Further improvements in the understanding of LQC

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Abstract. Loop Quantum Cosmology provides a successful quantization of isotropic and homogeneous flat universes with a massless, homogeneous scalar field as the matter content. Here, we propose a new ordering for the Hamiltonian constraint operator that facilitates the quantization of this model and makes the physical consequences much more transparent. In particular, our constraint is such that, in the gravitational sector, the zero volume state decouples, allowing us to get rid of the cosmological singularity already at a kinematical level, as well as to introduce a consistent densitization procedure for the constraint. Furthermore, the typical discretization of the spatial volume is achieved in superselection sectors which prove to be most suitable, with support on semilattices and where the basic functions that codify all the relevant information about the geometry have the expected Wheeler-DeWitt limit of standing waves. Thanks to these properties, we can demonstrate that the quantum bounce is generic for any physical state and superselection sector.

1. Introduction

Loop Quantum Cosmology (LQC) is an approach to construct a quantum description of homogeneous spacetimes using loop quantization techniques [1]. In the particular situation of a homogeneous, isotropic spacetime with flat spatial curvature coupled to a homogeneous, massless scalar field, the complete quantization was first studied in [2, 3, 4, 5]. Other systems with less symmetry have also been quantized using these techniques, including e.g. homogeneous Bianchi I models [6, 7].

In spite of the fact that the above works about the isotropic flat model succeeded in incorporating the effects of quantum geometry, some aspects of the quantization needed a deeper analysis. For instance, a rigorous densitization procedure of the Hamiltonian constraint was missing, and it was not clear whether and how one could attain superselection sectors that were simpler and with better properties. Actually, in previous studies of Bianchi I spacetimes in LQC [6], these questions and problems had been naturally solved. Inspired by those studies, we have adopted a new ordering for the representation of the gravitational Hamiltonian constraint of the isotropic case as an operator in the corresponding kinematic Hilbert space [8]. On the one hand, our constraint operator decouples the state which is the counterpart of the classical singularity in that Hilbert space (namely, the zero volume state), so that we can just remove it. Thanks to this, we are able to define a map between the solutions of the original Hamiltonian constraint and those of the (simpler) densitized one. On the other hand, the constraint (together with the physical observables) superselects sectors in the gravitational part of the kinematical Hilbert space with support on semilattices (where the zero volume state is not present) that are contained e.g. in the positive real line. The simple structure of these sectors considerably simplifies the analysis of the model. Indeed, when we consider the gravitational part of the quantum Hamiltonian constraint, we can fully determine the form and behavior of its generalized eigenfunctions. These admit a type of no-boundary description, and have the expected asymptotic limit, which agrees with the Wheeler-DeWitt (WDW) limit of the model. All these properties allow us to prove that the quantum bounce is generic and occurs for all superselection sectors.

2. The quantum Hamiltonian constraint

The phase space of the model is described by two pairs of conjugate variables. One pair describes the matter content: the homogeneous massless scalar field ϕ and its momentum p_ϕ, with $\{\phi, p_\phi\} = 1$. On the other hand, the geometry is represented in terms of a densitized triad and an Ashtekar-Barbero $su(2)$-connection. In the homogeneous and isotropic situation under analysis, and in a diagonal gauge, each of them is specified by a single coefficient, which we call p for the triad and c for the connection, with $\{c, p\} = 8\pi G\gamma/3$. Here, G is the Newton constant and γ the Immirzi parameter. Note that, owing to the homogeneity and in order to avoid divergences, one must restrict all the spatial integrals to a given cell, whose volume with respect to a Euclidean fiducial metric will be called V_0.

We introduce a standard quantization of the homogeneous scalar field. Regarding the geometry, we employ a loop quantization procedure. In LQC, and using the fiducial structure, one considers as basic variables fluxes of densitized triads through square surfaces, which can be identified essentially with p, and holonomies of the connection along straight lines of fiducial length $\bar{\mu}V_0^{1/3}$. The matrix elements of these holonomies, denoted by $\mathcal{N}_{\bar{\mu}}$, are given by exponential functions of ic. Besides, for these holonomies, we adopt an "improved dynamics" prescription. Then, the considered fiducial length must satisfy that $\bar{\mu}^{-1} = \sqrt{p/\Delta}$ (for more details see [4]). Here, Δ is the minimum non-zero area allowed quantum mechanically. The gravitational part of the configuration algebra Cyl_S is the algebra of almost periodic functions of c, which are identified with kets $|v\rangle$. The Cauchy completion with the discrete norm $\langle v'|v\rangle = \delta_{v'v}$ provides the gravitational part of the kinematical Hilbert space, $\mathcal{H}_{\text{kin}}^{\text{grav}}$. The action of the basic operators on the basis states $|v\rangle$ is given by $\hat{p}|v\rangle = \text{sign}(v)(2\pi\gamma G\hbar\sqrt{\Delta}|v|)^{2/3}|v\rangle$ and $\hat{\mathcal{N}}_{\bar{\mu}}|v\rangle = |v+1\rangle$. Here, $\hbar$ is the Planck constant. Using these basic operators, we can promote the classical Hamiltonian constraint into a quantum one. In particular, the $SU(2)$-curvature can be described in terms of holonomies forming a closed loop whose physical area coincides with the minimum area Δ (see [3, 4]). At this stage, we exploit the freedom in the factor ordering to introduce a new operator realization of the constraint, firstly studied in [6] and discussed in detail in [8]. Our Hamiltonian constraint is given by

$$\hat{C} := \left[\widehat{\frac{1}{V}}\right]^{1/2} \left(-\frac{6}{\gamma^2}\widehat{\Omega}^2 + 8\pi G\hat{p}_\phi^2\right) \left[\widehat{\frac{1}{V}}\right]^{1/2}, \tag{1}$$

where the gravitational part is the square of

$$\widehat{\Omega} := \frac{1}{4i\sqrt{\Delta}} \left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{-1/2} \widehat{\sqrt{|p|}} \left[\left(\hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}}\right) \widehat{\text{sign}(p)} + \widehat{\text{sign}(p)} \left(\hat{\mathcal{N}}_{2\bar{\mu}} - \hat{\mathcal{N}}_{-2\bar{\mu}}\right)\right] \widehat{\sqrt{|p|}} \left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{-1/2}. \tag{2}$$

The inverse volume operator $[\widehat{1/V}] := [\widehat{1/\sqrt{|p|}}]^3$ is regularized employing the so-called Thiemann's trick (see, e.g. [2]).

2.1. Densitization procedure

Our constraint operator $\hat{C}$ annihilates the zero volume state $|v = 0\rangle$ in the gravitational sector, and its action leaves invariant the orthogonal complement of $|v = 0\rangle$, denoted by $\widetilde{\mathcal{H}}_{\text{kin}}^{\text{grav}}$.

Then, the zero volume state has no contribution to the non-trivial solutions of the Hamiltonian constraint, and we can thus restrict the study to $\widetilde{\mathcal{H}}^{\rm grav}_{\rm kin}$. To this extent, the classical singularity is kinematically resolved. Since we have removed the kernel of the inverse volume operator, we can use $\widehat{[1/V]}^{-1}$ to define a bijection between the solutions of $\hat{C}$ and the solutions of its densitized version [8], given by

$$\hat{\mathcal{C}} := \left[\widehat{\frac{1}{V}}\right]^{-1/2} \hat{C} \left[\widehat{\frac{1}{V}}\right]^{-1/2} = -\frac{6}{\gamma^2}\widehat{\Omega}^2 + 8\pi G\hat{p}_\phi^2. \tag{3}$$

Let us call $\widetilde{\rm Cyl}_S$ the linear span of the states $|v\rangle$ with $v \neq 0$, whose Cauchy completion with respect to the discrete norm gives $\widetilde{\mathcal{H}}^{\rm grav}_{\rm kin}$. Since in general the solutions to the Hamiltonian constraint are not normalizable states in the kinematic Hilbert space, one should expect them to belong to a bigger space, like e.g. the algebraic dual of $\widetilde{\rm Cyl}_S$, namely $\widetilde{\rm Cyl}^*_S$. In that space, the map between the solutions $(\psi|$ of the constraint $\hat{C}$ and those of $\hat{\mathcal{C}}$, $(\psi'| = (\psi|\widehat{[1/V]}^{1/2}$, is indeed a bijection. Using $\hat{\mathcal{C}}$, we can directly recognize two commuting Dirac observables, $\widehat{\Omega}^2$ and $\hat{p}_\phi^2$, a fact that simplifies the resolution of the constraint.

2.2. Properties of $\widehat{\Omega}^2$

The gravitational constraint operator has an action of the form [8]

$$\widehat{\Omega}^2|v\rangle = F_-(v)|v+4\rangle + G(v)|v\rangle + F_+(v)|v-4\rangle, \tag{4}$$

so that it only connects states $|v\rangle$ separated by 4 units in the label v. Here, $F_\pm(v)$ and $G(v)$ are certain real functions such that $F_+(v) = 0$ if $v \in [-4, 0)$ and $F_-(v) = 0$ if $v \in (0, 4]$. In total, $\widehat{\Omega}^2$ only relates states with support on semilattices of the form $\mathcal{L}^\pm_{\tilde{\varepsilon}} := \{v = \pm(\tilde{\varepsilon} + 4n),\, n \in \mathbb{N}\}$, where $\tilde{\varepsilon} \in (0, 4]$. We note that the gravitational Hilbert subspaces corresponding to states with support on any of these semilattices are all separable. In the following, we will restrict the analysis of the quantum geometry to just one of these separable Hilbert spaces, e.g. one with $v > 0$. In [8] we demonstrated that the operator $\widehat{\Omega}^2$ defined in any of these superselection sectors turns out to be essentially self-adjoint, with a non-degenerate absolutely continuous spectrum equal to the positive real line. We will use the parameter $\lambda \in [0, \infty)$ to denote the generalized eigenvalues of $\widehat{\Omega}^2$.

2.3. Basis of generalized eigenfunctions

The generalized eigenstates $|e^{\tilde{\varepsilon}}_\lambda\rangle = \sum_{v\in\mathcal{L}^+_{\tilde{\varepsilon}}} e^{\tilde{\varepsilon}}_\lambda(v)|v\rangle$ are completely determined by a piece of initial data $e^{\tilde{\varepsilon}}_\lambda(\tilde{\varepsilon})$, which can be chosen positive. The resulting eigenfunctions are then real, since $\widehat{\Omega}^2$ is a real operator. The elements of this basis must be normalized to the Dirac delta, condition that fixes the norm of the initial datum. For large v, our operator tends to a differential operator which also possesses an absolutely continuous spectrum, but 2-fold degenerated. Its generalized eigenfunctions, denoted by $\underline{e}_{\pm\omega}(v)$, are given (up to a factor $1/\sqrt{v}$) by plane waves in the logarithm of v, each of them interpretable as expanding or contracting waves in the volume (once the dependence in the matter field is taken into account and the discussion restricted to positive-frequency solutions). In the regime $v \to \infty$, then, the eigenstates $e^{\tilde{\varepsilon}}_\lambda(v)$ must be linear combinations of these plane waves. Since the combination has to be real:

$$e^{\tilde{\varepsilon}}_\lambda(v) \to r\left\{\exp\left[i\phi_{\tilde{\varepsilon}}(\omega)\right]\,\underline{e}_\omega(v) + \exp\left[-i\phi_{\tilde{\varepsilon}}(\omega)\right]\,\underline{e}_{-\omega}(v)\right\}, \tag{5}$$

where $\underline{e}_\omega(v) = \exp\left(i\omega\ln v/\alpha\right)/\sqrt{2\pi\alpha v}$, $\omega = \sqrt{\lambda}$ and $\alpha = 4\pi\gamma G\hbar$. In addition, r is a real number and $\phi_{\tilde{\varepsilon}}(\omega)$ is certain real function (that in general varies slowly with ω).

The asymptotic limit of the eigenfunctions $e^{\tilde{\varepsilon}}_\lambda(v)$ has a clear physical interpretation: they behave as standing waves, with exactly the same contribution of incoming and outgoing flux.

2.4. Comparison and comments about different constraint operators

The operators chosen to represent the gravitational constraint in [3, 4, 5], denoted there as $\widehat{\Theta}$, have some qualitative differences with respect to ours. On the one hand, the superselection sectors in those works, in general, are lattices instead of semilattices. The absolutely continuous spectrum of $\widehat{\Theta}$ is 2-fold degenerate. The criterion suggested there to construct the generalized eigenfunctions needs the union of two lattices. Numerical analyses showed that, on each lattice and in the regime of semiclassical states peaked around classical trajectories, they behave as *almost* standing waves for $|v| \to \infty$, with different limits in each lattice (the phase of the almost standing waves is in principle different in each lattice). Even though this difference does not affect the norm of the considered physical states, the scenario changes drastically for more general regimes where the behavior as standing waves cannot be assured or the overlapping between contracting and expanding branches becomes non-negligible. Therefore, our proposal ensures a good behavior in regimes where the others commented cannot. On the other hand, there are some specific situations in which the operator $\widehat{\Theta}$ has essentially the same properties as $\widehat{\Omega}^2$. In more detail, let us consider the operator $\widehat{\Theta}$ of [4] (which is also one of the proposals in [3]) defined in the symmetric superselection sector that contains the $|v = \pm 2\rangle$ states. One can straightforwardly check that, in the difference eigenvalue equation corresponding to this constraint operator, some of the coefficients vanish if $v = \pm 2$. Something similar happens in the simplified model of [5] for $v = 0$. This fact leads to a decoupling between semiaxes. But this is so only in these exceptional situations, whereas the results of [8], confirming the standing wave behavior responsible of the quantum bounce, are valid in all circumstances and superselection sectors.

3. Conclusions

We have analyzed the flat Friedmann-Lemaître-Robertson-Walker model coupled to a massless scalar field in the context of LQC. We have proposed a new Hamiltonian constraint operator, based on a different factor ordering, which leads to a quantum resolution of the cosmological singularity already at the kinematical level. Employing this fact, we have been able to carry out a rigorous densitization of the constraint, facilitating the subsequent analysis of the system. Remarkably, the basic elements that codify the information about the geometry have support on semilattices, emerging from an initial section with non-zero volume. In this sense, they provide a no-boundary description. Furthermore, they behave as standing waves on macroscopic regions of the universe. These two properties indicate that any incoming flux evolves, through a quantum region, to outgoing flux and vice versa, for any physical state and in all superselection sectors. In other words, the intrinsic properties of the gravitational part of the constraint, inherited by its eigenfunctions, allow us to prove that the quantum bounce is generic. We have compared also the properties of our Hamiltonian constraint operator with those of previous proposals. Our operator is the only one which guaranties the phenomenon of a generic quantum bounce, independently of the particular state and superselection sector considered.

4. References

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