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Statistical Mechanics of Colliding Beams

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ABSTRACT

The observed performance of electron-positron colliding-beam storage rings is poorly understood theoretically. The problem of a storage ring's behavior is a particular instance of statistical mechanics in an external environment that varies periodically with time, in the limit of weak--and not necessarily isotropic--friction and (additive) noise. We suggest, as a practical starting point for a general theory of such problems, this ansatz: Phase space submanifolds (tori) with fixed canonical actions are manifolds of approximately equal probability density. Such an approach is especially well suited to analysis of the long-time effects of nonlinear resonance on storage ring behavior. We discuss formal consequences of this ansatz, and some associated conceptual difficulties. In an appendix, these issues are considered from the standpoint of "two-time" analysis. We also provide an example of a concrete application, closely related to Kramers' analysis of noise-induced barrier crossing. This paper is meant to be self-contained, so that it can be understood by readers outside the storage-ring community.

I. INTRODUCTION

Electron-positron colliding-beam storage rings are among the most valuable experimental instruments in modern High Energy Physics, and yet, in several crucial respects, their behavior is poorly understood theoretically. Most importantly, it has not yet been possible to consistently predict the (suitably normalized) rate (luminosity) at which a storage ring delivers hard e^+e^- collisions to within much better than a factor of ten, without the use of extensive computer simulation. This and related problems have been the subject of several symposia ([1], [2]). For a comprehensive and pedagogical review, see Ref. [3].

Most theoretical investigators of colliding beam instabilities have concentrated on the conservative or Hamiltonian parts of the interaction between electrons, positrons, and machine hardware. Particular attention has been paid to the Hamiltonian phenomenon of nonlinear resonance (to be explained in Section III), either as an isolated source of unstable behavior, or as the building block out of which more complicated chaotic instabilities are formed. Although this line of inquiry has generated some intuitively appealing qualitative pictures of the dominant processes in electron-positron storage rings, it has never produced a numerical prediction that can meaningfully be compared with experiment, because the full interaction between electrons, positrons, and machine hardware is not entirely Hamiltonian. Dissipative and noisy electromagnetic (synchrotron) radiation is emitted strongly by particles in stored lepton beams, so that the true state of such a system is necessarily a statistical balance between the destabilizing effects of resonance, and both the dissipative and diffusive effects of high-energy photon emission.[F1]

It is clearly desirable to develop a practical quantitative theory of the statistical mechanics of stored colliding beams that can draw as heavily as possible on the valid intuitions--and that can permit one decisively to evaluate the questionable ones--that have been developed during the last twenty years of Hamiltonian analysis. In this essay, we propose one possible starting point for such a theory, and discuss its formal structure and some of its problems. In two accompanying papers ([4], [5]), we shall take up some related technical questions.

An important step toward the development of practical techniques for the statistical mechanics of colliding beams was recently taken by Kheifets [6], who formulated a new perturbative representation of the colliding beam Fokker-Planck equation, with which he made progress toward accounting quantitatively for a number of experimental observations. Unfortunately, it appears difficult to understand Kheifets' numerical results at an intuitive level. The discussion in the present paper can be regarded as complementary to Kheifets' work.

The present paper is intended for two groups of readers. The first group consists of accelerator scientists, whom we wish to convince that there are still promising but untried directions for systematic study of colliding beam instabilities. The second group consists of applied mathematicians and mathematical physicists--particularly those knowledgeable in the areas of stochastic processes and nonlinear mechanics--whom we wish to interest in the problems of storage rings, especially those problems for which traditional textbook methods appear to be inadequate. For the sake of such readers, we have written this article in as self-contained and pedagogical a way as possible, consistent with publication in a research journal.

In order to explain our ideas, we will not need to provide a general introduction to the philosophy and formalism of storage ring design (see [3] and [7]), but we will need to make explicit a few basic facts, which we present now.

For our purposes, the most important fundamental result in storage ring theory is this: Electrons and positrons travel through storage rings (long thin pipes curved to close on themselves) in small bunches, and the simplest form (the so-called "weak-strong" limit [3]) of the system of equations that describes the oscillation of a stored electron or positron about the center of its bunch is

$$\ddot{x} + \gamma\dot{x} + \omega_x^2 x + \partial_x g(x,y,z,t) = \omega_x (2\gamma)^{1/2} \xi_x(t)$$

$$\ddot{y} + \gamma\dot{y} + \omega_y^2 y + \partial_y g(x,y,z,t) = \omega_y (2\gamma)^{1/2} \xi_y(t)$$

$$\ddot{z} + 2\gamma\dot{z} + \omega_z^2 z = \omega_z (4\gamma)^{1/2} \xi_z(t) \ . \qquad (1.1)$$

In this system of equations, the coordinate z refers to displacement from the bunch center along the bunch center's direction of motion ("beam axis"), and x and y refer to displacement transverse to the beam axis, within and perpendicular, respectively, to the plane of the storage ring (see Fig. 1).

The functions ξ in Eqs. (1.1) are independent sources of uncorrelated Gaussian white noise with unit delta-function variance, i.e.,

$$\langle \xi_x(t)\, \xi_x(t') \rangle = \delta(t-t')$$

$$\langle \xi_x(t)\, \xi_y(t') \rangle \equiv 0, \qquad (1.2)$$

etc. The normalization of the noise terms in (1.1) takes the form that it does because we have tacitly rescaled the coordinates so that

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = 1 \qquad (1.3)$$

in the equilibrium to which the beam distribution relaxes when the function g is set equal to zero.

The nonquadratic potential g describes collisions with the oppositely charged counter-rotating beam. In general, g is explicitly periodic in time. Physically, the repeat period T is the time it takes for a bunch center to circle the storage ring once. It is this explicit, rapid ($T\gamma \sim O(10^3)$) time dependence of g--in addition to the anisotropic character of damping and noise--that places system (1.1) beyond the reach of the usual textbook methods of statistical mechanics.

The g-terms in (1.1) are generally much smaller than the terms $\omega_x^2 x$ and $\omega_y^2 y$. Near x=y=0, for example, $|\partial_x g/\omega_x^2 x|$ and $|\partial_y g/\omega_y^2 y|$ are both typically $\lesssim O(10^{-3})$ (see[3]). Nevertheless, the potential g can have substantial cumulative effects: For example, $(\langle y^2 \rangle)^{1/2}$ is empirically observed to be as large as three, or more, when the collision terms are present (although $\langle x^2 \rangle$ and $\langle z^2 \rangle$ are typically almost unaffected.) One's inability to estimate luminosity accurately can in large measure be attributed to an inability to understand this enlargement of $\langle y^2 \rangle$ [3].

In order to understand the motivation for the main ideas in the present paper, it is important to recognize that in real storage rings the damping coefficient γ is small in comparison with other rates characteristic of (1.1). Specifically [3], γ is typically smaller than ω_x and ω_y by a factor of $O(10^4)$, and smaller than ω_z by a factor of $O(10^2)$, and therefore an electron or positron executes many "free" Hamiltonian[F2] oscillations in the time it takes for noise or damping to accumulate significantly.

In view of the slowness with which damping and noise appear to operate, it seems reasonable to guess that, after a long time, the phase space probability distribution for a particle subject to (1.1) should, in some approximation, smear itself out along the paths of the dynamical flow corresponding to the Hamiltonian system obtained by setting $\gamma = 0$. If this is so, we might expect to be able to approximate the probability distribution $P(x,v_x,y,v_y,z,v_z,t)$ by a function of a smaller number of variables, and this in turn would enable us to simplify the important task of solving the Fokker-Planck equation.

$$\frac{\partial P}{\partial t} + v_x \frac{\partial P}{\partial x} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z} - (\omega_x^2 x + \partial_x g) \frac{\partial P}{\partial v_x}$$

$$- (\omega_y^2 y + \partial_y g) \frac{\partial P}{\partial v_y} - (\omega_z^2 z) \frac{\partial P}{\partial v_z}$$

$$= \gamma \frac{\partial}{\partial v_x} (v_x P + \omega_x^2 \frac{\partial P}{\partial v_x}) + \gamma \frac{\partial}{\partial v_y} (v_y P + \omega_y^2 \frac{\partial P}{\partial v_y})$$

$$+ 2\gamma \frac{\partial}{\partial v_z} (v_z P + \omega_z^2 \frac{\partial P}{\partial v_z}) \qquad (1.4)$$

(The Fokker-Planck equation (1.4) determines the distribution P, which

determines the effective size of a beam, which directly determines [3] the rate at which a storage ring delivers particle-antiparticle collisions.)

This is the point of view that we shall try to develop in this paper. It is particularly suited to the analysis of systems dominated by a few nonlinear resonances, for the following reason: Under general circumstances, the approximation of the probability distribution P by a function of fewer than the full complement of seven variables is always conceivable in principle, but it requires information about the shape of the Hamiltonian flow that is difficult to obtain in practice. In the neighborhood of a nonlinear resonance, however, this information is readily available, because the Hamiltonian flow there can always be characterized in very simple terms [8], as we shall explain in Section III.

In the simpler context of underdamped nonlinear systems with only one degree of freedom and no explicit time dependence in the potential, the hypothesis that (in a sense we shall make precise shortly) probability smears itself out along the Hamiltonian flow was first articulated by Kramers more than forty years ago [9], and has since become a commonplace paradigm in the theory of stochastic processes. (For a survey of the literature, see Ref. [10].) In what follows, we shall refer to this hypothesis as "thermal averaging." We shall see that Kramers' original formulae have their likely counterparts in the present, more complicated setting; in trying to derive them, however, we shall encounter some unique difficulties.

This completes our preliminary discussion. Our detailed presentation, which begins in the next section, will be divided into four parts:

In the first part (Section II), we develop, through heuristic arguments, the formalism of thermal averaging in general, without specific reference to nonlinear resonance. We shall encounter in this section the first difficulty with this approach to statistical processes--the phenomenon of thermal resonance, which can undermine thermal averaging when various characteristic frequencies or integrals satisfy a linear equation with integer coefficients. A more systematic account of dynamical averaging and thermal resonance, based on the "two-time" method [20], is given in an appendix, at the end of this paper.

In the next part (Section III), we introduce the general theory and phenomenology of nonlinear resonance. (To distinguish nonlinear resonance from thermal resonance, it might be more appropriate to call it "mechanical resonance"; but we shall adhere to standard terminology.) In the specific case of systems with one dynamical variable, we shall explicitly write out the form taken by the expressions of Section II near a nonlinear resonance. This will reveal the second difficulty with thermal averaging--a number of apparent discontinuities of topological origin in the behavior of the probability distribution at characteristic phase space borders known [8] as resonance separatrices.

In Section IV, we shall discuss in greater detail phenomenological applications of thermal averaging to systems with only one dynamical variable, even though most workers agree that the enlargement of the beams in real storage rings is due to dynamical mechanisms that require

more than one active degree of freedom [11], [12]. We do this because at present there are many more procedures that we can conceive of carrying out (ignoring the problems of thermal resonance and separatrix discontinuity) in one dimension than in higher dimensions. These include the approximate description of systems dominated by more than one strong nonlinear resonance, the construction of steady-state distributions, and the evaluation of the rate at which a nonlinear resonance draws particles from the beam center before the steady state has been achieved. (Our theory of this last rate is closely related to Kramers' theory [9] of noise-induced barrier crossing.) It is hoped that this one-dimensional work can suggest directions for the further development of higher-dimensional methodology. As an illustration of our one-dimensional techniques, we shall apply them to some recently published computer-generated data [13] that exhibits features strikingly reminiscent of purely one-dimensional systems. Unfortunately, our simple one-dimensional model falls far short of explaining the observations in question, providing further confirmation that real storage ring problems are essentially higher-dimensional.

In the final part, Section V, we shall propose two ways in which the problem of thermal resonance might be investigated further. This section will be a preview of two papers ([4], [5]) that follow this one, each of which will develop one of these lines of inquiry in greater detail.

II. THERMAL AVERAGING IN GENERAL (HEURISTIC)

We must begin the discussion in this section by making explicit certain general facts about solutions to Eqs. (1.1):

According to the KAM theorem [21], as long as γ vanishes, and as long as the nonlinearity in $\vec{\nabla} g$ is not too great, most solutions to Eqs. (1.1) can be grouped into disjoint families, in each of which the Hamiltonian can be reduced to action-angle form. That is, in each such family, the solutions x(t), y(t), z(t) can be expressed as

$$\begin{aligned} x &= x(\vec{I},\vec{\theta},t) \\ \dot{x} \equiv v_x &= v_x(\vec{I},\vec{\theta},t), \end{aligned} \tag{2.1}$$

and similarly for y and z, where: x, v_x, y, v_y, z, v_z are 2π-periodic functions of the three components of the vector $\vec{\theta}$; the explicit time dependence of x, v_x, etc. is periodic in t, with a period equal to an integer multiple of the storage ring period T; and the implicit time dependence due to the Hamiltonian equations of motion is governed by

$$\begin{aligned} \dot{\vec{I}} &\equiv 0 \\ \dot{\vec{\theta}} &\equiv \vec{\omega}^1(\vec{I}), \end{aligned} \tag{2.2}$$

for some vector function $\vec{\omega}^1$. (The superscript "1" is to distinguish this vector from the triplet of angular frequencies (ω_x, ω_y, ω_z) appearing in (1.1).)

The set of three-component actions $\vec{I}$ that correspond to a single family is by definition connected, but does not necessarily fill all of three-dimensional space. By contrast, a single family of solutions necessarily encompasses all possible angle vectors $\vec{\theta}$.

It will help later on to have noted here that the transformation of variables $\{(x,y,z),\ (v_x,v_y,v_z)\} \to \{\vec{\theta},\vec{I}\}$ is canonical. In particular, it has unit Jacobian.[F3]

In words, Eqs. (2.1) and (2.2) say that the phase space point $(x,\dot{x},y,\dot{y},z,\dot{z})$ undergoes translation along one member of some smooth family of three-dimensional tori, while the tori themselves move and flex, according to the explicit periodic time-dependence in (2.1). We shall show in the next section that these tori can be described in a simple and universal way near regions of nonlinear resonance.

The first part of the hypothesis of thermal averaging is this: In the large-time limit, the phase space probability distribution $P(x,v_x,y,v_y,z,v_z,t)$ of a particle becomes approximately independent of the angular vector $\vec{\theta}$, within an action-angle family determined by Eqs. (1.1), i.e.,

$$P(x,v_x,\ldots,t) \equiv P(\vec{I},\vec{\theta},t) \simeq P(\vec{I},t). \tag{2.3}$$

One may interpret $P(\vec{I},t)$ as $(2\pi)^{-3}$ (or $(2\pi)^{-d}$, when the number of degrees of freedom is $d \neq 3$) times the density of probability in $\vec{I}$ space, because the canonical nature of the transformation (x, y, etc.) $\to (\vec{\theta},\vec{I})$ implies that the probability per d^3I is

$$\prod_{i=1}^{3} \left[\int_0^{2\pi} d\theta_i \right] P(\vec{I},\vec{\theta},t),$$

with or without (2.3).

The time evolution of the asymptotic form $P(\vec{I},t)$ is determined by a partial differential equation that we shall formulate shortly as the second part of the hypothesis.[F4] Before turning to this equation,

however, let us motivate Eq. (2.3) hueristically. For a more systematic treatment, the reader is referred to the Appendix. We feel strongly that the hueristic argument is important because it helps to show that the results of the systematic treatment are not artificial consequences of what may otherwise seem to be some arbitrariness in the starting point from which the systematic treatment proceeds.

We motivate Eq. (2.3) in three steps. First, we try to argue, with more precision than in the Introduction, that after enough time has passed, for small γ, any differential volume of probability in phase space becomes spread by the laws of motion (1.1) along orbits on the dynamical tori.

To do this, we begin with the observation that, in general, the vector function $\vec{\omega}^1$ depends nontrivially on $\vec{I}$. Thus, points initially close to one another on two close but distinct tori must become increasingly far apart as time passes. Accordingly, any differential volume of probability must become stretched out, as time passes, like a drop of ink in a river whose flow speed varies nontrivially with depth.[F5]

This is not yet what we want to show, however: Continuing the river metaphor, we want to show that the profile of the ink drop at any single depth spreads along the general flow direction, not just that the profiles at different depths move away from one another. It should be easy to see, however, that the former spreading is a consequence of the latter, because of the noise terms in (1.1), which ensure not only that probability flows along Hamiltonian tori, but also that it diffuses across tori; just as the ink drop diffuses up and down while the river's current draws it downstream.

In mathematical terms, we have just argued that an initial "dot" of probability centered at $\vec{I} = \vec{I}_0$ and $\vec{\theta} = \vec{\theta}_0$ evolves into a ribbon of probability clustered about the set of points with $\vec{I} = \vec{I}_0$ and $\vec{\theta} = \vec{\theta}_0 + \vec{\omega}^1(\vec{I}_0)\tau$, where the parameter τ ranges over an interval whose length grows steadily as time progresses. We want now to argue from this--as the second step in our motivation of (2.3)--that after some time, any phase-space volume of nonzero probability must spread to fully cover whatever torus it might intersect initially.

We can do this directly for any torus for which there is no triplet $\vec{n}$ of integers such that

$$\vec{n}\cdot\vec{\omega}^1(\vec{I}_0) = 0. \tag{2.4}$$

In this (noncommensurate) case, we know that the set of all points of the form $\vec{I} = \vec{I}_0$, $\vec{\theta} = \vec{\theta}_0 + \vec{\omega}^1(\vec{I}_0)\tau$ is dense in the torus $\vec{I} = \vec{I}_0$ [14], and therefore the steadily elongating ribbon of probability, necessarily of finite width because of diffusion, ought eventually to fill out the entire torus. When (2.4) is satisfied, by contrast, we cannot reason in this way; but since such commensurate tori, being defined by integers, are necessarily of total measure zero, continuity, together with diffusion across tori, might possibly ensure the same conclusion.

Equation (2.4) is our first example of thermal resonance, by which we shall mean in general an obstruction to our smoothing procedure localized on a countably dense set of points in phase space.[F6] The second example will occur in our formulation, below, of the partial differential equation that determines the evolution of $P(\vec{I},t)$. We are assuming here, and we shall assume again below, that continuity and

diffusion together tend to erase whatever structure thermal resonance might otherwise set up in probability distributions. As indicated in the Introduction, we shall propose in Section V some simpler settings in which this assumption might be more carefully scrutinized.

(Note that in systems with only one degree of freedom--in which case $\vec{\omega}^1$ has only one component--the commensurate condition (2.4) reduces to $\omega^1 = 0$, which can only be satisfied at isolated, non-dense, values of the one-component action I.)

For the last step in the hueristic argument that we have just interrupted, we must guess--as seems natural intuitively--that the process of torus-covering that we have just described tends to smooth out any dependence of the probability P on the internal torus coordinate triple $\vec{\theta}$, and thus we are led directly to (2.3). Deferring, as we have already indicated, a more systematic derivation to the Appendix, let us now take (2.3) for granted, and proceed to the differential equation that determines its evolution in time. Once again, we shall argue informally, leaving a more systematic treatment to the Appendix.

In order to see how the desired equation comes about, imagine substituting the ansatz (2.3) into the full Fokker-Planck equation (1.4). It should be clear that the result is explicitly $\vec{\theta}$-dependent, even if P itself is not, because of the $\vec{\theta}$-dependence of the variables x,y,z,v_x,v_y,v_z, according to Eq. (2.1). A differential volume of probability that streams approximately along a Hamiltonian orbit on a KAM torus sees this θ-dependence as giving rise to a rapid oscillation in the environment through which it passes. The explicit periodic time dependence expressed in (2.1) also contributes to this oscillation. It seems reasonable to suppose that one should average over this rapid

oscillation in order to obtain the persistent effects of the Fokker-Planck equation on the asymptotic form $P(\vec{I},t)$.

Let us now work out in detail how this averaging is to be done for one representative term in (1.4). The reasoning that we shall apply to this example will also be suitable for the remainder of the Fokker-Planck equation, whose averaged form we shall then be able to write out in full without further argumentation.

We first comment, however, that if the dominant contribution to the evolution of $P(\vec{I},t)$ is indeed determined by such averaging, it should be self-consistent to assume at the outset that $P(\vec{I},t)$, in its dependence on time, varies slowly compared to the oscillations being smoothed away. We shall adopt this assumption in what follows.

Consider now the sample term[F7]

$$v_x \frac{\partial P}{\partial x} = \left[v_x \frac{\partial I_i}{\partial x} \right] \frac{\partial P}{\partial I_i} . \tag{2.5}$$

In order to average this over the rapid oscillations encountered by a particle moving with the Hamiltonian flow, we should replace $\vec{\theta}$ in (2.5) by $\vec{\theta}_0 + t\vec{\omega}^1(\vec{I})$ for some arbitrary $\vec{\theta}_0$, and then average over a time interval of some intermediate length t_0, long with respect to the time scale of environmental oscillations, but short compared to the time scale over which $P(\vec{I},t)$ varies. Thus we advocate the replacement

$$v_x \frac{\partial P}{\partial x} (\vec{I},\vec{\theta},t) \to \frac{1}{t_0} \int_{t-t_0/2}^{t+t_0/2} dt' \left[v_x \frac{\partial P}{\partial x} \right] (\vec{I},\vec{\theta}_0+t'\vec{\omega}(\vec{I}),t')$$

$$\simeq \frac{\partial P}{\partial I_i}(\vec{I},t)\ \lim_{\tau\to\infty}\frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} dt' \left[v_x \frac{\partial I_i}{\partial x}\right](\vec{I},\vec{\theta}_0+t'\vec{\omega}(\vec{I}),t'). \qquad (2.6)$$

The passage to the final factorized expression in (2.6) follows from our assumption that P on the one hand, and on the other hand the variables x,y,z, etc., vary on very different time scales.

It is clearly important that the averaging in (2.6) result in an expression manifestly independent of the arbitrary initial vector $\vec{\theta}_0$, because our goal is a differential equation for a density function of $\vec{I}$ and t only. We can ensure this as long as there is <u>no</u> rational linear relation between the components of $\vec{\omega}^1$ and the fundamental angular frequency Ω that governs the explicit periodic time dependence in (2.1), i.e., as long as there are no integers $\{n_i\}$ and m such that

$$\vec{n}\cdot\vec{\omega}^1 + m\,\Omega = 0\ . \qquad (2.7)$$

In this case, when one expands the final integrand in (2.6) as a Fourier series in powers of the $\exp i\theta_j$, and of $\exp i\Omega t'$, and then makes the substitution $\vec{\theta} = \vec{\theta}_0 + t'\vec{\omega}^1(\vec{I})$, the only term in the multiple series that makes a nonzero contribution to the average is the term in which all the powers are zero. Thus we may write

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} dt' \left[v_x \frac{\partial I_i}{\partial x}\right](\vec{I},\ \vec{\theta}_0 + t'\vec{\omega}^1(\vec{I}),\ t')$$

$$= \left(\frac{2\pi}{\Omega}\right)^{-1} \int_0^{2\pi/\Omega} dt' \left\{ \prod_{j=1}^{3} \int_0^{2\pi} \frac{d\theta_j}{2\pi} \right\} \left[v_x \frac{\partial I_i}{\partial x} \right] (\vec{I},\vec{\theta},t')$$

$$\equiv < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial x} > , \qquad (2.8)$$

leaving us with an expression manifestly independent of $\vec{\theta}_0$, and also of t.

When some relation of the form (2.7) is satisfied, we cannot a priori factorize the averaging process as in (2.8), and thus we cannot show directly that the last expression in (2.6) is independent of $\vec{\theta}_0$. This is our second example of thermal resonance. It is tempting to suppose that since actions $\vec{I}$ that are resonant according to (2.7) are distributed densely among actions that are not--and since, collectively, they must have measure zero because they are determined by integers--continuity assures that (2.8) is appropriate for all $\vec{I}$. We shall assume in what follows that this is so. As already indicated, we defer a more critical discussion of this assumption to Section V, and to Refs. [4] and [5].

(In case the reader wishes to be convinced in advance that it is not self-evident that this assumption ought to be used freely in practical calculcations, we note here one example of a possible weakness: The factorized limit (2.8) may indeed apply to all values of $\vec{I}$, but perhaps it is not approached everywhere at the same rate. I.e., there might always be, as in the Gibbs phenomenon of Fourier analysis [15], a set of $\vec{I}$'s (perhaps a dense set) for which the value of the first expression in (2.8) is far from the value of the last expression,

even though the measure of this set might approach zero as time becomes infinite.)

When the full Fokker-Planck equation (1.4) is transformed according to the formal procedures that we have just introduced, the result is

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial I_i} < \oint \frac{d^3\theta}{(2\pi)^3} \left\{ \frac{\partial I_i}{\partial t} + v_x \frac{\partial I_i}{\partial x} + v_y \frac{\partial I_i}{\partial y} + v_z \frac{\partial I_i}{\partial z} \right.$$

$$\left. - (\omega_x^2 x + \partial_x g) \frac{\partial I_i}{\partial v_x} - (\omega_y^2 y + \partial_y g) \frac{\partial I_i}{\partial v_y} - (\omega_z^2 z) \frac{\partial I_i}{\partial v_z} \right\} >$$

$$= \gamma \left[P + \frac{\partial P}{\partial I_i} < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial v_x} > \right]$$

$$+ \gamma\omega_x^2 \left[\frac{\partial P}{\partial I_i} < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial^2 I_i}{\partial v_x^2} > + \frac{\partial^2 P}{\partial I_i \partial I_j} < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial I_i}{\partial v_x} \frac{\partial I_j}{\partial v_x} > \right]$$

+ similar expressions for y and z. (2.9)

In this equation, $\partial P/\partial t$ refers to the time derivative with $\vec{I}$ (and the trivial dependence on $\vec{\theta}$) fixed, in contrast to (1.4), where the time derivative is taken with x, v_x, etc. fixed. The time derivative of I_i in (2.9) is to be taken with x, v_x, etc. fixed.

We can simplify Eq. (2.9) in two ways:

First, we observe that the expression in curved brackets in (2.9) is equivalent to dI_i/dt when time evolution is governed by Eq. (1.1), with $\gamma = 0$, and when the v's are properly interpreted as velocities. However, dI_i/dt is identically zero under such conditions, according to Eq. (2.2). Therefore, we may completely ignore the summation on the left-hand side of (2.9).

Second, the right-hand side of Eq. (2.9) can be reexpressed as a perfect divergence, so that Eq. (2.9) can be made to have the structure of a continuity equation, just as the full Fokker-Planck Eq. (1.4) does. To show this, we prove below that

$$\frac{\partial}{\partial I_i} < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial v_x} > \equiv 1, \tag{2.10a}$$

$$\frac{\partial}{\partial I_j} < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial I_i}{\partial v_x} \frac{\partial I_j}{\partial v_x} > \equiv < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial^2 I_i}{\partial v_x^2} >, \tag{2.10b}$$

and similarly for y and z. From (2.10), and from the conclusion of the preceding paragraph, it follows that (2.9) reduces to

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial I_i} \left[P < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial v_x} > + \omega_x^2 \frac{\partial P}{\partial I_j} < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial I_i}{\partial v_x} \frac{\partial I_j}{\partial v_x} > \right]$$

$$+ \text{similar expressions for y and z.} \tag{2.11}$$

(It should be easy to see that when the system has an arbitrary number, d, of degrees of freedom (so that one has d x's and v's, and d components of $\vec{I}$ and $\vec{\theta}$), then Eq. (2.11) is still suitable, as long as $d^3\theta/(2\pi)^3$ is replaced by $d^d\theta/(2\pi)^d$.) Equation (2.11) generalizes Kramers' [9] Eq. (14).

To verify Eqs. (2.10), we use the identities

$$\frac{\partial I_i}{\partial v_k} = \frac{\partial x_k}{\partial \theta_i}$$

$$\frac{\partial I_i}{\partial x_k} = -\frac{\partial v_k}{\partial \theta_i}$$

$$\frac{\partial \theta_i}{\partial v_k} = -\frac{\partial x_k}{\partial I_i}$$

$$\frac{\partial \theta_i}{\partial x_k} = \frac{\partial v_k}{\partial I_i} \ , \tag{2.13}$$

which follow[F8] from the canonical nature of the variables $\{\vec{\theta}, \vec{I}\}$ and $\{\vec{x}, \vec{v}\}$. With the help of (2.13), Eq. (2.10a) can be established as follows:

$$\frac{\partial}{\partial I_i} < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial v_x} >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left(\frac{\partial v_x}{\partial I_i} \frac{\partial I_i}{\partial v_x} + v_x \cdot \frac{\partial}{\partial I_i} \frac{\partial I_i}{\partial v_x} \right) >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left(\frac{\partial v_x}{\partial I_i} \frac{\partial I_i}{\partial v_x} - \frac{\partial v_x}{\partial \theta_i} \frac{\partial x}{\partial I_i} + \frac{\partial}{\partial \theta_i} \left[v_x \frac{\partial x}{\partial I_i} \right] \right) >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left(\frac{\partial v_x}{\partial I_i} \frac{\partial I_i}{\partial v_x} + \frac{\partial v_x}{\partial \theta_i} \frac{\partial \theta_i}{\partial v_x} + \frac{\partial}{\partial \theta_i} \left[v_x \frac{\partial x}{\partial I_i} \right] \right) >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left(1 + \frac{\partial}{\partial \theta_i} \left[v_x \frac{\partial x}{\partial I_i} \right] \right) >$$

$$= 1. \tag{2.14}$$

(Note that we are able to discard integrals of perfect θ-derivative, because the arguments of the derivatives are 2π-periodic in the θ's.) We may verify (2.10b) in the same way:

$$\frac{\partial}{\partial I_j} < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial I_i}{\partial v_x} \frac{\partial I_j}{\partial v_x} >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left[\left(\frac{\partial}{\partial I_j} \frac{\partial I_i}{\partial v_x} \right) \frac{\partial I_j}{\partial v_x} + \frac{\partial I_i}{\partial v_x} \left(\frac{\partial}{\partial I_j} \frac{\partial I_j}{\partial v_x} \right) \right] >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left[\left(\frac{\partial x}{\partial \theta_j} \frac{\partial}{\partial I_j} - \frac{\partial x}{\partial I_j} \frac{\partial}{\partial \theta_j} \right) \frac{\partial x}{\partial \theta_i} + \frac{\partial}{\partial \theta_j} \left(\frac{\partial x}{\partial \theta_i} \frac{\partial x}{\partial I_j} \right) \right] >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \left(\frac{\partial I_j}{\partial v_x} \frac{\partial}{\partial I_j} + \frac{\partial \theta_j}{\partial v_x} \frac{\partial}{\partial \theta_j} \right) \frac{\partial I_i}{\partial v_x} >$$

$$= < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial^2 I_i}{\partial v_x^2} > . \qquad (2.15)$$

III. NONLINEAR RESONANCE AND THERMAL AVERAGING

In the first part of this section, we shall present a brief introduction to the theory of nonlinear resonance, in the specific context of system (1.1). In order to help justify the attention that we are paying to this general subject, we shall also provide an example of computer-generated data that shows, in a direct way, a nonlinear resonance in the process of rapidly drawing particles away from the center of a beam in an e^+e^- storage ring, thereby contributing to the enlargement of $<y^2>$. This example will appear again in Section IV as the object of a model application of thermal averaging.

We shall discuss the basic structure of thermal averaging near nonlinear resonance in the second part of this section.

1. General Discussion of Nonlinear Resonance

In the narrowest sense, the term "nonlinear resonance" (not to be confused with thermal resonance) refers specifically to structure that one encounters in solving Eqs. (1.1) (with $\gamma = 0$) with the potential g treated as a perturbation. In order to make this explicit, we shall follow custom [8] by first introducing action and angle variables for the system of harmonic oscillators corresponding to (1.1) with $\gamma = g = 0$, and then expanding the nonlinear perturbations $\partial_x g$ and $\partial_y g$ as Fourier series' in the oscillator angles, and in time.

The oscillator actions and angles $\vec{J}$ and $\vec{\phi}$ are defined by

$$J_x \equiv \frac{1}{2\omega_x} \left[\dot{x}^2 + x^2\omega_x^2\right]$$

$$\phi_x \equiv -\tan^{-1}\left[\dot{x}/x\omega_x\right] , \tag{3.1}$$

and similarly for y and z. In the absence of damping, noise, and collision terms, $\vec{J}$ and $\vec{\phi}$ vary with time according to

$$\dot{\vec{J}} = 0$$

$$\dot{\vec{\phi}} = \vec{\omega} , \tag{3.2}$$

where $\vec{\omega} \equiv (\omega_x, \omega_y, \omega_z)$.

Upon Fourier decomposition of the potential g, one finds the following expressions for the time evolution of J_x and ϕ_x in the presence of the collision terms:

$$\dot{J}_x = -\sum_{\vec{n},n} i n_x G_{\vec{n},n}(\vec{J}) \exp i\left[\vec{n}\cdot\vec{\phi} + \frac{2\pi n t}{T}\right],$$

$$\dot{\phi}_x = \omega_x + \sum_{\vec{n},n} \frac{\partial G_{\vec{n},n}(\vec{J})}{\partial J_x} \exp i\left[\vec{n}\cdot\vec{\phi} + \frac{2\pi n t}{T}\right]. \tag{3.3}$$

A similar formula holds for $\dot{J}_y$ and $\dot{\phi}_y$, while J_z and ϕ_z continue to evolve according to (3.2). The summation indices $\vec{n}$ and n in (3.3) are entirely integral. The Fourier coefficients G are defined by

$$G_{\vec{n},n}(\vec{J}) \equiv \int_0^T \frac{dt}{T} \oint \frac{d^3\phi}{(2\pi)^3} g\left[\left(\frac{2J_x}{\omega_x}\right)^{1/2}\cos\phi_x, \left(\frac{2J_y}{\omega_y}\right)^{1/2}\cos\phi_y, \left(\frac{2J_z}{\omega_z}\right)^{1/2}\cos\phi_z, t\right]$$

$$\cdot\exp - i\left[\vec{n}\cdot\vec{\phi} + \frac{2\pi n t}{T}\right]. \tag{3.4}$$

(For a discussion of a complication ignored for convenience in (3.3), see Ref. [3], Section IV.2.b.)

In substantial regions of phase space, one can ignore the summation terms that have nonzero $\vec{n}$ and n in Eqs. (3.3) and their y and z counterparts, because these terms oscillate rapidly, and therefore they make only small contributions to $\vec{J}$ and $\vec{\phi}$ upon integration. In such regions, the dominant collision-related contributions to $\vec{J}$ and $\vec{\phi}$ come from the terms with vanishing $\vec{n}$ and n, i.e.,

$$\dot{\vec{J}} \simeq 0,$$

$$\dot{\phi}_x \simeq \omega_x + \frac{\partial G_{\vec{0},0}}{\partial J_x},$$

$$\dot{\phi}_y \simeq \omega_y + \frac{\partial G_{\vec{o},o}}{\partial J_y},$$

$$\dot{\phi}_z = \omega_z. \tag{3.5}$$

Note that whenever (3.5) is applicable, we may identify $\vec{I}$ with $\vec{J}$ and $\vec{\theta}$ with $\vec{\phi}$, in which case the averaged Fokker-Planck equation (2.11) is insensitive to the nonlinear collision terms, $\partial_x g$ and $\partial_y g$, in the original equations of motion (1.1).

There can be substantial corrections to approximation (3.5)--and therefore substantial collision-related effects on beam distributions--near values of $\vec{J}$ at which a nonlinear resonance condition,

$$n_x\left(\omega_x + \frac{\partial G_{\vec{o},o}}{\partial J_x}\right) + n_y\left(\omega_y + \frac{\partial G_{\vec{o},o}}{\partial J_y}\right) + n_z\omega_z + \frac{2\pi n}{T} = 0\ , \tag{3.6}$$

is satisfied for some integral $\vec{n}$ and n, because in this case the $\pm(\vec{n},n)$ Fourier terms in (3.3) vary slowly, and can therefore make contributions to $\vec{J}$ and $\vec{\phi}$ that can accumulate in time. To be sure, the set of all $\vec{J}$ that solve an equation of the form (3.6) for some $\vec{n}$ and n should be dense in the set of all possible $\vec{J}$'s, and at first sight this must appear to undercut whatever validity one might otherwise ascribe to the distinction between nonresonant (corresponding to (3.5)) and resonant motion. However, one can assign widths [8] to regions of phase space in which any given Fourier term in (3.3) is resonant, and for all but a limited number of $\vec{n}$'s and n's, these widths are negligibly small for most practical purposes.

For the developments in the next subsection, and in Section IV, we shall need an approximate analytical description of an isolated strong nonlinear resonance (i.e., one whose width is not negligible). We shall present one shortly. However, for preparation, let us first discuss the general structure of strong nonlinear resonance in a qualitative, pictorial way.

For this purpose, consider Fig. 2, which represents schematically the phase plane of some single-resonance-dominated Hamiltonian system analogous to (1.1) (with $\gamma = 0$), but with only one degree of freedom, to be called y for definiteness. The closed curves represent the one-dimensional versions of the KAM tori at one instant of time. The region of strong resonance corresponds to the "island chain" of six sets of nested curves that do not enclose the origin. (Figure 2 is idealized in showing only one island chain. Several concentric such chains is common. Systems that exhibit two or more such strong resonances will be discussed explicitly in Section IV.1.) The resonance separatrix is the border between the orbit curves that enclose the origin and those that do not. The coexistence of families of orbit curves that have two distinct topologies in this way is characteristic of nonlinear resonance. When the system has an arbitrary number, d, of degrees of freedom, then d-dimensional families of d-dimensional KAM tori that do and do not nest upon the origin coexist in this way under the influence of nonlinear resonance. If the nonlinear collision terms were absent, there would be only one action-angle family and all its nesting tori would be centered at the origin.

The resonant pattern depicted in Fig. 1 would arise from Fourier coefficients having indices (n_y,n), with $n_y = \pm 6$, in the one-dimensional version of (3.3). The distance from the origin to the island centers would be given approximately by $(2J_y^\circ)^{1/2}$, where J_y° solves the one-dimensional version of the resonance condition (3.6).

Figure 3 (from Reference [13]) shows a concrete example of the effect of such a strongly resonant ^{F9} pattern, observed in a recent computer simulation [16] of an electron-positron storage ring. The particles in this simulation were subject to Eq. (1.1), and all three variables x, y, and z were included. The horizontal and vertical axes in this figure refer to $y/(2)^{1/2}$ and $\dot{y}/\omega_y(2)^{1/2}$, respectively. The points in this figure correspond to one particle's orbit, projected onto the vertical phase plane. The particular points shown here correspond to the phase space position of the particle at successive equally spaced time intervals of duration T, the repeat period of the collision potential g. The total equivalent real time elapsed is equal to about $(3.7)\cdot(2/\gamma)$, which would be about 3.7 frictional relaxation times if the collision terms were absent.

In this simulation, at least one test particle out of a statistical sample of less than one hundred [16] was observed to be drawn rapidly to the islands of the sixth order resonance in this way. In order to appreciate how remarkably rapid this is, one must realize that if there were no strong resonance (or any other comparable structure) so that the nonlinearities in g could effectively be ignored, one would have expected [7] that only one part of this model beam in about

$$4^{-1}\,(18)^{-2}\,\exp\frac{(18)^2}{2} \simeq 10^{67} \tag{3.7}$$

would reach the largest radius (18) achieved by any point in Fig. 2 within about four damping times. The rapid transport seen in this figure is the process that we shall try to account for quantitatively in Section IV.

We now proceed to outline the most common approximate analytical description of nonlinear resonance. Our goal will be the derivation of a small number of formulae that analytically characterize resonance islands and their immediate environs in a simple way. These formulae, especially the one that describes the separatrix, will be used prominently in the next subsection, and in Section IV.

The conventional approximation to Eq. (3.3) near a resonance of the $\pm(\vec{n},n)$ Fourier terms is ([8]; see also [3], Section IV.2.C)

$$\begin{aligned}\dot{J}_y &\simeq -in_y\, G_{\vec{n},n}(\vec{J}^\circ)\exp i[\vec{n}\cdot\vec{\phi} + \frac{2\pi nt}{T}] \\ &\quad + in_y\, G_{-\vec{n},-n}(\vec{J}^\circ)\exp i[-\vec{n}\cdot\vec{\phi} - \frac{2\pi nt}{T}] \\ &= 2n_y\left|G_{\vec{n},n}(\vec{J}^\circ)\right|\sin[\vec{n}\cdot\vec{\phi} + \frac{2\pi nt}{T} + \operatorname{Arg} G_{\vec{n},n}(\vec{J}^\circ)],\end{aligned} \tag{3.8}$$

$$\dot{\phi}_y \simeq \omega_y + \frac{\partial G_{\vec{o},o}}{\partial J_y}(\vec{J}^\circ) + \sum_{i=x,y,z}(J-J^\circ)_i\left[\frac{\partial^2 G_{\vec{o},o}}{\partial J_i \partial J_y}(\vec{J}^\circ)\right], \tag{3.9}$$

and similarly for x and z. The constant reference vector $\vec{J}^\circ$ is a solution of the resonance condition (3.6). The second equality in (3.8)

is possible because $G^{\rightarrow}_{n,n}$ and $G^{\rightarrow}_{-n,-n}$ must be complex conjugate, since (recall (3.4)) g must be real.

The general solution to this system of equations is not hard to derive [8]. However, when the number of degrees of freedom is greater than one, some intricate notation must be introduced. Since the structures to which we want to draw attention are already fully present when the number of degrees of freedom is just one, let us for simplicity ignore the dynamical variables x and z from now on. (We have chosen to retain the particular variable y because in Section IV we shall apply the formula below to the data shown in Fig. 3, which refers specifically to the vertical phase plane.) When necessary, we shall indicate what technical modifications are required when additional degrees of freedom are present.

When y is the only dynamical variable, Eqs. (3.9) and (3.10) reduce to

$$\dot{J}_y \simeq -n_y A \sin n_y [\phi_y - \omega_r t + a] ,$$

$$\dot{\phi}_y \simeq \omega_r + (J_y - J_y^{\circ}) B/n_y^2 . \qquad (3.10)$$

The definition of the parameters A, a, and B in terms of $G_{n_y,n}$ and $G_{o,o}$ at J_y^{0} is self-evident. In what follows, we shall assume without loss of generality that A and B have the same sign; this can always be arranged by a suitable shift of the offset phase a. We shall also assume without loss of generality that n_y is positive; this is simply a matter of convention, since the (n_y, n) and $(-n_y, -n)$ Fourier terms contribute to (3.10) with equal magnitude. The resonance frequency ω_r in (3.10) is defined by

$$\omega_r \equiv -\frac{2\pi}{T}\frac{n}{n_y} = \omega_y + \frac{dG_{o,o}}{dJ_y}(J_y^\circ). \tag{3.11}$$

The second equality in (3.11) follows from the resonance condition (3.6).

To make our work simpler, we define

$$\Phi \equiv n_y [\phi_y - \omega_r t + a] ,$$

$$\mathcal{J} \equiv (J_y - J_y^\circ)/n_y . \tag{3.12}$$

These variables are canonically conjugate, since ϕ_y and J_y are. The primary variables y and v_y have period $2\pi n_y$ as functions of Φ, since they have period 2π as functions of ϕ_y.

When Eqs. (3.10) are rewritten in terms of these new variables, one obtains the canonical equations of motion associated with the time independent Hamiltonian h(sgnB), where

$$h \equiv |B| \left[\frac{1}{2}\mathcal{J}^2 - \left(\frac{A}{B}\right)\cos\Phi\right] = \frac{1}{2}|B|\mathcal{J}^2 - |A|\cos\Phi. \tag{3.13}$$

This function is formally equivalent to the Hamiltonian of a simple gravitational pendulum with moment of inertia equal to $|B|^{-1}$, and with mass times length times gravitational constant equal to $|A|$. (The factor sgn $B = |B|/B$ has been deliberately included in the relation between h and the Hamiltonian that generates Eqs. (3.10), so that h can always have a properly positive kinetic term.) The pendulum also appears in a natural way when there are $d \geq 2$ degrees of freedom. In that case, one solves Eqs. (3.8) and (3.9) by linearly mixing the components of the oscillator action $\vec{J}$, and also the components of $\vec{\phi}$, so that (d-1) of the

resulting canonically conjugate pairs have time-independent actions, and angles linearly dependent on time, as in (3.2), while the one remaining pair forms a pendulum-like system, as in (3.10).

The pendulum analogy leads directly to the simple formulae that constitute our real objective, as follows: It should be intuitively apparent that the island curves in Figure 1 correspond to an equivalent pendulum that oscillates indefinitely about one of the potential minima $\Phi = 2\pi m$, where m is an arbitrary integer; while the curves that enclose the origin correspond to an equivalent pendulum that rotates indefinitely, increasing or decreasing by 2π during every rotation period. An oscillating pendulum means

$$-|A| \leq h < +|A|, \tag{3.14}$$

because at $h = -|A|$ the pendulum is in stable equilibrium, at rest pointing down, while at $h = +|A|$ the pendulum has just enough energy to come to rest pointing straight up, in unstable equilibrium. Similarly, a rotating pendulum means

$$+|A| < h. \tag{3.15}$$

Therefore also storage ring motion in a resonance island is equivalent to inequality (3.14), while storage ring motion just outside an island chain is equivalent to (3.15).

The separatrix must correspond to $h = +|A|$. According to the definition (3.13) of h, this can be rewritten in the simple form

$$\mathcal{J} = \pm 2 \left(\frac{A}{B}\right)^{1/2} \cos(\Phi/2) . \qquad (3.16)$$

Expressions (3.14) - (3.16) are the formulae that we wanted to derive.

2. Thermal Averaging Near Nonlinear Resonance

Our goals in this section are: to show how to rewrite the averaged Fokker-Planck equation (2.11) near nonlinear resonance in a form more suitable for calculation; and then to use this form to investigate the structure of the averaged Fokker-Planck equation near a separatrix. As before, we shall work within the specific context of one-dimensional systems, but our results are easily extended to systems with additional degrees of freedom.

In order to rewrite the averaged Fokker-Planck equation, we need information about the canonical action I. In general [17], we may self-consistently[F10] set $I = \oint \mathcal{J} d\Phi$, where $\oint$ indicates a contour integration along a closed KAM curve in the direction of Hamiltonian flow. In light of (3.10) - (3.13), this leads to

$$I = (2\pi)^{-1} \oint \mathcal{J} d\Phi = (2\pi)^{-1} (\mathrm{sgn}B) \left|\frac{2}{B}\right|^{1/2} \oint |d\Phi| \, |h + |A|\cos\Phi|^{1/2}$$

$$\equiv I(h). \qquad (3.17)$$

Unfortunately, I(h), unlike h itself, is not an elementary function of $\mathcal{J}$ and Φ. For this reason, the manipulations that follow will transform (2.11) from an equation for P(I,t) to an equation for P(h,t) ≡

$P(I(h),t)$. The substitution of h for I is a good thing, at least conceptually, because the action I is necessarily discontinuous in some way (see below) at the borders (separatrices) between distinct KAM orbit families, while h, according to (3.13), is manifestly smooth everywhere in phase space near the curve $J_y = J_y^\circ$.

To proceed with our revision of the averaged Fokker-Planck equation: We begin by substituting Eqs. (3.1) (with x replaced by y) and (3.12) and (3.13) into the first integral on the right-hand-side of (2.11) (suitably modified for a system with one degree of freedom) in order to obtain

$$< \oint \frac{d\theta}{2\pi} v_y \frac{\partial I}{\partial v_y} > = \frac{dI}{dh} \oint \frac{d\theta}{2\pi} < v_y \frac{\partial h}{\partial v_y} >$$

$$= \frac{dI}{dh} \oint \frac{d\theta}{2\pi} v_y \left[B\mathcal{J}\frac{\partial \mathcal{J}}{\partial v_y} + A(\sin\Phi) \frac{\partial \Phi}{\partial v_y}\right] \mathrm{sgn}B >$$

$$= \frac{dI}{dh} \oint \frac{d\theta}{2\pi} < v_y \left[B\mathcal{J}\frac{v_y}{n_y\omega_y} + A(\sin\Phi) \frac{n_y y}{2J_y}\right] \mathrm{sgn}B >$$

$$= \frac{dI}{dh} \oint \frac{d\theta}{2\pi} \lim_{\tau\to\infty} \int_o^\tau \frac{dt}{\tau} \left\{ \frac{2B\mathcal{J}J_y}{n_y} \sin^2 [\omega_r t - a + \Phi/n_y] \right.$$

$$\left. - \frac{n_y A}{2} \sin 2 [\omega_r t - a + \Phi/n_y] \right\} \mathrm{sgn}B$$

$$= \frac{dI}{dh} \oint \frac{d\theta}{2\pi} \frac{|B|}{n_y} \mathcal{J}(n_y\mathcal{J} + J_y^\circ). \qquad (3.18)$$

This can be simplified further by exploiting

$$\frac{dI}{dh}\,|B|\,\mathcal{J} = \frac{dI}{dh}\frac{\partial h}{\partial \mathcal{J}} = \frac{\partial I}{\partial \mathcal{J}} = \frac{\partial \Phi}{\partial \theta}\ . \tag{3.19}$$

(The last equality in (3.19) is just the first of identities (2.13) in a different notation.). Because of (3.18), we have finally

$$< \oint \frac{d\theta}{2\pi}\, v_y \frac{\partial I}{\partial v_y} > = \frac{1}{2\pi n_y} \oint (n_y \mathcal{J} + J_y^{\circ})\, d\Phi. \tag{3.20}$$

The notation $\oint$ on the right-hand-side indicates integration over a single closed curve of constant h in the Φ-$\mathcal{J}$ plane (wrapped into a cylinder by the identification of $\Phi = 0$ with $\Phi = 2\pi n_y$). The direction of the integration contour in (3.20) must be the direction of the Hamiltonian flow.

In a similar fashion, one can also derive

$$\begin{aligned} < \oint \frac{d\theta}{2\pi} \left(\frac{\partial I}{\partial v_y}\right)^2 > &= \left(\frac{dI}{dh}\right)^2 \oint \frac{d\theta}{2\pi} \frac{1}{\omega_y} \left[\frac{B^2}{{n_y}^2} \mathcal{J}^2 J_y + {n_y}^2 A^2 \frac{\sin^2\Phi}{4J_y}\right] \\ &= \frac{dI}{dh} \oint \frac{d\theta}{2\pi} \frac{1}{\omega_y} \left[\frac{B}{{n_y}^2} \mathcal{J} J_y \frac{\partial I}{\partial \mathcal{J}} + {n_y}^2 A \frac{\sin\Phi}{4J_y} \frac{\partial I}{\partial \Phi}\right] \mathrm{sgn}B \\ &= \frac{dI}{dh} \frac{1}{2\pi\omega_y} \oint \left[\frac{B}{{n_y}^2} \mathcal{J}(n_y\mathcal{J} + J_y^{\circ})\, d\Phi - {n_y}^2 A \frac{\sin\Phi}{4(n_y\mathcal{J} + J_y^{\circ})}\, d\mathcal{J}\right] \mathrm{sgn}B. \end{aligned} \tag{3.21}$$

When Eqs. (3.20) and (3.21) are substituted into (2.11) (modified for one degree of freedom), we obtain, after two more applications of the chain rule,

$$\frac{\partial}{\partial t}\left[2\pi \frac{dI}{dh} P\right] = \gamma \frac{\partial}{\partial h}\left\{\oint \frac{1}{n_y}(n_y \mathcal{J} + J_y^{\circ})\, d\Phi + \oint\left[\frac{B}{n_y^2}\mathcal{J}(n_y\mathcal{J} + J_y^{\circ})d\Phi - n_y^2 A \frac{\sin\Phi d\mathcal{J}}{4(n_y\mathcal{J} + J_y^{\circ})}\right](\mathrm{sgn}B)\,\omega_y \frac{\partial}{\partial h}\right\} P. \tag{3.22}$$

This is the desired revision of the Fokker-Planck equation. Note for future reference that the product $2\pi(dI/dh)P$ on the left-hand-side is the density of probability per unit length along the h axis within a single KAM orbit family, since $2\pi P$ is the density per unit length of I.

We want now to evaluate the coefficients in Eq. (3.22) at the resonance separatrix. This will reveal the discontinuities mentioned in the Introduction as the second difficulty connected with the idea of thermal averaging. These discontinuities come about because the curves to which the action I and the circuit integrals in (3.22) refer can approach the separatrix in three topologically distinct ways: (a) through origin-enclosing orbit curves that lie between the origin and the island chain; (b) through origin-enclosing curves that lie beyond the islands; and (c) through orbits that nest within a single resonant island. The parts of the separatrix that correspond to each of these distinct limits, in the particular case of Fig. 2, are shown in Figs. 4a-c. We shall soon see that each of these limits produces different limiting values for the circuit integrals on the right-hand side of (3.22), and leads to a divergence in dI/dh.

In view of Eq. (3.16), limit (a) must correspond to

$$J_y - J_y^\circ = n_y \mathcal{J} = -2n_y \left(\frac{A}{B}\right)^{1/2} |\cos(\Phi/2)|, \tag{3.23a}$$

and limit (b) must correspond to

$$n_y \mathcal{J} = +2n_y \left(\frac{A}{B}\right)^{1/2} |\cos(\Phi/2)|. \tag{3.23b}$$

The limit curve of type (c) is formed by joining the two sets (3.23a) and (3.23b), and then restricting Φ to lie between $(2m-1)\pi$ and $(2m+1)\pi$, for some integer m. It is clear that, in any calculation related to Eq. (3.22), circuit integrals around a type (c) part of the separatrix should be equal to n_y^{-1} times the sum of the circuit integrals around the type (a) and type (b) parts.

In these three limits, the first integral on the right-hand side of (3.2) takes the three distinct values

$$n_y^{-1} \oint (n_y \mathcal{J} + J_y^\circ)\, d\Phi$$

$$= (-\mathrm{sgn}B) n_y^{-1} \int_0^{2\pi n_y} \{J_y^\circ - 2|n_y| \left(\frac{A}{B}\right)^{1/2} |\cos(\Phi/2)|\} d\Phi$$

$$= (-\mathrm{sgn}B) n_y^{-1} \{2\pi n_y J_y^\circ - 2n_y \left(\frac{A}{B}\right)^{1/2} n_y \int_0^{2\pi} \sin\frac{\chi}{2}\, d\chi\}$$

$$= (-\mathrm{sgn}B) \left[2\pi J_y^\circ - 8n_y \left(\frac{A}{B}\right)^{1/2}\right] \tag{3.24a}$$

(limit (a)),

$$n_y^{-1}\oint(n_y\mathcal{J}+J_y^{\circ})d\Phi = \begin{cases} (+\mathrm{sgn}B)\left[2\pi J_y^{\circ} + 8n_y\left(\frac{A}{B}\right)^{1/2}\right] \\ \qquad (\text{limit (b)}) \qquad (3.24b) \\ \\ (\mathrm{sgn}B)\cdot\frac{8}{3}\cdot n_y\left(\frac{A}{B}\right)^{1/2} \\ \qquad (\text{limit (c)}). \qquad (3.24c) \end{cases}$$

The factor (-sgnB) enters the calculation in (3.24c) as the sign of the directed differential $d\Phi$ in the circuit integral on the left-hand side. This sign takes this particular value for the following reason: The sign of this $d\Phi$ is by definition the same as the sign of the time derivative $\dot{\Phi}$ in Hamiltonian flow. However, $\dot{\Phi} = \mathcal{J}B$ in Hamiltonian flow, and the sign of $\mathcal{J}$ is opposite that of n_y in limit (c), according to (3.23c). Therefore, sgn ($d\Phi$ in circuit integral) = sgn ($\mathcal{J}B$) = -sgn (Bn_y) = -sgnB. In the case of (3.24b), the corresponding factor is +sgnB.

In a similar way, we also have

$$\frac{B\mathrm{sgn}B}{n_y^2}\oint\mathcal{J}(n_y\mathcal{J}+J_y^{\circ})d\Phi = \begin{cases} \frac{4B}{n_y}\left(\frac{A}{B}\right)^{1/2}\{2J_y^{\circ}-\pi n_y\left(\frac{A}{B}\right)^{1/2}\} \\ \qquad (\text{limit (a)}), \qquad (3.25a) \\ \\ \frac{-4B}{n_y}\left(\frac{A}{B}\right)^{1/2}\{-2J_y^{\circ}-\pi n_y\left(\frac{A}{B}\right)^{1/2}\} \\ \qquad (\text{limit (b)}), \qquad (3.25b) \\ \\ \frac{8}{3}\frac{B}{n_y}\left(\frac{A}{B}\right)^{1/2}J_y^{\circ} \\ \qquad (\text{limit (c)}), \qquad (3.25c) \end{cases}$$

and

$$-\frac{n_y^2 A\,\mathrm{sgn}B}{4}\oint \frac{\sin\Phi}{(n_y\mathcal{J}+J_y^{\circ})}d\mathcal{J} = \begin{cases} -\frac{An_y^2}{2}\left[2R+\pi(R^2-\frac{1}{2})-2R(R^2-1)^{1/2}\cos^{-1}\left(\frac{-1}{R}\right)\right] \\ \qquad \text{(limit (a))}, \qquad (3.26a) \\ -\frac{An_y^2}{2}\left[2R-\pi(R^2-\frac{1}{2})+2R(R^2-1)^{1/2}\cos^{-1}\frac{1}{R}\right] \\ \qquad \text{(limit (b))}, \qquad (3.26b) \\ -\frac{An_y^2 R}{3}\left[1+(R^2-1)^{1/2}\left(\cos^{-1}\frac{1}{R}-\cos^{-1}\left(\frac{-1}{R}\right)\right)\right] \\ =-\frac{An_y^2 R}{3}\left[1+(R^2-1)^{1/2}(\pi-\cos^{-1}\frac{1}{R})\right] \\ \qquad \text{(limit (c))}. \qquad (3.26c) \end{cases}$$

The parameter R is defined by

$$R \equiv \frac{J_y^{\circ}}{2n_y}\cdot\left(\frac{B}{A}\right)^{1/2}. \qquad (3.27)$$

In deriving Eqs. (3.25), we have used formula 3.648.2 from the integral compendium of Gradshteyn and Ryzhik [18].

To complete these separatrix calculations, we have

$$2\pi\frac{dI}{dh} = \frac{(\mathrm{sgn}B)}{|2B|^{1/2}}\oint |d\Phi|\,|h+|A|\cos\Phi|^{-1/2}$$

$$= (\mathrm{sgn}B)\frac{n_y}{|2B|^{1/2}}\int_0^{2\pi} d\Phi\,[h+|A|\cos\Phi]^{-1/2}$$

$$\simeq -(\mathrm{sgn}B)\,\frac{n_y}{(AB)^{1/2}}\,\log\left(\frac{h-|A|}{|A|}\right)$$

$$\text{as } h \to |A|^{+} \text{ (limits (a) and (b))}, \tag{3.28}$$

$$2\pi\,\frac{dI}{dh} = \frac{(\mathrm{sgn}B)}{|2B|^{1/2}} \cdot 4 \cdot \int_0^{\cos^{-1}(-h/|A|)} d\chi\,[h + |A|\cos\chi]^{1/2}$$

$$\simeq -(\mathrm{sgn}B) \cdot \frac{2}{(AB)^{1/2}} \cdot \log\left(\frac{|A|-h}{|A|}\right)$$

$$\text{as } h \to |A|^{-} \text{ (limit (c))}. \tag{3.29}$$

What is the physical origin of the singular behavior evident in (3.24) - (3.29)? Two possibilities come immediately to mind:

First, it is possible that thermal averaging could be unsuitable near a resonance separatrix, because the averaging formalism neglects chaotic orbits, which tend to proliferate near separatrices [8].

Second, it is also possible that thermal averaging could be unsuitable near a resonance separatrix because orbits increasingly close to a separatrix spend indefinitely increasing amounts of time near the finitely many unstable equilibria at $\Phi = (2m+1)\pi$ (integer m). Thus, strictly speaking, near a separatrix, one must wait an infinite amount of time before nearby points on nearby orbits spread enough for the fundamental ansatz (2.3) of dynamical averaging to make sense.

Lacking a deeper understanding of these possibilities, let us now proceed to a discussion of solution methods for Eq. (3.22), assuming, as a working hypothesis, that this equation is suitable for application despite the topological singularities in its coefficient functions at $h = |A|$.

IV. SOLVING THE AVERAGED FOKKER-PLANCK EQUATION FOR ONE-DIMENSIONAL SYSTEMS

Equation (3.22) and its higher-dimensional analogues represent a considerable simplification of the original Fokker-Planck equation for $P(y,v_y,\ldots,t)$. Unfortunately, when the number of degrees of freedom is greater than one, even the dynamically averaged Fokker-Planck equation seems too complicated at this writing to be practically useful. (I have looked for variables in which the general averaged Fokker-Planck equation might be separable near nonlinear resonance, but have had no success.)

The situation is different when our system has only one degree of freedom. In this case, there are several things that we can conceive of doing, either to extend the range of problems to which Eq. (3.22) may in principle be applied, or to extract physical information from Eq. (3.22) in a quick and direct way.

We describe three such techniques in the first three parts of this section. It is hoped that they can suggest fruitful approaches to higher-dimensional problems. In the fourth part of this section, we apply one of these techniques to the data shown in Fig. 3.

1. More Than One Resonance

We have so far formulated two convenient approximations to Eq. (2.11) for two distinct but coexisting types of regions in the phase plane: Near a resonant island chain, we have Eq. (3.22); while between the chains, where the nontrivial Fourier terms in (3.3) can be ignored, we have the result of identifying I with J_y and θ with ϕ_y, i.e.,

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial J_y} J_y \left(1 + \omega_y^2 \frac{\partial}{\partial J_y}\right) P. \tag{4.1}$$

We want to show here that in fact (3.22) goes over smoothly to (4.1) when h becomes large, i.e., as one moves away from any particular resonant chain. We do this because it enables us to suggest that one might be able to construct simple approximations to evolving probability distributions throughout the whole of phase space, in the presence of two or more strong nonlinear resonances--corresponding to two or more distinct functions h--in the following way: Near each island chain, one solves for the evolution of P by using Eq. (3.22), into which one has inserted the parameters J_y°, n_y, A, B appropriate for that chain; well between the resonant chains one solves for P using (4.1); finally, one matches the large-h extrapolations of the near-resonance solutions to the inter-resonance solutions, using a continuity condition. Note that if (4.1) were not the large-h limit of (3.22), such a continuity condition could probably not be imposed self-consistently.

In brief, Eq. (3.22) has Eq. (4.1) as a limit because when h is large, curves of constant h are curves of approximately constant $J_y \simeq J_y^\circ \pm n_y (2h/|B|)^{1/2}$. This means that I, as defined by (3.17), satisfies

$$I = \frac{1}{2\pi} \oint \mathcal{J} \, d\Phi \simeq (\mathrm{sgn}B) \, |J_y - J_y^\circ|, \tag{4.2}$$

and that (recall $\mathrm{sgn}(d\Phi) = \mathrm{sgn}\, B\mathcal{J} = \mathrm{sgn}\,[Bn_y(J_y - J_y^\circ)]$)

$$n_y^{-1} \oint (n_y \mathcal{J} + J_y^\circ)\, d\Phi \simeq 2\pi\, J_y\, \mathrm{sgn}\,[B(J_y - J_y^\circ)]\ ,$$

$$n_y^{-2}\,|B| \oint \mathcal{J}(n_y \mathcal{J} + J_y^\circ)\, d\Phi \simeq n_y^{-2} B \cdot 2\pi J_y |J_y - J_y^\circ| \simeq 2\pi J_y \frac{dh}{dI}\ ,$$

$$(\tfrac{1}{4})\, n_y^{\,2}\, A(\mathrm{sgn}B) \oint \left(\frac{\sin\Phi}{n_y \mathcal{J} + J_y}\right) d\mathcal{J} \simeq 0. \qquad (4.3)$$

These in turn mean that for large h, (3.22) becomes

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial I} J_y \{\mathrm{sgn}\,[B(J_y - J_y^\circ)] + \omega_y^2 \frac{\partial}{\partial I}\} P$$

$$= \gamma \frac{\partial}{\partial J_y} J_y \,(1 + \omega_y^2 \frac{\partial}{\partial J_y})\, P, \qquad (4.4)$$

which is what we wanted to show.

A similar limit can easily be formulated and proved for systems with more than one degree of freedom. However, the matching procedure described above and made possible by this limit can in fact always be formulated self-consistently only when the dynamical system has just one degree of freedom. The reason for this is, as we now explain, to be found in condition (3.6), which determines the values of $\vec{J}$ near which a given Fourier term in (3.3) can be strongly resonant:

When a system has d degrees of freedom, the vector $\vec{J}$ has d components and therefore the set of solutions to (3.6) (suitably generalized from d = 3) for fixed $\vec{n}$ and n is a surface of dimension d - 1. It follows that when d is greater than one, two solution sets for two distinct (d + 1)-triples $(\vec{n},n)$ can (and very often do--see, for example, Ref. [12]) intersect in $\vec{J}$-space; and therefore phase space regions in which different Fourier terms are strongly resonant can (and

very often do) interpenetrate. This interpenetration is what subverts our matching procedure, which requires that there be extensive nonresonant free space between nonlinearly resonant regions.

Even when $d = 1$, of course, the matching procedure can be inapplicable when two concentric resonant island chains are so close in radius that there is no room for an asymptotic buffer between them, or so close that chaotic orbits can proliferate [8], precluding thermal averaging altogether.

2. Steady State

Within the framework of dynamical averaging, "steady state" can only sensibly mean $\partial P(\vec{I},t)/\partial t \equiv 0$. We are going to solve this equation exactly, and analyze the solution, for systems with one degree of freedom. Note that this steady state is not the same as $\partial P(x,v_x,\ldots,t)/\partial t \equiv 0$, because of the explicit time dependence in the definition, (2.1), that relates x, v_x, etc. to $\vec{I}$ and $\vec{\theta}$. Nor is it obviously the same as the statement that $P(x,v_x,\ldots,t)$ be periodic in time with repeat period equal to that (T) of Eqs. (1.1), although this is an aesthetically appealing possibility. We shall in fact argue below that in systems with just one degree of freedom, the steady state of thermal averaging is indeed periodic in this way. When the number of degrees of freedom exceeds one, however, we do not even know how to argue that a steady state exists at all.

For convenience, let us rewrite the thermally averaged Fokker-Planck Eq. (2.11) (for arbitrary dimension $d \neq 3$) schematically as

$$\frac{\partial P}{\partial t} = (2\pi)^{-d} \frac{\partial}{\partial I_i} K_i \equiv (2\pi)^{-d} \gamma \frac{\partial}{\partial I_i} \left[a_i(\vec{I}) + b_{ij}(\vec{I}) \frac{\partial}{\partial I_j} \right] P. \tag{4.5}$$

For d = 1, near a resonance, let us also rewrite (3.22) as

$$\frac{\partial}{\partial t} \left(\frac{dI}{dh} P \right) = (2\pi)^{-1} \frac{\partial}{\partial h} \tilde{K} \equiv \gamma (2\pi)^{-1} \frac{\partial}{\partial h} [\tilde{a}(h) + \tilde{b}(h) \frac{\partial}{\partial h}] P. \tag{4.6}$$

It is clear that $K(I,t) = \tilde{K}(h,t)$ in one dimension.

When the number of degrees of freedom is equal to one, the subscripts i and j can be omitted, and the condition $\partial P/\partial t = 0$ immediately integrated once:

$$\text{constant} = K = \gamma \left[a(I) + b(I) \frac{d}{dI} \right] P$$

$$\left(= \gamma \left[\tilde{a}(h) + \tilde{b}(h) \frac{d}{dh} \right] P \text{ near resonance} \right). \tag{4.7}$$

We shall determine the constant in (4.7) shortly. In order to do so, however, we shall need to be able to identify K with the current of probability along the I-line:

In general, for any d, it is tempting to interpret $\vec{K}$ as the current of probability in $\vec{I}$-space, since P is always $(2\pi)^{-d}$ times the density of probability in $\vec{I}$-space, so that (4.5) is manifestly in the form of a continuity equation. To make this rigorous for d>1, we would have to formulate a somewhat elaborate argument in order to rule out terms like

$$\left(\frac{\partial}{\partial I_j}\right) F_{ij}(\vec{I},P), \tag{4.8}$$

where F_{ij} is some antisymmetric matrix-valued function that could appear in the current while making no contribution whatsoever to its divergence. Fortunately, we shall only need to know the current for $d = 1$. In this case, we have only to rule out an additive constant in order to identify the current with K. But such a constant must be zero, because otherwise the current would not vanish in any region devoid of probability (i.e., with $P \equiv 0$), and that would be physically unreasonable.

We now show that in a closed system this time-independent current K must in fact equal zero, by the following topological argument: When the number d of degrees of freedom is equal to one, KAM tori--as closed curves--enclose fully planar subsets of the phase plane. (KAM tori cannot enclose anything when $d>1$. In general, only a $(2d-1)$ dimensional set can enclose part of a 2d dimensional phase space, while KAM tori only have dimension d.) Therefore, the current K must be the rate at which probability flows into or out of some (time dependent) closed finite-measure region of the phase plane. If $K \neq 0$, the total integrated probability in this region must become either negative, which is absurd, or infinite, which is equally absurd, since the total probability in all of phase must always remain equal to one.

It follows immediately, by integrating (4.7), that in the steady state, a closed one-dimensional system has, in a KAM family, the probability distribution

$$P = N \exp\left[-\int_{I_o}^{I} \frac{a(I')}{b(I')}\, dI'\right]$$

$$\left(= N \exp\left[-\int_{h_o}^{h} \frac{\bar{a}(h')}{\bar{b}(h')}\, dh'\right] \text{ near resonance}\right), \qquad (4.9)$$

where I_o and h_o are some arbitrary reference values, and N is a normalization scale.

As for d>1: It is tempting to guess by analogy that $\vec{K}$ vanishes in the steady state even when d>1. This is in principle not likely, for at least two reasons. First, there appears to be no geometrical structure that forces $\vec{K} = 0$ when d>1. Second, as far as I have been able to determine, the d first-order differential equations $K_i = 0$ (i=1,...,d) are in general not simultaneously integrable, unless all d degrees of freedom are uncoupled.

We have two observations to make concerning the one-dimensional solution (4.9).

First, the various constants N for the various coexisting action-angle families are determined by overall probability normalization, together with boundary conditions that relate the distributions in adjoining families. The most natural-seeming boundary condition (although we have no proof that it is correct) is continuity of P as a function of y and v_y. Near a resonance, this means that

$$\lim_{h \to |A|^+} P = \lim_{h \to |A|^-} P \,, \tag{4.10}$$

where the left-hand side can refer to the limit of orbit curves that enclose the origin either within or beyond the island chain, and the right-hand side can refer to the limit of the curves in any one of the n_y islands in the chain. An immediate consequence of (4.10) is that--at least within approximation (3.22)--the steady state probability distributions in all resonance islands in a single chain are identical (up to rotation) and therefore the entire probability distribution in the y-v_y plane repeats itself in time T, because so does the pattern of orbit curves. Another immediate consequence of (4.10) is that the probability P has a discontinuous derivative at the separatrix, since $\tilde{a}$ and $\tilde{b}$ are discontinuous at the separatrix, as we showed in Section III.

Second, within approximation (3.22), the solution (4.9) peaks at the nonlinear resonance centers ($\mathcal{J} = 0$, $\Phi = 2\pi m$). To see this, we need to know how $\tilde{a}$ and $\tilde{b}$ behave near a resonance center (i.e., $h = -|A|$). A straightforward calculation near $h = -|A|$ yields

$$\tilde{a}(h) \simeq \frac{(h+|A|)}{(AB)^{1/2}} \,(\text{sgn } B)$$

$$\tilde{b}(h) \simeq \frac{(h+|A|)}{(AB)^{1/2}} \left(\frac{BJ_y^{\circ}}{n_y^{\,2}} + \frac{n_y^{\,2} A}{4J_y^{\circ}} \right) \omega_y \,. \tag{4.11}$$

Therefore

$$\lim_{h\to-|A|} \frac{\tilde{a}(h)}{\delta(h)} = \omega_y^{-1} \left[\frac{|B|J_y^\circ}{n_y^2} + \frac{n_y^2|A|}{4J_y^\circ} \right]. \tag{4.12}$$

It follows that near $h = -|A|$, (4.9) is given approximately by

$$P \simeq (\text{constant}) \times \exp\left\{ -(h+|A|)\omega_y^{-1} \left[\frac{|B|J_y^\circ}{n_y^2} + \frac{n_y^2|A|}{4J_y^\circ} \right]^{-1} \right\}$$

$$\simeq (\text{constant})' \times \exp\left\{ -\frac{1}{2}\left[|B|\mathcal{J}^2 + |A|\,(\Phi - 2\pi m)^2 \right] \cdot \omega_y^{-1} \left[\frac{|B|J_y^\circ}{n_y^2} + \frac{n_y^2|A|}{4J_y^\circ} \right]^{-1} \right\}. \tag{4.13}$$

The last expression in (4.13) shows the promised peak explicitly.

One might have expected this peaking a priori, since the points $\mathcal{J} = 0$, $\Phi = 2\pi m$, are centers of stability--orbit curves in the islands oscillate directly about them and not about the origin--and we are conditioned by experience with the Botzmann factor exp-H/KT (which, however, does not apply to our explicitly time-dependent system) to look for peaks in probability distributions at stable points.

Is the beam enlargement seen in real storage rings due at least in part to such steady state resonance peaks away from the phase space origin? It is likely that other effects, most probably peculiar to nonlinear resonance in higher-dimensional systems (see, for example, the second paper cited in Ref. [12]; for a critique of this example, see Ref. [3], Section V.3) are much more important. It might seem desirable at this point to discuss numerical estimates of the steady-state populations of nonlinear resonance peaks in one-dimensional models of colliding beams. We shall forego this, however, because the available

computer data [13] with which we might compare such estimates is not extensive enough for this purpose. In the fourth part of this section, we present a numerical estimate of a kind that is much better suited to the data at hand.

3. Transport Flux

In this subsection, we explain how Eq. (4.5) permits us to obtain, under certain conditions, a simple solution to the following problem: Consider a sample of one-dimensional particles, all of whose initial conditions lie close to the origin in the phase plane. At some later time, what is the net outward flux of particles impinging on the border (i.e., part (a) of the separatrix) of the action-angle family that contains the origin?

Our simple formula for this rate is important for several reasons:

First, there are reasons to believe that our derivation of this formula might not be seriously subverted by the a priori concerns about separatrices that we articulated at the end of Section III. In part, this is because the coefficient dI/dh in Eq. (3.22) makes no explicit appearance in our derivation; thus, we need not worry about its divergence at the separatrix. In part, this is also because we have restricted the statement of the problem to the border of the origin's orbit family. If the points at which flux were to be calculated lay beyond this border, then particles that contributed to the flux would necessarily have to sample a discontinuity in the coefficients on the right-hand side of (3.22), because they would necessarily have to penetrate a separatrix at some time between leaving the origin and being

counted in the flux. The freedom to impose this kind of restriction distinguishes the problem of flux from the problem of the steady state: As we explained in the preceding subsection, the normalization of the steady state cannot be determined anywhere in phase space without information concerning separatrix boundary conditions, about which we are fundamentally uncertain at this time.

Finally, our formula for flux is much more suitable for application to the data shown in Fig. 3 than is our theory of the steady state. In order to apply the steady state analysis explained in the preceding subsection, we would have to compound the separatrix difficulties mentioned above with the uncertainty of an ergodic assumption with whose help we would try to infer the phase space distribution of an ensemble of particles from properties of the Brownian path of a single test particle. Such an ergodic assumption might not be necessary if we had comparably detailed data on the paths of many more test particles moving under the same external conditions in the computer simulations of Ref. [16]. Unfortunately, no such supplementary data appears to be available at this writing.

For our derivation, we now paraphrase an argument originally given by Kramers [9] in connection with the similar problem of noise-driven barrier crossing. I have chosen to present this derivation in full, rather than simply to quote the result with the proper citation, so that assumptions involved can be as clear as possible to readers who may be new to this kind of analysis.

We shall calculate the desired flux for times long enough that something resembling a slow but steady flow has been established between the origin and part (a) of the separatrix in question; but short enough

that particles that have crossed this border have not yet been able to cross back [F11] (so that we do not have to disentangle the competition between outflow and infall). We expect, intuitively, that the time needed to establish such a nearly steady leakage is also long enough that the distribution near the origin can relax, approximately, to the equilibrium Gaussian of far-from-resonance dynamics, i.e., $(2\pi\omega_y)^{-1}$ $\exp(-J_y/\omega_y)$.

In order to describe the flux at such times, it seems reasonable, following Kramers, to use a time-independent solution to Eq. (4.5), with the orbit curve in question treated as a perfectly absorbing barrier. If I_s ("s" stands for "separatrix") is the canonical action of the family border at which the flux is required, then "perfect absorber" is equivalent to the boundary condition $P(I_s) = 0$.

A time-independent P brings us back to equation (4.7). In accordance with the discussion in the preceding subsection, the absolute value of the constant K is the flux that we seek to determine. Whatever the value of K, the solution of (4.7) with $P(I_s) = 0$ is

$$P(I) = (K/\gamma) \int_{I_s}^{I} \frac{dI'}{b(I')} \exp\left[\int_{I}^{I'} \frac{a(I'')}{b(I'')}\, dI''\right] . \qquad (4.14)$$

One might think at this point that K is to be determined directly from the behavior of (4.14) near the phase-plane origin. In particular, let I_c ("c" stands for "center") be the canonical action corresponding to the origin; then one might expect that when $I = I_c$, the right-hand side of (4.14) should equal $(2\pi\omega_y)^{-1}$, the value of the resonance-free Gaussian at $J_y = 0$.

Unfortunately, the right-hand side of (4.14) is in fact singular at $I = I_c$. Indeed, near $I = I_c$, we may make the identifications $I - I_c \simeq J_y$, and $\theta \simeq \phi + wt$ for some frequency w, in which case

$$a(I) \simeq \left\langle \oint \frac{d\phi}{2\pi} v_y \frac{\partial J_y}{\partial v_y} \right\rangle = J_y \simeq I - I_c ,$$

$$b(I) \simeq \left\langle \oint \frac{d\phi}{2\pi} \left(\frac{\partial J}{\partial v_y}\right)^2 \right\rangle = J_y \omega_y \simeq (I - I_c)\, \omega_y . \qquad (4.15)$$

Thus, the right-hand side of (4.14) diverges logarithmically as $I \to I_c$.

This singularity certainly means that Eq. (4.14) cannot be used to describe $P(I,t)$ all the way down to $I = I_c$. Let us guess, however, following Kramers, that the region in which (4.14) is unsuitable for our purposes extends only from I_c to some I_c' close to I_c. We are thus choosing to view the singularity in (4.14) as only the result of an analytically appealing but physically artificial way of continuing an approximation beyond its proper domain of applicability.

(Why should this continuation have been singular in the first place? We can trace this problem to a topological defect in Eq. (4.7): If (4.7) really were valid down to $I = I_c$, then the region bounded by the orbit curve $I = I_c$ would be gaining or losing probability $|K| \neq 0$ per unit time. However, this orbit curve is just a single point, and a nonsingular distribution can only assign weight zero to the interior of a point. Thus $K \neq 0$ at $I = I_c$ implies a singularity (a point source or a point sink) in the probability distribution at $I = I_c$.)

Let us now optimistically suppose, again following Kramers, that I_c' is close enough to I_c that $P(I_c') \simeq (2\pi\omega_y)^{-1}$ is still a reasonable approximation. We then have, from (4.14),

$$K \simeq \left\{ 2\pi\omega_y \gamma^{-1} \int_{I_s}^{I_c'} \frac{dI'}{b(I')} \exp\left[\int_{I_c'}^{I'} \frac{a(I'')}{b(I'')} dI'' \right] \right\}^{-1} . \qquad (4.16)$$

To simplify (4.16), note first that the exponential on the right-hand side is inversely proportional, as a function of I', to the steady-state distribution (4.9). On this account we expect this exponential to be much larger on the outlying orbit $I' = I_s$ than at $I' = I_c'$, near the origin. For this reason, we propose simplifying K as follows: We replace the integrand in the dI' integration in (4.16) by its leading behavior for I' near I_s; and then we formally replace the dI' integration from $I' = I_c'$ to $I' = I_s$ by an integration from $-\infty$ to I_s, which we can perform analytically. In carrying out this procedure, we must optimistically assume that I_c', although close to I_c, is nevertheless far enough away that such a "separatrix-dominance" approximation is not undercut by the singularity of $1/b(I')$ near the origin.

To make the result of this procedure explicit, we change the integration variable near the separatrix from I' to h', since otherwise we must contend with a singularity in b, because

$$b(\text{separatrix}) = \delta(h=|A|) \cdot \left.\frac{dI}{dh}\right|_{h=|A|} = [\text{finite}] \cdot \infty = \infty. \qquad (4.17)$$

With this change of variable, the separatrix-dominance approximation to (4.16) is

$$K \simeq \left\{ \frac{2\pi\omega_y}{\gamma} \int_{h=|A|}^{h=+\infty} \frac{dh'}{\tilde{b}(|A|)} \exp\left[\int_{I_c'}^{I_s} \frac{a(I'')}{b(I'')}\, dI'' + (h'-|A|)\, \frac{\tilde{a}(|A|)}{\tilde{b}(|A|)} \right] \right\}^{-1}$$

$$= \frac{-\gamma\tilde{a}(|A|)}{2\pi\omega_y} \exp\left[-\int_{I_c'}^{I_s} \frac{a(I'')}{b(I'')}\, dI'' \right]. \qquad (4.18)$$

For a final simplification, note that the only integration that remains at the conclusion of (4.18) is no longer singular at the origin. Thus, we can now reinstate I_c in place of the imprecisely defined I_c', without significant distortion. The result is

$$K \simeq \frac{-\gamma\tilde{a}(|A|)}{2\pi\omega_y} \exp\left[-\int_{I_c}^{I_s} \frac{a(I'')}{b(I'')}\, dI'' \right]. \qquad (4.19)$$

The absolute value of (4.19) is the desired formula.

4. Phenomenological Calculation

In this subsection, we apply Eq. (4.19) to a one-dimensional idealization of the system corresponding to Fig. 3. For this purpose, we assume in what follows that the six conspicuous islands in Fig. 3 make up the only significant island chain in this system. We want to calculate the rate at which particles are expected to leak from the neighborhood of the origin to the separatrix of this supposedly unique chain.

We base our flux calculation on three simple properties of such an idealization, two model-independent and one model-dependent. The model-independent properties are these: In any one-dimensional model of

the system depicted in Fig. 3, equations of the form (3.10) should provide a reasonable approximation to the Hamiltonian dynamics near the nonlinear resonance islands; while far from the islands one should be able to set $\dot{J}_y \simeq 0$, $\dot{\phi}_y \simeq \omega^1(J_y)$ (i.e., resonance-free oscillation). The form of the transition between these two regions is the model-dependent property.

The reader, recalling part 1 of this section, might have expected that the near-resonance approximation could also be used to describe the transition region--the transition would then take place at values of J_y and ϕ_y for which curves of constant h were also curves of approximately constant J_y. Unfortunately, the structure evident in Fig. 3 is not favorable for this kind of ansatz. In particular, the origin, which in Fig. 3 appears to be well outside the resonant domain, happens also to sit on curves of constant h that would connect it with points that have unacceptable large values of J_y.

To make this more precise, one begins by estimating the parameters J_y° and (A/B) that characterize the effective pendulum energy h near the island chain of Fig. 3. We can extract estimates from this observation concerning the apparent separatrix in Fig. 3: The separatrix points that come closest to the origin appear to have $J_y \simeq [(7)^2/2]\omega_y$, while the farthest points on the separatrix appear to have $J_y \simeq [(18)^2/2]\omega_y$. According to Eqs. (3.12) and (3.16), these two numbers must be the same as

$$J_y = J_y^\circ \pm 2n_y \left(\frac{A}{B}\right)^{1/2} . \tag{4.20}$$

The index n_y is clearly equal to six, since there are six islands. Thus we learn from (4.20) that

$$J_y^\circ \simeq \frac{\omega_y}{2}\left[\frac{(18)^2}{2} + \frac{(7)^2}{2}\right] \simeq (93)\omega_y ,$$

$$\left(\frac{A}{B}\right)^{1/2} \simeq \frac{\omega_y}{24}\left[\frac{(18)^2}{2} - \frac{(7)^2}{2}\right] \simeq (5.7)\omega_y . \tag{4.21}$$

(For the sake of uniformity, we have made an arbitrary decision to report the numerical coefficients in (4.21), and in all subsequent computations, with two significant digits. Most of the computations that follow, however, are based on approximations so rough that the true error in our final result is undoubtedly much worse than one percent.)

From (4.21) we learn that curves of constant h can connect the origin, at $J_y = 0$, with points whose oscillator action J_y can be as large as

$$J_y^\circ - [(J_y^\circ)^2 - 4n_y^2 (\tfrac{A}{B})]^{1/2} \simeq 30\omega_y, \tag{4.22}$$

corresponding to

$$h = \frac{1}{2}|B|\left(\frac{J_y^\circ}{n_y}\right)^2 - |A| . \tag{4.23}$$

An oscillator action of $30\omega_y$ corresponds to a radius in Fig. 3 of $[2\cdot 30]^{1/2} \sim 8$, which is $8/(2)^{1/2} \simeq 5.6$ times the rms radius of the resonance-free Gaussian. These numbers make it clear that if one used

the near-resonance approximation to describe the transition region, it would be hard to see how the test particle in Fig. 2 could remain as close to the origin as it does for so long before it finally sets out for the island chain.

In order to accomodate to this situation, let us now introduce two parameters, J_y^t and h_t ("t" stands for "transition") to systematize our ignorance about the middle ground between the island chain and the vicinity of the origin. J_y^t is to be thought of as the largest value of J_y up to which $\dot{J}_y \simeq 0$ is an adequate approximation to the equations of motion. We suppose that near $J_y = J_y^t$, the system can also be described by a function h which is some sort of continuation of the near-resonance effective pendulum energy. Let h_t be the value of this continuation that corresponds roughly to J_y^t. We assume that the term $-|A|\cos\Phi$--which would otherwise give an h-curve its variation in J_y--has become negligible by the time h reaches h_t, mainly because of the natural J_y-dependence of the Fourier coefficient $G_{n_y,n}$ from which A is derived. (Indeed, in general [3], $G_{n_y,n} \sim J_y^{|n_y/2|}$ for small J_y.) Accordingly, we suppose, for definiteness, that h and J_y are related by

$$h \simeq \frac{1}{2}|B|\left(\frac{J_y - J_y^\circ}{n_y}\right) \tag{4.24}$$

near $h = h_t$.

What value are we to assign to J_y^t? At present, we can only set rather generous limits: At the low end, we must have $J_y^t \geq 0$, trivially. At the high end, it seems intuitively reasonable to suppose that J_y^t--which must correspond to an orbit curve that is nearly circular--must be no larger than $[(7)^2/2]\omega_y$, the lowest value of J_y

taken by any point on the apparent separatrix.

In what follows, we shall find it more convenient to refer to r, the ratio of J_y^t to J_y^o, than to J_y^t itself. In terms of r, the limits above are equivalent to

$$0 \leq r \lesssim \frac{(7)^2/2}{93} \simeq .26. \tag{4.25}$$

It is not hard to imagine ways to improve on the phenomenological parametrization that we have just described, or to improve on the associated numerical methodology that we shall explain below. We shall not discuss such improvements here, however. The crude calculation to be described below yields a result so far from the observed value of the flux in question that, in my opinion, no essentially one-dimensional model, no matter how refined, is likely to be able to account for Fig. 3 in a satisfactory way.[F12]

Let us now proceed directly to the flux calculation itself. We carry out this calculation in two parts. The easy part is the estimate of the prefactor $\tilde{a}(|A|)/2\pi\omega_y$ in Eq. (4.19). This is accomplished by combining Eqs. (3.24a) and (4.21), with the result

$$\lim_{(a)} \frac{|\tilde{a}(|A|)|}{2\pi\omega_y} \simeq 49 \ . \tag{4.26}$$

The hard part is the estimate of the integral in (4.19). To do this, we begin by decomposing the integral into two parts, referring, respectively, to the vicinity of the origin and to the vicinity of the island chain,

$$\int_{I_c}^{I_s} \frac{a(I")}{b(I")}\, dI" = \int_0^{J_y^t} \frac{a(J_y")}{b(J_y")}\, dJ_y" + \int_{h_t}^{|A|} \frac{\tilde{a}(h")}{\tilde{b}(h")}\, dh". \tag{4.27}$$

Next, we rewrite each of these parts separately as a function of r. This is very easy for the first integral on the right-hand side of (4.27)--we may use Eqs. (4.15), which apply here because of our assumptions concerning the form of the dynamics for $J_y < J_y^t$. The result is

$$\int_0^{J_y^t} \frac{a(J_y")}{b(J_y")}\, dJ_y" = \int_0^{J_y^t} \left(\frac{1}{\omega_y}\right) dJ_y" = \frac{J_y^t}{\omega_y} = \frac{J_y^o}{\omega_y} \cdot r \simeq (93)r, \tag{4.28}$$

where we have used (4.21) in generating the last approximate equality.

The corresponding transformation of the second integral on the right-hand side of (4.27) is done in several steps: First, we approximate it by trapezoidal interpolation, because at this writing we do not have enough information to proceed in a more sophisticated way. The result is

$$\int_{h_t}^{|A|} \frac{\tilde{a}(h")}{\tilde{b}(h")}\, dh" \simeq \left(\frac{|A|-h_t}{2}\right)\left(\frac{\tilde{a}(h_t)}{\tilde{b}(h_t)} + \frac{\tilde{a}(|A|)}{\tilde{b}(|A|)}\right). \tag{4.29}$$

We then reduce separately each of the factors on the right-hand side of (4.29), as follows:

For the first factor, we appeal to (4.24) and (4.21), with the result

$$\left(\frac{|A|-h_t}{2}\right) \simeq \frac{|B|}{2} \{\left(\frac{A}{B}\right) - \frac{1}{2} \left(\frac{J_y^\circ}{n_y}\right)^2 (r-1)^2\}$$

$$\simeq |B|\omega_y^2 [17 - (60)(r-1)^2]. \tag{4.30}$$

For the first term in the second factor, we appeal to (4.15), (4.24), and (4.21), with the result

$$\frac{\tilde{a}(h_t)}{\tilde{b}(h_t)} = \frac{a(J_y^t)}{b(J_y^t)} \left(\frac{\partial h}{\partial J_y}\right)^{-1} \Bigg|_{J_y=J_y^t}$$

$$= \left(\frac{1}{\omega_y}\right) \frac{n_y^2}{|B|(J_y^t - J_y^\circ)}$$

$$= [|B|\omega_y^2]^{-1} [n_y^2\omega_y/J_y^\circ]/(r-1)$$

$$\simeq [|B|\omega_y^2]^{-1} (.39)/(r-1). \tag{4.31}$$

For the second term in the second factor on the right-hand side of (4.29), we appeal to Eqs. (3.24a),(3.25a), and (3.26a), together with (4.21), with the result

$$\frac{\tilde{a}(|A|)}{\tilde{b}(|A|)} = \lim_{(a)} \frac{\tilde{a}}{\tilde{b}} \simeq -[|B|\omega_y^2]^{-1} \cdot (.37). \tag{4.32}$$

Combining Eqs. (4.27)-(4.32), we have, in toto,

$$\int_{I_c}^{I_s} \frac{a(I")}{b(I")}\, dI" \simeq (93)r + [(17)-(60)(r-1)^2] \cdot [(.39)(r-1)^{-1} - (.37)] . \tag{4.33}$$

In view of inequality (4.25), we learn from (4.33) that

$$33 \lesssim \int_{I_c}^{I_s} \frac{a(I")}{b(I")}\, dI" \lesssim 38. \tag{4.34}$$

Finally, combining (4.26) with (4.34) according to the absolute value of formula (4.19), we have our estimate of the desired rate: One particle in from

$$\left(\frac{1}{8\times 49}\right) \times \exp(33) \simeq O(10^{12}) \tag{4.35}$$

to

$$\left(\frac{1}{8\times 49}\right) \times \exp(38) \simeq O(10^{14}) \tag{4.36}$$

is expected to travel from near the origin to the apparent separatrix in Fig. 2 in what would be four relaxation times ($8/\gamma$) in the absence of collision effects. As promised, this is very far from what one actually observes. As we learned in Section III, the observed rate is in fact at least one particle in about 10^2 in roughly the same time period.

V. FURTHER STUDIES OF THERMAL RESONANCE

As explained in Section II, the problem of thermal resonance--which we have deliberately neglected in the last two sections--could subvert the formal probabalistic methods that we have just finished defining and exploring. With this in mind, we conclude this essay with a brief discussion of two additional classes of systems in which thermal resonance can be identified, but which are--at least in principle--easier to study than Eqs. (1.1). The ideas to be sketched here will be developed in much greater detail in two separate publications ([4], [5]).

1. Anharmonic Oscillator with Periodically Time-Dependent Friction [4]

The equation of a noisy anharmonic oscillator with periodically time-dependent friction is

$$\ddot{y} + \gamma(t)\dot{y} + \frac{d}{dy} U(y) = \lambda\, \xi(t), \tag{5.1}$$

where ξ is as defined in Section II, and λ is a constant. It is straightforward to adapt the arguments and manipulations in Section II (or in the Appendix) to this equation. One can easily formulate thermal averaging in this context, and can also easily identify the integrals that do not become independent of the canonical angle when a thermal resonance condition is satisfied.[F13] A notable feature of this equation is that the corresponding thermally averaged Fokker-Planck equation is the same--as one can easily verify--as the averaged Fokker-Planck equation that would be associated with Eq. (5.1) if $\gamma(t)$ were replaced by the time-independent value

$$\langle\gamma\rangle \equiv \frac{1}{T} \int_0^T \gamma(t)\, dt, \tag{5.2}$$

where T is the repeat period of γ. We shall explain shortly why this may be useful.

I was led to investigate this system by considering the semiclassical approximation to the path integral representation [19] of the probability distribution for system (1.1). In brief, this approximation relates the probability distribution to a path (three dimensional in the case of system (1.1)) that solves a variational fourth-order ordinary differential equation with time as the independent variable. In the case of (5.1), this variational equation is

$$\left(\frac{d^2}{dt^2} - \frac{d}{dt}\,\gamma(t) + \frac{d^2U}{dy^2}\right)\left(\ddot{y} + \gamma(t)\dot{y} + \frac{dU}{dy}\right) = 0\ . \tag{5.3}$$

This approximation is an appealing device for studying thermal resonance because one can hope that such a path might, in some sense, encounter a finite number of strong thermal resonances sequentially, and this might be easier to analyze than the full probability distribution, which could display thermal resonant effects in several locations simultaneously. In the simplest version of this conjectural scenario, thermal resonances would be localized events between which the path would proceed in some simple, easily characterized fashion.

Unfortunately, in the case of Eqs. (1.1) or (1.4) it is difficult to guess what the simple inter-resonance propagation law should be. This (apart from the obvious simplicity of a time-independent potential

U(y)) is what makes Eq. (5.1) so attractive as an alternative context in which to study thermal resonance. In the case of Eq. (5.1), it is easy to make such a guess: One can guess that the semiclassical path propagates between thermal resonances in some fashion closely related to a solution of Eq. (5.3) with $\gamma(t)$ replaced by $\langle\gamma\rangle$, since, as we have seen, this same replacement is mandated when thermal averaging is permissible.

Although it is not immediately apparent, Eq. (5.3) can actually be solved in closed form when the friction coefficient is independent of time, at least in the limit of small γ but long ($\gtrsim O(1/\gamma)$) time, which is the limit of interest to storage ring designers. This solution is the main result in Ref. [4]. It is a necessary first step towards a practical sequential analysis of thermal resonances. As explained in [4], however, more work needs to be done before it becomes clear that subsequent steps can be realized.

2. Linear Oscillator with Periodically Time-Dependent Parameters [5]

This system is defined by the equation

$$\ddot{y} + \gamma(t)\dot{y} + K(t)y = \lambda(t)\,\xi(t), \tag{5.4}$$

where now all three functions γ, K, and λ can depend periodically on time, with common period T. This is an appealing setting in which to study our problem, since linearity permits many otherwise difficult calculations to be carried out exactly.

Strictly speaking, thermal averaging as defined in Section II is difficult to formulate in this context because there is no "differential smearing" when $\gamma = \lambda = 0$--the frequency ω^1 in a linear system is

independent of the canonical action, so all points in phase space rotate about the origin together. This does not mean, however, that there is no analogue of thermal resonance in linear systems. Indeed, a formal argument similar to that given in the appendix of the present paper shows [5] that as long as ω^1 is not an integral multiple of 1/2 $(2\pi/T) = \pi/T$, the ansatz (2.3) applies to the steady state of Eq. (5.4)--i.e., to the solution of (5.4)'s Fokker-Planck equation that is periodic in time, with period T; and this argument can break down at $\omega^1 = n\pi/T$ in a manner very similar (see [5], Eq. (2.7)) to the manner in which the manipulations in Section II break down at thermal resonance.

Because of this similarity, perhaps one can learn something about thermal resonance in general from the answers to questions such as: How close must ω^1 be to some $n\pi/T$ in order that the steady state of (5.4) depart significantly from the result of dynamical averaging? What do such departures look like?

In principle, one can obtain this information directly from an exact expression that can be derived in general for the steady-state distribution of a linear system. In practice, however, this approach tends to be rather complicated algebraically.

The main result of Ref. [5] is a simpler approach, involving the leading terms in a systematic expansion of the steady state about a system with $\omega^1 = n\pi/T$. These terms can be calculated in a simple manner that is quite similar in spirit to the self-consistency argument in the appendix of the present paper.

With this method, the questions posed above can be answered in a straightforward way. It will turn out that in most respects these answers are not especially illuminating as far as nonlinear systems are

concerned, except, perhaps, for this observation: In general, departures from the results of dynamical averaging are small unless, for some n, $(\omega^1-(n\pi/T))$ is less than either $O(\gamma)$ (see [5], Eqs. (3.14), (3.17), and (B.13)) or $O(\gamma^{1/2})$ (see [5], Eqs. (4.10)-(4.12), (4.15), and (B.23)), depending on the circumstances, in which cases the departures are substantial.[F14] This leads us to wonder: In a nonlinear system like (1.1), is a thermal resonance a localized but perhaps prominent feature of diameter $O(\gamma)$ or $O(\gamma^{1/2})$ in the space of the canonical frequency vector $\vec{\omega}^1$ (and therefore in canonical action space, since $\vec{\omega}^1$ is a function of $\vec{I}$)? At present we have no way of testing this conjecture, or of convincingly formulating likely alternatives.

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Appendix: A Systematic Theory of Thermal Averaging

In this appendix, we try to provide a systematic basis for Eqs. (2.3) and (2.11), which in Section II were obtained only in a hueristic way. We shall still need here to make assumptions that we cannot prove; by contrast with Section II, however, we shall be making them within a precisely defined context.

In a nutshell, the basic philosophy underlying the arguments in Section II was this: For small γ, one may think of system (1.1) in terms of the coexistence of two distinct processes operating on two widely separated time scales: Hamiltonian circulation along KAM tori, which gives rise (with a little help, on small scales, from diffusion) to the smeared ansatz $P(\vec{I},t)$; and slow, long-range drift across tori, which gives $P(\vec{I},t)$ its time dependence, as governed by (2.11).

To make this philosophy precise, we try to apply a technique known as "two-timing", first introduced by Keller [20]. (Once the appropriate notation has been explained, we shall indicate briefly how our type of system differs from those for which this technique was originally formulated.)

Thus, we write the probability distribution as a perturbation series

$$P(\vec{I},\vec{\theta},t,s) = P_0(\vec{I},\vec{\theta},t,s) + \gamma P_1(\vec{I},\vec{\theta},t,s) + \ldots \quad . \tag{A.1}$$

The first time parameter t is qualitatively to parametrize Hamiltonian circulation, while the second time parameter s is qualitatively to describe frictional and diffusive drift. In intermediate stages of the following analysis we shall treat t and s as independent; at the end, however, one is to set

$$s = \gamma t. \tag{A.2}$$

Our specific aim is to prove that if thermal resonance can be ignored, then

$$P_o(\vec{I},\vec{\theta},t,s) = \Pi\ (\vec{I},s), \tag{A.3}$$

where the function Π satisfies Eq. (2.11) with t replaced by s/γ. This will provide us with the desired systematic basis for thermal averaging, because then Eqs. (2.3) and (2.11) follow directly from the natural assumption that if γ is small, then $P(\vec{I},\vec{\theta},t,s)$ can be approximated by the leading term in the perturbative expansion (A.1). Corrections to thermal averaging are obtained in a systematic way by calculating higher order terms in expansion (A.1).

Our analysis of P_0 begins with the Fokker-Planck equation for $P_o(\vec{I},\vec{\theta},t,s)$, which is obtained from (1.4) through the replacement

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial s} \ . \tag{A.4}$$

When Eq. (1.4) is modified accordingly, and then written in terms of the right-hand side of (A.1) and decomposed according to powers of γ, the result is

$$L_0 P_0 = 0, \tag{A.5a}$$

$$L_0 P_1 = L_1 P_0, \tag{A.5b}$$

$$L_0 P_2 = L_1 P_1, \text{ etc.}, \tag{A.5c}$$

where L_0 and $\gamma(L_1 + \partial/\partial s)$ are the same partial differential operators that appear on the left- and right-hand sides, respectively, of (1.4).

The desired properties of P_0 will be obtained from Eqs. (A.5) in two steps. The first step is the general solution of (A.5a). The second step is the derivation of conditions that this solution must satisfy in order that (A.5b) be self-consistent as an equation for P_1.

In order to solve (A.5a), we note that for any function $Q(\vec{I},\vec{\theta},t,s)$,

$$[L_0 Q]\ (\vec{I},\vec{\theta},t,s) = \frac{\partial Q}{\partial t} + \omega_i{}^1 \frac{\partial Q}{\partial \theta_i} \ , \tag{A.6}$$

because L_0 is nothing more than the (Lagrangian) derivative along Hamiltonian flow. In view of (A.6), the most general solution of Eq. (A.5a) is

$$P_0\ (\vec{I},\vec{\theta},t,s) = \tilde{P}_0\ (\vec{I},\vec{\theta}-t\vec{\omega}^1,s). \tag{A.7}$$

Beyond this, however, $L_0 P_0 = 0$ leaves P_0 unconstrained.

(This is where our systems differ from those for which two-timing is on the surest foundation. In the usual applications [20], the equation $L_0 P_0 = 0$ has only one solution, up to an overall s-dependent scale. Thermal resonance is the specific price that we pay for the additional freedom represented by (A.7).)

Most choices for $\tilde{P}_0$ are inadmissible because of a difficulty that shows up only when they are substituted into Eq. (A.5b). In such cases, (A.5b), as an equation for P_1, has no solutions that are consistent with the interpretation of t as an oscillation parameter distinct from the parameter s that characterizes secular drift. We now determine the conditions that $\tilde{P}_0$ must satisfy in order that this difficulty not appear. This will lead us directly to Eq. (A.3) and its associated partial differential equation.

To reveal the desired consistency conditions, let us substitute (A.7) into Eq. (A.5b) and integrate both sides with respect to oscillation time t along a Hamiltonian orbit. This yields

$$P_1(\vec{I},\vec{\theta}+(t'-t)\vec{\omega}^1,t',s) - P_1(\vec{I},\vec{\theta},t,s)$$

$$= \int_t^{t'} dt'' \left[\frac{\partial}{\partial t''} + \omega_i{}^1 \frac{\partial}{\partial \theta_i}\right] P_1$$

$$= \int_t^{t'} dt'' \,[L_0 P_1] = \int_t^{t'} dt'' \,[L_1 P_0]$$

$$= (t'-t) \left\{ - \frac{\partial \tilde{P}_0}{\partial s} + \tilde{P}_0 + \left[\frac{1}{t'-t}\int_t^{t'} dt'' \left(v_x \frac{\partial I_i}{\partial v_x} + \omega_x{}^2 \frac{\partial^2 I_i}{\partial v_x{}^2}\right)\right] \frac{\partial \tilde{P}_0}{\partial I_i} \right.$$

$$\left. + \left[\frac{\omega_x{}^2}{(t'-t)}\int_t^{t'} dt'' \left(\frac{\partial I_i}{\partial v_x}\frac{\partial I_j}{\partial v_x}\right)\right] \frac{\partial^2 \tilde{P}_0}{\partial I_i \partial I_j} \right. +$$

$$+\left[\frac{1}{t'-t}\int_t^{t'} dt''\left(v_x\frac{\partial}{\partial v_x}[\theta_i-t''\omega_i{}^1]+\omega_x{}^2\frac{\partial^2}{\partial v_x{}^2}[\theta_i-t\omega_i{}^1]\right)\right]\frac{\partial\tilde{P}_0}{\partial\theta_i}$$

$$+\left[\frac{\omega_x{}^2}{t'-t}\int_t^{t'} dt''\left(\frac{\partial}{\partial v_x}[\theta_i-t''\omega_i{}^1]\right)\left(\frac{\partial}{\partial v_x}[\theta_j-t''\omega_j{}^1]\right)\right]\frac{\partial^2\tilde{P}_0}{\partial\theta_i\partial\theta_j}$$

$$+\left[\frac{2\omega_x{}^2}{t'-t}\int_t^{t'} dt''\left(\frac{\partial I_i}{\partial v_x}\right)\left(\frac{\partial}{\partial v_x}[\theta_j-t''\omega_j{}^1]\right)\right]\frac{\partial^2\tilde{P}_0}{\partial I_i\partial\theta_j}$$

+ similar expressions with x replaced by y and by z. (A.8)

In Eq. (A.8), the action, angle, and two time arguments of each integrand are $\vec{I},\vec{\theta}+(t''-t)\vec{\omega}^1,t''$, and s, respectively. (Note that all but the first three integrands are in fact independent of s.) In the final decomposition of $\int L_1 P_0$, the arguments of $\tilde{P}_0$ in every term are $\vec{I},\vec{\theta}-t\vec{\omega}^1$, and s.

The desired constraints on $\tilde{P}_0$ follow from the observation that if t is qualitatively to parametrize oscillatory behavior, then the initial difference in (A.8) must indefinitely remain bounded as a function of t'. Such an observation is legitimate for this reason: The initial expression in (A.8) is the difference of two values of P_1 corresponding to two points on a single KAM torus at two different times. Since the torus depends periodically on time, these two points are forever restricted to a bounded domain in phase space. However, since t parametrizes oscillation, one intuitively expects P_1, as a function of t, to vary in a bounded way in bounded domains of phase space. Thus, the difference in question is bounded because each of its two constituent terms is separately bounded.

To extract the desired constraints, we now proceed as follows: The observation above implies that in the limit $t' \to \infty$, the expression in curved brackets in (A.8) must approach zero. In this limit, the dominant part of this expression is

$$\left[\frac{\omega_x^2}{t'-t} \int_t^{t'} dt'' \, (t'')^2 \left(\frac{\partial \omega_i^{\,1}}{\partial v_x} \frac{\partial \omega_j^{\,1}}{\partial v_x} \right) \right.$$

$$\left. + \text{similar expressions with x replaced by y and by z} \right] \frac{\partial^2 \tilde{P}_0}{\partial \theta_i \partial \theta_j}$$

$$= \left\{ \frac{1}{3} (t'-t)^2 \left[\omega_x^2 < \oint \frac{d^3\theta}{(2\pi)} \left(\frac{\partial \omega_i^{\,1}}{\partial v_x} \frac{\partial \omega_j^{\,1}}{\partial v_x} \right) > \right. \right.$$

$$\left. + \text{similar expressions with x replaced by y and by z} \right]$$

$$\left. + 0 \, (t'-t) \right\} \frac{\partial^2 \tilde{P}_0}{\partial \theta_i \partial \theta_j} \, . \qquad \text{(A.9)}$$

Note that we have neglected thermal resonance in obtaining this equality. If a thermal resonance condition (2.7) were satisfied, the coefficient within the curved brackets would depend nontrivially on $\vec{\theta}$. It follows from (A.9) that consistency of the oscillation interpretation of t implies

$$0 = \left[\omega_x^2 < \oint \frac{d^3\theta}{(2\pi)^3} \left(\frac{\partial \omega_i^{\,1}}{\partial v_x} \frac{\partial \omega_j^{\,1}}{\partial v_x} \right) > + \text{similar expressions} \right.$$

$$\left. \text{with x replaced by y and by z} \right] \frac{\partial^2 \tilde{P}_0}{\partial \theta_i \partial \theta_j}$$

$$\equiv C_{ij} \frac{\partial^2 \tilde{P}_0}{\partial\theta_i \partial\theta_j} , \qquad (A.10)$$

where the matrix C depends only on $\vec{I}$, and not on $\vec{\theta}$ or t (or, trivially, s) as long as thermal resonance can be neglected.

For all but a measure-zero set of $\vec{I}$'s, we expect that the only solutions to (A.10) satisfy

$$\partial \tilde{P}_0 / \partial\theta_i = 0, \qquad (A.11)$$

for all i. If there were some other solutions, we would conclude from Fourier decomposition of $\tilde{P}_0$ that

$$0 = C_{ij} n_i n_j \qquad (A.12)$$

for some nontrivial multiplet $\vec{n}$ of integers. This is possible for only a measure-theoretically exceptional set of $\vec{I}$'s, however, because all possible equations of the form (A.12) define, taken together, only a countable array of (d-1)-dimensional subsets of the d-dimensional $\vec{I}$-space. If thermal resonance could not be neglected in obtaining the coefficient of $(t'-t)^2$ in (A.9), then we could not arrive at (A.11) in this way because then C_{ij} would depend nontrivially on $\vec{\theta}$, so that Fourier decomposition of $\tilde{P}_0$ would by itself tell us nothing.

With (A.11) we have met our first goal, Eq. (A.3). The remaining goal follows upon substitution of (A.11) back into (A.8) and again requiring that the expression in curved brackets vanishes. When we can neglect thermal resonance of the form (2.7), the result is

$$0 = -\frac{\partial \Pi}{\partial s} + \left\{ \left[\Pi + < \oint \frac{d^3\theta}{(2\pi)^3} v_x \frac{\partial I_i}{\partial v_x} > \frac{\partial \Pi}{\partial I_i} \right] + \right.$$

$$+ \omega_x^2 \left[< \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial^2 I_i}{\partial v_x^2} > \frac{\partial \Pi}{\partial I_i} + < \oint \frac{d^3\theta}{(2\pi)^3} \frac{\partial I_i}{\partial v_x} \frac{\partial I_j}{\partial v_x} > \frac{\partial^2 \Pi}{\partial I_i \partial I_j} \right]$$

$$\left. + \text{similar expressions with x replaced by y and by z} \right\} . \quad \text{(A.13)}$$

With the help of Eqs. (2.10), one sees easily that this is the same as Eq. (2.11), with t replaced by s/γ, which is what we wanted to show.

FOOTNOTES

F1. Radiation is not emitted strongly by beams in modern proton/antiproton storage rings. The relaxation time associated with radiation is measured in milliseconds for modern high-energy e^+e^- storage rings, and in from days to months for the current generation of operating or planned $p/\bar{p}$ storage rings. The processes that determine the sizes and shapes of stored hadronic beams are quite different and more complicated in many respects than the processes that determine the configurations of stored e^+e^- beams. For this reason, and also because colliding-beam problems are experimentally less severe in $p/\bar{p}$ than in e^+e^- systems [3], we shall not consider proton/antiproton colliders in this paper.

F2. Strictly speaking, the system of Eqs. (1.1), with γ set equal to zero, is not in manifestly Hamiltonian form, because the third (z) equation lacks a term $\partial_z g$. However, we can easily show that this system is a <u>limit</u> of a sequence of Hamiltonian systems, as follows: Define $p_x = \dot{x}$, etc. Define $H_a(p_x,p_y,x,y,z,t) \equiv 1/2\ (p_x^2 + p_y^2 + \omega_x^2 x^2 + \omega_y^2 y^2) + g(x,y,z,t)$, and $H_b(p_z,z) \equiv 1/2\ (p_z^2 + \omega_z^2 z^2)$. Introduce a fictitious parameter ϵ and let $Q_x \equiv \epsilon x$, $Q_y \equiv \epsilon y$, $P_x \equiv \epsilon p_x$, $P_y \equiv \epsilon p_y$. Then Eqs. (1) are those derived canonically from the Hamiltonian $\epsilon H_a\ (P_x/\epsilon, P_y/\epsilon, Q_x/\epsilon, Q_y/\epsilon, z, t) + H_b(p_z,z)$, in the limit that all of $\epsilon, P_x, P_y, Q_x, Q_y$ approach zero, with P_x/ϵ, P_y/ϵ, Q_x/ϵ, Q_y/ϵ fixed.

F3. Orbits that cannot be placed in such families are chaotic, in a sense made precise in [8]. We shall say very little about chaotic orbits in the present essay, even though (see [3]) theorists have often tried to ascribe the problems of storage rings to chaotic effects. Hamiltonian chaos on a large scale usually arises in the domain of phase space between two resonant (action-angle) regions whose relative separation falls below some characteristic threshold ("resonance overlap," see [8]). For this reason, it seems to us that a practical theory of probability distributions near resonance is a prerequisite for a fully general and intuitive statistical methodology for colliding beams, even when chaotic phenomena invalidate the simple action-angle picture.

F4. The appropriate one-dimensional analogue of (2.3) actually does not appear in Kramers' 1940 paper [9]. Kramers makes only the somewhat weaker statement that the angle average $\int_0^{2\pi} (d\theta/2\pi)P(I,\theta,t)$ satisfies the differential equation (2.11). We feel it is important to recognize that a stronger result is possible. For an alternative proof of (2.3) for the type of system treated by Kramers, see Ref. [4].

F5. The alert reader will realize, as the argument progresses, that it is important for the spreading to be able to proceed along the flow direction $\vec{\omega}^1$. Mathematically, this means that for some infinitesimal vector $\vec{\delta}$, we must have $(\vec{\delta}\cdot\vec{\nabla})\vec{\omega}^1$ parallel to $\vec{\omega}^1$. This is in fact not possible for system (1.1) because the Hamiltonian part of the third (z) equation, which does not involve x or y, is the equation of a simple harmonic oscillator. Thus, in this case, one component (call it the third) of $\vec{\omega}^1$ must be independent of $\vec{I}$

(in fact, identically equal to ω_z), and therefore $(\vec{\delta}\cdot\vec{\nabla})\vec{\omega}^1$ cannot have a third component to parallel that of $\vec{\omega}^1$ itself. Strictly speaking, then, the argument to which this footnote refers leads not to Eq. (2.3), for the case of system (1.1), but instead to the statement $P(\vec{I},\vec{\theta},t) \simeq P(\vec{I},\theta_3,t)$. However, the dependence on I_3 and θ_3, the action and angle variables of the x-y-independent z-oscillator, is determined entirely by the third equation in (1.1), and therefore for very large times, the probability density P, as a function of I_3 and θ_3, must become proportional to the Maxwell-Boltzmann equilibrium Gaussian $\exp(-I_3/\omega_z)$, which is manifestly independent of θ_3, with or without the arguments in this section. There need be no concern about the rate at which this Gaussian is approached, since we may imagine, as is reasonable, that the colliding beams have time to reach some sort of noninteracting equilibrium before they are brought into collision, and therefore that the θ_3-independent z-Gaussian is already preestablished by the time the g-terms in (1.1) are "turned on."

Note, incidentally, that none of this would have to make sense if the explicit time dependence in (2.1) were not constrained to be periodic. This periodicity ensures that the apparent separating of points on different tori actually reflects real separating in x-v_x-etc. space, and not an artifact of an artificial choice of coordinates.

F6. Actually, our hueristic argument, although suitable enough in spirit, is too naive in leading us to conclude that (2.4) is the proper integer-based obstruction to smearing the probability about

dynamical tori. The correct such condition is identified in the Appendix (Eq. (A.2)).

F7. Throughout this paper, we adhere to the convention that summation is implicit in repeated dummy indices.

F8. To prove (2.13), one can compare the chain-rule identities $\delta_{ik} = [(\partial I_i/\partial v_j)\ (\partial v_j/\partial I_k) + (\partial I_i/\partial x_j)\ (\partial x_j/\partial I_k)]$, etc., with the Poisson-bracket identities $\delta_{ik} = [(\partial I_i/\partial v_j)\ (\partial \theta_k/\partial x_j) - (\partial I_i/\partial x_j)\ (\partial \theta_k/\partial v_j)]$, etc., using the fact that the Jacobian matrix, schematically

$$\begin{pmatrix} \partial I/\partial v & \partial I/\partial x \\ \partial \theta/\partial v & \partial \theta/\partial x \end{pmatrix} ,$$

has nonzero (in fact unit [17]) determinant.

F9. From now on, we use "resonance" and "nonlinear resonance" interchangeably, unless otherwise indicated.

F10. One can easily show that in a one-dimensional system, the action variable of an orbit family is uniquely determined up to sign and an additive constant. The angle variable is uniquely determined up to the same sign, as well as an additive function of I, and a term $2\pi mt/T$, for appropriate integer m (this preserves the periodicity of (2.1)). In the various discussions that follow, we shall be freely redefining I according to momentary convenience. However, since only signs and additive constants are involved, this should not cause any serious confusion.

F11. If we had wanted the flux across an arbitrary orbit curve in the family of the origin, this would be an unrealistic requirement, since there would be nothing to prevent a particle from turning

around immediately after reaching the curve in question. This objection does not apply to part (a) of the separatrix, however. Once a particle crosses limit (a), it is temporarily prevented from recrossing by two processes: (1) Hamiltonian circulation, which removes the particle from the vicinity of limit (a) to the vicinity of limit (b); and (2) friction, which causes a particle to fall in toward a resonance island's stable center. These expectations appear to be supported by the data shown in Fig. 3.

F12. Among higher-dimensional processes, the intervention of a single resonance with $n_x \neq 0$ or $n_z \neq 0$ would be the simplest possible explanation for the rapid transport seen in Fig. 3. This might become visually apparent in a graph that showed a different projection of six-dimensional phase space (see for example the second article in Ref. [12]). Unfortunately, no such alternative graphs appear to be available at this time for the particular simulation that corresponds to Fig. 3.

F13. We shall need to know in [4] that when the potential U is symmetric, then in fact thermal resonance is present only when the integer n (the one-dimensional reduction of $\vec{n}$ in (2.7)) is even. The reason is as follows: Thermal resonance for (5.1) comes about when the angular frequency Ω of γ is rationally related to the angular frequency of $v\partial I/\partial v$ when evaluated along the orbit of a particle following the Hamiltonian flow. But when U is symmetric, then both v and $\partial I/\partial v$ change sign in time $1/2(2\pi/\omega^1) = \pi/\omega^1$. Therefore the true frequency of $v\partial I/\partial v$ is not ω^1 but $2\omega^1$.

F14. This result assumes, as in Eqs. (1.1), that the parameter λ is treated as if formally proportional to $(\gamma)^{1/2}$.

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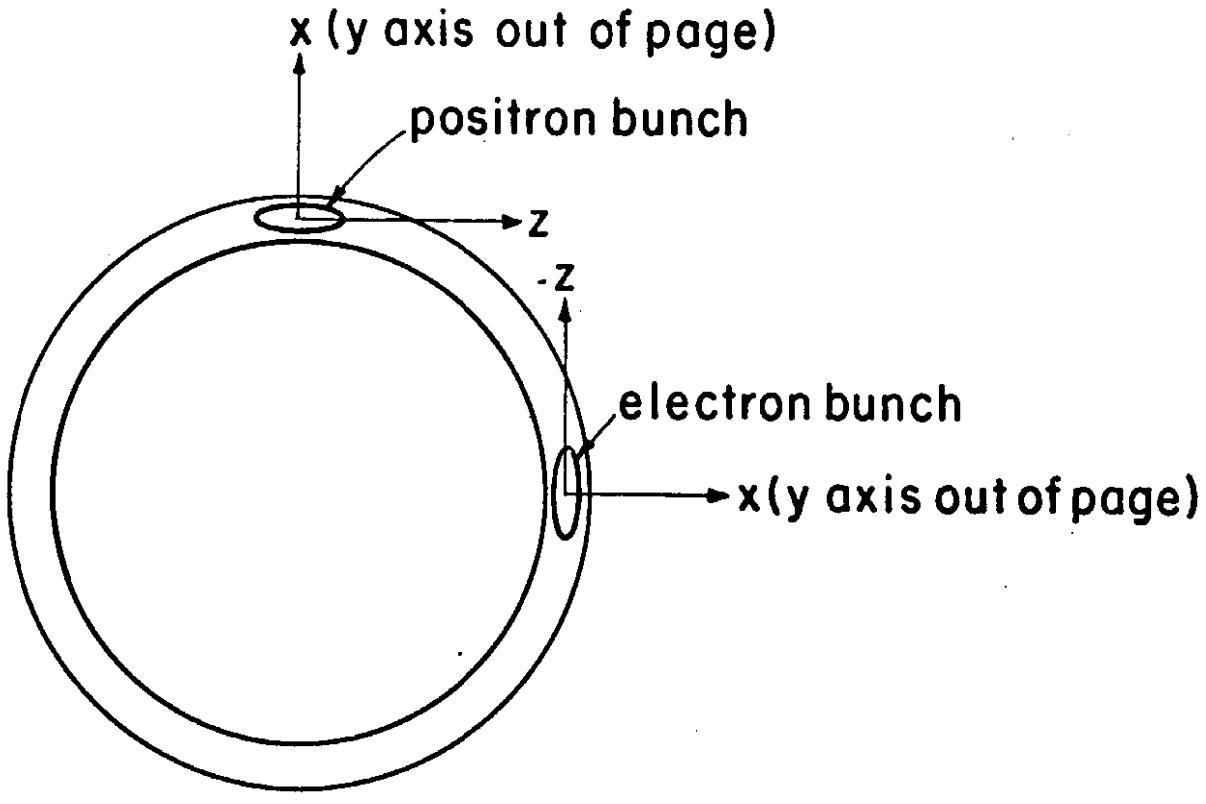
FIGURE CAPTIONS

1. Schematic representation of an electron-positron storage ring, showing beam bunches and moving coordinate axes.
2. KAM curves of a one-dimensional colliding beam model at a single moment in time. The separatrix is represented by dashes.
3. (From Ref. [13].) Vertically projected path of one test particle from the computer simulation described in Ref. [16]. Horizontal and vertical axes correspond to $y/(2)^{1/2}$ and $\dot{y}/\omega_y(2)^{1/2}$ respectively. The distance markers divide the axes into segments of length one. The points are snapshots of the particle's position at equally spaced time intervals of duration T. Figure 3a shows the first 10,348 such points (the initial point is near the origin); Figure 3b shows the succeeding 619 points. The particle stays within the central dark region for about 8000T before proceeding to the satellite islands. The damping time in this simulation is 3000T. The three islands apparent in Fig. 3a are in reality all images of a single island that revolves about the origin at $(70/3)\cdot 2\pi$ radians per storage ring period. Similarly for Fig. 3b.
4. Results of approaching the separatrix in Fig. 1 in three different ways. (a) Limit of orbit curves that lie between the origin and the island chain. (b) Limit of orbit curves that lie beyond the island chain. (c) Limit of orbit curves within an island.


x (y axis out of page)
positron bunch
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electron bunch
x(y axis out of page)

Figure 1

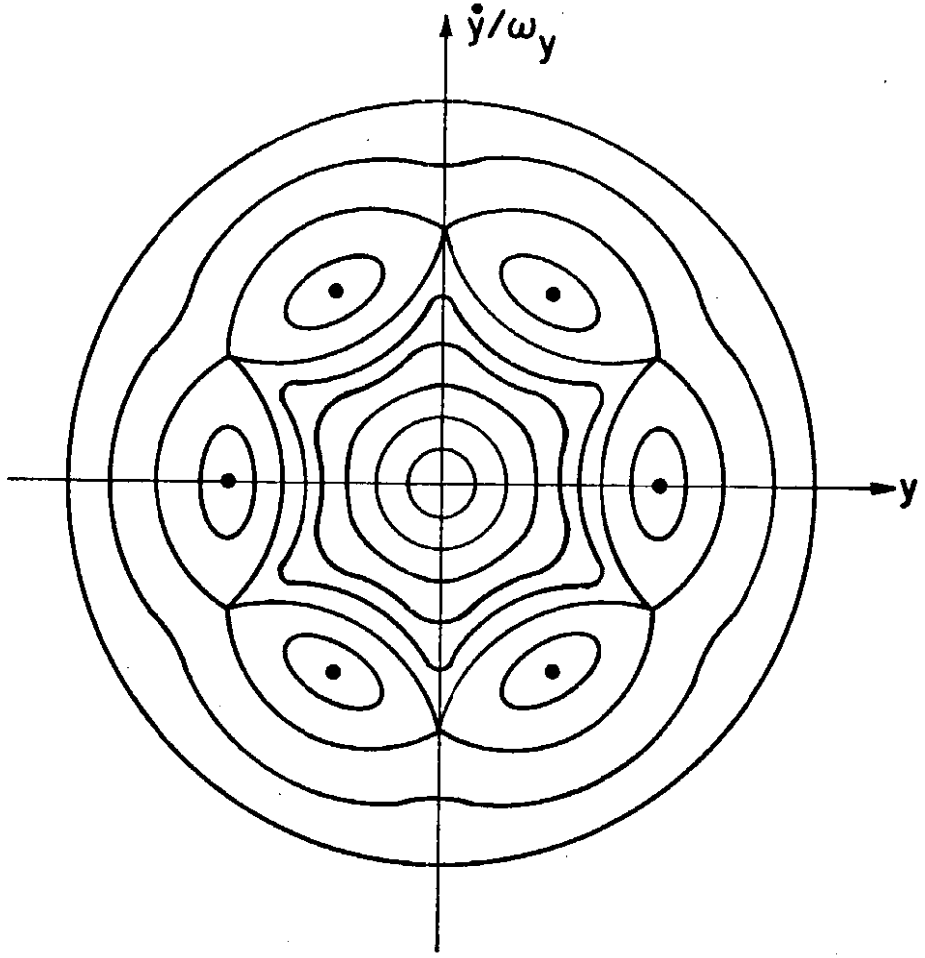

Figure 2

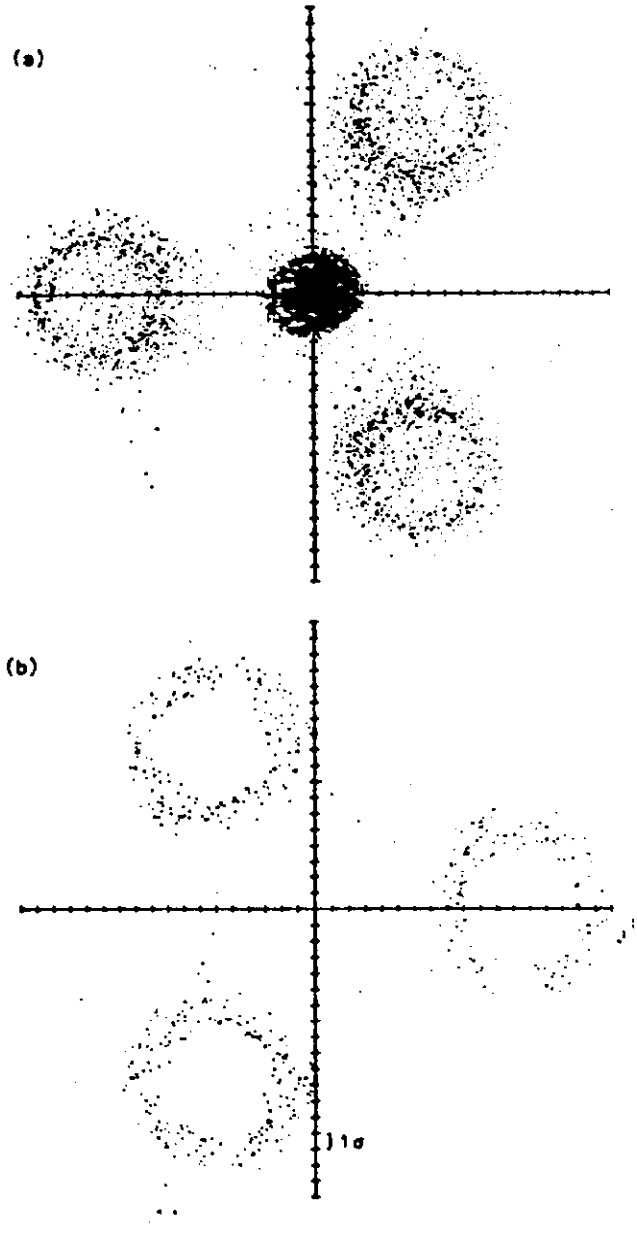

Figure 3

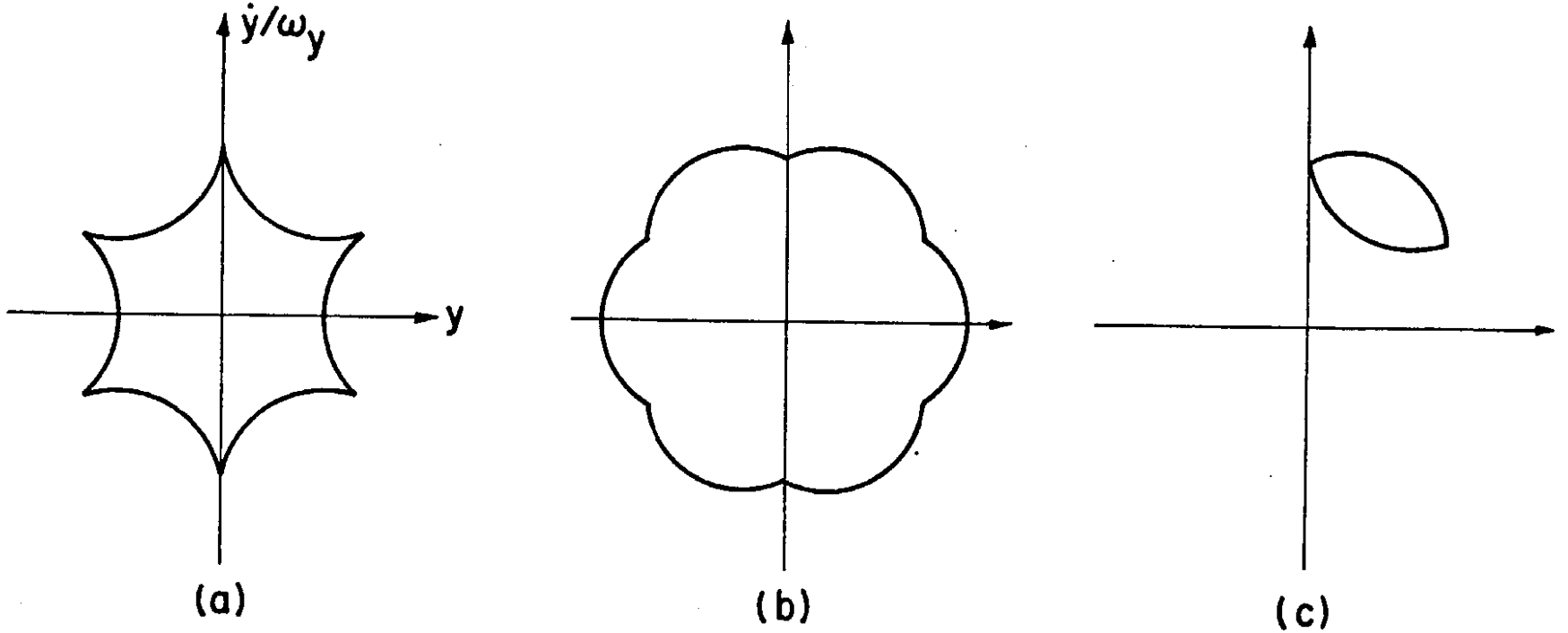

Figure 4