the homogeneity relation $W_\alpha(\lambda^q\phi^\beta) = \lambda^{1-q_\alpha}W_\alpha(\phi^\beta)$ and simplify this to

$$\overline{\boldsymbol{Q}}_W = \chi^\alpha \oint \frac{dz}{2\pi i} z^{\widetilde{\nu}_\alpha} W_\alpha\left(z^{\nu_\beta}(\phi^\beta + z^{-1}\rho^{\dagger\beta})\right) + \overline{\chi}^{\dagger\alpha} \oint \frac{dz}{2\pi i} z^{\widetilde{\nu}_\alpha - 1} W_\alpha\left(z^{\nu_\beta}(\phi^\beta + z^{-1}\rho^{\dagger\beta})\right) \ . \tag{2.91}$$

2.5.4 Comments on CPT

The spectrum we obtain should be invariant under CPT. This means that for any massless state with charge $(\boldsymbol{q}, \overline{\boldsymbol{q}})$ in the k sector we should find a massless state with charge $(-\boldsymbol{q}, -\overline{\boldsymbol{q}})$ in the $2N-k$ sector. In this section we will discuss how this works for sectors with odd k. To avoid additional notational elaborations we will make the simplifying assumption that $\widetilde{\nu} < 0$ for all fields.[25] As we will now argue, CPT invariance essentially reduces to Serre duality for Dolbeault cohomology on B, as well as a natural dual action of $\overline{\boldsymbol{Q}}_W$.

A pairing on the Hilbert spaces

The two-point function in the CFT is a natural pairing between the conjugate sectors respecting charge conservation and pairing states with the same energy, and given the quantum orbifold symmetry we expect that the Hilbert spaces of states in the $|k\rangle$ and $|2N-k\rangle$ sectors are dual to each other in this way.

From the expressions above it is clear that the vacua satisfy

$$E_{|2N-k\rangle} = E_{|k\rangle}; \qquad (\boldsymbol{q}_{|2N-k\rangle}, \overline{\boldsymbol{q}}_{|2N-k\rangle}) = (-\boldsymbol{q}_{|k\rangle}, d - \overline{\boldsymbol{q}}_{|k\rangle}) \tag{2.92}$$

while the moding in the conjugate sectors is related by

$$\nu_\alpha \leftrightarrow 1 - \nu_\alpha; \qquad \widetilde{\nu}_\alpha \leftrightarrow -1 - \widetilde{\nu}_\alpha \ . \tag{2.93}$$

[25] When this is not the case there are, as in the (R,R) sectors, χ and $\overline{\chi}$ zero-modes. It should be possible to extend the CPT discussion to these situations as well.

This implies that the fields $\phi^{i'}$ for which $\tau = 1/2$ in the k sector have $\tau = 3/2$ in the conjugate $2N-k$ sector, and vice versa, so that we have

$$\begin{aligned} \boldsymbol{Y}_k &= \operatorname{tot}(\oplus_{i'} L_{i'} \to B) & L_{|k\rangle} &= \otimes_A L_A^* \\ \boldsymbol{Y}_{2N-k} &= \operatorname{tot}(\oplus L_A \to B) & L_{|2N-k\rangle} &= \otimes_{i'} L_{i'}^* \ . \end{aligned} \tag{2.94}$$

In particular $L_{|k\rangle} \otimes L_{|2N-k\rangle} = K_B^*$. For any state with weight h and charge $(\boldsymbol{q}, \overline{\boldsymbol{q}})$ in the k sector, we can find a state with the same weight and charge $(-\boldsymbol{q}, d - \overline{\boldsymbol{q}})$ in the $2N-k$ sector by exchanging the oscillator excitations according to

$$y^\alpha \leftrightarrow \rho_\alpha \qquad \chi^\alpha \leftrightarrow \overline{\chi}_\alpha \ . \tag{2.95}$$

This is enough to show that at the level of left-moving oscillators the two-point function leads to a pairing between the state spaces defined above, which respects $\boldsymbol{q}$ and violates $\overline{\boldsymbol{q}}$ by d. If we denote $\mathcal{H}^{tlm}_{\mathbf{t},\mathbf{r}}(k) = \Gamma(\mathcal{E}^{tlm}_{\mathbf{t},\mathbf{r}}(k))$, then the pairing takes the form

$$\begin{aligned} \mathcal{H}^{1\oplus 2\oplus 3lm}{}_{\mathbf{t},\mathbf{r}}(k) \times \mathcal{H}^{1\oplus 2\oplus 3ml}{}_{\mathbf{r},\mathbf{t}}(2N-k) &\to \mathbb{C} \\ \mathcal{H}^{4lm}_{\mathbf{t},\mathbf{r}}(k) \times \mathcal{H}^{5ml}_{\mathbf{r},\mathbf{t}}(2N-k) &\to \mathbb{C} \ , \end{aligned} \tag{2.96}$$

$\overline{\boldsymbol{Q}}_0$ and Serre duality

The pairing descends to $\overline{\boldsymbol{Q}}_0$ cohomology, and in a reasonable physical theory this must be nondegenerate. This will be the case if

$$H^\bullet_{\overline{\boldsymbol{Q}}_0}\left(\mathcal{H}^{lm}_k(\mathbf{t},\mathbf{r})\right) = \left[H^{d-\bullet}_{\overline{\boldsymbol{Q}}_0}\left(\mathcal{H}^{ml}_{2N-k}(\mathbf{r},\mathbf{t})\right)\right]^* \ . \tag{2.97}$$

For the first line in (2.96), in which $\overline{\boldsymbol{Q}}_0$ acts as $-\bar{\partial}$, this is in fact equivalent to Serre duality. For simplicity let's see first how this works in $\mathcal{H}^{111}_{\mathbf{r},\mathbf{t}}$. The fine grading on $H^\bullet(T_{\boldsymbol{Y}_k})$ can be obtained from the long exact sequence (LES) following from the short exact sequence (SES) (A.29)

$$0 \longrightarrow \oplus_i (\pi_k^* L_i)_{\boldsymbol{r}+\boldsymbol{x}_i} \longrightarrow (T_{\boldsymbol{Y}_k})_{\boldsymbol{r}} \longrightarrow (\pi_k^* T_B)_{\boldsymbol{r}} \longrightarrow 0 \ \ , \tag{2.98}$$

which we here encounter twisted by a vector bundle (so still exact) as

$$0 \longrightarrow \oplus_i (\pi_k^* L_i)_{\boldsymbol{r}+\boldsymbol{x}_i} \otimes \widehat{V}_{\mathbf{t}} \longrightarrow (T_{\mathbf{Y}_k})_{\boldsymbol{r}} \otimes \widehat{V}_{\mathbf{t}} \longrightarrow (\pi_k^* T_B)_{\boldsymbol{r}} \otimes \widehat{V}_{\mathbf{t}} \longrightarrow 0 \ , \tag{2.99}$$

where

$$\widehat{V}_{\mathbf{t}} = \pi_k^* \left[\oplus_B \left(\otimes_A (L_A^{t_A+1}) \right) \right] \ . \tag{2.100}$$

The bundles on either end of the SES are pulled back from B, and we can use (A.20) to compute their cohomology. Thus

$$H_{\mathbf{r}}^{\bullet} \left(\mathbf{Y}_k, \oplus_i (\pi_k^* L_i)_{\boldsymbol{r}+\boldsymbol{x}_i} \otimes \widehat{V}_{\mathbf{t}} \right) = H^{\bullet} \left(B, \oplus_{i,B} \left(\otimes_A (L_A^{t_A+1}) \otimes \left(\otimes_j (L_j^*)^{r_j} \right) \right) \right) \ , \tag{2.101}$$

while

$$H_{\mathbf{r}}^{\bullet} \left(\mathbf{Y}_k, (\pi_k^* T_B)_{\boldsymbol{r}} \otimes \widehat{V}_{\mathbf{t}} \right) = H^{\bullet} \left(B, T_B \otimes \left(\oplus_B \left(\otimes_A (L_A^{t_A+1}) \otimes \left(\otimes_j (L_j^*)^{r_j} \right) \right) \right) \right) \ . \tag{2.102}$$

Recalling that $K_B = \otimes_\alpha L_\alpha$, these are Serre dual, respectively, to

$$\begin{aligned} &H^{d-\bullet} \left(B, \oplus_{i,B} \left(\otimes_A (L_A^*)^{t_A} \otimes \left(\otimes_j (L_j)^{r_j+1} \right) \right) \right) \\ &= H^{d-\bullet} \left(\mathbf{Y}_{2N-k}, \oplus_{i,B} \left(\pi_{2N-k}^* (L_A^*)_{\mathbf{t}-\mathbf{y}_A} \otimes \left(\otimes_j (\hat{L}_j^{r_j+1}) \right) \right) \right) \end{aligned} \tag{2.103}$$

and

$$\begin{aligned} &H^{d-\bullet} \left(B, T_B^* \otimes \left(\oplus_B \left(\otimes_A (L_A^*)^{t_A} \otimes \left(\otimes_j (L_j^{r_j+1}) \right) \right) \right) \right) \\ &= H^{d-\bullet} \left(\mathbf{Y}_{2N-k}, \left(\pi_{2N-k}^* T_B^* \right)_{\mathbf{t}} \otimes \left(\oplus_j (\pi_{2N-k}^* (L_j)^{r_j+1}) \right) \right) \ . \end{aligned} \tag{2.104}$$

Inserting this result into the dual LES we find

$$H^{\bullet}((T_{\mathbf{Y}_k})_{\boldsymbol{r}} \otimes \widehat{V}_{\mathbf{t}}) = \left[H^{d-\bullet}((T_{\mathbf{Y}_{2N-k}})_{\mathbf{t}} \otimes \widehat{V}_{\mathbf{r}}) \right]^* \tag{2.105}$$

with a suitable natural definition for $\widehat{V}_{\mathbf{r}}$.

Higher powers of the tangent/cotangent bundles are fine graded by recursively using the same SES and the dual, so recursively applying this argument we find that Serre duality implies CPT in the sense above whenever we can use $\overline{\boldsymbol{Q}}_0 = -\bar{\partial}$. This argument will fail if nontrivial obstruction classes arise in (2.87), because no such obstruction can arise for the dual states in $\mathcal{H}^4$. We conclude that in reasonable physical theories there will be no nontrivial obstructions in the twisted sectors.

$\overline{\boldsymbol{Q}}_W$ and CPT

Given that the cohomology of $\overline{\boldsymbol{Q}}_0$ produces a spectrum consistent with CPT, we can also show that the action of $\overline{\boldsymbol{Q}}_W$ is consistent with this. Consider a monomial in W_α that contributes to $\overline{\boldsymbol{Q}}_W$ in the k sector a term

$$\chi^\alpha \prod_\beta \left[(\phi^\beta)^{m_\beta}(\rho^{\dagger\beta})^{n_\beta}\right] \ . \tag{2.106}$$

This means that

$$\sum_\beta \left[\nu_{\beta,k}(m_\beta + n_\beta) - n_\beta\right] = -\widetilde{\nu}_{\alpha,k} - 1 \ . \tag{2.107}$$

Using (2.93) we see that this implies

$$\sum_\beta \left[\nu_{\beta,2N-k}(m_\beta + n_\beta) - m_\beta\right] = \sum_\beta \left[-\nu_{\beta,k}(m_\beta + n_\beta) + n_\beta\right] = \widetilde{\nu}_{\alpha,k} + 1 = -\widetilde{\nu}_{\alpha,2N-k} \ , \tag{2.108}$$

which means that the same monomial contributes a term

$$\overline{\chi}^{\dagger\alpha} \prod_\beta \left[(\phi^\beta)^{n_\beta}(\rho^{\dagger\beta})^{m_\beta}\right] \tag{2.109}$$

to $\overline{\boldsymbol{Q}}_W$ in the $2N-k$ sector. This acts in precisely the appropriately dual way on the states as mapped above, showing that CPT is maintained as a symmetry after taking $\overline{\boldsymbol{Q}}_W$ cohomology.

2.6 Examples

In this section we will apply the techniques developed in the previous sections to a number of hybrid examples. In each case we will focus on characterizing first order deformations that preserve (0,2) superconformal invariance and the $\mathfrak{e}_8 \oplus \mathfrak{e}_6$ spacetime gauge symmetry.

The infinitesimal deformations which preserve (2,2) symmetry parametrize the tangent space of the (2,2) moduli space. They are not obstructed and in a large radius limit are identified with complex structure and complexified Kähler moduli of the CY. There is a well-known correspondence between the (2,2) moduli and the $\mathfrak{e}_6$-charged matter, and we will borrow the large radius notation by denoting the number of chiral $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s in the hybrid computation by $h^{1,1}$ and $h^{2,1}$ respectively.

More interesting are the deformations which only preserve (0,2) superconformal invariance. The computation of the number of massless gauge singlets associated to these deformations, which we indicate as $\mathcal{M}$, is the main goal of this section. These singlets arise in (NS,R), i.e. the odd k sectors. In the following we will compute $\mathcal{M}$ in three examples that illustrate a number of technical and conceptual points.

1. For the first example we choose the simplest possible base, i.e. $B = \mathbb{P}^1$. This is a good warm-up for more difficult cases and is of interest in its own right since the model can be found as a phase of a GLSM without a large radius limit in its Kähler moduli space. In fact, it can be shown [6] that $h^{1,1} = 1$, and the only other phase is a LGO.

2. In the second example we describe a model in the broader orbi-bundle set-up with $B = \mathbb{P}^3$. It will be clear that most of our discussion above was restricted to the case in which X is a sum of line bundles solely for ease of exposition. This example also give us a chance to compute a higher order differential (it will turn out to be zero).

3. In the last example we consider the case in which one of the line bundles defining X is positive, and $B = \mathbb{F}_0$ is not a projective space.

While our construction does not depend on a GLSM embedding, all of these models do arise as phases of a GLSM. That gives us the possibility to compare the hybrid spectrum with the spectrum known in other phases. What we discover is that while in the hybrid limit extra singlets appear at a particular complex structure or

Table 2.3: Quantum numbers for the $X = \mathcal{O}(-2) \oplus \mathcal{O}^{\oplus 4} \to \mathbb{P}^1$ model.

k	$E_{\|k\rangle}$	$\boldsymbol{q}_{\|k\rangle}$	$\overline{\boldsymbol{q}}_{\|k\rangle}$	ℓ_k	ν_a, ν_1	$\widetilde{\nu}_a, \widetilde{\nu}_1$
0	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	$0, 0$	$0, 0$
1	-1	0	$-\frac{3}{2}$	0	$\frac{1}{8}, \frac{1}{4}$	$-\frac{3}{8}, -\frac{1}{4}$
2	0	$\frac{1}{2}$	$-\frac{3}{2}$	-2	$\frac{1}{4}, \frac{1}{2}$	$-\frac{3}{4}, -\frac{1}{2}$
3	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	-2	$\frac{3}{8}, \frac{3}{4}$	$-\frac{1}{8}, -\frac{3}{4}$
4	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}, 0$	$-\frac{1}{2}, 0$

	ϕ^i, ϕ^1	ρ_i, ρ_1	χ^i, χ^1	$\overline{\chi}_i, \overline{\chi}_1$
$\boldsymbol{q}$	$\frac{1}{4}, \frac{1}{2}$	$-\frac{1}{4}, -\frac{1}{2}$	$-\frac{3}{4}, -\frac{1}{2}$	$\frac{3}{4}, \frac{1}{2}$
$\overline{\boldsymbol{q}}$	$\frac{1}{4}, \frac{1}{2}$	$-\frac{1}{4}, -\frac{1}{2}$	$\frac{1}{4}, \frac{1}{2}$	$-\frac{1}{4}, -\frac{1}{2}$

Kähler form, there is no evidence of world-sheet instanton corrections to masses of $\mathfrak{e}_6$ singlets.

2.6.1 A hybrid with no large radius

We begin with the model $X = \mathcal{O}(-2) \oplus \mathcal{O}^{\oplus 4}$ and $B = \mathbb{P}^1$ with superpotential

$$W = \sum_{p=0}^{2} F_{[2p]} (\phi^1)^p. \tag{2.110}$$

Some notational clarifications are in order: it is convenient to distinguish between the trivial and non-trivial fiber indices, so let $a, b = 2, \ldots, 5$; moreover, let $F_{[d]}$ be a generic polynomial of degree $4 - d$ in the ϕ^a's, whose coefficients belong to $H^0(\mathbb{P}^1, \mathcal{O}(d))$. The left- and right-moving charges for the fields and the quantum numbers of the twisted ground states are summarized in table 2.3.

The orbifold action $\Gamma = \mathbb{Z}_8$ introduces 7 twisted sectors; because of CPT invariance to compute the number of massless $\mathfrak{e}_6$-singlets it is sufficient to study the $k = 1$ and $k = 3$ sectors.

$k = 1$ sector

The E_1 stage of the spectral sequence is obtained by taking $H_{\overline{\boldsymbol{Q}}_0}(\mathcal{H})$ as described in section 2.4.4 and we reproduce here the result, where the subscripts denote the dimension of the respective cohomology groups

$$
E_1^{p,u}: \quad \begin{array}{ccccc}
H^1(\boldsymbol{Y}, B_{1,0,0})_3 & \xrightarrow{\overline{\boldsymbol{Q}}_W} & \begin{array}{c} H^1(\boldsymbol{Y}, B_{0,0,1})_{10} \\ \oplus \\ H^1(\boldsymbol{Y}, B_{1,1,0})_{63} \end{array} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^1(\boldsymbol{Y}, B_{0,1,0})_{35} \\
 & & & & \\
 & & \begin{array}{c} H^0(\boldsymbol{Y}, B_{0,0,1})_{20} \\ \oplus \\ H^0(\boldsymbol{Y}, B_{1,1,0})_{17} \end{array} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, B_{0,1,0})_{176} \\
 \hline
-\frac{5}{2} & & -\frac{3}{2} & & -\frac{1}{2}
\end{array}
\tag{2.111}
$$

The lowest row of the sequence provides an example of the universal structure of currents we indicated above in (2.60), and for generic W the kernel is one-dimensional, corresponding to the $\mathrm{U}(1)_{\mathrm{L}}$ symmetry. By choosing a particular form of the superpotential (2.110) we can increase $\ker \overline{\boldsymbol{Q}}_W$, and the additional vectors correspond to an enhanced symmetry at the special locus in the moduli space.

In order to compute the cohomology of the top row of (2.111) let us list all the states contributing at $E_1^{p,1}$:

$$
V\rho_1\overline{\chi}_1|1\rangle_3 \xrightarrow{\quad\overline{\boldsymbol{Q}}_W\quad} \begin{array}{c}
H_{[2]}\overline{\chi}_1\chi^I|1\rangle_{30} \\ \oplus \\
G_{[1]}\overline{\chi}_b\chi^I|1\rangle_{16} \\ \oplus \\
G_{[1]}\overline{\chi}_1\chi^b|1\rangle_{16} \\ \oplus \\
\Phi_I\phi^1\overline{\chi}_1\chi^I|1\rangle_1 \\ \oplus \\
G_{[2]}\rho_1|1\rangle_{10} \\ \oplus \\
\Psi_I\partial_z y^I|1\rangle_1
\end{array} \xrightarrow{\quad\overline{\boldsymbol{Q}}_W\quad} G_{[4]}\chi^I|1\rangle_{35}
\tag{2.112}
$$

where $G_{[d]}$ and $H_{[d]}$ are generic polynomials of degree d in the ϕ^a's with coefficients in $H^1(\mathbb{P}^1, \mathcal{O}(-2))$ and $H^1(\mathbb{P}^1, \mathcal{O}(-4))$, respectively, while $\Psi_I, \Phi_I \in H^1(\mathbb{P}^1, \mathcal{O}(-2))$.

First, consider the map on the left. We have the state $V\rho_1\overline{\chi}_1|1\rangle$ where $V \in H^1(\boldsymbol{Y}, T_{\boldsymbol{Y}} \otimes T_{\boldsymbol{Y}}) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-4))$. Under $\overline{\boldsymbol{Q}}_W$ it maps to

$$\overline{\boldsymbol{Q}}_W V\rho_1\overline{\chi}_1|1\rangle = V\left(\partial_1 W\rho_1 + \partial_{11}W\chi^1\overline{\chi}_1 + \partial_{1I}W\chi^I\overline{\chi}_1\right)|1\rangle + V\left(\partial_{1a}W\chi^a\overline{\chi}_1\right)|1\rangle\ . \tag{2.113}$$

Since $\partial_1 W, \partial_{1a}W \in \Gamma(\mathbb{P}^1, \mathcal{O}(2))$, it follows that $V\partial_1 W, V\partial_{1a}W \in H^1(\mathbb{P}^1, \mathcal{O}(-2))$. To compute the dimension of the cokernel of this map we first note that if we restrict the superpotential to its Fermat form, namely $W = \sum_{i=2}^5(\phi^i)^4 + S_{[4]}(\phi^1)^2$, we have

$$\overline{\boldsymbol{Q}}_W V\rho_1\overline{\chi}_1|1\rangle = 2V\left(\phi^1 S_{[4]}\rho_1 + S_{[4]}\chi^1\overline{\chi}_1 + \phi^1\partial_I S_{[4]}\chi^I\overline{\chi}_1\right)|1\rangle\ . \tag{2.114}$$

Since $V\phi^1 S_{[4]} \in H^1(\mathbb{P}^1, \mathcal{O}(-2))$ and $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$ the kernel at Fermat is 2-dimensional.

Adding to W a term of the form $S_{[2]}\phi^2\phi^3\phi^1 + T_{[2]}\phi^4\phi^5\phi^1$, where $S_{[2]}, T_{[2]} \in \Gamma(B, \mathcal{O}(2))$, we find that (2.113) reads

$$\begin{aligned}\overline{\boldsymbol{Q}}_W V\rho_1\overline{\chi}_1|1\rangle = \underbrace{V\left(\partial_1 W\rho_1 + \partial_{11}W\chi^1\overline{\chi}_1 + \partial_{1I}W\chi^I\overline{\chi}_1\right)|1\rangle}_{\text{Fermat}} \\ + VS_{[2]}\left(\phi^2\chi^3 + \phi^3\chi^2\right)\overline{\chi}_1|1\rangle + VT_{[2]}\left(\phi^4\chi^5 + \phi^5\chi^4\right)\overline{\chi}_1|1\rangle\ ,\end{aligned} \tag{2.115}$$

and the map is injective for W generic enough. Now, for the map on the right in (2.112) we have

$$\begin{aligned}\overline{\boldsymbol{Q}}_W\left(\Psi_{ab}\rho_1 + \Psi_{ab,I}\chi^I\overline{\chi}_1\right)\phi^a\phi^b|1\rangle &= \partial_\alpha\left(\Psi_{ab}\partial_1 W\right)\chi^\alpha\phi^a\phi^b|1\rangle \\ \overline{\boldsymbol{Q}}_W\Sigma_{abI}\chi^I\overline{\chi}_1\phi^a\phi^b|1\rangle &= -\Sigma_{abI}\chi^I\partial_1 W\phi^a\phi^b|1\rangle \\ \overline{\boldsymbol{Q}}_W V_a^b\phi^a\chi^I\overline{\chi}_b|1\rangle &= -V_a^b\phi^a\partial_b W\chi^I|1\rangle \\ \overline{\boldsymbol{Q}}_W\Phi_I\phi^1\chi^I\overline{\chi}_1|1\rangle &= -\Phi_I\phi^1\partial_1 W\chi^I|1\rangle\end{aligned} \tag{2.116}$$

The cokernel of this map is thus any object of the form $\Psi_{abcd}\phi^a\phi^b\phi^c\phi^d\chi^I|1\rangle$ for $\Psi_{abcd} \in H^1(\mathbb{P}^1, \mathcal{O}(-2))$, which cannot be written as $\partial_1 W\phi^a\phi^b\chi^I|1\rangle$ or $\phi^a\partial_b W\chi^I|1\rangle$.

We find a 9-dimensional space. Thus, the E_2 stage of the spectral sequence is

$$
E_2^{p,u}: \quad \begin{array}{c|cccc}
U & & & & \\
 & 0 & \mathbb{C}^{45} & \mathbb{C}^{9} & \\
 & & \mathbb{C}^{1} & \mathbb{C}^{139} & \\
\hline
 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & p
\end{array}
\tag{2.117}
$$

and obviously all higher differentials vanish. Hence the spectral sequence degenerates already at this stage, $E_\infty = E_2$. Thus, in this sector we count $45 + 139 = 184$ chiral and 9 anti-chiral $\mathfrak{e}_6$-singlets.

$k = 3$ sector

The $k = 3$ ground state has a non-trivial vacuum bundle $L_{|3\rangle} = \mathcal{O}(2)$ and, as discussed in section 2.5, we must distinguish between light and heavy fields. In particular we have $A = 1$, $i' = 2, \ldots, 5$, $\alpha' = (I, i')$, while the geometry is determined by $\boldsymbol{Y}_3$, the total space of $\mathcal{O}^{\oplus 4} \xrightarrow{\pi_3} \mathbb{P}^1$. The expansion of $\overline{\boldsymbol{Q}}_W$ in this sector takes the form

$$
\overline{\boldsymbol{Q}}_W = \overline{\chi}^{A\dagger}\partial_A W + \chi^A \partial_{Ai} W \rho^{i\dagger} + \overline{\chi}^{\alpha'\dagger}\partial_{\alpha' i} W \rho^{i\dagger} + \chi^{\alpha'}\partial_{\alpha' ij} W \rho^{i\dagger}\rho^{j\dagger} \ . \tag{2.118}
$$

The E_1 stage of the spectral sequence is given by

$$
E_1^{p,u}: \quad \begin{array}{cc|c}
 & & U \\
H^1(\boldsymbol{Y}_3, \pi_3^*(L_6^*) \otimes \wedge^2 T_{\boldsymbol{Y}_3})_{18} \xrightarrow{\ \overline{\boldsymbol{Q}}_W\ } & H^1(\boldsymbol{Y}_3, T_{\boldsymbol{Y}_3})_{16} & \\
0 & \begin{array}{c} H^0(\boldsymbol{Y}_3, \wedge^2 T_{\boldsymbol{Y}_3} \otimes T^*_{\boldsymbol{Y}_3})_6 \\ \oplus \\ H^0(\boldsymbol{Y}_3, T_{\boldsymbol{Y}_3})_1 \end{array} & \\
\hline
-\frac{3}{2} & -\frac{1}{2} & p
\end{array}
\tag{2.119}
$$

Now, the only non-trivial map is at $U = 1$, where

$$
\overline{\boldsymbol{Q}}_W V^{ab}\rho_1 \overline{\chi}_a \overline{\chi}_b |3\rangle = 2V^{ab}\rho_1 \partial_{1a} W \overline{\chi}_b |3\rangle \neq 0 \ . \tag{2.120}
$$

The RHS never vanishes, giving a 6-dimensional image. Hence, the spectral sequence degenerates at the E_2 term

$$E_2^{p,u}: \quad \begin{array}{c|cc|l} & & & U \uparrow \\ & \mathbb{C}^{12} & \mathbb{C}^{10} & \\ & 0 & \mathbb{C}^{7} & \\ \hline & -\frac{3}{2} & -\frac{1}{2} & \to p \end{array} \tag{2.121}$$

Hence we count 19 chiral and 10 anti-chiral states for a total of 222 $\mathfrak{e}_6$ singlets. By similar methods we compute $h^{2,1} = 61$ and $h^{1,1} = 1$, yielding $\mathcal{M} = 160$.

2.6.2 The orbi-bundle

Now we present an example in which X is not a sum of line bundles, but a more general orbi-bundle. Let us take $B = \mathbb{P}^3$ and $X = \mathcal{O}(-5/2) \oplus \mathcal{O}(-3/2)$ along with the quasi-homogeneous superpotential

$$W = S_5(\phi^1)^2 + S_4\phi^1\phi^2 + S_3(\phi^2)^2, \tag{2.122}$$

where $S_d \in H^0(B, \mathcal{O}(d))$. The ground state quantum numbers and charges of the fields are given in table 2.4, and to find the singlets we need only consider the first twisted sector.

Table 2.4: Quantum numbers for the $X = \mathcal{O}(-5/2) \oplus \mathcal{O}(-3/2) \to \mathbb{P}^3$ model.

k	$E_{\lvert k\rangle}$	$\boldsymbol{q}_{\lvert k\rangle}$	$\overline{\boldsymbol{q}}_{\lvert k\rangle}$	ℓ_k	ν_i	$\widetilde{\nu}_i$
0	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	0	0
1	-1	0	$-\frac{3}{2}$	0	$\frac{1}{4}$	$-\frac{1}{4}$
2	0	$\frac{1}{2}$	$-\frac{3}{2}$	-4	$\frac{1}{2}$	$-\frac{1}{2}$

	ϕ^i	ρ_i	χ^i	$\overline{\chi}_i$
$\boldsymbol{q}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\overline{\boldsymbol{q}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

The first stage of the spectral sequence is

$$E_1^{p,u}: \quad \begin{array}{c|cccc} U & & & & \\ & \begin{matrix} H^3(\boldsymbol{Y}, B_{1,0,1})_6 \\ \oplus \\ H^3(\boldsymbol{Y}, B_{2,1,0})_{15} \end{matrix} & & & \\ & 0 & 0 & & \\ & 0 & H^1(\boldsymbol{Y}, B_{1,1,0})_2 & 0 & \\ & 0 & \begin{matrix} H^0(\boldsymbol{Y}, B_{0,0,1})_{21} \\ \oplus \\ H^0(\boldsymbol{Y}, B_{1,1,0})_6 \end{matrix} & \xrightarrow{\overline{\boldsymbol{Q}}_W} H^0(\boldsymbol{Y}, B_{0,1,0})_{295} & \\ \hline & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & p \end{array} \tag{2.123}$$

The bottom row is the only place where we can have cokernel, and for generic superpotential we find $\dim\ker \overline{\boldsymbol{Q}}_W = 1$.

Thus, the E_2 stage of the spectral sequence is

$$E_2^{p,u}: \quad \begin{array}{cccc|cc} & & & U & & \\ \mathbb{C}^{21} & & & & & \\ 0 & 0 & & & & \\ 0 & \mathbb{C}^2 & 0 & & & \\ 0 & \mathbb{C} & \mathbb{C}^{269} & & 0 & \\ \hline -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & & \frac{1}{2} & p \end{array} \tag{2.124}$$

All higher differentials vanish, and the spectral sequence degenerates at the E_2 term. We then count 271 chiral and 21 antichiral states corresponding to massless $\mathfrak{e}_6$ singlets. We also computed by similar methods the number of charged singlets, $h^{2,1} = 90$ and $h^{1,1} = 2$, corresponding to the (2,2) moduli, which we can subtract from the total number of neutral singlets to find $\mathcal{M} = 200$.

A higher differential ?

It is worth noting that the spectral sequence for computing the number of $\mathbf{1}_2 \subset \mathbf{27}$ states degenerates only at the E_4 term, giving us an example of a possible higher differential. At zero energy and $\boldsymbol{q} = 2$ we have

$$
E_1^{p,u}: \quad
\begin{array}{c|cccc}
U & & & & \\
 & 0 & & & \\
 & H^2(\boldsymbol{Y}, B^1_{3,0,0})_1 & 0 & & \\
 & 0 & 0 & 0 & \\
 & 0 & H^0(\boldsymbol{Y}, B_{2,0,0})_{120} \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, B_{1,0,0})_{905} \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, B_{0,0,0})_{875} \\
\hline
p & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array}
\tag{2.125}
$$

Trivially $d_2 = 0$, thus $E_3 = E_2$, but there is one more map we have to compute, in fact

$$
E_3^{p,u}: \quad
\begin{tikzcd}
0 & & & \\
\mathbb{C} \arrow[dddrrr, "d_3"] & 0 & & \\
0 & 0 & 0 & \\
0 & 0 & 0 & \mathbb{C}^{90} \\
-\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2}
\end{tikzcd}
\tag{2.126}
$$

Let us recall that an element $b \in \mathcal{H}$ represents a cohomology class in E_3 if there exist $c_1, c_2 \in \mathcal{H}$ such that

$$\overline{\boldsymbol{Q}}_0 b = 0 \ , \qquad \overline{\boldsymbol{Q}}_W b = \overline{\boldsymbol{Q}}_0 c_1 \ , \qquad \overline{\boldsymbol{Q}}_W c_1 = \overline{\boldsymbol{Q}}_0 c_2 \ , \tag{2.127}$$

and d_3 on the cohomology class $[b]_3$ is given by

$$d_3[b]_3 = [\overline{\boldsymbol{Q}}_W c_2]_3 \ . \tag{2.128}$$

Thus, we just chase down the state $\overline{\eta}^{\overline{J}}\overline{\eta}^{\overline{K}} V^{I}_{\overline{JK}} \overline{\chi}_1\overline{\chi}_2\overline{\chi}_I|1\rangle \in E_3^{-5/2,3}$ as prescribed in (2.128)

$$\begin{tikzcd}
0 & 0 & & \\
\overline{\eta}^{\overline{J}}\overline{\eta}^{\overline{K}} V^{I}_{\overline{JK}} \overline{\chi}_1\overline{\chi}_2\overline{\chi}_I|1\rangle \arrow[u, "\overline{\boldsymbol{Q}}_0"] \arrow[r, "\overline{\boldsymbol{Q}}_W"] & \overline{\eta}^{\overline{J}}\overline{\eta}^{\overline{K}} V^{I}_{\overline{JK}} \epsilon^{\alpha\beta\gamma}\partial_\alpha W \overline{\chi}_\beta\overline{\chi}_\gamma|1\rangle \arrow[u, "\overline{\boldsymbol{Q}}_0"] & & \\
 & \overline{\eta}^{\overline{J}} S^{I}_{\overline{J}} \epsilon^{\alpha\beta\gamma}\partial_\alpha W \overline{\chi}_\beta\overline{\chi}_\gamma|1\rangle \arrow[u, "\overline{\boldsymbol{Q}}_0"] \arrow[r, "\overline{\boldsymbol{Q}}_W"] & \overline{\eta}^{\overline{J}} S^{I}_{\overline{J}} \epsilon^{\alpha\beta\gamma}\partial_\alpha W \partial_{[\beta} W \overline{\chi}_{\gamma]}|1\rangle & \\
 & & R^I \epsilon^{\alpha\beta\gamma}\partial_\alpha W \partial_{[\beta} W \overline{\chi}_{\gamma]}|1\rangle \arrow[u, "\overline{\boldsymbol{Q}}_0"] \arrow[r, "\overline{\boldsymbol{Q}}_W"] & 0 \ .
\end{tikzcd} \tag{2.129}$$

The coefficients satisfy

$$V^{I}_{\overline{JK}} = -(\bar{\partial} S)^{I}_{\overline{JK}} \ , \qquad S^{I}_{\overline{J}} = -(\bar{\partial} R)^{I}_{\overline{J}} \ . \tag{2.130}$$

We just showed that d_3, while in principle allowed, vanishes, and the spectral sequence degenerates at the $E_4 = E_2$ term. In this sector we count $h^{2,1} = 90$ and $h^{1,1} = 1$ and the "missing" Kähler modulus is to be found in the $k = 3$ sector, as expected.

2.6.3 A positive line bundle

For our last example we consider $X = \mathcal{O}(-3,-3) \oplus \mathcal{O}(1,1)$ and $B = \mathbb{F}_0$. The novelty here is that we allow a positive line bundle over a non-projective base.

A non degenerate superpotential is given by

$$W = (\phi^1)^4 S_{[12,12]} + (\phi^1)^3 \phi^2 S_{[8,8]} + (\phi^1 \phi^2)^2 S_{[4,4]} + \phi^1 (\phi^2)^3 S_{[0,0]} \ , \tag{2.131}$$

where $S_{[m,n]} \in \Gamma(\mathbb{F}_0, \mathcal{O}(m,n))$ and the quantum numbers for this theory are listed in table 2.5. Studying the (R,R) sectors we find $h^{1,1} = 3$ and $h^{2,1} = 243$, and to count the remaining $\mathfrak{e}_6$ singlets we need to consider the $k = 1$ and $k = 3$ sectors.

$k = 1$ *sector*

In the first twisted sector the spectral sequence at $\boldsymbol{q} = 0$ is

$$E_1^{p,u} : \quad \begin{array}{ccc|l}
 & & & U \uparrow \\
\begin{matrix} H^2(\boldsymbol{Y}, B_{1,1,0})_{39} \\ \oplus \\ H^2(\boldsymbol{Y}, B_{0,0,1})_9 \end{matrix} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^2(\boldsymbol{Y}, B_{0,1,0})_{39} & \\
 & & & \\
H^1(\boldsymbol{Y}, B_{1,1,0})_{10} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^1(\boldsymbol{Y}, B_{0,1,0})_2 & \\
 & & & \\
\begin{matrix} H^0(\boldsymbol{Y}, B_{1,1,0})_{63} \\ \oplus \\ H^0(\boldsymbol{Y}, B_{0,0,1})_{27} \end{matrix} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, B_{0,1,0})_{825} & \\
\hline
-\frac{3}{2} & & -\frac{1}{2} & \quad p \rightarrow
\end{array} \tag{2.132}$$

It is not hard to verify that both the maps $\overline{\boldsymbol{Q}}_W\big|_{U=1}$ and $\overline{\boldsymbol{Q}}_W\big|_{U=2}$ are surjective for sufficiently generic W, and as we already saw in the discussion about the general $k = 1$ sector, there is only one state at $\overline{\boldsymbol{q}} = -\frac{3}{2}, u = 0$ in the kernel of $\overline{\boldsymbol{Q}}_W$. The spectral sequence degenerates at the E_2 term

$$E_2^{p,u} : \quad \begin{array}{cc|l}
 & & U \uparrow \\
\mathbb{C}^9 & & \\
\mathbb{C}^8 & 0 & \\
\mathbb{C} & \mathbb{C}^{736} & \\
\hline
-\frac{3}{2} & -\frac{1}{2} & \quad p \rightarrow
\end{array} \tag{2.133}$$

Thus we count 744 chiral and 9 anti-chiral massless $\mathfrak{e}_6$-singlets and one vector.

Table 2.5: Quantum numbers for the $X = \mathcal{O}(-3,-3) \oplus \mathcal{O}(1,1) \to \mathbb{F}_0$ model.

k	$E_{\vert k\rangle}$	$\boldsymbol{q}_{\vert k\rangle}$	$\overline{\boldsymbol{q}}_{\vert k\rangle}$	ℓ_k	ν_i	$\widetilde{\nu}_i$
0	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	0	0
1	-1	0	$-\frac{3}{2}$	0	$\frac{1}{8}$	$-\frac{3}{8}$
2	0	$\frac{1}{2}$	$-\frac{3}{2}$	$(-2,-2)$	$\frac{1}{4}$	$-\frac{3}{4}$
3	$-\frac{3}{4}$	$-\frac{1}{2}$	-1	0	$\frac{3}{8}$	$-\frac{1}{8}$
4	0	-1	-1	$(-2,-2)$	$\frac{1}{2}$	$-\frac{1}{2}$

	ϕ^i	ρ_i	χ^i	$\overline{\chi}_i$
$\boldsymbol{q}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$
$\overline{\boldsymbol{q}}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

$k = 3$ sector

In the $k = 3$ sector all the fields are "light", $L_{|3\rangle}$ is trivial, and the geometry is again encoded in the full $\boldsymbol{Y}_3 = \boldsymbol{Y}$. The spectral sequence starts then as

$$E_1^{p,u} : \quad \begin{array}{ccc|}
\begin{array}{c} H^2(\boldsymbol{Y}, \mathrm{Sym}^2\, T_{\boldsymbol{Y}})_{27} \\ \oplus \\ H^2(\boldsymbol{Y}, \wedge^2 T_{\boldsymbol{Y}} \otimes T^*_{\boldsymbol{Y}})_6 \end{array} & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, \mathcal{O})_1 \\
0 & & 0 \\
H^0(\boldsymbol{Y}, \mathrm{Sym}^2\, T_{\boldsymbol{Y}})_9 & \xrightarrow{\overline{\boldsymbol{Q}}_W} & H^0(\boldsymbol{Y}, \mathcal{O})_{58} \\ \hline
-\frac{3}{2} & & -\frac{1}{2}
\end{array} \; p \tag{2.134}$$

It can be shown that the map $\overline{\boldsymbol{Q}}_W\big|_{U=0}$ is injective while the map $\overline{\boldsymbol{Q}}_W\big|_{U=2}$ is surjective. Therefore the second stage of the spectral sequence is

$$E_2^{p,u} : \quad \begin{array}{cc|}
\mathbb{C}^{32} & 0 \\
0 & 0 \\
0 & \mathbb{C}^{49} \\ \hline
-\frac{3}{2} & -\frac{1}{2}
\end{array} \; p \tag{2.135}$$

The spectral sequence degenerates at the E_2 term, and we find 49 chiral and 32 anti-chiral singlets.

Summarizing, we count 834 massless chiral $\mathfrak{e}_6$-singlets and once we subtract the moduli we obtain $\mathcal{M} = 588$.

2.7 Discussion

We have described a class of perturbative vacua for heterotic string compactifications and a limit in which their properties are computable. We have illustrated these computations in models with (2,2) world-sheet supersymmetry, although the methods clearly extend to more general (0,2) theories.

Our class of (2,2) models fits in with a number of other constructions. To describe this we proceed in increasing dimension d of the base B and assume this is Fano. For $d = 1$ this means $B = \mathbb{P}^1$ and the $c = 6$ LGO theory on the fiber determines a one-parameter family of K3 compactifications. Models with no large radius limit in the Kähler moduli space, such as the first example in section 2.6, are obtained when the monodromies of the family are not simultaneously geometrical in any duality frame. It seems likely that any such model would be obtained as a limit in some GLSM, but we have not shown this.

For $d = 2$ the base is a del Pezzo surface and the $c = 3$ LGO theory on the fiber can be interpreted as determining in Weierstrass form an elliptic fibration over B. This can be smooth if the discriminant is nonsingular in B, in which case the model will have a large-radius limit. It is not clear how to construct a GLSM embedding for a hybrid with a non-toric base.

For $d = 3$ there are many possible choices for B, but the $c = 0$ LGO theory is quadratic, and hence appears to be trivial. Since the fiber fields are massive at generic points on the base, one might think the low-energy theory would be a NLSM with target space B, but this cannot be correct, as this would not be conformally invariant. This naïve discussion omits the orbifold action. Since at low energy there are no excitations in the fiber direction, one can try [2] to describe the resulting model as a NLSM with target space a double cover of B branched over the singular locus of $W(y, \phi)$ *considered as a function of ϕ only*. This leads to a geometric interpretation of the limiting point we called the hybrid limit. It is not directly related to a symplectic quotient construction and, if the model has a large-radius limit, it is not birational to the target space at this limit. The relationship between the two descriptions is unclear. It would be interesting to study, among other things, the behavior of the D-brane spectrum and moduli in a type-IIA compactification near such a hybrid limit.

The models we have studied have been "good" hybrids, in which the R-symmetry does not act on the base. Limiting points of GLSMs often produce hybrids for which this does not hold. The hybrid limit for "good" hybrids is expected to lie at infinite distance in the moduli space of SCFTs; it should be possible to determine the approximate moduli space metric in the hybrid limit. We expect that the approximation should improve as the hybrid limit is approached and the distance to the hybrid limit deep in the Kähler cone of B will diverge. It would be interesting to verify this in detail. In [8] "pseudo" hybrids were defined as hybrid limits lying at finite distance; the behavior of the D-brane spectrum near these limits was found to be quite different from that expected near a "good" hybrid. It seems natural to conjecture that "good" hybrids and "true" (not "pseudo") hybrid limits coincide.

Although we focused on models with (2,2) world-sheet supersymmetry, the methods extend naturally to a much larger class of models with (0,2) supersymmetry. This larger class presents an array of interesting questions. As a first foray in this direction, the massless $\mathfrak{e}_6$ singlets in (NS,R) sectors belong to (anti-) chiral multiplets containing massless scalars. Expectation values for these represent marginal deformations of the world-sheet SCFT preserving (0,2) supersymmetry. We do not at present have effective techniques to determine which of these are exactly marginal, and the structure of the moduli space of (0,2) SCFTs is still largely unknown.

In general one expects [35, 40] that away from the hybrid limit the (0,2) models we construct will be destabilized by world-sheet instantons wrapping cycles in B. In some classes of models this expectation has been thwarted, and the anticipated corrections are absent [91, 16, 17]. Even in cases in which no known argument precludes such corrections they have been found less generally than one might expect [11, 7]. It would be very interesting to investigate this issue in the context of hybrid models, in which the structure of the relevant instantons – associated to rational curves in B rather than in a Calabi–Yau threefold, may provide a simpler context for their study.

More generally, we can construct (0,2) hybrid models that are not deformations of (2,2) models by taking the left-moving fermions to be sections of a holomorphic bundle $\mathcal{E} \to \boldsymbol{Y}$ and a (0,2) superpotential given by a section $J \in \Gamma(\mathcal{E}^*)$ with $J^{-1}(0) = B$. It is to be expected that most such models will not have a limit in which they are described by a (0,2) NLSM or one in which they reduce to a (0,2) LGO theory, so that these will determine a large class of new perturbative vacua of the heterotic string.

3

Accidents in (0,2) Landau-Ginzburg models

3.1 Introduction

The construction and classification of conformal field theories (CFTs) plays a key role in modern quantum field theory. One approach is to solve the conformal bootstrap.[1] Another approach that has proven useful is to study the low energy (or IR) limits of renormalization group (RG) flows from known CFTs. This has challenges of its own, since the IR dynamics often involves emergent degrees of freedom and interactions.

Nevertheless, as already indicated in the seminal work of [100, 63, 75, 93], it is often possible to identify certain classes of operators and their OPEs and correlators of an IR CFT with corresponding objects in terms of the UV degrees of freedom. This is especially useful when the UV theory is asymptotically free, since then perturbative computations can provide information about a non-trivial CFT without a notion of a weak coupling. The identification of UV and IR data is simplified when some amount of supersymmetry is preserved along the RG flow: SUSY constraints lead to

[1] This is difficult in practice and can be carried out analytically only in theories with enormously enlarged symmetries like the W_N algebras of the minimal models [22]. Recently progress has been made with numeric techniques, for example in applications to the three-dimensional Ising model [48, 47].

well-known simplifications such as the relation between dimensions and R-charges of chiral operators and non-renormalization theorems. For instance, in two dimensional theories with (2,2) SUSY these simplifications are responsible for many well-known phenomena such as mirror symmetry and the Calabi-Yau (CY) / Landau-Ginzburg (LG) correspondence [56], and the identification of UV and IR data is a key tool in exploration and exploitation of these two-dimensional gems.

Such techniques rely on the assumption that accidental symmetries that might emerge in the IR limit do not invalidate the identification of operators in the IR with their UV avatars. This assumption is well-tested in (2,2) theories but is also often applied to theories with only (0,2) supersymmetry. For instance, it is key to various gauged linear sigma model constructions of (0,2) CFTs corresponding to heterotic string vacua [97, 41, 91, 39].

In this chapter we show that the assumption cannot be taken for granted in (0,2) theories, and the resulting "accidents" have drastic consequences for the IR physics and the relation between UV parameters and IR data. The examples we consider are (0,2) Landau-Ginzburg theories, and we identify a class of accidental symmetries of (0,2) LG RG flows by studying the space of F-term UV couplings modulo field redefinitions. We find that these accidental symmetries significantly modify the analysis of the IR theory. For instance, the spectrum of chiral operators and even the IR central charge are in general modified. This invalidates certain UV theories from giving good models for (0,2) SCFTs appropriate for a heterotic string vacuum—we examine an example taken from [41].

A classic (2,2) example

To describe the challenges of (0,2) accidents more precisely, it is useful to review the successes of the (2,2) theories. Consider the quintic (2,2) LG model with chiral superpotential

$$W = \alpha_0 X_0^5 + \alpha_1 X_1^5 + \cdots + \alpha_4 X_4^5 - 5\alpha_5 X_0 X_1 \cdots X_4 \ .$$

Here the X_i are (2,2) chiral superfields and the α_a are complex parameters. This theory flows to a $c = \bar{c} = 9$ (2,2) SCFT.[2] The complex parameter

$$\psi = \alpha_5^5 (\alpha_0 \alpha_1 \cdots \alpha_4)^{-1}$$

is invariant under $\mathbb{C}^*$ rescalings of the chiral superfields X_i and labels a one-parameter family of IR CFTs. At generic values of ψ the IR fixed point is a well-behaved CFT, and small changes in ψ correspond to small marginal deformations of the CFT, where "small" refers to the distance in the Zamolodchikov metric. At special values of ψ the CFT can become singular. For instance, $\psi = 1$ is a finite distance singularity —the analogue of a conifold point. This can be detected in the UV description: the theory develops a family of supersymmetric vacua with $X_i = \text{const.}$, and these signal a non-compact CFT: a theory with a continuum spectrum of conformal dimensions. Another point, $\psi = \infty$ is an infinite-distance singularity.

The quotient of the UV parameter space by field redefinitions is a complicated object [9, 33] with singularities and non-separated points. For instance, we can take the limit $\alpha_0 \to 0$ and $\alpha_5 \to 0$ while keeping ψ constant so as to obtain a product of four minimal models coupled to a free chiral superfield X_0, with $c = \bar{c} = 3(3 + \frac{2}{5})$. Fortunately, all such bad points are singular CFTs. The bad points corresponding to various infinite distance singularities and "wrong" central charges are easily identified in terms of the UV data: they all correspond to singular superpotentials with a continuum of supersymmetric vacua and therefore a continuum of states in the IR CFT. Away from such points the quotient is sensible and describes (2,2) marginal deformations of the IR theory.

[2] A $\mathbb{Z}_5$ orbifold of the theory describes a (2,2) non-linear sigma model with target space the quintic CY hypersurface in $\mathbb{P}^4$ at a special value of the complexified Kähler parameter. A complex structure parameter of the geometry is then related to the LG parameter ψ.

Similar considerations apply to more general (2,2) gauged linear sigma models, and with the parametrization of the smooth CFTs in terms of the UV parameters in hand, localization and topological field theory techniques can be used to compute certain correlators and chiral spectra in the CFT in terms of the weakly coupled UV Lagrangian.

(0,2) challenges

As we will show in section 3.3, the situation is more delicate in (0,2) theories, even in the relatively simple class of LG models (we review these in section 3.2). The essential difference is that we lack the simple diagnostic we had for a "bad" point in (2,2) theories. It is not sufficient to exclude UV parameters that lead to flat directions in the potential, and the identification of UV parameters with marginal deformations of the CFT requires (at least) a study of loci with enhanced symmetry. Unlike in (2,2) examples, accidental symmetries can emerge for non-singular UV potentials, thereby complicating the description of IR physics in terms of the UV data. Unlike in (2,2) theories a family of smooth UV potentials with each potential preserving the same R-symmetry along the RG flow need not correspond to a family of CFTs related by truly marginal deformations.

Fortunately, at least in (0,2) LG models it appears that we have enough control to identify accidental symmetries and special loci in the parameter space by generalizing the (2,2) paradigm of parametrizing the IR fixed points by the space of UV parameters modulo field redefinitions. This uncovers a rich structure of (0,2) RG flows and of the space of marginal deformations of (0,2) fixed points and will undoubtedly play a role in quantitatively descriptions of (0,2) moduli spaces.

Once we have identified a (0,2) LG theory with some particular IR fixed point, it is useful to develop the correspondence between deformations of the UV Lagrangian and (0,2) conformal perturbation theory. In section 3.4 we describe some properties of (0,2) conformal perturbation theory independent of any embedding of the CFT in a critical heterotic string. This section can be read independently from the rest of the chapter. We use these observations in section 3.5 to describe a conjecture for the global structure of the moduli space of (0,2) SCFTs with expected central charge in terms of the UV data for what we term *plain* (0,2) LG models.

3.2 A glance at (0,2) Landau-Ginzburg theories

We begin with a quick review of (0,2) LG theories following [41, 66, 79].[3] We work in Euclidean signature and a (0,2) superspace with coordinates $(z, \overline{z}, \theta, \overline{\theta})$. The UV theory consists of n bosonic chiral multiplets $\Phi_i = \phi_i + \ldots$, and N fermionic chiral multiplets $\Gamma^A = \gamma^A + \ldots$, as well as their conjugate anti-chiral multiplets. These are given a free kinetic term and a (0,2) SUSY potential term as interactions:

$$\mathcal{L}_{\text{int}} = \int d\theta\, \mathcal{W} + \text{h.c.}\ , \qquad\qquad \mathcal{W} = m_0 \textstyle\sum_A \Gamma^A J_A(\Phi)\ , \tag{3.1}$$

where m_0 has mass dimension 1 and the J_A are polynomial in the Φ_i. This is the simplest example of a (0,2) SUSY asymptotically free theory: for energies $E \gg m_0$ the theory is well-described by the set of free fields. Conversely, when $E \ll m_0$ the interactions become important and lead to non-trivial IR dynamics that depend on n, N, as well as the choice of ideal $\boldsymbol{J} = \langle J_1, \ldots, J_N \rangle \subset \mathbb{C}[\Phi_1, \ldots, \Phi_n]$. What can we say about the IR limit of this theory?

A basic constraint comes from the gravitational anomaly. In the UV the central charges are easy to determine: each Φ multiplet contains a complex boson and a right-moving Weyl fermion, while each Γ contains a left-moving Weyl fermion and an auxiliary field. Hence, we have $c_{UV} = 2n + N$, $\overline{c}_{UV} = 3n$. The RG flow induced by $\mathcal{W}$ will decrease the central charges, but since it is Lorentz-invariant, it will preserve the difference $c - \overline{c} = N - n$.

[3] Our superspace conventions are those of [79].

Another basic property of the theory is the set of global symmetries. The free theory has a large global symmetry that commutes with (0,2) SUSY: namely, $\mathrm{U}(n) \times \mathrm{U}(N)$ rotations of the chiral superfields. In addition we have the R-symmetry that rotates θ and leaves the lowest components of the superfields ϕ_i and γ^A invariant. The interactions break these symmetries. For completely generic J_A the remaining symmetry just $\mathrm{U}(1)^0_{\mathrm{R}}$ – an R-symmetry that assigns charge $+1$ to θ and γ^A and charge 0 to ϕ_i.

Properties of the superpotential

A key feature of (0,2) LG theories is that the holomorphic superpotential obeys the same non-renormalization properties as the, perhaps more familiar N=1 d=4 Wess-Zumino model's superpotential. The kinetic term, on the other hand, is a full superspace derivative and will receive complicated corrections along the RG flow. However, just as in (2,2) theories, we expect these corrections to be irrelevant provided that the fields Φ and Γ all acquire non-trivial scaling dimensions. In order to relate these scaling dimensions to properties of the UV theory, we will assume that the interactions preserve an additional global U(1) symmetry, which we will call $\mathrm{U}(1)_{\mathrm{L}}$, under which Γ^A have charges Q_A, while the Φ_i carry charges q_i. This will be the case if and only if the ideal is quasi-homogeneous, i.e.

$$J_A(t^{q_i}\Phi_i) = t^{-Q_A} J_A(\Phi) \tag{3.2}$$

for all $t \in \mathbb{C}^*$. We will demand that the ideal is zero-dimensional, i.e. $J_A(\Phi) = 0$ for all A if and only if $\Phi = 0$. If it is not then the theory necessarily has a non-compact set of supersymmetric vacua labeled by vevs of the bosonic fields. We will call such superpotentials singular. We are interested in "compact" CFTs and exclude this possibility.[4]

[4] A CFT is compact if its spectrum is such that for every fixed real Δ there is a finite number of fields with dimension less than Δ.

Another important property of the superpotential is that typically some of its parameters are not, in fact, F-terms. To see this, consider a perturbation of the form $\delta\mathcal{W} = \sum_A \Gamma^A \delta J_A$ around a theory with $\mathcal{W}_0 = \sum_A \Gamma^A J_A$. In the undeformed theory, if we assume canonical kinetic terms, the equations of motion read

$$\overline{\mathcal{D}\Gamma}^A = J_A(\Phi) , \qquad \overline{\mathcal{D}}\partial_z\overline{\Phi}_i = \sum_A \Gamma^A \frac{\partial J_A}{\partial \Phi_i} , \tag{3.3}$$

where $\overline{\mathcal{D}}$ is the antichiral superspace derivative $\overline{\mathcal{D}} = \partial_{\overline{\theta}} + \theta\partial_{\overline{z}}$. A more general kinetic term leads to more complicated expressions under the $\overline{\mathcal{D}}$ derivative of the left-hand sides of the equations.[5] Hence, a first order deformation of $\mathcal{W}$ of the form

$$\delta J_A = \sum_B M_A^B(\Phi) J_B(\Phi) + \sum_i \frac{\partial J_A}{\partial \Phi_i} F_i(\Phi) \tag{3.4}$$

is equivalent up to equations of motion to a D-term deformation.

The LG assumption that the D-terms are irrelevant along the flow implies that any two UV theories with superpotentials related by a holomorphic field redefinition lead to the same IR fixed point. Hence, any two UV superpotentials that are related by a holomorphic field redefinition belong to the same universality class.

(2,2) LG theories

The (0,2) theory will have an enhanced left-moving SUSY when $N = n$, so that in the free limit we can combine (Γ^i, Φ_i) into (2,2) chiral multiplets X^i, and when $J_i = \partial W/\partial \Phi_i$ for some potential W. In that case, we can rewrite the theory in a manifestly (2,2) supersymmetric fashion with a chiral superpotential $W(X)$. The quasi-homogeneity conditions set $Q_i = q_i - 1$, and the resulting central charge is given by the famous

$$\overline{c} = 3\sum_i (1 - 2q_i) . \tag{3.5}$$

[5] In a NLSM such total derivatives are more subtle than in this LG setting, as they are usually only sensible patch by patch in target space. Indeed, marginal deformations of NLSMs are such F-terms that cannot be globally recast as D-terms [18, 81].

IR consequences of UV symmetries

Returning to the more general (0,2) setting, if we assume that $\mathrm{U}(1)^0_{\mathrm{R}}$ and $\mathrm{U}(1)_{\mathrm{L}}$ are the only symmetries along the whole RG flow, then we can determine the linear combination of charges corresponding to the IR R-symmetry $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ as well as those of a left-moving $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}}$. The charges of the latter are fixed up to normalization by the quasi-homogeneity condition, and the normalization is fixed by

$$-\sum_A Q_A - \sum_i q_i = \sum_A Q_A^2 - \sum_i q_i^2 \ . \tag{3.6}$$

This ensures that $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}}$ and $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ have no mixed anomalies and become, respectively, left-moving and right-moving Kac-Moody symmetries in the IR theory. The central charge is determined from the two-point function of the $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ current. The result is

$$\bar{c} = 3(r+n-N) \ , \qquad\qquad r = -\sum_A Q_A - \sum_i q_i \ . \tag{3.7}$$

By studying the cohomology of the supercharge $\overline{Q}$ of the theory, we can also describe chiral operators and their charges. More details can be found in [79], but for our purposes it will be sufficient to note the charges and corresponding dimensions of ϕ_i and γ^A. Denoting the $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}}$ and $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ charges by, respectively, $\boldsymbol{q}$ and $\overline{\boldsymbol{q}}$, we have

	ϕ_i	γ^A
$\boldsymbol{q}$	q_i	Q_A
$\overline{\boldsymbol{q}}$	q_i	$1+Q_A$
h	$\frac{q_i}{2}$	$\frac{2+Q_A}{2}$
$\overline{h}$	$\frac{q_i}{2}$	$\frac{1+Q_A}{2}$

Since these are chiral operators, the right-moving weights are determined in the usual fashion $\overline{h} = \overline{\boldsymbol{q}}/2$, and the left-moving weights are fixed since RG flow preserves the spin of the operators.

This structure determines many properties of the IR theory such as the elliptic genus [66] and the topological heterotic ring [79]. As for (2,2) theories, there is also a simple prescription for using orbifolds of such (0,2) LG theories to build spacetime SUSY heterotic string vacua [41, 29]. For instance, the elliptic genus is given by [66]

$$\begin{aligned} Z(\tau, z) &= \mathrm{Tr}_{RR}(-)^F y^{J^{\mathrm{IR}}_{L0}} e^{2\pi i \tau H_L} e^{-2\pi i \bar{\tau} H_R} \\ &= i^{N-n} e^{2\pi i \tau (N-n)/12} y^{-r/2} \left[\chi(y) + O(e^{2\pi i \tau})\right] \ , \end{aligned} \tag{3.8}$$

where $y = e^{2\pi i z}$, and

$$\chi(y) = \left. \frac{\prod_A (1 - y^{-Q_A})}{\prod_i (1 - y^{q_i})} \right|_{y^{\mathrm{integer}}} \ . \tag{3.9}$$

The remaining τ-dependent terms are determined by modular properties of $Z(\tau, z)$.

Enhanced symmetry and c-extremization

For special values of the superpotential the UV theory will acquire enhanced symmetries that commute with the (0,2) SUSY algebra. In two dimensions these cannot be spontaneously broken, and, as in four dimensions, the abelian component $\mathrm{U}(1)^M$ can mix with $\mathrm{U}(1)^0_{\mathrm{R}}$ and $\mathrm{U}(1)_{\mathrm{L}}$ symmetries. Fortunately, as in the four-dimensional case we can still find candidate $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}}$ and $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ symmetries by applying the analogue of a-maximization [59] known as c-extremization [23]. We can summarize the results of [23] as follows. Let $\mathcal{J}_0$ denote the $\mathrm{U}(1)^0_{\mathrm{R}}$ R-symmetry current, and let $\mathcal{J}_\alpha$, $\alpha = 1, \ldots, M$ be the currents for $\mathrm{U}(1)^M$. Assuming that the correct $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ symmetry is a linear combination of $\mathcal{J}_0$ and the $\mathcal{J}_\alpha$, [23] construct the trial current and trial central charge

$$\mathcal{J} = \mathcal{J}_0 + \sum_\alpha t^\alpha \mathcal{J}_\alpha \ , \qquad \tfrac{1}{3}\overline{C} = n - N + 2\sum_\alpha t_\alpha K^\alpha + \sum_{\alpha,\beta} t^\alpha t^\beta L_{\alpha\beta} \ , \tag{3.10}$$

where

$$K^\alpha = -\sum_A Q^\alpha_A - \sum_i q^\alpha_i \ , \qquad L^{\alpha\beta} = \sum_i q^\alpha_i q^\beta_i - \sum_A Q^\alpha_A Q^\beta_A \ . \tag{3.11}$$

Here Q^α_A and q^α_i denote the $\mathrm{U}(1)^\alpha$ charges of Γ^A and Φ_i, respectively. The $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ is then identified by extremizing $\overline{C}$ with respect to t_α, leading to $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}}$ charges

$$q_i = \sum_\alpha q^\alpha_i t_{\alpha *} \ , \qquad\qquad Q_A = \sum_\alpha Q^\alpha_A t_{\alpha *} \ , \tag{3.12}$$

where $t_\alpha = t_{\alpha *}$ is the extremum point. The central charge is then also fixed as $\overline{c} = \overline{C}(t_*)$.

The symmetric form L has a real spectrum, and the sign of an eigenvalue has the following significance in the IR theory. We decompose the UV currents according to the sign of the eigenvalues as $\mathcal{J}_\alpha \to \{\mathcal{J}^+, \mathcal{J}^0, \mathcal{J}^-\}$. If we assume that there are no accidental symmetries in the IR, then unitarity of the SCFT implies that in the IR the $\mathcal{J}^+$ currents must correspond to right-moving Kac-Moody (KM) currents and the $\mathcal{J}^-$ must flow to left-moving KM currents. Finally, the $\mathcal{J}^0$ must decouple from the SCFT degrees of freedom. The last point has two consequences: on one hand, we should treat a theory with $\ker L \neq 0$ with some care; on the other hand, if we can be certain that the IR limit is nevertheless a unitary CFT, we can without loss of generality restrict to symmetries orthogonal to $\ker L$.

In typical examples of (0,2) LG theories L is negative definite; we do not know of a non-singular model where L has a positive eigenvalue. In fact, as far as the extremization procedure goes, symmetries corresponding to the positive eigenspace of L cannot be broken in the SCFT. More precisely, a UV deformation away from an RG trajectory with a "positive" symmetry is irrelevant — in the IR the "positive" symmetry will be restored. To understand this, we consider the change in the extremized central charge upon breaking a symmetry. Assuming $\ker L = 0$, the extremum central charge is

$$\tfrac{1}{3}\overline{C}_0 = n - N - K^T L^{-1} K \ . \tag{3.13}$$

Now suppose we change parameters so that some of the symmetries are broken. We can characterize the unbroken symmetries by a a vector v^α, so that the unbroken symmetries satisfy $t^T v = 0$. The modified extremization is then easily carried out with the aid of a Lagrange multiplier s:

$$\tfrac{1}{3}\overline{C}_v(t,s) = n - N + 2t^T K + t^T L t + 2st^T v \ . \tag{3.14}$$

Extremizing with respect to t leads to

$$\tfrac{1}{3}\overline{C}_v(s) = \tfrac{1}{3}\overline{C}_0 - 2sv^T L^{-1} K - s^2 v^T L^{-1} v \ . \tag{3.15}$$

This may be extremized for s if and only if $v^T L^{-1} v \neq 0$, in which case we obtain

$$\tfrac{1}{3}\overline{C}_v = \tfrac{1}{3}\overline{C}_0 + \frac{(v^T L^{-1} K)^2}{v^T L^{-1} v} \ . \tag{3.16}$$

The first observation is that the deformation changes the IR central charge if and only if the original symmetry, with charges determined from $t_* = -L^{-1}K$ is broken. Next, we see that if in addition v belongs to the positive eigenspace of L, then the central charge of the deformed theory is strictly greater than that of the undeformed theory — this means the deformation must be irrelevant in the IR, and we expect the deformed theory to flow to the original undeformed fixed point. Once we eliminate these irrelevant deformations from the parameter space, the symmetries corresponding to the positive eigenvalues of L are never broken, and we can restrict to v in the negative eigenspace of L.

We stress that in all examples we considered L is negative definite. In that case (3.16) shows that when a deformation breaks a symmetry the central charge changes if and only if the deformation breaks the R-symmetry, and whenever that happens the central charge decreases.

Constraints on UV data

The structure relating UV and IR physics sketched above assumes that for a given set of charges (q_i, Q_A) there exists a non-singular potential with a zero-dimensional ideal $\boldsymbol{J}$ and of course that $\mathrm{U}(1)_\mathrm{L}$ and $\mathrm{U}(1)^0_\mathrm{R}$ are the only symmetries all along the RG flow. Both of these are non-trivial assumptions.

It is an open problem to classify all sets of charges consistent with (3.7) and some fixed $\overline{c}$ that can be realized by a non-singular $\boldsymbol{J}$.[6] Demanding that $\chi(y)$ is a polynomial rules out many choices of charges, but while being a necessary condition, it is not sufficient to show that there exists a zero-dimensional $\boldsymbol{J}$ realizing the charge assignment.

The second assumption, which amounts to the statement that there are no accidental symmetries in the IR, also leads to some necessary conditions. For instance, just as in N=1 $d = 4$ SQCD [89], violation of unitarity bounds on the charges can indicate an inconsistency in the assumption. In particular, we have the unitarity bounds

$$0 < q_i \leqslant \overline{c}/3 \ , \qquad 0 < (1 + Q_A) \qquad \sum_A (1 + Q_A) \leqslant \overline{c}/3 \ . \tag{3.17}$$

These arise by demanding that ϕ_i, γ^A, and $\prod_A \gamma^A$ are chiral primary operators of a unitary N=2 superconformal algebra. The latter is particularly strong and eliminates many possible candidate charges.[7] While these criteria are important and will certainly play a role in any attempt to classify (0,2) LG theories, they are not sufficient to rule out accidents.

3.3 Accidents

Having reviewed the basic structure of (0,2) LG theories, we will now study it in a few examples that will illustrate some of the subtleties in their analysis.

[6] This should be contrasted with (2,2) LG models, for which such a classification exists [68, 67] and yields a finite set of quasi-homogeneous potentials at fixed central charge.

[7] In the (2,2) case this translates to the known bound $\sum_i q_i \leqslant n/3$ [69].

3.3.1 Accidents in (2,2) Landau-Ginzburg orbifolds

There are two familiar examples of accidental symmetries in (2,2) flows. Consider LG orbifolds with potentials

$$W_3 = X_1^3 + X_2^3 + X_3^3 - \psi X_1 X_2 X_3 \ , \quad W_4 = X_1^4 + X_2^4 + X_3^4 + X_4^4 - \psi X_1 X_2 X_3 X_4 \ , \tag{3.18}$$

For W_3 (W_4) we take the orbifold by $\mathbb{Z}_3 \subset \mathrm{U}(1)_{\mathrm{R}}$ ($\mathbb{Z}_4 \subset \mathrm{U}(1)_{\mathrm{R}}$). The endpoint of the flow in each case has accidental symmetries. In the case of W_3, which is a special point in the moduli space of a (2,2) compactification on T^2, there is an accidental N=2 Kac-Moody U(1) algebra for both left and right movers, corresponding to the isometries of the torus. In the case of W_4 the IR theory is actually a (4,4) SCFT, and there are additional currents that enhance $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ to $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$. Of course this is the case for any Landau-Ginzburg orbifold (or more generally linear sigma model) that corresponds to a locus in the moduli space of T^2 or K3 compactification.

3.3.2 A contrived (2,2) example

Consider a (2,2) LG theory with

$$W = X^3 + Y^4 \ . \tag{3.19}$$

There is a unique assignment of R-charge $\overline{\boldsymbol{q}}(X) = 1/3$ and $\overline{\boldsymbol{q}}(Y) = 1/4$, and the IR fixed point is the E_6 minimal model. On the other hand, we can make a field redefinition $\widetilde{X} = X - Y$ and $\widetilde{Y} = Y$. This is certainly non-singular and leads to a superpotential

$$\widetilde{W} = \widetilde{X}^3 + 3\widetilde{X}^2\widetilde{Y} + 3\widetilde{X}\widetilde{Y}^2 + \widetilde{Y}^3 + \widetilde{Y}^4 \ . \tag{3.20}$$

If we also perform the field redefinition in the kinetic terms, we have of course done nothing; however, if we assume the D-terms are indeed irrelevant, then taking standard kinetic terms and either W or $\widetilde{W}$ interactions should lead to the same IR fixed point. Unlike the original theory, the $\widetilde{W}$ theory has no manifest R-symmetry along the flow — the symmetry emerges accidentally in the IR.

The example is very contrived, but it illustrates the basic issue: field redefinitions can obscure the UV fields that should be identified with IR operators of some fixed scaling dimension. As we show in 3.3.5, if we restrict to (2,2) theories with a quasi-homogeneous potential, this ambiguity turns out to be harmless. As the next example shows, in (0,2) theories this is not the case.

3.3.3 A simple (0,2) example

Consider a theory with $N = 3$, $n = 2$ and superpotential

$$\mathcal{W}_0 = \begin{pmatrix} \Gamma^1 & \Gamma^2 & \Gamma^3 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \Phi_1^6 \\ \Phi_2^2 \\ \Phi_1^3 \Phi_2 \end{pmatrix} . \tag{3.21}$$

For generic values of the 9 parameters α the potential preserves a unique $\mathrm{U}(1)_\mathrm{L}$ symmetry, and normalizing the charges as in (3.6) leads to $r = 2$, $\overline{c} = 3$, and charge assignments

$$\begin{array}{cccc} & \Phi_1 & \Phi_2 & \Gamma^{1,2,3} \\ \boldsymbol{q} & \frac{1}{7} & \frac{3}{7} & -\frac{6}{7} \\ \overline{\boldsymbol{q}} & \frac{1}{7} & \frac{3}{7} & \frac{1}{7} \end{array} \tag{3.22}$$

To obtain a description of the parameter space of the IR theory we consider the α modulo field redefinitions consistent with (0,2) SUSY and the $\mathrm{U}(1)_\mathrm{L}$ symmetry:

$$\Gamma^A \to \sum_B \Gamma^B M_B^A , \qquad \Phi_1 \to x\Phi_1 , \qquad \Phi_2 \to y\Phi_2 + z\Phi_1^3 . \tag{3.23}$$

These transformations are invertible if and only if $M \in \mathrm{GL}(3, \mathbb{C})$ and $x, y \in \mathbb{C}^*$. The induced action on the Φ monomials is then

$$\begin{pmatrix} \Phi_1^6 \\ \Phi_2^2 \\ \Phi_1^3\Phi_2 \end{pmatrix} \to S \begin{pmatrix} \Phi_1^6 \\ \Phi_2^2 \\ \Phi_1^3\Phi_2 \end{pmatrix} , \qquad S = \begin{pmatrix} x^6 & 0 & 0 \\ x^3 z & x^3 y & 0 \\ z^2 & 2yz & y^2 \end{pmatrix} , \tag{3.24}$$

and hence the action on the parameters α is $\alpha \mapsto M\alpha S$.

A bit of algebra shows that every non-singular ideal $\boldsymbol{J}$ described by α is equivalent by a field redefinition to one of three superpotentials:

$$\begin{aligned} \mathcal{W}_1 &= \Gamma^1\Phi_1^6 + \Gamma^2\Phi_2^2 + \Gamma^3\Phi_1^3\Phi_2 \ , \\ \mathcal{W}_2 &= \Gamma^1(\Phi_1^6 + \Phi_2^2) + \Gamma^2\Phi_1^3\Phi_2 \ , \\ \mathcal{W}_3 &= \Gamma^1\Phi_1^6 + \Gamma^2\Phi_2^2 \ . \end{aligned} \tag{3.25}$$

The UV parameter space is stratified to three points, and we consider each in turn.

1. $\mathcal{W}_1$ has a $\mathrm{U}(1)^2$ global symmetry that acts independently on Φ_1 and Φ_2; extremization picks out the following charges.

	θ	Φ_1	Φ_2	Γ^1	Γ^2	Γ^3	
$\boldsymbol{q}$	0	$\frac{26}{167}$	$\frac{64}{167}$	$-\frac{156}{167}$	$-\frac{128}{167}$	$-\frac{142}{167}$	$\overline{c} = 3\left(1+\frac{2}{167}\right)$
$\overline{\boldsymbol{q}}$	1	$\frac{26}{167}$	$\frac{64}{167}$	$\frac{11}{167}$	$\frac{39}{167}$	$\frac{25}{167}$	

2. $\mathcal{W}_2$ has a free Γ^3 multiplet. The interacting part of the theory has no extra global symmetries and $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{L}} \times \mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$ charges

	θ	Φ_1	Φ_2	Γ^1	Γ^2	
$\boldsymbol{q}$	0	$\frac{4}{31}$	$\frac{12}{31}$	$-\frac{24}{31}$	$-\frac{24}{31}$	$\overline{c} = 3\left(1+\frac{1}{31}\right)$.
$\overline{\boldsymbol{q}}$	1	$\frac{4}{31}$	$\frac{12}{31}$	$\frac{7}{31}$	$\frac{7}{31}$	

3. $\mathcal{W}_3$ has a free Γ^3 multiplet, and the interacting part of the theory is a product of (2,2) minimal models with (2,2) superpotential $W = X_1^7 + X_2^3$ and charges

	θ	Φ_1	Φ_2	Γ^1	Γ^2	
$\boldsymbol{q}$	0	$\frac{1}{7}$	$\frac{1}{3}$	$-\frac{6}{7}$	$-\frac{2}{3}$	$\overline{c} = 3\left(1+\frac{1}{21}\right)$.
$\overline{\boldsymbol{q}}$	1	$\frac{1}{7}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{3}$	

If we assume that there are no accidental symmetries for the $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ theories, we obtain a consistent picture of the RG flows starting with the UV theory in (3.21).

There are three basins of attraction; each has a central charge $\overline{c} > 3$, a set of charges consistent with unitarity bounds and no marginal deformations. Moreover, we can construct interpolating RG flows $\mathcal{W}_3 \to \mathcal{W}_2 \to \mathcal{W}_1$ by adding relevant deformations to the superpotentials. However, $\mathcal{W}_1$ has no $\mathrm{U}(1)_{\mathrm{L}}$-invariant relevant deformations that make it flow to a putative $\overline{c} = 3$ theory described by $\mathcal{W}_0$.

We conclude that (0,2) LG RG flows have accidental symmetries, and identifying these is key in order to correctly pinpoint even basic properties of the IR theory. For instance, we see in the example at hand that no point in the UV parameter space leads to an IR theory with $\overline{c} = 3$ and $r = 2$.

3.3.4 Puzzles from enhanced symmetries

There are two questions that probably occur to our erudite reader. First, what's the big deal? One has to take account of field redefinitions when discussing the parameter space of a theory, and it seems that all we learned here is that the parameter space is smaller than one may have naively thought. Second, is it not perverse to discover some accidental symmetries associated to $\mathcal{W}_{1,2,3}$ versus $\mathcal{W}_0$ but then blithely assume that $\mathcal{W}_{1,2,3}$ do not themselves suffer from accidents?

There is a pragmatic answer to the second question: we assume there are no accidents unless we are able to identify some paradox in the putative description of the IR physics in terms of the UV parameters. In our example we find such a paradox: while a generic $\mathcal{W}$ has a unique global symmetry in the UV, there are special points with enhanced symmetries and a central charge that exceeds the putative $\overline{c} = 3$ of the generic $\mathcal{W}$! Once we take into account the accidental symmetries, we discover that the enhanced symmetries are unavoidable, and there is no $\overline{c} = 3$ theory that can be reached within the parameter space of these UV theories. It is also easy to construct paradoxical examples that would violate unitarity bounds unless one takes accidents into account [79].

The answer to the first question is contained in this pragmatic perspective. The "big deal" is that in the examples with which we are most familiar, namely the (2,2) LG theories, one never encounters these enhanced symmetry puzzles: although there are plenty of points with enhanced symmetries, these never mix with $\mathrm{U}(1)_{\mathrm{R}}^{\mathrm{IR}}$, and the central charge does not jump for any choice of non-singular (2,2) superpotential. We discuss this in detail in the next section.

3.3.5 Enhanced symmetries of (2,2) LG theories

Consider a (2,2) LG theory with a quasi-homogeneous (2,2) superpotential $W(X)$ obeying $W(t^{q_i} X_i) = tW(X)$. Unless W satisfies an independent quasi-homogeneity condition, the (2,2) R-symmetries are fixed uniquely, giving charge $\overline{\boldsymbol{q}} = q_i$ to Φ_i and Γ^i, where (Φ_i, Γ^i) are the (0,2) components of the (2,2) multiplet X_i. Without loss of generality we can restrict attention to $0 < q_i < 1/2$.[8] A special case occurs when we can split the fields $\{X_i\} \to \{X_a\} \cup \{X_p\}$ so that $W = W^1(X_a) + W^2(X_p)$. This leads to an enhanced symmetry, but the enhancement is very large: on both the left and right we obtain two $N = 2$ superconformal algebras with $\overline{c}_1$ and $\overline{c}_2$ that add up to the total $\overline{c}$. The enhanced right-moving U(1) symmetry is not part of an N=2 Kac-Moody algebra: there are two commuting N=2 superconformal algebras, and each U(1) is the lowest component of a different $N = 2$ algebra. Thinking of this theory as a (0,2) LG model and carrying out c-extremization leads to the same result for $\overline{c}$ and charges of the chiral fields.

We will now show that in non-singular (2,2) theories this is the only way that enhanced symmetries occur. Hence, there are no (2,2) accidents.

A necessary and sufficient condition to be able to perform the split $\{X_i\} \to \{X_a\} \cup \{X_p\}$ and $W = W^1(X_a) + W^2(X_p)$ is that the matrix of second derivatives

[8] We assume $q_i > 0$. In that case for a non-singular potential any fields with $q_i \geqslant 1/2$ can be eliminated by their equations of motion.

W_{ki} is block diagonal in the two sets of variables. [9] Since we understand the symmetry enhancement in that case, we assume that W_{ki} has no non-trivial block.[10] We will now show that no additional symmetry is possible when W is non-singular. The argument uses three facts.

1. A non-singular W can satisfy at most one linearly independent quasi-homogeneous relation. To see this, suppose the contrary. By taking linear combinations of two relations we arrive at $\sum_i \alpha_i X_i W_i = 0$. We can now split the fields X_i according to $\alpha_i > 0$, $\alpha_i < 0$, or $\alpha_i = 0$: $\{X_i\} \to \{Y_a\} \cup \{Z_s\} \cup \{U_\alpha\}$ and recast the relation as

$$\sum_a \beta_a Y_a W_a = \sum_s \gamma_s Z_s W_s \ , \tag{3.26}$$

where $\beta_a, \gamma_m > 0$. Without loss of generality we may assume $\beta_1 = 1 \geqslant \beta_a$ for $a \neq 1$. Every monomial in W that contains Y_1 must contain at least one Z. Hence, W will be singular unless $W \supset Y_1^m Z_s$ for some s, say $s = 1$. Similarly, $dW|_{Y=0}$ will be independent of Z_1 unless $W \supset Z_1^p Y_a$ for some a, which requires $\beta_a = \gamma_1 p = pm > 1$, where the last inequality follows since W has no quadratic terms in the fields. That is in contradiction with $\beta_a \leqslant 1$, so the theory must be singular.

2. Suppose we have a symmetry of the (2,2) theory that commutes with the (0,2) SUSY algebra. This means that there are charges Q'_i and q'_i such that

$$-Q'_i W_i = \sum_j q'_j X_j W_{ij} \qquad \Longrightarrow \qquad -Q'_i W_{ik} = q'_k W_{ki} + \sum_j q_j X_j W_{jik} \ .$$

Exchanging i and k in the second equation and taking the difference, we obtain

$$(Q'_k - q'_k) W_{ki} = W_{ki}(Q'_i - q'_i) \ .$$

[9] We use the shorthand $W_i = \partial W/\partial X_i$, $W_{ki} = \partial W/\partial X_k \partial X_i$, etc.

[10] Take the $n \times n$ matrix W_{ki} and set to 1 all non-zero components. The result is a symmetric matrix A_{ki} that is the adjacency matrix for a graph G on n nodes, with each $A_{ki} \neq 0$ specifying a path in the graph from node k to node i. The statement that there is no non-trivial block is simply that G is connected.

This means that whenever $W_{ki} \neq 0$ we need $Q'_k - q'_k = Q'_i - q'_i$.[11]

3. The (2,2) superpotential satisfies

$$(q'_i - Q'_i)W - \sum_j q'_j X_j W_j = U^i \ ,$$

where U^i is independent of X^i. This follows by integrating the quasi-homogeneity condition obeyed by W_i.

Using these observations, we now complete the argument as follows. Since W_{ki} does not contain a non-trivial block, we see from the second fact that for all k, i $Q'_k - q'_k = Q'_i - q'_i$. Combining this with the third fact, we find that W satisfies a quasi-homogeneity relation $W(t^{q'_j} X_j) = t^{q'_i - Q'_i} W(X)$; the first fact then implies that either $q'_i = cq_i$ and $q'_i - Q'_i = c$, or W is singular.

3.3.6 Subtleties for heterotic vacua

We have seen that the identification of UV parameters with a deformation space of an IR CFT, while reasonably well understood for (2,2) theories, is more subtle for (0,2) theories. The difference is that while in non-singular (2,2) theories enhanced symmetries are always associated to a decomposition of the UV theory into non-interacting components, this is not the case for (0,2) models. An enhanced symmetry of a (0,2) model does generically mix with the naive $\mathrm{U}(1)_\mathrm{R}$, so that the enhanced symmetry point has a different central charge from what one might expect naively. As illustrated by the example in section 3.3.3, the RG fixed points of a (0,2) model need not realize any CFT with the naive central charge.

There are situations where the consequences are more benign: there is a choice of UV parameters that leads to a CFT with the expected IR symmetries, but even then the identification of UV parameters with marginal deformations of the IR theory requires a careful study of the field redefinition orbits on the space of UV parameters.

[11] This is trivially satisfied for the usual (2,2) $\mathrm{U}(1)_\mathrm{L}$, where $Q_i = q_i - 1$.

The following familiar example illustrates the issue.

An SO(10) *heterotic Landau-Ginzburg orbifold*

Consider a (0,2) theory with the following field content and charge assignment

$$\begin{array}{c|cccc} & \theta & \Phi_{1,2} & \Phi_{3,\ldots,6} & \Gamma^{1,\ldots,7} \\ \boldsymbol{q} & 0 & \frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \\ \overline{\boldsymbol{q}} & 1 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{array} \tag{3.27}$$

It is easy to see that this symmetry leads to $r = 4$ and $\overline{c} = 9$. The orbifold of this theory by $e^{2\pi i J_0}$ is a candidate for an internal SCFT of an SO(10) heterotic vacuum. As described in [41] that does seem to be the case: the massless spectrum is organized into sensible SO(10) multiplets, and there is a reasonable large radius interpretation in terms of a rank 4 holomorphic bundle on a complete intersection CY manifold in $\mathbb{CP}^5_{111122}$. The generic superpotential for this theory is

$$\mathcal{W} = \sum_A \Gamma^A J_A \ , \tag{3.28}$$

where each J_A has charge $\boldsymbol{q} = 4/5$. We can choose the UV parameters of the theory to produce the following non-singular potential:

$$\mathcal{W}_1 = \Gamma^1\Phi_1^2 + \Gamma^2\Phi_2^2 + \sum_{i=3}^{6}\Gamma^i\Phi_i^4 + \Gamma^7 \times 0 \ . \tag{3.29}$$

This is a product of (2,2) minimal models and a free left-moving fermion. The resulting central charge is $\overline{c} = 3(3 + \frac{1}{15})$. Thus, this choice of UV parameters does not correspond to a point in the moduli space of the $\overline{c} = 9$ CFTs. Of course the orbit of field redefinitions of this point yields a large basin of attraction of UV theories that flow to the same CFT with $\overline{c} = 3(3 + \frac{1}{15})$. In this case we can identify another point that does lead to $\overline{c} = 9$:

$$\mathcal{W}_2 = \Gamma^1\Phi_1^2 + \Gamma^2\Phi_2^2 + \sum_{i=3}^{6}\Gamma^i\Phi_i^4 + \Gamma^7\Phi_1\Phi_2 \ . \tag{3.30}$$

While this superpotential still has a $\mathrm{U}(1)^6$ global symmetry, c-extremization leads to $\overline{c} = 9$ and R-charges as in the table above. Clearly there is a relevant deformation by $\Gamma^7\Phi_1\Phi_2$ that leads to an RG flow from the $\overline{c} = 3(3 + \frac{1}{15})$ theory to the $\overline{c} = 9$ CFT.

The general lesson is clear: field redefinitions stratify the space of UV parameters into orbits, and in general these orbits correspond to different IR fixed points that are not related by marginal deformations — in particular they can have different central charges. The orbits may or may not include an IR fixed point for which the manifest symmetry of the generic superpotential becomes the $\mathrm{U}(1)^{\mathrm{IR}}_{\mathrm{R}}$: in this example they do, while in that of section 3.3.3 they do not.

3.4 Marginal deformations of a unitary (2,0) SCFT

This section contains a number of results on (2,0) SCFTs. Many if not all of these are well-known in the context of heterotic compactifications, but the derivations given here are more general and give a useful alternative perspective.

3.4.1 Basic results

Consider a unitary compact (2,0) SCFT with the usual superconformal algebra generators $J(z)$, $G^{\pm}(z)$, and $T(z)$, with modes given respectively by J_n, $G^{\pm}_r$ and L_n.[12]

We will show that marginal Lorentz-invariant and supersymmetric deformations of this theory by a local operator take the form

$$\Delta S = \int d^2z \; \Delta\mathcal{L} \; , \qquad\qquad \Delta\mathcal{L} = \{G^{-}_{-1/2}, \mathcal{U}\} + \text{h.c.} \; , \tag{3.31}$$

where $\mathcal{U}$ is a chiral primary operator with $\mathrm{U}(1)_{\mathrm{L}}$-charge $\boldsymbol{q} = 1$ and weights $(h, \overline{h}) = (1/2, 1)$.[13] In string theory, where one considers (0,2) SCFTs with quantized $\boldsymbol{q}$

[12] While for many purposes it is very convenient to treat the supersymmetric side of the theory as anti-holomorphic, in the discussion that follows it leads to a great profusion of bars. Hence, in this section the SUSY side will be taken to be holomorphic.

[13] Some of the arguments given here were developed by IVM and MRP in collaboration with Ido Adam.

charges, this is a classic result [42]. Here we will apply the point of view developed for $N = 1$ $d = 4$ SCFTs [52] to arrive at the statement without any assumptions of charge integrality.

Constraints from supersymmetry

Without loss of generality we can consider deformations $\delta\mathcal{L} = \mathcal{O}(z, \overline{z})$ by a quasi-Virasoro primary operator $\mathcal{O}$, since a descendant would just be a total derivative. Lorentz invariance requires $\mathcal{O}$ to have spin 0, i.e. $h_{\mathcal{O}} = \overline{h}_{\mathcal{O}}$. In order for $\delta\mathcal{L}$ to be supersymmetric, we need $[G^{\mp}_{-1/2}, \mathcal{O}]$ to be a total derivative, i.e. $G^{\mp}_{-1/2}|\mathcal{O}\rangle = L_{-1}|\mathcal{M}^{\mp}\rangle$. Applying $G^{\pm}_{-1/2}$ to both sides of the equation and using the N=2 algebra, we obtain

$$G^{\pm}_{-1/2}G^{\mp}_{-1/2}|\mathcal{O}\rangle = L_{-1}G^{\pm}_{-1/2}|\mathcal{M}^{\mp}\rangle \ ,$$

$$\Longrightarrow L_{-1}\left[|\mathcal{O}\rangle - G^{+}_{-1/2}|\mathcal{M}^{-}\rangle - G^{-}_{-1/2}|\mathcal{M}^{+}\rangle\right] = 0 \ . \tag{3.32}$$

Hence, up to a constant multiple of the identity operator, which does not lead to a deformation of the theory, we can write $|\mathcal{O}\rangle$ as

$$|\mathcal{O}\rangle = G^{-}_{-1/2}|\mathcal{M}^{+}\rangle + G^{+}_{-1/2}|\mathcal{M}^{-}\rangle \ , \tag{3.33}$$

and hence, without loss of generality, any non-trivial deformation corresponds to a state

$$|\mathcal{O}\rangle = G^{-}_{-1/2}|\mathcal{U}\rangle + G^{+}_{-1/2}|\mathcal{V}\rangle + \left[G^{+}_{-1/2}G^{-}_{-1/2} - (1 + \tfrac{q_{\mathcal{K}}}{2h_{\mathcal{K}}})L_{-1}\right]|\mathcal{K}\rangle \ , \tag{3.34}$$

where $|\mathcal{U}\rangle$, $|\mathcal{V}\rangle$ and $|\mathcal{K}\rangle$ are all quasi-primary with respect to the N=2 superconformal algebra, i.e. annihilated by the lowering modes of the global N=2 algebra, L_1 and $G^{\pm}_{1/2}$. The linear combination of operators in the last term is fixed by $L_1|\mathcal{O}\rangle = 0$. The spins of the fields are

$$\overline{h}_{\mathcal{U}} - h_{\mathcal{U}} = 1/2 \ , \qquad \overline{h}_{\mathcal{V}} - h_{\mathcal{V}} = 1/2 \ , \qquad \overline{h}_{\mathcal{K}} - h_{\mathcal{K}} = 1 \ . \tag{3.35}$$

The remaining constraints from supersymmetry are

$$G^+_{-1/2}G^-_{-1/2}|\mathcal{U}\rangle = L_{-1}|X\rangle \ , \qquad\qquad G^-_{-1/2}G^+_{-1/2}|\mathcal{V}\rangle = L_{-1}|Y\rangle \tag{3.36}$$

for some states $|X\rangle$ and $|Y\rangle$. We will now show that the only solution to these equations is that $\mathcal{U}$ ($\mathcal{V}$) is a chiral primary (anti-chiral primary) state. It suffices to work out the constraint on $|\mathcal{U}\rangle$ — the one on $|\mathcal{V}\rangle$ follows by exchanging G^+ and G^-.

Without loss of generality we decompose

$$|X\rangle = a|\mathcal{U}\rangle + |\chi\rangle \ , \tag{3.37}$$

where a is real and $|\chi\rangle$ is orthogonal to $|\mathcal{U}\rangle$. The condition now becomes

$$(G^+_{-1/2}G^-_{-1/2} - aL_{-1})|\mathcal{U}\rangle = L_{-1}|\chi\rangle \ . \tag{3.38}$$

Applying $\langle\mathcal{U}|L_1$ to both sides and using orthogonality of $|\mathcal{U}\rangle$ and $|\chi\rangle$, we find $a = 1 + \frac{\boldsymbol{q}_\mathcal{U}}{2h_\mathcal{U}}$. Application of $\langle\chi|L_1$ to both sides shows $L_{-1}|\chi\rangle = 0$, so we are left with

$$G^+_{-1/2}G^-_{-1/2}|\mathcal{U}\rangle = (1 + \tfrac{\boldsymbol{q}_\mathcal{U}}{2h_\mathcal{U}})L_{-1}|\mathcal{U}\rangle \ . \tag{3.39}$$

Finally, applying $G^-_{-1/2}$, we find

$$(1 + \tfrac{1}{2h_\mathcal{U}})(2h_\mathcal{U} - \boldsymbol{q}_\mathcal{U}) = 0 \ . \tag{3.40}$$

The only solution of this equation consistent with unitarity is $2h_\mathcal{U} = \boldsymbol{q}_\mathcal{U}$, i.e. $|\mathcal{U}\rangle$ is a chiral primary state of the N=2 superconformal algebra.

Combining the preceding results and applying them to deformations by real operators, we conclude that real Lorentz-invariant supersymmetric deformations take the form

$$\mathcal{O}(z,\overline{z}) = \Big[\{G^-_{-1/2}, \mathcal{U}(z,\overline{z})\} + \text{h.c.}\Big] + \{G^+_{-1/2}, [G^-_{-1/2}, \mathcal{K}(z,\overline{z})]\} \ , \tag{3.41}$$

where $\mathcal{U}$ is a fermionic chiral primary operator with $\overline{h}_\mathcal{U} = \frac{1}{2} + h_\mathcal{U}$, $h_\mathcal{U} = \boldsymbol{q}_\mathcal{U}/2$, and $\mathcal{K}$ is a real bosonic quasi-primary operator with $\overline{h}_\mathcal{K} = 1 + h_\mathcal{K}$. As in four dimensions [52], we recognize the familiar superpotential and Kähler deformations.

Marginal operators

If we impose in addition that the perturbation is marginal, we obtain the constraints $\boldsymbol{q}_{\mathcal{U}} = 1$ and $h_{\mathcal{K}} = 0$. The latter implies that $L_{-1}|\mathcal{K}\rangle = 0$, i.e. $\mathcal{K}(\bar{z})$ is an anti-holomorphic conserved current that leads to a trivial deformation of the action. We arrive at the result (3.31).

3.4.2 A few consequences

The preceding analysis, when combined with some basic assumptions about superconformal perturbation theory, leads to important constraints on (2,0) SCFTs. The key feature is that we can use a (2,0) superspace to recast the marginal deformations into the form

$$\Delta S = \int d^2 z \int d\theta \; \alpha^i \mathcal{U}_i + \text{h.c.} \;, \tag{3.42}$$

where α^i denote the couplings and $\mathcal{U}_i$ are denote the chiral primary marginal fermi superfields. Assuming there exists a manifestly supersymmetric regularization scheme for conformal perturbation theory, the renormalized action at a renormalization scale μ must take the form

$$\begin{aligned} \Delta\mathcal{L}_{\text{ren}} = & \int d^2\theta \left[Z^a(\alpha, \bar{\alpha}; \mu) \bar{J}_a + \sum_A \mu^{2-d_A} \bar{K}_A \right] \\ & + \left\{ \int d\theta \left[(\alpha_i + \delta\alpha^i(\alpha;\mu))\mathcal{U}_i + \zeta^I(\alpha;\mu)\mathcal{U}_I \right] + \text{h.c.} \right\} . \end{aligned} \tag{3.43}$$

At the conformal point ($\alpha = 0$) the $\bar{J}_a$ and $\bar{K}_A$ are real operators of dimension $\Delta_a = 2$ and $\Delta_A > 2$, while the $\mathcal{U}_i$ and $\mathcal{U}_I$ are chiral primary operators with $\boldsymbol{q} = 1$ and $\boldsymbol{q} > 1$ respectively.[14] The first line is parallels the $N = 1$ $d = 4$ situation; however, the second line is new, following from the fact that the $\mathcal{U}_i \mathcal{U}_j$ OPE will in

[14] Compactness of the CFT ensures a gap in dimensions between $\bar{J}_a$ and $\bar{K}_A$, as well as between $\mathcal{U}_i$ and $\mathcal{U}_I$.

general have singular $\overline{z}$ dependence. Of course supersymmetry still requires that the renormalization of the superpotential should be holomorphic in the parameters.

Marginal irrelevance

Marginal deformations preserve the R-symmetry of the original SCFT. Hence, the unitarity bound $h_{\mathcal{U}} \geqslant \boldsymbol{q}/2$ implies that a marginal deformation is at worst marginally irrelevant and never marginally relevant.

D-terms and F-terms

Assuming that conformal perturbation theory is renormalizable, the terms involving the $\overline{K}_A$ and $\mathcal{U}_I$ do not arise, and scale invariance of the theory is equivalent to

$$D^a(\alpha,\overline{\alpha}) \equiv \mu\frac{\partial}{\partial\mu}Z^a = 0 \qquad \text{and} \qquad \overline{F}^i(\alpha) \equiv \mu\frac{\partial}{\partial\mu}\delta\alpha^i = 0. \tag{3.44}$$

A two-dimensional unitary compact scale-invariant theory is automatically conformal [87], so every deformation satisfying these "D-term" and "F-term" constraints is exactly marginal.

The "D-term" obstructions to marginality are exactly the same as in the $d = 4$ case studied in [52] — such a scale dependence requires the breaking of a global right-moving symmetry. This is easy to understand at leading order in conformal perturbation theory. In the presence of abelian currents $\overline{J}_a$, the OPE of $\mathcal{U}$ with its conjugate takes the form

$$\mathcal{U}_i(z,\overline{z})\overline{\mathcal{U}}_{\overline{\jmath}}(w,\overline{w}) \sim \frac{g_{i\overline{\jmath}}}{(z-w)(\overline{z}-\overline{w})^2} + \frac{g_{i\overline{\jmath}}q^i_a\overline{J}^a(w,\overline{w})}{(z-w)(\overline{z}-\overline{w})} + \dots, \tag{3.45}$$

where $\overline{J}^a = \gamma^{ab}\overline{J}_b$, and $z\overline{z}\langle\overline{J}_a(\overline{z})\overline{J}_b(0)\rangle = \gamma_{ab}$ in the undeformed theory. This leads to a logarithmic divergence in conformal perturbation theory proportional to

$$\int d^2w G^+_{-1/2}G^-_{-1/2} \int d^2z\frac{|\alpha^i|^2q^i_a\overline{J}^a(w,\overline{w})}{(z-w)(\overline{z}-\overline{w})} \sim \log\mu \times |\alpha^i|^2q^i_a\overline{J}^a(w,\overline{w}) \ , \tag{3.46}$$

which corresponds to the leading order D-term proportional to

$$D_a = \sum_i |\alpha^i|^2 q_a^i \ . \tag{3.47}$$

In applications to heterotic compactifications such a symmetry necessarily corresponds to a gauge boson in the space-time theory, and the space-time picture of the breaking is just the Higgs mechanism: the obstruction to marginality of a coupling α that breaks a right-moving symmetry is encoded in a space-time D-term potential.

We believe that in (0,2) LG models without an orbifold there are no F-term obstructions. The reason is simple: the free field UV presentation of the theory comes with the usual non-renormalization theorems for the superpotential, so the only divergences we expect to encounter will correspond to D-term counter-terms.

The two sources of obstruction are in one to one correspondence with the two ways in which a short chiral primary multiplet can combine into a long multiplet of (2,0) SUSY. Suppose we consider an infinitesimal (2,0) SUSY deformation under which a marginal chiral primary state $|\mathcal{U}\rangle$ acquires weights $(h,\overline{h}) = (\frac{1}{2}+\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$. In this case $|\mathcal{U}\rangle$ is no longer chiral primary, and by a choice of basis we can consider two separate cases:

$$G^+_{-1/2}|\mathcal{U}\rangle \neq 0 \ , \quad G^-_{1/2}|\mathcal{U}\rangle = 0 \ , \qquad \text{or} \quad G^+_{-1/2}|\mathcal{U}\rangle = 0 \ , \quad G^-_{1/2}|\mathcal{U}\rangle \neq 0 \ . \tag{3.48}$$

In other words, $|\mathcal{U}\rangle$ remains primary but is no longer chiral, or it remains chiral but fails to be primary. The first case corresponds to an F-term obstruction, where at $\epsilon = 0$ we have two chiral primary superfields $(\mathcal{U}, \mathcal{F})$ with $\boldsymbol{q}_{\mathcal{U}} = 1$ and $\boldsymbol{q}_{\mathcal{F}} = 2$ and $\overline{h} = 1$, while for $\epsilon > 0$ we find a complex long multiplet with lowest component $|\mathcal{U}\rangle$ and $G^+_{-1/2}|\mathcal{U}\rangle = \sqrt{\epsilon}|\mathcal{F}\rangle$.[15] The second case corresponds to a D-term obstruction, where at $\epsilon = 0$ we have chiral primary superfields $\mathcal{U}$, its anti-chiral conjugate $\overline{\mathcal{U}}$,

[15] In a $c = 9$ theory with spectral flow there is a canonical $\mathcal{F}$ for every $\mathcal{U}$ in the theory. Indeed, as observed in [12, 36, 42], the $\mathcal{F}$ (2,0) superfields can be used to construct vertex operators for the space-time auxiliary fields residing in chiral multiplets of the associated four-dimensional theory.

and a Kac-Moody current $\overline{J}$; for $\epsilon > 0$ we obtain a long real multiplet with lowest component $\overline{J}$ and descendants $G^+_{-1/2}|\overline{J}\rangle = \sqrt{\epsilon}|\mathcal{U}\rangle$ and $G^-_{-1/2}|\overline{J}\rangle = \sqrt{\epsilon}|\overline{\mathcal{U}}\rangle$.

In particular, we see that there are no F-term obstructions if the undeformed theory has no chiral primary operators with $\boldsymbol{q} = 2$ and $\overline{h} = 1$. This is the case, for instance, in every (2,0) SCFT with $c < 6$. If there are also no left-moving Kac-Moody symmetries then every (2,0) marginal deformation must remain exactly marginal. In appendix B.1 we mention a simple example illustrating an F-term obstruction at $c = 9$.

Kähler geometry of the moduli space

One can use the same reasoning as in [52] to argue that the space of truly marginal deformations of a (2,0) SCFT must be a Kähler manifold. This is because the D-term constraints and the quotient by global symmetries lead to a toric quotient on the space of marginal couplings, while the F-term constraints are manifestly holomorphic constraints, restricting the truly marginal directions to a Kähler subvariety of the toric variety. In heterotic compactification this can of course be argued either from the space-time heterotic supergravity or by using additional assumptions of a (2,0) SCFT with integral charges [86]. The argument given here is more direct and general.

Application to (2,2) theories

The case of a (2,2) SCFT and its (2,2)-preserving deformations is much simpler. There are two types of superpotential deformations: the chiral and the twisted chiral. The former corresponds to deformations by chiral primary (c,c) ring operators, while the latter by the (a,c) ring operators. Supersymmetry implies that twisted chiral parameters can never show up in the renormalized chiral superpotential and vice-versa. Moreover, the OPE of the (c,c) and (a,c) chiral primaries with themselves is non-singular, so that neither potential is corrected—there are no F-term obstructions to marginality. Hence, all marginal (c,c) and (a,c) symmetry-preserving deformations are truly marginal. This is again a familiar story in string applications [35, 42] .

Accidents beyond field redefinitions

We can now see that the field redefinitions of (0,2) LG theories do not describe all accidents. It is not the case that every direction transverse to field redefinition orbits corresponds to a marginal deformation of the IR theory. This is due to the possibility that marginal deformations of a (0,2) theory can turn out to be marginally irrelevant. In (0,2) LG theories this is due to D-term obstructions where a U(1) symmetry is broken by turning on operators with a definite sign of the U(1) charge. We give an example of this phenomenon in a well-known heterotic vacuum in appendix B.2.

3.4.3 Deformations and left-moving abelian currents

As a final application of the preceding results, we consider the interplay between deformations of a (2,0) SCFT and left-moving currents.

A (2,0) SCFT may possess a KM algebra on the SUSY side of the world-sheet in addition to the $\mathrm{U}(1)_{\mathrm{L}}$ current J_L in the $N = 2$ multiplet. Such structures are familiar from heterotic compactifications preserving 8 space-time supercharges in four dimensions — when realized geometrically these correspond to geometries $\pi : X \to K3$ — principal T^2 fibrations over a base $K3$ [20, 82]. In each such case we can use a Sugawara-like decomposition to decompose the $N = 2$ world-sheet superconformal algebra (SCA) into two commuting sets of generators, one associated to the KM algebra, and the other corresponding to the remaining degrees of freedom.

Suppose we have an abelian current algebra U(1) with current J_1. There are two ways that the decomposition can work. If J does not belong to a multiplet of the $\mathcal{A}_c$ SCA, then we must have a decomposition

$$\mathcal{A}_c = \mathcal{A}'_{c'} \oplus \mathcal{A}''_{c''} \ , \tag{3.49}$$

where $c = c' + c''$, and the lowest components of the N=2 multiplets of $\mathcal{A}'$ and $\mathcal{A}''$ are obtained by appropriate linear combinations of J_L and J_1. We are familiar with such examples from above: this happens whenever the LG theory decomposes into a product of two non-interacting theories.

If J does belong to a multiplet of $\mathcal{A}$, then it must be accompanied by a second U(1) Kac-Moody current J_2, as well as weight $h = 1/2$ operators ψ_1 and ψ_2. Together these arrange themselves into a well-known $c = 3$ unitary representation of $N = 2$:

$$J_L =: \boldsymbol{\psi}\overline{\boldsymbol{\psi}} : \ , \quad G^+ = \sqrt{2}\boldsymbol{\psi}\overline{\boldsymbol{\jmath}} \ , \quad G^- = \sqrt{2}\overline{\boldsymbol{\psi}}\boldsymbol{\jmath} \ , \quad T =: \boldsymbol{\jmath}\overline{\boldsymbol{\jmath}} : -\tfrac{1}{2}(: \overline{\boldsymbol{\psi}}\partial\boldsymbol{\psi} : + : \boldsymbol{\psi}\partial\overline{\boldsymbol{\psi}} :) \ , \tag{3.50}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\jmath}$ have the free-field OPEs

$$\overline{\boldsymbol{\psi}}(z)\boldsymbol{\psi}(w) \sim (z-w)^{-1}, \qquad \overline{\boldsymbol{\jmath}}(z)\boldsymbol{\jmath}(w) \sim (z-w)^{-2}. \tag{3.51}$$

This is equivalent to the holomorphic sector of a T^2 (1,0) non-linear sigma model, and we will call it $\mathcal{A}_3^{\text{free}}$.

There is a key difference between these two generalizations. In the first case, there are generally deformations that can break the extra left-moving symmetry — in (2,2) LG this happens when we move away from a Gepner point to a more generic theory. In the second case such breaking is impossible. To see this, we just need to apply what we learned about the structure of SUSY deformations in conformal perturbation theory. Since our algebra splits as

$$\mathcal{A}_c = \mathcal{A}'_{c-3} \oplus \mathcal{A}_3^{\text{free}} \ , \tag{3.52}$$

a marginal deformation has a similar decomposition

$$\mathcal{U} = \mathcal{U}' + S\boldsymbol{\psi} \ , \tag{3.53}$$

where $\mathcal{U}'$ is a chiral primary operator with $\overline{h} = 1$ and $\boldsymbol{q}' = 1$, while S is a (0,1) current. The deformation of the action is then

$$G^-_{-1/2} \cdot \mathcal{U} = G'^-_{-1/2} \cdot \mathcal{U}' + \sqrt{2}S\boldsymbol{\jmath} \ . \tag{3.54}$$

This is neutral with respect to J_L, $\boldsymbol{\jmath}$ and $\overline{\boldsymbol{\jmath}}$. More generally, any relevant deformation must be of the form $\mathcal{U} = \mathcal{U}'$ with $\boldsymbol{q}' < 1$.

3.5 Toric geometry of the deformation space

In the previous sections we saw that accidental symmetries play an important role in (0,2) Landau-Ginzburg theories, and more generally, in (0,2) SCFTs. In this section we will describe a conjecture that allows us to account for these accidents in a certain class of (0,2) LG theories. In that context our goal is to describe the moduli space $\mathcal{M}$ of IR fixed points corresponding to a class of UV data determined by a choice of charges q_i and Q_A which have the expected central charge

$$\overline{c} = 3(n - N + r) \ , \qquad\qquad r = -\sum_A Q_A - \sum_i q_i \ . \tag{3.55}$$

To do so, we need to perform two steps:

1. decompose the UV parameter space into orbits under the action of field redefinitions;

2. determine which orbits contribute to $\mathcal{M}$.

The result is expected to be a (typically singular) Kähler space. In general these are rather formidable tasks. The group of field redefinitions is rather large and the space of orbits is non-separable. A reasonable geometry can only emerge after implementing the second task. This involves excluding two types of orbits:

- Along a *discriminant locus* Δ in parameter space, the superpotential is singular. The discriminant will clearly be invariant under field redefinitions, and orbits contained in Δ will not contribute to $\mathcal{M}$.

- For some non-singular values of the parameters, the theory will have accidental symmetries in the IR. As we have seen, in some cases these symmetries will mix nontrivially with the R-symmetry and the central charge of the IR fixed point will be larger than $\overline{c}$. Thus, these orbits as well need to be excluded from $\mathcal{M}$.

In general, the second step is difficult even if one restricts attention to symmetries which act diagonally on the UV fields. Detecting the basin of attraction of some component of the IR moduli space with central charge $\bar{c}' > \bar{c}$ requires a determination of the R-symmetry along each such component to find which deformations away from this locus are in fact irrelevant.

3.5.1 The toric conjecture

There is a simpler version of both of these problems that may be tractable. The group of field redefinitions always contains an abelian subgroup, the complexification of the $U(1)^n \times U(1)^N$ subgroup of the global symmetry of the free kinetic terms, that corresponds to rescaling the chiral fields of the theory. In particular, if we write the most general superpotential in our class as

$$\mathcal{W} = \sum_A \Gamma^A \sum_{m \in \Delta_A} \alpha_{Am} \prod_i \Phi_i^{m_i} \ , \tag{3.56}$$

where

$$\Delta_A = \{ m \in \mathbb{Z}^n \ \ | \ \ \textstyle\sum_i m_i q_i = -Q_A \} \tag{3.57}$$

describes the lattice points in the Newton polytope for J_A, then the field redefinitions

$$\Phi_i \mapsto t_i \Phi_i \ , \qquad\qquad \Gamma^A \mapsto \tau_A \Gamma^A \tag{3.58}$$

lead to a $T_{\mathbb{C}} = (\mathbb{C}^*)^{N-n+1}$ action[16] on the space of UV parameters $Y = \mathbb{C}^{\sum_A |\Delta_A|}$

$$\alpha_{Am} \mapsto \tau_A \textstyle\prod_i t_i^{m_i} \times \alpha_{Am} \ . \tag{3.59}$$

We will refer to these as toric field redefinitions.

We now restrict attention to these toric actions in *both* of the tasks listed above. Namely, we decompose the parameter space into $T_{\mathbb{C}}$ orbits and exclude those orbits that either lie in Δ or exhibit accidental symmetries contained in $T_{\mathbb{C}}$ and lead to

[16] The rank of the $\mathbb{C}^*$ action is reduced by 1 due to the quasi-homogeneity of $\mathcal{W}$.

$\overline{c}' > \overline{c}$. The result, which we will call $\mathcal{M}_T$, will in some cases be equivalent to $\mathcal{M}$, but in general the two will differ. We will comment on this further below.

The action of the compact torus $T \subset T_{\mathbb{C}}$, given by restricting to $|t| = |\tau| = 1$, determines a moment map $\mu = (\lambda; \Lambda) : Y \to \mathbb{R}^{N+n}$ with

$$\lambda_i = \sum_A \sum_{m \in \Delta_A} m_i |\alpha_{Am}|^2 \ , \qquad\qquad \Lambda_A = \sum_{m \in \Delta_A} |\alpha_{Am}|^2 \ . \tag{3.60}$$

Quasi-homogeneity of $\mathcal{W}$ implies that the image lies in the hyperplane

$$\sum_i q_i \lambda_i + \sum_A Q_A \Lambda_A = 0 \ . \tag{3.61}$$

The image of μ is the intersection of this hyperplane with a cone, determined by the charges, inside the positive orthant in $\mathbb{R}^{N+n}$. This intersection is itself a cone $\widehat{\Sigma}$, of dimension $N + n - 1$. The level sets of μ determine a selection of orbits: generic orbits will be $N + n - 1$ dimensional, but the action will degenerate along points with a non-trivial stabilizer subgroup, leading to orbits of smaller dimension.[17] More precisely, the cone $\widehat{\Sigma}$ can be subdivided into a fan Σ, such that the collection of orbits containing a point for which $\mu(\alpha) = \mu^*$ is determined by the cone of Σ that contains μ^*. This is the *secondary fan* for the T action. We now have enough structure to state our conjecture.

Conjecture

The toric moduli space $\mathcal{M}_T$ is the complement of the discriminant subvariety Δ in a *toric variety*

$$V = \mu^{-1}(\lambda^*, \Lambda^*)/T \ , \tag{3.62}$$

where

$$\lambda_i^* = 1 - q_i \ , \qquad \Lambda_A^* = 1 + Q_A \ . \tag{3.63}$$

[17] There are orbifold singularities when the subgroup is discrete; we will focus on continuous stabilizer subgroups.

This is a rather strong statement, and we will not provide a complete proof but rather some evidence for it. Some ideas on a possible derivation are discussed in section 3.5.4. We will motivate the conjecture by combining our results and observations from above with some facts about toric varieties.

We should note a few important points. First, V may turn out to be empty. Second, while we claim that $V \backslash \Delta$ describes $\mathcal{M}_T$ as a variety, we do not make any statement about the relation between the Zamolodchikov metric on the space of marginal couplings and the metric on V obtained by the Kähler quotient. Finally, in this chapter we will be concerned with orbits of continuous field redefinitions. In general there will be additional discrete quotients that identify points in $\mathcal{M}_T$.

Combinatorics of the secondary fan

Codimension-one cones in Σ are associated with orbits containing a point at which a single $\mathbb{C}^* \subset T_{\mathbb{C}}$ is unbroken. More precisely, $G_{(q',Q')} = \mathbb{C}^* \subset T_{\mathbb{C}}$ acting with charges q'_i, Q'_A on the chiral superfields will fix points at which

$$|\alpha_{Am}|^2 \left(Q'_A + \textstyle\sum_i m_i q'_i\right) = 0 \qquad \text{for all} \;\; A,\; m \in \Delta_A \; . \tag{3.64}$$

The μ-image of the $T_{\mathbb{C}}$ orbits of such points will lie in a cone $\sigma_{(q',Q')}$ generated by the charge vectors of the α_{Am} fixed by $G_{(q',Q')}$. Thus, the codimension-one cones of Σ are determined by one-dimensional subgroups for which $\sigma_{(q',Q')}$ has dimension $N+n-2$ and lies in the hyperplane

$$\sum_A Q'_A \Lambda_A + \sum_i q'_i \lambda_i = 0 \; . \tag{3.65}$$

In terms of $\mathcal{W}$ the codimension-one cones of Σ correspond to subgroups for which we can write a (possibly singular) family of models fixed precisely by $(\mathbb{C}^*)^2$. Cones of higher codimension in the fan are boundaries of these cones and arise at the intersections of these hyperplanes.[18] Points in the interior of some cone of the fan (of

[18] Note that this does not imply that cones of higher codimension correspond to models with larger unbroken symmetry: values of μ at the intersection of two codimension-one cones can be in the

any codimension) lie in the image of a collection of orbits determined by that cone. Our conjecture is thus equivalent to the statement that the μ-image of the $T_{\mathbb{C}}$ orbits of models with central charge $\bar{c}$ intersects the cone σ^* containing μ^* in its interior.

A cone $\sigma \in \Sigma$ can be specified by its relation to the codimension-one cones $\sigma_{(q',Q')}$. For each of these, σ either lies inside $\sigma_{(q',Q')}$, in which case $\mu \in \sigma$ satisfy (3.65), or it lies on one side or the other, meaning (3.65) is satisfied as a strict inequality for all $\mu \in \sigma$. To prove our claim we thus need to show that orbits of points in parameter space corresponding to models with central charge $\bar{c}$ are precisely those containing in their image points satisfying the inequalities satisfied by μ^*. To do this we must consider all codimension-one cones of Σ. We classify these by the nature of the models exhibiting the enhanced symmetry.

3.5.2 Enhanced toric symmetries

Symmetries realized by a non-singular potential

Consider first the case of one-parameter subgroups of $T_{\mathbb{C}}$ for which the generic point in the locus they fix corresponds to a nonsingular model with a $\mathrm{U}(1)^2$ global symmetry. The IR R-symmetry can then be determined by $\bar{c}$-extremization as

$$\hat{q}_i = tq_i + sq'_i \ , \qquad\qquad \hat{Q}_A = tQ_A + sQ'_A \ , \tag{3.66}$$

where

$$L \begin{pmatrix} t-1 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_i q'_i\lambda^*_i + \sum_A Q'_A\Lambda^*_A \end{pmatrix} \ . \tag{3.67}$$

L is the negative-definite 2×2 matrix defined in (3.11) and (q, Q) are normalized as in (3.6). We now distinguish two situations.

<u>1. $\bar{c}' = \bar{c}$</u>. If

$$\sum_i q'_i\lambda^*_i + \sum_A Q'_A\Lambda^*_A = 0 \ , \tag{3.68}$$

---

image of two distinct $T_{\mathbb{C}}$ orbits, each of which is fixed by a different subgroup.

then the IR symmetry is given by $(q;Q)$, and the $T_{\mathbb{C}}$ orbit of the model with enhanced symmetry is a point in V. In this case, we can apply conformal perturbation theory to deformations of this theory. The symmetry-breaking couplings α_{Am} (those vanishing on the locus exhibiting enhanced symmetry) parameterize classically marginal deformations away from the symmetric theory. The analysis of section 4 shows that in fact some of these will be marginally irrelevant, and the moduli space is given to first order in the symmetry-breaking couplings by the vanishing of the D-term for the broken symmetry. We can write this explicitly here as

$$\begin{aligned} D &= \sum_{A,n\in\Delta_A} (Q'_A + \sum_i m_i q'_i)|\alpha_{Am}|^2 \\ &= \sum_i q'_i \lambda_i + \sum_A Q'_A \Lambda_A \ . \end{aligned} \tag{3.69}$$

This holds at leading order in conformal perturbation theory about the symmetric point, and our conjecture amounts here to the statement that higher order corrections do not qualitatively modify the structure of the symplectic quotient that leads to the variety V: while the metric may be modified, which orbits are kept and which are excluded is not changed by higher order corrections. This implies that points in V are $T_{\mathbb{C}}$ orbits containing points whose image under μ lies in the cone $\sigma_{(q',Q')}$. We see from (3.68) that this condition is satisfied by μ^*.

<u>2. $\bar{c}' > \bar{c}$.</u> If (3.68) is not satisfied, the central charge $\bar{c}'$ determined by extremization will be larger than $\bar{c}$, and the $T_{\mathbb{C}}$ orbit of the model with enhanced symmetry is not a point of V. Moreover, the symmetry-breaking parameters α_{Am} are not marginal couplings in this theory. Solving (3.67) we find

$$s = -\frac{r}{\det L}\left(\sum_i q'_i \lambda^*_i + \sum_A Q'_A \Lambda^*_A\right) \ . \tag{3.70}$$

Without loss of generality we can choose the sign of (q', Q') so that s is negative. Since by construction all our couplings are invariant under $(q;Q)$, and by assumption L is negative definite, the sign of the charge under the IR R-symmetry is then the opposite of the charge under (q',Q'). Thus, couplings α_{Am} for which $Q'_A + \sum_i m_i q'_i > 0$ will be *relevant* deformations of the model with enhanced symmetry, while couplings with the opposite charge will be irrelevant; couplings preserving the enhanced symmetry are marginal. The $T_{\mathbb{C}}$ orbits of points in parameter space corresponding to irrelevant deformations of the symmetric model will not be points in V: as discussed above they will exhibit an accidental symmetry in the IR and a central charge $\overline{c}'$. Orbits for which at least one relevant coupling is nonzero are characterized precisely by the fact that they contain points for which the moment map satisfies

$$\sum_i q'_i \lambda_i + \sum_A Q'_A \Lambda_A > 0 \ . \qquad (3.71)$$

This specifies one side of the hyperplane associated to the enhanced symmetry, and, as we have shown, this is the side on which the point μ^* lies.

Symmetries without a smooth realization

If every enhanced symmetry were realized by a non-singular $\mathcal{W}$ the discussion above would suffice. In general, however, there are codimension-one cones in Σ associated to one-parameter subgroups of $T_{\mathbb{C}}$ for which it is not possible to construct a non-singular $\mathcal{W}$ exhibiting the symmetry. In these cases the RG trajectories exhibiting the enhanced symmetry along the flow are singular, and we cannot use their properties to determine the local structure of the moduli space.

A simple example of this is given by the symmetry acting as $\Gamma^A \to \tau\Gamma^A$ for some A with all the other fields invariant. This fixes the locus $J_A = 0$ which will in general be singular (it will always be singular when $n = N$). In this case, the corresponding hyperplane is $\Lambda_A = 0$, and the associated codimension-one cone lies on the boundary of $\widehat{\Sigma}$.

More interesting is the case of a codimension-one cone in the interior of $\widehat{\Sigma}$ to which the methods of the previous section do not apply. Our conjecture here is that whenever the enhanced symmetry does not satisfy (3.68), non-singular models will only exist when at least one symmetry-breaking coupling whose charge under the broken symmetry is in accord with the sense of the inequality is non-zero. In parallel with the second discussion in the previous subsection, $T_{\mathbb{C}}$ orbits associated to points in V will be those containing points whose image under μ lies on the side of the hyperplane which contains the point μ^*.

There will also be codimension-one cones in Σ associated to one-parameter subgroups for which there is no non-singular model exhibiting the symmetry, but which satisfy (3.68). Here as well we can classify the symmetry-breaking couplings by their charge under the broken symmetry. In this case, we conjecture that non-singular models will have nonzero values for at least one coupling of *each* sign. Restricting to models with non-zero couplings of only one sign (as well as the neutral couplings) will produce a singular model. The space of $T_{\mathbb{C}}$ orbits associated to points in V in this case will not be toric. It can, however, be described as the complement of the symmetric locus (a component of Δ) in a (singular) toric variety. This contains orbits containing points whose image under μ lies *in* the cone $\sigma_{(q',Q')}$. When we exclude the singular symmetric locus here, we find precisely orbits that have nonzero symmetry-breaking couplings with both signs of the broken charge. The point μ^* clearly lies in this hyperplane.

3.5.3 Examples

A few examples may be helpful at this point. We proceed from a simple example for which our methods produce correctly the actual moduli space to models demonstrating their limitations.

A plain model

Consider first the class of models with $n = N = 2$ and charges

$$\begin{array}{ccccc} & \Phi_1 & \Phi_2 & \Gamma^1 & \Gamma^2 \\ \boldsymbol{q} & \frac{1}{4} & \frac{1}{6} & -\frac{3}{4} & -\frac{5}{6} \end{array} \tag{3.72}$$

and $\bar{c}/3 = r = 1 + \frac{1}{6}$. The most general superpotential is

$$\mathcal{W} = \Gamma^1 \left(\alpha_{11}\phi_1^3 + \alpha_{12}\phi_1\phi_2^3\right) + \Gamma^2 \left(\alpha_{21}\phi_2^5 + \alpha_{22}\phi_1^2\phi_2^2\right) \ , \tag{3.73}$$

and the discriminant is

$$\Delta = \alpha_{11}\alpha_{21}\left(\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22}\right) \ . \tag{3.74}$$

This is an example of what we call a *plain* model: the torus $T_{\mathbb{C}}$ includes all field redefinitions consistent with the symmetry, so our toric considerations will in fact generate the moduli space $\mathcal{M}$ itself.

The torus $T = \mathrm{U}(1)^3$ action on $\mathbb{C}^4$ is characterized by the charges and moment map components

$$\begin{array}{ccccc} D & \alpha_{11} & \alpha_{12} & \alpha_{21} & \alpha_{22} \\ \lambda_1 & 3 & 1 & 0 & 2 \\ \lambda_2 & 0 & 3 & 5 & 2 \\ \Lambda_1 & 1 & 1 & 0 & 0 \\ \Lambda_2 & 0 & 0 & 1 & 1 \end{array} \tag{3.75}$$

where the latter satisfy

$$3\lambda_1 + 2\lambda_2 = 9\Lambda_1 + 10\Lambda_2 \ . \tag{3.76}$$

There are six codimension-one cones in Σ. Only two of these are realized by non-singular models; the remaining four comprise the boundaries of $\widehat{\Sigma}$ given by $\Lambda_1 > 0$ and $\Lambda_2 > 0$, as well as $\lambda_2 - 2\Lambda_2 > 0$ and $\lambda_1 - \Lambda_1 > 0$.

There are two codimension-one cones in the interior of $\widehat{\Sigma}$. Consider first the symmetry $(q';Q') = (1,0;-3,0)$, which satisfies (3.68). The non-singular models realizing this symmetry have $\alpha_{12} = \alpha_{22} = 0$. In fact the model reduces to a product of two (2,2) minimal models and, as expected, the central charge is $\bar{c}$. The symmetry determined by $(q',Q') = (-1,1;-2,0)$, for which $\sum_i q'_i \lambda^*_i + \sum_A Q'_A \Lambda^*_A < 0$, fixes models with $\alpha_{11} = \alpha_{22} = 0$.[19] Under the broken symmetry, α_{11} is negatively charged and α_{22} positively charged. We see from (3.74) that, in accordance with the conjecture, non-singular models require a non-zero value for the negatively charged coupling.

The moduli space $\mathcal{M}$ is thus determined. We can fix two of the generators of $T_{\mathbb{C}}$ by setting $\alpha_{11} = \alpha_{21} = 1$, and the remaining couplings parameterize the toric variety $V = \mathbb{C}$ with invariant coordinate $z = \alpha_{12}\alpha_{22}$. The moduli space is $\mathcal{M} = V \backslash \widetilde{\Delta}$ where the discriminant reduces in these coordinates to $1 - z$.

A non-plain model

We can also consider the model with $n = N = 2$ and charges given by

$$\begin{array}{ccccc} & \Phi_1 & \Phi_2 & \Gamma^1 & \Gamma^2 \\ \boldsymbol{q} & \frac{46}{471} & \frac{115}{471} & -\frac{460}{471} & -\frac{230}{471} \end{array} \tag{3.77}$$

with $\bar{c}/3 = r = 1 + \frac{58}{471}$. The most general superpotential invariant under this symmetry is

$$\mathcal{W} = \Gamma^1(\alpha_{11}\Phi_1^{10} + \alpha_{12}\Phi_1^5\Phi_2^2 + \alpha_{13}\Phi_2^4) + \Gamma^2(\alpha_{21}\Phi_1^5 + \alpha_{22}\Phi_2^2) \ , \tag{3.78}$$

and the discriminant is

$$\Delta = \alpha_{11}\alpha_{22}^2 - \alpha_{12}\alpha_{21}\alpha_{22} + \alpha_{13}\alpha_{21}^2 \ . \tag{3.79}$$

The torus $T = \mathrm{U}(1)^3$ action on $\mathbb{C}^5$ is characterized by the charges and moment

[19] Note that this symmetry leads a 2×2 L matrix that is not negative-definite; however, the corresponding superpotential is singular.

map components

$$\begin{array}{cccccc} D & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{21} & \alpha_{22} \\ \lambda_1 & 10 & 5 & 0 & 5 & 0 \\ \lambda_2 & 0 & 2 & 4 & 0 & 2 \\ \Lambda_1 & 1 & 1 & 1 & 0 & 0 \\ \Lambda_2 & 0 & 0 & 0 & 1 & 1 \end{array} \tag{3.80}$$

where the latter satisfy

$$2\lambda_1 + 5\lambda_2 = 20\Lambda_1 + 10\Lambda_2 \ . \tag{3.81}$$

The cone $\widehat{\Sigma}$ is the intersection of this with the positive orthant. This is bounded, in this case, by the coordinate hyperplanes. There are four codimension-one cones in the interior of $\widehat{\Sigma}$ here, none of which satisfy (3.68).

The symmetries acting with charges $(q'; Q') = (1, 0; -5, -5)$ and $(q', Q') = (1, 0, -5, 0)$ are preserved by singular models, and non-singular models, as per the conjecture, lie in orbits containing points for which $\lambda_1 - 5\Lambda_1 - 5\Lambda_2 < 0 < \lambda_1 - 5\Lambda_1$. The symmetry acting with charges $(q', Q') = (1, 0; -10, 0)$ fixes the locus $\alpha_{12} = \alpha_{13} = \alpha_{21} = 0$ where we find a product of (2,2) minimal models: up to a rescaling

$$\mathcal{W}_1 = \Gamma^1\Phi_1^{10} + \Gamma^2\Phi_2^2 \ , \tag{3.82}$$

with central charge $\overline{c}_1/3 = 1 + \frac{5}{33} > r$. At this point, the operators associated to α_{12} and α_{13} are *irrelevant* but the operator associated to α_{21} is *relevant*. We conclude that models with $\alpha_{11}\alpha_{22} \neq 0$ and $\alpha_{21} = 0$ flow to this IR fixed point and orbits containing such models do not contribute to V. Orbits that do contribute have a point for which $\lambda_1 > 10\Lambda_1$.

The symmetry acting with charges $(q', Q') = (0, 1; -4, 0)$ fixes the locus $\alpha_{11} = \alpha_{12} = \alpha_{22} = 0$ where we find a product of (2,2) minimal models: up to a rescaling

$$\mathcal{W}_2 = \Gamma^1\Phi_2^4 + \Gamma^2\Phi_1^5 \ , \tag{3.83}$$

with central charge $\bar{c}_2/3 = 1 + \frac{4}{15} > \bar{c}_1/3$. At this point, the operators associated to α_{11} and α_{12} are *irrelevant* but the operator associated to α_{22} is *relevant*. We conclude that models with $\alpha_{13}\alpha_{21} \neq 0$ and $\alpha_{22} = 0$ flow to this IR fixed point and orbits containing such models do not contribute to V. The orbits that do contribute have a point for which $\lambda_2 > 4\Lambda_1$.

Our toric model V of the moduli space is thus determined here by the cone

$$\lambda_1 > 10\Lambda_1 \ , \qquad \lambda_2 > 4\Lambda_1 \ , \qquad \Lambda_1 > 0 \ . \tag{3.84}$$

Applying (3.63) we find that, as expected, the point

$$\mu^* = (\tfrac{435}{471}, \tfrac{356}{471}; \tfrac{11}{471}, \tfrac{241}{471}) \tag{3.85}$$

lies in this cone. Points in the preimage of this have α_{21} and α_{22} both non-zero. We can use two of our rescalings to fix $\alpha_{21} = \alpha_{22} = 1$, and under the remaining symmetry the three coefficients in J_1 transform homogeneously, so we have $V = \mathbb{P}^2$. Of course, this is an overparametrization. This is not a plain model, and we can use the remaining field redefinitions $\Gamma^2 \to \Gamma^2 + \Gamma^1(a\Phi_1^5 + b\Phi_2^2)$ to show that these theories flow to a unique IR fixed point. Not unrelated to this is the fact that there is no discriminant here: any point in $\mathbb{P}^2$ corresponds to a non-singular model.

A model with $N > n$

The model discussed in section 3.3.3 shows more of the limitations of toric methods. Here we have $Y = \mathbb{C}^9$ and $T_{\mathbb{C}} = (\mathbb{C}^*)^4$ acts on the couplings. $\widehat{\Sigma}$ is the intersection of $\lambda_1 + 3\lambda_2 = 6(\Lambda_1 + \Lambda_2 + \Lambda_3)$ with the positive orthant. There are a total of 18 codimension-one cones in the interior of $\widehat{\Sigma}$. Proceeding with our method we find a five-dimensional toric variety V determined by the moment map values $\mu^* = (\frac{6}{7}, \frac{4}{7}; \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$. This is a puzzle, since we found previously that there are no models in this class with $\bar{c} = 3$. The resolution is that the model

$$\mathcal{W}_3' = \Gamma^1\Phi_1^6 + \Gamma^2\Phi_2^2 + \Gamma^3\Phi_2^2 \ , \tag{3.86}$$

which is in the inverse image under μ of the point μ^*, is fixed by a U(1) rotation in the $\Gamma^{2,3}$ plane that is not contained in $T_{\mathbb{C}}$. This is a symmetry which arises as an accidental symmetry for all points in V, and is manifest for $\mathcal{W}_3'$. This mixes with the IR R-symmetry leading to the central charge found above. This phenomenon in which a Fermi field is in fact free in the IR can occur in non-plain models with $N > n$. For models with $N = n$ a model with a free Fermi field will be singular.

3.5.4 Summary and further thoughts

We have provided evidence for a strong conjecture on the structure of the space of $T_{\mathbb{C}}$ orbits contributing to V. For models in which these are the only field redefinitions consistent with the UV symmetry this produces the moduli space $\mathcal{M}$ of SCFTs with central charge $\bar{c}$. By analogy with studies of (0,2) GLSM parameter spaces [70, 80], we call these *plain* models. For models with larger groups of field redefinitions, our discussion is partial in two ways: we have overparametrized the moduli space, and we have failed, in general, to exclude the basins of attraction of models in which a symmetry in the complement of $T_{\mathbb{C}}$ mixes with the IR R-symmetry.

Our evidence, while suggestive, falls short of a derivation of the result. The key difficulties in a proof are twofold. First, the consequences of enhanced symmetries that are only realized by singular superpotentials are difficult to grasp, since we do not have conformal perturbation theory as a guide. For these our evidence is based on the analysis of many examples that all turned out to be consistent with the conjecture. The second difficulty lies in extending the leading order conformal perturbation theory result for enhanced symmetry loci with $\bar{c}' = \bar{c}$. It may be possible to improve this by a more detailed study of the combinatorial structures involved.

A more satisfactory derivation can be imagined, which proceeds by constructing a $\bar{c}$ function along the RG flow and showing that this can be written in terms of α

through the combinations forming $(\lambda; \Lambda)$, along the lines of [71, 72, 49].[20] In that work, global symmetries broken by couplings were incorporated into a-maximization in four-dimensional theories by imposing constraints on the space of symmetries over which one maximized a trial a-function. The Lagrange multipliers implementing the constraints could then be used to parameterize the flow. In our case the symmetry-breaking couplings are the superpotential couplings which break the global symmetry $U(1)^{N+n}$ of the (free) UV theory to $U(1)$ and one can introduce Lagrange multipliers to constrain the symmetries over which $\overline{c}$ is extremized. Of course, imposing $N + n - 1$ constraints is a formal procedure, because this is tantamount to specifying the outcome. However, if one proceeds formally, one finds an expression for $\overline{c}$ in terms of the Lagrange multipliers and the *values* of these at the extremum — which reproduces (3.7) — are precisely the values of the moment map given by (3.63). The relation between this formal result and the values of the moment map is not clear to us.

3.6 Outlook

The project described in this chapter began as an attempt to classify IR fixed points of (0,2) LG theories — a generalization of the results obtained for (2,2) LG theories in [68, 67] . This beautiful work shows that for fixed $c = \overline{c}$ there is a finite set of families of superpotentials $W(X_1, \ldots, X_n)$, or equivalently charges $\boldsymbol{q}(X_i)$ that lead to a non-singular (2,2) SCFT of desired central charge. Having the (0,2) generalization would be very useful: we would have a new class of heterotic vacua and more generally (0,2) SCFTs with many properties computable in terms of the simple UV description. These would naturally fit into the class of (0,2) gauged linear sigma models and could be used to produce a large class of hybrid models along the lines of [27].

[20] A conversation with D. Kutasov, in which he suggested this idea, was instrumental in leading us to the results of this section.

What we learned is that, in contrast to the (2,2) case, it is not enough to classify non-singular (0,2) potentials realizing a particular set of $\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R}$ charges. For instance, the model studied section 3.3.3 would naively realize a $c = 4$, $\overline{c} = 3$ (0,2) SCFT that could correspond to some rather exotic 8-dimensional heterotic vacuum. In fact no such IR fixed point is obtained for any choice of the UV parameters. This is a general lesson for building UV models of (0,2) SCFTs: a check of UV R-symmetry anomalies is not enough, and while the UV theory may well flow somewhere (i.e to an SCFT with $\overline{c} > 0$), it may wind up far (i.e. at infinite distance) from the expectations of the model builder. We expect this to be a general lesson applicable to the wider class of gauged linear sigma models. In exploring that latter point it should be interesting to study in detail GLSMs with LG phases that exhibit accidents and extrapolate their consequences to large radius geometries.

For a class of models—the plain LG theories—we were able to obtain a compelling conjecture for a global description of the (0,2) moduli space $\mathcal{M}$ realizing the expected central charge. While the resulting combinatorial structure is consistent with a case-by-case analysis of field redefinitions and their orbits in examples, we were not able to prove it in generality. Progress on both testing and proving the conjecture could be made by developing a better understanding of the combinatorial structure of quasi-homogeneous (0,2) superpotentials, as well as developing Lagrange multiplier techniques and trial $\overline{c}$ functions. A classification of plain LG theories seems achievable; this would yield a large playground to explore LG RG flows and could give hints to the more general classification problem. Finally, it should be illuminating to relate our work to studies of RG flows with redundant couplings, e.g. [21].

4

Worldsheet instantons and linear models

4.1 Introduction

A natural starting point for exploring the moduli space of (0,2) heterotic compactifications is the study of the geometry of holomorphic vector bundles $\mathcal{V}$ over Calabi-Yau (CY) manifolds M. Under suitable conditions such bundles determine, to all orders in α', a supersymmetric heterotic vacuum. For a long time it has been known [35] that worldsheet instantons wrapping rational curves in M in principle generate a potential which destabilizes the vacuum. In rather special cases, such as models with (2,2) supersymmetry [42] or some specially fine-tuned (0,2) models [38, 40], the correction terms vanish for each instanton separately, but this is not true in more generic models [24, 31, 7].

In this context, heterotic compactifications obtained as gauged linear sigma models (GLSMs) [97] have received special attention, as they are believed to be stable under worldsheet instantons, even in the generic case in which the contributions of individual instantons do not vanish. This claim is then a nontrivial vanishing theorem about the total contribution from each instanton class. This was first proposed in [91] and further studied in [17] [1]. This work suggests that in these models the corrections vanish even when the contributions of individual instantons do not. If true, this would guarantee the existence of a vast playground for tackling issues of (0,2) moduli spaces.

Essentially, these arguments rely on the fact that in heterotic vacua determined by (0,2) theories the space-time superpotential for gauge singlets can be determined by a correlator $\mathcal{C}$ computed in a (half-) twisted version of the model. Unlike the twisted versions of (2,2) theories, this is not a topological field theory, but the simplifications associated with the existence of a nilpotent scalar charge, such as the decoupling of exact operators from the correlators of closed operators, carry over to this case and show that $\mathcal{C}$ depends holomorphically on the relevant worldsheet couplings.

This holomorphy together with compactness arguments can be used to show that the correlator vanishes identically. The argument of [91] used the fact that the parameter space of the GLSM is compact (or has a natural compactification). $\mathcal{C}$ was shown to be a global section of a holomorphic bundle of negative curvature. If nonzero, this must exhibit poles, which in this theory arise from the finite-energy configurations with very large field values which occur at special loci in the parameter space. At these loci the model is indeed singular, but the large-field region can be studied semiclassically to demonstrate that these configurations do not lead to any singularities in $\mathcal{C}$. The absence of poles shows that this vanishes identically. This was pursued explicitly in a simple example, but the argument did not appear to rely on details of this example so seemed likely to generalize.

In turn, the argument of [17] relied on the compactness of an appropriate moduli space of instantons. More precisely, these authors used the fact that the contribution to $\mathcal{C}$ at any fixed instanton number can be related to a calculation in a model in which many of the worldsheet couplings vanish. In this model, the moduli space

[1] In [16] an argument along rather different lines was pursued.

of instantons is compact, and a zero-mode counting argument shows that the contribution to $\mathcal{C}$ vanishes. Here too, detailed calculations were done in simple examples but the argument seemed very robust and likely to hold in general.

In this chapter, we follow up on this work with a systematic study of the conditions under which the vanishing theorem of [17] applies. We find that in a generic gauged linear sigma model the argument that the moduli space of instantons is compact fails, reviving the question of whether these models are in fact destabilized by worldsheet instantons. We do not resolve this question. We are, however, able to construct an extensive class of models for which the argument holds – a sizable playground, if not as extensive as had been hoped.

The rest of the chapter is organized as follows. In Section 4.2 we review the construction of (0,2) linear models relevant for our analysis. In Section 4.3 we show that there exist models for which the vanishing of the space-time superpotential for gauge singlets is not guaranteed and we present an example in detail. In Section 4.4 we prove a vanishing theorem for a particular subclass of (0,2) linear models. In Section 4.5 we end with some implications of this work and future directions.

4.2 The linear model

Our tool for investigating the issue of instanton corrections in this chapter is the (0,2) gauged linear sigma model. For a suitably constructed bundle $\mathcal{V}$ on a CY space M presented as a complete intersection $H_A = 0$ in a Fano toric variety V,[2] the IR worldsheet dynamics is expected to be the same as that of an Abelian gauge theory with (0,2) supersymmetry. In this section we are going to review the construction of the (0,2) linear model [97] in order to establish notation and define the class of models we consider. More details can be found in appendix C.1.

The linear models we consider are gauge theories with gauge group $\mathrm{U}(1)^R$, along

[2] We recall that a variety V is Fano if and only if the anticanonical bundle K_V of V is ample.

with m neutral chiral supermultiplets we call $\Sigma_\mu = (\sigma_\mu, \lambda_{\mu,+})$. We couple these to a collection of charged supermultiplets determined by the geometric data:

$$\begin{array}{ccccccc} \text{fields} & P^\alpha & \Phi^i & \Gamma^I & \Lambda^A & S & \Xi \\ \mathrm{U}(1)^a & -m^a_\alpha & q^a_i & Q^a_I & -d^a_A & m^a - d^a & d^a - m^a \end{array} \tag{4.1}$$

where

$$m^a = \sum_\alpha m^a_\alpha \ , \qquad\qquad d^a = \sum_A d^a_A \ . \tag{4.2}$$

The n chiral multiplets $\Phi^i = (\phi^i, \psi^i)$ and their charges are determined by a presentation of V as a symplectic $\mathrm{U}(1)^R$ quotient. The model has Fayet-Iliopoulos D-terms whose values r^a correspond to the shift in the moment map for the $\mathrm{U}(1)^R$ action. The moduli space of classical vacua of a theory containing only these fields will be V when the r^a lie in a cone $\mathcal{K}_V$, the Kähler cone of V.

The N Fermi multiplets Γ^I, with lowest components the left-moving fermions γ^I, satisfy a chirality condition [3]

$$\overline{\mathcal{D}}\Gamma^I = \sqrt{2}E^I(\Sigma, \Phi) \ , \qquad\qquad E^I(\Sigma,\Phi) = \Sigma_\mu E^{I\mu}(\Phi) \ , \tag{4.3}$$

and their charges determine the bundle $\mathcal{E} \to V$ by the short exact sequence (SES)

$$0 \longrightarrow \oplus_\mu \mathcal{O} \xrightarrow{E^{I\mu}} \oplus_I \mathcal{O}(Q_I) \longrightarrow \mathcal{E} \longrightarrow 0 \ . \tag{4.4}$$

This collection of fields with these couplings comprises what we refer to as the V model [84]. It is not a conformal field theory, and will typically exhibit trivial IR behavior. The space of V models is parameterized by r^a complexified by θ angles as well as the coefficients of the maps $E^{I\mu}$.[4]

In general, there is a larger cone, which we call the *geometric cone* $\mathcal{K}_c$, in which the space of vacua has this character. More precisely, this is the cone in which the

[3] Unless otherwise specified, we use Einstein's summation convention throughout the chapter.

[4] These are in general subject to identifications, so this is an overparameterization.

V model as defined above has supersymmetric classical vacua. It is generated by q_i, and it is divided into phases. These correspond to subcones of $\mathcal{K}_c$, one of which is $\mathcal{K}_V$, separated by hyperplanes associated to U(1) subgroups of the gauge group which are unbroken at large ϕ.

To construct our superconformal theory we augment the V model by the k chiral multiplets $P^\alpha = (p^\alpha, \chi^\alpha)$. For $r \in \mathcal{K}_V$ the space of classical vacua for the theory including these is the total space $V^+ = \mathrm{tot}\,(\oplus_\alpha \mathcal{O}(-m_\alpha) \to V)$. We introduce as well L Fermi multiplets Λ^A, whose lowest components are the left-moving fermions η^A, satisfying a chirality condition

$$\overline{\mathcal{D}}\Lambda^A = \sqrt{2}E^A(P, \Sigma, \Phi) , \qquad E^A(P, \Sigma, \Phi) = \Sigma_\mu P^\alpha E_\alpha^{A\mu}(\Phi) . \tag{4.5}$$

The model with these fields and couplings will be referred to as the V^+ model. Like the V model, it will not in general be conformal. In addition to the parameters listed above, it is specified by the coefficients of the maps $E_\alpha^{A\mu}$.

The conformal model in which we are interested – the M model – is obtained from the V^+ model by adding a superpotential interaction

$$\int d\theta^+ \left(\Lambda^A H_A(\Phi) + \Gamma^I J_I(P, \Phi)\right)\big|_{\bar\theta^+=0} + \text{h.c.} , \tag{4.6}$$

where

$$J_I(P, \Phi) = P^\alpha J_{I\alpha}(\Phi) , \tag{4.7}$$

subject to the conditions

$$\sum_A H_A E_\alpha^{A\mu} + \sum_I J_{I\alpha} E^{\mu I} = 0 \qquad \forall \alpha, \mu , \tag{4.8}$$

required in order to preserve (0,2) supersymmetry. For $r \in \mathcal{K}_V$ and generic H_A, the space of classical vacua is the complete intersection $M = \{\phi \in V | H_A(\phi) = 0\}$. When this is nonsingular the Λ fermions all acquire a mass and the light left-moving fermions take values in the bundle $\mathcal{V} \to M$ defined by the restriction to M of the complex

$$0 \longrightarrow \oplus_\mu \mathcal{O} \xrightarrow{E^{I\mu}} \oplus_I \mathcal{O}(Q_I) \xrightarrow{J_{I\alpha}} \oplus_\alpha \mathcal{O}(m_\alpha) \longrightarrow 0 \ , \tag{4.9}$$

as $\mathcal{V} = \operatorname{Ker} J/\mathrm{Im}E$. We assume that E is everywhere injective and J everywhere surjective on M, and that $\mathcal{V}$ is a nonsingular, stable holomorphic vector bundle. These are the geometric models to which our methods apply. An intermediate step in the vanishing argument below involves setting $H = J = E = 0$ in the M model – we refer to this as the O model.

The chiral superfield S and the chiral Fermi multiplet Ξ are the "spectator" fields introduced in [39] to maintain the Kähler parameters r^a as RG invariant quantities. In fact, the counterterm by which r gets renormalized at one-loop is proportional to the sum of the gauge charges of the scalar fields in the theory. In our models this is in general nonzero (this is related to the fact that V^+ is not Calabi–Yau). Introducing S as above cancels this. The spectators earn their name because they interact via a superpotential

$$\int d\theta^+ \Xi\, S\big|_{\overline{\theta}^+=0} + \text{h.c.}\ . \tag{4.10}$$

This means these are massive fields and have no effect on the IR dynamics of the theory, so one might question the relevance of including them here. As we will see, in some cases accounting for their presence allows the argument to proceed where it might otherwise fail. This is, of course, a technical matter. If there are no instanton corrections in the presence of spectators there are none in their absence. This demonstrates an important caveat to our work, mentioned above. When the argument of [17] fails, we cannot assert that instanton corrections *do* destabilize the model, only that this particular argument that they do not is not valid.

Symmetries

The action (C.9)-(C.14) we have described is invariant under (0,2) SUSY and the gauge symmetry $\mathrm{U}(1)^R$, as well as a global $\mathrm{U}(1)_R \times \mathrm{U}(1)_L$ symmetry acting as

$$\begin{array}{ccccccccc} \text{fields} & P^\alpha & \Phi^i & \Gamma^I & \Lambda^A & S & \Xi & \Sigma_\mu & \Upsilon_a \\ \mathrm{U}(1)_L & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 0 \\ \mathrm{U}(1)_R & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array} \tag{4.11}$$

While $\mathrm{U}(1)_R$ is believed to be the R-symmetry of the SCFT to which our model is supposed to flow, the global $\mathrm{U}(1)_L$ symmetry is equally important for our purposes: in heterotic compactifications, it can be used to construct a left-moving spectral flow operator and it provides a linearly realized component of the space-time group. The action of global symmetries on charged fields is of course defined up to an arbitrary action of the gauge symmetry generators. We have here chosen a representative action that is manifestly unbroken in the classical vacua (when $r \in \mathcal{K}_V$) comprising M.

These symmetries are respected by the classical action, but are in general anomalous in the presence of non-trivial gauge fields. The anomalies vanish when the charges satisfy

$$\begin{aligned} d^a &= \sum_i q_i^a \ , \\ m^a &= \sum_I Q_I^a \ , \\ \sum_\alpha m_\alpha^a m_\alpha^b + \sum_i q_i^a q_i^b &= \sum_A d_A^a d_A^b + \sum_I Q_I^a Q_I^b \ . \end{aligned} \tag{4.12}$$

In terms of our geometric data, the first two conditions reflect the fact that

$$c_1(T_M) = c_1(\mathcal{V}) = 0 \ , \tag{4.13}$$

while the quadratic condition implies

$$\mathrm{ch}_2(T_M) = \mathrm{ch}_2(\mathcal{V}) \ . \tag{4.14}$$

Under these conditions, the M model is believed to flow at low energies to a nontrivial superconformal field theory which is in the same moduli space as the nonlinear sigma model determined by the pair $(M, \mathcal{V})$. Nonperturbative effects (worldsheet instantons) which can destroy conformal invariance are captured by GLSM gauge instantons, which are the subject of our investigation here.

4.3 The argument

Let us first review the argument prescribed in [17] for the vanishing of the instanton contributions to the superpotential W for space-time gauge singlets in a (0,2) linear sigma model.

The goal is to probe for a background space-time superpotential W. A simple and direct way to achieve this is to compute the correlator $\mathcal{C}_{abc} = \langle R_a R_b R_c \rangle$, where R_a is the vertex operator representative for the Kähler modulus $\mathcal{R}_a$ of V. In fact, for each instanton the exponential factor e^{I_0}, where I_0 is the instanton classical action, contains all the dependence on $\mathcal{R}_a$ [35, 37]. The correlator $\mathcal{C}_{abc}$ computes the third derivatives of W with respect to $\mathcal{R}_a$, thus it determines W up to quadratic terms in the $\mathcal{R}_a$. These terms are forbidden by standard α' non-renormalization theorems, hence $\mathcal{C}_{abc}$ determines W directly.

The computation is most easily done in the half-twisted model (see appendix C.2). In this model, the supercharge $\overline{\mathcal{Q}}_+$ becomes a nilpotent scalar symmetry generator, and correlators of $\overline{\mathcal{Q}}_+$-closed operators can be computed in its cohomology. On a genus-zero worldsheet, the twist can be realized by spectral flow insertions and calculations in the twisted model produce suitable correlators of the untwisted (physical) model.

In order to determine the linear model representative of the space-time mode R_a we restrict our attention to the (0,2) gauge multiplets. In fact, R_a appears in the linear model through a F-I term. Moreover, gauge singlets must have $q_L = 0$ and bosonic vertex operators have $q_R = 1$. Finally,

$$\overline{\mathcal{Q}}_+ \lambda_{a,-} = 0 \ , \qquad \mathcal{Q}_+ \lambda_{a,-} = \frac{1}{2}\left(D_a - i f_{a,01}\right) \tag{4.15}$$

determine $R_a = \lambda_{a,-}$.

The first step of the argument is to show that $\mathcal{C}_{abc}$ vanishes in the O model. The idea is that the theory without superpotential has a very large symmetry, $G = \mathrm{U}(1)^{\oplus(n+N+k+L+2)}$, where each matter superfield is rotated separately, and the vertex operators R_a are invariant under this. This symmetry is generically broken by superpotential couplings down to $\mathrm{U}(1)_L$. If the zero-mode path integral measure [5] in nontrivial topological sectors turns out not to be invariant under G, i.e. the symmetry is anomalous, then contributions to the invariant correlator $\mathcal{C}_{abc}$ from this sector will vanish. In practice, we follow [17] and construct a U(1) subgroup of G that is rendered anomalous in *all* topological sectors by the twisting procedure, demonstrating that $\mathcal{C}_{abc} = 0$ identically in the O model.

The second step uses the fact that $\mathcal{C}_{abc}$ depends *holomorphically* on J, H and E. One can then examine the contribution at arbitrary order in an expansion in these couplings. If there is no term that can possibly absorb the fermion zero-modes in the anomalous measure, then the correlator vanishes identically. This computation can be performed in each topological sector of the path integral.

In the untwisted model, the limiting point $J = H = E = 0$ is of course highly singular. Both σ and p acquire zero-modes and the space of classical vacua is non-compact. Such a singularity can invalidate the order-by-order calculation described

[5] We recall that the path integral for a correlator of $\overline{\mathcal{Q}}_+$-invariant operators localizes on fixed loci of $\overline{\mathcal{Q}}_+$, given by zero-modes.

above. The key observation of [17] is that in suitable examples these dangerous zero-modes are absent in the half-twisted model. For example, the bosons σ always have zero-modes, but in the twisted model (see appendix C.2) these fields acquire a spin and their zero-modes are absent. In general, as we shall see below, p zero-modes are not completely removed by the twisting.

Another approach was presented in [17], where the vanishing of $\mathcal{C}_{abc}$ at any instanton number follows from an appropriate counting of fermi zero-modes in the half-twisted model. This was applied in detail for heterotic compactifications described by half-linear sigma models, but it also extends to linear models as well. However, as pointed out above, the same assumption of compactness is required for this argument to be valid. For definiteness, we present our analysis of the linear model following the approach of [17] reviewed above, as our results will not depend on this choice.

4.3.1 The quintic

Let us review how all of this works for the linear model describing the deformations of the tangent bundle T_M over the quintic hypersurface M in $V = \mathbb{P}^4$. The gauge charges for the (2,2) multiplets $\Phi^i = (\Phi^i, \Gamma^i)$ and $P = (P, \Lambda)$ are

$$\begin{array}{ccccccc} \text{fields} & P & \Phi^1 & \Phi^2 & \Phi^3 & \Phi^4 & \Phi^5 \\ \mathrm{U}(1) & -5 & 1 & 1 & 1 & 1 & 1 \end{array} \tag{4.16}$$

The Kähler cone $\mathcal{K}_V = \mathcal{K}_c$ here is simply given by $r \geqslant 0$ and the relevant instantons are defined by $\mathcal{K}_V^\vee = \{n \geqslant 0\}$. The O model has as a target space the total space of the anticanonical bundle on V, $\operatorname{tot}(\mathcal{O}(-5) \to \mathbb{P}^4)$. First of all, we check that the moduli space of gauge instantons for this model is compact. Indeed, we verify that there are no holomorphic sections of

$$p \quad \leftrightarrow \quad \Gamma(K^{\frac{1}{2}} \otimes \mathcal{O}(-5n)) \ , \tag{4.17}$$

and thus p has no zero-modes. This, together with the fact that there are no zero-modes of σ shows that the space of zero-modes is compact in any topological sector. Next, by looking at the degree of the line bundles of the half-twisted model in (C.21) we see that the relevant fermions zero-modes are

$$\begin{array}{ccccc} \text{fields} & \overline{\psi}^i & \gamma^i & \overline{\eta} & \chi \\ \text{bundle} & \overline{\mathcal{O}}(n) & \mathcal{O}(n-1) & \mathcal{O}(5n) & \overline{\mathcal{O}}(5n-1) \\ \#\ \text{z.m.} & n+1 & n & 5n+1 & 5n \end{array} \tag{4.18}$$

The fermion contribution to the zero-mode path-integral measure is then given by

$$d\mu_F = d\overline{\lambda}_- d\overline{\eta} d\chi \prod_i d\overline{\psi}^i d\gamma^i \ . \tag{4.19}$$

Now, the O model is invariant under a symmetry $\mathrm{U}(1)_C$ which assigns charge $+1$ to the multiplets Φ^i and leaves everything else invariant. Under this symmetry the measure above transforms with charge $+5$. Hence, the correlator $\mathcal{C}$ vanishes in the O model. The holomorphic superpotential couplings are given by

$$\mathcal{L}_{\text{Yuk}}\big|_{\overline{J}=\overline{H}=\overline{E}=0} = -\overline{\gamma}^i E^i \lambda_+ + \gamma^i J_i \chi + \eta H_{,j} \psi^j \ , \tag{4.20}$$

where H is a quintic polynomial defining the hypersurface M, J_i are generic quartic polynomials and E^i are generic linear polynomials subject to (4.8). Clearly, each coupling transforms under $\mathrm{U}(1)_C$ with either charge $+5$ or is neutral. By the argument above the correlator $\mathcal{C}$ vanishes in the full theory and there are no instanton corrections to the space-time superpotential.

4.3.2 A counter-example

Let us consider a two-parameter model with the following charge assignments

$$\begin{array}{ccccccccccc} \text{fields} & \Phi^{1,2,3} & \Phi^{4,5} & \Phi^{6,7} & \Lambda^1 & \Lambda^2 & \Gamma^{1,2} & \Gamma^3 & \Gamma^{4,5,6,7} & P^1 & P^2 \\ \mathrm{U}(1)_1 & 1 & 1 & 0 & -3 & -2 & 2 & 1 & 0 & -4 & -1 \\ \mathrm{U}(1)_2 & 1 & 0 & 1 & -3 & -2 & 0 & 1 & 1 & -2 & -3 \end{array} \tag{4.21}$$

In the geometric phase it describes a complete intersection M of degree $(3,3)$ and $(2,2)$ hypersurfaces in the toric variety V defined by the charges

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} . \tag{4.22}$$

It is useful to write the maps defining the superpotential more explicitly. For ease of notation, let us denote $x = \{\phi^{1,2,3}\}$, $y = \{\phi^{4,5}\}$, $z = \{\phi^{6,7}\}$, as well as $\Gamma^{(1)} = \{\Gamma^{1,2}\}$, $\Gamma^{(2)} = \{\Gamma^{3}\}$, $\Gamma^{(3)} = \{\Gamma^{4,5,6,7}\}$, and a condensed notation in which, e.g. x^k denotes a generic homogeneous polynomial of degree k in $\phi^{1,2,3}$. With this notation the maps are given as

$$\begin{aligned} J_{(1)} &= p^1(x^2 + xyz + y^2z^2) \ , \\ J_{(2)} &= p^1(xy^2 + y^3z) + p^2z^2 \ , \\ J_{(3)} &= p^1(xy^3 + y^4z) + p^2(xz + yz^2) \ , \end{aligned} \tag{4.23}$$

while the equations defining the complete intersections are

$$\begin{aligned} H_1 &= x^3 + x^2yz + xy^2z^2 + y^3z^3 \ , \\ H_2 &= x^2 + xyz + y^2z^2 \ . \end{aligned} \tag{4.24}$$

The complete intersection M is realized in the cone $\mathcal{K}_V = \{r_1 > 0, r_1 - r_2 < 0\}$, where the irrelevant ideal is $B = (xy)(z)$. Since the z's are not both allowed to vanish and the coefficients in the expressions above are generic, we have that

$$\begin{gathered} \underbrace{(xz + yz^2)}_{\text{4 of these}} = x^3 + x^2yz + xy^2z^2 + y^3z^3 = x^2 + xyz + y^2z^2 = 0 \ , \\ \underbrace{(x^2 + xyz + y^2z^2)}_{\text{3 of these}} = xy^2 + y^3z = \underbrace{(xy^3 + y^4z)}_{\text{4 of these}} = x^3 + x^2yz + xy^2z^2 + y^3z^3 = 0 \ , \end{gathered} \tag{4.25}$$

have no solutions compatible with the ideal B. Thus $p^1 = p^2 = 0$ and there are no flat directions in this phase.

Now, note that $-p^2 \in \mathcal{K}_V$ but $-p^1 \notin \mathcal{K}_V$. Therefore there are instantons contributing for this phase for which p^1 develops zero-modes in the O model. We can see this explicitly. Gauge instantons in this model have instanton numbers $n_a \in \mathcal{K}_V^\vee$, i.e. $n_2 > 0$ and $n_1 + n_2 > 0$. From appendix C.2 we see that the zero-modes of p^1 are in one to one correspondence with holomorphic section of the bundle

$$p^1 \quad \leftrightarrow \quad \Gamma(K^{\frac{1}{2}} \otimes \mathcal{O}(-4n_1 - 2n_2)) \ , \tag{4.26}$$

and the number of such sections is non-zero when $2n_1 + n_2 < 0$. The subcone defined by $(2n_1 + n_2 < 0) \cap \mathcal{K}_V^\vee$ is non empty, and the moduli space of gauge instantons of the O model is not compact. The twisted O model calculation in these sectors is ill-defined and the argument from holomorphy does not exclude instanton corrections to $\mathcal{C}_{abc}$.

4.4 The vanishing theorem

The example of the previous section shows that a generic (0,2) GLSM is not protected from worldsheet instanton corrections. In this section we undertake the task of constructing a class of models for which the vanishing theorem holds. In fact, a necessary condition for the vanishing argument to apply is that there exists a cone $\mathcal{K}_V \subseteq \mathcal{K}_c$ such that

1. the M model defined in $\mathcal{K}_V$ is nonsingular;
2. the O model of the half-twisted theory has a compact moduli space of gauge instantons for any $n_a \in \mathcal{K}_V^\vee$.

Notice that as advertised above, due to the twist the bosons σ acquire a spin and do not have zero-modes. A quick inspection at the form of the E-couplings (4.3) and (4.5) implies that setting $E = 0$ does not lead to any singularities in the half-twisted theory.

4.4.1 O model gauge instanton moduli space

While the moduli space of gauge instantons for the V model is compact, as we have seen above, there can be unbounded zero-modes coming from the p^α fields. This occurs when, for a given subcone $\mathcal{K}_V \subseteq \mathcal{K}_c$ we have $m_\alpha \notin \mathcal{K}_V$ for some α. Hence, a necessary condition for the argument to work is that there exists a nonsingular subcone $\mathcal{K}_V \subseteq \mathcal{K}_c$ such that $m_\alpha \in \mathcal{K}_V$ $\forall \alpha$.

The discussion so far did not take into account the spectator boson s, whose expectation value is set to zero, and whose zero-modes could also be fatal for our assumption of compactness. In order to establish when this is the case, we need the following simple fact: the cone $\widehat{\mathcal{K}}_c$ defined by adjoining the vector $m - d$ to $\mathcal{K}_c$ is convex unless $d - m \in \mathcal{K}_c$. In fact, $\widehat{\mathcal{K}}_c$ fails to be convex if we can write

$$\sum_i \alpha_i q_i + \beta(m - d) = 0 \ , \tag{4.27}$$

with $\alpha_i, \beta \geqslant 0$ and not all vanishing. Because $\mathcal{K}_c$ is convex by assumption, we must have β strictly positive. This means $\beta(d - m) = \sum_i \alpha_i q_i$, i.e. $(d - m) \in \mathcal{K}_c$.

This little result suggests there are three separate cases we should consider:

1. $d - m \in \mathcal{K}_V$. By (C.21) s has no zero-modes. In fact, by looking at the degree
$$d_S = (m^a - d^a) n_a, \tag{4.28}$$
we have $d_S \leqslant 0$ $\forall n \in \mathcal{K}_V^\vee$.

2. $d - m \in \mathcal{K}_c$ but $d - m \notin \mathcal{K}_V$. In this case there exist $n \in \mathcal{K}_V^\vee$ such that $d_S > 0$ and s has zero-modes. The half-twisted O model develops s-flat directions and is therefore singular.

3. $d - m \notin \mathcal{K}_c$. In this case s always has zero-modes, but by the result above, together with the fact that a toric variety is compact if and only if the geometric cone is strongly convex, the moduli space of instantons for the O model is nevertheless compact.

We can now summarize the set of conditions we are going to assume for our vanishing theorem: there exists a nonsingular subcone $\mathcal{K}_V \subseteq \mathcal{K}_c$ such that the gauge charge vectors for the fields p^α and s satisfy

$$m_\alpha \in \mathcal{K}_V \quad \forall \alpha \ , \qquad\qquad d - m \in \mathcal{K}_V \quad \text{or} \quad d - m \notin \mathcal{K}_c \ . \tag{4.29}$$

4.4.2 A classical symmetry

For the remaining of this section we restrict our attention to models obeying the conditions above. To proceed with the argument we need to construct a suitable $\mathrm{U}(1)_C$ subgroup of the symmetry group of the O model. Let us choose the charges for the matter fields under this "classical" symmetry as

$$\begin{array}{ccccccc} \text{fields} & P^\alpha & \Phi^i & \Gamma^I & \Lambda^A & S & \Xi \\ \mathrm{U}(1)_C & 0 & q_i^C & Q_I^C & 0 & q_S^C & Q_\Xi^C \end{array} \tag{4.30}$$

while the gauge fields are invariant. This symmetry will be non-anomalous (before twisting) if

$$\sum_i q_i^a q_i^C + (m^a - d^a) q_S^C = \sum_I Q_I^a Q_I^C + (d^a - m^a) Q_\Xi^C \ , \tag{4.31}$$

for $a = 1, \ldots, R$.

The measure

First, let us look at the zero-mode contribution to the path integral measure. In particular, we are going to focus only on the fermionic part of the measure. In fact, the form of the maps (C.21), together with our assumption of compactness yield an exact balance between holomorphic and anti-holomorphic bosonic zero-modes. It is convenient to write the fermionic measure as

$$d\mu_F = d\mu_G d\mu_M d\mu_S \ , \tag{4.32}$$

where the three factors correspond to the measure for the gauge, matter and spectator fields respectively. From (C.20) it follows that the gauge measure is simply given by

$$d\mu_G = \prod_a d\overline{\lambda}_{-,a} \ . \tag{4.33}$$

For the matter fields we have

$$\begin{array}{lcccc} \text{fields} & \chi^\alpha & \overline{\chi}^\alpha & \psi^i & \overline{\psi}^i \\ \text{bundle} & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}(-d_\alpha) & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}(d_\alpha) & \overline{K} \otimes \overline{\mathcal{O}}\,(-d_i) & \overline{\mathcal{O}}\,(d_i) \\ \#\text{ z.m.} & \max(0,-d_\alpha) = -d_\alpha & \max(0,d_\alpha) = 0 & \max(0,-d_i-1) & \max(0,d_i+1) \end{array} \tag{4.34}$$

where we used the fact that $m_\alpha \in \mathcal{K}_V$ implies $d_\alpha \leqslant 0$, as well as

$$\begin{array}{lcccc} \text{fields} & \gamma^I & \overline{\gamma}^I & \eta^A & \overline{\eta}^A \\ \text{bundle} & K^{\frac{1}{2}} \otimes \mathcal{O}\,(D_I) & K^{\frac{1}{2}} \otimes \mathcal{O}\,(-D_I) & K \otimes \mathcal{O}\,(D_A) & \mathcal{O}\,(-D_A) \\ \#\text{ z.m.} & \max(0,D_I) & \max(0,-D_I) & \max(0,D_A-1) & \max(0,-D_A+1) \end{array} \tag{4.35}$$

The matter measure then reads

$$d\mu_M = \prod_\alpha d\chi^\alpha \prod_{i|d_i\geqslant 0} d\overline{\psi}^i \prod_{i|d_i<0} d\psi^i \prod_{I|D_I\geqslant 0} d\gamma^I \prod_{I|D_I<0} d\overline{\gamma}^I \prod_{A|D_A>0} d\eta^A \prod_{A|D_A\leqslant 0} d\overline{\eta}^A \ , \tag{4.36}$$

and it is easy to check that it is gauge-invariant.

Finally, for the spectators

$$\begin{array}{lcccc} \text{fields} & \xi_+ & \overline{\xi}_+ & \xi_- & \overline{\xi}_- \\ \text{bundle} & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}\,(-d_S) & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}\,(d_S) & K^{\frac{1}{2}} \otimes \mathcal{O}(-d_S) & K^{\frac{1}{2}} \otimes \mathcal{O}(d_S) \\ \#\text{ z.m.} & \max(0,-d_S) & \max(0,d_S) & \max(0,-d_S) & \max(0,d_S) \end{array} \tag{4.37}$$

Here we need to distinguish two cases, according to whether $d-m \in \mathcal{K}_V$ or $d-m \notin \mathcal{K}_c$, and we obtain

$$d\mu_S = \begin{cases} d\xi_+ d\xi_- & \text{if } d_S < 0 \ , \\ d\overline{\xi}_+ d\overline{\xi}_- & \text{if } d_S > 0 \ . \end{cases} \tag{4.38}$$

Of course, if $d_S = 0$ we simply ignore this factor.

Now we can finally determine how the measure transforms under the symmetry $\mathrm{U}(1)_C$ defined above. The gauge measure is invariant, while for the matter factor we obtain

$$q^C(d\mu_M) = \sum_{i|d_i \geqslant 0} (d_i + 1) q_i^C + \sum_{i|d_i<0} (-d_i - 1)(-q_i^C) + \sum_{I|D_I \geqslant 0} D_I(-Q_I^C) + \sum_{I|D_I<0} (-D_I) Q_I^C \ . \tag{4.39}$$

Finally, for the spectator measure in both cases of (4.38) we get

$$q^C(d\mu_S) = d_S(q_S^C + Q_\Xi^C) \ , \tag{4.40}$$

Let us observe at this point that a very simple solution to (4.31) is given by

$$q_i^C = Q_I^C = q_S^C = 1 \ , \qquad Q_\Xi^C = 0 \ , \tag{4.41}$$

where it is easy to verify that the equality holds by (4.12). Plugging these values into the expressions above we find that the total fermionic zero-mode measure in the twisted model transforms with charge $q^C(d\mu_F) = n$, where we recall that n is the number of one-dimensional cones of the fan Δ_V for the toric variety V, and in particular is strictly positive. Thus, the fermion zero-modes cause $\mathrm{U}(1)_C$ to be anomalous, and $\mathcal{C}_{abc}$ vanishes in the O model.

The superpotential couplings

Let us turn to the analysis of the superpotential couplings in the action. The relevant Yukawa couplings are

$$\mathcal{L}_{\mathrm{Yuk}}\big|_{\overline{J}=\overline{H}=\overline{E}=0} = -\overline{\gamma}^I E^I_{,\mu} \lambda_{\mu,+} + \gamma^I J_{I\alpha} \chi^\alpha + \eta^A H_{A,j} \psi^j \ , \tag{4.42}$$

where we have set $\sigma_\mu = p^\alpha = 0$, as they have no zero-modes. We immediately see that all couplings, when non-zero, have the following lower bounds on the charges

$$\begin{array}{cccc} \text{couplings} & \overline{\gamma}^I E^I_{,\mu} \lambda_{\mu,+} & \gamma^I J_{I\alpha} \chi^\alpha & \eta^A H_{A,j} \psi^j \\ \mathrm{U}(1)_C & \geqslant 0 & \geqslant 2 & \geqslant 1 \end{array} \tag{4.43}$$

In particular, we note that these values are all non-negative and therefore it is not possible to absorb the zero-modes in excess in the measure by bringing down fermion terms from the action. The correlator $\mathcal{C}_{abc}$ thus vanishes at all orders in the superpotential couplings, which concludes the proof that instantons do not contribute to the space-time superpotential in our class of models.

Note that we ignored the anti-holomorphic functions $\overline{J}$, $\overline{H}$ and $\overline{E}$ in (4.42). This is in fact legitimate since, as observed above, half-twisted correlators of $\overline{\mathcal{Q}}_+$-closed operators have a holomorphic dependence on J, H and E.

4.5 Outlook

In this chapter, we investigated the details of the elegant argument of [17] for the absence of instanton corrections to the space-time superpotential in heterotic compactifications based on (0,2) GLSMs. We have not been able to extend the argument to the most general case.

The immediate question raised is: are some of these vacua in fact destabilized by instantons? One clear way to resolve this would be to produce an argument that holds in more generality. It is possible, however, that no such argument can be found and that in fact instanton corrections do arise. One way to detect such corrections would be an indirect approach, in which properties of the solution, such as the dimension of the space of massless gauge-neutral scalar fields, are compared at different limiting points in the moduli space. A more direct approach would be to compute the instanton contributions explicitly. Perhaps the GLSM can provide a framework within which these calculations, which have proved difficult in general, are tractable.

On the other hand, we have now an extensive class of (0,2) models which are truly conformally invariant. These can be used to explore the moduli space of (0,2) theories without a (2,2) locus, extending recent work that has focused on deformations of

(2,2) models [76, 5, 45]. In particular, one could look for special loci, e.g. good hybrid models [27] or Landau-Ginzburg points and hope to learn something about the structure of the resulting theories. In particular, hybrid models could be a promising laboratory for explicit computations of worldsheet instantons, given the simpler structure of rational curves on the lower dimensional base instead than on a CY three-fold.

We have shown in Chapter 3 that other "bad" things can happen in (0,2) models [26]. In particular, it is shown, in the context of Landau-Ginzburg models, that the common assumption that accidental IR symmetries do not spoil the correspondence between operators in the IR and the ones in the UV is not guaranteed in (0,2) models. When this occurs, the structure of the conformal manifold is dramatically modified. There is a priori no reason that would prevent the same phenomenon from happening in a generic phase of a GLSM. For example, one could realize one of the "accidental" LG theories as a phase of a GLSM and study how this pathology is realized in the geometric phase. This could shed new light on the conditions for the data $(M, \mathcal{V})$ to lead to consistent heterotic backgrounds.

Appendix A

Hybrid geometry

A.1 An example

Let $B = \mathbb{P}^1$ and take $\boldsymbol{Y}$ to be the total space of $X = \mathcal{O}(-2) \to \mathbb{P}^1$. We cover $\boldsymbol{Y}$ by two patches $\mathcal{U}_u$ and $\mathcal{U}_v$, with local coordinates (u, ϕ_u) and (v, ϕ_v), respectively :

$$u = v^{-1}, \qquad \phi_u = v^2 \phi_v \qquad \text{on} \quad \mathcal{U}_u \cap \mathcal{U}_v = \mathbb{C}^*. \tag{A.1}$$

The projection $\pi : \boldsymbol{Y} \to B$ is simply $(u, \phi_u) \to u$ and $(v, \phi_v) \to v$ in the two patches. The transition function for $\sigma = \sigma_u^1 \partial_u + \sigma_u^2 \partial_{\phi_u}$, a section of $T_{\boldsymbol{Y}}$, is

$$\begin{pmatrix} \sigma_u^1 & \sigma_u^2 \end{pmatrix} = \begin{pmatrix} \sigma_v^1 & \sigma_v^2 \end{pmatrix} \begin{pmatrix} -v^{-2} & 2v\phi_v \\ 0 & v^2 \end{pmatrix} . \tag{A.2}$$

$T_{\boldsymbol{Y}}$ belongs to a family of rank 2 holomorphic bundles $\mathcal{V}_\epsilon \to \boldsymbol{Y}$ with transition function

$$\begin{pmatrix} \sigma_u^1 & \sigma_u^2 \end{pmatrix} = \begin{pmatrix} \sigma_v^1 & \sigma_v^2 \end{pmatrix} M_\epsilon , \qquad M_\epsilon \equiv \begin{pmatrix} -v^{-2} & 2\epsilon v\phi_v \\ 0 & v^2 \end{pmatrix} . \tag{A.3}$$

When $\epsilon = 0$ the bundle splits: $\mathcal{V}_{\epsilon=0} = \pi^*\mathcal{O}(2) \oplus \pi^*\mathcal{O}(-2)$; more generally $\mathcal{V}_\epsilon$ is an irreducible rank 2 bundle over $\boldsymbol{Y}$.

An example of a quasi-homogeneous superpotential depending on a parameter α is

$$W_u = (\alpha + u^8)\phi_u^4 \ , \qquad W_v = (\alpha v^8 + 1)\phi_v^4 \ . \tag{A.4}$$

Clearly $W_u = W_v$ on the overlap. Computing the gradient in the two patches, we obtain

$$dW_u = 8u^7\phi_u^4 du + 4(\alpha + u^8)\phi_u^3 d\phi_u \ , \qquad dW_v = 8\alpha v^7\phi_v^4 dv + 4(\alpha v^8 + 1)\phi_v^3 d\phi_v \ . \tag{A.5}$$

It is then easy to see that for $\alpha \neq 0$ we have $dW^{-1}(0) = B$. A more general superpotential respecting the same quasi-homogeneity is

$$W_u = S_u(u)\phi_u^4 \ , \qquad W_v = S_v(v)\phi_v^4 \ , \tag{A.6}$$

where $S_{u,v}$ is the restriction of $\Sigma \in H^0(B, \mathcal{O}(8))$ to $\mathcal{U}_{u,v}$. The potential condition is satisfied for generic choices of Σ.

We can see how the fibration affects the naive chiral ring R_p of the LG fiber theory over a point $p \in B$: $\dim R_p$ jumps in complex co-dimension 1 but stays finite if the potential condition is satisfied. In our example $R_u = \{1, \phi_u, \phi_u^2\}$ for $u^8 + \alpha \neq 0$, while at the 8 special points $R = \{1, \phi_u, \phi_u^2, \phi_u^3\}$. If $\alpha = 0$ then the potential condition is violated, and $\dim R_0 = \infty$.

A (0,2) deformation

Taking the left-moving bundle to be $\mathcal{E} = \mathcal{V}_\epsilon$, we obtain a class of (0,2) theories. The most general (0,2) superpotential that respects the same quasi-homogeneity as dW, $J \in \Gamma(\mathcal{E}^*)$, takes the form

$$J_u = \begin{pmatrix} T_u(u)\phi_u^4 \\ 4S_u(u)\phi_u^3 \end{pmatrix} , \qquad J_v = \begin{pmatrix} T_v(v)\phi_v^4 \\ 4S_v(v)\phi_v^3 \end{pmatrix} \ , \tag{A.7}$$

where S and T are holomorphic functions constrained by $J_u = M_\epsilon^{-1} J_v$ when $u \neq v$. $S_{u,v}$ are restrictions of Σ as above, while $T_{u,v}$ are given by

$$T_u(u) = -\left.\widetilde{\Sigma}\right|_u + 8\epsilon u^{-1}\left(S_u(u) - S_u(0)\right) \ , \qquad T_v(v) = \left.\widetilde{\Sigma}\right|_v + 8\epsilon v^7 S_u(0) \ , \tag{A.8}$$

where $\widetilde{\Sigma} \in H^0(B, \mathcal{O}(6))$. The potential condition is satisfied for generic Σ and $\widetilde{\Sigma}$. Setting $\epsilon = 1$ and $T_u = \partial_u S_u$, we recover the (2,2) potential from above. On the other hand, taking $\epsilon = 0$, we see that T is just given by restriction of holomorphic sections of $\mathcal{O}(6)$.

We can compare the number of holomorphic deformation parameters in the (2,2) or (0,2) superpotentials. W depends on 9 holomorphic parameters specifying section Σ. The more generic (0,2) superpotential J, on the other hand, depends on 16 parameters, independent of ϵ; as a check, we see that demanding that J is integrable to W reduces the parameters to 9.

Metrics for $\boldsymbol{Y}$ *and* $\mathcal{E}$

It is well known that $\boldsymbol{Y}$ admits an ALE Kähler Ricci-flat metric with Kähler potential [1]

$$K_{\rm CY} = \sqrt{1+L} + \frac{1}{2}\log\frac{\sqrt{1+L}-1}{\sqrt{1+L}+1} \ , \qquad L \equiv 4\phi\overline{\phi}(1+u\overline{u})^2 \ . \tag{A.9}$$

This is obviously well-defined with respect to the patching. To leading order in the fiber coordinates, we find that up to irrelevant constants

$$K_{\rm CY} = K + O(|\phi|^4), \qquad K = \log(1+u\overline{u}) + (1+u\overline{u})^2\phi\overline{\phi} \ . \tag{A.10}$$

K leads to a complete non-Ricci-flat metric on X:

$$g_X = \begin{pmatrix} g_{u\overline{u}} & g_{u\overline{\phi}} \\ g_{\phi\overline{u}} & g_{\phi\overline{\phi}} \end{pmatrix} = \begin{pmatrix} (1+u\overline{u})^{-2} + 2(1+2u\overline{u})\phi\overline{\phi} & 2\overline{u}\phi(1+u\overline{u}) \\ 2u\overline{\phi}(1+u\overline{u}) & (1+u\overline{u})^2 \end{pmatrix} . \tag{A.11}$$

To $O(|\phi|^4)$ this agrees with the Kähler metric obtained by symplectic reduction from $\mathbb{C}^3$.

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[1] Constructions of such metrics for line bundles over $\mathbb{P}^{n-1}$, which generalize the classic work of Eguchi and Hanson [46], go back to [32, 50]; [55] gives an elegant generalization for line bundles over symmetric spaces. These are also the only explicitly known ALE metrics with $\mathrm{SU}(n)$ $n \geqslant 3$ holonomy [60].

We can also endow $\mathcal{E}$ with a Hermitian metric. In our example with $\mathcal{E} = \mathcal{V}_\epsilon$, a convenient choice is

$$(\sigma, \tau) \equiv \sigma \mathcal{G} \overline{\tau}, \qquad \mathcal{G} = \begin{pmatrix} (1+u\overline{u})^{-2} + 2\epsilon\overline{\epsilon}(1+2u\overline{u})\phi\overline{\phi} & 2\epsilon\overline{u}\phi(1+u\overline{u}) \\ 2\overline{\epsilon}u\overline{\phi}(1+u\overline{u}) & (1+u\overline{u})^2 \end{pmatrix} . \tag{A.12}$$

Setting $\epsilon = 1$, we obtain a Hermitian, in fact Kähler, metric on $T_{\boldsymbol{Y}}$. Setting $\phi = 0$ we obtain the bundles restricted to B. As we might expect, $T_{\boldsymbol{Y}}|_B = \mathcal{V}_\epsilon|_B = \mathcal{O}(2) \oplus \mathcal{O}(-2)$.

The explicit Ricci-flat metric on $\boldsymbol{Y}$ is fairly complicated, and generalizations to other spaces are typically not available. Fortunately, we do not need the explicit form of the metric for our analysis: by construction the superpotential restricts low energy field configurations to B, and the details of the metric on $\boldsymbol{Y}$ away from the base become irrelevant to the IR physics.

A.2 Vertical Killing vectors

In this appendix we examine holomorphic vertical Killing vectors on $\boldsymbol{Y}$ and prove that with our assumptions they act homogeneously on the fiber directions.

Let $V = V^\alpha \frac{\partial}{\partial y^\alpha} + \text{c.c.}$ be an holomorphic vector field on $\boldsymbol{Y}$, i.e. $V^\alpha_{,\overline{\beta}} = 0$. Then the Killing equation for a Kähler metric $g_{\alpha\overline{\beta}}$ takes the form

$$\partial_\gamma(g_{\alpha\overline{\beta}} V^\alpha) + \partial_{\overline{\beta}}(g_{\gamma\overline{\alpha}} \overline{V}^{\overline{\alpha}}) = 0 \ . \tag{A.13}$$

Using the base/fiber decomposition $y^\alpha = (u^I, \phi^i)$, the hybrid metric has components

$$g_{I\overline{J}} = G_{I\overline{J}} + \phi h_{I\overline{J}} \overline{\phi}, \qquad g_{i\overline{J}} = h_{i\overline{m}\overline{J}} \overline{\phi}^{\overline{m}}, \qquad g_{I\overline{\jmath}} = \phi^m h_{m\overline{\jmath}I}, \qquad g_{i\overline{\jmath}} = h_{i\overline{\jmath}} \ . \tag{A.14}$$

Since V is vertical, we have $V = V^i \frac{\partial}{\partial \phi^i} + \text{c.c.}$, and a moment's thought shows that $V^i(u, \phi)$ transforms as a section of $\pi^*(X)$. In this case the Killing equation reduces to

$$\partial_\gamma(g_{i\overline{\beta}} V^i) + \partial_{\overline{\beta}}(g_{\gamma\overline{\imath}} \overline{V}^{\overline{\imath}}) = 0 \ , \tag{A.15}$$

and decomposing it further along base/fiber directions leads to two non-trivial conditions. First, from $\overline{\beta}, \gamma = \overline{\jmath}, k$ we obtain

$$\partial_k V^i + h^{\overline{\jmath} i} \partial_{\overline{\jmath}} \overline{V}^{\overline{\imath}}_{\overline{\jmath}} h_{k\overline{\imath}} = 0 \ . \tag{A.16}$$

Since h is ϕ-independent and $\partial_m \overline{V}^{\overline{\imath}}_{\overline{\jmath}} = 0$, we conclude that

$$V^i = A^i_k(u)\phi^k + B^i(u), \qquad \overline{A}^{\overline{\imath}}_{\overline{k}} = (A^i_k)^* = -h^{\overline{\imath} i} A^k_i h_{k\overline{k}} \ . \tag{A.17}$$

The latter restriction on $A \in H^0(B, X \otimes X^*)$, combined with its holomorphy leads to $D_J A = 0$. The remaining non-trivial conditions are obtained by taking $\overline{\beta}, \gamma = J, \overline{k}$ in the Killing equation, and they lead to $D_J B = 0$ for $B \in H^0(B, X)$.

So, we have learned that vertical automorphic Killing vectors are characterized by covariantly constant sections $A \in \Gamma(X \otimes X^*)$ and $B \in \Gamma(X)$, with the additional restriction

$$(A^i_k)^* = -h^{\overline{\imath} i} A^k_i h_{k\overline{k}} \ . \tag{A.18}$$

In fact, we can always shift away the global section B by a redefinition of the ϕ^i; moreover, for a generic choice of metric h the only solution for A is a diagonal anti-Hermitian u-independent matrix; demanding $\mathcal{L}_V W = W$ will fix the eigenvalues (up to an overall i) to be the charges q_i.

A.3 A little sheaf cohomology

In this section we present some useful results for reducing sheaf cohomology on $\boldsymbol{Y}$ to computations on the base B in the case that $X = \oplus_i L_i$. In order to compute $\overline{\boldsymbol{Q}}_0$ cohomology we need an effective method to evaluate

$$H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}, \pi^*(\mathcal{E}) \otimes \wedge^s T_{\boldsymbol{Y}} \otimes \wedge^t T^*_{\boldsymbol{Y}}), \tag{A.19}$$

where $\mathcal{E}$ is some bundle (or more generally sheaf) on B, and $\boldsymbol{r}$ is the restriction to fine grade $\boldsymbol{r}$. Recall that the grading $\boldsymbol{r} \in \mathbb{Z}^n$ assigns to every monomial $\prod_i \phi_i^{r_i}$ grade

$\boldsymbol{r} = (r_1, \ldots, r_n)$; in particular ϕ_i has grade $\boldsymbol{x}_i$ with $(\boldsymbol{x}_i)_j = \delta_{ij}$. Since $\boldsymbol{Y}$ is non-compact the grade restriction is necessary to obtain a well-posed counting problem. For instance, the structure sheaf $\mathcal{O}_{\boldsymbol{Y}}$ clearly has infinite-dimensional cohomology group $H^0(\boldsymbol{Y}, \mathcal{O}_{\boldsymbol{Y}})$.

Graded cohomology of a pulled-back sheaf

Suppose $s = t = 0$ in (A.19). As we now show,

$$H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}, \pi^*(\mathcal{E})) \simeq H^\bullet(B, \mathcal{E} \otimes \mathbb{L}_{\boldsymbol{r}}), \tag{A.20}$$

where $\mathbb{L}_{\boldsymbol{r}} \to B$ is the line bundle $\mathbb{L}_{\boldsymbol{r}} \equiv \otimes_i (L_i^*)^{r_i}$.

The proof follows from the basic geometry. First, to describe the line bundles $L_i \to B$, we work with a cover $\mathcal{U} = \{U_a\}$ for B with local coordinates u_a^I in each patch, so that on overlaps $U_{ab} \neq \varnothing$ sections of L_i satisfy

$$\lambda_b^i(u_b) = \lambda_a^i(u_a) g_{ab}^i(u_a) \ , \tag{A.21}$$

where the g_{ab}^i are the transition functions defining the bundle L_i. The sections σ_a of a sheaf $\mathcal{E} \to B$ satisfy

$$\sigma_b(u_b) = \sigma_a(u_a) G_{ab}(u_a) \ , \tag{A.22}$$

where the G_{ab} are the transition functions for $\mathcal{E}$, and sections of $\pi^*(\mathcal{E}) \to \boldsymbol{Y}$ patch as

$$\sigma_b(u_b, \phi_b) = \sigma_a(u_a, \phi_a) G_{ab}(u_a) \ , \tag{A.23}$$

with $\phi_b^i = \phi_a^i g_{ab}^i(u_a)$. Since the transition functions for $\pi^*(\mathcal{E})$ are identical to the transition functions for $\mathcal{E}$ over B, at fixed grade (A.23) takes the form

$$\prod_i (\phi_b^i)^{r_i} \xi_b(u_b) = \prod_i (\phi_a^i)^{r_i} \xi_a(u_a) G_{ab}(u_a) \iff \xi_b(u_b) = \xi_a(u_a) G_{ab}(u_a) \prod_i \left[g_{ab}^i(u_a) \right]^{-r_i} \ . \tag{A.24}$$

Hence the space of sections of $\pi^*(\mathcal{E})_{\boldsymbol{r}}$ over $\boldsymbol{Y}$ is isomorphic to the space of sections of $\mathcal{E} \otimes \mathbb{L}_{\boldsymbol{r}}$ over B. The grading is compatible with Čech cohomology (i.e. with defining chains for higher intersections $U_{a_1 \cdots a_k}$ and taking cohomology of the Čech differential), and (A.20) holds.

The tangent bundle

Having reduced the graded cohomology of a pull-back sheaf to a cohomology problem on the base, we now turn to the tangent bundle. This is of course not in general the pull-back of a sheaf from B, as we explictly saw in appendix A.1. However, $T_{\boldsymbol{Y}}$ fits into a short exact sequence

$$0 \longrightarrow \pi^*(X) \longrightarrow T_{\boldsymbol{Y}} \longrightarrow \pi^*(T_B) \longrightarrow 0 \ . \tag{A.25}$$

This is easy to see explicitly. In an open neighborhood U_a a vector field Σ takes the form

$$\Sigma_a = V_a \frac{\partial}{\partial u_a} + \nu_a \frac{\partial}{\partial \phi_a}, \tag{A.26}$$

and on overlaps U_{ab}

$$V_b = V_a \tfrac{\partial u_b}{\partial u_a}, \qquad \nu_b = \nu_a g_{ab} + \phi_a \mathcal{L}_V g_{ab} \ , \tag{A.27}$$

where g_{ab} are the transition functions for X. Hence, we see that a section ν of X lifts to a section of $T_{\boldsymbol{Y}}$ with $V = 0$, while a section of $T_{\boldsymbol{Y}}$ at $\phi = 0$ yields a section of T_B.

This short exact sequence can be decomposed with respect to the fine grading. Working again in the case $X = \oplus_i L_i$, the transition functions for sections of $T_{\boldsymbol{Y}}$ can be written explicitly as

$$(\sigma_b^0, \sigma_b^1, \ldots, \sigma_b^n) = (\sigma_a^0, \sigma_a^1, \ldots, \sigma_a^n) \begin{pmatrix} \frac{\partial u_b}{\partial u_a} & \phi_a^1 \partial g_{ab}^1 & \phi_a^2 \partial g_{ab}^2 & \cdots & \phi_a^n \partial g_{ab}^n \\ 0 & g_{ab}^1 & 0 & \cdots & 0 \\ 0 & 0 & g_{ab}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{ab}^n \end{pmatrix} . \tag{A.28}$$

Hence, sections of $T_{\boldsymbol{Y}}$ also admit a fine grading, which we define

$$(\Sigma)_{\boldsymbol{r}} \equiv (\sigma^0_{\boldsymbol{r}}, \sigma^1_{\boldsymbol{r}+\boldsymbol{x}_1}, \sigma^2_{\boldsymbol{r}+\boldsymbol{x}_2}, \ldots, \sigma^n_{\boldsymbol{r}+\boldsymbol{x}_n}) \ . \tag{A.29}$$

This means the short exact sequence for $T_{\boldsymbol{Y}}$ can be decomposed according to $\boldsymbol{r}$ as

$$0 \longrightarrow \oplus_i (\pi^* L_i)_{\boldsymbol{r}+\boldsymbol{x}_i} \longrightarrow (T_{\boldsymbol{Y}})_{\boldsymbol{r}} \longrightarrow (\pi^* T_B)_{\boldsymbol{r}} \longrightarrow 0 \ . \tag{A.30}$$

Using the induced long exact sequence on cohomology, together with (A.20), we can evaluate $H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}, T_{\boldsymbol{Y}})$. Taking appropriate products one can generalize this result to compute all desired cohomology groups in (A.19).

We should mention one small subtlety in grading the sections of $T_{\boldsymbol{Y}}$: from (A.29) we see that there can be non-trivial contributions for $r_i = -1$. More precisely, $(T_{\boldsymbol{Y}})_{\boldsymbol{r}} = 0$ whenever any $r_i < -1$ or $r_i = r_j = -1$, and if a single $r_i = -1$ we have

$$(T_{\boldsymbol{Y}})_{\boldsymbol{r}} = (\pi^* L_i)_{\boldsymbol{r}+\boldsymbol{x}_i} \ , \tag{A.31}$$

in which case $H^\bullet_{\boldsymbol{r}}(\boldsymbol{Y}, T_{\boldsymbol{Y}}) = H^\bullet(B, \mathbb{L}_{\boldsymbol{r}})$.

Application to $X = \mathcal{O}(-2)$ and $B = \mathbb{P}^1$

In this case the grading is one-dimensional $\boldsymbol{r} = (r)$, the grading bundle is $\mathbb{L}_s = (\mathcal{O}(-2)^*)^s = \mathcal{O}(2s)$, and for any $r \geqslant 0$ the structure sheaf cohomology is

$$H^0_r(\boldsymbol{Y}, \mathcal{O}_{\boldsymbol{Y}}) = H^0(B, \mathcal{O}(2r)) \simeq \mathbb{C}^{2r+1} \ , \quad H^q_r(\boldsymbol{Y}, \mathcal{O}_{\boldsymbol{Y}}) = 0, \qquad \text{for } q > 0 \ . \tag{A.32}$$

For the tangent sheaf the short exact sequence

$$0 \longrightarrow (\pi^*\mathcal{O}(-2))_{r+1} \longrightarrow (T_{\boldsymbol{Y}})_r \longrightarrow (\pi^*\mathcal{O}(2))_r \longrightarrow 0 \tag{A.33}$$

leads to the long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(B,\mathcal{O}(2r)) \arrow[r] & H^0_r(\boldsymbol{Y},T_{\boldsymbol{Y}}) \arrow[r] & H^0(B,\mathcal{O}(2r+2)) \arrow[dlll, out=-90, in=90] \\ H^1(B,\mathcal{O}(2r)) \arrow[r] & H^1_r(\boldsymbol{Y},T_{\boldsymbol{Y}}) \arrow[r] & H^1(B,\mathcal{O}(2r+2)) \arrow[r] & 0 \end{tikzcd} \tag{A.34}$$

At grade 0 we obtain

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0_0(\boldsymbol{Y}, T_{\boldsymbol{Y}}) \longrightarrow \mathbb{C}^3 \tag{A.35}$$

$$0 \longrightarrow H^1_0(\boldsymbol{Y}, T_{\boldsymbol{Y}}) \longrightarrow 0 \longrightarrow 0$$

Hence, $H^0_0(\boldsymbol{Y}, T_{\boldsymbol{Y}}) \simeq \mathbb{C}^4$, and $H^1_0(\boldsymbol{Y}, T_{\boldsymbol{Y}}) = 0$. More generally, for any non-negative grade

$$H^0_r(\boldsymbol{Y}, T_{\boldsymbol{Y}}) = H^0_r(B, \mathcal{O}(2r)) \oplus H^0_r(B, \mathcal{O}(2r+2)) \simeq \mathbb{C}^{4r+4}, \qquad H^1_r(\boldsymbol{Y}, T_{\boldsymbol{Y}}) = 0\ . \tag{A.36}$$

A note on horizontal representatives

In order to evaluate $\overline{\boldsymbol{Q}}_0$ cohomology we needed to study the finely graded Dolbeault cohomology of horizontal forms on $\boldsymbol{Y}$ valued in a holomorphic sheaf $\mathcal{F}$. One might wonder what is the relationship between these horizontal forms and more general Dolbeault classes in $H^{(0,u)}_{\bar{\partial}}(\boldsymbol{Y}, \mathcal{F})$. In fact, every such class has a horizontal representative, which is why our results on finely graded cohomology describe horizontal Dolbeault cohomolgy as well. This is rather intuitive, since the fiber space is simply $\mathbb{C}^n$ (or $\mathbb{C}^n/\Gamma$ for orbi-bundles), but for completeness we give a sketch of the proof.[2]

The statement is trivial at $u = 0$, so we consider $u = 1$. Let $\tau \in \ker \bar{\partial} \cap \Omega^{(0,1)}(\boldsymbol{Y}, \mathcal{O}_{\boldsymbol{Y}})$. In any patch U_a we have

$$\tau_a = \omega_{a\bar{I}} d\overline{u}^{\bar{I}}_a + \sigma_{a\bar{\imath}} d\overline{\phi}^{\bar{\imath}}_a\ . \tag{A.37}$$

We define $\eta_a(u_a, \overline{u}_a, \phi_a, \overline{\phi}_\alpha)$ via the line integral

$$\eta_\alpha = \int_0^{\overline{\phi}_a} d\overline{z}^{\bar{\imath}} \sigma_{a\bar{\imath}}(u_a, \overline{u}_a, \phi_a, \overline{z})\ . \tag{A.38}$$

Since $\bar{\partial}\tau = 0$ implies $\sigma_{a\bar{\imath},\bar{\jmath}} = \sigma_{a\bar{\jmath},\bar{\imath}}$, the line integral does not depend on the choice of contour from 0 to $\overline{\phi}$; moreover, a change of variables $\overline{z}^{\bar{\imath}} = \overline{g}^{\bar{\imath}}_{ba}\overline{y}^{\bar{\imath}}$ in the integral shows that $\eta_a = \eta_b$ on any $U_{ab} \neq \varnothing$, so that η patches to a function on $\boldsymbol{Y}$. Therefore $\tau' = \tau - \bar{\partial}\eta$ is a (0,1) horizontal form, and a moment's thought shows that $\bar{\partial}\tau' = 0$ implies that it has a holomorphic dependence on the fiber coordinates.

One can generalize the argument to $u > 1$ and more general holomorphic sheaf

[2] This essentially follows the standard proof [30] that $H^k_{dR}(\mathbb{R}^n, \mathbb{R}) = 0$ for $k > 0$.

$\mathcal{F} \to \boldsymbol{Y}$. The analogous construction yields η, a section of $\Omega^{(0,u-1)}(\boldsymbol{Y}, \mathcal{F})$, such that $\tau' = \tau - \bar{\partial}\eta$ is a horizontal representative of $[\tau] \in H^{(0,u)}_{\bar{\partial}}(\boldsymbol{Y}, \mathcal{F})$.

A.4 Massless spectrum of a (0,2) CY NLSM

In this appendix we apply the first-order techniques developed in section 2.4.3 to marginal deformations of (0,2) NLSMs with CY target space B and a left-moving $\mathrm{SU}(n)$ bundle $\mathcal{V}$. We assume $\mathrm{ch}_2(\mathcal{V}) = \mathrm{ch}_2(T_B)$ and $\mathcal{V}$ is a stable bundle. This ensures that the NLSM is conformally invariant to all orders in α' perturbation theory [96, 35]. Our techniques allow us to determine the massless spectrum to all orders in α'. The results for the (R,R) sector and for the gauge-charged matter are exactly the same as those obtained by a Born-Oppenheimer approach in [40]. However, the massless gauge-neutral chiral matter has to our best knowledge not been studied directly in the NLSM. The first-order formulation of $\overline{\boldsymbol{Q}}_0$ cohomology turns out to be perfectly suited to this task and should be thought of as a first step in systematically including any non-perturbative world-sheet effects.

In parallel with the analysis of the $k = 1$ sector in section 2.5.2, we first list the operators that can give rise to massless singlets. We need to slightly alter our notation in comparison to the $T_B = \mathcal{V}$ analysis of section 2.5.2; just in this appendix we use $I, J, \ldots$ for tangent/cotangent indices, while the α, β indices will refer to sections of the left-moving bundle $\mathcal{V}$ and its dual $\mathcal{V}^*$. We will continute to denote the bosonic coordinates by $y, \overline{y}$. Thus, χ^α ($\overline{\chi}_\alpha$) transforms as a section of the pullback of $\mathcal{V}$ ($\mathcal{V}^*$). In particular, the χ kinetic term is

$$2\pi L \supset \overline{\chi}_\alpha \overline{D}_{\bar{z}} \chi^\alpha = \overline{\chi}_\alpha (\bar{\partial}_{\bar{z}} \chi^\alpha + \bar{\partial}_{\bar{z}} y^I \mathcal{A}^\alpha_{I\beta} \chi^\beta) \ , \tag{A.39}$$

where $\mathcal{A}$ is a HYM connection on $\mathcal{V}$ with traceless curvature $\mathcal{F} = \bar{\partial}\mathcal{A}$.

Using the connection, we can easily describe the full set of operators that can give rise to gauge-netural massless states in the (NS,R) sector (we ignore the universal gravitino and dilatino states and drop the normal ordering):

$$\mathcal{O}^4(z) = \Psi^4_I \partial y^I, \qquad \mathcal{O}^{5+6}(z) = \Psi^{5\alpha}_{\beta} \overline{\chi}_\alpha \chi^\beta + \Psi^{6I}(\rho_I - \mathcal{A}^\alpha_{I\beta}\overline{\chi}_\alpha \chi^\beta) \ . \tag{A.40}$$

As in our discussion of states in the $k = 1$ sector we suppressed the expansion of each of these in $\overline{\eta}$; taking that into account the wavefunctions correspond to the following bundles:

$$\Psi^4 \in \Gamma(\oplus_u \Omega^{(0,u)}(T^*_B)) \ , \qquad \Psi^5 \in \Gamma(\oplus_u \Omega^{(0,u)}(\mathrm{End}\,\mathcal{V})) \ , \qquad \Psi^6 \in \Gamma(\oplus_u \Omega^{(0,u)}(T_B)) \ . \tag{A.41}$$

These states are $\overline{\boldsymbol{Q}}_0$ closed iff Ψ^4, Ψ^5 and Ψ^6 are $\bar{\partial}$-closed and

$$\mathrm{obs}(\Psi^6) + \bar{\partial}\Psi^5 = 0 \ , \tag{A.42}$$

where $\mathrm{obs}(\Psi^6_u)$ is a (0,u+1) $\bar{\partial}$-closed form valued in (traceless) endomorphisms of $\mathcal{V}$

$$\mathrm{obs}(\Psi^6)^\alpha_{\beta \overline{J}_0 \cdots \overline{J}_u} \equiv \Psi^{6I}_{[\overline{J}_1 \cdots \overline{J}_u} \mathcal{F}^\alpha_{\overline{J}_0] I\beta} \ . \tag{A.43}$$

Taking cohomology, $[\mathrm{obs}(\Psi^6_u)] \in H^{u+1}(B, \mathrm{End}\,\mathcal{V})$. As explained in [3], at $u = 1$ this is the Atiyah class [13]—an obstruction to extending infinitesimal complex structure deformations of the base B to infinitesimal complex structure deformations of the holomorphic bundle $\mathcal{V} \to B$. Thus, our states fit into the complex

$$\begin{array}{ccccccc} \mathcal{O}^4_0 & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \mathcal{O}^4_1 & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \mathcal{O}^4_2 & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \mathcal{O}^4_3 \\ \mathcal{O}^5_0 & & \mathcal{O}^5_1 & & \mathcal{O}^5_2 & & \mathcal{O}^5_3 \\ \oplus & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \oplus & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \oplus & \xrightarrow{\overline{\boldsymbol{Q}}_0} & \oplus \\ \mathcal{O}^6_0 & & \mathcal{O}^6_1 & & \mathcal{O}^6_2 & & \mathcal{O}^6_3 \end{array} \tag{A.44}$$

Taking $\overline{\boldsymbol{Q}}_0$ cohomology we find

$$\begin{array}{ccccccc} 0 & & H^1(T^*) & \xrightarrow{0} & H^2(T^*) & & 0 \\ H^0(\mathrm{End}\,\mathcal{V}) & & H^1(\mathrm{End}\,\mathcal{V}) & & H^2(\mathrm{End}\,\mathcal{V}) & & H^3(\mathrm{End}\,\mathcal{V}) \\ \oplus & \xrightarrow{\mathrm{obs}_0} & \oplus & \xrightarrow{\mathrm{obs}_1} & \oplus & \xrightarrow{\mathrm{obs}_2} & \oplus \\ H^0(T) & & H^1(T) & & H^2(T) & & H^3(T) \end{array} \ . \tag{A.45}$$

For traceless $\operatorname{End}\mathcal{V}$ on the CY 3-fold B

$$H^0(B, \operatorname{End}\mathcal{V}) = H^3(B, \operatorname{End}\mathcal{V}) = 0\ , \qquad H^2(\operatorname{End}\mathcal{V}) \simeq \overline{H^1(B, \operatorname{End}\mathcal{V})}\ , \tag{A.46}$$

so that the complex reduces to

$$\begin{array}{ccccc} 0 & H^1(T^*) & \xrightarrow{0} & H^2(T^*) & 0 \\ \\ & H^1(\operatorname{End}\mathcal{V}) & & H^2(\operatorname{End}\mathcal{V}) & \\ 0 & \oplus & \xrightarrow{\mathrm{obs}_1} & \oplus & 0 \\ & H^1(T) & & H^2(T) & \end{array} \tag{A.47}$$

The only Atiyah obstructions arise in $H^1(B,T) \to H^2(B, \operatorname{End}\mathcal{V})$, and hence there are

$$h^1(T^*) + h^1(T) + h^1(\operatorname{End}\mathcal{V}) - \dim\ker \mathrm{obs}_1 \tag{A.48}$$

massless gauge-neutral singlets.

The patient reader who has made it to this last appendix may perhaps be aware that in a (0,2) NLSM with a tree-level H-flux there are additional obstructions similar to the $H^1(B,T) \to H^2(B, \operatorname{End}\mathcal{V})$ map just discussed [81]. The B-field coupling will alter the η equations of motion and lead to H-flux appearing in $\overline{\boldsymbol{Q}}_0 \cdot \rho$, and we expect that including this contribution should reproduce the result of [81]. It would be useful to check that in detail.

Appendix B

Obstructions to marginal couplings

B.1 An F-term obstruction

In this section we give an example, taken from [11], that illustrates both D-term and F-term obstructions to marginal couplings. The setting is a (2,2) LG orbifold (LGO) compactification of the heterotic string with a superpotential

$$W = X_0^4 + X_1^4 + X_2^4 + X_3^8 + X_4^8 + \psi X_0 X_1 X_2 X_3 X_4 + \epsilon \Delta W \ . \tag{B.1}$$

Here ψ and ϵ are parameters and ΔW is a generic polynomial with $\boldsymbol{q} = 1$. Marginal (2,0) deformations of the LGO correspond to massless E_6-neutral space-time chiral multiplets. We can compute the massless spectrum exactly as a function of the complex parameters in the superpotential using the technique developed in [61]. This leads to the following results.

1. Setting $\psi = \epsilon = 0$ leads to a $\mathrm{U}(1)^4$ right-moving Kac-Moody algebra and 298 marginal (2,0) deformations. We now turn on the (2,2)-preserving ψ and ϵ deformations and investigate what happens to the remaining (2,0) deformations. From above we know that at worst the marginal (2,0) deformations can become marginally irrelevant.

2. With $\psi \neq 0$ but $\epsilon = 0$ the $\mathrm{U}(1)^4$ symmetry is broken, and the number of marginal (2,0) deformations is $298 - 4 - 6 = 288$. While 4 of the 10 marginally irrelevant deformations are associated to the broken symmetries the 6 others are not.

3. Finally, turning on $\epsilon \neq 0$ does not break any continuous symmetries, but the number of marginal (2,0) deformations decreases to $288 - 6 = 282$.

Note however, that all the singlets lifted by F-terms correspond to twisted sectors of the LGO. This is consistent with there being no F-term obstructions in pure LG theories.

B.2 A D-term obstruction in a heterotic vacuum

Consider now the (2,2) quintic LG theory coupled to a free left-moving fermion with

$$\mathcal{W} = \sum_{i=1}^{5} \Gamma^i J_i(\Phi) = \sum_{i=1}^{5} \Gamma^i (\Phi_i^4 + \psi \prod_{j \neq i} \Phi_j) + \Gamma^6 \times 0 \ . \tag{B.2}$$

The interacting fields have their usual $\mathrm{U}(1)_\mathrm{L}$ charges $q_i = 1/5$ and $Q_i = -4/5$, and for $\psi \neq 0$ there are no extra $\mathrm{U}(1)$ symmetries in addition to $\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R} \times \mathrm{U}(1)_6$, where $\mathrm{U}(1)_6$ is the symmetry associated to the free Γ^6. This flows to a conformal field theory with $r = 4$ and $\overline{c} = 9$, and we can now consider deformations of the IR theory from the general perspective of deforming by chiral primary operators. In the (2,2) theory we have a good understanding of the map between the IR chiral primary marginal operators and the UV data, so we can identify the marginal deformations of the IR theory with the space of possible $\mathcal{W}$ modulo field redefinitions. If we keep Γ^6 free, we find a 301–dimensional space of deformations.

We can also include deformations of the form $\Gamma^6 J_6$, where J_6 is some generic degree 5 polynomial. Although these break the $\mathrm{U}(1)_6$ symmetry, they preserve the central charge and the $\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R}$ quantum numbers of the fields. In particular,

Γ^6 has the quantum numbers of a free field. Including these J_6 deformations yields a 402-dimensional space of marginal deformations away from the (2,2) $r = 4$ $\bar{c} = 9$ fixed point.

Are all of these 402 marginal deformations exactly marginal? While all of the 301 deformations of the J_i are truly marginal, the 101 extra deformations associated to $J_6 \neq 0$ are marginally irrelevant. This is completely clear from the conformal perturbation theory discussion we gave in the text. All of these break the $\mathrm{U}(1)_6$ symmetry, and every symmetry-breaking coupling has the same sign of $\mathrm{U}(1)_6$ charge. Let us now see how the same result is recovered from a heterotic space-time perspective.

Heterotic insights

The $\mathbb{Z}_5$ orbifold of the LG theory just described, combined with an appropriate heterotic GSO projection leads to a well-understood heterotic vacuum: the LG point in the moduli space of the quintic compactification with standard embedding. The massless fields of the resulting space-time N=1 $d = 4$ supergravity theory consist of the supergravity multiplet, the axio-dilaton chiral multiplet, the $\mathfrak{e}_6 \oplus \mathfrak{e}_8$ vector multiplets, 326 gauge-neutral chiral multiplets, and a $\mathfrak{e}_6$ charged chiral spectrum $\mathbf{27} \oplus \overline{\mathbf{27}}^{\oplus 101}$. The 301 deformations of the J_i described above correspond to $\mathfrak{e}_6$-preserving marginal deformations in the untwisted sector of the orbifold. These remain truly marginal for any value of the Kähler modulus (itself in a twisted sector), and at large radius they are the 101 complex structure deformations of the CY quintic, as well as 200 of the 224 deformations of the tangent bundle. As reviewed in [77], there are many arguments for why these deformations are truly marginal.

The 101 deformations associated with the J_6 couplings also have a simple space-time interpretation: they correspond to $\mathfrak{so}(10)$-singlet components of the $\overline{\mathbf{27}}^{\oplus 101}$. Turning on these deformations corresponds to Higgsing $\mathfrak{e}_6 \to \mathfrak{so}(10)$. This makes it obvious that the deformations are marginally irrelevant. Under the decomposition of $\mathfrak{e}_6 \supset \mathfrak{so}(10) \oplus \mathfrak{u}(1)$, we have

$$\overline{\mathbf{27}} = \overline{\mathbf{16}}_{-1/2} \oplus \mathbf{10}_1 \oplus \mathbf{1}_{-2} \ . \tag{B.3}$$

The $\mathfrak{so}(10)$ singlets all have charge -2 under the broken $\mathfrak{u}(1)$, and hence have a D-term space-time potential. This is an example of a "D-term" obstruction to a marginal coupling being exactly marginal.

Since the deformation only involves world-sheet fields in untwisted sector of the orbifold, it is clear by the orbifold inheritance principle that this obstruction to marginality lifts to the un-orbifolded quintic LG model and matches the conformal perturbation theory result. In the orbifold theory it is possible to find exactly marginal deformations that Higgs $\mathfrak{e}_6 \to \mathfrak{so}(10)$ [34], but they involve an interplay between marginal couplings in twisted and untwisted sectors [77].

Appendix C

GLSMs details

C.1 Linear model conventions

C.1.1 (0,2) superspace

We work in (0,2) superspace[1] with coordinate $x^{\pm}, \theta^+, \overline{\theta}^+$. The supercharges are given by

$$\mathcal{Q}_+ = \frac{\partial}{\partial\theta^+} + i\overline{\theta}^+\nabla_+ \ , \qquad\qquad \overline{\mathcal{Q}}_+ = -\frac{\partial}{\partial\overline{\theta}^+} - i\theta^+\nabla_+ \ , \tag{C.1}$$

where $\partial_+ = \partial/\partial x^+$ and ∇_+ is the covariant gauge derivative. We also have the superderivatives

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+\nabla_+ \ , \qquad\qquad \overline{\mathcal{D}}_+ = -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+\nabla_+ \ . \tag{C.2}$$

The non-trivial anti-commutation relations are

$$\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} = -2i\nabla_+ \ , \qquad\qquad \{\mathcal{D}_+, \overline{\mathcal{D}}_+\} = 2i\nabla_+ \ . \tag{C.3}$$

C.1.2 Field content

There are two types of multiplets in the (0,2) models we consider in this work.

[1] More details may be found in [94].

1. Gauge fields multiplets. We have

$$V_{a,-} = v_{a,-} - 2i\theta^+\overline{\lambda}_{a,-} - 2i\overline{\theta}^+\lambda_{a,-} + 2\theta^+\overline{\theta}^+ D_a \ ,$$
$$\Sigma_\mu = \sigma_\mu + \sqrt{2}\theta^+\lambda_{\mu,+} - i\theta^+\overline{\theta}^+\partial_+\sigma_\mu \ , \tag{C.4}$$

where $a = 1,\ldots,R$ and $\mu = 1,\ldots,m$. The multiplets Σ_μ are neutral chiral multiplets which in (2,2) theories [2] combine with the (0,2) gauge multiplets into (2,2) gauge multiplets. The twisted chiral gauge invariant field strength is defined as

$$\begin{aligned}\Upsilon_a &= [\overline{\mathcal{D}}_+, \nabla_-] \\ &= i\overline{\mathcal{D}}_+ V_{a,-} + \theta^+\nabla_- v_{a,+} \\ &= -2\lambda_{a,-} - i\theta^+(D_a - if_{a,01}) - i\theta^+\overline{\theta}^+\partial_+\lambda_{a,-} \ .\end{aligned} \tag{C.5}$$

2. Matter multiplets. Here we have bosonic chiral (anti-chiral) multiplets

$$\begin{aligned}P^\alpha &= p^\alpha + \sqrt{2}\theta^+\chi^\alpha - i\theta^+\overline{\theta}^+\nabla_+ p^\alpha \ , & \overline{P}^\alpha &= \overline{p}^\alpha - \sqrt{2}\overline{\theta}^+\overline{\chi}^\alpha + i\theta^+\overline{\theta}^+\nabla_+\overline{p}^\alpha \ , \\ \Phi^i &= \phi^i + \sqrt{2}\theta^+\psi^i - i\theta^+\overline{\theta}^+\nabla_+\phi^i \ , & \overline{\Phi}^i &= \overline{\phi}^i - \sqrt{2}\overline{\theta}^+\overline{\psi}^i + i\theta^+\overline{\theta}^+\nabla_+\overline{\phi}^i \ , \\ S &= s + \sqrt{2}\theta^+\xi_+ - i\theta^+\overline{\theta}^+\nabla_+ s \ , & \overline{S} &= \overline{s} - \sqrt{2}\overline{\theta}^+\overline{\xi}_+ + i\theta^+\overline{\theta}^+\nabla_+\overline{s} \ ,\end{aligned} \tag{C.6}$$

where $\alpha = 1,\ldots,k$ and $i = 1,\ldots,n$. We also have fermionic matter multiplets, which we again divide into three groups

$$\begin{aligned}\Gamma^I &= \gamma^I - \sqrt{2}\theta^+ G^I - i\theta^+\overline{\theta}^+\nabla_+\gamma^I - \sqrt{2}\overline{\theta}^+ E^I(\Phi,\Sigma) \ , \\ \Lambda^A &= \eta^I - \sqrt{2}\theta^+ F^A - i\theta^+\overline{\theta}^+\nabla_+\eta^A - \sqrt{2}\overline{\theta}^+ E^A(P,\Phi,\Sigma) \ , \\ \Xi &= \xi_- - \sqrt{2}\theta^+ K - i\theta^+\overline{\theta}^+\nabla_+\xi_- \ ,\end{aligned} \tag{C.7}$$

[2] In (2,2) theories we have $R = m$.

as well as their complex conjugate

$$\begin{aligned}
\overline{\Gamma}^I &= \overline{\gamma}^I - \sqrt{2}\overline{\theta}^+\overline{G}^I + i\theta^+\overline{\theta}^+\nabla_+\overline{\gamma}^I - \sqrt{2}\theta^+\overline{E}^I(\overline{\Phi},\overline{\Sigma}) \ , \\
\overline{\Lambda}^A &= \overline{\eta}^I - \sqrt{2}\overline{\theta}^+\overline{F}^A + i\theta^+\overline{\theta}^+\nabla_+\overline{\eta}^A - \sqrt{2}\theta^+\overline{E}^A(\overline{P},\overline{\Phi},\overline{\Sigma}) \ , \\
\overline{\Xi} &= \overline{\xi}_- - \sqrt{2}\overline{\theta}^+\overline{K} + i\theta^+\overline{\theta}^+\nabla_+\overline{\xi}_- \ .
\end{aligned} \tag{C.8}$$

Here the indices are $I = 1\ldots,N$ and $A = 1,\ldots,L$.

C.1.3 The action

Let us list the various terms that appear in the action for the (0,2) linear models we consider in this work. We have the kinetic term for the gauge fields[3]

$$\mathcal{L}_{G,K} = \frac{1}{8e^2}\int d^2\theta^+ \,\mathrm{Tr}\,\overline{\Upsilon}_a\Upsilon_a = \frac{1}{2e^2}\left[2i\overline{\lambda}_{a,-}\partial_+\lambda_{a,-} + D_a^2 + f_{a,01}^2\right] \ , \tag{C.9}$$

as well as the kinetic term for the Σ_μ fields

$$\mathcal{L}_{\Sigma,K} = \frac{i}{2e^2}\int d^2\theta^+\Sigma_\mu\nabla_-\Sigma_\mu = \frac{1}{e^2}\left[\partial_+\overline{\sigma}_\mu\partial_-\sigma_\mu + i\overline{\lambda}_{\mu,+}\partial_-\lambda_{\mu,+}\right] \ . \tag{C.10}$$

Then we have the kinetic terms for the various matter fields. These are given as

$$\begin{aligned}
\mathcal{L}_{\Phi,K} = \frac{i}{2}\int d^2\theta^+\overline{\Phi}^i\nabla_-\Phi^i &= \frac{1}{2}\left(\nabla_+\overline{\phi}^i\nabla_-\phi^i + \nabla_-\overline{\phi}^i\nabla_+\phi^i\right) + i\overline{\psi}^i\nabla_-\psi^i \\
&\quad + i\sqrt{2}q_i^a\left(\overline{\psi}^i\overline{\lambda}_{a,-}\phi^i - \overline{\phi}^i\lambda_{a,-}\psi^i\right) + q_i^a D_a\overline{\phi}^i\phi^i \ , \\
\mathcal{L}_{\Gamma,K} = \frac{1}{2}\int d^2\theta^+\overline{\Gamma}^I\Gamma^I &= i\overline{\gamma}^I\nabla_+\gamma^I + \overline{G}^I G^I - \overline{E}^I E^I \\
&\quad - \overline{\gamma}^I E^I_{,j}\psi^j - \overline{\gamma}^I E^I_{,\mu}\lambda_{\mu,+} - \overline{E}^I_{,j}\overline{\psi}^j\gamma^I - \overline{E}^I_{,\mu}\overline{\lambda}_{\mu,+}\gamma^I \ ,
\end{aligned} \tag{C.11}$$

[3] For simplicity, we have set equal all gauge coupling constants.

and similarly

$$
\begin{aligned}
\mathcal{L}_{P,K} &= \frac{i}{2}\int d^2\theta^+ \overline{P}^\alpha \nabla_- P^\alpha = \frac{1}{2}\left(\nabla_+\overline{p}^\alpha\nabla_- p^\alpha + \nabla_-\overline{p}^\alpha\nabla_+ p^\alpha\right) + i\overline{\chi}^\alpha\nabla_-\chi^\alpha \\
&\qquad\qquad\qquad - i\sqrt{2}m^a_\alpha\left(\overline{\chi}^\alpha\overline{\lambda}_{a,-}p^\alpha - \overline{p}^\alpha\lambda_{a,-}\chi^\alpha\right) - m^a_\alpha D_a\overline{p}^\alpha p^\alpha \ , \\
\mathcal{L}_{\Lambda,K} &= \frac{1}{2}\int d^2\theta^+ \overline{\Lambda}^A\Lambda^A = i\overline{\eta}^A\nabla_+\eta^A + \overline{F}^A F^A - \overline{E}^A E^A \\
&\qquad\qquad - \overline{\eta}^A E^A_{,j}\psi^j - \overline{\eta}^A E^A_{,\mu}\lambda_{\mu,+} - \overline{E}^A_{,j}\overline{\psi}^j\eta^A - \overline{E}^A_{,\mu}\overline{\lambda}_{\mu,+}\eta^A \ , \\
\mathcal{L}_{S,K} &= \frac{i}{2}\int d^2\theta^+ \overline{S}\nabla_- S = \frac{1}{2}\left(\nabla_+\overline{s}\nabla_- s + \nabla_-\overline{s}\nabla_+ s\right) + i\overline{\xi}_+\nabla_-\xi_+ \\
&\qquad\qquad + i\sqrt{2}(m^a - d^a)\left(\overline{\xi}_+\overline{\lambda}_{a,-}s - \lambda_{a,-}\xi_+\overline{s}\right) + (m^a - d^a)D_a\overline{s}s \ , \\
\mathcal{L}_{\Xi,K} &= \frac{1}{2}\int d^2\theta^+ \overline{\Xi}\Xi = i\overline{\xi}_-\nabla_+\xi_- + \overline{K}K \ .
\end{aligned}
\tag{C.12}
$$

The Fayet-Iliopoulos terms action arises as a linear twisted superpotential for the twisted chiral fields Υ_a

$$
\mathcal{L}_{\text{F-I}} = \tfrac{1}{4}\int d\theta^+ \Upsilon_a\tau^a\big|_{\overline{\theta}^+=0} + \text{h.c.} = -D_a r^a + \frac{\theta^a}{2\pi}f_{a,01} \ , \tag{C.13}
$$

where $\tau^a = ir^a + \theta^a/2\pi$ are the complexified F-I parameters. Finally, the matter superpotential is a sum of three terms

$$
\begin{aligned}
\mathcal{L}_J &= -\frac{1}{\sqrt{2}}\int d\theta^+ \Gamma^I J_I(P,\Phi)\big|_{\overline{\theta}^+=0} + \text{h.c.} = G^I p^\alpha J_{I\alpha} + \gamma^I p^\alpha J_{I\alpha,j}\psi^j + \gamma^I J_{I\alpha}\chi^\alpha + \text{h.c.} \ , \\
\mathcal{L}_H &= -\frac{1}{\sqrt{2}}\int d\theta^+ \Lambda^A H_I(\Phi)\big|_{\overline{\theta}^+=0} + \text{h.c.} = F^A H_A + \eta^A H_{A,j}\psi^j + \text{h.c.} \ , \\
\mathcal{L}_S &= -\frac{1}{\sqrt{2}}\int d\theta^+ \Xi S\big|_{\overline{\theta}^+=0} + \text{h.c.} = Ks + \xi_-\xi_+ + \text{h.c.} \ .
\end{aligned}
\tag{C.14}
$$

The last term explicitly shows that all the excitations of the spectator fields are massive and they do not affect the low energy physics. In (C.14) we implemented the form for the superpotential (4.7).

C.2 The half-twist

In order to probe for a background space-time superpotential W it is convenient to half-twist the model, that is we twist by $J_H = J_R/2$, where J_R is the generator of the right-moving R-symmetry. We implement this by redefining the Lorentz generator J_L as

$$J'_L = J_L - J_R/2 \ . \tag{C.15}$$

Explicitly, for the gauge fields we have

$$\begin{array}{ccccccc} \text{fields} & \sigma_\mu & \overline{\sigma}_\mu & \lambda_{+,\mu} & \overline{\lambda}_{+,\mu} & \lambda_{-,a} & \overline{\lambda}_{-,a} \\ J_L & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ J'_L & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 & 0 \end{array} \tag{C.16}$$

while for the matter fields we have instead

$$\begin{array}{ccccccccc} \text{fields} & p^\alpha & \overline{p}^\alpha & \phi^i & \overline{\phi}^i & \chi^\alpha & \overline{\chi}^\alpha & \psi^i & \overline{\psi}^i \\ J_L & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ J'_L & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{array} \tag{C.17}$$

and

$$\begin{array}{ccccccccccc} \text{fields} & \eta^A & \overline{\eta}^A & \gamma^I & \overline{\gamma}^I & s & \overline{s} & \xi_+ & \overline{\xi}_+ & \xi_- & \overline{\xi}_- \\ J_L & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ J'_L & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array} \tag{C.18}$$

In the twisted model the supercharge $\overline{\mathcal{Q}}_+$ becomes a worldsheet scalar. $\overline{\mathcal{Q}}_+$-exact operators will decouple from the correlators of $\overline{\mathcal{Q}}_+$-closed fields, to which we restrict our attention. In particular, the kinetic terms for all fields are $\overline{\mathcal{Q}}_+$-exact up to a topological term determined by the gauge bundle on the world-sheet $\Sigma = \mathbb{P}^1$ via the instanton numbers

$$n_a = -\frac{1}{2\pi}\int f_{a,01} \ . \tag{C.19}$$

The integral over field configurations breaks up into a sum over topological sectors indexed by n_a. For $r \in \mathcal{K}_V$, these lie in $\mathcal{K}_V^\vee$, and the classical action weights the contribution of each sector by $\prod_a q_a^{n_a}$ where $q_a = e^{-2\pi r_a + i\theta_a}$. Extracting this topological contribution we can perform the computation within each topological sector semiclassically, and the path integral reduces to an integral over the zero modes of the fields.

The space of zero modes to which the path integral reduces in each sector can be represented as the space of (anti-) holomorphic sections of appropriate line bundles over Σ. Explicitly, the gauge fields take values in

$$\begin{array}{cccccc} \sigma_a & \leftrightarrow & K^{\frac{1}{2}} & \overline{\sigma}_a & \leftrightarrow & \overline{K}^{\frac{1}{2}} \\ \lambda_{+,\mu} & \leftrightarrow & \overline{K}^{\frac{1}{2}} & \overline{\lambda}_{+,\mu} & \leftrightarrow & \overline{K}^{\frac{1}{2}} \\ \lambda_{-,a} & \leftrightarrow & K & \overline{\lambda}_{-,a} & \leftrightarrow & \mathcal{O} \end{array} \tag{C.20}$$

where $K = \mathcal{O}(-2)$ is the canonical bundle. For the matter fields we have instead

$$\begin{array}{cccccc} p^\alpha & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}(d_\alpha) & \overline{p}^\alpha & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}(d_\alpha) \\ \phi^i & \leftrightarrow & \mathcal{O}(d_i) & \phi^i & \leftrightarrow & \overline{\mathcal{O}}(d_i) \\ \psi^i & \leftrightarrow & \overline{K} \otimes \overline{\mathcal{O}}(-d_i) & \overline{\psi}^i & \leftrightarrow & \overline{\mathcal{O}}(d_i) \\ \gamma^I & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}(D_I) & \overline{\gamma}^I & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}(-D_I) \\ \eta^A & \leftrightarrow & K \otimes \mathcal{O}(D_A) & \overline{\eta}^A & \leftrightarrow & \mathcal{O}(-D_A) \\ \chi^\alpha & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}(-d_\alpha) & \overline{\chi}^\alpha & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}(d_\alpha) \\ s & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}\left(d_S\right) & \overline{s} & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}\left(d_S\right) \\ \xi_+ & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}\left(-d_S\right) & \overline{\xi}_+ & \leftrightarrow & \overline{K}^{\frac{1}{2}} \otimes \overline{\mathcal{O}}\left(d_S\right) \\ \xi_- & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}\left(-d_S\right) & \overline{\xi}_- & \leftrightarrow & K^{\frac{1}{2}} \otimes \mathcal{O}\left(d_S\right) \end{array} \tag{C.21}$$

where the various degrees are defined as

$$d_\alpha = -m_\alpha^a n_a \ , \quad d_i = q_i^a n_a \ , \quad D_I = Q_I^a n_a \ , \quad D_A = -d_A^a n_a \ , \quad d_S = (m^a - d^a) n_a \ . \tag{C.22}$$

Note that it turned out to be convenient to use a hermitian metric on the appropriate bundles on $\mathbb{P}^1$ to redefine some of the fields [64]. By examining the half-twisted action it is possible to show that the couplings $\overline{\tau}_a$ as well as $\overline{H}$, $\overline{J}$ and $\overline{E}$ only appear in $\overline{\mathcal{Q}}_+$-exact terms. One very important consequence of this for us is that in the half-twisted theory, correlators of $\overline{\mathcal{Q}}_+$-closed operators are holomorphic in J, H and E, thus for the purpose of our computations we can set $\overline{J} = \overline{H} = \overline{E} = 0$.

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