An alternative subtraction scheme using Nagy-Soper dipoles

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Abstract

In this thesis we address an alternative subtraction scheme in high energy colliders at Next-to-Leading Order (NLO) QCD calculations. In particular, we focus on the treatment of real radiation contributions in the soft and collinear limits. After UV-renormalization, the remaining infrared singularities appearing both in the real radiation and in the virtual contributions can be regularized using dipole subtraction method. In this scheme, dipoles are based on the momentum mapping and on the splitting functions derived from an improved parton shower formulation with quantum interference effects. In our new scheme, we employ a slightly altered momentum mapping such that the number of subtraction terms is greatly reduced in comparison with the standard Catani-Seymour scheme. In addition, the new scheme also facilitates the matching of NLO calculations with parton showers using the same splitting functions. We also achieve the complete integrations of the splitting functions over an unresolved one parton phase space, obtaining the correct soft and collinear singularity structures that are necessary to cancel the soft divergences in the virtual contributions. We discuss the general framework setup of the scheme as well as some scattering processes at colliders; we find complete agreement with the results in the widely used Catani-Seymour dipole subtraction scheme.

Contents

Chapter 1

A brief review of QCD

1.1 Introduction

In elementary particle physics, the fundamental theory of the strong interaction is described by Quantum Chromodynamics (QCD). It describes the interactions between quarks and gluons, and in particular how they bind together to form hadrons ($e.g.$ the proton and the neutron). QCD is a quantum field theory within a special class described by non-abelian gauge theory [1] (or sometimes called Yang-Mills gauge theory). It is based on the $SU(N)$ gauge group. Understanding how to use non-abelian gauge theory, combined with the parton model, led to the development of QCD and it is now a very well established theory in the sense that QCD predictions have successfully accounted for all the strong interaction experiments observed at colliders, in particular the phenomena of hadronic jet structure in e^+e^- annihilation, the Drell-Yan process and heavy quark production [100]. QCD has two peculiar properties, which differ from Quantum Electrodynamics (QED)/electroweak interactions and which also reveal its uniqueness.

- **Asymptotic freedom:** this means that at very high energies, the strong force (also called the colour force) of quarks and gluons is so weak that they behave almost as free particles when the quarks or gluons are really close to each other. This phenomenon is called asymptotic freedom and it is due to the fact that the strong running coupling constant $\alpha_s(Q^2)$ depends on the energy scale Q; $\alpha_s(Q^2)$ becomes weaker as the scale Q increases[1]. To check this, one must determine the running of the coupling constant α_s, which is governed by the renormalization group equation,

$$\beta(\alpha_s) \;=\; Q^2\,\frac{\partial \alpha_s}{\partial Q^2}$$

[1] In contrast to QED where the coupling α becomes strong at high energies.

if $\beta < 0$, the theory is asymptotically free. The asymptotic freedom of QCD also explains why we can apply a perturbative approach to explore the structure of matter at short distances (= high energies). This prediction of QCD was first discovered in the early 1970s by H. David Politzer [2] and by David J. Gross and Frank Wilczek [3]. For this work they were awarded the 2004 Nobel Prize in Physics.

- **Colour confinement:** the strong force of quarks becomes stronger when the distance increases, which implies low energy. So it would take an infinite amount of energy to move apart two quarks; they are always confined inside hadrons. Confinement is widely believed to be true (although analytically unproven yet; a lot of QCD theorems are based on assumptions) in the sense that no free quark and gluon degrees of freedom have been observed at colliders[2].

The recent progress in the understanding of strong interactions has been due to the comparison between precise higher order perturbative QCD calculations and accurate experimental data. The perturbative calculation of jet cross sections is based on the QCD improved parton model picture [4–6], which has been made rigorous since the discovery of asymptotic freedom. In this model a hard scattering process between two hadrons can be thought of as an interaction between the quarks and gluons, which are the constituents of the incoming hadrons. Much of the techniques of perturbative QCD derive from the well known methods of QED apart from the fact that QCD is a non-abelian gauge theory; however, there are still big differences between the two theories.

Importantly because the quanta of QCD, quarks and gluons, the analogues of electrons and photons in QED, are always bound into hadrons and not observed as free particles at colliders. At low energies, confinement effects dominate and non-perturbative approaches become more important. The most widely used method is lattice QCD. At very high energies, however, one still cannot avoid confinement effects due to the fact that the asymptotic incoming and outgoing partons consist of hadrons. For certain quantities, factorization theorems [7–9] allow the two scales to be appropriately separated, and the low energy pieces can be treated by parametrizations, model calculations or factoring them into the parton distribution functions (PDF). The remaining quantities involve only high momentum transfers (and therefore short distances and short times) and is insensitive to long distances behaviour of QCD. Thus, these quantities are calculable in perturbation theory because of asymptotic freedom. The factorization theorem states that the short distance behaviour of parton scattering (the hard part) does not interfere with the long distance process that turns partons (quarks and gluons) into hadrons, hadronization. This factorization property can be proved to be valid to all orders in perturbation theory.

[2]Note, that all hadrons are colourless (or colour singlets) and only colour singlet states can be observed as free particles; *i.e.* we never observe a free quark/gluon since quark/gluon carries colour charge.

Another important ingredient of perturbative QCD is infrared safety, which is the guiding principle of higher order perturbative calculations. In general, we consider any quantity that is infrared finite. That is to say, infrared safe quantities do not depend on the long distance behaviour of QCD (except for the long distance effects that are factored into the parton distribution functions), so that it makes sense to calculate any quantity in perturbative QCD. The proof of infrared safety comes from the KLN theorem [10, 11], which is a fundamental quantum mechanical result and which provides the theoretical argument to the problem of collinear and infrared singularities due to massless charged particles. It states that fully inclusive measurements, which sum over all degenerate initial and final states, are free from infrared divergences.

In this thesis we will only deal with the physics of hard processes at Next-to-Leading Order precision, in which case the infrared divergences have to be treated carefully. The properties of any given hadron species will be irrelevant because of factorization theorems. This is why jet production is computed as simple parton scattering. The probability that partons will produce hadrons is unity.

1.2 Colour $SU(3)_C$ and quark confinement

First, let us review the addition of angular momenta in Quantum Mechanics. Addition of two spin-$1/2$ particles $j_A = j_B = 1/2$ has total spin $J = 0$ or $J = 1$. Symbolically, we have[3]

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \tag{1.1}$$

Now, combining a third spin-$1/2$ particle, we have

$$(\mathbf{2} \otimes \mathbf{2}) \otimes \mathbf{2} = (\mathbf{1} \oplus \mathbf{3}) \otimes \mathbf{2} = (\mathbf{1} \otimes \mathbf{2}) \oplus (\mathbf{3} \otimes \mathbf{2}) = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \tag{1.2}$$

At the end, we have a quartet of spin $3/2$ and two doublets of spin $1/2$. The quarks in the spin-$3/2$ baryons are in a symmetrical state of spin, space and $SU(3)_f$ flavour degrees of freedom, *e.g.* we consider the pion-nucleon resonance[4] Δ^{++} with spin-$3/2$

$$\left| \Delta^{++}, J_3 = 3/2 \right\rangle = |\, u\uparrow, u\uparrow, u\uparrow \rangle \tag{1.3}$$

Here, J_3 is the third component of the total angular momentum for Δ^{++} and the arrow represents the spin aligned up. However, this state is not allowed because the wave function has to be totally antisymmetric under interchange of any of the two quarks due to the Fermi-Dirac statistics. To

[3] Here, we use the dimension (*i.e.* the size of multiplet $2\,J + 1$) to label the irreducible representation.
[4] Δ^{++} is made of three u-quarks.

reconcile the baryon spectrum and the Fermi-Dirac statistics, one can introduce an additional quantum number, called colour [121, 122]. Hence, we can construct colour singlet states

$$\mid M > \sim \mid q_a\, \bar{q}_a > \quad \text{or} \quad \mid M > = \frac{1}{\sqrt{3}}\, \delta^{ab} \mid q_a\, \bar{q}_b > \quad \text{and} \quad \mid B > \sim \epsilon^{abc} \mid q_a\, q_b\, q_c >$$

for mesons[5] and baryons[6], respectively, where ϵ^{abc} is the totally antisymmetric tensor and $a, b, c\,(=\, 1, 2, 3\,)$ represent three colours of quarks. It is then easy to construct the totally antisymmetric wave function for Δ^{++}

$$\left|\, \Delta^{++}, J_3 \,=\, 3/2 \,\right\rangle \,=\, \epsilon_{abc} \left|\, u^a \uparrow, u^b \uparrow, u^c \uparrow \right\rangle \tag{1.4}$$

The state Eq. (1.4) is then a singlet $\mathbf{1}$ representation of $SU(3)_C$. Next, we consider the confinement effect in QCD. Quark confinement is directly related to the fact that quarks (gluons) are coloured quanta and hence cannot be observed in nature. All physical hadrons are colourless. In order to construct colour singlet states we have to pick out a singlet representation in the decomposition of the product of two/three quarks into irreducible representation. For meson state, we have[7]

$$\mathbf{3} \otimes \bar{\mathbf{3}} \,=\, \mathbf{1} \oplus \mathbf{8} \tag{1.5}$$

For baryon state, we have

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \,(\,\bar{\mathbf{3}} \oplus \mathbf{6}\,) \otimes \mathbf{3} \,=\, (\,\bar{\mathbf{3}} \otimes \mathbf{3}\,) \,\oplus\, (\,\mathbf{6} \otimes \mathbf{3}\,) \,=\, \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{1.6}$$

Diquark and four-quark states belong to colour nonsinglets:

$$\begin{aligned}
\mid q\, q > : &\quad \mathbf{3} \otimes \mathbf{3} \,=\, \bar{\mathbf{3}} \oplus \mathbf{6} \\
\mid q\, q\, q\, q > : &\quad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \,=\, \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \bar{\mathbf{6}} \oplus \bar{\mathbf{6}} \oplus \mathbf{15} \oplus \mathbf{15} \oplus \mathbf{15} \oplus \mathbf{15}' \\
\mid \bar{q}\, \bar{q} > : &\quad \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \,=\, \mathbf{3} \oplus \bar{\mathbf{6}} \\
\mid \bar{q}\, \bar{q}\, \bar{q}\, \bar{q} > : &\quad \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \,=\, \bar{\mathbf{3}} \oplus \bar{\mathbf{3}} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \mathbf{6} \oplus \overline{\mathbf{15}} \oplus \overline{\mathbf{15}} \oplus \overline{\mathbf{15}} \oplus \overline{\mathbf{15}}'
\end{aligned} \tag{1.7}$$

Only $\mid q\, \bar{q} >$ and $\mid q\, q\, q >$ states belong to colour singlets. The conjecture that only colour singlet states can be observed is the same as that of the quark confinement.

1.3 QCD Lagrangian

Strong interactions between quarks and gluons are described by non-abelian local gauge theory and $SU(3)_C$ is the gauge group. Each quark field (flavour) forms a triplet in the fundamental

[5] Mesons are quark-antiquark bound states.

[6] Baryons are bound states of three quarks (antiquarks).

[7] Quark field belongs to the fundamental $\mathbf{3}$ representation of $SU(3)_C$ and antiquark is then assigned to the complex-conjugate $\bar{\mathbf{3}}$ representation. The $\mathbf{6}$ is symmetric and $\bar{\mathbf{3}}$ is antisymmetric under interchange of the two quarks.

representation of $SU(3)_C$

$$q_a = \begin{pmatrix} q_{\text{red}} \\ q_{\text{blue}} \\ q_{\text{green}} \end{pmatrix} , \qquad (a = 1, 2, 3) \tag{1.8}$$

and eight gluon fields G^A_μ form an octet in the adjoint representation (defined to have the same dimensions as the gauge group). The index A runs over the eight colour degrees of freedom of the gluon field ($A = 1, \cdots, 8$). The QCD Lagrangian density is given by

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{\text{ghost}} \tag{1.9}$$

where the classical Lagrangian density is

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4} G^A_{\mu\nu} G^{\mu\nu}_A + \sum_{\text{flavours}} \bar{q}_a \left(i \not{D} - m \right)_{ab} q_b \tag{1.10}$$

which describes the interaction of spin-$\frac{1}{2}$ quarks of mass m and massless spin-1 gluons. $\not{D}$ in Eq. (1.10) is a symbolic notation for $\gamma_\mu D^\mu$ and the spinor indices of γ_μ and q_a have been suppressed. The sum in Eq. (1.10) runs over the n_f different flavours of quarks ($= u, d, c, s, t, b$). We follow the standard notation with metric given by $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and set $\hbar = c = 1$. The gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} \tag{1.11}$$

$G^A_{\mu\nu}$ is the field strength tensor, which can be derived from the gluon field G^A_μ

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g f^{ABC} G^B_\mu G^C_\nu , \qquad (A, B, C = 1, \cdots, 8) \tag{1.12}$$

where g is the strength of the strong coupling constant[8] between coloured quanta (quarks and gluons). It is the third term on the right-hand-side of Eq. (1.12) that gives rise to cubic and quartic gauge boson (gluon) self-interactions[9]. Note that the mass terms $m^2\, G^A_\mu\, G^\mu_A$ for the gauge bosons (gluons) are not gauge invariant! Gauge bosons of unbroken non-abelian gauge theory are massless. Gauge invariance combined with renormalizability (absence of higher powers of fields and covariant derivatives in Lagrangian) determines gauge boson/fermion couplings and gauge boson self-interactions. In order to preserve the renormalizability of QCD, each term in the Lagrangian has to have mass dimension four. It follows, that the dimensions of the fields q_a and G^A_μ are 3/2 and 1, respectively. f^{ABC} are the structure constants of the $SU(3)_C$ colour group. D is the covariant derivative, which acts on triplet and octet fields according to

$$\begin{aligned} (D_\mu)_{ab} &= \partial_\mu \delta_{ab} + ig \left(t^C G^C_\mu \right)_{ab} \\ (D_\mu)_{AB} &= \partial_\mu \delta_{AB} + ig \left(T^C G^C_\mu \right)_{AB} \end{aligned} \tag{1.13}$$

[8] Note that the notations g and g_s can be exchanged with each other without making any difference in the later discussions.

[9] This term leads to the property of asymptotic freedom at the end.

where t and T are matrices in the fundamental and adjoint representations of $SU(3)_C$ colour group respectively.

$$\begin{aligned} \left[t^A, t^B\right] &= i\, f^{ABC}\, t^C \\ \left[T^A, T^B\right] &= i\, f^{ABC}\, T^C \\ \left(T^A\right)_{BC} &= -\,i\, f^{ABC} \end{aligned} \tag{1.14}$$

The generators t^A can be represented by the eight Gell-Mann matrices. These matrices are hermitian and traceless,

$$t^A = \frac{1}{2}\,\lambda^A \tag{1.15}$$

with

$$\begin{gathered} \lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{gathered} \tag{1.16}$$

The colour charges of the $SU(N)$ matrices can be chosen to be (see Fig. 1.1)

$$\begin{aligned} \mathrm{Tr}\; t^A\, t^B &= T_R\, \delta^{AB}, \quad T_R = \frac{1}{2}\,, \quad \sum_A t^A_{ab}\, t^A_{bc} = C_F\, \delta_{ac}, \quad C_F = \frac{N^2-1}{2\,N} \\ \mathrm{Tr}\, T^C\, T^D &= \sum_{A,B} f^{ABC}\, f^{ABD} = C_A\, \delta^{CD}, \quad C_A = N \end{aligned} \tag{1.17}$$

The colour charge is directly related to the Casimir operator $C_i = T^2 = T^A\, T^A$ where, $\left(T^A\right)_{BC} \equiv -\,i\, f^{ABC}$ if i is a gluon and $T^A_{ab} \equiv t^A_{ab}\ (-\,t^A_{ba})$ if i is a quark (antiquark). The Casimir operator commutes with all group generators, *i.e.*

$$\left[T^B, T^A\, T^A\right] = i\, f^{BAC}\, T^C\, T^A + i\, f^{BAC}\, T^A\, T^C = i\, \underbrace{f^{BAC}}_{\text{antisymmetric}}\, \underbrace{\left\{T^C, T^A\right\}}_{\text{symmetric}} = 0 \tag{1.18}$$

Hence, T^2 is an invariant of the algebra. For the specific case of $SU(3)_C$ we have

$$C_F = \frac{4}{3}, \qquad C_A = 3 \tag{1.19}$$

For the anticommutator of the t matrices in the fundamental representation, we have

$$\left\{t^A, t^B\right\} = \frac{1}{N}\,\delta^{AB}\, I + d^{ABC}\, t^C\,, \quad \sum_{A,B} d^{ABC}\, d^{ABD} = \frac{N^2-4}{N}\,\delta^{CD}, \quad d^{AAC} \equiv 0 \tag{1.20}$$

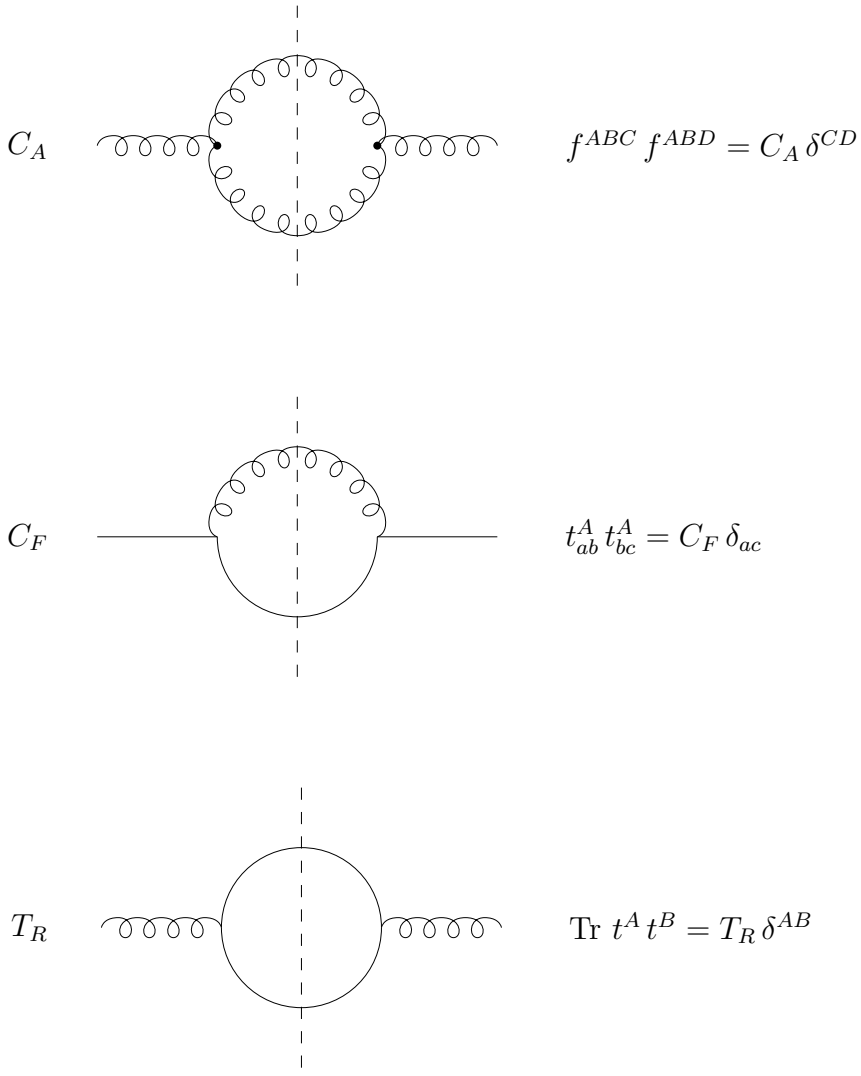


Figure 1.1: The definitions of the colour charges in the $SU(N)$ gauge group. Repeated indices are summed over the $N^2 - 1(N)$ values of $A, B, C(a, b, c)$ of the adjoint (fundamental) representation. Here, the curl line means gluon field and solid line means quark field. We can also define $C_i = C_F = (N^2 - 1)/(2\,N)$ if i is a quark or antiquark and $C_i = C_A = N$ if i is a gluon.

1.4 Local gauge invariance

Eq. (1.10) is invariant under local gauge transformations. This means the parameters $\theta^A(x)$ which characterize the transformation depend on the spacetime coordinates and we can redefine the quark fields independently at every spacetime point, without changing the physics.

$$q_a(x) \to q'_a(x) = \Omega(x)_{ab}\, q_b(x), \quad \Omega(x) \;=\; \exp\{i\, t^A \theta^A(x)\}, \quad \Omega^\dagger\, \Omega = \Omega\, \Omega^\dagger = 1 \tag{1.21}$$

If Ω is unitary, the t^A are hermitian matrices, called group generators. They generate infinitesimal transformations around the unit matrix element of the Lie group. For $SU(N)$ matrix (unitary and $\det \Omega = 1$), there are $N^2 - 1$ traceless, hermitian generators $t^A = 1/2\, \lambda^A$.

$$\Omega(x) = 1 + i\, t^A\, \theta^A(x) + \mathcal{O}(\theta^2) \tag{1.22}$$

The covariant derivative transforms as

$$D_\mu\, q(x) \quad \to \quad D'_\mu\, q'(x) \;\equiv\; \Omega(x) D_\mu\, q(x) \tag{1.23}$$

Here we have omitted the colour labels of the quark fields. We can use Eq. (1.23) to derive the transformation property of the gluon field $G_\mu(x)$

$$\begin{aligned} D'_\mu\, q'(x) \;&=\; \left(\partial_\mu + i\, g\, t \cdot G'_\mu\right) \Omega(x) q(x) \\ &=\; \left(\partial_\mu\, \Omega(x)\right) q(x) + \Omega(x) \partial_\mu q(x) + i\, g\, t \cdot G'_\mu\, \Omega(x) q(x) \end{aligned} \tag{1.24}$$

where

$$t \cdot G_\mu \;=\; \sum_A t^A\, G^A_\mu \tag{1.25}$$

Thus we find

$$t \cdot G'_\mu \;=\; \Omega(x)\, t \cdot G_\mu\, \Omega^{-1}(x) + \frac{i}{g}\, \left(\partial_\mu\, \Omega(x)\right)\, \Omega^{-1}(x) \tag{1.26}$$

which in terms of the infinitesimal parameters $\theta(x)$ can be rewritten as

$$G'^A_\mu \;=\; G^A_\mu - f^{ABC}\, \theta^B\, G^C_\mu - \frac{1}{g} \partial_\mu \theta^A + \mathcal{O}(\theta^2) \tag{1.27}$$

The third term is similar to the abelian case. The second term is specific to the non-abelian gauge theory. Introducing the generators T^A in the adjoint representation of the gauge group, one may write the infinitesimal transformation as

$$\delta\, G^A_\mu \;=\; -\frac{1}{g} (D_\mu)_{AB}\, \theta^B, \qquad G'^A_\mu \;=\; G^A_\mu + \delta\, G^A_\mu \tag{1.28}$$

It is straightforward to show that the transformation property of the field strength tensor $G_{\mu\nu}$ is[10]

$$t \cdot G_{\mu\nu}(x) \;\rightarrow\; t \cdot G'_{\mu\nu}(x) \;=\; \Omega(x)\, t \cdot G_{\mu\nu}(x)\, \Omega^{-1}(x) \;, \tag{1.29}$$

which may be derived using the relation

$$[D_\mu, D_\nu] \;=\; i\, g\, t \cdot G_{\mu\nu} \;. \tag{1.30}$$

1.5 Renormalization

The running coupling constant and renormalization group equation (RGE)

Suppose A is a dimensionless quantity which depends on a single energy scale Q. By assumption the scale Q is much bigger than another mass scale: $Q^2 \gg m^2$. In the limit $m \to 0$, A is independent of Q by dimensional analysis.

$$A \;=\; A(Q^2/m^2, \alpha_s) \quad \longrightarrow \quad A(\alpha_s) \qquad \text{as} \quad m \to 0 \tag{1.31}$$

After quantization, the theory must be renormalized due to the presence of ultraviolet (UV) divergences. Hence an arbitrary mass scale μ has to be introduced.

$$A \quad \longrightarrow \quad A(Q^2/\mu^2, \alpha_s) \qquad \text{after quantization} \tag{1.32}$$

The scale μ is arbitrary, and physical results cannot depend on it. Mathematically, the μ independence of A may be expressed by

$$\mu^2 \frac{d}{d\mu^2} A(Q^2/\mu^2, \alpha_s) \;=\; \mu^2 \left(\frac{\partial}{\partial \mu^2} + \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) A \;=\; 0 \tag{1.33}$$

This is a renormalization group equation (RGE). In order to solve RGE, one defines

$$t \;=\; \ln Q^2/\mu^2, \quad \beta(\alpha_s) \;=\; \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \tag{1.34}$$

Using

$$\frac{\partial}{\partial \mu^2} \;=\; \frac{\partial t}{\partial \mu^2} \frac{\partial}{\partial t} \tag{1.35}$$

we have

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) A \;=\; 0 \tag{1.36}$$

[10] In contrast to QED, the field strength tensor is not gauge invariant in QCD, because of the gluon self-interactions. However, the trace $\mathrm{Tr}\, G_{\mu\nu}\, G^{\mu\nu}$ is gauge invariant. Also, the gluon fields are coloured quanta, in contrast to QED, where the photons are electrically neutral.

The strong running coupling $\alpha_s(Q^2)$ is then introduced

$$\boxed{t \,=\, \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}, \quad \alpha_s(\mu^2) \,\equiv\, \alpha_s} \tag{1.37}$$

We can then take derivatives with respect to t and α_s (the two independent variables) on both sides of Eq. (1.37). By taking d/dt we obtain

$$1 \,=\, \frac{1}{\beta(\alpha_s(Q^2))} \frac{\partial \alpha_s(Q^2)}{\partial t}, \qquad \frac{\partial \alpha_s}{\partial t} \,=\, 0 \tag{1.38}$$

By taking $d/d\alpha_s$ we obtain

$$0 \,=\, \frac{1}{\beta(\alpha_s(Q^2))} \frac{\partial \alpha_s(Q^2)}{\partial \alpha_s} - \frac{1}{\beta(\alpha_s)} \tag{1.39}$$

These two relations show explicit the dependence of the running coupling on t and α_s:

$$\frac{\partial \alpha_s(Q^2)}{\partial t} \,=\, \beta(\alpha_s(Q^2)), \quad \frac{\partial \alpha_s(Q^2)}{\partial \alpha_s} \,=\, \frac{\beta(\alpha_s(Q^2))}{\beta(\alpha_s)} \tag{1.40}$$

from which it follows that

$$A(Q^2/\mu^2, \alpha_s) \,=\, A(1, \alpha_s(Q^2)) \tag{1.41}$$

is a solution of Eq. (1.36). Thus, the scale dependence of A is known if that of $\alpha_s(Q^2)$ is known.

The β function

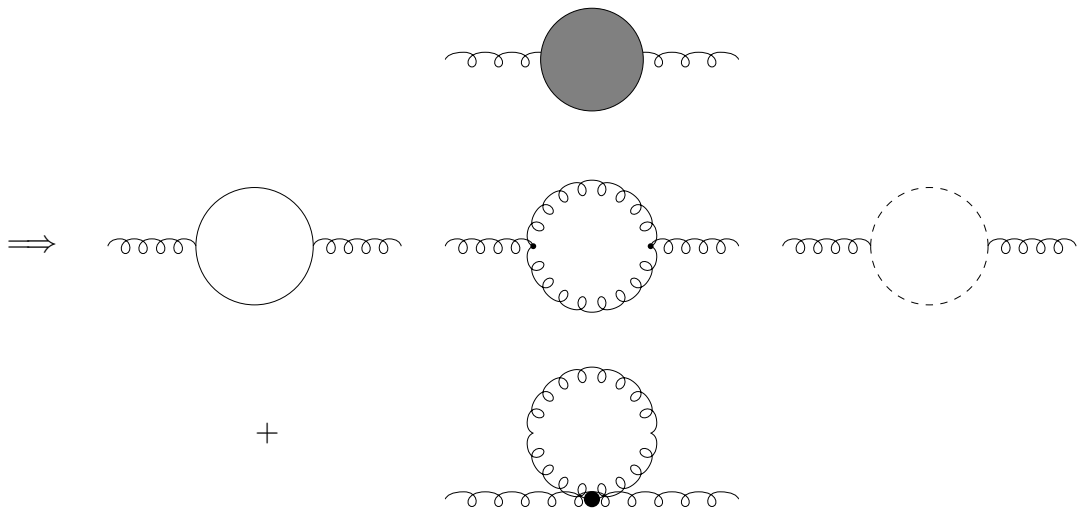
Instead of discussing different UV renormalization schemes, let us compute a simple renormalization scale dependent parameter: the running strong coupling α_s. In QCD, the differential equation for the strong coupling α_s is

$$\beta(\alpha_s) \,=\, Q^2 \frac{\partial \alpha_s}{\partial Q^2} \tag{1.42}$$

Thus α_s is a function of the energy scale at which it is evaluated and runs according to the β function, which at the one-loop level is given by

$$\boxed{\beta(\alpha_s) \,=\, -\frac{1}{2\,\pi}\,\beta_0\,\alpha_s^2 + \cdots, \qquad \beta_0 \,=\, \frac{11}{6}\,C_A - \frac{2}{3}\,n_f\,T_R} \tag{1.43}$$

where $(\cdots)$ represents the terms beyond the one-loop level. In the following we will explain in detail how Eq. (1.43) is computed. Let us consider bottom pair production at the LHC: $q\bar{q} \to b\bar{b}$. The Feynman diagram is an s-channel off-shell gluon. The physical parameters we can



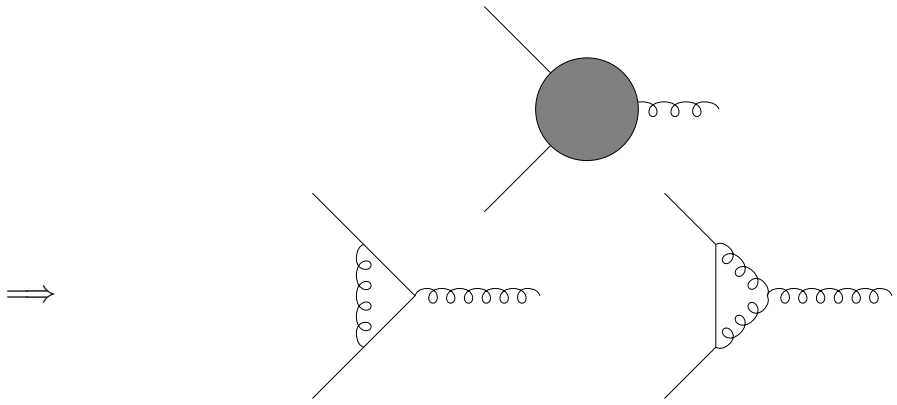


Figure 1.2: The one-loop contributions to the renormalization counterterms for the gluon and fermion self-energies and the gluon-fermion vertex. The curl line means gluon field, solid line means quark field and dash line means ghost field.

renormalize in this process are the strong coupling α_s and the bottom quark mass. Wave function renormalization constants are not physical. Here we assume that all quarks are massless. To compute the β function, one has to calculate three types of virtual diagrams (see Fig. 1.2): the internal gluon self-energies with a renormalization constant Z_A, the external quark self-energies with a renormalization constant Z_ψ and the gluon-fermion vertex $Z_{A\overline{\psi}\psi}$. The strong coupling renormalization constant Z_g is related to $Z_{A\overline{\psi}\psi}$, Z_A and Z_ψ by

$$Z_{A\overline{\psi}\psi} \;=\; Z_g\, Z_A^{1/2}\, Z_\psi \tag{1.44}$$

where[11]

$$\begin{aligned} Z_A &= 1 + \frac{\alpha_s}{4\,\pi}\,\left(\frac{5}{3}\,C_A - \frac{4}{3}\,n_f\,T_R\right)\,\Gamma(\epsilon)\,\mu^{-2\epsilon} \\ Z_\psi &= 1 - \frac{\alpha_s}{4\,\pi}\,C_F\,\Gamma(\epsilon)\,\mu^{-2\epsilon} \\ Z_{A\overline{\psi}\psi} &= 1 + \frac{\alpha_s}{4\,\pi}\,\left(C_F + C_A\right)\,\Gamma(\epsilon)\,\mu^{-2\epsilon} \end{aligned} \tag{1.45}$$

For the gluon self-energies (see the first block of Fig. 1.2) the fermion loop contribution gives

$$\boxed{i\,\left(q^2\,g^{\mu\nu} - q^\mu q^\nu\right)\,\delta^{AB}\,\left(-\frac{\alpha_s}{4\,\pi}\,\frac{4}{3}\,n_f\,T_R\,\mu^{-2\epsilon}\,\frac{1}{\epsilon} + \cdots\right)} \tag{1.46}$$

while the rest of the diagrams give

$$\boxed{i\,\left(q^2\,g^{\mu\nu} - q^\mu q^\nu\right)\,\delta^{AB}\,\left(\frac{\alpha_s}{4\,\pi}\,\frac{5}{3}\,C_A\,\mu^{-2\epsilon}\,\frac{1}{\epsilon} + \cdots\right)} \tag{1.47}$$

For the fermion self-energies (see the second block of Fig. 1.2), we have

$$\boxed{i\,\frac{\alpha_s}{4\,\pi}\,\not{p}\,C_F\,\mu^{-2\epsilon}\,\frac{1}{\epsilon} + \cdots} \tag{1.48}$$

and for the gluon-fermion vertex (see the last block of Fig. 1.2), we have

$$\boxed{i\,\frac{\alpha_s}{4\,\pi}\,g_s\,t^A\,\gamma^\mu\,\left(C_F + C_A\right)\,\mu^{-2\epsilon}\,\frac{1}{\epsilon} + \cdots} \tag{1.49}$$

We define the function $\tilde{\beta}(g_s)$ via the relation $\beta(\alpha_s) \;=\; g_s\,\tilde{\beta}(g_s)/(4\,\pi)$, where α_s and g_s are related by $g_s^2 \;=\; 4\,\pi\,\alpha_s$. Hence, the result for $\tilde{\beta}(g_s)$ is[12]

$$\tilde{\beta}(g_s) \;=\; (-2)\,\frac{g_s^3}{(4\,\pi)^2}\,\left[\left(C_F + C_A\right) - C_F + \frac{1}{2}\,\left(\frac{5}{3}\,C_A - \frac{4}{3}\,n_f\,T_R\right)\right] \tag{1.50}$$

[11] Here we calculate in $d = 4 - 2\,\epsilon$ dimensions.

[12] Here we have changed the notation for the strong coupling constant by g_s, which we denoted with g in the last section.

or equivalently

$$\beta(\alpha_s) \;=\; -\frac{\alpha_s^2}{2\,\pi}\,\beta_0 \tag{1.51}$$

The coupling constant α_s has an expression, which relates two different scales: Q^2 at which α_s is calculated and the renormalization scale μ^2. At leading order in the perturbative expansion, we can solve Eq. (1.42) with Eq. (1.43) to obtain

$$\alpha_s(Q^2) \;=\; \frac{\alpha_s(\mu^2)}{1+\alpha_s(\mu^2)\,b_0\,\ln\left(Q^2/\mu^2\right)}, \qquad b_0 \;=\; \frac{\beta_0}{2\,\pi} \tag{1.52}$$

In QCD with $n_f \leq 16$ (or $\beta \;<\; 0$) the strong coupling α_s becomes weaker as the energy scale increases. This is known as asymptotic freedom. This behaviour of asymptotic freedom is in contrast to QED where the coupling becomes strong at high energies. In QED the β function is

$$\beta_{\rm QED}(\alpha) \;=\; \frac{1}{3\,\pi}\,\alpha^2 + \cdots \tag{1.53}$$

1.6 Parton branching at Next-to-Leading Order

In this section we will discuss in detail infrared and collinear singularity[13] that we used throughout the main part of this thesis. Fig. 1.3 shows the kinematics and notation for the splitting of parton a into b and c in the final state, *e.g.* a virtual quark splits into a real quark plus a real gluon.

For the propagator we have[14]

$$\text{propagator} \;=\; \frac{1}{(p+k)^2-m^2} \;=\; \frac{1}{2\,p\cdot k} \;=\; \frac{1}{2\,E_b\,E_c\,(1\;-\;A\,\cos\vartheta)} \tag{1.54}$$

We have to take the square of the amplitude and integrate over the final state phase space, all together, we get $d^3\vec{k}/E_b\cdot 1/E_b^2 \;\sim\; E_b\,dE_b\cdot 1/E_b^2 \;\sim\; dE_b/E_b$. When E_b goes to zero this corresponds to a soft singularity. For $m \;\to\; 0$ we have $A \;\to\; 1$ and $(1 \;-\; A\,\cos\vartheta)$ vanishes at $\cos\vartheta \;=\; 1$. This corresponds to a collinear mass singularity. However, infrared (soft plus

[13] We will explain this in more detail in Section 5.5 and Section 5.6, where we discussed soft and collinear photon radiations. The generalization to soft and collinear gluon radiations in QCD is straightforward, one can simply replace photon field with gluon field and take the colour factors and QCD coupling constant g_s into account, *i.e.* we make the following substitution for vertex:

$$-\,i\,e\,Q_f\,\gamma_\mu \;\to\; -\,i\,g_s\,t^A_{ab}\,\gamma_\mu\,.$$

[14] Here we choose $p \;=\; (\,E_c\,,\,\vec{p}_c\,)\,, k \;=\; (\,E_b\,,\,\vec{p}_b\,)$, hence $|\vec{p}_c| \;=\; E_c\,\sqrt{1\;-\;m^2/E_c^2} \;\equiv\; E_c\,A$ and $|\vec{p}_b| \;=\; E_b$. Note also that the notation p_b and k can be exchanged with each other without making any difference; *i.e.* we will use both notations interchangeably without being noticed. Similarly for p_c and p.

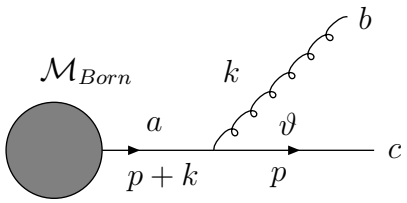


Figure 1.3: A final-state parton branching.

collinear) singularities cancel, if we add virtual diagrams (see *e.g.* Fig. 1.2). This is a consequence of the KLN theorem [10, 11].

In QCD there is a generic property, that the real emission amplitude ($(m+1)$-parton phase space) can be factorized into a Born-level amplitude (m-parton phase space) in the soft and collinear limits based on the factorization theorems

$$\boxed{\mathcal{M}_{m+1} \approx v_\ell \cdot \mathcal{M}_m} \tag{1.55}$$

where v_ℓ means the singular factor. One can also refer to Eq. (5.116) in appendix for soft-gluon approximation and Eq. (5.133) in appendix for collinear gluon emission[15], *e.g.* for a collinear final-state qqg parton branching (see Fig. 1.3) we have

$$\begin{aligned}\mathcal{M}_{m+1} &\sim \bar{u}(p)\left[-i\,g_s\,t^A_{ab}\,\gamma_\mu\right]\varepsilon^{*\mu}(k)\,\frac{i\not{p}_a}{p_a^2}\,\mathcal{A}(p_a) \sim \frac{1}{t}\,\bar{u}(p)\left[g_s\,t^A_{ab}\,\gamma_\mu\right]\varepsilon^{*\mu}(k)\sum_{\text{spins}} u(p_a)\,\underbrace{\bar{u}(p_a)\,\mathcal{A}(p_a)}_{=\mathcal{M}_m} \\ &\sim \sum_{\text{spins}}\frac{1}{t}\,\bar{u}(p)\,\left[g_s\,t^A_{ab}\,\gamma_\mu\right]\,\varepsilon^{*\mu}(k)\,u(p_a)\,\mathcal{M}_m \end{aligned} \tag{1.56}$$

where $\bar{u}(p)$ is the spinor of the fermion and $\mathcal{A}(p_a)$ the remaining part of the amplitude. Hence[16]

$$|v_\ell|^2 \sim \frac{1}{t}\,P_{qq}(z) \quad \text{with} \quad P_{qq}(z) = C_F\,\frac{1+z^2}{1-z} \tag{1.57}$$

[15] In Section 5.5 and Section 5.6 we discussed photon radiations; for the gluon radiations we simply make the following substitution for vertex:

$$-i\,e\,Q_f\,\gamma_\mu \;\to\; -i\,g_s\,t^A_{ab}\,\gamma_\mu\,.$$

[16] Here $P_{qq}(z)$ is the spin-averaged splitting function [6]. In the collinear limit, the radiation of additional partons is described by universal splitting functions, independent of the hard interaction. The Eq. (1.57) shows that the $m \to m+1$ amplitude factorizes into a soft/collinear part $1/t$ from the propagator, the splitting function $P_{qq}(z)$ and a hard $m \to m$ amplitude without IR singularities.

Here we have assumed that[17]

$$p_a^2 \,=\, (p_b + p_c)^2 \,\equiv\, t \,\gg\, p_b^2\,,\, p_c^2 \tag{1.58}$$

and the energy fraction is defined by

$$z \,=\, E_b/E_a \,=\, 1 - E_c/E_a \tag{1.59}$$

Hence for small angles we obtain

$$t \,\equiv\, p_a^2 \,=\, 2\,E_b\,E_c\,(1-\cos\vartheta) \,=\, z\,(1-z)\,E_a^2\,\vartheta^2 \tag{1.60}$$

The limit where p_b and p_c become collinear can be precisely defined using Sudakov decomposition

$$\begin{aligned}
p_b &= z\,p_a + p_T + \zeta_b\,n \\
p_c &= (1-z)\,p_a - p_T + \zeta_c\,n \\
2\,p_b\cdot p_c &= -\frac{p_T^2}{z\,(1-z)}\,, \qquad p_T \to 0 \\
p_b^2 = 0 \;&\Rightarrow\; \zeta_b \,=\, -\frac{p_T^2}{2\,z\,n\cdot p_a} \\
p_c^2 = 0 \;&\Rightarrow\; \zeta_c \,=\, -\frac{p_T^2}{2\,(1-z)\,n\cdot p_a}
\end{aligned} \tag{1.61}$$

where $p_T\cdot p_a \,=\, p_T\cdot n \,=\, n^2 \,=\, 0$. In Eq. (1.61), n^μ is an auxiliary lightlike vector, that is necessary to specify the transverse component p_T ($p_T^2 \,<\, 0$). We can split the angle ϑ for massless partons according to Fig. 5.2 of [109]

$$\vartheta \,=\, \vartheta_b + \vartheta_c \quad \text{and} \quad \frac{\vartheta_b}{\vartheta_c} \,=\, \frac{p_T}{E_b}\left(\frac{p_T}{E_c}\right)^{-1} \,=\, \frac{1-z}{z} \quad\Leftrightarrow\quad \vartheta \,=\, \frac{\vartheta_b}{1-z} \,=\, \frac{\vartheta_c}{z} \tag{1.62}$$

In order to calculate the cross section for the various splitting processes, we need to know the factorization of the phase space from $(m+1)$-parton phase space into m-parton phase space. We consider the multi-particle phase space decomposition (see also appendix). For the branching of parton a into $b+c$, we can decompose the collinear phase space according to[18]

$$\begin{aligned}
dPS_{m+1} &= dPS_m\,(2\,\pi)^4\,\delta^4\,(p_a - p_b - p_c)\;\frac{d^3\vec{p}_b}{(2\,\pi)^3\,2\,E_b}\,\frac{d^3\vec{p}_c}{(2\,\pi)^3\,2\,E_c}\,\frac{dp_a^2}{2\,\pi} \\
&= dPS_m\,\frac{1}{(2\,\pi)^3}\,\frac{1}{4\,E_b\,E_c}\,\delta^4\,(p_a - p_b - p_c)\;d^3\vec{p}_b\,d^3\vec{p}_c\,dt \\
&= dPS_m\,\frac{1}{2\,(2\,\pi)^3}\int E_b\,dE_b\,\vartheta_b\,d\vartheta_b\,d\varphi\,dt\,\frac{dz}{1-z}\,\delta(t - E_b\,E_c\,\vartheta^2)\,\delta(z - E_b/E_a) \\
&= dPS_m\,\frac{1}{4\,(2\,\pi)^3}\,dt\,dz\,d\varphi
\end{aligned} \tag{1.63}$$

[17] Here we choose $p_a \,=\, (\,E_a\,,\,\vec{p}_a\,)$.

[18] $d^3\vec{p}_c \,=\, d^3\vec{p}_a$ at fixed $\vec{p}_b$, $d^3\vec{p}_b \,=\, E_b^2\,dE_b\,\vartheta_b\,d\vartheta_b\,d\varphi$ and $\sin\vartheta_b \,\sim\, \vartheta_b$.

where φ is the azimuthal angle. Adding the matrix elements to this factorization of the collinear phase space we can derive the cross section for one collinear emission

$$\begin{aligned} d\sigma_{m+1} &\sim |\mathcal{M}_{m+1}|^2 \, dPS_{m+1} \\ &= |\mathcal{M}_{m+1}|^2 \, dPS_m \, \frac{1}{4\,(2\,\pi)^3} \, dt \, dz \, d\varphi \\ &= |\mathcal{M}_{m+1}|^2 \, dPS_m \, \frac{1}{4\,(2\,\pi)^2} \, dt \, dz \qquad \text{spherically symmetric} \\ &= \frac{2\,g_s^2}{t} \, P_{ab}(z) \, |\mathcal{M}_m|^2 \, dPS_m \, \frac{dt\,dz}{16\,\pi^2} \quad \text{assuming} \quad \boxed{|\mathcal{M}_{m+1}|^2 \;=\; \frac{2\,g_s^2}{t} \, P_{ab}(z) \, |\mathcal{M}_m|^2} \end{aligned} \tag{1.64}$$

Here we have neglected the initial-state flux factor $\mathcal{F}$. Using $d\sigma_m \sim |\mathcal{M}_m|^2 \, dPS_m$ we can write the most general form of Eq. (1.64)

$$\boxed{d\sigma_{m+1} \;=\; d\sigma_m \, \frac{dt}{t} \, dz \, \frac{\alpha_s}{2\,\pi} \, P_{ab}(z)} \tag{1.65}$$

The Eq. (1.65) means that we can calculate the $(m+1)$-particle cross section from the m-particle cross section convoluted with the universal splitting functions $P_{ab}(z)$. In Chapter 2 we will discuss various branching processes, *e.g.* a quark splitting into a quark and a gluon (qqg), a gluon splitting into two quarks $(gq\bar{q})$, a gluon splitting into two gluons (ggg) and derive alternative splitting functions in the final (initial) states using a slightly different kinematics and momentum mapping. Our new subtraction scheme is based on these improved splitting functions.

Chapter 2

Nagy-Soper dipoles

2.1 Motivation

The main topic of this thesis is the calculation of QCD cross sections in high energy hadron colliders or lepton colliders at Next-to-Leading Order (NLO) accuracy. For the LHC we will be faced with complex hadronic scatterings with many particles in the final state and we need to understand the standard model (SM) predictions precisely in order to dig out any signal from physics beyond the SM (BSM). Therefore, processes have in general to be calculated at least to NLO precision. Another reason why we bother with higher order calculations is that the naïve parton model picture corresponds to the Leading-Order (LO) approximation; and the LO calculations only predict the rough order of magnitude of a given cross section due to poor convergence of perturbative expansion; there is still a strong dependence on the unphysical input scales (renormalization and factorization scales). NLO QCD calculations can help us to reduce dependence on the renormalization scale for observables including $\alpha_s(\mu_F^2)$, which at the end leads to stable predictions for the cross sections. In higher order calculations, we have to consider real emission corrections and virtual contributions. Furthermore we will often be facing two different sources of singularities: ultraviolet (UV) divergences and infrared (IR) divergences; the UV singularities, which are only present in the virtual diagrams, can be removed by the standard renormalization procedure and infrared singularities (soft and collinear), which instead can appear both in the real and in the virtual contributions, also cancel when we sum over real and virtual contributions. That is a consequence of the KLN theorem. In general, when we compute cross sections with initial state hadrons; there are still left-over collinear singularities, which need to be factorized into the universal and process independent parton distribution functions (PDFs). As a result, cross sections are finite at the parton level order by order in perturbation theory. Recent progress for results at NLO have been available for all $2 \to 2$ and $2 \to 3$, and for some $2 \to 4$ processes at hadron colliders.

There are, generally speaking, two types of algorithms widely used for dealing with the infrared divergences in NLO QCD calculations: the phase space slicing and the subtraction methods. Now suppose that we are interested in calculating the integral

$$I = \lim_{\epsilon\to 0}\left(\int_0^1 \frac{dx}{x}\, x^{-\epsilon}\, M(x) + \frac{1}{\epsilon}\, M_0\right) \tag{2.1}$$

where $M(x)$ is a complicated function which is nonsingular at $x = 0$; and it depends on the hard scattering processes. The first term on the right-hand side can be thought of the contribution from real graphs and the second term plays the role of the contribution from virtual diagrams. Here x is analogous to the energy of a gluon, or to the angle between two partons. There is a singularity at $x = 0$. Using dimensional regularization by lowering the dimension to $d = 4 - 2\,\epsilon$, this singularity is regularized by a factor $x^{-\epsilon}$. The integral is divergent as $\epsilon \to 0$, but the divergence is cancelled by the second term $+\,(1/\epsilon)\, M_0$, which is a result of the KLN theorem. In this case KLN theorem also ensures that $\lim_{x\to 0} M(x) = M_0$. In practice, the function $M(x)$ could be very complicated for an increasing number of external particles in the final state. The question is how to calculate the value of I numerically (and efficiently) if the function $M(x)$ is too complicated such that an analytic computation cannot be allowed.

- **Slicing:** the widely used method is called the phase space slicing method. Introducing an arbitrary cutoff δ (we choose $\delta \ll 1$ and $\delta \gg |\epsilon|$), one can split the integration region into two pieces: $0 < x < \delta$ and $\delta < x < 1$. For the region $0 < x < \delta$, we can use the simple approximation that $M(x) \to M(0)$. This gives

$$\begin{aligned} I &= \int_\delta^1 \frac{dx}{x^{1+\epsilon}}\, M(x) + \int_0^\delta \frac{dx}{x^{1+\epsilon}}\, M(x) + \frac{1}{\epsilon}\, M_0 \;\simeq\; \int_\delta^1 \frac{dx}{x}\, M(x) + \frac{1}{\epsilon}\,\left(1 - \delta^{-\epsilon}\right)\, M_0 \\ &\simeq \int_\delta^1 \frac{dx}{x}\, M(x) + \ln(\delta)\, M_0 \end{aligned} \tag{2.2}$$

  Now the first integration can be integrated numerically in the respective Monte Carlo program. As long as δ is small, the result will be independent of δ. The details of this method are explained by Baer, Ohnemus and Owens [12] in the context of a calculation of photoproduction of jets. This method has also been applied to NLO calculations of three-jet cross sections in e^+e^- annihilation [13, 14].

- **Subtraction:** the idea was first used for QCD calculations of jet structure in e^+e^- annihilation by R. K. Ellis, Ross and Terrano [15], and later by Z. Kunszt and P. Nason [16]. The basic idea is that one can write

$$\begin{aligned} I &= \int_0^1 \frac{dx}{x^{1+\epsilon}}\, M(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}}\, M_0 + \int_0^1 \frac{dx}{x^{1+\epsilon}}\, M_0 + \frac{1}{\epsilon}\, M_0 \\ &= \int_0^1 dx\, \frac{M(x) - M_0}{x^{1+\epsilon}} + \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon}\right)\, M_0 \;\simeq\; \int_0^1 \frac{dx}{x}\, \left[M(x) - M_0\right] \end{aligned} \tag{2.3}$$

The integration can now be performed numerically by Monte Carlo integration. In summary, both the phase space slicing and the subtraction algorithms provide the foundation for setting up a Monte Carlo program, which can be used to implement arbitrary higher order QCD calculations in a given process. As for the subtraction algorithm, a general NLO formalism has been applied to calculate three-jet cross sections in e^+e^- annihilation and cross sections up to two-jet production in the final state at hadron colliders [17–19]; the algorithm of [17] has been modified to deal with three-jet cross sections at next-to-leading order [20]. This formalism is also applicable to n-jet production in e^+e^- annihilation and in hadron collisions. The treatment of massive partons has also been considered in the case of heavy quark correlations in hadron collisions at next-to-leading order [20, 21].

In recent years, an important calculational tool for the implementation of NLO QCD corrections in Monte Carlo style programs are dipole subtraction schemes [22–25]. The key point for the dipole subtraction method is that the QCD squared real-emission matrix element can be factorized into Born matrix element in the soft and collinear limits based on the factorization properties of QCD matrix elements [26, 27]. Dipole subtraction schemes introduce local counterterms, which mimic the behaviour of the real-emission matrix element in the singular limits. After standard UV-renormalization, the soft and collinear singularities then cancel when the integrated subtraction terms are added to the virtual cross section. Hence, the results to the NLO cross section are finite and the further phase-space integrations can be performed numerically by Monte Carlo techniques.

The various schemes [22,23,25] differ in the phase-space momentum mapping, which relates LO and NLO kinematics. In the standard scheme of Catani and Seymour [22], the universal local counterterms need to be re-calculated for each emitter/spectator pair[1]. Therefore, this scheme suffers from a large number of momentum mappings needed to evaluate the subtraction terms. Basically, the number of momentum mappings scales like N^3 for a LO $2 \to N$ process. This scaling leads to a rapidly rising number of momentum mappings for a large number of external particles in the final state. Following an approach suggested by Zoltan Nagy and Dave Soper [28–30], we employ a subtraction scheme with a slightly altered momentum mapping, such that the number of kinematic transformations is greatly reduced. Basically, the number of mappings scales like N^2 for a LO $2 \to N$ process, thereby reducing the number of matrix element computations by a factor of N. In addition, the dipole subtraction terms in this alternative scheme are based on splitting functions which have been proposed in the context of an improved parton shower formulation including quantum interference effects. Hence, the new scheme facilitates numerical implementations of higher order corrections in Monte Carlo Event Generators and also allows for easy matching with a parton shower using the same splitting functions.

[1]The number of momentum mappings = number of emitters.

We begin in Section 2.2 by giving a brief overview of the general subtraction procedure. In Section 2.3 we discuss the general framework setup and the momentum mapping between m- and $(m+1)$-particle phase space. In Section 2.4 we will give the explicit expressions of splitting functions for each process in both the initial and final states, as well as the eikonal splitting functions and soft splitting functions. In Section 2.5 we will show the complete integrated splitting functions including collinear and soft integrals. In Chapter 3 we will show our first applications to NLO processes at hadron and lepton colliders. Finally we will summarize in Chapter 4.

2.2 General structure of the NLO cross section and subtraction procedure

2.2.1 The general subtraction procedure

In this Section we explain the general subtraction procedure for calculating NLO cross sections at lepton and hadron colliders. Suppose that we want to calculate the jet quantity σ at NLO accuracy

$$\sigma \;=\; \sigma^{LO} + \sigma^{NLO} \tag{2.4}$$

Suppose also that there are m partons in the final state at LO, then we have

$$\sigma^{LO} \;=\; \int_m d\sigma^B \tag{2.5}$$

Here, $d\sigma^B$ is the Born-level cross section, which can be symbolically written as

$$d\sigma^B \;=\; dPS_m \; |\mathcal{M}_m|^2 \; F_J^{(m)} \tag{2.6}$$

where dPS_m denotes the phase space of m particles in four dimensions, $\mathcal{M}_m$ is the matrix element and $F_J^{(m)}$ is a function of cuts defining the jet observables, which we will discuss in Section 2.2.3. By definition, the LO cross section is finite so that Eq. (2.5) can be integrated (analytically or numerically) in four dimensions.

At NLO, we have to consider both the real and virtual contributions. There are $m+1$ partons in the final state for the real emission and m partons in the final state for the virtual one-loop correction. So we can write

$$\sigma^{NLO} \;=\; \int d\sigma^{NLO} \;=\; \int_{m+1} d\sigma^R \;+\; \int_m d\sigma^V \tag{2.7}$$

The first integral on the right-hand side of Eq. (2.7) is the contribution from real diagrams, which contains IR divergences, and the second integral on the right-hand side of Eq. (2.7) is the contribution from virtual diagrams, which contains both UV and IR divergences. A traditional way

for dealing with IR singularities is by introducing an infinitesimal regulator, *e.g.* by lowering the dimension to $d = 4 - 2\,\epsilon$, the so-called dimensional regularization scheme, in which the Feynman diagrams are computed in d dimensions and the singularities in the integral can be extracted as double (soft and collinear) poles $1/\epsilon^2$ and single (soft, collinear or UV) poles $1/\epsilon$. Here we suppose that one has already performed the renormalization procedure in $d\sigma^V$ so that all its UV singularities have been removed. This way, the analytic cancellation of the respective divergent parts for fully inclusive measurements is straightforward. However, numerical implementations of parts containing infinitesimal regulators for multi-particle processes proved to be challenging. In subtraction schemes, the difficulty is circumvented by introducing universal local counterterms (or dipole terms), which mimic the behaviour of the squared real emission matrix elements in the singular regions; adding back the respective one particle integrated counterparts to the virtual contributions results in finite integrands for both real contribution ($m+1$-particle phase space) and virtual correction (m-particle phase space). Symbolically, we write

$$d\sigma^{NLO} \;=\; \left[d\sigma^R - d\sigma^A\right] \;+\; \left[d\sigma^A \;+\; d\sigma^V\right] \tag{2.8}$$

where $d\sigma^A$ is regarded as a local counterterm (or dipole), which mimics the singular behaviour of $d\sigma^R$. In the next step we introduce the phase space integration. One can safely perform the limit $\epsilon \to 0$ in the first term on the right-hand side of Eq. (2.8) by definition. For the second term on the right-hand side of Eq. (2.8), one can carry out analytically the integrated dipole term $d\sigma^A$ over the emitted one parton phase space, leading to the poles in ϵ that are necessary to cancel the soft singularities in virtual one-loop cross section. Thus we can perform the limit $\epsilon \to 0$ after all the divergences are cancelled. The final structure of the calculation is given by

$$\begin{aligned}
\sigma^{NLO} \;&=\; \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^A\right]}_{\text{finite}} + \underbrace{\int_{m+1} d\sigma^A + \int_m d\sigma^V}_{\text{finite}} \\
&=\; \int_{m+1} \left[d\sigma^R_{\epsilon=0} - d\sigma^A_{\epsilon=0}\right] + \int_m \left[\int_1 d\sigma^A + d\sigma^V\right]_{\epsilon=0}
\end{aligned} \tag{2.9}$$

Both integrands are now finite, meaning that we can perform all integrations numerically in the respective Monte Carlo program. The explicit expressions of the cross section σ for m and $m+1$ particle contributions to the total NLO cross section are given by

$$\begin{aligned}
\int_m \left[d\sigma^B \;+\; d\sigma^V \;+\; \int_1 d\sigma^A\right] &= \int dPS_m \left[|\mathcal{M}_m|^2 \;+\; |\mathcal{M}_m|^2_{\text{one-loop}} \;+\; \sum_\ell \mathfrak{B}_\ell \; |\mathcal{M}_m|^2\right] \\
\int_{m+1} \left[d\sigma^R - d\sigma^A\right] &= \int dPS_{m+1} \left[|\mathcal{M}_{m+1}|^2 \;-\; \sum_\ell \mathcal{D}_\ell \; |\mathcal{M}_m|^2\right]
\end{aligned} \tag{2.10}$$

Here, $\epsilon \to 0$ in the $m+1$ particle phase space is always understood. Convolution with jet functions then ensures that the σ is infrared and collinear safe and that the Born-level contribution

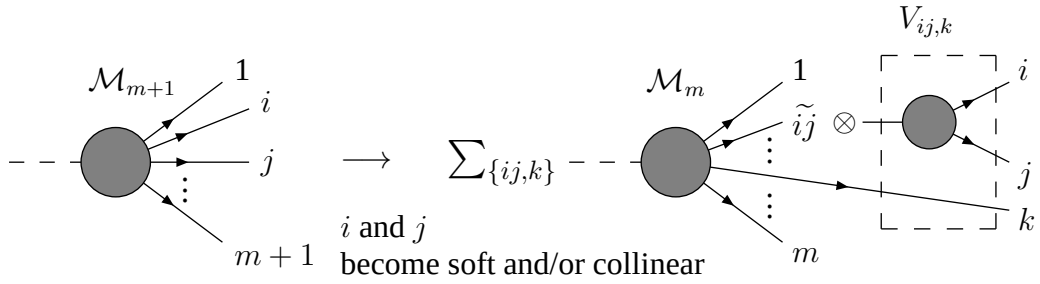


Figure 2.1: Dipole factorization procedure [22]. When the partons i and j become soft and/or collinear, the $(m+1)$-parton matrix element factorizes into the m-parton matrix element times a singular factor $V_{ij,k}$ which in our notation is denoted with $v_\ell(\{\hat{p}\}_{m+1})$.

is well defined. In Eq. (2.10), the sum goes over all local counterterms needed to match the complete singularity structure of the real emission contribution. For each singular limit, *i.e.* when two partons become collinear or when one parton becomes soft, the real emission matrix element factorizes into the Born-level matrix element according to[2] (Fig. 2.1)

$$\mathcal{M}_{m+1}(\{\hat{p}\}_{m+1}) \longrightarrow \sum_\ell v_\ell(\{\hat{p}\}_{m+1}) \otimes \mathcal{M}_m(\{p\}_m) \qquad \text{and} \qquad \mathcal{D}_\ell \propto v_\ell^2 \tag{2.11}$$

where $\mathcal{D}_\ell$ denote the dipoles containing the respective singularity structure. Here $\mathcal{D}_\ell$ is just a symbolic notation. The explicit expressions for each splitting process will be given in Section 2.4. The symbol $\otimes$ represents properly defined phase-space, spin and colour convolutions. $\hat{p}$ and p represent momenta in $(m+1)$- and m-parton phase spaces, respectively. As $|\mathcal{M}_{m+1}|^2$ and $|\mathcal{M}_m|^2$ live in different phase spaces, a mapping of the respective momenta from $(m+1)$- to m-particle phase space needs to be introduced, which is defined by a mapping function F_{map} according to $p = F_{\text{map}}(\hat{p})$. $\mathcal{D}_\ell$ and $\mathfrak{B}_\ell$ are related by $\mathfrak{B}_\ell = \int d\zeta_p \, \mathcal{D}_\ell$, where $d\zeta_p$ is an unresolved one parton integration measure. In summary, any subtraction scheme needs to fulfill the following requirements:

- dipole subtraction terms $\mathcal{D}_\ell$ must match the behavior of the real emission matrix element in each soft and collinear region.
- the mapping function F_{map} guarantees total energy momentum conservation as well as the on-shell condition for all external particles both before and after the mapping.

[2] In Fig. 2.1 we follow Catani-Seymour's notation to explain the dipole factorization procedure; it is worth mentioning that in this thesis we used ℓ for the mother parton instead of $\widetilde{ij}$, which Catani and Seymour used.

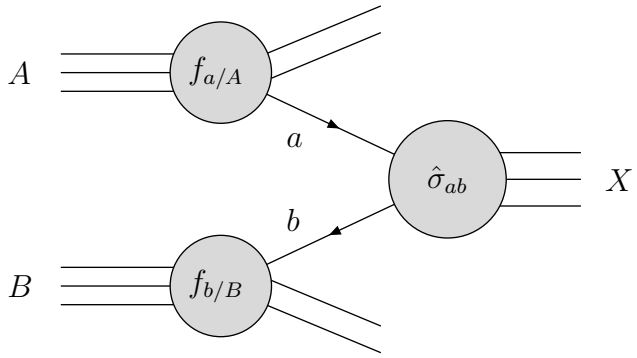


Figure 2.2: Hadron collision.

- subtraction terms have to lead to correct IR poles when carrying out the analytical integration over the one parton phase space in d dimensions that are necessary to cancel the soft singularities in virtual one-loop matrix element.

2.2.2 Generalization to hadron collisions

Now we consider the cross sections in hadron collisions (Fig. 2.2). In the case of processes with two initial-state hadrons A and B carrying momenta p_A and p_B, respectively, the calculation of the QCD cross sections must be convoluted with parton distribution functions (PDFs):

$$\sigma(p_A, p_B) = \sum_{a,\,b} \int_0^1 d\eta_a \, f_{a/A}(\eta_a, \mu_F^2) \int_0^1 d\eta_b \, f_{b/B}(\eta_b, \mu_F^2) \left[\sigma_{ab}^{LO}(p_a, p_b) + \sigma_{ab}^{NLO}(p_a, p_b, \mu_F^2)\right] + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^n}{Q^n}\right) \tag{2.12}$$

where $p_a = \eta_a \, p_A$ and $p_b = \eta_b \, p_B$ are parton momenta, while η_a and η_b are the momentum fractions of the partons. The parton distribution functions $f_{a/A}(\eta_a, \mu_F^2)$ gives the probability of finding parton a inside hadron A with momentum fraction η_a when the hadron is probed at the scale μ_F. In general, the hard scattering cross sections σ^{LO} and σ^{NLO} depend on $\alpha_s(\mu_F^2)$ and the ratio Q^2/μ_F^2. The parameter μ_F is the factorization scale at which long distance physics (PDFs) and short distance process (hard scattering cross sections) can be separately treated. The scale μ_F is arbitrary that is introduced in order to renormalize the UV divergences after quantization,

and physical results cannot depend on it. The corresponding parton level cross sections are:

$$\begin{aligned}\sigma_{ab}^{LO}(p_a,p_b) &= \int_m d\sigma_{ab}^B(p_a,p_b) \\ \sigma_{ab}^{NLO}(p_a,p_b,\mu_F^2) &= \int_{m+1} d\sigma_{ab}^R(p_a,p_b) + \int_m d\sigma_{ab}^V(p_a,p_b) + \int_m d\sigma_{ab}^C(p_a,p_b,\mu_F^2)\end{aligned} \tag{2.13}$$

In general, the initial-state collinear singularities in hadron collisions do not cancel. However, they are universal and process-independent in all orders in perturbation theory. Therefore they can be cancelled by universal collinear counterterms $d\sigma_{ab}^C$, which are generated by the renormalization of the PDFs of the incoming particles. The explicit form is given by[3]:

$$\begin{aligned}\int_m d\sigma_{ab}^C(p_a,p_b,\mu_F^2) = &\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\sum_c \int_0^1 dx \int_m d\sigma_{cb}^B(xp_a,p_b)\,\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon} P^{ac}(x) \\ &+ \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\sum_c \int_0^1 dx \int_m d\sigma_{ac}^B(p_a,xp_b)\,\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon} P^{bc}(x)\end{aligned} \tag{2.14}$$

Here, the $P^{ab}(x)$ are the Altarelli-Parisi kernels in four dimensions. The collinear counterterm depends on the factorization scheme. Here we have chosen the most commonly used $\overline{\text{MS}}$ scheme. As in the case of UV renormalization, the full hadronic cross section is unaltered under a change of the factorization scheme, provided that the PDFs are also suitably changed.

For processes with incoming hadrons, the subtraction procedure is applied to $\sigma_{ab}^{NLO}(p_a,p_b,\mu_F^2)$ as previously described and we can write it as follows

$$\begin{aligned}\sigma_{ab}^{NLO}(p_a,p_b,\mu_F^2) = &\int_{m+1}\left[d\sigma_{ab}^R(p_a,p_b) - d\sigma_{ab}^A(p_a,p_b)\right] \\ &+ \int_m \left[\int d\sigma_{ab}^V(p_a,p_b) + \int_1 d\sigma_{ab}^A(p_a,p_b) + d\sigma_{ab}^C(p_a,p_b,\mu_F^2)\right]_{\epsilon=0}\end{aligned} \tag{2.15}$$

where $\int_1 d\sigma_{ab}^A + d\sigma_{ab}^C$ can be written as

$$\begin{aligned}&\int_m \left[\int_1 d\sigma_{ab}^A(p_a,p_b) + d\sigma_{ab}^C(p_a,p_b,\mu_F^2)\right] \\ &= \int_m d\sigma_{ab}^B(p_a,p_b)\otimes I(\epsilon) + \int_0^1 dx \int_m d\sigma_{ab}^B(xp_a,p_b)\otimes\left[K^a(xp_a) + P(x,\mu_F^2)\right] \\ &+ \int_0^1 dx \int_m d\sigma_{ab}^B(p_a,xp_b)\otimes\left[K^b(xp_b) + P(x,\mu_F^2)\right]\end{aligned} \tag{2.16}$$

This equation defines the insertion operators $I(\epsilon), K(x), P(x,\mu_F^2)$ at the cross section level, where we follow the standard notation introduced in [22]. The symbol $\otimes$ denotes all possible spin, colour and flavour correlations. Eq. (2.16) can be divided into two parts: the first part

[3]Here the splitting functions P_{ab} and P^{ab} can be exchanged with each other without making any difference, *i.e.* $P_{ab} = P^{ab}$ in the later discussions.

is the universal insertion operator $I(\epsilon)$, which contains all the poles in ϵ that are necessary to cancel the soft singularities in the virtual cross section. The universal insertion operator has the LO kinematics. The second part consists of the finite pieces that are left over after absorbing the initial-state collinear singularities into a redefinition of the parton distribution functions at NLO. It involves an additional one dimensional integration over the momentum fraction x of an incoming parton with the LO cross sections and the x-dependent structure functions.

2.2.3 Observable-dependent formulation of the subtraction method

The jet observables which we are interested in should be well defined experimentally or theoretically in such a way, that the cross sections are infrared and collinear safe. In general, the jet function may contain θ functions (which define cuts and corresponding cross sections) and δ functions (which define differential cross sections). To be more specific, we consider the following expressions for Born-level and the corresponding NLO cross sections

$$\begin{aligned}
\sigma^{LO} &= \int dPS_m(p_1,\cdots,p_m)\ |\mathcal{M}_m(p_1,\cdots,p_m)|^2\ F_J^{(m)}(p_1,\cdots,p_m) \\
\sigma^{NLO} &= \int dPS_{m+1}(p_1,\cdots,p_{m+1})\ |\mathcal{M}_{m+1}(p_1,\cdots,p_{m+1})|^2\ F_J^{(m+1)}(p_1,\cdots,p_{m+1}) \\
&\quad + \int dPS_m(p_1,\cdots,p_m)\ |\mathcal{M}_m(p_1,\cdots,p_m)|^2_{\text{one-loop}}\ F_J^{(m)}(p_1,\cdots,p_m)
\end{aligned} \tag{2.17}$$

where $|\mathcal{M}_m|^2$, $|\mathcal{M}_{m+1}|^2$ and $|\mathcal{M}_m|^2_{\text{one-loop}}$ are the squared LO matrix element, the squared NLO real emission matrix element and the squared NLO virtual matrix element, respectively. There is a formal requirement on the jet function $F_J^{(m)}$. For cases where two partons become collinear or where one parton becomes soft, the function $F_J^{(m+1)}$ reduces to $F_J^{(m)}$, *i.e.* in the soft and collinear limits, the jet function fulfils the following properties

$$\begin{aligned}
F_J^{(m+1)}(p_1,\cdots,p_j=\lambda\,q,\cdots,p_{m+1}) &\to F_J^{(m)}(p_1,\cdots,p_{m+1}) \quad \text{if} \quad \lambda\to 0 \\
F_J^{(m+1)}(p_1,..,p_i,..,p_j,..,p_{m+1}) &\to F_J^{(m)}(p_1,..,p,..,p_{m+1}) \quad \text{if} \quad p_i\to z\,p,\ p_j\to(1-z)\,p \\
F_J^{(m)}(p_1,\cdots,p_m) &\to 0 \quad \text{if} \quad p_i\cdot p_j\to 0
\end{aligned} \tag{2.18}$$

The first two conditions of Eq. (2.18) define the essential property of the jet function that the jet observable has to be infrared and collinear safe for any number m of partons in the final state, *i.e.* to any order in QCD perturbation theory. The last condition of Eq. (2.18) guarantees that the Born-level cross section is well defined. To summarise, we require that

$$F_J^{(m+1)}\ \to\ F_J^{(m)}$$

in the singular limits.

2.3 Scheme setup and momentum mapping

In the scheme presented here, the NLO dipole subtraction terms are derived from the splitting functions introduced in [28]. The subtraction scheme is based on a physical picture that partons can split or join. For parton splitting, one of the m partons splits, producing $m+1$ partons in the final state. For parton joining, two partons can join, reducing the $m+1$ partons back into m partons. Parton splitting is needed to describe a parton shower, while parton joining is relevant in NLO QCD calculations. A momentum mapping function is needed to relate different phase spaces, and the mappings from $m+1$ to m partons needed correspond to the inverse transformation of the respective shower m to $m+1$ partons mappings.

In the following, the partons are labelled by an index that takes values a or b for the initial state partons and $1, 2, \cdots, m$ for the final state partons. We will denote an m parton state by momenta $\{p\}_m$. The partons will be labelled by $\{a, b, 1, 2, \cdots, m\}$ with momenta

$$\{p_a, p_b, p_1, p_2, \cdots, p_m\}$$

After the splitting, we have an $m+1$ parton state with momenta $\{\hat{p}\}_{m+1}$. The momenta are labelled by

$$\{\hat{p}_a, \hat{p}_b, \hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{m+1}\}$$

Now suppose that partons ℓ and j are produced by the splitting of a mother parton. Here the mother parton is in an m-parton state $\{p\}_m$, while partons ℓ and j are in an $(m+1)$-parton state $\{\hat{p}\}_{m+1}$. There are two situations. For a final state splitting, ℓ and j are in the $\{\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{m+1}\}$ configuration. The mother parton emerges from the hard interaction and splits into partons ℓ and j. In this case, the momentum of the mother parton is labelled by $p_\ell \in \{p_1, p_2, \cdots, p_m\}$ while the momenta of partons ℓ and j are labelled by $\hat{p}_\ell$ and $\hat{p}_j$, respectively. For an initial state splitting, j is in the $\{\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{m+1}\}$ configuration and ℓ is in the $\{\hat{p}_a, \hat{p}_b\}$ configuration. Parton ℓ splits into parton j and an initial state parton that enters the hard interaction. Hence the momentum of mother parton is labelled by p_a (or p_b) while the momenta of partons ℓ and j are labelled by $\hat{p}_a$ (or $\hat{p}_b$) and $\hat{p}_j$, respectively. Our description for an initial state splitting follows backwards evolution, in which the initial state parton that enters the hard interaction is the mother parton. In summary, the notation is that mother parton ℓ splits into partons with labels ℓ and j for both initial and final state splittings, while the other partons keep their labels. Following this rule, our convention throughout will be that in an $(m+1)$-parton state, $\hat{p}_\ell$ is the emitter, $\hat{p}_j$ the emitted parton, and $\hat{p}_k$ the spectator[4].

The momenta $\{\hat{p}\}_{m+1}$ after splitting are determined by the momenta $\{p\}_m$ and a momentum splitting variable ζ_p, which defines the momenta of the daughter partons. Here we consider an

[4] In contrast to [22], in our case a spectator only needs to be specified if $\hat{p}_j$ denotes a gluon.

example: when we join two partons ℓ and j, the momentum of mother parton, p_ℓ, is approximately $p_\ell \approx \hat{p}_\ell + \hat{p}_j$ for a final state splitting, and $p_\ell \approx \hat{p}_\ell - \hat{p}_j$ for an initial state splitting. As stated previously, hatted and unhatted momenta correspond to $m+1$ and m-parton phase spaces, respectively. The momenta of the other partons, which we can denote by p_n, are approximately unchanged, meaning that $p_n \approx \hat{p}_n$ for $n \notin \{\ell, j\}$. However, these relations cannot be exact because the momenta after parton joining should be on shell. In order for the mother parton momentum p_ℓ to be on-shell, we have to take some momenta from the spectators, so $p_n \neq \hat{p}_n$ and $p_\ell \neq \hat{p}_\ell \pm \hat{p}_j$. So the definition of a mapping $\{p\}_m \leftrightarrow \{\hat{p}\}_{m+1}$ should guarantee that all external partons are on-shell, as well as total energy momentum conservation.

There are many ways to define the momentum mapping. The most widely used scheme is that of Catani and Seymour [22]. This scheme may be called a local mapping. They define a emitter/spectator pair, the momentum fraction goes to one additional parton only. Hence, the momenta of the remaining partons are left unaltered. In this scheme, the momentum mapping follows the rule that two partons are mapped into three partons according to

$$(p_\ell, p_k) \leftrightarrow (\hat{p}_\ell, \hat{p}_j, \hat{p}_k)$$

for each emitter/spectator pair. Here the color connection between a spectator and an emitter has to be considered. The antenna factorization [31, 32] also uses a local mapping. In the new scheme, we apply a global mapping, in which the mapping takes all the partons into account at once when going from $(m+1)$- to m-particle phase space, instead of separately summing over all possible emitter/spectator pairs. We will restrict our expressions to subtractions on the parton level and to the massless case; details on convolution with PDFs are given in [28].

In the following subsections, we will first describe the final state splitting, then continue with the more complicated initial state splitting case. For the final state splitting we first show how $\{\hat{p}\}_{m+1}$ is obtained from $\{p\}_m$ and $\{\zeta_p\}$, then we reverse the transformation from $\{\hat{p}\}_{m+1}$ to $\{p\}_m$ and $\{\zeta_p\}$. In the same way we will derive the initial state splitting.

2.3.1 Splitting a final state parton

Parton splitting

We will neglect parton masses in the kinematics. Suppose the mother parton ℓ with momentum p_ℓ emerges from the hard interaction and splits into daughter partons ℓ and j with momenta $\hat{p}_\ell$ and $\hat{p}_j$, respectively. The on-shell condition ensures that $p_\ell^2 = \hat{p}_\ell^2 = \hat{p}_j^2 = 0$. We always have $(\hat{p}_\ell + \hat{p}_j)^2 \geq 0$. For a final state splitting, we can always leave the momenta of the initial state partons unchanged:

$$p_a = \hat{p}_a, \quad p_b = \hat{p}_b \tag{2.19}$$

Let Q be the total momentum of the final state partons

$$Q \equiv \sum_{n=1}^{m} p_n = p_a + p_b \tag{2.20}$$

Here the momenta of the incoming partons remain the same, hence $\hat{Q} = \hat{p}_a + \hat{p}_b$ is the same as Q. We define

$$a_\ell = \frac{Q^2}{2\, p_\ell \cdot Q} \tag{2.21}$$

Note that $a_\ell \geq 1$. The transformation also has to keep all of the momenta on-shell. In order to define the momentum mapping for the final state splitting, we first parametrize the total momenta of the two daugther partons $\hat{p}_\ell$ and $\hat{p}_j$ according to

$$P_\ell = \hat{p}_\ell + \hat{p}_j = \lambda\, p_\ell + \frac{1 - \lambda + y}{2\, a_\ell}\, Q \tag{2.22}$$

There are two parameters in this definition: y and λ. y is the measure for the virtuality of the splitting and λ is a function of y that we will determine later. In order for the mother parton momentum p_ℓ to be on-shell as well as to preserve momentum conservation, we have to take some momentum fractions from the final state spectators. In contrast to the Catani-Seymour scheme, in which only a single spectator parton in the final state donates the needed momentum, we can choose that all of the final state spectator partons, except partons ℓ and j, donate a momentum fraction. So the needed momentum from all of the final state spectators can be obtained by a Lorentz transformation, which relates the momenta after and before the splitting,

$$\hat{p}_n^\mu = \Lambda(\hat{K}, K)^\mu{}_\nu\, p_n^\nu, \quad n \notin \{\ell, j = m+1\} \tag{2.23}$$

Here, K is the total momentum of the final state spectators before the splitting

$$K = Q - p_\ell \tag{2.24}$$

and $\hat{K}$ is the total momentum of the final state spectators after the splitting

$$\hat{K} = Q - P_\ell \tag{2.25}$$

Since each final state spectator is changed by a Lorentz transformation, we have

$$\hat{K}^\mu = \Lambda^\mu{}_\nu\, K^\nu \tag{2.26}$$

The Lorentz transformation is given by

$$\Lambda(\hat{K}, K)^\mu{}_\nu = g^\mu{}_\nu - \frac{2\,(\hat{K} + K)^\mu\,(\hat{K} + K)_\nu}{(\hat{K} + K)^2} + \frac{2\,\hat{K}^\mu\, K_\nu}{K^2} \tag{2.27}$$

which automatically implies

$$\hat{K}^2 \;=\; K^2 \tag{2.28}$$

In a particular case in which there are only two massless partons in the final state, this corresponds to $a_\ell \;=\; 1$ and $K^2 \;=\; 0$. In this case, an alternative representation of the Lorentz transformation has to be introduced so that the boost in Eq. (2.27) still remains well defined when $K^2 \;=\; 0$.

$$\Lambda(\hat{K},K)^\mu{}_\nu \;=\; g^\mu{}_\nu + \left(\frac{K\cdot n}{\hat{K}\cdot n} - 1\right) n^\mu\,\bar{n}_\nu + \left(\frac{\hat{K}\cdot n}{K\cdot n} - 1\right) \bar{n}^\mu\, n_\nu \tag{2.29}$$

where n and $\bar{n}$ are lightlike vectors in the Q-p_ℓ plane with $n\cdot\bar{n} \;=\; 1$ and $(p_\ell\cdot n/p_\ell\cdot\bar{n}) < (Q\cdot n/Q\cdot\bar{n})$.

The parameters λ and y can be determined from Eq. (2.28).

$$\lambda \;=\; \sqrt{(1+y)^2 - 4\,a_\ell\,y}, \qquad y \;=\; \frac{\hat{p}_\ell\cdot\hat{p}_j}{p_\ell\cdot Q} \tag{2.30}$$

There is a maximum value of y corresponding to $\lambda = 0$

$$y_{\text{max}} \;=\; \left(\sqrt{a_\ell} - \sqrt{a_\ell - 1}\right)^2 \;=\; 2\,a_\ell \;-\; 1 \;-\; 2\,\sqrt{a_\ell\,(a_\ell-1)} \tag{2.31}$$

Another important relation, connecting hatted and unhatted quantities, is given by

$$p_\ell\cdot Q \;=\; (\hat{p}_\ell + \hat{p}_j)\cdot Q - \hat{p}_\ell\cdot\hat{p}_j \tag{2.32}$$

which can also be rewritten as

$$2\,P_\ell\cdot Q \;=\; (1+y)\,2\,p_\ell\cdot Q \tag{2.33}$$

As stated previously, the momenta $\{\hat{p}\}_{m+1}$ after splitting are determined by the momenta $\{p\}_m$ and a momentum splitting variable ζ_p. It will be convenient to define the daughter parton momenta by

$$\zeta_p \;\equiv\; (\hat{p}_\ell, \hat{p}_j) \tag{2.34}$$

Hence, the momentum mapping from the m to the $m+1$ particle phase space is given by a transformation[5]

$$\boxed{\{\hat{p},\hat{f}\}_{m+1} \;=\; R_\ell\left(\{\{p,f\}_m, \{\zeta_p,\zeta_f\}\}\right)} \tag{2.35}$$

where f denotes the flavour of each parton: $f \;\in\; \{g,u,\bar{u},d,\bar{d},\cdots\}$. Here, the splitting variable ζ_f is given by the flavours of the daughter partons, so we have

$$\zeta_f \;=\; (\hat{f}_\ell, \hat{f}_j) \tag{2.36}$$

[5]More precisely, after the splitting $\{\hat{p},\hat{f}\}_{m+1}$ is determined from $\{p,f\}_m$ and splitting variable $\{\zeta_p,\zeta_f\}$, where ζ_p denotes the daughter momenta and ζ_f denotes the daughter flavours.

The flavours of the spectator partons remain unchanged

$$\hat{f}_n \;=\; f_n, \quad n \notin \{\ell, j = m+1\} \tag{2.37}$$

while the flavour of the mother parton f_ℓ obeys

$$\hat{f}_\ell + \hat{f}_j \;=\; f_\ell \tag{2.38}$$

e.g. if the mother parton ℓ is a quark/antiquark, then we have $\zeta_f \;=\; (q/\bar{q}, g)$. If the mother parton ℓ is a gluon, then ζ_f can be a pair of gluons $\zeta_f \;=\; (g, g)$, which corresponds to $g \to g\,g$ splitting, or any choice of quark/antiquark flavours $\zeta_f \;=\; (q, \bar{q})$, which corresponds to $g \to q\,\bar{q}$ splitting[6].

Parton joining

There is an inverse transformation of Eq. (2.35), which maps the $(m+1)$-parton momenta to the m-parton momenta. So we start with $\{\hat{p}\}_{m+1}$ and determine $\{p\}_m$ and $\{\zeta_p\}$. The splitting variable for the momenta is given by the momenta of the daughter partons $\zeta_p \;\equiv\; (\hat{p}_\ell, \hat{p}_j)$. Let Q be the total momentum of the final state partons:

$$Q \;=\; \sum_{n=1}^{m+1} \hat{p}_n \;=\; \hat{p}_a + \hat{p}_b \tag{2.39}$$

One can determine the lightlike momentum p_ℓ by rearranging Eq. (2.22)

$$p_\ell \;=\; \frac{1}{\lambda}\,(\hat{p}_\ell + \hat{p}_j) - \frac{1-\lambda+y}{2\,\lambda\,a_\ell}\,Q \tag{2.40}$$

Here, the parameter a_ℓ still depends on the mother parton momentum p_ℓ, which can be mapped into her daughters $\hat{p}_\ell$ and $\hat{p}_j$ through Eq. (2.32). Now we need the inverse Lorentz transformation to Eq. (2.23). All non-emitting final state spectators are mapped using the following Lorentz transformation. From $K \;=\; Q \;-\; p_\ell$ and $\hat{K} \;=\; Q - P_\ell$, we have

$$p_n^\mu \;=\; \Lambda(K, \hat{K})^\mu{}_\nu\, \hat{p}_n^\nu, \quad n \notin \{\ell, j = m+1\} \tag{2.41}$$

where $\Lambda(K, \hat{K})^\mu{}_\nu$ can be obtained using Eq. (2.27) with $\hat{K}$ and K interchanged. Now the inverse transformation from the $m+1$ to the m particle phase space is given by

$$\boxed{\{\{p, f\}_m, \{\zeta_p, \zeta_f\}\} \;=\; Q_\ell\,(\{\hat{p}, \hat{f}\}_{m+1})} \tag{2.42}$$

[6]For final state $q \to q\,g$ or $\bar{q} \to \bar{q}\,g$ splittings, we use j for the label of the gluon. For a final state $g \to q\,\bar{q}$ splitting, we use j for the label of the $\bar{q}$.

The transformation of the flavours is similar to the case of parton splitting. The splitting variable ζ_f is given by the flavours of the daughter partons $\zeta_f = (\hat{f}_\ell, \hat{f}_j)$. The flavour of the mother parton f_ℓ is given by

$$f_\ell = \hat{f}_\ell + \hat{f}_j \tag{2.43}$$

with the rule of adding flavours, $q + g = q$ and $q + \bar{q} = g$. The flavours of the spectators remain unchanged

$$f_n = \hat{f}_n, \quad n \notin \{\ell, j = m + 1\} \tag{2.44}$$

In summary, it is the transformation R_ℓ that is needed to describe a parton shower, while Q_ℓ is needed to describe the NLO QCD calculations.

The integration measure for final state splitting

In order to calculate the various splitting processes and extract the correct singularities in ϵ, we need to know the factorization of the phase space from $m + 1$ to m partons

$$\int d\{\hat{p}, \hat{f}\}_{m+1}\; g(\{\hat{p}, \hat{f}\}_{m+1}) = \int d\{p, f\}_m\; d\zeta_p\, g(\{\hat{p}, \hat{f}\}_{m+1}) \tag{2.45}$$

where $g(\{\hat{p}, \hat{f}\}_{m+1})$ is an arbitrary function. The definition of the unresolved one parton integration measure is [28]

$$\begin{aligned} d\zeta_p = {} & dy\, \theta(y_{\text{min}} < y < y_{\text{max}})\, \lambda^{d-3}\, \frac{p_\ell \cdot Q}{\pi}\, \frac{d^d\hat{p}_\ell}{(2\,\pi)^d}\, 2\,\pi\, \delta^+(\hat{p}_\ell^2)\, \frac{d^d\hat{p}_j}{(2\,\pi)^d}\, 2\,\pi\, \delta^+(\hat{p}_j^2) \\ & \times (2\,\pi)^d\, \delta^d\left(\hat{p}_\ell + \hat{p}_j - \lambda\, p_\ell - \frac{1 - \lambda + y}{2\, a_\ell}\, Q\right) \end{aligned} \tag{2.46}$$

Here, $y_{\text{min}} = 0$ for massless partons and y_{max} is given by Eq. (2.31). The final expressions in terms of the integration variables (see Section 2.4.3) can be found in Section 5.2.1.

2.3.2 Splitting an initial state parton

Parton splitting

For an initial state splitting, we follow the backwards evolution description that an initial state daughter parton ℓ splits into an initial state mother parton that enters the hard part of the process and a final state daughter parton j when going forward in time, *i.e.* $\hat{p}_a \to p_a + \hat{p}_j$. For simplicity we have chosen the convention that

$$\ell = a \tag{2.47}$$

For the case $\ell = b$, we just simply interchange $a \leftrightarrow b$. Here the final state daughter parton j is in $\{\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{m+1}\}$ configuration. In this subsection, we will describe how $\{\hat{p}\}_{m+1}$ is determined from $\{p\}_m$ and a splitting variable $\{\zeta_p\}$. To describe the initial state partons momenta, we start by assuming the incoming hadron momenta p_A and p_B to be massless, so

$$p_A^2 = p_B^2 = 0 \quad \text{and} \quad 2\, p_A \cdot p_B = s \tag{2.48}$$

where s is the center-of-mass energy of incoming hadrons. We will restrict our expressions to massless partons in the kinematics, so $p_a^2 = p_b^2 = \hat{p}_a^2 = \hat{p}_j^2 = 0$, where

$$\begin{aligned} p_a &= \eta_a\, p_A \\ p_b &= \eta_b\, p_B \\ \hat{p}_a &= \hat{\eta}_a\, p_A \end{aligned} \tag{2.49}$$

Here, η_a and η_b are momentum fractions of the incoming partons before the splitting and $\hat{\eta}_a$ is the momentum fraction after the splitting. We choose the momentum fraction of parton b to remain unaltered

$$\hat{\eta}_b = \eta_b \tag{2.50}$$

The momentum fraction $\hat{\eta}_a$ after the splitting will be determined by the final-state daughter parton $\hat{p}_j$. As in the case of a final-state splitting, the relation $\hat{p}_a \approx p_a + \hat{p}_j$ cannot hold, as the momenta after parton splitting should be on-shell. In order to allow all partons to be on-shell, while conserving momentum, we choose to take the needed momenta from the final state spectator partons. This can be achieved by a Lorentz transformation as in the case of a final state splitting

$$\hat{p}_n^\mu = \Lambda(\hat{K}, K)^\mu{}_\nu\, p_n^\nu, \quad n \in \{1, \cdots, m\} \quad \text{and} \quad n \neq j \tag{2.51}$$

Here, K is the total momentum of the final state spectators before the splitting

$$K = p_a + p_b \tag{2.52}$$

and $\hat{K}$ is the total momentum of the final state spectators after the splitting

$$\hat{K} = \hat{p}_a + p_b - \hat{p}_j = \hat{Q} - \hat{p}_j, \quad \hat{Q} = \hat{p}_a + p_b \tag{2.53}$$

Since each final state spectator is changed by a Lorentz transformation, we have

$$\hat{K}^\mu = \Lambda^\mu{}_\nu\, K^\nu \tag{2.54}$$

with the Lorentz transformation given by Eq. (2.27). In order that K is related to $\hat{K}$ by a Lorentz transformation, we need $K^2 = \hat{K}^2$. Hence, we get

$$K^2 = \alpha\, \eta_a = \hat{K}^2 \tag{2.55}$$

where

$$\alpha \;=\; \eta_b\, s \tag{2.56}$$

From this it follows that

$$\begin{aligned} \eta_a\,\eta_b\, s &= (\hat{p}_a + p_b - \hat{p}_j)^2 \\ &= \hat{\eta}_a\,\eta_b\, s - 2\,[\hat{\eta}_a\, p_A\cdot\hat{p}_j + \eta_b\, p_B\cdot\hat{p}_j] \end{aligned} \tag{2.57}$$

which defines

$$\hat{\eta}_a \;=\; \frac{\eta_a\,\eta_b\, s + 2\,\eta_b\, p_B\cdot\hat{p}_j}{\eta_b\, s - 2\, p_A\cdot\hat{p}_j} \leq 1 \tag{2.58}$$

A consequence of Eq. (2.58) is

$$\eta_a < \hat{\eta}_a < 1 \tag{2.59}$$

Again, we can introduce the splitting variable ζ_p that defines the momenta of the daughter partons:

$$\zeta_p \equiv (\hat{p}_a, \hat{p}_j) \tag{2.60}$$

so that $\{p\}_m$ together with ζ_p determines $\{\hat{p}\}_{m+1}$. There is a transformation R_a relating m to the $m+1$ particle phase space,

$$\boxed{\{\hat{p},\hat{f}\}_{m+1} \;=\; R_a\left(\{\{p,f\}_m, \{\zeta_p,\zeta_f\}\}\right)} \tag{2.61}$$

Here, the splitting variable ζ_f is given by the flavours of the daughter partons, so we have

$$\zeta_f \;=\; (\hat{f}_a, \hat{f}_j) \tag{2.62}$$

The flavours of the spectator partons remain unchanged

$$\hat{f}_n \;=\; f_n, \quad n \notin \{a, j = m+1\} \tag{2.63}$$

while the flavours of the daughter partons $\hat{f}_a$ and $\hat{f}_j$ obey

$$\hat{f}_a + \hat{f}_j \;=\; f_a \tag{2.64}$$

Parton joining

Now we consider the inverse transformation needed in the subtraction, meaning that we combine an initial-state parton with a final-state parton into a mother parton that enters the hard part. So we start with $\{\hat{p}\}_{m+1}$ and determine $\{p\}_m$ and $\{\zeta_p\}$.

The splitting variable for the momenta is given by the momenta of the daughter partons

$$\zeta_p \;\equiv\; (\hat{p}_a, \hat{p}_j) \tag{2.65}$$

One can determine η_a by solving $K^2 = \hat{K}^2$, so we can express η_a in terms of $\hat{\eta}_a$. This gives

$$\eta_a \;=\; \frac{\hat{K}^2}{\alpha} \;=\; \frac{\hat{\eta}_a\,\eta_b\,s - 2\,[\hat{\eta}_a\,p_A\cdot\hat{p}_j + \eta_b\,p_B\cdot\hat{p}_j]}{\eta_b\,s} \tag{2.66}$$

Once we have η_a, we can construct $K \;=\; p_a + p_b$. Further we need the inverse Lorentz transformation of Eq. (2.51). From K and $\hat{K}$, we have

$$p_n^\mu \;=\; \Lambda(K,\hat{K})^\mu{}_\nu\,\hat{p}_n^\nu, \quad n\in\{1,\cdots,m\} \quad \text{and} \quad n\neq j \tag{2.67}$$

where $\Lambda(K,\hat{K})^\mu{}_\nu$ can be obtained using Eq. (2.27) with $\hat{K}$ and K interchanged. The mapping for the initial state parton a is

$$p_a \;=\; \left(1 - \frac{\hat{p}_j\cdot\hat{Q}}{\hat{p}_a\cdot p_b}\right)\,\hat{p}_a \tag{2.68}$$

Now the inverse transformation from the $(m+1)$- to the m-particle phase space is given by

$$\boxed{\{\{p,f\}_m,\{\zeta_p,\zeta_f\}\} \;=\; Q_a\,(\{\hat{p},\hat{f}\}_{m+1})} \tag{2.69}$$

The transformation of the flavours is similar to the case of parton splitting. The splitting variable ζ_f is given by the flavours of the daughter partons $\zeta_f \;=\; (\hat{f}_a,\hat{f}_j)$ in the sense of backwards evolution. The flavour of the mother parton f_a is given by

$$f_a \;=\; \hat{f}_a + \hat{f}_j \tag{2.70}$$

The flavours of the spectators remain unchanged

$$f_n \;=\; \hat{f}_n, \quad n\notin\{a, j=m+1\} \tag{2.71}$$

The integration measure for initial state splitting

The phase space factorization from $m+1$ to m partons takes a similar form as in the final state splitting, *i.e.* we have again

$$\int d\{\hat{p},\hat{f}\}_{m+1}\;g(\{\hat{p},\hat{f}\}_{m+1}) \;=\; \int d\{p,f\}_m\;d\zeta_p\,g(\{\hat{p},\hat{f}\}_{m+1}) \tag{2.72}$$

where $g(\{\hat{p},\hat{f}\}_{m+1})$ is an arbitrary function. The definition of the unresolved one parton integration measure is [28]

$$d\zeta_p = \;\frac{d^d\hat{p}_j}{(2\,\pi)^d}\,2\,\pi\,\delta^+(\hat{p}_j^2)\,\frac{\alpha}{\hat{\alpha}} \tag{2.73}$$

Here, $\hat{\alpha} \;=\; \eta_b\,s - 2\,p_A\cdot\hat{p}_j$. The factor $\alpha/\hat{\alpha}$ is just the derivative $d\hat{\eta}_a/d\eta_a$ calculated from the relation $\hat{K}^2 \;=\; K^2$. The final expressions in terms of the integration variables (see Section 2.4.4) can be found in Section 5.2.2.

2.4 Splitting functions

In the Nagy-Soper dipole subtraction scheme, the dipoles are based on splitting functions [28,29] that will be used to generate the shower. The new subtraction scheme also allows for easy matching with a parton shower using the same splitting functions in the spin-averaged approximation. In addition, the use of the dipoles as splitting functions in the shower, when combined with NLO calculations, simplifies the treatment of double counting.

Consider for a moment the QCD scattering amplitude without spin, colour or flavours. In the singular limit the transition amplitude factorizes according to

$$\mid \mathcal{M}_{m+1}(\{\hat{p}\}_{m+1}) > \approx \; v_\ell(\{\hat{p}\}_{m+1}) \cdot \mid \mathcal{M}_m(\{p\}_m) > \tag{2.74}$$

where $v_\ell(\{\hat{p}\}_{m+1})$ is the splitting amplitude proportional to $1/\hat{p}_\ell \cdot \hat{p}_j$. The momentum mapping between $\{\hat{p}\}_{m+1}$ and $\{p\}_m$ is described in Section 2.3. In reality, we have soft as well as collinear singularities in QCD, we also have to consider spin, colour and parton flavours. We will describe splitting amplitudes in more detail including all these factors in the next subsection.

It has been known, that scattering amplitudes can be factorized out in a general way by using the factorization properties of QCD amplitudes in the soft and collinear limits [31–35]. At the next-to-next-to-leading order and beyond, the factorization of QCD scattering amplitudes can be found in [36].

2.4.1 Definition of the splitting amplitudes

The splitting functions described in [28,29] are based on the spin dependent splitting amplitudes. The QCD scattering amplitude for $m+1$ partons is a vector in colour $\otimes$ spin space.

$$\mid \mathcal{M}(\{\hat{p},\hat{f}\}_{m+1}) > \tag{2.75}$$

when two partons ℓ and j are almost collinear, this amplitude becomes

$$\mid \mathcal{M}(\{\hat{p},\hat{f}\}_{m+1}) > \sim \mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) > \tag{2.76}$$

In the limit, that parton j becomes soft, then all of the $\mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) >$ amplitudes contribute, and we have ($\mid \mathcal{M}_\ell \rangle$ is to be defined in Eq. (2.78))

$$\mid \mathcal{M}(\{\hat{p},\hat{f}\}_{m+1}) > \sim \sum_\ell \mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) > \tag{2.77}$$

After splitting parton ℓ, the amplitude $\mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) >$ can be factorized into a splitting operator times the m-parton matrix element according to

$$\mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) > = \; t^\dagger_\ell(f_\ell \to \hat{f}_\ell + \hat{f}_j)\, V^\dagger_\ell(\{\hat{p},\hat{f}\}_{m+1}) \; \mid \mathcal{M}(\{p,f\}_m) > \tag{2.78}$$

Here, the Born amplitude for producing m partons is evaluated at momenta and flavours $\{p, f\}_m$ determined from $\{\hat{p}, \hat{f}\}_{m+1}$ according to the transformation $Q_\ell(\{\hat{p}, \hat{f}\}_{m+1})$ in Eq. (2.42) or Eq. (2.69). $V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1})$ is an operator acting on the spin part of the colour $\otimes$ spin space, while $t_\ell^\dagger(f_\ell \to \hat{f}_\ell + \hat{f}_j)$ is an operator acting on the colour part of the colour $\otimes$ spin space. The spin-dependent splitting operator can be described in the spin space $\mid \{s\}_m >$,

$$< \{\hat{s}\}_{m+1} \mid V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1}) \mid \{s\}_m > \tag{2.79}$$

we can take Eq. (2.79) to be diagonal

$$\boxed{< \{\hat{s}\}_{m+1} \mid V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1}) \mid \{s\}_m > = \left(\prod_{n \notin \{\ell, j=m+1\}} \delta_{\hat{s}_n, s_n} \right) v_\ell(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell)} \tag{2.80}$$

Here, the splitting amplitudes $v_\ell(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell)$ can be derived from the QCD vertices, spinors and polarization vectors for on-shell partons. We can use the mother momentum p_ℓ to describe the splitting amplitudes v_ℓ, and the relation between p_ℓ and $\{\hat{p}, \hat{f}\}_{m+1}$ is given by the transformation $Q_\ell(\{\hat{p}, \hat{f}\}_{m+1})$.

In the following, we will illustrate splitting amplitudes by giving some examples. First we consider the case of a $g \to g\,g$ splitting. The ggg QCD vertex is given by

$$v^{\alpha\beta\gamma}(p_a, p_b, p_c) \;=\; g^{\alpha\beta}(p_a - p_b)^\gamma + g^{\beta\gamma}(p_b - p_c)^\alpha + g^{\gamma\alpha}(p_c - p_a)^\beta \tag{2.81}$$

In order to define the propagator for the gluon field properly, we have to make a choice of gauge. The choice

$$\mathcal{L}_{\text{gauge-fixing}} \;=\; -\frac{1}{2\,\lambda}\,(n^\mu G_\mu^A)^2 \tag{2.82}$$

is called the axial gauge, in which we introduce an additional vector n. Here the parameter λ will break the gauge invariance of the theory, however, the physical results will be independent of λ at the end. The advantage of the axial gauge is that ghost fields are not required. Now we consider a special case in which $\lambda \;=\; 0$ and $n^2 \;=\; 0$. This is called the light-cone gauge. We define

$$D^{\mu\nu}(P, n) \;=\; -g^{\mu\nu} + \frac{P^\mu\, n^\nu + n^\mu\, P^\nu}{P \cdot n} \tag{2.83}$$

In the limit $P^2 \to 0$ we have

$$n^\mu\, D_{\mu\nu}(P, n) \;=\; 0, \qquad P^\mu\, D_{\mu\nu}(P, n) \;=\; 0 \tag{2.84}$$

In the axial gauge $n \cdot G^A \;=\; 0$, only two physical polarization states propagate, which are orthogonal to n and P. Hence we can use the propagator for an off-shell gluon

$$\frac{D^{\mu\nu}(\hat{p}_\ell + \hat{p}_j, n_\ell)}{(\hat{p}_\ell + \hat{p}_j)^2} \tag{2.85}$$

where the lightlike vector n_ℓ is chosen to lie in the plane of p_ℓ and Q (see [28, 29]),

$$n_\ell \;=\; Q - a_\ell\, p_\ell \;=\; \frac{1+y+\lambda}{2\,\lambda}\,Q - \frac{a_\ell}{\lambda}(\hat{p}_\ell + \hat{p}_j) \tag{2.86}$$

for the final state splitting. More generally, the vector n_ℓ has the following form for both initial and final state splittings,

$$n_\ell \;=\; \begin{cases} p_B & \text{for} \quad \ell \,=\, a \\ p_A & \text{for} \quad \ell \,=\, b \\ Q - a_\ell\, p_\ell & \text{for} \quad \ell \,\in\, \{1, \cdots, m\} \end{cases} \tag{2.87}$$

In the case of a final-state $g \to g\,g$ splitting (Fig. 2.3), we derive

$$\begin{aligned} v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) = \frac{\sqrt{4\pi\alpha_s}}{2\,\hat{p}_j\cdot\hat{p}_\ell}\,\varepsilon_\alpha(\hat{p}_j,\hat{s}_j,\hat{Q})^*\,\varepsilon_\beta(\hat{p}_\ell,\hat{s}_\ell,\hat{Q})^*\varepsilon^\nu(p_\ell,s_\ell,\hat{Q}) \\ \times\, v^{\alpha\beta\gamma}(\hat{p}_j,\hat{p}_\ell,-\hat{p}_j-\hat{p}_\ell)\,D_{\gamma\nu}(\hat{p}_\ell+\hat{p}_j,n_\ell) \end{aligned} \tag{2.88}$$

In the case of an initial-state $g \to g\,g$ splitting (Fig. 2.4), we have

$$\begin{aligned} v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) = -\frac{\sqrt{4\pi\alpha_s}}{2\,\hat{p}_j\cdot\hat{p}_\ell}\,\varepsilon_\alpha(\hat{p}_j,\hat{s}_j,\hat{Q})^*\,\varepsilon_\beta(\hat{p}_\ell,\hat{s}_\ell,\hat{Q})\varepsilon^\nu(p_\ell,s_\ell,\hat{Q})^* \\ \times\, v^{\alpha\beta\gamma}(\hat{p}_j,-\hat{p}_\ell,\hat{p}_\ell-\hat{p}_j)\,D_{\gamma\nu}(\hat{p}_\ell-\hat{p}_j,n_\ell) \end{aligned} \tag{2.89}$$

Here we have the exact QCD vertex and the off-shell gluon propagator in the axial gauge $n\cdot G^A \,=\, 0$ for both initial- and final-state ggg splittings. In order for the mother gluon to be on-shell, we can make an approximation so that the mother gluon is projected onto the physical degrees of freedom as it emerges from the hard matrix element. This projection is contained in the on-shell polarization vector $\varepsilon(p_\ell, s_\ell, \hat{Q})$.

For a final-state $q \to q\,g$ splitting (Fig. 2.5), we derive

$$v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) \;=\; \sqrt{4\pi\alpha_s}\,\varepsilon_\mu(\hat{p}_j,\hat{s}_j,\hat{Q})^*\,\frac{\overline{U}(\hat{p}_\ell,\hat{s}_\ell)\,\gamma^\mu\,(\not{\hat{p}}_\ell+\not{\hat{p}}_j)\,\not{n}_\ell\,U(p_\ell,s_\ell)}{(\hat{p}_\ell+\hat{p}_j)^2\,2p_\ell\cdot n_\ell} \tag{2.90}$$

For an initial-state $q \to q\,g$ splitting (Fig. 2.6), we have

$$v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) \;=\; -\sqrt{4\pi\alpha_s}\,\varepsilon_\mu(\hat{p}_j,\hat{s}_j,\hat{Q})^*\,\frac{\overline{U}(p_\ell,s_\ell)\,\not{n}_\ell\,(\not{\hat{p}}_\ell-\not{\hat{p}}_j)\,\gamma^\mu\,U(\hat{p}_\ell,\hat{s}_\ell)}{(\hat{p}_\ell-\hat{p}_j)^2\,2p_\ell\cdot n_\ell} \tag{2.91}$$

Here, $\overline{U}$ and U denote Dirac spinors for the quark fields satisfying $(\not{q}-m)\,U(\vec{q},s) \;=\; 0$ and $\overline{U}(\vec{q},s)\,(\not{q}-m) \;=\; 0$. The Dirac spinors for the antiquark fields obey $(\not{q}+m)\,V(\vec{q},s) \;=\; 0$ and $\overline{V}(\vec{q},s)\,(\not{q}+m) \;=\; 0$. They are normalized to

$$\begin{aligned} \overline{U}(\vec{p},s)\,\gamma^\mu\,U(\vec{p},s) &\;=\; 2\,p^\mu, \\ \overline{V}(\vec{p},s)\,\gamma^\mu\,V(\vec{p},s) &\;=\; 2\,p^\mu \end{aligned} \tag{2.92}$$

The projection operators are

$$\not{q} + m \;=\; \sum_s U(\vec{q},s)\,\overline{U}(\vec{q},s), \qquad \not{q} - m \;=\; \sum_s V(\vec{q},s)\,\overline{V}(\vec{q},s) \tag{2.93}$$

The polarization vector ε_μ for the gluon field is defined in a timelike axial gauge so that

$$\hat{p}_j \cdot \varepsilon \;=\; \hat{Q} \cdot \varepsilon \;=\; 0, \qquad \hat{Q} \;=\; Q \tag{2.94}$$

For other flavour choices, the results can be found in [28]. For the splitting functions, we have to square the splitting amplitudes. In the new formalism, there are two sorts of splitting functions: direct splitting functions (Fig. 2.7) and interference splitting functions (Fig. 2.8). The direct splitting functions correspond to the scattering amplitude $\mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) >$ for a parton ℓ to split times its complex conjugate scattering amplitude $< \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) \mid$ for that same parton ℓ to split, while the interference splitting functions correspond to the interference between the scattering amplitude $\mid \mathcal{M}_\ell(\{\hat{p},\hat{f}\}_{m+1}) >$ for a parton ℓ to split into partons ℓ and j and the complex-conjugate scattering amplitude $< \mathcal{M}_k(\{\hat{p},\hat{f}\}_{m+1}) \mid$ for another parton k to split into partons with labels k and j. These functions generate leading singularities when parton j is a soft gluon.

The direct splitting function is the product of a splitting amplitude v_ℓ and its complex-conjugate splitting amplitude v_ℓ^*, so we have

$$v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)\,v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)^* \tag{2.95}$$

which, after summing over the daughter parton spins and averaging over the mother parton spins, leads to the spin averaged splitting functions

$$\overline{W}_{\ell\ell} \;=\; \frac{1}{2}\sum_{\hat{s}_\ell,\hat{s}_j,s_\ell} \Big| v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)\Big|^2 \tag{2.96}$$

2.4.2 Eikonal factor

When parton j is a gluon, there is a common result in the limit $\hat{p}_j \to 0$, in which the splitting amplitude v_ℓ can be replaced by the eikonal approximation,

$$v_\ell^{\text{eikonal}}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) \;=\; \sqrt{4\pi\alpha_s}\,\delta_{\hat{s}_\ell,s_\ell}\,\frac{\varepsilon(\hat{p}_j,\hat{s}_j,\hat{Q})^*\cdot\hat{p}_\ell}{\hat{p}_j\cdot\hat{p}_\ell} \tag{2.97}$$

The eikonal approximation of the spin-averaged splitting functions $\overline{W}_{\ell\ell}$ is then

$$\overline{W}_{\ell\ell}^{\text{eikonal}} \;=\; 4\,\pi\,\alpha_s\,\frac{\hat{p}_\ell\cdot D(\hat{p}_j,\hat{Q})\cdot\hat{p}_\ell}{(\hat{p}_j\cdot\hat{p}_\ell)^2} \tag{2.98}$$

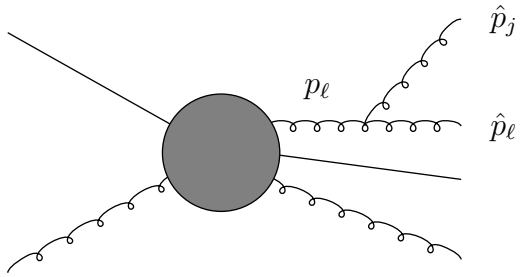


Figure 2.3: A final-state ggg splitting.

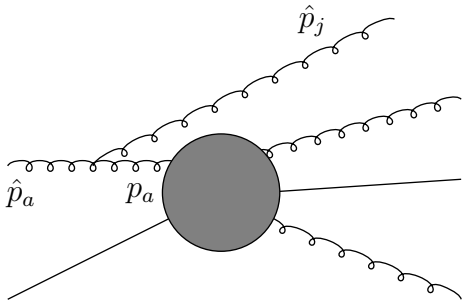


Figure 2.4: An initial-state ggg splitting.

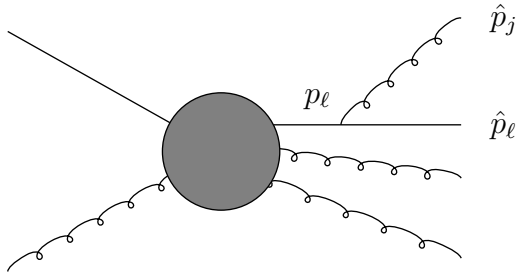


Figure 2.5: A final-state qqg splitting.

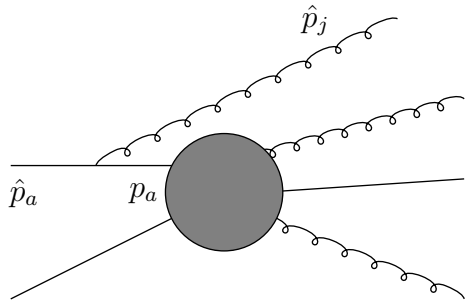


Figure 2.6: An initial-state qqg splitting.

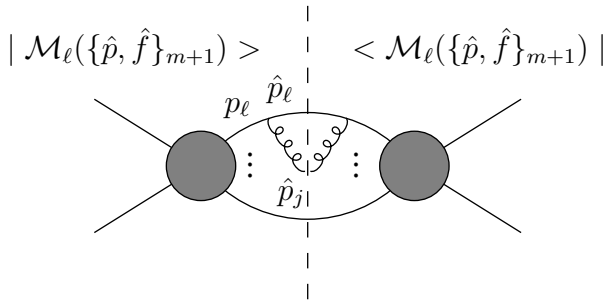


Figure 2.7: Collinear diagram: parton j is emitted from parton ℓ in the scattering amplitude $| \mathcal{M}_\ell(\{\hat{p}, \hat{f}\}_{m+1}) >$ and parton j is emitted from that same parton ℓ in the complex-conjugate scattering amplitude $< \mathcal{M}_\ell(\{\hat{p}, \hat{f}\}_{m+1}) |$.

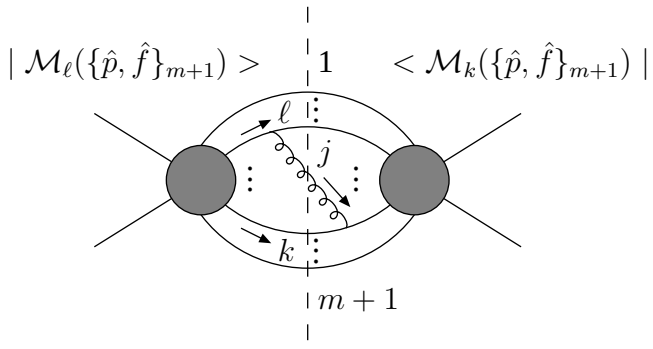


Figure 2.8: Soft diagram: parton j is emitted from parton ℓ in the scattering amplitude and parton j is emitted from parton k in the complex-conjugate scattering amplitude.

Here, $D_{\mu\nu}$ is the transverse projection tensor for an off-shell gluon, which is given by

$$D_{\mu\nu}(\hat{p}_j, \hat{Q}) \;=\; -g_{\mu\nu} + \frac{\hat{p}_j^{\,\mu}\,\hat{Q}^{\nu} + \hat{Q}^{\mu}\,\hat{p}_j^{\,\nu}}{\hat{p}_j \cdot \hat{Q}} - \frac{\hat{Q}^2\,\hat{p}_j^{\,\mu}\,\hat{p}_j^{\,\nu}}{(\hat{p}_j \cdot \hat{Q})^2} \tag{2.99}$$

It will be convenient if we define a dimensionless function F:

$$F \;=\; \frac{\hat{p}_\ell \cdot \hat{p}_j}{4\,\pi\,\alpha_s}\,\overline{W}_{\ell\ell} \tag{2.100}$$

then as the gluon j becomes soft, $\hat{p}_j \to 0$, we can define the function F_{eik} :

$$F_{\text{eik}} \;=\; \frac{\hat{p}_\ell \cdot \hat{p}_j}{4\,\pi\,\alpha_s}\,\overline{W}_{\ell\ell}^{\text{eikonal}} \;=\; \frac{\hat{p}_\ell \cdot D(\hat{p}_j, \hat{Q}) \cdot \hat{p}_\ell}{\hat{p}_\ell \cdot \hat{p}_j} \;=\; \frac{2\,\hat{p}_\ell \cdot Q}{\hat{p}_j \cdot Q} - \frac{Q^2\,\hat{p}_\ell \cdot \hat{p}_j}{(\hat{p}_j \cdot Q)^2} \tag{2.101}$$

We can also introduce the notation v_{eik}^2 or $\bar{v}_{\text{eik}}^2$, which we will use throughout in the following discussions, and which is defined through the spin-averaged splitting function $\overline{W}_{\ell\ell}^{\text{eikonal}}$

$$\overline{W}_{\ell\ell}^{\text{eikonal}} \;=\; \bar{v}_{\text{eik}}^2 \;\;=\;\; \begin{cases} \frac{1}{2}\,v_{\text{eik}}^2 & \text{for} \quad \text{quarks} \\ \\ \frac{1}{2\,(1-\epsilon)}\,v_{\text{eik}}^2 & \text{for} \quad \text{gluons} \end{cases} \tag{2.102}$$

The prefactor $1/2$ is the quark spin (or helicity) degrees of freedom, while the factor $1/2/(1-\epsilon)$ is the gluon spin degrees of freedom. More precisely, if we consider both the colour and spin average, we can introduce a notation $1/\omega(a)$, which is defined by

$$\omega(a) \;\;=\;\; \begin{cases} 2\,N & \text{if} \quad a \,=\, q \\ \\ 2\,(1-\epsilon)\,D_A & \text{if} \quad a \,=\, g \end{cases} \tag{2.103}$$

$\omega(a)$ is the number of colour and spin degrees of freedom for the flavour a. $D_A \,=\, N^2 - 1$ is the dimension of the adjoint representation of the $SU(N)$ colour group, while N is the dimension of the fundamental representation of the colour group, *e.g.* for $SU(3)_C$ we have $D_A \,=\, 8$ colour degrees of freedom of the gluon field in the adjoint representation and $N \,=\, 3$ colour degrees of freedom of the quark field in the fundamental representation of $SU(3)_C$. In principle we also have to consider δ^{ab} or δ^{AB} (which equals 3 or 8) in the quark or gluon propagators, where the indices $a, b \,=\, 1, 2, 3$ and $A, B \,=\, 1, \cdots, 8$. In practice all these factors are included in the matrix element.

In Eq. (2.102), v_{eik}^2 denotes the spin-unaveraged splitting function. So we have

$$v_{\text{eik}}^2 \;\;=\;\; 4\,\pi\,\alpha_s \times \begin{cases} \left[2 \,/\, (\hat{p}_\ell \cdot \hat{p}_j)^2\right]\, \hat{p}_\ell \cdot D(\hat{p}_j,\, \hat{Q}) \cdot \hat{p}_\ell & \text{for} \quad \text{quarks} \\ \\ \left[2\,(1-\epsilon) \,/\, (\hat{p}_\ell \cdot \hat{p}_j)^2\right]\, \hat{p}_\ell \cdot D(\hat{p}_j,\, \hat{Q}) \cdot \hat{p}_\ell & \text{for} \quad \text{gluons} \end{cases} \tag{2.104}$$

For the interference (soft) term, we obtain

$$\Delta W \;=\; \overline{W}_{\ell\ell}^{\text{eikonal}} - \overline{W}_{\ell k} \;=\; \begin{cases} \frac{1}{2}\,(v_{\text{eik}}^2 - v_{\text{soft}}^2) & \text{for quarks} \\ \\ \frac{1}{2\,(1-\epsilon)}\,v_{\text{eik}}^2 - \frac{1}{2}\,v_{\text{soft}}^2 & \text{for gluons} \end{cases} \tag{2.105}$$

where the soft term is given by $\overline{W}_{\ell k} \,=\, 4\,\pi\,\alpha_s\,2\,A_{\ell k}\,\hat{p}_\ell \cdot D(\hat{p}_j, \hat{Q}) \cdot \hat{p}_k / (\hat{p}_j \cdot \hat{p}_\ell\,\hat{p}_j \cdot \hat{p}_k)$ and $A_{\ell k}$ is to be defined precisely in Eq. (2.162) (see also Section 2.4.5).

2.4.3 Collinear splitting functions: final state splittings

In this and the next subsections, we will give the explicit expressions of splitting function for each process. In the following, we will remove the common factor $4\,\pi\alpha_s$, which we will add it back in the end. We also ignore the colour factors for a moment, which we will include when we consider the integrated splitting functions. For the splitting functions in $(m+1)$-parton phase spaces we always work in four dimensions, meaning that we can safely put $d = 4$ (or $\epsilon = 0$) in the following discussions. We will also include the eikonal factor for the collinear splitting functions if the emitted parton j is a gluon. The eikonal splitting function will turn out to be important when we incorporate the soft gluon interference diagrams. We will explain this in more detail in Section 2.4.5.

Kinematics: integration variables

We can introduce the two integration variables

$$x \,=\, \frac{\hat{p}_j \cdot Q}{P_\ell \cdot Q}, \qquad z \,=\, \frac{\hat{p}_j \cdot n_\ell}{P_\ell \cdot n_\ell} \tag{2.106}$$

Hence we could express most dot products in terms of $x,\, y\,, z\,,\, \lambda,\, a_\ell$. The parameters a_ℓ, λ and y have been given in Section 2.3. Note, however, that these are not all independent variables; a_ℓ depends on the kinematics before splitting/after recombination respectively, and $\lambda \,=\, \lambda(a_\ell, y)$. The variable x is given by

$$x \,=\, \frac{\lambda}{1+y}\,z + \frac{2\,a_\ell\,y}{(1+y)\,(1+y+\lambda)} \tag{2.107}$$

So we are left with $a_\ell,\, y,\, z$ as free variables; we will use y and z in the integration. If we want to eliminate the x dependence and go back to y and z as integration variables and keep a_ℓ as a fixed parameter coming from the m-particle phase space, we can introduce

$$x_0 \,=\, \frac{1-\lambda+y}{1+\lambda+y} \,=\, x_0(a_\ell, y) \tag{2.108}$$

Using

$$\gamma \;=\; \frac{1}{2}\,(1+y+\lambda) \tag{2.109}$$

we obtain a couple of useful relations, *e.g.*

$$\begin{aligned} x_0 \;&=\; \frac{y\,a_\ell}{\gamma^2}, \quad 1-x_0 \;=\; \frac{\lambda}{\gamma}, \quad 1+x_0 \;=\; \frac{1+y}{\gamma} \\ x\,(1+y) \;&=\; \gamma\,[x_0 + z\,(1-x_0)]\,, \quad (1-x)\,(1+y) \;=\; \gamma\,[1-z\,(1-x_0)] \end{aligned} \tag{2.110}$$

When integrating the interference terms, we need to make use of an additional integration variable parametrizing the angle between emitter and spectator. To parametrize this in a Lorentz invariant way, we introduce the variables

$$\boxed{v \;=\; \frac{\hat{p}_j\cdot\hat{p}_k}{\hat{p}_k\cdot P_\ell}, \qquad \bar{z} \;=\; \frac{\hat{p}_k\cdot n_\ell\,\hat{p}_\ell\cdot\hat{p}_j}{\hat{p}_k\cdot P_\ell\,n_\ell\cdot P_\ell} \;=\; \frac{y}{\gamma}\,\frac{\hat{p}_k\cdot n_\ell}{\hat{p}_k\cdot P_\ell} \;=\; \frac{y}{\gamma}\,\tilde{a}}} \tag{2.111}$$

and

$$\hat{p}_k\cdot Q \;=\; \frac{\hat{p}_k\cdot P_\ell}{\gamma}\,(\lambda\,\tilde{a}+a_\ell) \tag{2.112}$$

with

$$\boxed{\tilde{a} \;=\; \tilde{a}(y) \;=\; \frac{\hat{p}_k\cdot n_\ell}{\hat{p}_k\cdot P_\ell}} \tag{2.113}$$

It is convenient to introduce the angles θ, θ_k and φ in the integration measure such that in the center of mass system we have[7]

$$\hat{p}_j = A_j \begin{pmatrix} 1 \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \qquad \hat{p}_k = A_k \begin{pmatrix} 1 \\ \sin\theta_k \\ 0 \\ \cos\theta_k \end{pmatrix}, \qquad P_\ell \;=\; P \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A_j = \frac{P}{2} \tag{2.114}$$

In the integration measure, we will need an additional change of variable,

$$\boxed{v' \;=\; \frac{1}{2}\,(1-\cos\varphi) = \frac{v-v_{\text{min}}}{v_{\text{max}}-v_{\text{min}}}} \tag{2.115}$$

where

$$\begin{aligned} v \;&=\; \frac{1}{2}\,(1-\sin\theta\sin\theta_k\cos\varphi-\cos\theta\cos\theta_k) \\ v_{\text{max}} \;&=\; \frac{1}{2}\,(1+\sin\theta\sin\theta_k-\cos\theta\cos\theta_k) \\ v_{\text{min}} \;&=\; \frac{1}{2}\,(1-\sin\theta\sin\theta_k-\cos\theta\cos\theta_k) \end{aligned} \tag{2.116}$$

[7] In the reference frame where P_ℓ is at rest (center of mass frame), we have $\hat{p}_j \;=\; (\,p_j^0, \vec{p}_j\,) \;=\; (\,p_j^0, p_j^0\,\hat{u}\,)$ and $\hat{p}_\ell \;=\; (\,p_\ell^0, -\vec{p}_j\,) \;=\; (\,p_\ell^0, -p_j^0\,\hat{u}\,) \;=\; (\,p_j^0, -p_j^0\,\hat{u}\,)$. Hence, the 4-vectors $\hat{p}_j$ and $\hat{p}_\ell$ have the same energy content in the massless case: $A_j \;=\; p_j^0 \;=\; p_\ell^0 \;=\; P/2$. Here, the unit vector is $\hat{u} \;=\; \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$.

We can also write $\bar{z}$ and z in terms of the angles θ_k and θ,

$$\bar{z} \;=\; \frac{1}{2}(1-\cos\theta_k), \qquad z \;=\; \frac{1}{2}(1-\cos\theta) \tag{2.117}$$

Finally we give a list of useful relations for dot products of vectors, which we have used throughout the dipole calculations,

$$\begin{aligned}
P_\ell \cdot Q &= (1+y)\, p_\ell \cdot Q \\
\hat{p}_j \cdot Q &= x\,(1+y)\, p_\ell \cdot Q \\
\hat{p}_\ell \cdot Q &= (1-x)\,(1+y)\, p_\ell \cdot Q \\
P_\ell \cdot p_\ell &= \frac{1+y-\lambda}{2\,a_\ell}\, p_\ell \cdot Q \\
P_\ell \cdot n_\ell &= \frac{1+y+\lambda}{2}\, p_\ell \cdot Q \\
p_\ell \cdot \hat{p}_j &= \frac{1}{\lambda}\left(y - \frac{1-\lambda+y}{2\,a_\ell}\, x\,(1+y)\right) p_\ell \cdot Q \\
p_\ell \cdot \hat{p}_\ell &= \frac{1}{\lambda}\left(y - \frac{1-\lambda+y}{2\,a_\ell}\,(1-x)\,(1+y)\right) p_\ell \cdot Q
\end{aligned} \tag{2.118}$$

The subprocess qqg splitting

First we consider the collinear qqg splitting function in the final state. The $\bar{q}\bar{q}g$ final state splitting function is the same for massless quarks. The dipole including the eikonal splitting function is given by

$$\boxed{v^2_{qqg} - v^2_{\text{eik}} \;=\; \frac{2}{y\,(p_\ell \cdot Q)} \left\{ \frac{(\lambda-1+y)^2 + 4\,y}{4\,\lambda}\, F_{\text{eik}} + \frac{d-2}{4}\, z\;(1+y+\lambda) \right\}} \tag{2.119}$$

which, in terms of the momenta, can be rewritten as

$$\begin{aligned}
v^2_{qqg} - v^2_{\text{eik}} = \frac{2}{\hat{p}_\ell \cdot \hat{p}_j} \Bigg\{ \Bigg[\left(\frac{P_\ell \cdot n_\ell}{p_\ell \cdot Q} - 1 \right) + \frac{a_\ell\, \hat{p}_\ell \cdot \hat{p}_j}{\sqrt{(p_\ell \cdot Q)^2 + (\hat{p}_\ell \cdot \hat{p}_j)^2 + 2\, p_\ell \cdot Q\, \hat{p}_\ell \cdot \hat{p}_j\,(1-2\,a_\ell)}} \Bigg] \\
\times \left(\frac{2\,\hat{p}_\ell \cdot Q}{\hat{p}_j \cdot Q} - \frac{Q^2\, \hat{p}_\ell \cdot \hat{p}_j}{(\hat{p}_j \cdot Q)^2} \right) + 2 \left(\frac{d-2}{4} \right) \left(\frac{\hat{p}_j \cdot Q - a_\ell\, p_\ell \cdot \hat{p}_j}{p_\ell \cdot Q} \right) \Bigg\}
\end{aligned} \tag{2.120}$$

where

$$F_{\text{eik}} \;=\; 2\,\frac{1-x}{x} - \frac{2\,a_\ell\, y}{x^2\,(1+y)^2} \tag{2.121}$$

The dipole $v^2_{qqg} - v^2_{\text{eik}}$ denotes the spin-unaveraged splitting function. After averaging over the incoming parton spins gives an additional factor $1/2$. The relation, connecting hatted and unhatted quantities, is given in Eq. (2.32). In Eq. (2.119), only the collinear singularity is left

over after subtracting the eikonal factor. Note that $\hat{Q} = Q$ for the final state splittings and $P_\ell \cdot n_\ell = (\hat{p}_\ell + \hat{p}_j) \cdot Q - a_\ell \, (\hat{p}_\ell + \hat{p}_j) \cdot p_\ell$ is reduced to $p_\ell \cdot Q$ if $a_\ell = 1$. It corresponds to the case in which there are only two massless partons in the final state. If we want to eliminate x and switch to x_0 and z as variables[8], we obtain

$$F_{\text{eik}} \, = \, 2 \left(-1 + \frac{1 + x_0}{x_0 + z\,(1 - x_0)} - \frac{x_0}{\left(x_0 + z\,(1 - x_0)\right)^2} \right) \tag{2.122}$$

Note also that as $y \to 0$ (which means $x = z$), F turns into the standard Altarelli-Parisi splitting function,

$$F_{AP}(z) \, = \, \frac{1 + (1 - z)^2}{z} \tag{2.123}$$

The subprocess $gq\bar{q}$ splitting

Only the collinear singularity is present for the $gq\bar{q}$ splitting, so we do not subtract the eikonal splitting function in this case. Hence the dipole for the massless quarks is given by

$$\boxed{v^2_{gq\bar{q}} \, = \, \frac{2}{y \, p_\ell \cdot Q} \, \left(1 - \epsilon - 2\,z\,(1 - z)\right)} \tag{2.124}$$

which, in terms of the momenta, can be rewritten as

$$v^2_{gq\bar{q}} = \frac{1}{\hat{p}_\ell \cdot \hat{p}_j}(d - 4) + \frac{2}{\left(\hat{p}_\ell \cdot \hat{p}_j\right)^2} \left\{ \frac{\hat{p}_\ell \cdot Q \, \hat{p}_j \cdot p_\ell + \hat{p}_j \cdot Q \, \hat{p}_\ell \cdot p_\ell}{p_\ell \cdot Q} - \frac{Q^2 \, \hat{p}_\ell \cdot p_\ell \, \hat{p}_j \cdot p_\ell}{\left(p_\ell \cdot Q\right)^2} \right\} \tag{2.125}$$

Averaging over the incoming parton spins gives an additional factor $1/2/(1 - \epsilon)$. For massless quarks, the $g\bar{q}q$ splitting function is the same.

The subprocess ggg splitting

For the ggg splitting, we have to do something slightly different because there are two identical gluons in the final state. We have to make sure that only the daughter parton j generates a singularity. The unaveraged splitting function is given by

$$v^2_{ggg} \, = \, \frac{1}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \, \left\{ (d - 2) \, \left[\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell + \hat{p}_j \cdot D_\ell \cdot \hat{p}_j \right] - k_\perp^2 \, \text{Tr}\left[D_\ell \cdot D_j \right] \right\} \tag{2.126}$$

[8] Here x_0 is a function of a_ℓ and y, *i.e.* $x_0 \, = \, x_0(a_\ell, y)$.

where $D_j = D(\hat{p}_j, \hat{Q})$, $D_\ell = D(\hat{p}_\ell, \hat{Q})$ and

$$\begin{aligned}
\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell &= \frac{2\,y\,p_\ell \cdot Q}{x_0 + z\,(1-x_0)} \left[1 - z\,(1-x_0) - \frac{x_0}{x_0 + z\,(1-x_0)}\right] \\
\hat{p}_j \cdot D_\ell \cdot \hat{p}_j &= \frac{2\,y\,p_\ell \cdot Q}{1 - z\,(1-x_0)} \left[x_0 + z\,(1-x_0) - \frac{x_0}{1 - z\,(1-x_0)}\right] \\
k_\perp^2 &= -2\,y\,z\,(1-z)\,p_\ell \cdot Q \\
\mathrm{Tr}\,[D_\ell \cdot D_j] &= d - 2 - 2\,\Delta + \Delta^2 \\
\Delta &= \frac{Q^2\,\hat{p}_\ell \cdot \hat{p}_j}{\hat{p}_\ell \cdot Q\,\hat{p}_j \cdot Q} = \frac{2\,x_0}{(x_0 + z\,(1-x_0))\,(1 - z\,(1-x_0))}
\end{aligned} \tag{2.127}$$

Here, the additional statistical factor $1/2$ in Eq. (2.126) appears only for a final state ggg splitting because the two gluons are identical. Instead of using Eq. (2.126) as a dipole subtraction term, however, we will use a slightly modified splitting function in order to well separate the singularities in the ggg splitting final state. Then we add a term $v'^2_{ggg} = v_2^2 - v_3^2$ to Eq. (2.126), where v_2^2 and v_3^2 are defined in [29]. The additional term v'^2_{ggg} with the roles of the two daughter gluons ℓ and j interchanged does not change the result. This way, there is a singularity when daughter gluon j becomes soft, but not when daughter gluon ℓ becomes soft. The additional term v'^2_{ggg} is given by

$$v'^2_{ggg} = v_2^2 - v_3^2 = \frac{d-2}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2}\,[\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell - \hat{p}_j \cdot D_\ell \cdot \hat{p}_j] \tag{2.128}$$

In the end, we obtain

$$\begin{aligned}
\tilde{v}^2_{ggg} &= v^2_{ggg} + v'^2_{ggg} = \frac{1}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2}\,\left\{2\,(d-2)\,\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell - k_\perp^2\,\mathrm{Tr}\,[D_\ell \cdot D_j]\right\} \\
&= \frac{2\,(1-\epsilon)}{(\hat{p}_\ell \cdot \hat{p}_j)^2}\,\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell - \frac{k_\perp^2}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2}\,\mathrm{Tr}\,[D_\ell \cdot D_j]
\end{aligned} \tag{2.129}$$

Hence, the dipole, including the eikonal splitting function, is given by

$$\boxed{\tilde{v}^2_{ggg} - v^2_{\text{eik}} = -\frac{k_\perp^2}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2}\,\mathrm{Tr}\,[D_\ell \cdot D_j] = \frac{z\,(1-z)}{y\,p_\ell \cdot \hat{Q}}\,\left[d - 2 - 2\,\Delta + \Delta^2\right]} \tag{2.130}$$

which, in terms of the momenta, can be rewritten as

$$\tilde{v}^2_{ggg} - v^2_{\text{eik}} = \frac{(\hat{p}_j \cdot Q - a_\ell\,\hat{p}_j \cdot p_\ell)\,(\hat{p}_\ell \cdot Q - a_\ell\,\hat{p}_\ell \cdot p_\ell)}{(P_\ell \cdot n_\ell)^2\;\hat{p}_\ell \cdot \hat{p}_j}\,\left\{2 - 2\frac{\hat{p}_j \cdot \hat{p}_\ell\,Q^2}{\hat{p}_\ell \cdot Q\,\hat{p}_j \cdot Q} + \left(\frac{\hat{p}_j \cdot \hat{p}_\ell\,Q^2}{\hat{p}_\ell \cdot Q\,\hat{p}_j \cdot Q}\right)^2\right\} \tag{2.131}$$

Averaging over the incoming parton spins gives an additional factor $1/2/(1-\epsilon)$.

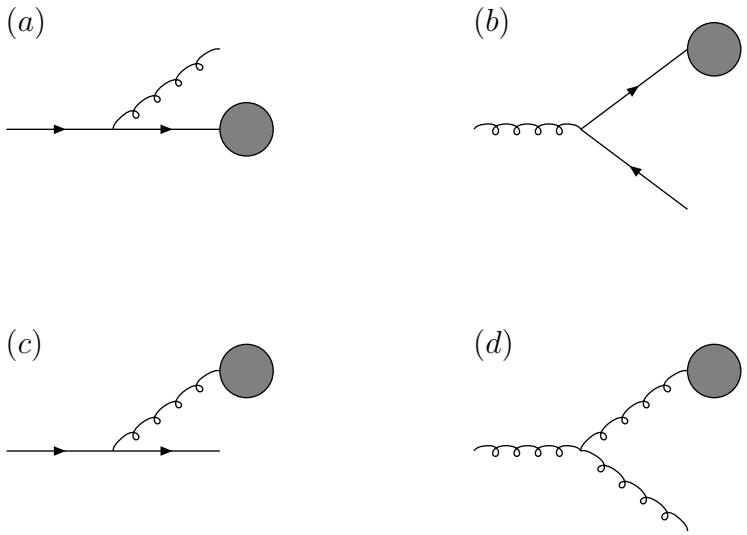


Figure 2.9: Initial-state splitting processes.

2.4.4 Collinear splitting functions: initial state splittings

In this subsection, we will again ignore the common factor $4\,\pi\alpha_s$ and the colour factors for a moment. In $(m+1)$-parton phase spaces we always work in four dimensions. The eikonal factor will be considered if the emitted parton j is a gluon. For the initial state splittings, there is an additional scattering process in which the daughter quark comes in and splits a mother gluon that enters the hard interaction (see Fig. 2.9).

Kinematics: integration variables

For initial state splitting, we use the following integration variables:

$$\begin{aligned} x &= \frac{\hat{p}_a\cdot p_b - \hat{p}_a\cdot\hat{p}_j - p_b\cdot\hat{p}_j}{\hat{p}_a\cdot p_b} = 1 - \frac{2\,p_A\cdot\hat{p}_j}{\eta_b\,s} - \frac{2\,p_B\cdot\hat{p}_j}{\hat{\eta}_a\,s} = \frac{\eta_a}{\hat{\eta}_a} \\ y &= \frac{\hat{p}_a\cdot\hat{p}_j}{\hat{p}_a\cdot p_b} = \frac{\hat{\eta}_a\,p_A\cdot\hat{p}_j}{\frac{1}{2}\,\hat{\eta}_a\,\eta_b\,s} = \frac{2\,p_A\cdot\hat{p}_j}{\eta_b\,s} \end{aligned} \tag{2.132}$$

Turning the equations above around, we then obtain

$$p_A \cdot \hat{p}_j \;=\; \frac{y\,\eta_b\,s}{2}, \quad p_B \cdot \hat{p}_j \;=\; \frac{\eta_a\,s}{2\,x}\,(1-x-y), \quad \hat{p}_j \cdot \hat{Q} \;=\; \frac{\eta_a\,\eta_b\,s}{2\,x}\,(1-x) \tag{2.133}$$

and

$$\boxed{\hat{\eta}_a \;=\; \frac{\eta_a}{x}, \qquad 0 < \hat{\eta}_a < 1 \qquad \Rightarrow \quad x > \eta_a} \tag{2.134}$$

Here, $\hat{\eta}_a$ and η_a are momentum fractions after and before the splittings. As in the final state interference integrals, we also need to parametrize the additional angle which appears in interference terms. We define

$$\boxed{v \;=\; \frac{\hat{\eta}_a\,\eta_b\,s\,\hat{p}_j \cdot \hat{p}_k}{2\,\hat{p}_k \cdot \hat{Q}\,\hat{p}_j \cdot \hat{Q}} \;=\; \frac{1}{1-x}\,\frac{\hat{p}_j \cdot \hat{p}_k}{\hat{p}_k \cdot \hat{Q}}} \tag{2.135}$$

and

$$\boxed{\bar{z} \;=\; \frac{\hat{p}_a \cdot \hat{p}_k}{\hat{p}_k \cdot \hat{Q}}, \qquad y' \;=\; \frac{y}{1-x}} \tag{2.136}$$

As in the final state interference case, it is convenient to introduce the angles θ, θ_k and φ in the integration measure such that in the center of mass system we have

$$\hat{p}_a \;=\; \frac{\sqrt{\hat{s}}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{p}_j \;=\; A_j \begin{pmatrix} 1 \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \qquad \hat{p}_k \;=\; A_k \begin{pmatrix} 1 \\ \sin\theta_k \\ 0 \\ \cos\theta_k \end{pmatrix} \tag{2.137}$$

In the integration measure, we will need an additional change of variable,

$$\boxed{v' \;=\; \frac{1}{2}\,(1-\cos\varphi) = \frac{v - v_{\text{min}}}{v_{\text{max}} - v_{\text{min}}}} \tag{2.138}$$

where

$$\begin{aligned} v \;&=\; \frac{1}{2}\,(1 - \sin\theta\sin\theta_k\cos\varphi - \cos\theta\cos\theta_k) \\ v_{\text{max}} \;&=\; \frac{1}{2}\,(1 + \sin\theta\sin\theta_k - \cos\theta\cos\theta_k) \\ v_{\text{min}} \;&=\; \frac{1}{2}\,(1 - \sin\theta\sin\theta_k - \cos\theta\cos\theta_k) \end{aligned} \tag{2.139}$$

We can also write $\bar{z}$ and y' in terms of the angles θ_k and θ,

$$\bar{z} \;=\; \frac{1}{2}(1-\cos\theta_k), \qquad y' \;=\; \frac{1}{2}(1-\cos\theta) \tag{2.140}$$

The subprocess $qq(x)g$ splitting

We assume that the gluon j is emitted from the parton a. For massless quarks, the unaveraged $qq(x)g$ splitting function is given by

$$v^2_{qqg} \;=\; (d-2)\,\frac{2\,p_B\cdot\hat{p}_j}{\eta_a\,\hat{\eta}_a\,s\,(p_A\cdot\hat{p}_j)} \;+\; \frac{2\,\eta_b^2\,p_B\cdot\hat{p}_j\,s}{(\hat{p}_j\cdot\hat{Q})^2\,p_A\cdot\hat{p}_j} \tag{2.141}$$

where $s \;=\; 2\,p_A\cdot p_B$. Here, $\hat{\eta}_a$ and η_a are momentum fractions after and before the splittings. With our kinematics, the momentum fraction of parton b remains unchanged, *i.e.* $\hat{\eta}_b \;=\; \eta_b$. We denote the total momentum of the final state partons before the splitting by $Q \;=\; p_a + p_b$ and after the splitting by $\hat{Q} \;=\; \hat{p}_a + p_b$. In the splitting function, we choose the lightlike vector to be $n_a \;=\; p_B$ and it will be convenient to define $P_a \;=\; \hat{p}_a - \hat{p}_j$. Using the variables x and y (defined in previous subsection), Eq. (2.141) can be rewritten as

$$v^2_{qqg} \;=\; \frac{2\,(1-x-y)}{\eta_a\,\eta_b\,y\,s}\,\left(d-2+\frac{4\,x}{(1-x)^2}\right) \tag{2.142}$$

We also define the eikonal approximation for soft gluon emission as discussed in Section 2.4.2,

$$F_{\text{eik}} \;=\; \frac{\hat{p}_a\cdot D(\hat{p}_j,\hat{Q})\cdot\hat{p}_a}{\hat{p}_a\cdot\hat{p}_j} \tag{2.143}$$

or equivalently we can write the unaveraged eikonal splitting function as

$$v^2_{\text{eik}} \;=\; \frac{2\,\eta_b^2\,p_B\cdot\hat{p}_j\,s}{(\hat{p}_j\cdot\hat{Q})^2\,p_A\cdot\hat{p}_j} \;=\; \frac{8\,x\,(1-x-y)}{(1-x)^2\,\eta_a\,\eta_b\,y\,s} \tag{2.144}$$

Using the variables x and y, the unaveraged splitting function including the eikonal factor is given by

$$\boxed{v^2_{qqg} \;-\; v^2_{\text{eik}} \;=\; \frac{(d-2)\,2\,p_B\cdot\hat{p}_j}{\eta_a\,\hat{\eta}_a\,s\,(p_A\cdot\hat{p}_j)} \;=\; \frac{(d-2)\,2\,p_b\cdot\hat{p}_j}{x\,\hat{Q}^2\,(\hat{p}_a\cdot\hat{p}_j)} \;=\; \frac{(d-2)\,2\,(1-x-y)}{\eta_a\,\eta_b\,y\,s}} \tag{2.145}$$

Averaging over the incoming parton spins gives an additional factor $1/2$. For massless quarks, the $\bar{q}\bar{q}(x)g$ splitting function is the same.

The subprocess $qqg(x)$ splitting

For the initial $qqg(x)$ splitting process, in which the daughter quark with momentum $\hat{p}_a$ comes in and splits a mother gluon with momentum p_a which enters the hard interaction, only the collinear

singularity is present. Hence, we do not subtract the eikonal splitting function in this case. The unaveraged $qqg(x)$ splitting function for massless quarks is given by

$$\boxed{v^2_{qqg} = \frac{2\,(d-2)\,x}{\eta_a\,\eta_b\,y\,s} + \frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,s\,y\,(x+y)^2}} \tag{2.146}$$

which, in terms of the momenta, can be rewritten as

$$\begin{aligned} v^2_{qqg} &= \frac{d-2}{\hat{\eta}_a\,p_A\cdot\hat{p}_j} + \frac{2}{\eta_b\,p_B\cdot P_a}\left\{-1+\frac{1}{\hat{\eta}_a\,p_A\cdot\hat{p}_j}\left(\hat{p}_j\cdot\hat{Q}+\eta_b\,p_B\cdot\hat{p}_j+\frac{2\,\eta_b\,(p_B\cdot\hat{p}_j)^2}{p_B\cdot P_a}\right)\right\} \\ &= \frac{d-2}{\hat{p}_a\cdot\hat{p}_j} + \frac{2}{p_b\cdot P_a}\left\{-1+\frac{1}{\hat{p}_a\cdot\hat{p}_j}\left(\hat{p}_j\cdot\hat{Q}+\,p_b\cdot\hat{p}_j+\frac{2\,(p_b\cdot\hat{p}_j)^2}{p_b\cdot P_a}\right)\right\} \end{aligned} \tag{2.147}$$

Averaging over the incoming parton spins gives an additional factor $1/2$. For massless quarks, the $\bar{q}\bar{q}g(x)$ splitting function is the same.

The subprocess $gq\bar{q}$ splitting

Only the collinear singularity is present for the $gq\bar{q}$ splitting, so we do not subtract the eikonal splitting function in this case. The unaveraged splitting function for massless quarks is given by

$$\boxed{v^2_{gq\bar{q}} = \frac{2\,(d-2)}{\eta_a\,\eta_b\,y\,s} + \frac{8\,(x+y)}{\eta_a\,\eta_b\,y\,s}\,(x+y-1)} \tag{2.148}$$

which, in terms of the momenta, can be rewritten as

$$\begin{aligned} v^2_{gq\bar{q}} &= \frac{d-2}{\eta_a\,p_A\cdot\hat{p}_j} + \frac{4}{\eta_a\,\eta_b\,s}\left\{1-\frac{1}{\hat{\eta}_a\,p_A\cdot\hat{p}_j}\left(\hat{p}_j\cdot\hat{Q}+\eta_b\,p_B\cdot\hat{p}_j\right)+\frac{4\,p_B\cdot\hat{p}_j}{\hat{\eta}_a\,s}\left(-1+\frac{\hat{p}_j\cdot\hat{Q}}{\hat{\eta}_a\,p_A\cdot\hat{p}_j}\right)\right\} \\ &= \frac{d-2}{x\,\hat{p}_a\cdot\hat{p}_j} + \frac{4}{x\,\hat{Q}^2}\left\{1-\frac{1}{\hat{p}_a\cdot\hat{p}_j}\left(\hat{p}_j\cdot\hat{Q}+\,p_b\cdot\hat{p}_j\right)+\frac{4\,p_b\cdot\hat{p}_j}{\hat{Q}^2}\left(-1+\frac{\hat{p}_j\cdot\hat{Q}}{\hat{p}_a\cdot\hat{p}_j}\right)\right\} \end{aligned} \tag{2.149}$$

Averaging over the incoming parton spins gives an additional factor $1/2/(1-\epsilon)$. For massless quarks, the $g\bar{q}q$ splitting function is the same.

The subprocess ggg splitting

The unaveraged ggg splitting function is given by

$$\begin{aligned} v^2_{ggg} \;=\; & \frac{1}{(\hat{p}_a\cdot\hat{p}_j)^2}\left\{2\,(1-\epsilon)\,\frac{\hat{\eta}_a^2\,\eta_b^2\,s\,p_A\cdot\hat{p}_j\,p_B\cdot\hat{p}_j}{(\hat{p}_j\cdot\hat{Q})^2}\right. \\ & \left.+\,\frac{4\,p_A\cdot\hat{p}_j\,p_B\cdot\hat{p}_j}{s}\left[2\,(1-\epsilon)\,\left(1+\frac{\hat{\eta}_a^2\,s^2}{4\,(p_B\cdot P_a)^2}\right)-\frac{s\,\hat{\eta}_a^2\,p_A\cdot\hat{p}_j}{p_B\cdot P_a\,\hat{p}_j\cdot\hat{Q}}\right]\right\} \end{aligned} \tag{2.150}$$

which, in terms of the variables x and y, can be rewritten as

$$v_{ggg}^2 \;=\; \frac{4\,x\,(1-x-y)}{y\,\eta_a\,\eta_b\,s}\,(d-2)\left[1\,+\,\frac{1}{(1-x)^2}\,+\,\frac{1}{(x+y)^2}\right]-\frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,s\,(1-x)\,(x+y)} \tag{2.151}$$

If we include the eikonal factor, we obtain

$$\boxed{v_{ggg}^2\,-\,v_{\rm eik}^2\,=\,\frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,y\,s}\,\left[(1-\epsilon)\,\left(1\,+\,\frac{1}{(x+y)^2}\right)-\,\frac{\epsilon}{(1-x)^2}\right]-\frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,s\,(1-x)\,(x+y)}} \tag{2.152}$$

Averaging over the incoming parton spins gives an additional factor $1/2/(1-\epsilon)$. Using the variables $\hat{p}_a$, $\hat{p}_j$, p_b and $\hat{Q}$, we then have

$$\begin{aligned} v_{ggg}^2\,-\,v_{\rm eik}^2 \;&=\frac{4\,p_A\cdot\hat{p}_j\,p_B\cdot\hat{p}_j}{s\,(\hat{p}_a\cdot\hat{p}_j)^2}\left[2\,\left(1+\frac{\hat{\eta}_a^2\,s^2}{4\,(p_B\cdot P_a)^2}\right)-\frac{s\,\hat{\eta}_a^2\,p_A\cdot\hat{p}_j}{p_B\cdot P_a\,\hat{p}_j\cdot\hat{Q}}\right] \\ &=\frac{4\,\hat{p}_a\cdot\hat{p}_j\,p_b\cdot\hat{p}_j}{\hat{Q}^2\,(\hat{p}_a\cdot\hat{p}_j)^2}\left[2\,\left(1+\frac{\hat{Q}^4}{4\,(p_b\cdot P_a)^2}\right)-\frac{\hat{Q}^2\,\hat{p}_a\cdot\hat{p}_j}{p_b\cdot P_a\,\hat{p}_j\cdot\hat{Q}}\right] \end{aligned} \tag{2.153}$$

This is the dipole subtraction term for $(m+1)$-parton phase spaces, so we can safely put $d=4$.

2.4.5 Soft splitting functions

We have discussed the spin-averaged splitting functions $\overline{W}_{\ell\ell}$ in which the parton j is emitted from the emitter ℓ in the scattering amplitude and parton j is emitted from that same emitter ℓ in the complex-conjugate scattering amplitude. In higher-order QCD calculations, double poles in splitting functions only arise if the emitted parton j is a gluon. In this case, interference diagrams between different emitters have to be taken into account. This means that the emitted parton j can be emitted from emitter ℓ in the amplitude and parton j can also be emitted from different emitter k in the complex-conjugate amplitude (Fig. 2.8). The interference splitting function is then

$$v_\ell(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)\,v_k(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_k,s_k)^*\,\delta_{\hat{s}_\ell,s_\ell}\,\delta_{\hat{s}_k,s_k} \tag{2.154}$$

The splitting function Eq. (2.154) contains a singularity when the emitted gluon j is soft. However when gluon j is collinear with parton ℓ or parton k, it does not contribute a leading singularity. In the special case that $\hat{p}_j$ is soft, or possibly soft and collinear with $\hat{p}_\ell$, a simpler splitting amplitude can be used. When $\hat{p}_j$ is soft, we have

$$\mid\mathcal{M}(\{\hat{p},\hat{f}\}_{m+1})>\sim\sum_\ell\mid\mathcal{M}_\ell^{\rm soft}(\{\hat{p},\hat{f}\}_{m+1})> \tag{2.155}$$

where

$$\mid\mathcal{M}_\ell^{\rm soft}(\{\hat{p},\hat{f}\}_{m+1})>=\,t_\ell^\dagger(f_\ell\to\hat{f}_\ell+\hat{f}_j)\,V_\ell^{\dagger\,\rm soft}(\{\hat{p},\hat{f}\}_{m+1})\;\mid\mathcal{M}(\{p,f\}_m)> \tag{2.156}$$

The spin dependent splitting operator can be described in the spin space $\mid \{s\}_m >$,

$$< \{\hat{s}\}_{m+1} \mid V_\ell^{\dagger\,\text{soft}}(\{\hat{p},\hat{f}\}_{m+1}) \mid \{s\}_m > = \left(\prod_{n\notin\{\ell,j=m+1\}} \delta_{\hat{s}_n,s_n} \right) v_\ell^{\text{soft}}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) \tag{2.157}$$

If parton j is a quark or antiquark, $v_\ell^{\text{soft}} = 0$. When parton j is a gluon, we can use a simple eikonal approximation to the splitting amplitude,

$$v_\ell^{\text{soft}}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell) = \sqrt{4\pi\alpha_s}\,\delta_{\hat{s}_\ell,s_\ell}\,\frac{\varepsilon(\hat{p}_j,\hat{s}_j,\hat{Q})^*\cdot\hat{p}_\ell}{\hat{p}_j\cdot\hat{p}_\ell} \tag{2.158}$$

Having used the eikonal approximation, the interference splitting function becomes

$$\overline{W}_{\ell k} \sim v_\ell^{\text{soft}}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)\, v_k^{\text{soft}}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_k,s_k)^*\,\delta_{\hat{s}_\ell,s_\ell}\,\delta_{\hat{s}_k,s_k} \tag{2.159}$$

In the Nagy-Soper scheme, we split the collinear and soft parts of the respective spin-averaged splitting functions according to

$$\overline{W}_{\ell\ell} - \overline{W}_{\ell k} = \left(\overline{W}_{\ell\ell} - \overline{W}_{\ell\ell}^{\text{eikonal}}\right) + \left(\overline{W}_{\ell\ell}^{\text{eikonal}} - \overline{W}_{\ell k}\right) \tag{2.160}$$

where the spin-averaged soft splitting function $\overline{W}_{\ell k}$ is given by

$$\overline{W}_{\ell k} = 4\,\pi\,\alpha_s\,2\,A_{\ell k}\frac{\hat{p}_\ell\cdot D(\hat{p}_j,\hat{Q})\cdot\hat{p}_k}{\hat{p}_j\cdot\hat{p}_\ell\,\hat{p}_j\cdot\hat{p}_k} \tag{2.161}$$

Here, $A_{\ell k}$ is the partitioning weight function, which in principle can depend on the momenta $\{\hat{p}\}_{m+1}$. It specifies how the two interference diagrams in Fig. 2.8 are partitioned into separate terms. In [28], the default value is taken to be $A_{\ell k} = A_{k\ell} = 1/2$. We can also define the partitioning function as function of momenta $A_{\ell k}(\{\hat{p}\}_{m+1})$ and $A_{k\ell}(\{\hat{p}\}_{m+1})$, so another choice would be

$$A_{\ell k}(\{\hat{p}\}_{m+1}) = \frac{B_{\ell k}(\{\hat{p}\}_{m+1})}{B_{\ell k}(\{\hat{p}\}_{m+1}) + B_{k\ell}(\{\hat{p}\}_{m+1})} \tag{2.162}$$

where

$$B_{\ell k}(\{\hat{p}\}_{m+1}) = \frac{\hat{p}_j\cdot\hat{p}_k}{\hat{p}_j\cdot\hat{p}_\ell}\,\hat{p}_\ell\cdot D(\hat{p}_j,\hat{Q})\cdot\hat{p}_\ell \tag{2.163}$$

Here, $D(\hat{p}_j,\hat{Q})$ is defined in Eq. (2.99). The partitioning functions are non-negative and obey $A_{\ell k}(\{\hat{p}\}_{m+1}) + A_{k\ell}(\{\hat{p}\}_{m+1}) = 1$.

One of the advantages from dividing the splitting functions into two parts is that the two terms $\overline{W}_{\ell\ell} - \overline{W}_{\ell\ell}^{\text{eikonal}}$ and $\overline{W}_{\ell\ell}^{\text{eikonal}} - \overline{W}_{\ell k}$ are positive, and thus we can use these splitting functions as dipoles to construct a parton shower Monte Carlo program without needing separate partitioning

weight function. Another result from splitting Eq. (2.160) into two pieces is that the first term of Eq. (2.160) only contains a collinear singularity, while the second term of Eq. (2.160) has both the soft singularity and the soft⊗collinear singularity.

The second part of Eq. (2.160) can be then expressed in terms of dipole partitioning functions $A'_{\ell k}$, which is given in [30]:

$$\overline{W}_{\ell\ell}^{\text{eikonal}} - \overline{W}_{\ell k} \;=\; 4\,\pi\,\alpha_s\,A'_{\ell k}\,\frac{-\hat{P}^2_{\ell k}}{(\hat{p}_j\cdot\hat{p}_\ell\,\hat{p}_j\cdot\hat{p}_k)^2} \tag{2.164}$$

where $\hat{P}_{\ell k} \;=\; \hat{p}_j\cdot\hat{p}_\ell\,\hat{p}_k - \hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell$. Several choices for $A'_{\ell k}$ have been proposed in [30]. All results given here have been obtained using Eq. (7.12) therein, which is given by

$$A'_{\ell k}(\{\hat{p}\}_{m+1}) \;=\; \frac{\hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}}{\hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q} + \hat{p}_j\cdot\hat{p}_\ell\,\hat{p}_k\cdot\hat{Q}} \tag{2.165}$$

The partitioning weight function $A'_{\ell k}$ also obeys the relation $A'_{\ell k}(\{\hat{p}\}_{m+1}) + A'_{k\ell}(\{\hat{p}\}_{m+1}) = 1$. The general form of the interference spin-averaged splitting function is then given by

$$\boxed{\Delta W \;=\; \overline{W}_{\ell\ell}^{\text{eikonal}} - \overline{W}_{\ell k} \;=\; \frac{2\,\hat{p}_\ell\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}}{\hat{p}_\ell\cdot\hat{p}_j\,\left(\hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q} + \hat{p}_\ell\cdot\hat{p}_j\,\hat{p}_k\cdot\hat{Q}\right)}} \tag{2.166}$$

Here, we have removed the common factor $4\,\pi\alpha_s$, which we will add back in the end. The only singularity in Eq. (2.166) arises from the factor $\hat{p}_\ell\cdot\hat{p}_j$ in the denominator. The interference term is constructed in such a way that it vanishes for $\hat{p}_j\cdot\hat{p}_k \;\to\; 0$. We also assume that the variables considered are such that they are finite for $p_\ell\cdot p_k \;\to\; 0$, *i.e.* singularities arising in this limit should be taken care of by the definition of the jet function. The interference term only needs to be considered if the emitted parton j is a gluon. If parton j is a quark or antiquark, this term vanishes.

Eq. (2.166) corresponds to interference between final states. It is worth mentioning that if we replace $\hat{p}_\ell$ by $\hat{p}_a$ in Eq. (2.166), then the interference spin-averaged splitting function corresponds to interference between initial and final states,

$$\begin{aligned}
\Delta W_{ak} \;&=\; \frac{2p_A\cdot\hat{p}_k\,\eta_b\,s}{p_A\cdot\hat{p}_j\,\left(\hat{p}_j\cdot\hat{p}_k\,\eta_b\,s + 2\,p_A\cdot\hat{p}_j\,\hat{p}_k\cdot\hat{Q}\right)} \\
&=\; \frac{2\,\hat{p}_a\cdot\hat{p}_k\,\hat{Q}^2}{\hat{p}_a\cdot\hat{p}_j\,\left(\hat{p}_j\cdot\hat{p}_k\,\hat{Q}^2 + 2\,\hat{p}_a\cdot\hat{p}_j\,\hat{p}_k\cdot\hat{Q}\right)} \\
&=\; \frac{4\,\bar{z}\,x}{\eta_a\,\eta_b\,s\,y'\,(1-x)^2\,(v+y')}
\end{aligned} \tag{2.167}$$

where the variables $\bar{z}$, y' , x and v are defined in Section 2.4.4. The Eq. (2.167) can be further reduced by the following replacement

$$\hat{p}_k \quad \to \quad p_b \qquad \text{or} \qquad \bar{z} = \frac{\hat{p}_a \cdot \hat{p}_k}{\hat{p}_k \cdot \hat{Q}} \quad \to \quad 1 \tag{2.168}$$

in which case the soft splitting function Eq. (2.166) corresponds to interference between initial states:

$$\boxed{\Delta W_{ab} \;=\; \frac{\eta_b\, s}{p_A \cdot \hat{p}_j\, \hat{p}_j \cdot \hat{Q}} \;=\; \frac{\hat{Q}^2}{\hat{p}_a \cdot \hat{p}_j\, \hat{p}_j \cdot \hat{Q}} \;=\; \frac{4\,x}{(1-x)\,\eta_a\,\eta_b\,y\,s}} \tag{2.169}$$

Again, if we replace $\hat{p}_k$ by $\hat{p}_a$ in Eq. (2.166), then the interference splitting function corresponds to interference between final and initial states (see *e.g.* Eq. (2.204) or Eq. (2.205)).

2.5 Integrated splitting functions

In this section we will list all the integrated splitting functions, which are needed for the m-parton phase spaces. The integrated splitting functions contain all the singularities in ϵ that are necessary to cancel the poles in one-loop virtual matrix element. In order to achieve the cancellation of singularities we have to define good parametrizations in the integration measures which we will discuss in Section 5.2. We will consider the collinear integrals in both the initial state and final state splittings as well as the interference terms.

For the initial and final state collinear integrals we have been using the colour algebra relations defined by

$$\boxed{\mathrm{T}_\ell^2 \;\equiv\; C_i} \tag{2.170}$$

where $C_i = C_A\,(C_F)$ in the ggg (qqg) splitting. For the splitting process $gq\bar{q}$ we use the colour charge T_R. For the soft splitting functions the appropriate colour charges are given by

$$\left(-\sum_{k\neq\ell} \mathrm{T}_\ell \cdot \mathrm{T}_k \right) \;\equiv\; C_i \tag{2.171}$$

We will discuss this issue in more detail in Section 5.3. Note that parton ℓ is the emitter and parton k is the spectator.

2.5.1 Collinear integrals: final state splittings

The subprocess qqg

After subtracting the eikonal factor, which will be combined with the interference term, the collinear part of the integrated splitting function is given by

$$\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{qqg}-v^2_{\text{eik}}\right]\;=\;\frac{\alpha_s}{4\,\pi}\,C_F\,\left(\frac{4\,\pi\,\mu^2}{2\,p_\ell\cdot Q}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\,\times$$
$$\left\{-\frac{1}{\epsilon}\,-2\,\left[\,2\,a_\ell+1+2\,(a_\ell-1)^2\,\ln(a_\ell-1)\,+\,2\,a_\ell\,(2-a_\ell)\ln\,a_\ell\,-4\,(a_\ell-1)\,y_{\text{max}}\right.\right.$$
$$\left.+\,y^2_{\text{max}}\,+\,2\,\ln\,y_{\text{max}}\,\right]\,+\,\frac{1}{2}\,\left[2\,(1-2\,a_\ell)\,y_{\text{max}}\,+\,\frac{1}{2}\,y^2_{\text{max}}\,+\,\ln\,y_{\text{max}}\,-\,\frac{7}{2}\right.$$
$$\left.\left.+\,a_\ell\,+\,(a^2_\ell-1)\,\ln(a_\ell-1)\,-\,a^2_\ell\,\ln\,a_\ell\,\right]\,+4\,I_{\text{fin}}(a_\ell)\,\right\} \tag{2.172}$$

where

$$\boxed{I_{\text{fin}}(a_\ell)\;=\;-\int_0^{y_{\text{max}}}dy\,\frac{(\lambda-1+y)^2+4\,y}{4y}\,\frac{(1+x_0)\ln\,x_0}{1-x_0}} \tag{2.173}$$

For $a_\ell=1$, this simplifies to

$$\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{qqg}-v^2_{\text{eik}}\right]=\frac{\alpha_s}{4\,\pi}\,C_F\,\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\,\left(-\frac{1}{\epsilon}-14+\frac{4\pi^2}{3}+\mathcal{O}(\epsilon)\right) \tag{2.174}$$

which has been used in dijet production at NLO.

The subprocess $gq\bar{q}$

Including all prefactors, we obtain

$$\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,T_R\,\mu^{2\epsilon}\int d\zeta_p\,v^2_{gq\bar{q}}\;=$$
$$\frac{\alpha_s}{\pi}\,T_R\,\left(\frac{4\,\pi\,\mu^2}{2\,p_\ell\cdot Q}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\,\left[-\frac{1}{3\,\epsilon}\,-\,\frac{8}{9}\,+\,\frac{1}{3}\,\left[\,(a_\ell-1)\,\ln(a_\ell-1)-a_\ell\,\ln\,a_\ell\,\right]\right] \tag{2.175}$$

For $a_\ell=1$, we exactly reproduce the result in [22]. Note that the first two terms in Eq. (2.175) are exactly the same as in [22]; differences in the finite terms stem from the difference in the $(m+1)$- to m-parton momentum matching.

The subprocess ggg

Including all prefactors, we obtain

$$\frac{4\,\pi\,\alpha_s}{2\,(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\left[\tilde{v}^2_{ggg}-v^2_{\text{eik}}\right]\;=\;\frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{4\,\pi\,\mu^2}{2\,p_\ell\cdot Q}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\left[-\frac{1}{6\,\epsilon}\,-\,\frac{4}{9}\,+\,\frac{1}{6}\,\Big[\,(a_\ell-1)\,\ln(a_\ell-1)\,-\,a_\ell\ln\,a_\ell\,\Big]\,+\,I_{\text{fin}}(a_\ell)\right] \tag{2.176}$$

where

$$I_{\text{fin}}(a_\ell)\;=\;-2\,a_\ell\int_0^{y_{\text{max}}}\frac{dy}{\lambda\,(1+y)^2}\left\{\left[(1+y)^2+2\,a_\ell\,y\right]\,+\,4\,a_\ell\,\frac{y\,\ln\,x_0}{\lambda\,(1+y)}\,\left[(1+y)^2-a_\ell\,y\right]\right\}$$
$$\stackrel{a_\ell\neq 1}{=}\;a_\ell\left\{1-\sqrt{a_\ell}\,\ln\left(\frac{\sqrt{a_\ell}+1}{\sqrt{a_\ell}-1}\right)-\ln\left(\frac{a_\ell}{a_\ell-1}\right)\,+\,8\,a_\ell\int_0^{y_{\text{max}}}dy\,\frac{y\,\ln\,x_0}{\lambda^2\,(1+y)^3}\,\left[a_\ell\,y-(1+y)^2\right]\right\}$$
$$\stackrel{a_\ell=1}{=}\;-\frac{3}{8}\pi^2\,+\,\frac{7}{2} \tag{2.177}$$

2.5.2 Collinear integrals: initial state splittings

The subprocess $qq(x)g$

Using the variables defined in Section 2.4.4, the unaveraged $qq(x)g$ splitting function is given by

$$\boxed{v^2_{qqg}\;=\;\frac{2\,(1-x-y)}{\eta_a\,\eta_b\,y\,s}\left(d-2+\frac{4\,x}{(1-x)^2}\right)} \tag{2.178}$$

Including all prefactors, we obtain

$$\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,v^2_{qqg}$$
$$=\,\frac{\alpha_s}{2\,\pi}\,C_F\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\int_0^1\frac{dx}{x}\left[\delta(1-x)\left(\frac{1}{\epsilon^2}+\frac{5}{2\,\epsilon}\right)-\frac{1}{\epsilon}\left(\frac{1+x^2}{1-x}\right)_+\,+\,\mathfrak{B}(x,\epsilon^0)\right] \tag{2.179}$$

where

$$\begin{aligned}\mathfrak{B}(x,\epsilon^0)\;&=\;-(1-x)\,\ln x+2\,(1-x)\,\ln(1-x)+4\,x\,\left(\frac{\ln(1-x)}{1-x}\right)_+\\&-\;\frac{2\,x\,(1+\ln x)}{(1-x)_+}+2\,\delta(1-x)\left(1-\frac{\pi^2}{12}\right)\end{aligned} \tag{2.180}$$

The leftover factor $1/x$ will be reabsorbed into the flux factor. Note that we should take the eikonal splitting function into account, when we consider the interference term. The integrated eikonal splitting function will be discussed later in more detail.

The subprocess $qqg(x)$

Using the variables defined in Section 2.4.4, the unaveraged $qqg(x)$ splitting function is given by

$$\boxed{v^2_{qqg} \;=\; \frac{2\,(d-2)\,x}{\eta_a\,\eta_b\,y\,s} \;+\; \frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,s\,y\,(x+y)^2}} \tag{2.181}$$

Including all prefactors, we obtain

$$\begin{aligned}&\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,v^2_{qqg}\\ &=\frac{\alpha_s}{2\,\pi}\,C_F\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\int_0^1\frac{dx}{x}\left[-\frac{1}{\epsilon}\,\left(\frac{1+(1-x)^2}{x}\right)+\mathfrak{B}(x,\epsilon^0)\right]\end{aligned} \tag{2.182}$$

where

$$\mathfrak{B}(x,\epsilon^0) \;=\; \frac{x^2-2\,(1-x)}{x}-x\,\ln x+2\,\ln(1-x)\,\left(\frac{1+(1-x)^2}{x}\right) \tag{2.183}$$

The leftover factor $1/x$ will be reabsorbed into the flux factor.

The subprocess $gq\bar{q}$

The unaveraged $gq\bar{q}$ splitting function is given by

$$\boxed{v^2_{gq\bar{q}} \;=\; \frac{2\,(d-2)}{\eta_a\,\eta_b\,y\,s} \;+\; \frac{8\,(x+y)}{\eta_a\,\eta_b\,y\,s}\;(x+y-1)} \tag{2.184}$$

Including all prefactors, the integrated splitting function is given by

$$\begin{aligned}&\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,T_R\,\mu^{2\epsilon}\int d\zeta_p\,v^2_{gq\bar{q}}\\ &=\frac{\alpha_s}{2\,\pi}\,T_R\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\int_0^1\frac{dx}{x}\left[-\frac{1}{\epsilon}\,\left[x^2\,+\,(1-x)^2\right]\,+\,\mathfrak{B}(x,\epsilon^0)\right]\end{aligned} \tag{2.185}$$

where

$$\mathfrak{B}(x,\epsilon^0) \;=\; 6\,x-5\,x^2-1\,+\left[2\,\ln(1-x)\,-\,\ln\,x\right]\left[x^2\,+\,(1-x)^2\right] \tag{2.186}$$

The leftover factor $1/x$ will be reabsorbed into the flux factor.

The subprocess ggg

The unaveraged ggg splitting function is given by

$$\boxed{v_{ggg}^2 \;=\; \frac{4\,x\,(1-x-y)}{y\,\eta_a\,\eta_b\,s}\,(d-2)\left[1\;+\;\frac{1}{(1-x)^2}\;+\;\frac{1}{(x+y)^2}\right]-\frac{8\,x\,(1-x-y)}{\eta_a\,\eta_b\,s\,(1-x)\,(x+y)}} \tag{2.187}$$

Including all prefactors, the integrated splitting function is given by

$$\begin{aligned}&\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,v_{ggg}^2 \;=\; \frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\\ &\times\int_0^1\frac{dx}{x}\left[\frac{1}{\epsilon^2}\,\delta(1-x)+\frac{1}{\epsilon}\left[\delta(1-x)-2\,\left(\frac{x}{(1-x)_+}+x\,(1-x)+\frac{1-x}{x}\right)\right]+\mathfrak{B}(x,\epsilon^0)\right]\end{aligned} \tag{2.188}$$

where

$$\begin{aligned}\mathfrak{B}(x,\epsilon^0) = {}& 4\,x\left(\frac{\ln(1-x)}{1-x}\right)_+ - 2\,x(1-x)\ln x + 4\,(1-x)\ln(1-x)\left(\frac{1+x^2}{x}\right)\\ &+2\left(x^2-\frac{1-x}{x}-\frac{x}{(1-x)_+}\right)+2\,\left(1-\frac{\pi^2}{12}\right)\delta(1-x)\end{aligned} \tag{2.189}$$

The leftover factor $1/x$ will be reabsorbed into the flux factor.

The subprocess ggg with eikonal splitting function

Now we consider the eikonal factor. Using the variables defined in Section 2.4.4, the averaged eikonal splitting function is given by (see also Section 2.4.2)

$$\boxed{\bar{v}_{\text{eik}}^2 \;=\; \frac{4\,x\,(1-x-y)}{(1-x)^2\,\eta_a\,\eta_b\,y\,s}} \tag{2.190}$$

Integrating it out, we obtain

$$\begin{aligned}&4\,\pi\,\alpha_s\,C_i\,\mu^{2\epsilon}\int d\zeta_p\,\bar{v}_{\text{eik}}^2 \;=\; \frac{\alpha_s}{2\,\pi}\,C_i\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\\ &\times\int_0^1 dx\left[\frac{1}{\epsilon^2}\,\delta(1-x)+\frac{1}{\epsilon}\left(\delta(1-x)-\frac{2}{(1-x)_+}\right)+2\,\left(1-\frac{\pi^2}{12}\right)\delta(1-x)\right.\\ &\left.-2\,\frac{1+\ln x}{(1-x)_+}+4\left(\frac{\ln(1-x)}{1-x}\right)_+\right]\end{aligned} \tag{2.191}$$

Here, C_i is given by Eq. (2.170). Hence the collinear ggg splitting including eikonal factor is given by

$$\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\left[v_{ggg}^2-\,v_{\text{eik}}^2\right]\,=\,\frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\,\int_0^1\frac{dx}{x}\left[-\frac{2}{\epsilon}\left(\frac{x}{(1-x)_+}+x(1-x)+\frac{1-x}{x}\right)+\frac{1}{\epsilon}\frac{2\,x}{(1-x)_+}+\mathfrak{A}(x,\epsilon^0)\right] \tag{2.192}$$

where

$$\mathfrak{A}(x,\epsilon^0)=-2\,x\,(1-x)\ln x+4\,(1-x)\ln(1-x)\left(\frac{1+x^2}{x}\right)+2\,\left(x^2-\frac{1-x}{x}\right)+2\,x\,\frac{\ln x}{(1-x)_+} \tag{2.193}$$

The complete integrated ggg splitting function including collinear and interference terms (see Section 2.5.3) is then given by

$$\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\left[v_{ggg}^2-\,v_{\text{eik}}^2\right]+4\,\pi\,\alpha_s\,C_A\,\mu^{2\epsilon}\,\int d\zeta_p\,\Delta W_{ab}=\,\frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\int_0^1\frac{dx}{x}\left[\left(\frac{1}{\epsilon^2}-\frac{\pi^2}{6}\right)\,\delta(1-x)-\,\frac{2}{\epsilon}\left(\frac{x}{(1-x)_+}+x(1-x)+\frac{1-x}{x}\right)+\mathfrak{J}(x,\epsilon^0)\right] \tag{2.194}$$

where

$$\mathfrak{J}(x,\epsilon^0)=4\,x\left(\frac{\ln(1-x)}{1-x}\right)_+-2\,x\,(1-x)\ln x+4(1-x)\ln(1-x)\left(\frac{1+x^2}{x}\right)+2\left(x^2-\frac{1-x}{x}\right) \tag{2.195}$$

2.5.3 Interference between initial states

After adding back the eikonal factor, which we have subtracted in the collinear integrals, the interference part of the integrated splitting function (see Eq. (2.169)) including all prefactors is given by

$$4\,\pi\,\alpha_s\,C_i\,\mu^{2\epsilon}\,\int d\zeta_p\,\Delta W_{ab}\,=\,\frac{\alpha_s}{2\,\pi}\,C_i\,\left(\frac{4\,\pi\,\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\int_0^1\,dx\,\left\{\frac{1}{\epsilon^2}\,\delta(1-x)-\frac{2}{\epsilon}\,\frac{1}{(1-x)_+}-\frac{\pi^2}{6}\,\delta(1-x)-2\,\frac{\ln x}{(1-x)_+}+4\,\left(\frac{\ln(1-x)}{1-x}\right)_+\right\} \tag{2.196}$$

where C_i is given by Eq. (2.171). Note that the individual part of the unaveraged soft splitting function is given by

$$\boxed{v_{\text{soft}}^2\,=\,-16\,\frac{x\,(1-x-y)^2}{\eta_a\,\eta_b\,s\,(1-x)^2\,\,(y^2+(1-x-y)^2)}} \tag{2.197}$$

Integrating it out, we obtain

$$\frac{4\pi\alpha_s}{2}\,C_i\,\mu^{2\epsilon}\int d\zeta_p\,v_{\text{soft}}^2$$
$$=\frac{\alpha_s}{2\,\pi}\,C_i\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\int_0^1 dx\,\left[\frac{1}{\epsilon}\,\delta(1-x)+2\,\delta(1-x)-\frac{2}{(1-x)_+}\right] \tag{2.198}$$

2.5.4 Interference between initial and final states

The initial-final state interference splitting function is given by Eq. (2.167), which is derived from Eq. (2.166) by the replacement $\hat{p}_\ell \to \hat{p}_a$. Using the initial state integration measure given by Eq. (5.43), we obtain

$$4\,\pi\,\alpha_s\,C_i\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W_{ak} \;=\; \frac{\alpha_s}{2\,\pi}\,C_i\,\left(\frac{4\pi\mu^2}{\eta_a\,\eta_b\,s}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\int_0^1 dx\,\left\{\frac{1}{\epsilon^2}\,\delta(1-x)-\frac{1}{\epsilon}\left[\frac{2}{(1-x)_+}+\delta(1-x)\,\ln z_0\right]+4\left(\frac{\ln(1-x)}{1-x}\right)_+-\frac{2\ln x}{(1-x)_+}\right.$$
$$\left.+\ln(16)\,\ln z_0\,\delta(1-x)+\frac{2}{\pi}\,I_{\text{fin}}^a(x,\bar{z})\right\} \tag{2.199}$$

where C_i is given by Eq. (2.171). In Eq. (2.167), $\bar{z}=0$ corresponds to a singularity in the m-particle phase space. This singularity should be excluded by an appropriate jet function definition since we only consider infrared safe observables. The function $\bar{z}=\bar{z}(x,y')$ in Eq. (2.167) contains Lorentz-transformed variables. Only in the limit of $x\to 1$ (soft limit) or $y'\to 0$ (soft or collinear limit) this dependence disappears; in this case the $\bar{z}$ is denoted by z_0. Hence we have

$$\boxed{\bar{z}\;=\;\bar{z}(x,y'),\quad z_0\;=\;\bar{z}(1,y')=\bar{z}(x,0),\quad 0<z_0<1} \tag{2.200}$$

The finite term I_{fin}^a is given by

$$I_{\text{fin}}^a=I_{\text{fin}}^a(x,\bar{z})=\pi\,\delta(1-x)\left\{-\ln 4\,\ln z_0-\frac{1}{8}\,\left[2\,\text{Li}_2\left(\frac{z_0-1}{z_0}\right)-\ln^2 z_0\right]\right.$$
$$\left.+\int_0^1 dy\,\frac{z_0}{y\sqrt{4\,y^2\,(1-z_0)+z_0^2}}\,\ln\frac{2\sqrt{4\,y^2\,(1-z_0)+z_0^2}\,\sqrt{1-y}}{2\,y\,(1-z_0)+z_0+\sqrt{4\,y^2\,(1-z_0)+z_0^2}}\right\}$$
$$+\frac{1}{(1-x)_+}\int_0^1\frac{dy}{y}\left\{\left[\int_0^1\frac{dv}{\sqrt{v\,(1-v)}}\,\frac{\bar{z}(x,y)}{N(x,y,v,\bar{z})}\right]-\pi\right\} \tag{2.201}$$

with

$$N(x,y,v,\bar{z})\;=\;(4v-2)\sqrt{y\,(1-y)\,\bar{z}\,(1-\bar{z})}+2\,y+\bar{z}-2\,y\,\bar{z} \tag{2.202}$$

Here, v and y are dummy variables, which we denoted with v' and y' previously. So the function $\bar{z}$ in Eq. (2.201) is now a function of x and y. Note that we can only use Eq. (2.200) in the

singular limits ($x \to 1$ or $y' \to 0$) where the dependence on v' disappears. For the finite parts, *i.e.* Eq. (2.201), we need to use the original definition of $\bar{z}$ given by Eq. (2.136)

$$\bar{z} \;=\; \frac{\hat{p}_a \cdot \hat{p}_k}{\hat{p}_k \cdot \hat{Q}}$$

Additionally we use the original definition of v given by Eq. (2.135)

$$v \;=\; \frac{\hat{\eta}_a\, \eta_b\, s\, \hat{p}_j \cdot \hat{p}_k}{2\, \hat{p}_k \cdot \hat{Q}\, \hat{p}_j \cdot \hat{Q}} \;=\; \frac{1}{1-x}\, \frac{\hat{p}_j \cdot \hat{p}_k}{\hat{p}_k \cdot \hat{Q}}$$

where $\hat{p}_k$ needs to be calculated using the Lorentz transformation: $\hat{p}_k^{\mu} \;=\; \Lambda(\hat{K}, K)^{\mu}{}_{\nu}\, p_k^{\nu}$ and $\hat{p}_j$ is parametrized according to Sudhakov parametrization[9]. In the singular limits, we obtain $\hat{p}_k \;\to\; p_k$. For $\bar{z} \;\to\; 1$ (which means $\hat{p}_k \;\to\; p_b$), we get

$$\boxed{I^a_{\text{fin}} \;=\; -\frac{\pi^3}{12}\, \delta(1-x)} \tag{2.203}$$

in which case the initial-final state integrated splitting function is reduced to initial-initial state interference term (see Section 2.5.3).

2.5.5 Interference between final (final and initial) states

The final-final and final-initial state interference terms have the same structure. The Eq. (2.166) corresponds to final-final state interference term. If we replace $\hat{p}_k$ by $\hat{p}_a$ in Eq. (2.166), then the interference splitting function corresponds to final-initial state interference term. For $a_\ell = 1$ (which corresponds to only two particles in the final state), there are two situations: $\tilde{a} \;=\; 0$ and $\tilde{a} \;\neq\; 0$, which corresponds to final-final and final-initial state splittings, respectively. Using the variables defined in Section 2.4.3, the interference splitting function Eq. (2.166) can be rewritten as

$$\Delta W = \frac{1}{y\; (p_\ell \cdot Q)} \left(\frac{2\,(1-v)\,(1-z\,(1-x_0))}{v\,(1-z\,(1-x_0)) + x_0 \left[\lambda\, \frac{\tilde{a}}{a_\ell} + 1\right]} \right) \tag{2.204}$$

where $\tilde{a} \;=\; \tilde{a}(y)$ and the parameter v is given by Eq. (5.23). We can split this function into a singular and a non-singular part, leading to

$$\Delta W \;=\; \frac{1}{y\; (x_0 + z\,(1-x_0))}\, \Delta W_{\text{fin}} = \frac{1}{x\, y}\, \Delta W_{\text{fin}} \tag{2.205}$$

with

$$z = \frac{x - x_0}{1 - x_0} \tag{2.206}$$

[9] See also the discussions in Section 5.2.1 and Section 5.2.2.

and

$$\begin{aligned}\Delta W_{\text{fin}} &= \frac{2}{(p_\ell\cdot Q)}\left(\frac{(x_0+z\,(1-x_0))\;(1-v)\,(1-z\,(1-x_0))}{v\,(1-z\,(1-x_0))+x_0\left[\lambda\,\frac{\tilde{a}}{a_\ell}+1\right]}\right)\\ &= \frac{2}{(p_\ell\cdot Q)}\,(x_0+z\,(1-x_0))\left[\frac{(1-z\,(1-x_0))+x_0\left[\lambda\,\frac{\tilde{a}}{a_\ell}+1\right]}{v\,(1-z\,(1-x_0))+x_0\left[\lambda\,\frac{\tilde{a}}{a_\ell}+1\right]}-1\right]\\ &= \frac{2}{(p_\ell\cdot Q)}\,x\,\left[\frac{1-x+x_0\left[\lambda\,\frac{\tilde{a}}{a_\ell}+2\right]}{v\,(1+x_0-x)+x_0\left[\lambda\,\frac{\tilde{a}}{a_\ell}+1\right]}-1\right]\end{aligned}\tag{2.207}$$

In the following we will discuss $a_\ell = 1$ and $a_\ell \neq 1$ cases, respectively.

Simplified case: $a_\ell = 1$ and $\tilde{a} = 0$ (Interference between final states)

If $\hat{p}_k$ is the final state spectator, it is straightforward to show that $\tilde{a} = 0$ from Eq. (2.112). Hence, $\tilde{a} = 0$ corresponds to final-final state splitting. The averaged splitting function is then given by

$$\Delta W = \frac{1}{ux^2}\frac{2}{p_\ell\cdot Q}\,x\,\left(\frac{(1-x)+ux\,[(1-ux)\tilde{a}+2]}{v(1+ux-x)+ux\,[(1-ux)\tilde{a}+1]}-1\right) \;=\; \frac{1}{ux^2}\,\Delta W_{\text{fin}}\tag{2.208}$$

where ΔW_{fin} satisfies the following limits

$$\begin{aligned}\lim_{u\to 0}\Delta W_{\text{fin}} &= \frac{2}{p_\ell\cdot Q}\,(1-x),\\ \lim_{x\to 0}\Delta W_{\text{fin}} &= \frac{2}{p_\ell\cdot Q}\,\frac{1}{v_r(x=0)+u\,(a_0+1)},\\ \lim_{u\to 0,x\to 0}\Delta W_{\text{fin}} &= \frac{2}{p_\ell\cdot Q}\end{aligned}\tag{2.209}$$

and

$$\boxed{\lim_{x\to 0}\tilde{a} = \lim_{u\to 0}\tilde{a} = a_0}\tag{2.210}$$

which for final-final state interference term equals zero ($a_0 = 0$). The function $v_r(x=0)$ is given by

$$\lim_{x\to 0} v = x\,v_r(x=0) = x\,\left[(4v'-2)\sqrt{u(1-u)\,a_0}+1-u+u\,a_0\right]\tag{2.211}$$

Including all prefactors, the integrated splitting function is given by

$$4\pi\alpha_s\,C_i\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W \;=\; \frac{\alpha_s}{\pi}\,C_i\,\left(\frac{4\pi\mu^2}{2p_\ell\cdot Q}\right)^{\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\left(\frac{1}{2\,\epsilon^2}+\frac{1}{\epsilon}+3-\frac{\pi^2}{4}+\mathcal{O}(\epsilon)\right)\tag{2.212}$$

where C_i is given by Eq. (2.171).

Simplified case: $a_\ell = 1$ and $\tilde{a} \neq 0$ (Interference between final and initial states)

Now, if $\hat{p}_k$ is the initial state parton ($\hat{p}_k$ is replaced by $\hat{p}_a$), then

$$\tilde{a} = \frac{\hat{p}_a \cdot \hat{p}_k}{(1-y)\,\hat{p}_a \cdot (\hat{p}_\ell + \hat{p}_j)} \neq 0 \tag{2.213}$$

It follows that, $\tilde{a} \neq 0$ corresponds to final-initial state interference term. Including all prefactors, the integrated splitting function is

$$\begin{aligned} 4\pi\alpha_s\, C_i\, \mu^{2\epsilon} \int d\zeta_p\, \Delta W = \frac{\alpha_s}{\pi}\, C_i \left(\frac{4\pi\mu^2}{2\,p_\ell \cdot Q}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left[1 + \frac{1}{2}\ln(1+a_0)\right] \right. \\ \left. -\frac{\pi^2}{6} + 3 - 2\ln 2 \ln(1+a_0) + \frac{1}{\pi}\left(I^b_{\text{fin}} + I^c_{\text{fin}}\right) \right\} \end{aligned} \tag{2.214}$$

where C_i is given by Eq. (2.171) and the finite terms are given by

$$\begin{aligned} I^b_{\text{fin}}(a_0) &= \frac{\pi}{2}\int_0^1 \frac{du}{u} \left[2\ln 2 + \frac{1}{\sqrt{1+4a_0(1+a_0)u^2}} \ln\left[\frac{(1-u)}{\left(1+2a_0u+\sqrt{1+4a_0(1+a_0)u^2}\right)^2} \right] \right] \\ &\quad + \frac{\pi}{2}\left(2\ln 2\ln(1+a_0) + \frac{1}{2}\ln^2(1+a_0) + \frac{5}{2}\text{Li}_2\left(\frac{a_0}{a_0+1}\right) - \frac{1}{2}\text{Li}_2\left[\left(\frac{a_0}{a_0+1}\right)^2\right] \right) \\ I^c_{\text{fin}}(\tilde{a}) &= \int_0^1 \frac{du}{u}\int_0^1 \frac{dx}{x}\int_0^1 dv'\, \left[v'(1-v')\right]^{-\frac{1}{2}} \\ &\quad \times \left\{ \frac{x\,(1-x+u\,x\,[(1-u\,x)\,\tilde{a}\,+\,2])}{v\,[1+u\,x-x]\,+\,u\,x\,[(1-u\,x)\,\tilde{a}+1]} - \frac{1}{1+2\,a_0\,u\,+(4v'-2)\sqrt{u\,(1-u)\,a_0}} \right\} \\ &= \pi\int_0^1 \frac{du}{u}\int_0^1 \frac{dx}{x} \left\{ \frac{x\,(1-x+u\,x\,[(1-u\,x)\,\tilde{a}\,+\,2])}{k(u,x,\tilde{a})} - \frac{1}{\sqrt{1+4\,a_0\,u^2\,(1+a_0)}} \right\} \end{aligned} \tag{2.215}$$

We have introduced

$$\begin{aligned} k^2(u,x,\tilde{a}) &= \left[(1+u\,x\,-x)(z-\bar{z})\,+\,u\,x\,\left((1-u\,x)\,\tilde{a}+1\right)\right]^2 \\ &\quad +\,4\,u\,x\,\bar{z}\,(1-z)\,(1+u\,x-x)\,\left((1-ux)\,\tilde{a}+1\right) \end{aligned} \tag{2.216}$$

and

$$z = \frac{x\,(1-u)}{1-u\,x},\ \bar{z} = u\,x\,\tilde{a},\ \tilde{a} = \frac{p_a \cdot n_\ell}{p_a \cdot p_\ell\,+\,y\,p_a \cdot n_\ell},\ a_0 = \tilde{a}(y=0) = \frac{p_a \cdot n_\ell}{p_a \cdot p_\ell} \tag{2.217}$$

Here, the parameter v is given by Eq. (5.23). For $\tilde{a} = 0$, the finite terms I^b_{fin} and I^c_{fin} can be reduced to

$$\boxed{I^b_{\text{fin}} = -\frac{\pi^3}{12}, \qquad I^c_{\text{fin}} = 0} \tag{2.218}$$

and we of course obtain the result in the last subsection. All leftover integrals are finite in the limits $u \to 0$ and $x \to 0$.

General case: $a_\ell \neq 1$

We will now consider the integrated splitting function for $a_\ell \neq 1$. We have again the factorization into a finite and a singular term (see Eq. (2.205)), with the limits of the finite term ΔW_{fin} being given by

$$\begin{aligned}
\lim_{u\to 0} \Delta W_{\text{fin}} &= \frac{2}{p_\ell \cdot \hat{Q}}\,(1-x), \\
\lim_{x\to 0} \Delta W_{\text{fin}} &= \frac{2}{p_\ell \cdot \hat{Q}}\,\frac{1}{v_r(x=0) + u\left(\frac{a_0}{a_\ell}+1\right)}, \\
\lim_{u\to 0, x\to 0} \Delta W_{\text{fin}} &= \frac{2}{p_\ell \cdot \hat{Q}}
\end{aligned} \tag{2.219}$$

Here, the parameter $v_r(x=0)$ is

$$v_r(x=0) \;=\; (4v'-2)\,\sqrt{u(1-u)\frac{a_0}{a_\ell}} + 1 + u\,\left(\frac{a_0}{a_\ell} - 1\right) \tag{2.220}$$

Including all prefactors, the integrated splitting function is

$$\begin{aligned}
4\pi\alpha_s\, C_i\, \mu^{2\epsilon} \int d\zeta_p\, \Delta W = \frac{\alpha_s}{\pi}\, C_i\, \left(\frac{4\pi\mu^2}{2\,p_\ell\cdot\hat{Q}}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \Bigg\{ & \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left[1 + \frac{1}{2}\ln\left(a_\ell + a_0\right)\right] \\
& - \frac{\pi^2}{6} + 3 - 2\,\ln 2\,\ln(a_\ell + a_0) + \frac{1}{\pi}\left[I^b_{\text{fin}}\left(\frac{a_0}{a_\ell}\right) + I^d_{\text{fin}} + I^e_{\text{fin}}\right] \\
& + \ln a_\ell \left[2\ln 2 - \frac{1}{4}\ln a_\ell + \frac{1}{2}\ln\left(a_\ell + a_0\right) + 1\right]\Bigg\}
\end{aligned} \tag{2.221}$$

where C_i is given by Eq. (2.171) and the finite terms are

$$\begin{aligned}
I^d_{\text{fin}} = & \int_0^1 \frac{du}{u} \int_0^1 \frac{dx}{x} \int_0^1 dv'\; \left[v'\,(1-v')\right]^{-\frac{1}{2}} \\
& \times \left\{ \gamma\, x\, \left[\frac{1 - x + x_0\,\left[\lambda\,\frac{\tilde{a}}{a_\ell} \,+\, 2\right]}{v\,\left[1 + x_0 - x\right] \,+\, x_0\,\left[\lambda\,\frac{\tilde{a}}{a_\ell} + 1\right]} - 1\right] + x \,-\, \frac{1}{v_r(x\,=\,0) + u\left(\frac{a_0}{a_\ell}+1\right)}\right\} \\
I^e_{\text{fin}} = & \;\pi \int_0^1 dx\, \left(\frac{1-x}{x}\right)\, \ln\left[\frac{\delta\, a_\ell}{x}\right]
\end{aligned} \tag{2.222}$$

with v now being given by

$$v \;=\; \frac{1}{1-x_0}\,\Big\{(4v'-2)\,[(1-x)\,(x-x_0)\,\bar{z}(1-\bar{z})]^{\frac{1}{2}} + x \;-\; x_0 + \bar{z}\,(1-x_0) \;-2\,\bar{z}\,(x-x_0)\Big\} \tag{2.223}$$

We use Eq. (2.113) for $\tilde{a}$ and Eq. (2.111) for $\bar{z}$. The spectator $\hat{p}_k$ needs to be calculated using the Lorentz transformation

$$\hat{p}_k^{\mu} \;=\; \Lambda(\hat{K}, K)^{\mu}{}_{\nu}\, p_k^{\nu} \quad .$$

Note that for the initial-state splitting, we need to use the original definition of v. The problem for the initial state is that an additional angle appears in the Lorentz transformation; this is not the case for the final state, where we take P_ℓ as given by Eq. (2.22), without any reference to the additional angle and/or use of $\hat{p}_j$. Hence, we never need to use the Sudhakov parametrization for $\hat{p}_j$ in the final state [10].

The finite term $I^b_{\text{fin}}\,(a_0/a_\ell)$ in Eq. (2.221) means that a_0 is now being replaced by a_0/a_ℓ in $I^b_{\text{fin}}(a_0)$, which is already given in the last subsection. For $a_\ell = 1$, $I^e_{\text{fin}} = 0$ and $I^d_{\text{fin}} = I^c_{\text{fin}}$. Hence, we obtain the result in the last subsection. All leftover integrals are finite in the limits $u \to 0$ and $x \to 0$. Note also that the treatment of interference terms significantly differs from [22]. Here, our choice of momentum mapping leads to more complicated integrated finite terms, which we choose to evaluate numerically.

[10] See also the discussions in Section 5.2.1 and Section 5.2.2.

Chapter 3

Applications

For Nagy-Soper scheme, all collinear as well as singular parts of the soft splitting functions have been tested. In this chapter, we give a numerical comparison for Drell-Yan process at NLO using [22] as well as the Nagy-Soper scheme. We also give the analytic result of our splitting functions when applied to dijet production at lepton colliders, as well as the Higgs production at hadron colliders and decay.

3.1 Single W production

3.1.1 Tree level

We start with a simple process: single W production at hadron collider. The W production provides one of the cleanest processes with a large cross section at the Tevatron and at the LHC. This process is not only suited for a precise determination of the W boson mass M_W, it also yields valuable information on the parton structure of the proton. The QCD NLO calculations have been available in the literatures for some time [37–40].

The cross section for W production at hadron collider is

$$A + B \rightarrow W^{\pm} + \text{anything} \tag{3.1}$$

The parton level subprocess in this case is (Fig. 3.1)

$$q\,\bar{q}' \rightarrow W^{+} \tag{3.2}$$

where q is a quark with charge $2/3$ (or an antiquark with charge $1/3$) from hadron A and $\bar{q}'$ is an antiquark with charge $1/3$ (or a quark with charge $2/3$) from hadron B. Labelling the momenta

by $q(p_2)\,\bar{q}'(p_1) \to W^+(P)$, we have

$$\mathcal{M} \;=\; -i\,V_{qq'}\,\frac{g}{\sqrt{2}}\,\varepsilon_\alpha^{\lambda *}(P)\,\bar{v}(p_1)\,\gamma^\alpha\,\frac{1}{2}\,\left(1-\gamma^5\right)\,u(p_2) \tag{3.3}$$

Neglecting the quark masses and squaring the matrix element gives

$$\frac{1}{4}\sum_{\text{spins}}\mid \mathcal{M}\mid^2 = \;\frac{g^2}{4}\;\mid V_{qq'}\mid^2\;\left(p_1\cdot p_2 + 2\,\frac{p_1\cdot P\,p_2\cdot P}{M_W^2}\right) \tag{3.4}$$

where we have averaged over the incoming parton spins. For the massive on-shell vector boson, we use

$$\sum_\sigma \varepsilon_\sigma^{\mu *}(P)\,\varepsilon_\sigma^\nu(P) \;=\; -g^{\mu\nu} + \frac{P^\mu\,P^\nu}{M_W^2} \tag{3.5}$$

Using now $P \;=\; p_1 + p_2$ and $P^2 \;=\; M_W^2$, we get

$$\frac{1}{4}\sum_{\text{spins}}\mid \mathcal{M}\mid^2 = \;\frac{g^2}{4}\;\mid V_{qq'}\mid^2\;M_W^2 \tag{3.6}$$

If we average over the parton colours, we obtain an additional factor $3 \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{3}$ leading to

$$\mid \mathcal{M}_B\mid^2 = \;\frac{1}{4}\,\frac{1}{9}\sum_{\text{spins, colours}}\mid \mathcal{M}\mid^2 = \;\frac{g^2}{12}\;\mid V_{qq'}\mid^2\;M_W^2 \tag{3.7}$$

The one-particle phase space is given by

$$\int dPS_1 \;=\; \int \frac{d^3\vec{P}}{(2\pi)^3\,2\,P^0}\,(2\pi)^4\,\delta^4(P-p_1-p_2) \;=\; 2\,\pi\,\delta^+(\hat{s}-M_W^2) \tag{3.8}$$

Here, $\hat{s}$ is the center of mass energy of the colliding partons.

3.1.2 Real emission, virtual correction and dipole subtraction

There are two subprocesses needed to be considered (Fig. 3.2 and Fig. 3.3): $q\bar{q} \to g\,W$ and $qg \to q\,W$. For the subprocess $q\bar{q} \to g\,W$, a gluon can be emitted from either the incoming quark or the incoming antiquark. Labelling the momenta by $q(p)\,\bar{q}(p') \to g(k')\,W(k)$, we have

$$\begin{aligned} i\,\mathcal{M} = {} & \frac{-i\,g\,g_s}{\sqrt{2}}\,t_{ij}^A\,\varepsilon_\mu^*(k')\,\tilde{\varepsilon}_\nu^*(k)\,V_{q\bar{q}} \\ & \times \bar{v}(p')\,\left[\gamma^\mu\,\frac{(\not{p}'-\not{k}')}{(p'-k')^2+i\epsilon}\,\gamma^\nu\,\frac{1}{2}\,(1-\gamma^5) - \gamma^\nu\,\frac{1}{2}\,(1-\gamma^5)\,\frac{(\not{p}-\not{k}')}{(p-k')^2+i\epsilon}\,\gamma^\mu\right]\,u(p) \end{aligned} \tag{3.9}$$

Figure 3.1: LO and virtual diagrams.

Figure 3.2: Real emission diagrams: $q\bar{q} \to g\,W$

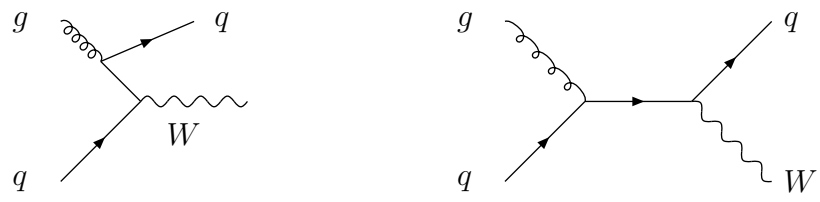


Figure 3.3: Real emission diagrams: $qg \to q\,W$

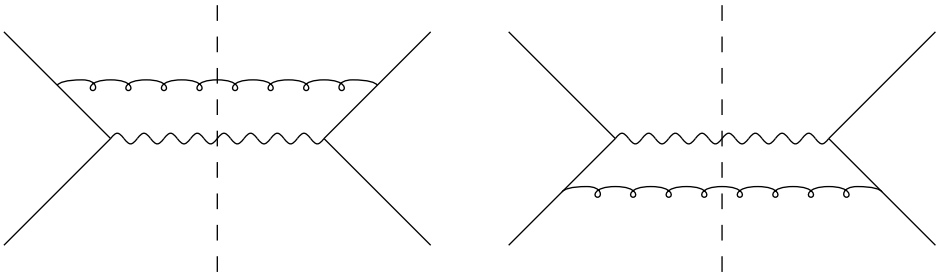


Figure 3.4: Collinear splittings.

Here we have neglected the quark/antiquark masses, the ε_μ and $\tilde{\varepsilon}_\mu$ denote the polarization vectors of gluon and W boson, respectively. They satisfy

$$\begin{aligned} \sum_\sigma \varepsilon_\sigma^{\mu *}(k')\,\varepsilon_\sigma^{\rho}(k') &= -g^{\mu\rho} \\ \sum_\tau \tilde{\varepsilon}_\tau^{\nu *}(k)\,\tilde{\varepsilon}_\tau^{\sigma}(k) &= -g^{\nu\sigma} + \frac{k^\nu\,k^\sigma}{M_W^2} \end{aligned} \tag{3.10}$$

Squaring the amplitude and averaging over the parton spins and colours, we obtain the matrix element for NLO real emission process $q\bar{q} \to g\,W$

$$\begin{aligned} \mid \mathcal{M}_R \mid^2 = \frac{1}{4}\frac{1}{9} \sum_{\text{spins, colours}} \mid \mathcal{M} \mid^2 &= \pi\,\alpha_s\,\sqrt{2}\,G_F\,M_W^2 \mid V_{qq'} \mid^2 \frac{32}{9}\,\frac{\hat{t}^2 + \hat{u}^2 + 2\,M_W^2\,\hat{s}}{\hat{t}\,\hat{u}} \\ &= \frac{8}{9}\,g^2\,\pi\,\alpha_s \mid V_{qq'} \mid^2 \frac{\hat{t}^2 + \hat{u}^2 + 2\,M_W^2\,\hat{s}}{\hat{t}\,\hat{u}} \end{aligned} \tag{3.11}$$

where $\hat{s}, \hat{t}, \hat{u}$ are the Mandelstam variables. There are both soft and collinear singularities corresponding to $\hat{t} \to 0$ and $\hat{u} \to 0$. The coupling constant g and the mass of gauge boson M_W are related to the Fermi coupling constant G_F by

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,M_W^2}} \tag{3.12}$$

The two-particle phase space is given by

$$\int dPS_2 = \int \frac{d^3\vec{k}}{(2\,\pi)^3\,2\,k_0}\,\frac{d^3\vec{k'}}{(2\,\pi)^3\,2\,k_0'}\,(2\,\pi)^4\,\delta^4(Q - k - k') \tag{3.13}$$

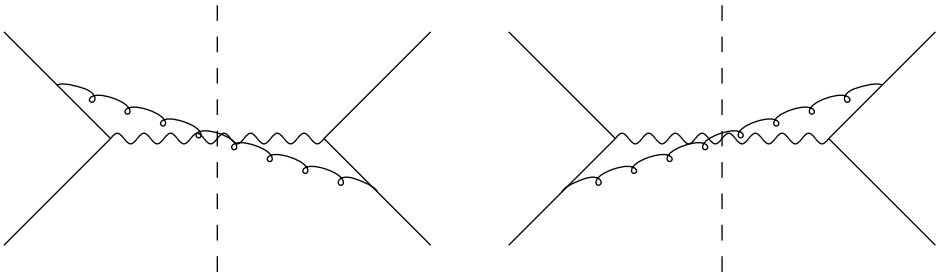


Figure 3.5: Interference splittings.

Nagy-Soper dipoles

The calculations of the subtracted splitting functions contain two dipole contributions: $\mathcal{D}_{qqg}$ and $\mathcal{D}_{\bar{q}\bar{q}g}$, each of which contains both collinear and soft splittings (Fig. 3.4 and Fig. 3.5). Their explicit expressions are given in Eq. (2.141) and Eq. (2.161). We find

$$\mathcal{D}_{qqg} \;=\; \frac{\hat{s}}{M_W^2}\,\left[\frac{4\,\hat{u}}{\hat{s}\,\hat{t}} + \frac{8\,(\hat{s}+\hat{u})}{\hat{t}\,(\hat{t}+\hat{u})} - \frac{8\,\hat{s}}{(\hat{t}+\hat{u})^2}\right] + \frac{16\,\hat{s}\,\hat{u}^2}{(\hat{t}^2+\hat{u}^2)\,(\hat{t}+\hat{u})^2} \tag{3.14}$$

The dipole contribution $\mathcal{D}_{\bar{q}\bar{q}g}$ can be obtained from Eq. (3.14) by the replacement $\hat{t} \leftrightarrow \hat{u}$. The final expression for the two-particle cross section is given by

$$\begin{aligned}\sigma^{NLO\,\{2\}} &= \int_2 \left[d\sigma^R_{\epsilon=0} - d\sigma^A_{\epsilon=0}\right] \\ &= \frac{1}{2\,\hat{s}} \int dPS_2 \left\{ \mid \mathcal{M}_R \mid^2 - \left(\frac{4\,\pi\,\alpha_s}{2}\right) C_F\,(\mathcal{D}_{qqg} + \mathcal{D}_{\bar{q}\bar{q}g}) \mid \mathcal{M}_B \mid^2 \right\}\end{aligned} \tag{3.15}$$

Eq. (3.15) is completely finite after subtracting the dipoles. For the m-parton phase space we use the results of the integrations of the splitting functions over the emitted one-parton phase space. All the collinear and soft integrals are given in Section 2.5. Using Eq. (2.16), we have

$$\begin{aligned}&\int_1 d\sigma^B_{ab}(xp_a,p_b)\left\{\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{qqg} - v^2_{\text{eik}}\right] + 4\,\pi\,\alpha_s\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W_{ab}\right\} \\ &+ \int_1 d\sigma^B_{ab}(p_a,xp_b)\left\{\frac{4\pi\alpha_s}{2}\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{qqg} - v^2_{\text{eik}}\right] + 4\,\pi\,\alpha_s\,C_F\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W_{ab}\right\} \\ &+ \int_1 d\sigma^C_{ab}(p_a,p_b,\mu_F^2) \\ &= \int_1 d\sigma^B_{ab}(p_a,p_b)\otimes I(\epsilon) + \int_0^1 dx \int_1 d\sigma^B_{ab}(xp_a,p_b)\otimes\left[K^a(xp_a) + P(x,\mu_F^2)\right] \\ &+ \quad (a \leftrightarrow b)\end{aligned} \tag{3.16}$$

where the universal collinear counter terms for any finite hard scattering m-parton cross section are defined by Eq. (2.14). The corresponding I, K and P terms are given by

$$\begin{aligned}
I(\epsilon) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right) \\
K^a(xp_a) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left[-(1-x)\ln x + 2(1-x)\ln(1-x) + 4x \left(\frac{\ln(1-x)}{1-x}\right)_+ \right. \\
&\quad \left. - \frac{2x\ln x}{(1-x)_+} - \left(\frac{1+x^2}{1-x}\right)_+ \ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right) \right] \\
P(x,\mu_F^2) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{1+x^2}{1-x}\right)_+ \ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right)
\end{aligned} \tag{3.17}$$

The plus prescription is defined by

$$\int_0^1 dx\, f(x)\, F(x)_+ \;=\; \int_0^1 dx\; [f(x) - f(1)]\; F(x) \tag{3.18}$$

or alternatively,

$$\begin{aligned}
F(x)_+ &= \lim_{\epsilon\to 0} \left[\theta(1-x-\epsilon)\, F(x) - \delta(1-x) \int_0^{1-\epsilon} dx'\, F(x') \right] \\
&= F(x) - \delta(1-x) \int_0^1 dx'\, F(x') \quad \text{with} \quad \theta(1-x) = 1
\end{aligned} \tag{3.19}$$

From that it follows immediately that

$$\int_0^1 dx F(x)_+ = 0 \tag{3.20}$$

and

$$P_{qq}(x) \;=\; C_F \left(\frac{1+x^2}{1-x}\right)_+ \;=\; C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right], \qquad C_F \;=\; \frac{4}{3} \tag{3.21}$$

appearing in the K terms is the famous Altarelli-Parisi splitting function. The virtual matrix element in the $\overline{\text{MS}}$ renormalization scheme is given by

$$\mid \mathcal{M}_V \mid^2 = \mid \mathcal{M}_B \mid^2 \; \frac{\alpha_s}{2\,\pi}\, C_F \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right\} \tag{3.22}$$

We see that the singularities cancel each other between $\mid \mathcal{M}_V \mid^2$ and $I(\epsilon)$ as they must.

Now we consider the gluon induced process $qg \to q\,W$. The matrix element can be obtained from Eq. (3.11) by crossing symmetry. If we include the colours and spins, we obtain

$$\mid \mathcal{M}_R \mid^2 = \frac{1}{4}\,\frac{1}{24} \sum_{\text{spins, colours}} \mid \mathcal{M} \mid^2 = \frac{1}{3}\, g^2 \pi\, \alpha_s \;\mid V_{qq'} \mid^2 \frac{\hat{s}^2 + \hat{u}^2 + 2\, M_W^2\, \hat{t}}{-\hat{s}\,\hat{u}} \tag{3.23}$$

It is worth mentioning that no soft singularity is present in the process $qg \to q\,W$; $\hat{u} \to 0$ corresponds to collinear singularity. There is only one dipole needed in this case $g \to q\bar{q}$, which in terms of Mandelstam variables, is given by

$$\mathcal{D}_{gq\bar{q}} = -\left(\frac{4}{M_W^2}\right)\frac{\hat{s}^2 + 2\,\hat{s}\,\hat{t} + 2\,\hat{t}^2}{\hat{s}\,\hat{u}} \tag{3.24}$$

We found that

$$\begin{aligned}\sigma^{NLO\{2\}} = \int_2 \left[d\sigma^R_{\epsilon=0} - d\sigma^A_{\epsilon=0}\right] &= \frac{1}{2\,\hat{s}}\int dPS_2\left\{\mid \mathcal{M}_R\mid^2 - \left(\frac{4\,\pi\,\alpha_s}{2}\right) T_R\,\mathcal{D}_{gq\bar{q}} \mid \mathcal{M}_B\mid^2\right\} \\ &= \frac{1}{2\,\hat{s}}\int dPS_2\left\{-\frac{1}{3}\,g^2\,\pi\,\alpha_s\frac{(2\,\hat{t}+\hat{u})}{\hat{s}}\right\}\end{aligned} \tag{3.25}$$

The collinear singularity appearing in m-parton phase spaces will be absorbed into PDFs, when we combine the integrated splitting function with the collinear counter term Eq. (2.14).

$$\begin{aligned}\int_1 d\sigma^B_{ab}(xp_a,p_b)\left\{\frac{4\,\pi\,\alpha_s}{2\,(1-\epsilon)}\,T_R\,\mu^{2\epsilon}\int d\zeta_p\,v^2_{gq\bar{q}}\right\} + \int_1 d\sigma^C_{ab}(p_a,p_b,\mu_F^2) \\ = \int_0^1 dx\int_1 d\sigma^B_{ab}(xp_a,p_b)\otimes\left[K(xp_a)+P(x,\mu_F^2)\right]\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}K(xp_a) &= \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\Big\{\,T_R(6x-5x^2-1) + [2\ln(1-x)-\ln(x)]\,P_{qg}(x) \\ &- P_{qg}(x)\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right)\Big\} \\ P(x,\mu_F^2) &= \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}P_{qg}(x)\ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right) \\ P_{qg}(x) &= T_R\left[x^2+(1-x)^2\right], \qquad T_R = \frac{1}{2}\end{aligned} \tag{3.27}$$

and $P_{qg}(x)$ is the standard Altarelli-Parisi splitting function.

Catani-Seymour dipoles

Using Catani-Seymour dipoles, the K and P terms for process $qg \to q\,W$ are given by

$$\begin{aligned}K(xp_a) = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\Big\{&2P_{qg}(x)\ln(1-x) + T_R 2x(1-x) - P_{qg}(x)\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right) \\ &- P_{qg}(x)\ln x\Big\} \\ P(x,\mu_F^2) = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}&P_{qg}(x)\ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right)\end{aligned} \tag{3.28}$$

and the I, K and P terms for process $q\bar{q} \to g\,W$ are given by

$$
\begin{aligned}
I(\epsilon) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right) \\
K^a(xp_a) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left[4\left(\frac{\ln(1-x)}{1-x}\right)_+ + (1-x) - 2(1+x)\ln(1-x)\right. \\
&\quad \left. - \left(\frac{1+x^2}{1-x}\right)_+ \ln\left(\frac{4\pi\mu^2}{2xp_a \cdot p_b}\right) - \left(\frac{1+x^2}{1-x}\right)_+ \ln x\right] \\
P(x,\mu_F^2) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{1+x^2}{1-x}\right)_+ \ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right)
\end{aligned}
\tag{3.29}
$$

Results

In summary, we see that the pole structures in $I(\epsilon)$ terms are equivalent between Catani-Seymour's scheme and Nagy-Soper's scheme as they should. The finite terms in K and P are shifted around due to different momentum mapping. However the final results are the same. For a comparison, we calculated single W production for a pp initial state at NLO, using both the scheme in [22] as well as Nagy-Soper scheme, including PDFs[1] (CTEQ6M [101]) and varying the hadronic center of mass energy of the process. Here we have used routines from the CUBA library (Vegas) [99] as a Monte Carlo algorithm to implement our numerical phase space calculations. Fig. 3.6 shows the relative difference between the two implemented schemes. We see, that the numerical differences are on the permill level and consistent with zero. Fig. 3.7 plots the NLO corrections to single W production at the LHC as a function of the hardonic center of mass energy.

3.2 Dijet production in e^+e^- annihilation

Next we consider dijet production at NLO. The LO and NLO diagrams of dijet production are shown in Fig. 3.8 and Fig. 3.9. The kinematics of two-jet production is defined as follows: The variables are $x_i = 2\,p_i \cdot Q/Q^2$, $y_{ij} = s_{ij}/Q^2$ and $s_{ij} = (p_i + p_j)^2$. The squared expression of Q corresponds to the square of the center of mass energy and p_i is the momentum of any QCD parton in the final state. They are related by $x_1 = 1 - y_{23}$, $x_2 = 1 - y_{13}$ and $x_3 = 1 - y_{12}$. We can choose the center of mass frame in which $Q = \left(\sqrt{s}, \vec{0}\right)$ and $p_i = (E_i, \vec{p}_i)$, hence $\sum_i E_i = \sqrt{s}$ and $\sum_i \vec{p}_i = \vec{0}$. $\sqrt{s}$ is the center of mass energy. It is straightforward to show that $x_1 + x_2 + x_3 = 2$.

[1] *c.f.* [102].

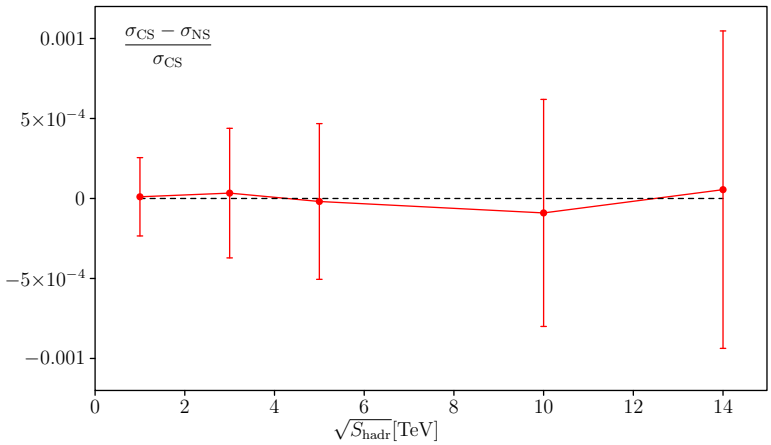


Figure 3.6: Relative difference between NLO corrections to single W production using Catani-Seymour and Nagy-Soper dipoles respectively, as a function of the hardonic center of mass energy. The results agree on sub-permil level. Additionally the numerical integration errors are shown.

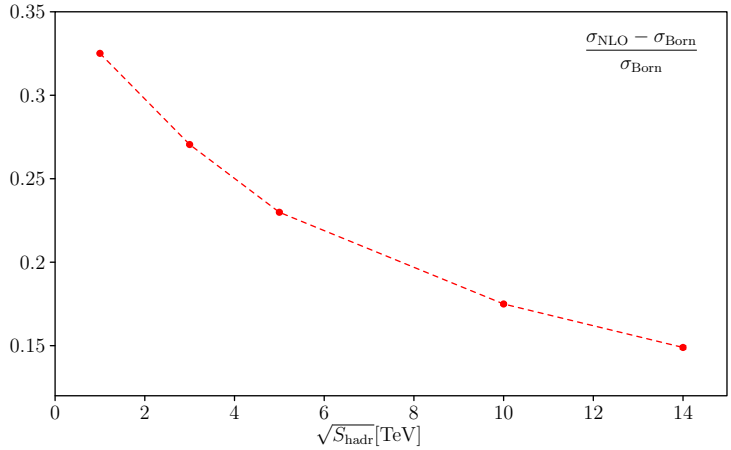


Figure 3.7: NLO corrections $(\sigma_{\mathrm{NLO}} - \sigma_{\mathrm{Born}})/\sigma_{\mathrm{Born}}$ to single W production at the LHC as a function of the hardonic center of mass energy. The result was obtained using the CTEQ6M parton distribution function [101].

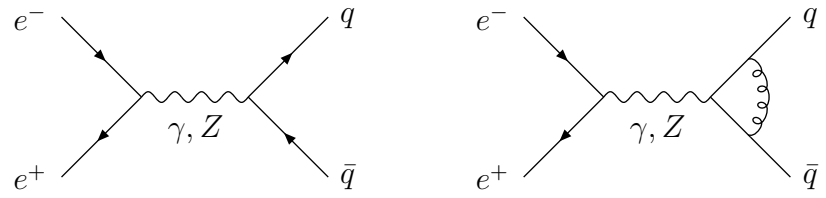


Figure 3.8: LO and virtual diagrams.

The LO contribution is the parton model process $e^+e^- \to q(p_1)\,\bar{q}(p_2)$ with matrix element $\mathcal{M}_2$. We average over the event orientation in the LO process. In this case, the momentum dependence of the Born contribution vanishes. The NLO real emission process is

$$e^+\,e^- \;\to\; \gamma^*(Q) \;\to\; q(p_1)\,\bar{q}(p_2)\,g(p_3) \tag{3.30}$$

The scattering amplitude for $e^+(p')e^-(p) \to q(p_1)\,\bar{q}(p_2)\,g(p_3)$ is

$$i\mathcal{M} \;=\; -\frac{e^2\,g_s}{Q^2}\,t^A_{ij}\,Q_q\,\varepsilon^{\lambda *}(p_3)\,\bar{v}(p')\,\gamma_\mu\,u(p)\,\left\{\bar{u}(p_1)\,\gamma_\lambda\,\frac{1}{\not{p}_1+\not{p}_3}\,\gamma^\mu\,v(p_2) - \bar{u}(p_1)\,\gamma^\mu\,\frac{1}{\not{p}_2+\not{p}_3}\,\gamma_\lambda\,v(p_2)\right\} \tag{3.31}$$

where $Q = p_1+p_2+p_3$ and $\varepsilon^\lambda(p_3)$ is the polarization vector of the gluon. Squaring the scattering amplitude gives then

$$\frac{1}{4}\sum \mid \mathcal{M} \mid^2 = \; -\frac{1}{4}\frac{e^4\,g_s^2}{Q^4}\left(t^A_{ij}\right)^2\;Q_q^2\;\mathrm{Tr}(\not{p}'\,\gamma^\mu\,\not{p}\,\gamma^\nu)\;\mathrm{Tr}(\not{p}_1\,\Lambda_{\lambda\mu}\,\not{p}_2\,\Lambda_{\nu\lambda}) \tag{3.32}$$

where

$$\begin{aligned}\Lambda_{\lambda\mu} &= \left(\gamma_\lambda\,\frac{-1}{\not{p}_1+\not{p}_3}\,\gamma_\mu + \gamma_\mu\,\frac{1}{\not{p}_2+\not{p}_3}\,\gamma_\lambda\right)\\ &= \left(\frac{-1}{Q^2\,(1-x_2)}\,\gamma_\lambda\,(\not{p}_1+\not{p}_3)\,\gamma_\mu + \frac{1}{Q^2\,(1-x_1)}\,\gamma_\mu\,(\not{p}_2+\not{p}_3)\,\gamma_\lambda\right)\end{aligned} \tag{3.33}$$

After a straightforward calculation we find the matrix element $\mathcal{M}_3(p_1,p_2,p_3)$

$$|\mathcal{M}_3(p_1,p_2,p_3)|^2 \;=\; C_F\,\frac{8\,\pi\,\alpha_s}{Q^2}\,\frac{x_1^2+x_2^2}{(1-x_1)\,(1-x_2)}\;|\mathcal{M}_2|^2 \tag{3.34}$$

The final-state parton momenta are denoted by p_i, where,

$$\begin{aligned}s_{12} &= (p_1+p_2)^2\,, \quad s_{13} = (p_1+p_3)^2\,, \quad s_{23} = (p_2+p_3)^2\\ s_{123} &= (p_1+p_2+p_3)^2 = s_{12}+s_{13}+s_{23} = Q^2\\ y_{12} &= s_{12}/Q^2, \quad y_{13} = s_{13}/Q^2, \quad y_{23} = s_{23}/Q^2, \quad y_{12}+y_{13}+y_{23} = 1\end{aligned} \tag{3.35}$$

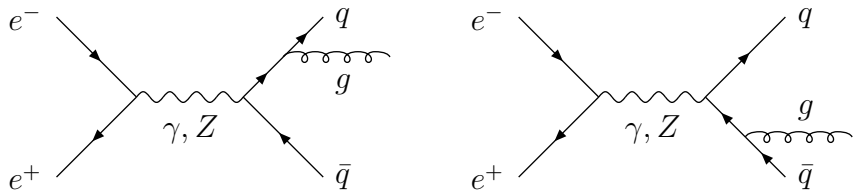


Figure 3.9: Real emission diagrams: $e^+e^- \to q\bar{q}\,g$.

or equivalently we have

$$s_{ij} = (p_i + p_j)^2\,, \quad s_{ijk} = (p_i + p_j + p_k)^2\,, \quad y_{ij} = s_{ij}/Q^2, \quad (i < j < k) \tag{3.36}$$

For the two-parton phase space integration, we follow the Catani-Seymour's convention, which is slightly different from the standard convention up to a factor of $8\,\pi$. Hence

$$\int_{CS} dPS_2 \;=\; \int dy_{12}\,\delta(1-y_{12}) \;=\; 8\,\pi \int dPS_2 \tag{3.37}$$

and the LO cross section is given by

$$\begin{aligned}\sigma^{LO} &= \int_{CS} dPS_2\; |\mathcal{M}_2|^2\; F_J^{(2)}(p_1,p_2) \\ &= |\mathcal{M}_2|^2 \int dy_{12}\,\delta(1-y_{12})\,F_J^{(2)}(p_1,p_2) \quad \to \quad |\mathcal{M}_2|^2\end{aligned} \tag{3.38}$$

The three-parton phase space is given by

$$\begin{aligned}\int_{CS} dPS_3 &= \frac{Q^2}{16\,\pi^2} \int_0^1 dx_1 \int_0^1 dx_2\,\theta(x_1+x_2-1) \;=\; 8\,\pi \int dPS_3 \\ &= \frac{Q^2}{16\,\pi^2} \int_0^1 dy_{23} \int_0^1 dy_{13}\,\theta(1-y_{23}-y_{13})\end{aligned} \tag{3.39}$$

The calculation of the subtracted splitting functions contains two dipole contributions: $\mathcal{D}_{qqg}$ and $\mathcal{D}_{\bar{q}\bar{q}g}$, each of which contains both collinear and soft contributions (see Fig. 3.10). Their definition is given in Eq. (2.119) and Eq. (2.161). We find that

$$\begin{aligned}\mathcal{D}_{qqg} = \frac{4}{\hat{Q}^2}\Bigg\{&\left(\frac{1}{x_2}\right)\left[2\left(\frac{x_1}{2-x_1-x_2} - \frac{1-x_2}{(2-x_1-x_2)^2}\right) + \frac{1-x_1}{1-x_2}\right] \\ &+ 2\left(\frac{x_1+x_2-1}{1-x_2}\right)\frac{x_1}{(1-x_1)\,x_1 + (1-x_2)\,x_2}\Bigg\}\end{aligned} \tag{3.40}$$

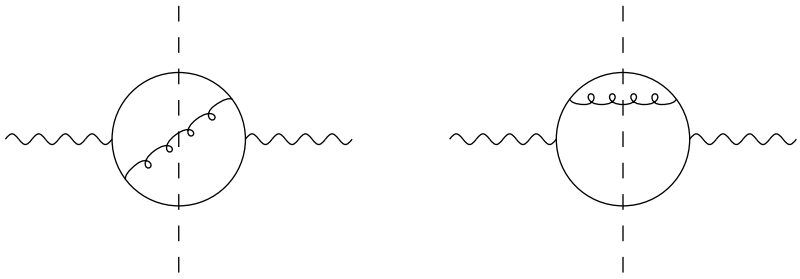


Figure 3.10: Soft and collinear diagrams.

where x_1, x_2 and x_3 are defined by

$$x_1 \,=\, \frac{2\,\hat{p}_\ell \cdot \hat{Q}}{\hat{Q}^2}, \quad x_2 \,=\, \frac{2\,\hat{p}_k \cdot \hat{Q}}{\hat{Q}^2}, \quad x_3 \,=\, \frac{2\,\hat{p}_j \cdot \hat{Q}}{\hat{Q}^2}, \quad \hat{Q} \,=\hat{p}_\ell + \hat{p}_j + \hat{p}_k = Q \tag{3.41}$$

The dipole contribution $\mathcal{D}_{\bar{q}qg}$ can be obtained from Eq. (3.40) by the replacement $x_1 \leftrightarrow x_2$. The final expression for the three-parton cross section is given by

$$\begin{aligned}
\sigma^{NLO\,\{3\}} &= \int_3 \left[d\sigma^R_{\epsilon=0} - d\sigma^A_{\epsilon=0}\right] \\
&= \int_{CS} dPS_3 \Big\{ \mid \mathcal{M}_3(p_1,p_2,p_3) \mid^2 F_J^{(3)}(p_1,p_2,p_3) \\
&- \left(\frac{4\pi\alpha_s}{2}\right) C_F \left(\mathcal{D}_{qqg}\, F_J^{(2)}(\tilde{p}_{13},\tilde{p}_2) + \mathcal{D}_{\bar{q}\bar{q}g}\, F_J^{(2)}(\tilde{p}_{23},\tilde{p}_1)\right) \mid \mathcal{M}_2 \mid^2 \Big\} \\
&= \left(\frac{\alpha_s}{2\,\pi}\, C_F\right) \mid \mathcal{M}_2 \mid^2 \int_0^1 dx_1\, dx_2\, \theta(x_1+x_2-1) \left\{ \frac{x_1^2+x_2^2}{(1-x_1)\,(1-x_2)}\, F_J^{(3)}(p_1,p_2,p_3) \right. \\
&- \left[\left(\frac{1}{x_2}\right)\left(2\left(\frac{x_1}{2-x_1-x_2} - \frac{1-x_2}{(2-x_1-x_2)^2}\right) + \frac{1-x_1}{1-x_2}\right) \right. \\
&+ \left. 2\left(\frac{x_1+x_2-1}{1-x_2}\right)\frac{x_1}{(1-x_1)x_1+(1-x_2)x_2}\right] F_J^{(2)}(\tilde{p}_{13},\tilde{p}_2) \\
&- \left[\left(\frac{1}{x_1}\right)\left(2\left(\frac{x_2}{2-x_1-x_2} - \frac{1-x_1}{(2-x_1-x_2)^2}\right) + \frac{1-x_2}{1-x_1}\right) \right. \\
&+ \left. \left. 2\left(\frac{x_1+x_2-1}{1-x_1}\right)\frac{x_2}{(1-x_1)x_1+(1-x_2)x_2}\right] F_J^{(2)}(\tilde{p}_{23},\tilde{p}_1) \right\}
\end{aligned} \tag{3.42}$$

which for any infrared safe observable (implying that $F_J^{(3)} \to F_J^{(2)}$ as x_i approaches 1) is finite and which for $F_J^{(3)} = F_J^{(2)} \,=\, 1$ can be reduced to

$$\sigma^{NLO\,\{3\}} \,=\, \frac{\alpha_s}{2\,\pi}\, C_F \left(\frac{23}{2} - \frac{4}{3}\pi^2\right) \sigma^{LO} \tag{3.43}$$

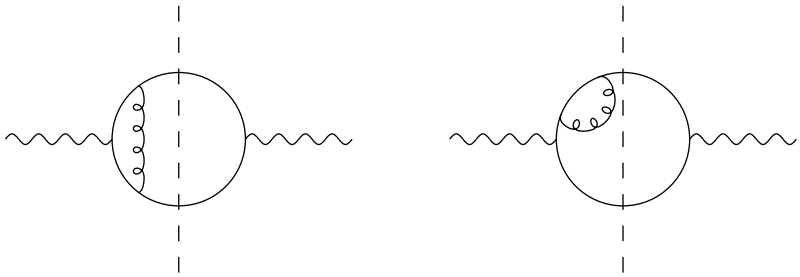


Figure 3.11: Vertex and quark self energy diagrams.

where we have used the formulae

$$\begin{aligned}
\int_0^1 \int_{1-x_1}^1 dx_1\, dx_2 \frac{(1-x_1)^3}{x_2\,(2-x_1-x_2)^2} &= \frac{1}{6}\,(-29+3\,\pi^2) \\
\int_0^1 \int_{1-x_1}^1 dx_1\, dx_2 \frac{(1-x_1)\,(1-x_2)^2}{x_2\,(2-x_1-x_2)^2} &= \frac{1}{12}\,(-19+2\,\pi^2) \\
\int_0^1 \int_{1-x_1}^1 dx_1\, dx_2 \frac{2\,x_1\,(1-x_1)}{(2-x_1-x_2)^2} &= \frac{2}{3}
\end{aligned}$$

Now we have to use the results of the integrated splitting functions for the m-parton phase space contributions which, in general, can be devided into two pieces: soft integral and collinear integral. In the end we have to combine the integrated splitting functions with the virtual cross section leading to finite result. The one-loop matrix element (Fig. 3.11) in the $\overline{\rm MS}$ renormalization scheme is given by Eq. (3.22), and the collinear and soft integrals can be looked up in Section 2.5. Combining these contributions, we obtain a finite ($\epsilon \to 0$) expression for the two-parton cross section:

$$\begin{aligned}
\sigma^{NLO\,\{2\}} &= \int_2 \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0} \\
&= \int_{CS} dPS_2 \left\{ \mid \mathcal{M}_V \mid^2 + 2 \mid \mathcal{M}_2 \mid^2 \right. \\
&\times \left. \left[\frac{4\pi\alpha_s}{2}\, C_F\, \mu^{2\epsilon} \int d\zeta_p \left[v_{qqg}^2 - v_{\rm eik}^2 \right] + 4\pi\alpha_s\, C_F\, \mu^{2\epsilon} \int d\zeta_p\, \Delta W \right] \right\} F_J^{(2)}(p_1,p_2) \\
&= \frac{\alpha_s}{2\,\pi}\, C_F \left(-10 + \frac{4\pi^2}{3} \right) \mid \mathcal{M}_2 \mid^2 \int dy_{12}\, \delta(1-y_{12})\, F_J^{(2)}(p_1,p_2) \\
&= \frac{\alpha_s}{2\,\pi}\, C_F \left(-10 + \frac{4}{3}\pi^2 \right) \sigma^{LO}
\end{aligned} \tag{3.44}$$

$2\,E_g/\sqrt{s}$	$3\cdot 10^{-4}$	$3\cdot 10^{-8}$
σ_{real} [pb]	$-0.0051499031 \pm 0.0000051413$	$-0.0051496084 \pm 0.0000051484$

Table 3.1: Variation of the real radiation subtracted cross section σ_{real} for two different values of E_g. If $2\,E_g/\sqrt{s} < 3\cdot 10^{-4}$, the result is the same. Here $\sqrt{s} = 500$ GeV.

Results

Summing Eq. (3.42) and Eq. (3.44), it is straightforward to show that the total NLO cross section (for $F_J^{(3)} = F_J^{(2)} = 1$) agrees with the well known result [22, 109],

$$\sigma^{NLO} \;=\; \sigma^{NLO\,\{2\}} + \sigma^{NLO\,\{3\}} \;=\; \frac{3}{4}\,\frac{\alpha_s}{\pi}\,C_F\,\sigma^{LO} \tag{3.45}$$

where σ^{LO} is given by

$$\boxed{\sigma^{LO} \;=\; \sigma_0 \;=\; \frac{4\,\pi\,\alpha^2}{3\,Q^2}\,\left(3\sum_{q=1}^{n_f} Q_q^2\right)} \tag{3.46}$$

Note, that here we can set jet functions being equal to one safely. Non-unit jet functions need to account for the mappings from $m+1$ to m-phase space.

We also show our numerical results for dijet production. For the numerical computation we used the following parameters[2]

$$\alpha_s \;=\; 0.118055085612548\,, \quad \alpha \;=\; 7.54677226134035754\times 10^{-3}$$

and vary the center of mass energy. The Fig. 3.12 plots the relative difference between the two implemented schemes. We see that the schemes are equivalent with agreement on the permill level and consistent with zero[3]. Fig. 3.13 and Fig. 3.14 plot a comparison of analytical result and results using Nagy-Soper and Catani-Seymour dipoles, respectively. We also show a comparison of analytical and numerical results using Nagy-Soper dipoles for the real emission subtracted cross section (see Fig. 3.15). Here we used routines from the CUBA library [99] as a Monte Carlo algorithm to implement our numerical evaluations[4].

[2]Here we set quark flavour to be d quark and hence $Q_q = -\frac{1}{3}$.

[3]We also found that the number of total iterations of Vegas Monte Carlo integrations for the Nagy-Soper scheme is less than the Catani-Seymour scheme under the same accuracy, *e.g.* 259 for NS scheme and 498 for CS scheme. So we are confident that for a multi-particle process in the final state the CPU run time for NS scheme would be much less than the CS scheme.

[4]We also performed the Lorentz boost using our routines to check the Lorentz invariance of the real emission matrix element and all the dipoles; they completely agree with each other before and after the boost.

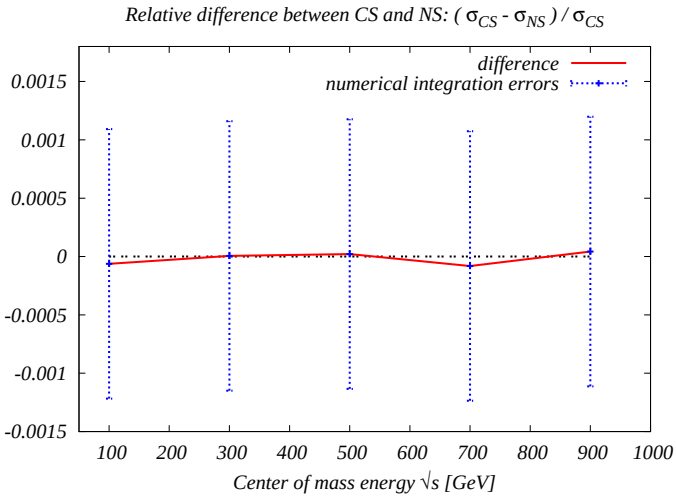


Figure 3.12: Relative difference between NLO corrections to dijet production using Catani-Seymour and Nagy-Soper dipoles respectively, as a function of the center of mass energy. The results agree on sub-permil level, shown are the numerical integration errors.

We have also checked, that the final result is insensitive to variations of the gluon energy, if the ratio $2\,E_g/\sqrt{s}$ is below $3 \cdot 10^{-4}$. The Table 3.1 shows the real radiation subtracted cross section σ_{real}, where we vary the ratio between $3 \cdot 10^{-4}$ and $3 \cdot 10^{-8}$. Here we choose $\sqrt{s} = 500$ GeV.

3.3 Higgs production in gluon-gluon fusion: $gg \to H$

One of the most crucial experimental challenges for present and future high-energy physics is to search for Higgs boson, which is a fundamental ingredient of the Standard Model (SM). The discovery of Higgs boson will also enable us to well establish the Higgs mechanism, one of the cornerstones of the SM. The Higgs mechanism can not only explain the source of Electroweak Symmetry Breaking (EWSB) but also the generation of elementary particle masses.

For Higgs boson masses up to 700 GeV the dominant production process in the Standard Model is gluon-gluon fusion $gg \to H$ via a virtual top quark loop. Higher mass values may also be realized in extensions of the Beyond Standard Model, *e.g.* Supersymmetry [100]. In the following we will re-calculate QCD radiative corrections of $\mathcal{O}(\alpha_s^3)$ to Higgs boson production in hadronic collisions using both the Catani-Seymour and Nagy-Soper dipoles. These include the one-loop virtual contributions (Fig. 3.16) to the lowest-order process $gg \to H$ as well as real

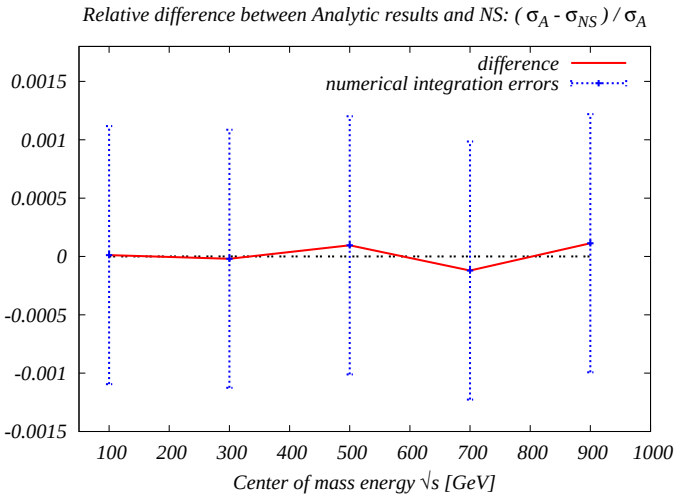


Figure 3.13: Analytic result vs Nagy-Soper scheme.

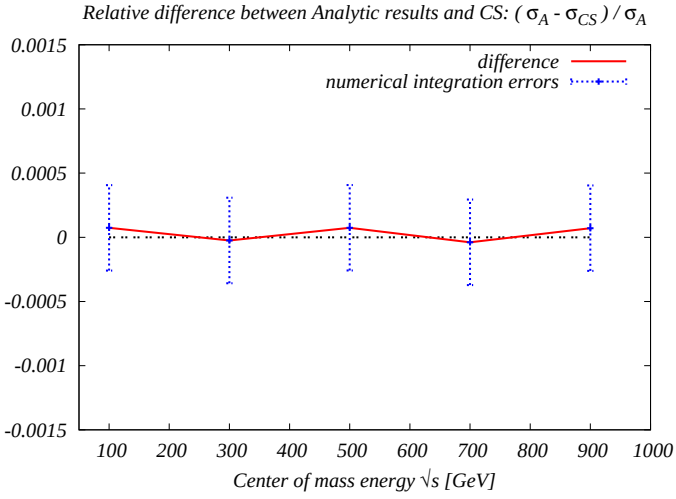


Figure 3.14: Analytic result vs Catani-Seymour scheme.

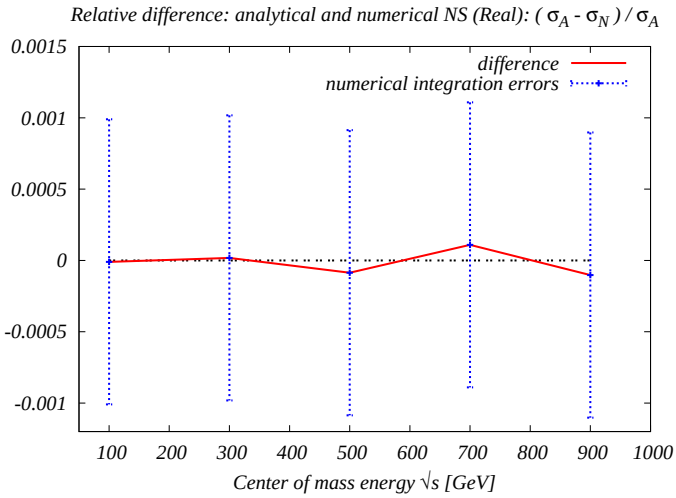


Figure 3.15: Nagy-Soper scheme: a comparison of analytical and numerical results for the real subtracted cross section.

gluon emissions (Fig. 3.17)

$$gg \to g\,H, \quad qg \to q\,H, \quad q\bar{q} \to g\,H \tag{3.47}$$

The corrections increase the LO cross section by approximately 50 %. This correction is larger than the corresponding correction to the Drell-Yan process. This is related to the integer/fractional colour charges of gluons/quarks. The lowest-order cross section has been available for some time [44–47],

$$\hat{\sigma}_0(gg \to H) \;=\; \frac{\alpha_s^2}{\pi}\frac{M_H^2}{256\,v^2}\;|A|^2\;\delta\left(\hat{s} - M_H^2\right) \tag{3.48}$$

where

$$\begin{aligned} |A|^2 &= \left|\sum_q \tau_q\left(1 + (1-\tau_q)\,f(\tau_q)\right)\right|^2 \\ \tau_q &= \frac{4\,M_q^2}{M_H^2}, \quad v^2 = \frac{4\,M_W^2}{g^2} = \frac{1}{\sqrt{2}\,G_F} = (246\,\text{GeV})^2 \end{aligned} \tag{3.49}$$

and

$$f(\tau_q) \;=\; \begin{cases} \left[\sin^{-1}\left(\sqrt{1/\tau_q}\right)\right]^2 & \text{if} \quad \tau_q \geq 1 \\ -\frac{1}{4}\left[\ln\left(\frac{1+\sqrt{1-\tau_q}}{1-\sqrt{1-\tau_q}}\right) - i\pi\right]^2 & \text{if} \quad \tau_q < 1 \end{cases} \tag{3.50}$$

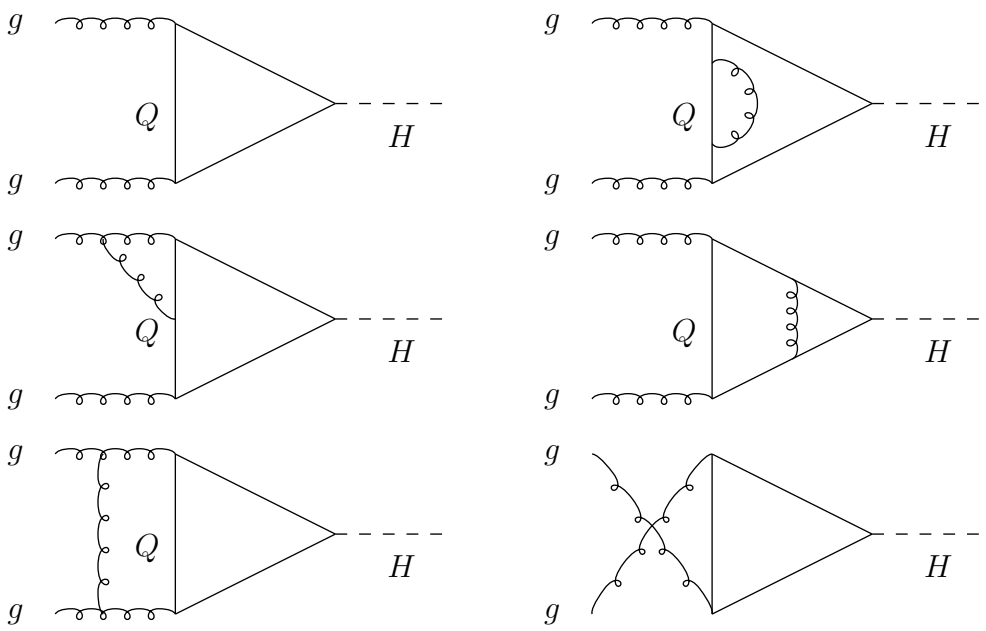


Figure 3.16: Generic diagrams for the Higgs production in gluon-gluon collisions $gg \to H$ at LO and its one loop virtual corrections.

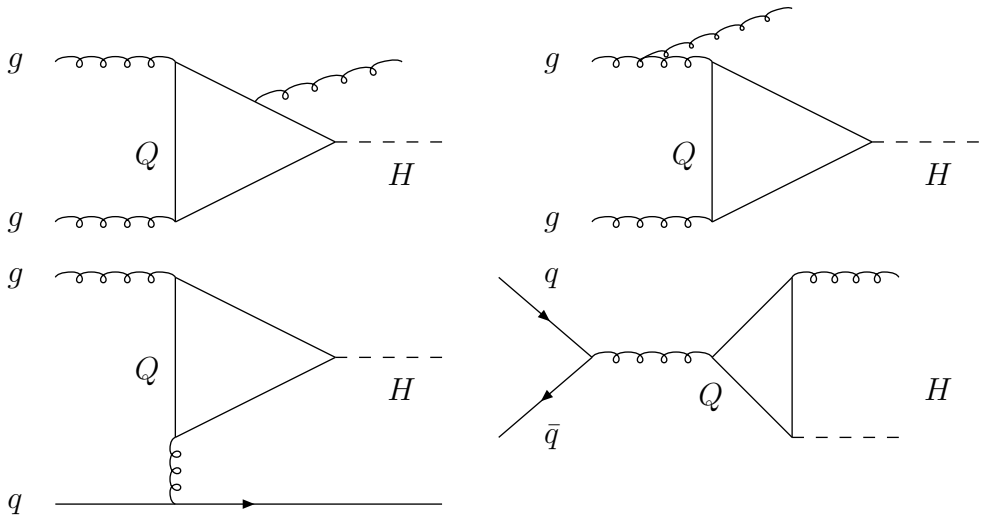


Figure 3.17: Generic diagrams for real corrections: $gg \to gH$, $qg \to qH$ and $q\bar{q} \to gH$.

The Lagrangian of ggH coupling has a closed form

$$\boxed{\mathcal{L}_{ggH} = \frac{\alpha_s}{12\,\pi} G^A_{\mu\nu} G^{\mu\nu}_A \ln\left(1 + \frac{H}{v}\right)} \tag{3.51}$$

with

$$\ln(1+x) = -\sum_{n=1} \frac{(-x)^n}{n} \tag{3.52}$$

where $G^A_{\mu\nu}$ is the gluon field strength tensor. Eq. (3.51) is very convenient for simple calculations, but for $gg +$ jets production it only holds in the limit that all jet momenta are much smaller than top quark mass. It also becomes problematic in the $gg \to HH$ or $gg \to HHH$ processes close to threshold, where the momenta of slow-moving Higgs bosons lead to an additional scale in the process.

In the limit that the top quark mass is infinitely large, $\tau_q \to \infty$, $A \to \frac{2}{3}$ and

$$\hat{\sigma}_0(gg \to H) \to \frac{\alpha_s^2}{\pi} \frac{M_H^2}{576\,v^2} \delta\left(\hat{s} - M_H^2\right) \tag{3.53}$$

In this thesis we will consider only heavy quark limit (Fig. 3.18, Fig. 3.19 and Fig. 3.20). When the momentum transfer to the Higgs boson is small, or equivalently in the limit where $M_{\text{top}} \gg M_H$, the cross section to $\mathcal{O}(\alpha_s^3)$ can be obtained from the effective Lagrangian [48–50]

$$\mathcal{L}_{\text{eff}} = \frac{\alpha_s}{12\,\pi\,v} H\, G^A_{\mu\nu}\, G^{\mu\nu}_A \tag{3.54}$$

The full NLO QCD cross section contains

$$\int d\sigma^{NLO} = \int d\sigma_{q\bar{q}\to gH} + \int d\sigma_{qg\to qH} + \int d\sigma_{gg\to gH} + \int d\sigma^V \tag{3.55}$$

In the following we will discuss each subprocess and its corresponding dipoles. The NLO QCD calculations have already been available in the literatures for some time [51–55].

3.3.1 The subprocess $q\bar{q} \to gH$

Using the effective Lagrangian Eq. (3.54), the matrix element for NLO real emission process $q\bar{q} \to gH$ in four dimensions is found to be

$$|\mathcal{M}(q\bar{q} \to gH)|^2 = \frac{16}{9} \frac{\alpha_s^3}{\pi v^2} \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}} \tag{3.56}$$

The cross section for this process is completely finite. The spin and colour averages yield an additional factor which equals $1/2 \times 1/2 \times 1/3 \times 1/3 = 1/36$. It is straightforward to integrate

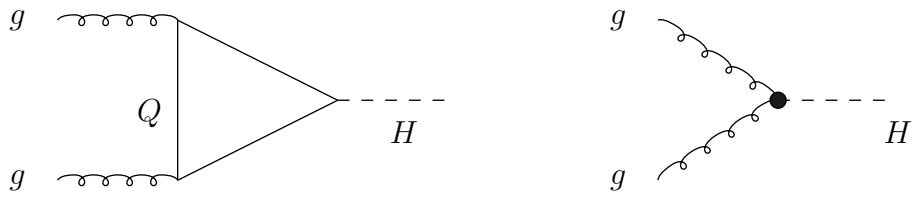


Figure 3.18: $gg \to H$: Heavy quark limit.

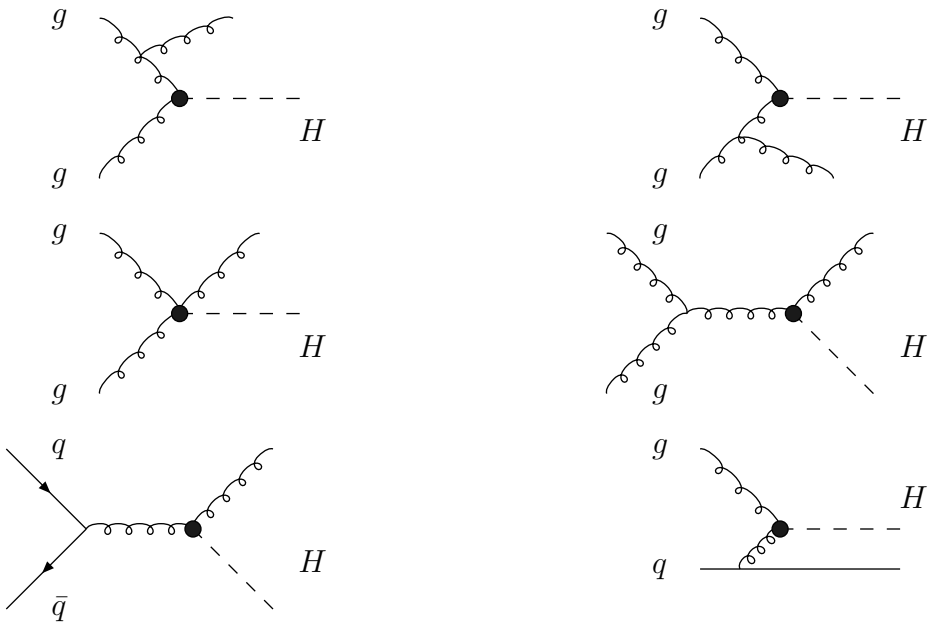


Figure 3.19: Heavy quark limit for $gg \to gH$, $qg \to qH$ and $q\bar{q} \to gH$.

over the phase space to obtain the cross section.

$$\begin{aligned}\sigma(q\bar{q} \to gH) &= \frac{1}{2\hat{s}} \int dPS_2 \, |\mathcal{M}(q\bar{q} \to gH)|^2 \\ &= \frac{1}{486} \left(\frac{\alpha_s^3}{\pi^2 \, v^2}\right) \left(1 - \frac{M_H^2}{\hat{s}}\right)^3 \end{aligned} \tag{3.57}$$

The two-particle phase space integration is given by

$$dPS_2 = \frac{1}{32\,\pi^2} \left(1 - \frac{M_H^2}{\hat{s}}\right) d\Omega \tag{3.58}$$

provided that

$$\hat{s} > M_H^2, \qquad \hat{s} = \eta_a \, \eta_b \, s \tag{3.59}$$

Here, η_a and η_b are the momentum fractions of the incoming partons. The parameter $\hat{s}$ is the center of mass energy of the colliding partons, while s is the center of mass energy of incoming hadrons.

3.3.2 The subprocess $qg \to qH$

Nagy-Soper dipoles

The matrix element for NLO real emission process $qg \to qH$ in four dimensions can be obtained by crossing from that for $q\bar{q} \to gH$. We find

$$|\mathcal{M}(qg \to qH)|^2 = -\frac{16}{9} \, \frac{\alpha_s^3}{\pi \, v^2} \, \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}} \tag{3.60}$$

which has a singularity when $\hat{t} \to 0$ (collinear singularity). The spin and colour averages yield an additional factor $1/2 \times 1/2/(1-\epsilon) \times 1/3 \times 1/8 = 1/96/(1-\epsilon)$. This singularity can be regularized by subtracting the dipole (defined in Section 2.4) which, in terms of the Mandelstam variables, is given by

$$\mathcal{D}_{qqg} = -\frac{4}{\hat{t}} \, \frac{\hat{s}^2 + \hat{u}^2}{(\hat{s} + \hat{u})^2} \tag{3.61}$$

Hence we obtain

$$\frac{4\,\pi\,\alpha_s}{2} \, C_F \, \mathcal{D}_{qqg} \, |\mathcal{M}_{LO}|^2 = -\frac{1}{54} \left(\frac{\alpha_s^3}{\pi \, v^2}\right) \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}} - \frac{1}{54} \left(\frac{\alpha_s^3}{\pi \, v^2}\right) \frac{(\hat{s}^2 + \hat{u}^2) \, \left[\hat{t} + 2\,(\hat{s} + \hat{u})\right]}{(\hat{s} + \hat{u})^2} \tag{3.62}$$

where the lowest-order matrix element is given by

$$|\mathcal{M}_{LO}|^2 = \frac{\alpha_s^2 \, M_H^4}{576\,\pi^2 \, v^2} \, \frac{1}{(1-\epsilon)} \tag{3.63}$$

The subtracted cross section is

$$\begin{aligned}\sigma(qg \to qH) &= \frac{1}{2\,\hat{s}} \int dPS_2 \left\{ |\mathcal{M}(qg \to qH)|^2 - \frac{4\,\pi\,\alpha_s}{2}\, C_F\, \mathcal{D}_{qqg}\, |\mathcal{M}_{LO}|^2 \right\}_{NS} \\ &= \frac{1}{1728\,\hat{s}} \left(\frac{\alpha_s^3}{\pi^2\,v^2}\right) \left(1 - \frac{M_H^2}{\hat{s}}\right) \left(3\,M_H^2 + \hat{s} + 4\,\hat{s} \ln\left(\frac{\hat{s}}{M_H^2}\right)\right) \end{aligned} \tag{3.64}$$

The collinear singularity appearing in m-parton phase spaces will be absorbed into PDFs when we combine the integrated splitting function with the collinear counter term Eq. (2.14).

$$\begin{aligned}&\int_1 d\sigma_{ab}^B(xp_a, p_b) \left\{ \frac{4\,\pi\,\alpha_s}{2} C_F\, \mu^{2\,\epsilon} \int d\zeta_p\, v_{qqg}^2 \right\} + \int_1 d\sigma_{ab}^C(p_a, p_b, \mu_F^2) \\ &\qquad = \int_0^1 dx \int_1 d\sigma_{ab}^B(xp_a, p_b) \otimes \left[K(xp_a) + P(x, \mu_F^2) \right] \end{aligned} \tag{3.65}$$

where

$$\begin{aligned} K(xp_a) &= \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \left[\frac{x^2 - 2(1-x)}{x} - x \ln x \right] C_F + 2 \ln(1-x) P_{gq}(x) \right. \\ &\quad \left. - P_{gq}(x) \ln\left(\frac{4\pi\mu^2}{2xp_a \cdot p_b}\right) \right\} \\ P(x, \mu_F^2) &= \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} P_{gq}(x) \ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right) \\ P_{gq}(x) &= C_F\, \frac{1 + (1-x)^2}{x}, \qquad C_F = \frac{4}{3} \end{aligned} \tag{3.66}$$

Here $P_{gq}(x)$ denotes the standard Altarelli-Parisi splitting function.

The collinear integrated splitting function is given in Section 2.5. It is worth mentioning that no soft singularity is present in the process $qg \to qH$. To this order, the appropriate scale at which to calculate α_s is not determined. We can take $\alpha_s = \alpha_s(\mu^2)$, where μ is an arbitrary renormalization scale. The hadronic cross section is independent of μ to $\mathcal{O}(\alpha_s^3)$.

Catani-Seymour dipoles

Using Catani-Seymour's scheme the dipole subtraction term is

$$\frac{4\,\pi\,\alpha_s}{2}\, C_F\, \mathcal{D}_{qqg}\, |\mathcal{M}_{LO}|^2 = -\frac{1}{54} \left(\frac{\alpha_s^3}{\pi\,v^2}\right) \frac{M_H^4 + 2\,\hat{s}\,(\hat{s} - M_H^2)}{\hat{t}} \tag{3.67}$$

and the subtracted cross section is given by

$$\begin{aligned}\sigma(qg \to qH) &= \frac{1}{2\,\hat{s}} \int dPS_2 \left\{ |\mathcal{M}(qg \to qH)|^2 - \frac{4\,\pi\,\alpha_s}{2}\, C_F\, \mathcal{D}_{qqg}\, |\mathcal{M}_{LO}|^2 \right\}_{CS} \\ &= -\frac{1}{576} \left(\frac{\alpha_s^3}{\pi^2\,v^2}\right) \left(1 - \frac{M_H^2}{\hat{s}}\right)^2 \end{aligned} \tag{3.68}$$

The K and P terms are

$$K(xp_a) = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left\{C_F x + [2\ln(1-x) - \ln x]\, P_{gq}(x) - P_{gq}(x)\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right)\right\}$$
$$P(x,\mu_F^2) = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}P_{gq}(x)\ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right) \tag{3.69}$$

3.3.3 The subprocess $gg \to gH$

To calculate the QCD corrections to the inclusive production of the Higgs boson from $gg \to H$, we also need the real contributions from $gg \to gH$. The matrix element for NLO real emission process in four dimensions is given by [56, 57]

$$|\mathcal{M}(gg \to gH)|^2 \;=\; \frac{\alpha_s^3}{v^2}\frac{32}{3\,\pi}\,\frac{M_H^8 + \hat{s}^4 + \hat{t}^4 + \hat{u}^4}{\hat{s}\,\hat{t}\,\hat{u}} \tag{3.70}$$

Collinear and soft singularities come from $\hat{t} \to 0$ or $\hat{u} \to 0$. The spin and colour averages yield an additional factor $1/2/(1-\epsilon) \times 1/2/(1-\epsilon) \times 1/8 \times 1/8 = 1/256/(1-\epsilon)^2$.

Nagy-Soper dipoles

The subtracted dipole term contains both t-channel and u-channel contributions. Their explicit expressions are defined in Section 2.4. We find

$$\mathcal{D}_{ggg} \;=\; \frac{8\,\hat{u}}{\hat{t}}\left\{\frac{\hat{s}}{\left(\hat{t}+\hat{u}\right)^2} + \frac{1}{\hat{s}}\left[\frac{\left(\hat{s}+\hat{u}\right)^2+\hat{s}^2}{\left(\hat{s}+\hat{u}\right)^2} - \frac{\hat{s}\,\hat{t}}{\left(\hat{s}+\hat{u}\right)\,\left(\hat{t}+\hat{u}\right)}\right]\right\} + (\hat{t}\leftrightarrow\hat{u}) + \frac{16\,\hat{s}}{\left(\hat{t}+\hat{u}\right)^2} \tag{3.71}$$

It is straightforward to integrate over the phase space to obtain the subtracted cross section

$$\begin{aligned}\sigma(gg\to gH) &= \frac{1}{2\,\hat{s}}\int dPS_2\left\{|\mathcal{M}(gg\to gH)|^2 - \frac{4\,\pi\,\alpha_s}{2}\,C_A\,\mathcal{D}_{ggg}\;|\mathcal{M}_{LO}|^2\right\}_{NS}\\ &= \frac{1}{384\,\hat{s}}\left(\frac{\alpha_s^3}{\pi^2\,v^2}\right)\left(1-\frac{M_H^2}{\hat{s}}\right)\left\{\frac{4\,\hat{s}\,(2\,M_H^4 - 2\,M_H^2\,\hat{s}+\hat{s}^2)}{(M_H^2-\hat{s})^2}\ln\left(\frac{\hat{s}}{M_H^2}\right)\right.\\ &\quad\left. + \frac{M_H^4 + 34\,M_H^2\,\hat{s}+\hat{s}^2}{3\,\hat{s}} + \frac{4\,M_H^2\,\hat{s}}{M_H^2-\hat{s}}\right\}\end{aligned} \tag{3.72}$$

Next we have to consider the m-parton phase space contributions. All collinear and soft integrals can be looked up in Section 2.5. Using Eq. (2.16), we find

$$\begin{aligned}
&\int_1 d\sigma^B_{ab}(xp_a,p_b)\left\{\frac{4\pi\alpha_s}{2(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{ggg}-\,v^2_{\text{eik}}\right]+4\,\pi\,\alpha_s\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W_{ab}\right\}\\
&+\int_1 d\sigma^B_{ab}(p_a,xp_b)\left\{\frac{4\pi\alpha_s}{2(1-\epsilon)}\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\left[v^2_{ggg}-\,v^2_{\text{eik}}\right]+4\,\pi\,\alpha_s\,C_A\,\mu^{2\epsilon}\int d\zeta_p\,\Delta W_{ab}\right\}\\
&+\int_1 d\sigma^C_{ab}(p_a,p_b,\mu_F^2)\\
&=\int_1 d\sigma^B_{ab}(p_a,p_b)\otimes I(\epsilon)+\frac{\alpha_s}{2\,\pi}\,\frac{1}{\Gamma(1-\epsilon)}\int_1 d\sigma^B_{ab}(p_a,p_b)\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\left(\frac{2}{\epsilon}\right)\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right)\\
&+\int_0^1 dx\int_1 d\sigma^B_{ab}(xp_a,p_b)\otimes\left[K^a(xp_a)+P(x,\mu_F^2)\right]+\quad(a\leftrightarrow b)
\end{aligned}\tag{3.73}$$

Here, we always keep in mind that the factor $1/(1-\epsilon)$ is already included in the soft terms. The universal collinear counter terms are defined by Eq. (2.14). The standard Altarelli-Parisi splitting function $P_{gg}(x)$ is now

$$P_{gg}(x)\;=\;2\,C_A\,\left(\frac{x}{(1-x)_+}+x(1-x)+\frac{1-x}{x}\right)+\delta(1-x)\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right),\quad C_A=3\tag{3.74}$$

The corresponding I, K and P terms are given by

$$\begin{aligned}
I(\epsilon)&=\frac{\alpha_s}{2\pi}C_A\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left(\frac{2}{\epsilon^2}-\frac{\pi^2}{3}+\mathcal{O}(\epsilon)\right)\\
K^a(xp_a)&=\frac{\alpha_s}{2\pi}C_A\frac{1}{\Gamma(1-\epsilon)}\left\{4x\left(\frac{\ln(1-x)}{1-x}\right)_+-2x(1-x)\ln x+4(1-x)\ln(1-x)\left(\frac{1+x^2}{x}\right)\right.\\
&\qquad\left.+\,2\left(x^2-\frac{1-x}{x}\right)-2\left(\frac{x}{(1-x)_+}+x(1-x)+\frac{1-x}{x}\right)\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right)\right\}\\
P(x,\mu_F^2)&=\frac{\alpha_s}{2\pi}C_A\frac{1}{\Gamma(1-\epsilon)}\left\{2\left(\frac{x}{(1-x)_+}+x(1-x)+\frac{1-x}{x}\right)\ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right)\right\}
\end{aligned}\tag{3.75}$$

Catani-Seymour dipoles

Using Catani-Seymour's scheme the dipole subtraction term is

$$\frac{4\,\pi\,\alpha_s}{2}\,C_A\,\mathcal{D}_{ggg}\,|\mathcal{M}_{LO}|^2\;=\;-\frac{1}{12}\,\left(\frac{\alpha_s^3}{\pi\,v^2}\right)\,\left(\frac{1}{\hat{t}}+\frac{1}{\hat{u}}\right)\,\left\{\hat{s}\,(\hat{s}-M_H^2)+\frac{\hat{s}\,M_H^4}{\hat{s}-M_H^2}+\frac{M_H^4\,(\hat{s}-M_H^2)}{\hat{s}}\right\}\tag{3.76}$$

Figure 3.20: Heavy quark limit for virtual corrections.

and the subtracted cross section is given by

$$\begin{aligned}\sigma(gg \to gH) &= \frac{1}{2\,\hat{s}} \int dPS_2 \left\{ |\mathcal{M}(gg \to gH)|^2 - \frac{4\,\pi\,\alpha_s}{2}\, C_A\, \mathcal{D}_{ggg}\, |\mathcal{M}_{LO}|^2 \right\}_{CS} \\ &= -\frac{11}{1152} \left(\frac{\alpha_s^3}{\pi^2\, v^2}\right) \left(1 - \frac{M_H^2}{\hat{s}}\right)^3 \end{aligned} \tag{3.77}$$

The I, K and P terms are

$$\begin{aligned} I(\epsilon) &= \frac{\alpha_s}{2\pi} C_A \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} - \frac{\pi^2}{3} + \mathcal{O}(\epsilon)\right) \\ K^a(xp_a) &= \frac{\alpha_s}{2\pi} C_A \frac{1}{\Gamma(1-\epsilon)} \left\{ 4 \left(\frac{\ln(1-x)}{1-x}\right)_+ + 4 \left(-1 + x(1-x) + \frac{1-x}{x}\right) \ln(1-x) \right. \\ &\quad - 2 \left(\frac{x}{(1-x)_+} + x(1-x) + \frac{1-x}{x}\right) \ln\left(\frac{4\pi\mu^2}{2xp_a \cdot p_b}\right) \\ &\quad \left. - 2 \left(\frac{x}{(1-x)_+} + x(1-x) + \frac{1-x}{x}\right) \ln x \right\} \\ P(x, \mu_F^2) &= \frac{\alpha_s}{2\pi} C_A \frac{1}{\Gamma(1-\epsilon)} \left\{ 2 \left(\frac{x}{(1-x)_+} + x(1-x) + \frac{1-x}{x}\right) \ln\left(\frac{4\pi\mu^2}{\mu_F^2}\right) \right\} \end{aligned} \tag{3.78}$$

3.3.4 One-loop virtual corrections

Now we compute the one-loop virtual matrix element (Fig. 3.20) in the $\overline{\text{MS}}$ renormalization scheme. In the heavy quark limit $M_{\text{top}} \to \infty$ the Higgs boson couples to the trace of the energy momentum tensor [48–50, 58, 59]

$$\theta \;\equiv\; \theta^{\mu}{}_{\mu} \;=\; \partial^{\mu} s_{\mu} \;=\; \frac{\beta(g_s)}{2\,g_s}\, G^A_{\mu\nu}\, G_A^{\mu\nu} \;+ (1+\delta)\, M_{\text{top}}\, \bar{t}\, t \tag{3.79}$$

where s_μ is the scale current. The $(1+\delta)$ term arises from the low energy theorem [53, 60–62]. Since the Higgs coupling to heavy fermions is $M_{\text{top}}\,(1+H/v)\,\bar{t}\,t$, the counterterm for the Higgs Yukawa interaction is fixed in terms of the fermion mass and wave function renormalization. We have $\delta = 2\,\alpha_s/\pi$. Hence, in the heavy quark limit $M_{\text{top}} \to \infty$ we have

$$\mathcal{L}_{\text{eff}} \;=\; \frac{H}{2\,v}\,\frac{\beta(g_s)}{g_s\,(1+\delta)}\,G^A_{\mu\nu}\,G^{\mu\nu}_A \tag{3.80}$$

Since the ggH coupling results from heavy quark loops, only processes with heavy fermions contribute to the β function. The heavy fermion contribution to the QCD β function is [63, 64]

$$\left.\frac{\beta(g_s)}{g_s}\right|_{\text{heavy fermions}} \;=\; N_H\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{1}{3}+\frac{\alpha_s}{\pi}\,\frac{19}{12}\right) \tag{3.81}$$

Here, N_H denotes the number of heavy fermions. Therefore, to second order

$$\mathcal{L}_{\text{eff}} \;=\; \frac{\alpha_s}{12\,\pi\,v}\,H\,G^A_{\mu\nu}\,G^{\mu\nu}_A\,\left(1+\frac{11}{4}\,\frac{\alpha_s}{\pi}\right) \tag{3.82}$$

As a consequence of the non-abelian gauge invariance, the Lagrangian Eq. (3.82) can not only describe the Hgg coupling, but also the $Hggg$ and $Hgggg$ interactions (see Fig. 3.21). After a tedious calculation the one-loop virtual matrix element is given by [51, 53]

$$|\mathcal{M}_V|^2 \;=\; |\mathcal{M}_{LO}|^2\;\frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{4\pi\mu^2}{M_H^2}\right)^{\epsilon}\,\Gamma(1+\epsilon)\left(-\frac{1}{\epsilon^2}+\frac{2}{3}\pi^2+\mathcal{O}(\epsilon)\right)\times 2 \tag{3.83}$$

Hence, we obtain

$$\int_1 d\sigma^B_{ab}(p_a,p_b)\otimes I(\epsilon)+\int d\sigma^V \;=\; \int_1 d\sigma^B_{ab}(p_a,p_b)\,\frac{\alpha_s}{2\,\pi}\,C_A\,\left(\frac{2\,\pi^2}{3}\right) \tag{3.84}$$

The leftover $1/\epsilon$ pole can be regularized by performing charge renormalization. The charge counterterm in the $\overline{\text{MS}}$ renormalization scheme is (see *e.g.* [104, 106])

$$\sigma_{\text{ch}} \;=\; (4\,Z_g)\;\hat{\sigma}_0(gg\to H) \tag{3.85}$$

where

$$\begin{aligned} Z_g \;&=\; -\frac{\alpha_s}{2\,\epsilon}\,\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\,b_0\,\Gamma(1+\epsilon)\,\left(\frac{\mu_F^2}{\mu^2}\right)^{\epsilon} \\ &=\; -\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right)\,\frac{\alpha_s}{4\,\pi}\,\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\,\Gamma(1+\epsilon)\,\left(\frac{1}{\epsilon}+\ln\frac{\mu_F^2}{\mu^2}\right), \\ b_0 \;&=\; \frac{1}{2\,\pi}\,\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right) \end{aligned} \tag{3.86}$$

and n_f is the number of light quarks. So we write for the charge renormalization cross section

$$\sigma_{\text{ch}} = \frac{\alpha_s}{2\,\pi}\frac{1}{\Gamma(1-\epsilon)}\int_1 d\sigma^B_{ab}(p_a,p_b)\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\left(-\frac{2}{\epsilon}-2\ln\frac{\mu_F^2}{\mu^2}\right)\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right) \quad (3.87)$$

Now if we combine real contributions, virtual contributions, charge renormalization and collinear counter terms the physical cross section for $gg \to H$ is completely finite.

In summary, we see, that the pole structures in $I(\epsilon)$ terms are equivalent between Catani-Seymour's scheme and Nagy-Soper's scheme as they should. The finite terms in K and P are shifted around due to different momentum mapping. However the final numerical/analytical results are the same.

Result

In this subsection, we again summarize the results we obtain using Nagy-Soper dipoles from the different subprocesses of the Higgs production. The results shown here include one-particle phase space cross sections, two-particle phase space subtracted cross sections, collinear counterterms, virtual contributions, charge renormalization as well as the effective Lagrangian correction[5].

$$\begin{aligned}\sigma(gg\to H) &= \sigma(q\bar{q}\to gH)\\ &\quad+\Delta\sigma(qg\to qH)+\sigma^C(qg\to qH) && \text{quark induced}\\ &\quad+\Delta\sigma(gg\to gH)+\sigma^C(gg\to gH)+\sigma^V+\sigma_{\text{ch}}+\sigma_{\text{eff}} && \text{gluon induced}\end{aligned} \quad (3.88)$$

where

$$\begin{aligned}\Delta\sigma_{q\bar{q}} &= \sigma(q\bar{q}\to gH) = \frac{1}{486}\left(\frac{\alpha_s^3}{\pi^2\,v^2}\right)(1-z)^3\\ \Delta\sigma_{qg} &= \Delta\sigma(qg\to qH)+\sigma^C(qg\to qH)\\ &= \frac{\alpha_s}{\pi}\,\sigma_0\left\{-1+2\,z-\frac{1}{3}\,z^2-\frac{1}{2}\,z\,P_{gq}(z)\left[\ln\frac{Q^2}{\hat{s}}-2\,\ln(1-z)\right]\right\}\\ \Delta\sigma_{gg} &= \Delta\sigma(gg\to gH)+\sigma^C(gg\to gH)+\sigma^V+\sigma_{\text{ch}}+\sigma_{\text{eff}}\\ &= \frac{\alpha_s}{\pi}\,\sigma_0\left\{\delta(1-z)\left[\frac{11}{2}+\pi^2+\left(\frac{11}{6}\,C_A-\frac{2}{3}\,n_f\,T_R\right)\ln\left(\frac{\mu^2}{\mu_F^2}\right)\right]\right\}\\ &\quad+\frac{\alpha_s}{\pi}\,\sigma_0\left\{12\left[\left(\frac{\ln(1-z)}{1-z}\right)_+-z\,[2-z\,(1-z)]\ln(1-z)\right]\right.\\ &\qquad\left.-\frac{11}{2}\,(1-z)^3-z\,P_{gg}(z)\ln\frac{Q^2}{\hat{s}}\right\}\end{aligned} \quad (3.89)$$

[5]See talks given by F.J. Petriello at PSI Zuoz Summerschool 2010: *http://ltpth.web.psi.ch/zuoz2010/index.htm* and Fourth Graduate School in Physics at Colliders (Torino) 2009: *http://www.ph.unito.it/dft/scuola09/*

with

$$\boxed{\sigma_{\text{eff}} = \hat{\sigma}_0 \, \frac{\alpha_s}{\pi} \, \frac{11}{2} \, , \quad \hat{\sigma}_0 = \sigma_0 \, \delta(1-z) \, , \quad \sigma_0 = \frac{\alpha_s^2}{576 \, \pi \, v^2} \quad \text{and} \quad z = \frac{M_H^2}{\hat{s}}}$$

Here, $\Delta\sigma(qg \to qH)$ and $\Delta\sigma(gg \to gH)$ in Eq. (3.88) and Eq. (3.89) mean the sum of one-particle and two-particle phase space contributions. We find total agreement with the results in [51,53]. In the following sections we will show that the results of Nagy-Soper scheme, Catani-Seymour scheme and literature [51, 53] for the process $gg \to H$ are identical.

3.3.5 Proof: Nagy-Soper scheme and Catani-Seymour scheme

In this section, we prove that the results of Nagy-Soper scheme and Catani-Seymour scheme for the process $gg \to H$ are equivalent. We only compare with real emission subtracted terms and K terms. The remaining I and P terms are exactly the same in both schemes.

The subprocess $gg \to gH$

The real emission subtracted cross section including PDFs for Catani-Seymour scheme is given by

$$\begin{aligned} R_{CS} &= \int_0^1 d\eta \int_0^1 d\bar{\eta} \, g(\eta, Q^2) \, g(\bar{\eta}, Q^2) \, \theta(\eta \, \bar{\eta} \, s - M_H^2) \, \left[-\frac{11}{1152} \left(\frac{\alpha_s^3}{\pi^2 \, v^2} \right) \left(1 - \frac{M_H^2}{\hat{s}} \right)^3 \right] \\ &= \int_{\tau_H}^1 d\tau \, \frac{d\mathcal{L}^{gg}}{d\tau} \left[-\frac{11}{1152} \left(\frac{\alpha_s^3}{\pi^2 \, v^2} \right) \left(1 - \frac{M_H^2}{\hat{s}} \right)^3 \right] , \quad \tau = \eta \, \bar{\eta}, \; \hat{s} = \tau \, s > M_H^2 \, , \; \tau_H = \frac{M_H^2}{s} \end{aligned}$$

where the differential parton luminosities are defined by

$$\begin{aligned} \frac{d\mathcal{L}^{gg}}{d\tau} &= g \otimes g(\tau, Q^2) \\ \frac{d\mathcal{L}^{gq}}{d\tau} &= g \otimes q(\tau, Q^2) + q \otimes g(\tau, Q^2) \\ \frac{d\mathcal{L}^{q\bar{q}}}{d\tau} &= q \otimes \bar{q}(\tau, Q^2) + \bar{q} \otimes q(\tau, Q^2) \end{aligned} \tag{3.90}$$

and the notation $\otimes$ is given by

$$f \otimes g \left(x, \mu^2 \right) \, = \, \int_x^1 \frac{dz}{z} \; f \left(z, \mu^2 \right) \; g \left(\frac{x}{z}, \mu^2 \right) \tag{3.91}$$

The variables η and $\bar{\eta}$ denote momentum fractions of the partons. Furthermore, $\hat{s}$ and s are the center-of-mass energies of partons and hadrons, respectively, while $q(\bar{q})$ and g in Eq. (3.90) are

PDFs of quark (antiquark) and gluon. The PDFs belong to the long-distance physics process of the scattering and hence belong to the non-perturbative part. But their evolutions follow the master equation of QCD: DGLAP equation, by which we can predict how PDFs evolve from one scale to another scale.

$$\boxed{t\frac{d}{dt}q_a(x,t) \;=\; \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{d\xi}{\xi}P_{ab}\left(\frac{x}{\xi},\alpha_s(t)\right)q_b(\xi,t)} \tag{3.92}$$

where $t=\mu^2$ and P_{ab} is the splitting function. More generally, the DGLAP equation in $(2\,n_f{+}1)$-dimension in the flavour space (flavour = quarks, antiquarks and gluons) is

$$t\,\frac{\partial}{\partial t}\begin{pmatrix} q_i(x,t) \\ g(x,t)\end{pmatrix} \;=\; \frac{\alpha_s(t)}{2\,\pi}\sum_{q_j,\bar{q}_j}\int_x^1 \frac{d\xi}{\xi}\begin{pmatrix} P_{q_iq_j}\left(\frac{x}{\xi},\alpha_s(t)\right) & P_{q_ig}\left(\frac{x}{\xi},\alpha_s(t)\right) \\ P_{gq_j}\left(\frac{x}{\xi},\alpha_s(t)\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s(t)\right)\end{pmatrix}\begin{pmatrix} q_j(\xi,t) \\ g(\xi,t)\end{pmatrix} \tag{3.93}$$

The real emission subtracted cross section including PDFs for Nagy-Soper scheme is given by

$$\begin{aligned} R_{NS} = \int_{\tau_H}^1 d\tau\,\frac{d\mathcal{L}^{gg}}{d\tau}\left[\frac{1}{384\,\hat{s}}\left(\frac{\alpha_s^3}{\pi^2\,v^2}\right)\left(1-\frac{M_H^2}{\hat{s}}\right)\left\{\frac{4\,\hat{s}\,(2\,M_H^4-2\,M_H^2\,\hat{s}+\hat{s}^2)}{(M_H^2-\hat{s})^2}\ln\left(\frac{\hat{s}}{M_H^2}\right)\right.\right. \\ \left.\left. +\;\frac{M_H^4+34\,M_H^2\,\hat{s}+\hat{s}^2}{3\,\hat{s}}+\frac{4\,M_H^2\,\hat{s}}{M_H^2-\hat{s}}\right\}\right] \end{aligned} \tag{3.94}$$

Define $\Delta K = K_{CS} - K_{NS}$ and we found

$$\Delta K^a = \frac{\alpha_s}{2\,\pi}\,C_A\,(-2)\left\{\left(\frac{x}{1-x}+\frac{1-x}{x}\right)\ln x+\left(x^2-\frac{1-x}{x}\right)\right\}$$

Convoluting former expression with PDFs and an additional one-dimensional integration yield

$$\begin{aligned} \int\Delta K^a+\int\Delta K^b = \int_{\tau_H}^1 d\tau\,\frac{d\mathcal{L}^{gg}}{d\tau}\left[-\frac{6}{576}\left(\frac{\alpha_s^3}{\pi^2\,v^2}\right)\left(\frac{M_H^2}{\hat{s}}\right)\left\{\left(\frac{M_H^2}{\hat{s}}\right)^2-\frac{\hat{s}}{M_H^2}\left(1-\frac{M_H^2}{\hat{s}}\right)\right.\right. \\ \left.\left. -\;\left(\frac{\hat{s}}{M_H^2}\left(1-\frac{M_H^2}{\hat{s}}\right)+\frac{M_H^2}{\hat{s}\,(1-M_H^2/\hat{s})}\right)\ln\frac{\hat{s}}{M_H^2}\right\}\right] \end{aligned} \tag{3.95}$$

Hence, it is straightforward to show that $\int\Delta K^a+\int\Delta K^b+R_{CS}-R_{NS}=0$, where the symbolic notation $\int\Delta K$ simply means the convolution with PDFs and an additional one-dimensional integration.

The subprocess $qg \to qH$

The real emission subtracted cross section including PDFs for Catani-Seymour scheme is given by

$$R_{CS} = \sum_{q,\bar{q}}\int_{\tau_H}^1 d\tau\,\frac{d\mathcal{L}^{gq}}{d\tau}\left[-\frac{1}{576}\left(\frac{\alpha_s^3}{\pi^2\,v^2}\right)\left(1-\frac{M_H^2}{\hat{s}}\right)^2\right] \tag{3.96}$$

The real emission subtracted cross section including PDFs for Nagy-Soper scheme is given by

$$R_{NS} = \sum_{q,\bar{q}} \int_{\tau_H}^{1} d\tau \, \frac{d\mathcal{L}^{gq}}{d\tau} \left[\frac{1}{1728\,\hat{s}} \left(\frac{\alpha_s^3}{\pi^2\,v^2} \right) \left(1 - \frac{M_H^2}{\hat{s}} \right) \left(3\,M_H^2 + \hat{s} + 4\,\hat{s} \ln \left(\frac{\hat{s}}{M_H^2} \right) \right) \right] \tag{3.97}$$

From the definition of $\Delta K = K_{CS} - K_{NS}$ it follows that

$$\Delta K = \frac{\alpha_s}{2\,\pi}\, C_F\, \frac{2\,(1-x)}{x}\, (1 - \ln x)$$

Convolution with PDFs and an additional one-dimensional integration yield

$$\int \Delta K = \sum_{q,\bar{q}} \int_{\tau_H}^{1} d\tau \, \frac{d\mathcal{L}^{gq}}{d\tau} \left[\frac{1}{576}\, \frac{4}{3} \left(\frac{\alpha_s^3}{\pi^2\,v^2} \right) \left(1 - \frac{M_H^2}{\hat{s}} \right) \left(1 + \ln \frac{\hat{s}}{M_H^2} \right) \right] \tag{3.98}$$

Hence, it is straightforward to show that $\int \Delta K + R_{CS} - R_{NS} = 0$.

3.3.6 Proof: Catani-Seymour scheme and literature results

In this section, we show that the results of Catani-Seymour scheme for the process $gg \to H$ are identical to literature results [51,53] and hence we complete our proofs that CS, NS and literature results agree with each other. Here, we only compare with the real emission part, K term and P term; terms proportional to δ function are straightforward and hence we will not list the proofs.

The subprocess $gg \to gH$

In the limit that the top quark mass is infinitely large, the parton level cross section is given by[6]

$$\hat{\sigma}_0(gg \to H) = \frac{\alpha_s^2}{\pi}\, \frac{M_H^2}{576\,v^2}\, \delta\left(\hat{s} - M_H^2\right) = \frac{\alpha_s^2}{576\,\pi\,v^2}\, \delta\,(1-z) = \sigma_0\, \delta\,(1-z) \tag{3.99}$$

with

$$\sigma_0 = \frac{\alpha_s^2}{576\,\pi\,v^2} \quad \text{and} \quad z = \frac{M_H^2}{\hat{s}} \tag{3.100}$$

The real emission subtracted cross section is then

$$\sigma(gg \to gH) = -\frac{11}{1152} \left(\frac{\alpha_s^3}{\pi^2\,v^2} \right) (1-z)^3 = \frac{\alpha_s}{\pi}\, \sigma_0 \left[-\frac{11}{2}\, (1-z)^3 \right] \tag{3.101}$$

[6]Here we put a hat on cross section σ_0 in order to indicate that we only work on parton level.

Next we consider the K term and the P term contributions. They can be written as

$$\begin{aligned}
&\int_0^1 dx \int d\sigma^B(x) \otimes [K(x) + P(x)] \times 2 \;=\; \int_0^1 dx \left[\sigma_0\, M_H^2\, \delta\left(x\hat{s} - M_H^2\right)\right] \otimes [K(x) + P(x)] \times 2 \\
&= \sigma_0\, \frac{M_H^2}{\hat{s}}\, [K(x) + P(x)] \times 2\,, \quad \text{with} \quad x = \frac{M_H^2}{\hat{s}} \\
&= \frac{\alpha_s}{\pi}\, \sigma_0 \left\{ 12 \left[x \left(\frac{\ln(1-x)}{1-x} \right)_+ - x\, [2 - x\,(1-x)] \ln(1-x) + \ln(1-x) \right] - x\, P_{gg}(x) \ln \frac{Q^2}{\hat{s}} \right\}
\end{aligned} \tag{3.102}$$

where we have added an extra term

$$0 \;=\; -\,\delta(1-x) \left(\frac{11}{6}\, C_A - \frac{2}{3}\, n_f\, T_R \right) \ln x \tag{3.103}$$

in Eq. (3.102) and we have also used the plus prescription in such a way that

$$x \left(\frac{\ln(1-x)}{1-x} \right)_+ + \ln(1-x) \quad \to \quad \left(\frac{\ln(1-x)}{1-x} \right)_+ \tag{3.104}$$

Here, the test function is simply the LO cross section. Combining Eq. (3.101) and Eq. (3.102), we obtain Eq. (10) of [53].

The subprocess $\boldsymbol{qg \to qH}$

The real emission subtracted cross section is

$$\sigma(qg \to qH) \;=\; \frac{\alpha_s}{\pi}\, \sigma_0 \left(-1 + 2\, z - z^2 \right) \tag{3.105}$$

The K term and the P term contributions are given by

$$\int_0^1 dx \int d\sigma^B(x) \otimes [K(x) + P(x)] \;=\; \frac{\alpha_s}{\pi}\, \sigma_0 \left\{ \frac{2}{3}\, z^2 - \frac{1}{2}\, z\, P_{gq}(z) \left[\ln \frac{Q^2}{\hat{s}} - 2\, \ln(1-z) \right] \right\} \tag{3.106}$$

Here, $x = z = \dfrac{M_H^2}{\hat{s}}$. Combining Eq. (3.105) and Eq. (3.106), we obtain Eq. (11) of [53].

3.4 Higgs decay: $\boldsymbol{H \to gg}$

A closely related problem to the process $gg \to H$ at NLO is the QCD radiative corrections to the gluonic decay modes of the Higgs boson. Two different emissions appear at NLO: either the emission of an additional gluon or the splitting of one gluon into a $q\bar{q}$ pair.

$$H \to g\, g\, g, \qquad H \to g\, q\, \bar{q} \tag{3.107}$$

for which we can test the final pieces of splitting processes in the final state using Nagy-Soper dipoles: $g \to gg$ and $g \to q\bar{q}$. The lowest-order, real emissions and virtual diagrams are shown in Fig. 3.21 and Fig. 3.22. The NLO QCD calculations have already been available in the literatures for some time [53–55].

3.4.1 The subprocess $H \to gq\bar{q}$

We consider the heavy quark limit $M_{\text{top}} \gg M_H$. Using again the effective Lagrangian shown in Eq. (3.54), the lowest-order matrix element for $H(Q) \to g(p_1)g(p_2)$, which includes a symmetry factor $1/2!$ for identical gluons in the final state, is given by

$$|\mathcal{M}_{LO}|^2 \;=\; \frac{2}{9}\,\frac{\alpha_s^2\,M_H^4}{\pi^2\,v^2}(1-\epsilon) \tag{3.108}$$

Here, we calculate in $d = 4 - 2\,\epsilon$ dimensions. The matrix element for $H(Q) \to g(p_1)q(p_2)\bar{q}(p_3)$ is

$$\begin{aligned} |\mathcal{M}(H \to gq\bar{q})|^2 &= \frac{16}{9}\,\frac{\alpha_s^3}{\pi\,v^2}\frac{(p_1+p_2)^4+(p_1+p_3)^4}{(p_2+p_3)^2} \\ &= \frac{16}{9}\,\frac{\alpha_s^3}{\pi\,v^2}\hat{Q}^2\frac{(x_1+x_2-1)^2+(1-x_2)^2}{(1-x_1)} \end{aligned} \tag{3.109}$$

where x_1, x_2 and x_3 are defined by

$$\begin{aligned} x_1 &= \frac{2\hat{p}_k\cdot\hat{Q}}{\hat{Q}^2}, \qquad \hat{p}_k \;\to\; p_1, \qquad \hat{Q}^2 = Q^2 = M_H^2 \\ x_2 &= \frac{2\hat{p}_\ell\cdot\hat{Q}}{\hat{Q}^2}, \qquad \hat{p}_\ell \;\to\; p_2 \\ x_3 &= \frac{2\hat{p}_j\cdot\hat{Q}}{\hat{Q}^2}, \qquad \hat{p}_j \;\to\; p_3 \end{aligned} \tag{3.110}$$

The collinear singularity arises from $x_1 \to 1$. Introducing the dipole $\mathcal{D}_{gq\bar{q}}$, defined in Section 2.4, we find

$$\begin{aligned} &\frac{4\,\pi\,\alpha_s}{2}\,T_R\,\mathcal{D}_{gq\bar{q}}\;|\mathcal{M}_{LO}|^2\;\times 2 \\ &= \frac{4\,\pi\,\alpha_s}{2}\,T_R\;|\mathcal{M}_{LO}|^2\;\frac{2}{(\hat{p}_\ell\cdot\hat{p}_j)^2}\left\{\frac{\hat{p}_\ell\cdot Q\,\hat{p}_j\cdot p_\ell+\hat{p}_j\cdot Q\,\hat{p}_\ell\cdot p_\ell}{p_\ell\cdot Q}-\frac{Q^2\,\hat{p}_\ell\cdot p_\ell\,\hat{p}_j\cdot p_\ell}{(p_\ell\cdot Q)^2}\right\}\times 2 \\ &= \frac{8}{9}\frac{\alpha_s^3}{\pi\,v^2}\,\hat{Q}^2\,\frac{1}{1-x_1}\left\{\frac{x_2\,(x_1+x_2-1)+(2-x_1-x_2)\,(1-x_2)}{x_1}\right. \\ &\qquad\left.-2\,\frac{(1-x_1)\,(1-x_2)\,(x_1+x_2-1)}{x_1^2}\right\}+(x_2\leftrightarrow x_3) \end{aligned} \tag{3.111}$$

Here, the dot products $\hat{p}_j \cdot p_\ell$ and $\hat{p}_\ell \cdot p_\ell$ are related to λ, x, y and a_ℓ via

$$\begin{aligned} p_\ell \cdot \hat{p}_j &= \frac{1}{\lambda}\left(y - \frac{1-\lambda+y}{2\,a_\ell}\,x\,(1+y)\right) p_\ell \cdot Q \\ p_\ell \cdot \hat{p}_\ell &= \frac{1}{\lambda}\left(y - \frac{1-\lambda+y}{2\,a_\ell}\,(1-x)\,(1+y)\right) p_\ell \cdot Q \end{aligned} \tag{3.112}$$

where λ, x, y and a_ℓ are defined in Section 2.3 and Section 2.4. It is worth mentioning that the second term in Eq. (3.111) is exactly the same as the first term, so we can just simply multiply by a factor of 2 in the end. Integrating over the three particle phase space and summing over final state quark flavours yields the subtracted decay rate

$$\begin{aligned} d\Gamma(H \to gq\bar{q}) &= \frac{1}{2\,M_H}\int dPS_3 \sum_q \left(|\mathcal{M}(H \to gq\bar{q})|^2 - \frac{4\,\pi\,\alpha_s}{2}\,T_R\,\mathcal{D}_{gq\bar{q}}\,|\mathcal{M}_{LO}|^2\,\times 2\right) \\ &= \frac{G_F\,M_H^3\,\alpha_s^2}{36\,\sqrt{2}\,\pi^3}\left(-\frac{5}{18}\,n_f\right)\frac{\alpha_s}{\pi} \end{aligned} \tag{3.113}$$

Next we use the results of the integrated splitting functions for the m-parton phase space contributions (Section 2.5), which in the case of $gq\bar{q}$ splitting only involves collinear integral. Hence, the integrated $gq\bar{q}$ splitting function, which sums over final state quark flavours, is given by

$$\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\,T_R\,\mu^{2\epsilon}\sum_q \int d\zeta_p v_{gq\bar{q}}^2\,\times 2 \;=\; \frac{\alpha_s}{2\,\pi}\,T_R\left(\frac{4\pi\mu^2}{2p_\ell\cdot\hat{Q}}\right)^{\epsilon}\frac{1}{\Gamma(1-\epsilon)}\,n_f\left(-\frac{2}{3\,\epsilon}-\frac{16}{9}\right)\times 2 \quad . \tag{3.114}$$

3.4.2 The subprocess $\boldsymbol{H \to ggg}$

The matrix element for $H(Q) \to g(p_1)g(p_2)g(p_3)$, which includes a symmetry factor $1/3!$ for identical gluons in the final state, is given by

$$\begin{aligned} &|\mathcal{M}(H \to ggg)|^2 \\ &= \frac{\alpha_s^3}{\pi\,v^2}\frac{32}{3}\frac{1}{3!}\frac{[(p_1+p_2)^2+(p_1+p_3)^2+(p_2+p_3)^2]^4+(p_1+p_2)^8+(p_1+p_3)^8+(p_2+p_3)^8}{(p_1+p_2)^2(p_1+p_3)^2(p_2+p_3)^2} \\ &= \frac{\alpha_s^3}{\pi\,v^2}\frac{32}{3}\frac{1}{3!}\left\{\frac{2s_{123}^2 s_{12}}{s_{13}s_{23}}+\frac{2s_{123}^2 s_{13}}{s_{12}s_{23}}+\frac{2s_{123}^2 s_{23}}{s_{12}s_{13}}+\frac{2s_{12}s_{13}}{s_{23}}+\frac{2s_{12}s_{23}}{s_{13}}+\frac{2s_{13}s_{23}}{s_{12}}+8s_{123}\right\} \end{aligned} \tag{3.115}$$

where the first three terms correspond to soft singularities (interference terms) and the next three terms correspond to collinear singularities. Since there are three identical gluons in the final

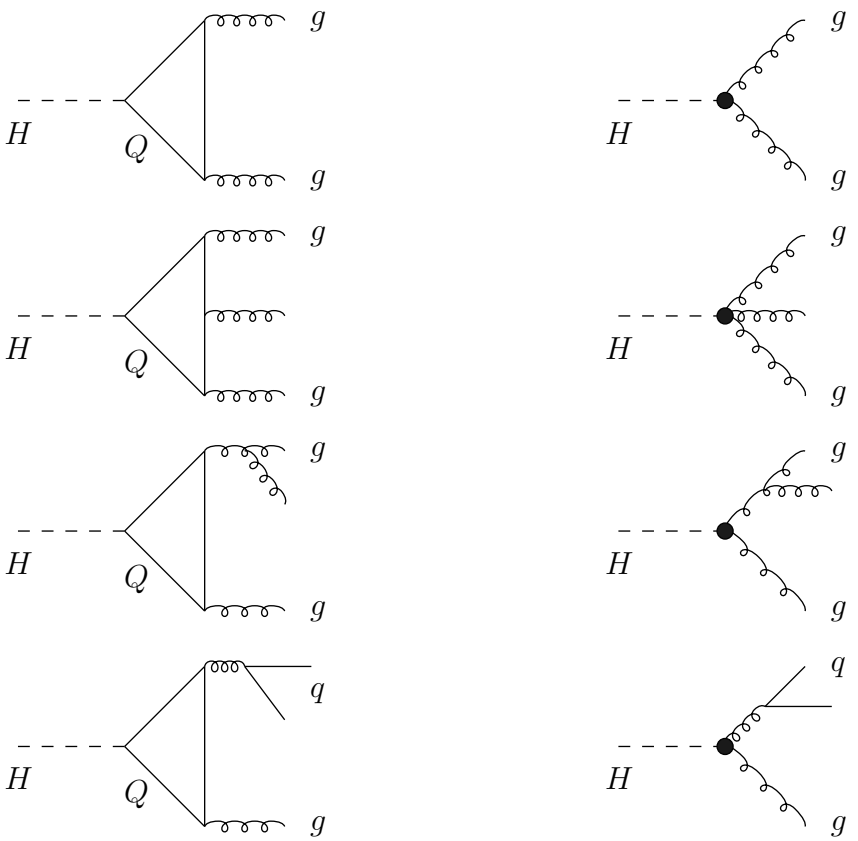


Figure 3.21: Decays of the Higgs boson: LO diagrams and $H \to ggg$, $H \to gq\bar{q}$ diagrams.

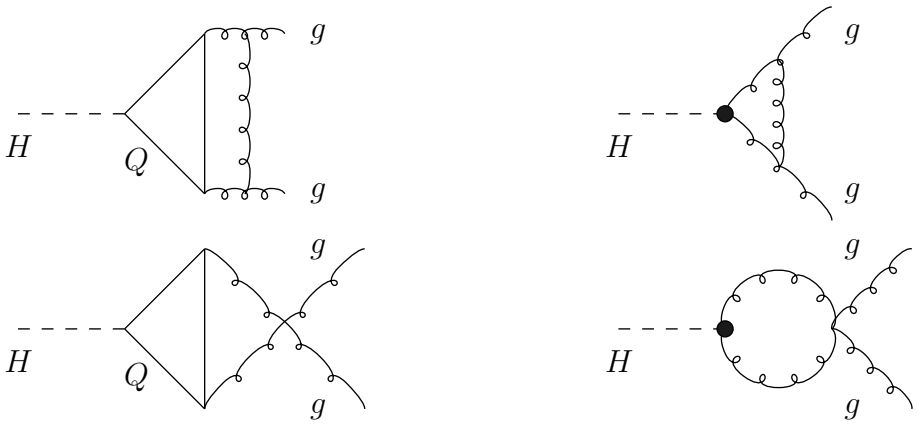


Figure 3.22: Decays of the Higgs boson $H \to gg$: virtual diagrams.

state, the labelling of the gluon, denoted with $\hat{p}_j$ $(j = 1, 2, 3)$, is arbitrary. So we have to take all combinations into account:

$$
\begin{array}{lll}
A: & \hat{p}_j = p_1 \to \text{emitted gluon} & p_2/p_3 \to \text{emitter/spectator (or spectator/emitter)} \\
B: & \hat{p}_j = p_2 \to \text{emitted gluon} & p_1/p_3 \to \text{emitter/spectator (or spectator/emitter)} \\
C: & \hat{p}_j = p_3 \to \text{emitted gluon} & p_1/p_2 \to \text{emitter/spectator (or spectator/emitter)}
\end{array}
$$

so the dipoles corresponding to soft part can be written as three different configurations A, B, C

$$
\begin{aligned}
\mathcal{D}_s &= \frac{4\,\pi\,\alpha_s}{2}\,C_A\,|\mathcal{M}_{LO}|^2\,\frac{1}{3}\,\left(v_s^2(A)+v_s^2(B)+v_s^2(C)\right)\,\times 2 \\
&= \frac{\alpha_s^3}{\pi\,v^2}\frac{32}{3}\frac{1}{3!}\left\{\frac{2s_{123}^2 s_{12}}{s_{13}s_{23}}+\frac{2s_{123}^2 s_{13}}{s_{12}s_{23}}+\frac{2s_{123}^2 s_{23}}{s_{12}s_{13}}\right\}
\end{aligned}
\tag{3.116}
$$

where

$$
v_s^2 \;=\; v_{\text{eik}}^2 - v_{\text{soft}}^2 \;=\; \frac{4\,\hat{p}_\ell\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}}{\left(\hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}+\hat{p}_\ell\cdot\hat{p}_j\,\hat{p}_k\cdot\hat{Q}\right)\,\hat{p}_\ell\cdot\hat{p}_j}, \quad (\ell,j,k=1,2,3) \tag{3.117}
$$

and factor of 2 in Eq. (3.116) is present due to the fact that emitters and spectators are interchangeable. The dipole corresponding to collinear part is

$$
\begin{aligned}
\mathcal{D}_c &= \frac{4\,\pi\,\alpha_s}{2}\,C_A\,|\mathcal{M}_{LO}|^2\,\frac{1}{3}\,\left(v_c^2(A)+v_c^2(B)+v_c^2(C)\right)\,\times 2 \\
&= \frac{4}{3}\frac{\alpha_s^3}{\pi\,v^2}\,\hat{Q}^4\,\frac{1}{3}\,\left\{\frac{4s_{12}s_{13}}{s_{23}(s_{12}+s_{13})^2}\left[2-\frac{4s_{23}\hat{Q}^2}{(s_{12}+s_{23})(s_{13}+s_{23})}+\left[\frac{2s_{23}\hat{Q}^2}{(s_{12}+s_{23})(s_{13}+s_{23})}\right]^2\right]\right. \\
&+\frac{4s_{12}s_{23}}{s_{13}(s_{12}+s_{23})^2}\left[2-\frac{4s_{13}\hat{Q}^2}{(s_{12}+s_{13})(s_{13}+s_{23})}+\left[\frac{2s_{13}\hat{Q}^2}{(s_{12}+s_{13})(s_{13}+s_{23})}\right]^2\right] \\
&\left.+\frac{4s_{13}s_{23}}{s_{12}(s_{13}+s_{23})^2}\left[2-\frac{4s_{12}\hat{Q}^2}{(s_{12}+s_{23})(s_{12}+s_{13})}+\left[\frac{2s_{12}\hat{Q}^2}{(s_{12}+s_{23})(s_{12}+s_{13})}\right]^2\right]\right\}
\end{aligned}
\tag{3.118}
$$

where

$$
\begin{aligned}
v_c^2 &= \tilde{v}_{ggg}^2 - v_{\text{eik}}^2 \\
&= \frac{(\hat{p}_j\cdot Q-\hat{p}_j\cdot p_\ell)\,(\hat{p}_\ell\cdot Q-\hat{p}_\ell\cdot p_\ell)}{(p_\ell\cdot Q)^2\,\hat{p}_\ell\cdot\hat{p}_j}\left\{2-2\,\frac{\hat{p}_j\cdot\hat{p}_\ell\,Q^2}{\hat{p}_\ell\cdot Q\,\hat{p}_j\cdot Q}+\left(\frac{\hat{p}_j\cdot\hat{p}_\ell\,Q^2}{\hat{p}_\ell\cdot Q\,\hat{p}_j\cdot Q}\right)^2\right\} \\
&(\ell,j,k=1,2,3)
\end{aligned}
\tag{3.119}
$$

Here, factor of 2 is present for the same reason as explained above. Integrating over the three particle phase space yields the subtracted decay rate

$$\begin{aligned} d\Gamma(H \to ggg) &= \frac{1}{2\,M_H} \int dPS_3 \, \left(|\mathcal{M}(H \to ggg)|^2 - \mathcal{D}_c - \mathcal{D}_s \right) \\ &= \frac{G_F\, M_H^3\, \alpha_s^2}{36\,\sqrt{2}\,\pi^3} \left(\frac{-214 + 27\,\pi^2}{24} \right) \frac{\alpha_s}{\pi} \end{aligned} \tag{3.120}$$

Next we have to consider the m-parton phase space contributions, the integrated splitting functions in the case of ggg splitting involve both the collinear and soft integrals (see Section 2.5) which are given by

$$\begin{aligned} & \left(\frac{4\pi\alpha_s}{2\,(1-\epsilon)}\, C_A\, \mu^{2\epsilon} \int d\zeta_p \, \left[\tilde{v}_{ggg}^2 - v_{\text{eik}}^2 \right] + 4\,\pi\,\alpha_s\, C_A\, \mu^{2\epsilon} \int d\zeta_p\, \Delta W \right) \times 2 \\ = \; & \frac{\alpha_s}{2\,\pi}\, C_A \left(\frac{4\pi\mu^2}{2 p_\ell \cdot \hat{Q}} \right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{2}{\epsilon^2} + \frac{11}{3\,\epsilon} + \frac{163}{9} - \frac{7}{4}\pi^2 \right) \end{aligned} \tag{3.121}$$

Here, the $1/(1-\epsilon)$ is already included in the soft term. The virtual matrix element for $H \to gg$ in the $\overline{\text{MS}}$ scheme is given in Eq. (3.83). Now, if we combine the real emission contributions $H \to gq\bar{q},\; H \to ggg$ and virtual contribution, the $1/\epsilon^2$ pole is cancelled

$$\begin{aligned} & d\Gamma(H \to gg(g) + gq\bar{q}) \\ & = \frac{1}{2\,M_H} \int dPS_2 \left\{ |\mathcal{M}_V|^2 + \sum_q \int d\zeta_p\, \mathcal{D}(gq\bar{q})\; |\mathcal{M}_{LO}|^2 + \int d\zeta_p\, \mathcal{D}(ggg)\; |\mathcal{M}_{LO}|^2 \right\} \\ & + d\Gamma(H \to ggg) + d\Gamma(H \to gq\bar{q}) \\ & = \Gamma_{LO} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{\alpha_s}{2\,\pi} \right) \beta_0\, \frac{2}{\epsilon} + \Gamma_{LO} \left(\frac{73}{4} - \frac{7}{6}\, n_f \right) \left(\frac{\alpha_s}{\pi} \right) \end{aligned} \tag{3.122}$$

and the leftover $1/\epsilon$ pole has to be renormalized by performing charge renormalization. Here, $\mathcal{D}(gq\bar{q})$ and $\mathcal{D}(ggg)$ are symbolic notations for splitting functions with appropriate prefactor neglected, and their explicit expressions are given in Eq. (3.114) and Eq. (3.121). The charge counterterm in the $\overline{\text{MS}}$ renormalization scheme is

$$\Gamma_{\text{ch}} \;=\; (4\,Z_g)\; \Gamma_{LO}(H \to gg) \tag{3.123}$$

where

$$Z_g \;=\; -\frac{\alpha_s}{2\,\epsilon}\; (4\,\pi)^{\epsilon}\; b_0\, \Gamma(1+\epsilon), \quad b_0 \;=\; \frac{1}{2\,\pi} \left(\frac{11}{6}\, C_A - \frac{2}{3}\, n_f\, T_R \right) \;=\; \frac{1}{2\pi}\, \beta_0 \tag{3.124}$$

and n_f is the number of light quarks. Hence

$$\Gamma_{\text{ch}} \;=\; -\Gamma_{LO}\, \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^{\epsilon} \left(\frac{\alpha_s}{\pi} \right) \beta_0\, \frac{1}{\epsilon} - \Gamma_{LO} \left(\frac{\alpha_s}{\pi} \right) \beta_0\, \ln \frac{\mu_F^2}{\mu^2} \tag{3.125}$$

If we treat everything equivalently in the effective theory, we need to add the effective Lagrangian correction term, which is given by[7]

$$\boxed{\Gamma_{LO}\,\frac{\alpha_s}{\pi}\,\frac{11}{2}} \tag{3.126}$$

Result

Combining all contributions to the total decay rate for $H \to gg$ at NLO, the result is completely finite

$$\begin{aligned} & \Gamma_{LO} + \Gamma(H \to gg(g) + gq\bar{q}) + \Gamma_{\text{ch}} \\ = \;& \Gamma_{LO}\left[1 + \left[\left(\frac{95}{4} - \frac{7}{6}\,n_f\right) + \left(\frac{11}{6}\,C_A - \frac{2}{3}\,n_f\,T_R\right)\ln\frac{\mu^2}{\mu_F^2}\right]\frac{\alpha_s\,(M_H^2)}{\pi}\right] \end{aligned} \tag{3.127}$$

where

$$\Gamma_{LO}(H \to gg) \;=\; \frac{G_F\,M_H^3\,\alpha_s^2}{36\,\sqrt{2}\,\pi^3} \tag{3.128}$$

The result shown here is in agreement with [53].

[7] See talks given by F.J. Petriello at PSI Zuoz Summerschool 2010: *http://ltpth.web.psi.ch/zuoz2010/index.htm* and Fourth Graduate School in Physics at Colliders (Torino) 2009: *http://www.ph.unito.it/dft/scuola09/*

Chapter 4

Conclusions

In this thesis we have proposed an alternative subtraction method at NLO QCD calculations. The traditional way of regularizing the infrared singularities in both one-loop diagrams and real radiation corrections in the context of dimensional regularization scheme is straightforward for the analytic cancellation of the respective divergences. However, numerical implementations for multi-particle processes prove to be challenging.

The subtraction method provides a way of achieving this. The goal of subtraction formalism is to extract infrared divergences from real radiation matrix elements in order to combine them with the one-loop virtual contributions. The key observation for the subtraction scheme is that a $(m+1)$-parton matrix element can be factorized into a m-parton matrix element multiplied by the generalised splitting functions (which contain the singularity structure of the $(m+1)$-parton matrix element) in the soft and collinear limits based on the factorization theorems (see Eq. (4.1)). An important message is that the splitting functions are universal and process independent; this means we calculate them once and for all. The symbolic expression is

$$\mathcal{M}_{m+1}(\{\hat{p}\}_{m+1}) \approx v_\ell(\{\hat{p}\}_{m+1}) \cdot \mathcal{M}_m(\{p\}_m) \ . \tag{4.1}$$

In this new scheme, dipoles ($\approx |v_\ell|^2$) are based on the momentum mappings and on the splitting functions derived from an improved parton shower formulation with quantum interference [28]. Momentum mappings must guarantee total energy momentum conservation as well as the on-shell condition for all external partons both before and after the mappings. One important feature of our scheme is that we use a global momentum mapping in which the mapping takes all the partons into account at once when going from $(m+1)$- to m-particle phase space, instead of separately summing over all possible emitter/spectator pairs like Catani-Seymour scheme. As a result, the number of dipole terms is greatly smaller than the standard subtraction schemes. Another essential point of our formalism is that we split the collinear and soft (based on the eikonal approximation) splitting functions according to Eq. (2.160) in such a way that the two

terms $\overline{W}_{\ell\ell} - \overline{W}^{\text{eikonal}}_{\ell\ell}$ and $\overline{W}^{\text{eikonal}}_{\ell\ell} - \overline{W}_{\ell k}$ are positive and hence we can use these splitting functions as dipole subtraction terms to construct a parton shower Monte Carlo program. The use of the shower splitting functions as dipoles also significantly facilitates the matching of NLO calculations with the corresponding parton shower.

We have also achieved the analytical integrations of the splitting functions over an unresolved one parton integration measure, obtaining the correct soft and collinear singularities in ϵ that are necessary to cancel the soft divergences in the virtual diagrams.

To establish our formalism we have investigated some simple processes at colliders with up to two massive/massless particles in the final state. We have presented all subtraction terms and their integrated splitting functions and have applied our scheme to a variety of well-known processes at NLO, showing that the singular behaviours of the shower splitting functions indeed match the behaviours of real radiation matrix elements and one-loop contributions in the soft and collinear limits. In more detail[1], we have investigated single W production at hadron colliders (initial-state $qq(x)g$ and $gq\bar{q}$ collinear splittings, interference between initial states), dijet production at lepton colliders (final-state qqg collinear splittings, interference between final states), Higgs production at hadron colliders (initial-state $qqg(x)$ and ggg collinear splittings, interference between initial states), Higgs decay to two gluons (final-state $gq\bar{q}$ and ggg collinear splittings, interference between final states). The discussions about interference between initial and final states or interference between final and initial states can be found in [87] in which we have used deep-inelastic scattering (DIS) process. In all cases, we have reproduced the results from the literature and have shown that our implementation agrees with results obtained using the Catani-Seymour scheme.

In this thesis, we have demonstrated that the global momentum mappings combined with the shower splitting functions as dipoles indeed can be used as the subtraction terms for some simple processes; the advantages of the two features will become apparent when applying to more involved multi-parton processes at NLO or matching the NLO calculations with the corresponding improved parton shower [2]. Due to the different momentum mapping prescription, our scheme leads to more complicated finite parts of the integrated splitting functions when considering processes with three or more final-state partons, and an example of the general case ($a_\ell \neq 1$) for the final-state splitting function $g \to q\,\bar{q}$ has been presented in [86]. Nevertheless all these finite parts can be integrated numerically in the respective Monte Carlo program. A generic application to a more non-trivial scattering process is still work in progress. However, we are confident that we will obtain some interesting results for multi-parton final states using our new scheme.

[1] *c.f.* Table 4.1

[2] We note that the work [89] is to implement the new scheme into the Helac Event Generator [90].

Splitting function	*Process*
Initial state:	
$qq(x)g$	single W production, DIS
$gq\bar{q}$	single W production
$qqg(x)$	Higgs production
ggg	Higgs production
interference between initial states	single W production, Higgs production
interference between initial and final states	DIS
Final state:	
qqg	Dijet production, DIS
$gq\bar{q}$	Higgs decay
ggg	Higgs decay
interference between final states	Dijet production, Higgs decay
interference between final and initial states	DIS

Table 4.1: List of all splitting functions presented in Chapter 2 and test processes used for the scheme validation in Chapter 3 and [87].

Chapter 5

Appendix

5.1 Useful mathematical formulae

Here are some formulae that I find useful from time to time.

5.1.1 Gamma function, Beta function and Hypergeometric function

Gamma function

We make frequent use of the Euler Γ-function, which can be defined by the convergent integral

$$\Gamma(z) \;=\; \int_0^\infty dt\, t^{z-1}\, e^{-t}, \qquad \mathcal{R}e\{z\} > 0 \tag{5.1}$$

Integration by parts can confirm the identity

$$\Gamma(1+z) \;=\; z\,\Gamma(z) \tag{5.2}$$

Hence, for positive, integer values of z, we have

$$\Gamma(z) \;=\; (z-1)! \tag{5.3}$$

Eq. (5.2) can also be used to shift the argument and define the Γ-function when $\mathcal{R}e\{z\} < 0$. This shows that there are simple poles at $z \;=\; 0, -1, -2, \cdots$. The following expansion is also useful

$$\Gamma(1+\epsilon) \;=\; 1 - \gamma_E\,\epsilon + \left(\frac{\pi^2}{12} + \frac{1}{2}\,\gamma_E^2\right)\,\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{5.4}$$

where $\gamma_E \;=\; 0.577\,215\,664\,901\,53\cdots$ is the Euler-Mascheroni constant.

Beta function

We will also sometimes use the β-function integral, which is defined by

$$B(1+m,1+n) \;=\; \int_0^1 dx\, x^m\,(1-x)^n \;=\; \frac{\Gamma(1+m)\,\Gamma(1+n)}{\Gamma(2+m+n)}, \qquad \mathcal{R}e\{m,n\} > -1 \tag{5.5}$$

Hypergeometric function ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$

Next we list a couple of useful relations for the Hypergeometric functions; further details can be found in [119, 120].

$$\begin{aligned} &{}_pF_q(a_1,\cdots,a_{k-1},a_k,a_{k+1},\cdots,a_p;b_1,\cdots,b_{m-1},a_k,b_{m+1},\cdots,b_q;x) \\ &= {}_{p-1}F_{q-1}(a_1,\cdots,a_{k-1},a_{k+1},\cdots,a_p;b_1,\cdots,b_{m-1},b_{m+1},\cdots,b_q;x) \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\int_0^1 dx\, x^{m-1}\,(1-x)^{n-1}\,{}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;xt) \\ &= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\,{}_{p+1}F_{q+1}(m,a_1,\cdots,a_p;m+n,b_1,\cdots,b_q;t) \end{aligned} \tag{5.7}$$

$$\begin{aligned} {}_2F_1(a_1,a_2;b;x) &= \frac{\Gamma(b)}{\Gamma(a_2)\Gamma(b-a_2)}\int_0^1 dt\, t^{a_2-1}(1-t)^{b-a_2-1}(1-xt)^{-a_1} \\ {}_2F_1(a_1,a_2;b;1) &= \frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)} \end{aligned} \tag{5.8}$$

5.1.2 Dilogarithm function

The dilogarithm function can be defined by the sum

$$\mathrm{Li}_2(z) \;=\; \frac{z^1}{1^2}+\frac{z^2}{2^2}+\frac{z^3}{3^2}+\cdots \;=\; \sum_{n=1}^{\infty}\frac{z^n}{n^2} \tag{5.9}$$

or the integral

$$\mathrm{Li}_2(z) \;=\; -\int_0^1 dt\,\frac{\ln(1-t\,z)}{t} = -\int_0^z dt\,\frac{\ln(1-t)}{t} \tag{5.10}$$

The derivative of the dilogarithm function is given by

$$\frac{d}{dz}\,\mathrm{Li}_2(z) \;=\; -\,\frac{\ln(1-z)}{z} \tag{5.11}$$

The major functional equations for the dilogarithm function are given by

$$\begin{aligned}
\mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) - \ln(z)\,\ln(1-z) + \zeta(2) \\
\mathrm{Li}_2\left(\frac{1}{z}\right) &= -\mathrm{Li}_2(z) - \frac{1}{2}\,\ln^2(-z) - \zeta(2) \\
\mathrm{Li}_2\left(\frac{1}{1-z}\right) &= \mathrm{Li}_2(z) + \ln(1-z)\,\ln(-z) - \frac{1}{2}\,\ln^2(1-z) + \zeta(2) \\
\mathrm{Li}_2\left(-\frac{1-z}{z}\right) &= \mathrm{Li}_2(z) + \ln(z)\,\ln(1-z) - \frac{1}{2}\,\ln^2(z) - \zeta(2) \\
\mathrm{Li}_2\left(-\frac{z}{1-z}\right) &= -\mathrm{Li}_2(z) - \frac{1}{2}\,\ln^2(1-z) \\
\mathrm{Li}_2(z^2) &= 2\,[\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)]
\end{aligned} \tag{5.12}$$

where the Riemann zeta function is defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{5.13}$$

The values of $\zeta(n)$ for small positive integer values of n are

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(3) &= 1.202\,056\,903\,159\,59\cdots \\
\zeta(4) &= \frac{\pi^4}{90}
\end{aligned} \tag{5.14}$$

5.1.3 The volume element in d dimensions

We consider general formula of the volume element in d dimensions

$$dV_d \;=\; d^d r \;=\; r^{d-1}\,dr\,d\Omega_d \qquad (\text{Euclidean space}) \tag{5.15}$$

where the area element in Euclidean space is

$$\begin{aligned}
d\,\Omega_d = \prod_{\ell=1}^{d-1} \sin^{d-1-\ell}\theta_\ell\,d\,\theta_\ell \;&=\; \int_0^\pi d\,\theta_1 \sin^{d-2}\theta_1 \cdots \int_0^\pi d\,\theta_{d-2} \sin\theta_{d-2} \int_0^{2\,\pi} d\,\theta_{d-1} \\
&= \int_0^{2\,\pi} d\,\phi \prod_{\ell=1}^{d-2} \int_0^\pi d\,\theta_\ell \sin^\ell\theta_\ell \quad \text{where} \quad \theta \in [0,\pi], \quad \phi \in [0, 2\,\pi]
\end{aligned} \tag{5.16}$$

and the following relations hold

$$\int_0^\pi d\,\theta\,\sin^\ell\theta \;=\; \sqrt{\pi}\,\frac{\Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+2}{2}\right)}, \qquad \prod_{\ell=1}^{d-2} \int_0^\pi d\,\theta_\ell\,\sin^\ell\theta_\ell \;=\; \frac{\pi^{\frac{d}{2}-1}}{\Gamma\,(d/2)} \tag{5.17}$$

d	$\Gamma(d/2)$	$\int d\,\Omega_d$
1	$\sqrt{\pi}$	2
2	1	$2\,\pi$
3	$\sqrt{\pi}/2$	$4\,\pi$
4	1	$2\,\pi^2$

Table 5.1: The values of $\Gamma(d/2)$ and $d\,\Omega_d$.

The relation between $d\,\Omega_{d-1}$ and $d\,\Omega_{d-2}$ is given by

$$\begin{aligned} d\,\Omega_{d-1} &= d\,\Omega_{d-2} \int_0^\pi d\theta\, \sin^{d-3}\theta \;=\; d\,\Omega_{d-2} \int_{-1}^1 d\cos\theta \left(1-\cos^2\theta\right)^{\frac{d-4}{2}} \\ &= \int_0^{2\,\pi} d\,\phi \prod_{\ell=1}^{d-3} \int_0^\pi d\,\theta_\ell \sin^\ell\theta_\ell \end{aligned} \tag{5.18}$$

Finally, we give the values of $\Gamma(d/2)$ and $d\,\Omega_d$ for $d \;=\; 1, \cdots, 4$ in Table 5.1.

5.2 Integration measures

In this subsection we will first derive the integration measures for both the initial state and final state splittings. In order to extract the correct pole structures we have to define good parametrizations of the kinematics. Then we will also list master integrals and general formulae that are used to extract singularities and finite terms in Section 2.5.

5.2.1 Final state splitting

Integration measure: $a_\ell \;=\; 1$

With the kinematics as defined in Section 2.4.3, we obtain the integration measure

$$d\zeta_p \;=\; \frac{(2\,p_\ell\cdot Q)^{1-\epsilon}}{16\,\pi^2}\,\frac{(4\,\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^{y_{\max}} dy\, y^{-\epsilon}\,\lambda^{1-2\,\epsilon} \int_0^1 dz\; \left[z\,(1-z)\right]^{-\epsilon} \int \frac{d^{d-2}\,\Omega}{\Omega_{d-2}} \tag{5.19}$$

for the integration in $d \;=\; 4-2\,\epsilon$ dimensions. Here,

$$y_{\max} \;=\; \left(\sqrt{a_\ell}-\sqrt{a_\ell-1}\right)^2 \;=\; 2\,a_\ell \;-\; 1 \;-\; 2\,\sqrt{a_\ell\,(a_\ell-1)} \tag{5.20}$$

For the integration of the interference terms, we need to consider the additional azimuthal angle of the emitted particle. Hence we keep the second angular integration,

$$\begin{aligned} d\Omega_{d-2} &= d\Omega_{d-3} \int_{-1}^{1} d\cos\varphi \left(1-\cos^2\varphi\right)^{-\frac{(1+2\epsilon)}{2}} \\ &= \frac{2^{1-2\epsilon}\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \int_0^1 dv' \left[v'(1-v')\right]^{-\frac{(1+2\epsilon)}{2}}, \qquad \int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma\left(d/2\right)} \end{aligned} \tag{5.21}$$

where φ is now the azimuthal angle of the emitted parton. So the integration measure becomes

$$d\zeta_p = \frac{(2\,p_\ell\cdot\hat{Q})^{1-\epsilon}}{16} \frac{\pi^{-\frac{5}{2}+\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \int_0^{y_{\max}} dy\, y^{-\epsilon}\,\lambda^{1-2\,\epsilon} \int_0^1 dz\, \left[z\,(1-z)\right]^{-\epsilon} \int_0^1 dv'\, \left[v'\,(1-v')\right]^{-\frac{(1+2\epsilon)}{2}} \tag{5.22}$$

In the integration, we will use v' as a variable; however, the splitting functions are written in terms of $v = v[\bar{z}(y), z, v']$

$$\boxed{v = (4v'-2)\left[z(1-z)\bar{z}(1-\bar{z})\right]^{\frac{1}{2}} + z + \bar{z} - 2\,z\,\bar{z}} \tag{5.23}$$

Note that Eq. (5.23) has been derived in a specific frame which generally does not coincide with the frame following in the inverse transform from m- to $(m+1)$-particle phase space. In general, v is defined in Eq. (2.111), *i.e.*

$$v = \frac{\hat{p}_j\cdot\hat{p}_k}{\hat{p}_k\cdot P_\ell} \quad ,$$

where $\hat{p}_k$ needs to be calculated using the Lorentz transformation

$$\hat{p}_k^\mu = \Lambda(\hat{K}, K)^\mu{}_\nu\, p_k^\nu \quad .$$

We now consider the special case in which $a_\ell = 1$ (which corresponds to $\lambda = 1-y$). We start with the integral

$$I = \int_0^1 dy\, y^{-\epsilon}\,(1-y)^{1-2\epsilon} \int_0^1 dz\, \left[z\,(1-z)\right]^{-\epsilon} \tag{5.24}$$

We now make a change of variable such that

$$x = y + z\,(1-y) \tag{5.25}$$

thus we have

$$I = \int_0^1 dy\, y^{-\epsilon} \int_y^1 dx\, \left[(x-y)\,(1-x)\right]^{-\epsilon} \tag{5.26}$$

We now change the integration order

$$\int_0^1 dy \int_y^1 dx \quad \longrightarrow \quad \int_0^1 dx \int_0^x dy$$

and obtain

$$I \;=\; \int_0^1 dx \int_0^x dy\, y^{-\epsilon}\, \left[(x-y)\,(1-x)\right]^{-\epsilon} \tag{5.27}$$

We now make a change of variable once again

$$u \;=\; \frac{y}{x}$$

finally leads to

$$I \;=\; \int_0^1 dx\, x^{1-2\epsilon}\,(1-x)^{-\epsilon} \int_0^1 du\, u^{-\epsilon}\,(1-u)^{-\epsilon} \tag{5.28}$$

Now the integration measure becomes

$$d\zeta_p \;=\; \frac{(2p_\ell\cdot\hat{Q})^{1-\epsilon}}{16}\frac{\pi^{-\frac{5}{2}+\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \int_0^1 du\, u^{-\epsilon}\,(1-u)^{-\epsilon} \int_0^1 dx\, x^{1-2\epsilon}\,(1-x)^{-\epsilon} \int_0^1 dv'\,\left[v'\,(1-v')\right]^{-\frac{(1+2\epsilon)}{2}} \tag{5.29}$$

Here u and x can be expressed in terms of momenta, we have

$$\boxed{x \;=\; \frac{\hat{p}_j\cdot Q}{\gamma\, p_\ell\cdot Q}, \qquad u \;=\; \gamma\,\frac{\hat{p}_\ell\cdot\hat{p}_j}{\hat{p}_j\cdot Q}} \tag{5.30}$$

For $a_\ell \;=\; 1$ case we have $\gamma \;=\; 1$. It follows immediately that x is purely soft variable and u is purely collinear variable. Note that the collinear integrations do not depend on v' (or the azimuthal angle φ), while the soft integrations do depend on v' or φ.

Integration measure: general case $a_\ell \;\neq\; 1$

Again, we start with the integral

$$I \;=\; \int_0^{y_{\max}} dy\, y^{-\epsilon}\,\lambda^{1-2\epsilon} \int_0^1 dz\,\left[z\,(1-z)\right]^{-\epsilon} \tag{5.31}$$

The first variable transformation is defined via

$$z \;=\; \frac{x-x_0}{1-x_0} \tag{5.32}$$

leading to

$$I \;=\; \int_0^{y_{\max}} dy\, y^{-\epsilon}\,\gamma^{1-2\,\epsilon} \int_{x_0}^1 dx\,\left[(1-x)\,(x-x_0)\right]^{-\epsilon} \tag{5.33}$$

Next we use the rescaling parameter δ defined as

$$1+\delta \;=\; \frac{a_\ell - \sqrt{a_\ell^2 - a_\ell\,\frac{4\,x}{(1+x)^2}}}{2\,x}\,(1+x)^2 \;\geq\; 1 \tag{5.34}$$

and redefine

$$y = \delta\, u \tag{5.35}$$

We then obtain

$$I \;=\; \int_0^1 du\, u^{-\epsilon} \int_0^1 dx\, \delta^{1-\epsilon}\, \gamma^{1-2\epsilon}\; [(1-x)\,(x-x_0)]^{-\epsilon} \tag{5.36}$$

Finally the complete integration measure is

$$\begin{aligned} d\zeta_p \;=\; & \frac{(2\,p_\ell\cdot\hat{Q})^{1-\epsilon}}{16}\,\frac{\pi^{-\frac{5}{2}+\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \\ & \times \int_0^1 du\, u^{-\epsilon} \int_0^1 dx\, \delta^{1-\epsilon}\, \gamma^{1-2\,\epsilon}\; [(1-x)\,(x-x_0)]^{-\epsilon} \int_0^1 dv'\; [v'\,(1-v')]^{-\frac{1+2\,\epsilon}{2}} \end{aligned} \tag{5.37}$$

For $a_\ell = 1$ we of course obtain the result in the last subsection.

5.2.2 Initial state splitting

Integration measure

For the initial state splitting, the integration measure is given by

$$d\zeta_p = \frac{d^d\hat{p}_j}{(2\,\pi)^d}\,2\,\pi\,\delta^+(\hat{p}_j^2)\,\frac{\alpha}{\hat{\alpha}} \;=\; \frac{d^d\hat{p}_j}{(2\,\pi)^d}\,2\,\pi\,\delta^+(\hat{p}_j^2)\,\frac{d\hat{\eta}_a}{d\eta_a} \tag{5.38}$$

where $\alpha = \eta_b s$ and $\hat{\alpha} \;=\; \eta_b\, s - 2\,p_A\cdot\hat{p}_j$. The factor $\alpha/\hat{\alpha}$ is just the derivative $d\hat{\eta}_a/d\eta_a$ calculated from the relation $\hat{K}^2 \;=\; K^2$. Using the identity

$$d^d p\,\delta^+(p^2) \;=\; \frac{1}{2}\,(p^0)^{d-3}\,dp^0\,d\Omega_{d-2}\,d\cos\theta\,\left(1-\cos^2\theta\right)^{\frac{d-4}{2}} \tag{5.39}$$

We then obtain

$$\begin{aligned} d\zeta_p = dx\,dy & \frac{(\eta_a\,\eta_b\,s)^{1-\epsilon}x^{\epsilon-1}}{\Gamma(1-\epsilon)(4\,\pi)^{2-\epsilon}}\,(1-x)^{-2\epsilon}\left[\frac{y}{1-x}\left(1-\frac{y}{1-x}\right)\right]^{-\epsilon}\,\frac{1}{x}\,\theta(x-\eta_a) \\ & \times\,\theta(x\,(1-x))\,\theta(y)\,\theta\left(1-\frac{y}{1-x}\right) \end{aligned} \tag{5.40}$$

where we calculated in $d \;=\; 4-2\,\epsilon$ dimensions. However, note that the change of variables in η_a corresponds to

$$\boxed{\int_0^1 d\hat{\eta}_a \;=\; \frac{1}{x}\int_0^x d\eta_a} \tag{5.41}$$

so we have to impose the condition $x > \eta_a$; alternatively, we can not make the change of variables and keep $\hat{\eta}_a = \eta_a/x$ as the integration variable; this is the approach followed in [22]. For comparison, we will instead use

$$\boxed{d\zeta_p \to d\zeta_p \frac{1}{x}\theta(x-\eta_a)} \tag{5.42}$$

For the integration of the interference terms, we need to keep the additional azimuthal angle φ of the emitted particle in $d\Omega_{d-2}$; see Eq. (5.21). Using the variables as defined in Section 2.4.4, we obtain for the integration measure including the additional angle,

$$\begin{aligned} d\zeta_p = dx\,dy\,dv' \frac{(\eta_a\,\eta_b\,s)^{1-\epsilon}x^{\epsilon-1}}{(4\,\pi)^2}(1-x)^{-2\epsilon}\left[\frac{y}{1-x}\left(1-\frac{y}{1-x}\right)\right]^{-\epsilon} \\ \times \frac{\pi^{\epsilon-\frac{1}{2}}}{\Gamma\left(\frac{1-2\epsilon}{2}\right)}\left(v'\,(1-v')\right)^{-\frac{(1+2\epsilon)}{2}}\,\frac{1}{x}\,\theta(x-\eta_a) \\ \times \theta(x\,(1-x))\,\theta(y)\,\theta\left(1-\frac{y}{1-x}\right)\,\theta(v'\,(1-v')) \end{aligned} \tag{5.43}$$

In the integration, we will use v' as a variable; however, the splitting functions are written in terms of $v = v(\bar{z}, y', v')$

$$\boxed{v = (4v'-2)\left[y'\,(1-y')\,\bar{z}\,(1-\bar{z})\right]^{\frac{1}{2}} + y' + \bar{z} - 2\,y'\,\bar{z}} \tag{5.44}$$

hence we have

$$\begin{aligned} v_{\text{max}} &= y' + \bar{z} - 2y'\bar{z} + 2\left[y'(1-y')\bar{z}(1-\bar{z})\right]^{\frac{1}{2}} \\ v_{\text{min}} &= y' + \bar{z} - 2y'\bar{z} - 2\left[y'(1-y')\bar{z}(1-\bar{z})\right]^{\frac{1}{2}} \end{aligned} \tag{5.45}$$

Note, however, that we can only use Eq. (5.44) in the singular limits ($x \to 1$ or $y' \to 0$) where the dependence on v' disappears; for the finite parts, we need to use the original definition given by Eq. (2.135)

$$v = \frac{\hat{\eta}_a\,\eta_b\,s\,\hat{p}_j\cdot\hat{p}_k}{2\,\hat{p}_k\cdot\hat{Q}\,\hat{p}_j\cdot\hat{Q}} = \frac{1}{1-x}\,\frac{\hat{p}_j\cdot\hat{p}_k}{\hat{p}_k\cdot\hat{Q}}$$

where $\hat{p}_k$ needs to be calculated using the Lorentz transformation

$$\hat{p}_k^{\mu} = \Lambda(x,y')^{\mu}{}_{\nu}\,p_k^{\nu} = \Lambda(\hat{K},K)^{\mu}{}_{\nu}\,p_k^{\nu} \quad . \tag{5.46}$$

In the limits $x \to 1$ or $y' \to 0$, we obtain $\hat{p}_k \to p_k$; and $\hat{p}_j$ is parametrized according to Sudhakov parametrization (*e.g.* [109]). The change of variables defined in Section 2.4.4 always requires that the integration over v' is performed before the integration over y in Eq. (5.43). It is worth mentioning that $\bar{z}$ is not integrated out and therefore still depends on x, y' and v' through the Lorentz transformation of $\hat{p}_k$. Also note that the collinear integrations do not depend on v' (or the azimuthal angle φ), while the soft integrations do depend on v' or φ.

5.2.3 Master integrals

Interference between initial and final states

The general master integrals appearing in Section 2.5.4 are

$$\int_0^1 \frac{dy}{y}\left\{\frac{z_0}{\sqrt{4y^2(1-z_0)+z_0^2}}-1\right\}=\ln z_0 \tag{5.47}$$

and

$$\begin{aligned}&\int_0^1 dv\,\frac{\left[v\,(1-v)\right]^{-\frac{1+2\,\epsilon}{2}}}{(4\,v-2)\left[y\,(1-y)z\,(1-z)\right]^{\frac{1}{2}}+2y+z-2yz}\\&=\frac{\pi}{\sqrt{4y^2(1-z)+z^2}}\left\{1-2\,\epsilon\ln\left[\frac{\sqrt{4y^2(1-z)+z^2}}{2\left(2y+z-2yz+\sqrt{4y^2(1-z)+z^2}\right)}\right]\right\}+\mathcal{O}(\epsilon^2)\end{aligned} \tag{5.48}$$

Interference between final (final and initial) states

The general master integrals appearing in Section 2.5.5 are

$$\begin{aligned}&\int_0^1 \frac{du}{u}\frac{1}{\sqrt{1+4a_0(1+a_0)u^2}}\ln\left(1+4a_0(1+a_0)u^2\right)\\&=-\left[\frac{\pi^2}{12}+\ln(1+2a_0)\,\ln\left[2(1+a_0)\right]+\mathrm{Li}_2\left(-(1+2a_0)\right)+\mathrm{Li}_2\left(-2a_0\right)\right]\end{aligned} \tag{5.49}$$

$$\begin{aligned}\int_0^1 dv\,\frac{\ln[v(1-v)]}{\sqrt{v(1-v)}(av+b)}&=\frac{2\pi}{\sqrt{b}\sqrt{a+b}}\ln\frac{\sqrt{b}\sqrt{a+b}}{\left(\sqrt{a+b}+\sqrt{b}\right)^2}\\\int_\epsilon^1 du\,\frac{\ln u}{u\sqrt{1+bu^2}}&=\frac{1}{2}\ln^2\left(\frac{2}{\sqrt{1+b}+1}\right)+\frac{1}{2}\mathrm{Li}_2\left(\frac{\sqrt{1+b}-1}{\sqrt{1+b}+1}\right)-\frac{1}{2}\ln^2\epsilon\\\int_\epsilon^1 du\,\frac{\ln u}{u}&=-\frac{1}{2}\ln^2\epsilon\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}&\int_0^1 dx\,\frac{\ln\left(1+ax^2\right)}{x\sqrt{1+ax^2}}\\&=-\left[\frac{\pi^2}{12}+\frac{1}{2}\,\ln(a+1)\,\ln\left[\sqrt{a+1}+1\right]+\mathrm{Li}_2\left(-\sqrt{a+1}\right)+\mathrm{Li}_2\left(1-\sqrt{a+1}\right)\right]\end{aligned} \tag{5.51}$$

Here, ϵ in Eq. (5.50) is just an infinitesimal parameter.

5.2.4 Pole extractions

Single poles

For a function having a single pole, *e.g.* $x = 0$, we use

$$\int_0^1 dx\, x^{-(1+\epsilon)} g(x) = \int_0^1 dx\, x^{-(1+\epsilon)} \left[g(x) - g(0)\right] - \frac{g(0)}{\epsilon} \tag{5.52}$$

where $g(x)$ is nonsingular function at $x = 0$. For a general case where we integrate only up to x_{max} instead of 1, the equation above becomes

$$\int_0^{x_{\text{max}}} dx\, x^{-(1+\epsilon)} g(x) = \int_0^{x_{\text{max}}} dx\, x^{-(1+\epsilon)} \left[g(x) - g(0)\right] - \frac{g(0)}{\epsilon} + g(0) \ln x_{\text{max}} \tag{5.53}$$

Double poles

In order to extract the double poles, *e.g.* $x = 0$ and $y = 0$, we consider the following integral

$$I \;=\; \int_0^1 dx \int_0^1 dy\, \frac{1}{x^{1+\epsilon}} \frac{1}{y^{1+\epsilon'}}\, g(x,y) \tag{5.54}$$

where $g(x,y)$ is nonsingular function at $x = 0$ and $y = 0$. We use Eq. (5.52) twice and obtain the general formula

$$\begin{aligned} I = & \int_0^1 dx \int_0^1 dy\, \frac{1}{x^{1+\epsilon}} \frac{1}{y^{1+\epsilon'}} \left[g(x,y) - g(x,0) - g(0,y) + g(0,0)\right] \\ & - \frac{1}{\epsilon'} \int_0^1 dx\, \frac{1}{x^{1+\epsilon}} \left[g(x,0) - g(0,0)\right] - \frac{1}{\epsilon} \int_0^1 dy\, \frac{1}{y^{1+\epsilon'}} \left[g(0,y) - g(0,0)\right] + \frac{g(0,0)}{\epsilon\epsilon'} \end{aligned} \tag{5.55}$$

5.3 Colour algebra

Notations

In this section we will give a brief descriptions about the manipulations of colour algebra; we will follow Catani-Seymour's notations very closely. Here we will only consider the processes that involve the final state QCD partons; in the case of processes that involve the initial state QCD partons please refer to [22]. First, it will be convenient to introduce a basis $\{\mid c_1 \cdots c_m > \otimes \mid s_1 \cdots s_m > \}$ in colour + helicity space in such a way that

$$\mathcal{M}_m^{c_1 \cdots c_m\,,\, s_1 \cdots s_m}(p_1, \cdots, p_m) \;=\; \{ < c_1 \cdots c_m \mid \otimes < s_1 \cdots s_m \mid \} \mid 1, \cdots, m >_m \tag{5.56}$$

where $\mid 1,\cdots,m >_m$ is a vector in colour + helicity space; $\{c_1\cdots c_m\}$ and $\{s_1\cdots s_m\}$ are colour indices (for gluons the values take $1\cdots N^2-1$, while the values take $1\cdots N$ for quarks or antiquarks) and spin indices (the values take $\mu = 1,\cdots,d-2$ for gluons and $s = 1,2$ for massless fermions), respectively. According to this notation, the matrix element squared (summed over final-state colours and spins) can be written as

$$|\,\mathcal{M}_m|^2 \;=\; {}_m\!<1,\cdots,m|1,\cdots,m>_m \tag{5.57}$$

It is useful to define the square of colour-correlated tree-amplitudes according to

$$\begin{aligned} |\,\mathcal{M}_m^{i,k}\,|^2 &= {}_m\!<1,\cdots,m\,|\,\mathrm{T}_i\cdot\mathrm{T}_k\,|\,1,\cdots,m>_m \\ &= \left[\mathcal{M}_m^{A_1\cdots B_i\cdots B_k\cdots A_m}(p_1,\cdots,p_m)\right]^*\,\left(T^C\right)_{B_iA_i}\,\left(T^C\right)_{B_kA_k}\,\mathcal{M}_m^{A_1\cdots A_i\cdots A_k\cdots A_m}(p_1,\cdots,p_m) \end{aligned} \tag{5.58}$$

where we have associated a colour charge T_i with the emission of a gluon from each parton i. Here we follow the notations in Chapter 1 where $\left(T^A\right)_{BC} \equiv -\,i\,f^{ABC}$ (colour-charge matrix in the adjoint representation) if the emitting parton i is a gluon and $T^A_{ab} \equiv t^A_{ab}$ (colour-charge matrix in the fundamental representation) if the emitting parton i is a quark; if the emitting parton i is a antiquark, then we have $T^A_{ab} \equiv \bar{t}^A_{ab} = -\,t^A_{ba}$). It is straightforward to check that the colour-charge algebra obeys

$$\mathrm{T}_i\cdot\mathrm{T}_j \;=\; \mathrm{T}_j\cdot\mathrm{T}_i \qquad \text{if } i\neq j, \qquad \mathrm{T}_i^2 \;=\; C_i \tag{5.59}$$

where C_i is the Casimir operator, *i.e.* , $C_i = C_F = (N^2-1)/(2\,N)$ if i is a quark or antiquark and $C_i = C_A = N$ if i is a gluon. Each vector $\mid 1,\cdots,m>_m$ is a colour-singlet state, therefore colour conservation is

$$\sum_{i=1}^{m}\,\mathrm{T}_i\;\mid 1,\cdots,m>_m\,=\,0 \tag{5.60}$$

Examples

In this subsection we will practice with the simplest cases of colour algebra. For the cases with two or three partons, the colour algebra can be computed in factorized form. First, we consider the case with two partons. Using colour conservation relation, we have

$$\mathrm{T}_1\cdot\mathrm{T}_2\,|1,2>=\,-\mathrm{T}_1\cdot\mathrm{T}_1\,|1,2>=\,-\mathrm{T}_1^2\,|1,2>=\,-\mathrm{T}_2^2\,|1,2> \tag{5.61}$$

so that all the charge operators $\{\,\mathrm{T}_1^2,\mathrm{T}_2^2,-\mathrm{T}_1\cdot\mathrm{T}_2\,\}$ are factorizable in terms of the Casimir operator. Now we consider the case with three partons, using colour conservation, we have

$$0\;=\;\left(\sum_{i=1}^{3}\mathrm{T}_i\right)^2\;|1,2,3>=\;\left(\mathrm{T}_1^2+\mathrm{T}_2^2+\mathrm{T}_3^2+2\,\mathrm{T}_1\cdot\mathrm{T}_2+2\,\mathrm{T}_1\cdot\mathrm{T}_3+2\,\mathrm{T}_2\cdot\mathrm{T}_3\right)\;|1,2,3> \tag{5.62}$$

and

$$(\mathrm{T}_1 \cdot \mathrm{T}_2 + \mathrm{T}_1 \cdot \mathrm{T}_3)\ |1,2,3> = \ -\mathrm{T}_1^2\,|1,2,3> \tag{5.63}$$

Combining these two equations we obtain

$$2\,\mathrm{T}_2 \cdot \mathrm{T}_3\,|1,2,3> = \ \left(\mathrm{T}_1^2 - \mathrm{T}_2^2 - \mathrm{T}_3^2\right)\ |1,2,3> \tag{5.64}$$

and similarly for $\mathrm{T}_1 \cdot \mathrm{T}_3$ and $\mathrm{T}_1 \cdot \mathrm{T}_2$. Hence, all the charge operators are factorizable in terms of linear combinations of the Casimir operators $C_1,\ C_2$ and C_3.

The colour algebra does not factorize any longer when the total number n of partons is $n \geq 4$, *e.g.* , if $n = 4$ we have

$$\mathrm{T}_i^2\,|1,2,3,4> = \ C_i\,|1,2,3,4>, \qquad i = 1, \cdots, 4 \tag{5.65}$$

and

$$\mathrm{T}_i \cdot \sum_{j=1}^{4} \mathrm{T}_j\,|1,2,3,4> = \ 0, \qquad i = 1, \cdots, 4 \tag{5.66}$$

in order to single out two independent charge operators, we can write

$$\begin{aligned}
\mathrm{T}_3 \cdot \mathrm{T}_4\,|1,2,3,4> &= \left[\frac{1}{2}(C_1 + C_2 - C_3 - C_4) + \mathrm{T}_1 \cdot \mathrm{T}_2\right]\ |1,2,3,4>, \\
\mathrm{T}_2 \cdot \mathrm{T}_4\,|1,2,3,4> &= \left[\frac{1}{2}(C_1 + C_3 - C_2 - C_4) + \mathrm{T}_1 \cdot \mathrm{T}_3\right]\ |1,2,3,4>, \\
\mathrm{T}_2 \cdot \mathrm{T}_3\,|1,2,3,4> &= \left[\frac{1}{2}(C_4 - C_1 - C_2 - C_3) - \mathrm{T}_1 \cdot \mathrm{T}_2 - \mathrm{T}_1 \cdot \mathrm{T}_3\right]\ |1,2,3,4>, \\
\mathrm{T}_1 \cdot \mathrm{T}_4\,|1,2,3,4> &= -\left[C_1 + \mathrm{T}_1 \cdot \mathrm{T}_2 + \mathrm{T}_1 \cdot \mathrm{T}_3\right]\ |1,2,3,4>
\end{aligned} \tag{5.67}$$

and express all the charge operators in terms of Casimir invariants and $\mathrm{T}_1 \cdot \mathrm{T}_2$ and $\mathrm{T}_1 \cdot \mathrm{T}_3$.

5.4 Phase space integration

In this section, we present the parameterization of the n-particle phase space that we use to evaluate the cross sections. The n-body phase space in d dimensions is

$$\begin{aligned}
dPS_n &= (2\,\pi)^d\,\delta^d\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^d p_i}{(2\,\pi)^{d-1}}\,\delta^+(p_i^2 - m_i^2) \\
&= (2\,\pi)^d\,\delta^d\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^{d-1}\vec{p}_i}{(2\,\pi)^{d-1}\,2\,E_i}
\end{aligned} \tag{5.68}$$

where $\delta^+(q^2-m^2)=\delta(q^2-m^2)\,\theta(q^0)$ ensures that we only consider positive-energy particles. The case of $n=1$ is particularly simple

$$dPS_1 \;=\; 2\,\pi\,\delta^+(Q^2-m^2)|_{Q=p_a+p_b} \tag{5.69}$$

For the two-body phase space ($n=2$), Eq. (5.68) reduces to

$$\begin{aligned} dPS_2(Q\to p_1+p_2) &= \int \frac{d^{d-1}\vec{p}_1}{(2\,\pi)^{d-1}\,2\,E_1}\int \frac{d^{d-1}\vec{p}_2}{(2\,\pi)^{d-1}\,2\,E_2}\,(2\,\pi)^d\,\delta^d(Q-p_1-p_2) \\ &= \frac{1}{4\,(2\,\pi)^{d-2}}\int \frac{d^{d-1}\vec{p}_1}{E_1\,E_2}\underbrace{\int d^{d-1}\vec{p}_2\,\delta^{d-1}(\vec{Q}-\vec{p}_1-\vec{p}_2)}_{=1}\,\delta(Q^0-E_1-E_2) \\ &= \frac{1}{4\,(2\,\pi)^{d-2}}\int \frac{d^{d-1}\vec{p}_1}{E_1\,E_2}\,\delta(Q^0-E_1-E_2) \end{aligned} \tag{5.70}$$

where $E_i=\sqrt{\vec{p}_i\cdot\vec{p}_i+m_i^2}$. We now consider the massive case in which $p_i^2 \;=\; m_i^2\,(i=1,2)$. Using the identities

$$\int dp_2^0\,\delta\left(p_2^0-\sqrt{\vec{p}_2\cdot\vec{p}_2+m_2^2}\right)=1 \quad\text{and}\quad \delta(p_2^2-m_2^2)\,\theta(p_2^0)=\frac{1}{2\,p_2^0}\,\delta\left(p_2^0-\sqrt{\vec{p}_2\cdot\vec{p}_2+m_2^2}\right) \tag{5.71}$$

Hence we have

$$\boxed{\frac{d^3\vec{q}}{2\,q^0} \;=\; d^4q\,\delta(q^2-m^2)\,\theta(q^0) \quad\text{with}\quad q\;=\;p_2} \tag{5.72}$$

Using

$$d^3\vec{p}_1=\vec{p}_1\cdot\vec{p}_1\,d|\vec{p}_1|\,d\Omega\,,\qquad d\Omega \;=\; d\cos\theta\,d\phi,\qquad |\vec{p}_1|\,d|\vec{p}_1| \;=\; p_1^0\,dp_1^0 \tag{5.73}$$

then we obtain

$$\begin{aligned} dPS_2 &= \frac{1}{8\,\pi^2}\,|\vec{p}_1|\,dp_1^0\,d\Omega\,\theta\left(\sqrt{s}-p_1^0\right)\,\delta\left(p_1^0-\frac{s+m_1^2-m_2^2}{2\,\sqrt{s}}\right)\frac{1}{2\,\sqrt{s}} \\ &= \frac{1}{32\,\pi^2}\,\frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{s}\,d\Omega\,\theta\left(\sqrt{s}-\frac{s+m_1^2-m_2^2}{2\,\sqrt{s}}\right) \end{aligned} \tag{5.74}$$

with

$$|\vec{p}_1|=\frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{2\,\sqrt{s}} \quad\text{and}\quad \lambda(x,y,z) \;=\; x^2+y^2+z^2-2\,(x\,y+y\,z+z\,x) \tag{5.75}$$

Here we work in the CM frame in which $Q=(\sqrt{s},\vec{0}\,)$. If only one of the two final state particles is massive with mass $m_1 \;=\; m\,(m_2 \;=\; 0)$, then the two-particle phase space integration is

$$dPS_2 \;=\; \frac{1}{32\,\pi^2}\left(1-\frac{m^2}{s}\right)\,d\Omega,\quad \text{provided that}\quad s>m^2 \tag{5.76}$$

Multiparticle phase space decomposition

Now consider a system Z with particles of momenta $p_1, p_2, \cdots, p_n$ in the final state. We partition the final state particles into two subsystems X and Y:

$$Z \longrightarrow X(p_1 p_2 \cdots p_j) + Y(p_{j+1} \cdots p_n) \tag{5.77}$$

The n-body phase space can be decomposed as follows:

$$\begin{aligned} dPS_n(Z \to p_1 p_2 \cdots p_n) = \; & dPS_2(Z \to X\,Y)\, \frac{dm_X^2}{2\,\pi}\, \frac{dm_Y^2}{2\,\pi} \\ & \cdot dPS_j(X \to p_1 p_2 \cdots p_j)\, dPS_{n-j}(Y \to p_{j+1} \cdots p_n) \end{aligned} \tag{5.78}$$

where m_X and m_Y are resonant masses of the decaying particles X and Y, respectively. The ranges of the invariant mass m_X and m_Y are

$$\sum_{i=1}^{j} m_i \;\le\; m_X\,, \qquad \sum_{i=j+1}^{n} m_i \;\le\; m_Y\,, \qquad m_X + m_Y \le m_Z \tag{5.79}$$

Hence a particle decays into another two particles (with resonance of masses m_X and m_Y, respectively) which later decay into j and $n-j$ particles, respectively. The subsystems X and Y can be further reduced recursively until we obtain the products of two-particle phase spaces. In the special case where Y is a single particle p_n, then we have

$$\begin{aligned} & dPS_n(Z \to p_1 p_2 \cdots p_n) = \; dPS_2(Z \to X\,Y) \;\cdot\; \frac{dm_X^2}{2\,\pi} \;\cdot dPS_{n-1}(X \to p_1 p_2 \cdots p_{n-1}) \\ & m_1 + m_2 + \cdots + m_{n-1} \le m_X \le m_Z - m_n \end{aligned} \tag{5.80}$$

We now consider the parametrization of the three-particle phase space [97]

$$Z \longrightarrow X(p_1 p_3) + Y(p_2) \tag{5.81}$$

In this case Y is a single particle p_2. We decompose the three-particle phase space into two two-particle phase spaces as follows

$$dPS_3(Z \to p_1 p_2 p_3) = \; dPS_2(Z \to X\,Y) \;\cdot\; \frac{ds_{13}}{2\,\pi} \;\cdot dPS_2(X \to p_1 p_3) \tag{5.82}$$

where $s_{ij} = (p_i + p_j)^2 = p_{ij}^2$ and $p_1 + p_2 + p_3 = Q$. We work in the CM frame in which $Q = (\sqrt{s}, \vec{0}\,)$, hence we have $\vec{p}_{13} = -\vec{p}_2$ and $|\vec{p}_{13}| = 1/2\,\sqrt{\lambda(s, s_{13}, m_2^2)/s}$. Using

$$\begin{aligned} dPS_2(Z \to X\,Y) &= \frac{1}{32\,\pi^2}\, \frac{\sqrt{\lambda(s, s_{13}, m_2^2)}}{s}\, d\cos\theta_{13}\, d\phi_{13}\, \theta\left(\sqrt{s} - \frac{s + s_{13} - m_2^2}{2\,\sqrt{s}}\right), \\ dPS_2(X \to p_1 p_3) &= \frac{1}{(4\,\pi)^2\, |\vec{p}_{13}|}\, dp_1^0\, d\phi_1 \end{aligned} \tag{5.83}$$

Thus we obtain

$$\boxed{dPS_3(Z \to p_1 p_2 p_3) \;=\; \frac{1}{(2\,\pi)^5}\,\frac{s}{32}\,d\cos\theta_{13}\,d\phi_{13}\,d\phi_1\,dx_1\,dx_2} \tag{5.84}$$

where θ_{13}, ϕ_{13} and ϕ_1 are Euler angles; θ_{13} and ϕ_{13} are the polar and azimuthal angles of $\vec{p}_{13}$, respectively, and ϕ_1 is the azimuthal angle of $\vec{p}_1$ with respect to the axis pointing along $\vec{p}_{13}$. If we integrate out the Euler angles then the phase space depends only on x_1 and x_2. We now parametrize the four-particle phase space according to

$$Z \;\longrightarrow\; X(p_1 p_2) + Y(p_3 p_4) \tag{5.85}$$

We decompose the four-particle phase space into three two-particle phase spaces as follows

$$dPS_4(Z \to p_1 p_2 p_3 p_4) = dPS_2(Z \to X\,Y)\;\cdot\;\frac{ds_{12}}{2\,\pi}\,\frac{ds_{34}}{2\,\pi}\;\cdot dPS_2(X \to p_1 p_2)\,dPS_2(Y \to p_3 p_4) \tag{5.86}$$

where $p_1 + p_2 + p_3 + p_4 \;=\; Q$. We work in the CM frame and choose the x axis arbitrarily, hence we have $\vec{p}_{12} \;=\; -\vec{p}_{34}$ and $|\vec{p}_{12}| \;=\; (1/2)\,\sqrt{\lambda(s, s_{12}, s_{34})/s}$. Using

$$\begin{aligned}
dPS_2(Z \to X\,Y) &= \frac{1}{32\,\pi^2}\,\frac{\sqrt{\lambda(s, s_{12}, s_{34})}}{s}\,d\Omega_{12}\,\theta\left(\sqrt{s} - \frac{s + s_{12} - s_{34}}{2\,\sqrt{s}}\right), \\
dPS_2(X \to p_1 p_2) &= \frac{1}{(4\,\pi)^2\,|\vec{p}_{12}|}\,dp_1^0\,d\phi_1, \\
dPS_2(Y \to p_3 p_4) &= \frac{1}{(4\,\pi)^2\,|\vec{p}_{34}|}\,dp_3^0\,d\phi_3
\end{aligned} \tag{5.87}$$

where $d\Omega_{12} \;=\; d\cos\theta_{12}\,d\phi_{12}$; θ_{12} and ϕ_{12} are the polar and azimuthal angles of $\vec{p}_{12}$, respectively, and $\phi_i\,(i \;=\; 1, 3)$ is the azimuthal angle of $\vec{p}_i$ with respect to the axis pointing along $\vec{p}_{ij}$, hence we obtain

$$\boxed{dPS_4(Z \to p_1 p_2 p_3 p_4) \;=\; \frac{8}{(4\,\pi)^8\sqrt{\lambda(s, s_{12}, s_{34})}}\,ds_{12}\,ds_{34}\,d\cos\theta_{12}\,d\phi_{12}\,dp_1^0\,d\phi_1\,dp_3^0\,d\phi_3} \tag{5.88}$$

The integration over ϕ_{12} is trivial due to the azimuthal symmetry. We can choose the coordinate system in such a way that $\vec{p}_{12}$ points along the z axis and $\vec{p}_1$ lies in the x-z plane, hence we have $\theta \;=\; \theta_{12}$ and $\phi \;=\; \pi - \phi_1$. Introducing $y_{ij} \;=\; s_{ij}/Q^2$ and $x_i \;=\; 2\,p_i \cdot Q/Q^2$, we thus have

$$dPS_4(Z \to p_1 p_2 p_3 p_4) \;=\; \frac{s^2}{(4\,\pi)^7\sqrt{\lambda(1, y_{12}, y_{34})}}\,dy_{12}\,dy_{34}\,dx_1\,dx_3\,d\phi_3\,d\cos\theta\,d\phi \tag{5.89}$$

Here the limit of integration variable ϕ is $-\pi < \phi < \pi$; the scalar products we need are

$$\begin{aligned}
p_1 \cdot p_3 &= \frac{s}{4}\, x_1\, x_3\, [1 - (\sin\theta_1\ \sin\theta_3\ \cos\phi_3 - \cos\theta_1\ \cos\theta_3)], \\
p_1 \cdot p_4 &= \frac{s}{2}\, (x_1 - y_{12}) - p_1 \cdot p_3, \\
p_2 \cdot p_3 &= \frac{s}{2}\, (x_3 - y_{34}) - p_1 \cdot p_3, \\
p_2 \cdot p_4 &= \frac{s}{2}\, (1 - x_1 - x_3) + p_1 \cdot p_3
\end{aligned} \tag{5.90}$$

which in terms of the dimensionless quantities y_{ij} can be rewritten as

$$\begin{aligned}
y_{13} &= \frac{1}{2}\, x_1\, x_3\, [\, 1 - (\sin\theta_1\ \sin\theta_3\ \cos\phi_3 - \cos\theta_1\ \cos\theta_3)\,], \\
y_{14} &= x_1 - y_{12} - y_{13}, \\
y_{23} &= x_3 - y_{34} - y_{13}, \\
y_{24} &= 1 - x_1 - x_3 + y_{13}
\end{aligned} \tag{5.91}$$

where θ_i is the angle enclosed between $\vec{p}_{ij}$ and $\vec{p}_i$. It is determined by

$$\cos\theta_i \;=\; \frac{x_i\,(1 + y_{ij} - y_{kl}) - 2\, y_{ij}}{x_i\, \sqrt{\lambda(1, y_{ij}, y_{kl})}} \tag{5.92}$$

with $(i,j), (k,l) = (1,2), (3,4)$ and $(i,j) \neq (k,l)$. Furthermore, we have

$$x_j \;=\; 1 - x_i + y_{ij} - y_{kl} \tag{5.93}$$

The limits of integration boundary are

$$\begin{cases} 0 \;<\; y_{12} \;<\; 1 & x_i^- \;<\; x_i \;<\; x_i^+ \\ 0 \;<\; y_{34} \;<\; (1 - \sqrt{y_{12}})^2 & 0 \;<\; \phi_3 \;<\; 2\,\pi \end{cases} \tag{5.94}$$

where

$$x_i^{\pm} \;=\; \frac{1}{2}\, \left[\, (1 + y_{ij} - y_{kl}) \;\pm\; \sqrt{\lambda(1, y_{ij}, y_{kl})}\, \right] \tag{5.95}$$

The four-particle phase space with massive particles is discussed in [97]. Finally we summarize this subsection by writing the three-particle and four-particle phase spaces in terms of kinematic invariants $s_{ij} = 2\, p_i \cdot p_j$ and the d-dimensional hypersphere $d\Omega_d$. Using Eq. (5.68), the three-particle phase space is [98]

$$\begin{aligned}
dPS_3 \;=& (2\,\pi)^{3-2\,d}\, 2^{-1-d}\, (Q^2)^{\frac{2-d}{2}}\, d\Omega_{d-1}\, d\Omega_{d-2}\ (s_{12}\, s_{13}\, s_{23})^{\frac{d-4}{2}} \\
& \cdot ds_{12}\, ds_{13}\, ds_{23}\, \delta\left(Q^2 - s_{12} - s_{13} - s_{23}\right)
\end{aligned} \tag{5.96}$$

and the four-particle phase space is

$$dPS_4 = (2\,\pi)^{4-3\,d}\,(Q^2)^{1-\frac{d}{2}}\,2^{1-2\,d}(-\Delta_4)^{\frac{d-5}{2}}\,\theta(-\Delta_4)\,\delta(Q^2 - s_{12} - s_{13} - s_{14} - s_{23} - s_{24} - s_{34}) \cdot d\Omega_{d-1}\,d\Omega_{d-2}\,d\Omega_{d-3}\,ds_{12}\,ds_{13}\,ds_{14}\,ds_{23}\,ds_{24}\,ds_{34} \tag{5.97}$$

where the Gram determinant Δ_4 is given by $+\,\Delta_4 \,=\, \lambda(s_{12}s_{34}, s_{13}s_{24}, s_{14}s_{23})$ and λ is the Källen function $\lambda(x,y,z) \,=\, x^2 + y^2 + z^2 - 2\,(x\,y + y\,z + z\,x)$. If we take the unit matrix element and integrate out the three-particle and four-particle phase spaces, respectively, we obtain

$$\int_{CS} dPS_3 = 8\,\pi \int dPS_3 \,=\, \frac{Q^2}{32\,\pi^2}\,, \qquad \int_{CS} dPS_4 = 8\,\pi \int dPS_4 \,=\, \frac{Q^4\,S}{3072\,\pi^4} \tag{5.98}$$

An alternative parametrization of four-particle phase space is [15]

$$\int dPS_4 \,=\, A_0\,\left(\frac{1}{8\,\pi^2}\right)^2\,\frac{1}{16}\,S\,Q^2 \int dy_{123} \int dy_{134} \int dy_{13}\,\theta(y_{13})\,\theta(y_{123}y_{134} - y_{13}) \cdot \theta(y_{13} + 1 - y_{123} - y_{134}) \cdot \int_0^1 dv \,\cdot\, \frac{1}{\pi} \int_0^\pi d\theta' \tag{5.99}$$

Here, $A_0 \,=\, Q^2/(2\,\pi)$, $y_{ijk} \,=\, s_{ijk}/Q^2$ and S is the symmetry factor. The variables v and θ' in Eq. (5.99) will be given in the next subsection. The lower limit of the y_{13} integration is specified by the θ functions

$$\theta(y_{13})\,\theta(y_{13} + 1 - y_{123} - y_{134}) \,\to\, \theta(y_{13})\,\theta(1 - y_{123} - y_{134}) \,+\, \theta(y_{123} + y_{134} - 1)\,\theta(y_{13} + 1 - y_{123} - y_{134}) \tag{5.100}$$

so the range of the y_{13} integration can be split so that

$$\int dy_{13} \,\to\, \theta(1 - y_{123} - y_{134}) \int_0^{y_{123}\,y_{134}} dy_{13} + \theta(y_{123} + y_{134} - 1) \int_{y_{123}+y_{134}-1}^{y_{123}\,y_{134}} dy_{13} \tag{5.101}$$

Parametrizations of the four-momenta: four-particle phase space

In order to calculate the momentum mappings between $(m+1)$- and m-particle phase space in the subtraction terms (dipoles), we need to find the explicit expressions of the four-momenta p_1, p_2, p_3 and p_4 in terms of the integration variables y_{12}, y_{34}, x_1, x_3 and ϕ_3. We can choose that $\vec{p}_{12}$ points along the z axis and $\vec{p}_1$ lies in the x-z plane. Here we work in the CM frame, *i.e.* $\vec{p}_{12} \,=\, |\vec{p}_{12}|\,\hat{z}$, $\vec{p}_{34} \,=\, -\,|\vec{p}_{12}|\,\hat{z}$ and hence $\vec{p}_{12} \,=\, -\vec{p}_{34}$ is fulfilled.

$$Q \,=\, \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_1 \,=\, \frac{\sqrt{s}}{2}\,x_1 \begin{pmatrix} 1 \\ \sin\theta_1 \\ 0 \\ \cos\theta_1 \end{pmatrix}, \quad p_3 \,=\, \frac{\sqrt{s}}{2}\,x_3 \begin{pmatrix} 1 \\ \sin\theta_3\cos\phi_3 \\ \sin\theta_3\sin\phi_3 \\ -\,\cos\theta_3 \end{pmatrix} \tag{5.102}$$

where θ_i is the angle enclosed between $\vec{p}_{ij}$ and $\vec{p}_i$, and ϕ_3 is the azimuthal angle of $\vec{p}_3$ with respect to the axis pointing along $\vec{p}_{34}$. Note that there is a minus sign in the z-component of $\vec{p}_3$ reflecting the fact that the z-component of $\vec{p}_3$ is in the direction of $-z$ axis, while the x-component and y-component of $\vec{p}_3$ point along the $+x$ axis and $+y$ axis, respectively.

$$p_{12} \;=\; \begin{pmatrix} p_{12}^0 \\ 0 \\ 0 \\ |\vec{p}_{12}| \end{pmatrix}, \quad p_{34} \;=\; \begin{pmatrix} p_{34}^0 \\ 0 \\ 0 \\ -|\vec{p}_{12}| \end{pmatrix} \tag{5.103}$$

We can write $|\vec{p}_{12}|$ in terms of y_{ij}, *i.e.*

$$\boxed{|\vec{p}_{12}| \;=\; \frac{\sqrt{s}}{2}\sqrt{\lambda(1,y_{12},y_{34})}} \tag{5.104}$$

Using Eq. (5.102), Eq. (5.103) and the identities

$$x_1 + x_2 \;=\; 1 - y_{34} + y_{12}\,, \qquad x_3 + x_4 \;=\; 1 - y_{12} + y_{34} \tag{5.105}$$

we obtain

$$p_2 \;=\; \frac{\sqrt{s}}{2}\begin{pmatrix} 1 - y_{34} + y_{12} - x_1 \\ -\,x_1\,\sin\theta_1 \\ 0 \\ \sqrt{\lambda(1,y_{12},y_{34})} - \,x_1\,\cos\theta_1 \end{pmatrix}, \quad p_4 \;=\; \frac{\sqrt{s}}{2}\begin{pmatrix} 1 - y_{12} + y_{34} - x_3 \\ -\,x_3\sin\theta_3\cos\phi_3 \\ -\,x_3\sin\theta_3\sin\phi_3 \\ -\sqrt{\lambda(1,y_{12},y_{34})} + \,x_3\cos\theta_3 \end{pmatrix} \tag{5.106}$$

Eq. (5.102) and Eq. (5.106) will be useful when we compute the dipoles. An alternative parametrizations of the four-momenta is given by [15][1]

$$p_1 \;=\sqrt{s_{13}}/2\,(1,\sin\theta\sin\theta',\sin\theta\cos\theta',\cos\theta)\,, \qquad p_2 \;=(s_{123}-s_{13})/(2\,\sqrt{s_{13}})\,(1,0,0,1)$$

$$p_3 \;=\sqrt{s_{13}}/2\,(1,-\sin\theta\sin\theta',-\sin\theta\cos\theta',-\cos\theta)\,, \quad p_4 \;=(s_{134}-s_{13})/(2\,\sqrt{s_{13}})\,(1,0,\sin\beta,\cos\beta) \tag{5.107}$$

Parametrizations of the four-momenta: three-particle phase space

We will do the same thing for the momentum mappings in three-particle phase space. We need to find the explicit expressions of the four-momenta p_1, p_2 and p_3 in terms of the integration

[1]Here, we set up the reference frame where $p_{13} \;=\; p_1 + p_3$ is at rest. We shall refer to this as the $1-3$ system. The parameters v and θ are related by $v \;=\; 1/2\,(1-\cos\theta)$ and the variable $\cos\beta$ is determined by using energy-momentum conservation

$$\frac{1}{2}\,(1-\cos\beta) \;=\; \frac{s_{13}\,(Q^2 - s_{123} - s_{134} + s_{13})}{(s_{123}-s_{13})\,(s_{134}-s_{13})}$$

variables y_{13} and y_{23}. We can choose that $\vec{p}_3$ points along the z axis and $\vec{p}_1$ lies in the x-z plane. Here we work in the CM frame, hence

$$p_1 \;=\; \frac{\sqrt{s}}{2}\,x_1 \begin{pmatrix} 1 \\ \sin\theta_{13} \\ 0 \\ \cos\theta_{13} \end{pmatrix}, \quad p_2 \;=\; \frac{\sqrt{s}}{2}\,x_2 \begin{pmatrix} 1 \\ \sin\theta_{23}\cos\phi \\ \sin\theta_{23}\sin\phi \\ \cos\theta_{23} \end{pmatrix}, \quad p_3 \;=\; \frac{\sqrt{s}}{2}\,x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.108}$$

where θ_{13} is the angle enclosed between $\vec{p}_1$ and $\vec{p}_3$, θ_{23} is the angle enclosed between $\vec{p}_2$ and $\vec{p}_3$ and ϕ is the azimuthal angle of $\vec{p}_2$ with respect to the z axis[2]. We find

$$\begin{aligned} y_{12} \;&=\; \frac{1}{2}\,x_1\,x_2\,\left(1-\sin\theta_{13}\,\sin\theta_{23}\,\cos\phi-\cos\theta_{13}\,\cos\theta_{23}\right) \\ y_{13} \;&=\; \frac{1}{2}\,x_1\,x_3\,(1-\cos\theta_{13})\;, \qquad y_{23} \;=\; \frac{1}{2}\,x_2\,x_3\,(1-\cos\theta_{23}) \end{aligned} \tag{5.109}$$

Using the identity: $x_3 \;=\; y_{13}+y_{23} \;=\; 1-y_{12}$, we then obtain

$$\begin{aligned} \cos\theta_{13} \;&=\; 1-\frac{2\,y_{13}}{(1-y_{23})\,(y_{13}+y_{23})}\;, \qquad \cos\theta_{23} \;=\; 1-\frac{2\,y_{23}}{(1-y_{13})\,(y_{13}+y_{23})} \\ \cos\phi \;&=\; \frac{1}{\sin\theta_{13}\,\sin\theta_{23}}\left[1-\cos\theta_{13}\,\cos\theta_{23}-\frac{2\,(1-y_{13}-y_{23})}{(1-y_{13})\,(1-y_{23})}\right] \end{aligned} \tag{5.110}$$

5.5 Soft photon radiation

In this section, we will discuss the emission of a soft photon in the final state[3]. We will first derive the amplitude for a soft photon emitted from a single outgoing fermion; then we sum over all external fermions, obtaining the amplitude for a single soft photon emitted from all external particles. Finally we generalize to the emission of an arbitrary number of photons in the soft limit. The emission of a photon from an incoming particle (initial state radiation) can be dealt with in the same way. In this section, we only discuss final state radiation.

Final state radiation

The LO amplitude can be written as

$$\mathcal{M}_{Born}(p) \;=\; \bar{u}(p)\,\mathcal{A}(p) \tag{5.111}$$

[2]Here we have used slightly different notations for the angles.

[3]For the gluon radiation in QCD we just simply replace photon field with gluon field and take the colour factors and QCD coupling constant g_s into account, *i.e.* we make the following substitution for vertex:

$$-\,i\,e\,Q_f\,\gamma_\mu \;\rightarrow\; -\,i\,g_s\,t^A_{ab}\,\gamma_\mu$$

and hence $e\,Q_f \;\rightarrow\; g_s\,t^A_{ab}$ where t^A_{ab} are generators of $SU(3)_C$ in the fundamental representation.

where $\bar{u}(p)$ is the spinor of the fermion and $\mathcal{A}(p)$ the remaining part (hard part) of the amplitude. The amplitude for the emission of a photon with momentum k in final state (see Fig. 5.1) reads

$$i\,\mathcal{M}(p,k) = \bar{u}(p)\,\left[-i\,e\,Q_f\,\gamma_\mu\right]\,\varepsilon^{*\mu}(k)\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\,\epsilon}\,\mathcal{A}(p+k) \tag{5.112}$$

In the soft-photon approximation, *i.e.* $k \to 0$, we have

$$i\,\mathcal{M}(p,k) \approx e\,Q_f\,\bar{u}(p)\,\frac{\gamma_\mu\,(\not{p}+m)}{2\,p\cdot k+i\,\epsilon}\,\mathcal{A}(p)\,\varepsilon^{*\mu}(k) \tag{5.113}$$

Using the identity

$$\boxed{\gamma_\mu\,(\not{p}+m)\;=\;2\,p_\mu-\not{p}\,\gamma_\mu+m\,\gamma_\mu} \tag{5.114}$$

and Dirac equation $\bar{u}(p)\,(\not{p}-m)\;=\;0$, Eq. (5.113) can be rewritten as[4]

$$i\,\mathcal{M}(p,k) \approx \frac{2\,e\,Q_f\,p_\mu\,\varepsilon^{*\mu}(k)}{2\,p\cdot k+i\,\epsilon}\,\bar{u}(p)\,\mathcal{A}(p)\;\approx\;\left[\frac{2\,e\,Q_f\,p\cdot\varepsilon^*(k)}{2\,p\cdot k+i\,\epsilon}\right]\,\mathcal{M}_{Born}(p) \tag{5.115}$$

For a photon emitted from an outgoing scalar particle, the result is the same, *i.e.* the result is independent of the spin of the charged particle. Spin-dependent terms are IR regular.

The emission of a photon from an incoming particle can be dealt with in the same way; for the initial state radiation the charged particle has momentum $p-k$ instead of $p+k$.

If we sum over all external particles, we obtain the amplitude for the emission of a single soft photon, *i.e.* $k \to 0$

$$i\,\mathcal{M}(p_j,k)\overset{k\to 0}{\approx}\sum_\ell\,\left[\frac{2\,e\,Q_\ell\,p_\ell\cdot\varepsilon^*(k)}{2\,\eta_\ell\,p_\ell\cdot k+i\,\epsilon}\right]\,\mathcal{M}_{Born}(p_j) \tag{5.116}$$

where Q_ℓ and p_ℓ are the charges and momenta of the ℓth external particle; $\eta_\ell\;=\;1$ for outgoing particles (final state radiation) and $\eta_\ell\;=\;-1$ for incoming particles (initial state radiation). Here p_j means the emitter that emits a soft photon.

The generalization to multi-photon emissions is straightforward. Let us now consider the emission of an arbitrary number of photons; the corresponding result can be derived by induction and as a consequence, soft photons are emitted independently. The amplitude for the emission of n soft photons with momenta $k_1,k_2,\cdots,k_n$ in the limit $k_i\;\to\;0$ is given by

$$i\,\mathcal{M}(p_j,k_i)\overset{k_i\to 0}{\approx}\prod_{m=1}^{n}\,\left[\sum_\ell\,\frac{2\,e\,Q_\ell\,p_\ell\cdot\varepsilon^*(k_m)}{2\,\eta_\ell\,p_\ell\cdot k_m+i\,\epsilon}\right]\,\mathcal{M}_{Born}(p_j) \tag{5.117}$$

[4]For QCD scattering the prefactor of $\mathcal{M}_{Born}(p)$ is actually the eikonal factor, which corresponds to a soft gluon emission, *i.e.* $k\;\to\;0$.

5.6 Collinear photon radiation

In this section, we will discuss the emission of a photon in the collinear limit[5]; we will derive the matrix elements of the emission of a collinear photon for both final state and initial state radiations.

5.6.1 Final state radiation

The Born matrix element is given by (see Fig. 5.2)

$$|\mathcal{M}_{Born}(p)|^2 \;=\; |\bar{u}(p)\,\mathcal{A}(p)|^2 \;=\; \overline{\mathcal{A}}(p)\,u(p)\,\bar{u}(p)\,\mathcal{A}(p) \;=\; \overline{\mathcal{A}}(p)\,(\not{p}+m)\,\mathcal{A}(p) \tag{5.118}$$

where the projection operator is

$$\sum_{\text{spins}} u(p)\,\bar{u}(p) \;=\; \not{p}+m \tag{5.119}$$

and $\bar{u}(p)$ is the spinor of the fermion; $\mathcal{A}(p)$ the remaining part (hard part) of the amplitude. Here the spin indices have been suppressed and sum over spins (polarizations) is also implicitly understood. Let us now consider the emission of a collinear photon with momentum k (see Fig. 5.3). The corresponding matrix element is given by

$$\begin{aligned}
|\mathcal{M}(p',k)|^2 &= \left|\bar{u}(p')\;[-i\,e\,Q_f\,\gamma_\mu]\;\varepsilon^{*\mu}(k)\,\frac{i(\not{p}+m)}{p^2-m^2+i\,\epsilon}\,\mathcal{A}(p)\right|^2 \\
&= e^2\,Q_f^2\,\left|\bar{u}(p')\,\gamma_\nu\,\varepsilon^{*\nu}(k)\,\frac{(\not{p}+m)}{p^2-m^2+i\,\epsilon}\,\mathcal{A}(p)\right|^2 \\
&= \frac{e^2\,Q_f^2}{(p^2-m^2)^2}\;|\bar{u}(p')\,\gamma_\mu\,\varepsilon^{*\mu}(k)\,(\not{p}+m)\,\mathcal{A}(p)|^2 \\
&= \frac{e^2\,Q_f^2}{(p^2-m^2)^2}\,\overline{\mathcal{A}}(p)\,(\not{p}+m)\,\gamma_\nu\,\varepsilon^{\nu}(k)\,u(p')\,\bar{u}(p')\,\gamma_\mu\,\varepsilon^{*\mu}(k)\,(\not{p}+m)\,\mathcal{A}(p) \\
&= \frac{e^2\,Q_f^2}{(p^2-m^2)^2}\,\overline{\mathcal{A}}(p)\,(\not{p}+m)\,\gamma_\nu\,u(p')\,\bar{u}(p')\,\gamma_\mu\,(\not{p}+m)\,\mathcal{A}(p)\,\varepsilon^{*\mu}(k)\,\varepsilon^{\nu}(k) \\
&= \frac{e^2\,Q_f^2}{(p^2-m^2)^2}\,\overline{\mathcal{A}}(p)\,(\not{p}+m)\,\gamma_\nu\,(\not{p}'+m)\,\gamma_\mu\,(\not{p}+m)\,\mathcal{A}(p)\,d^{\mu\nu}(k)
\end{aligned} \tag{5.120}$$

[5]Similarly for the gluon radiation we replace photon field with gluon field and make the following substitution for vertex:

$$-i\,e\,Q_f\,\gamma_\mu \;\rightarrow\; -i\,g_s\,t^A_{ab}\,\gamma_\mu$$

and hence

$$e^2\,Q_f^2 \;\rightarrow\; g_s^2\,t^A_{ad}\,t^A_{db}$$

.

where we have used Eq. (5.119) and

$$d^{\mu\nu}(k) \;=\; \sum_{\rm pol} \varepsilon^{*\mu}(k)\,\varepsilon^{\nu}(k) \;=\; -\,g^{\mu\nu} \;+\; \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n} \tag{5.121}$$

again the spin indices have been suppressed and sum over spins (polarizations) is also implicitly understood. The four vector n and $d^{\mu\nu}(k)$ satisfy the light-cone gauge conditions

$$\begin{aligned} &n^2 \;=\; 0\,, \quad -\,g^{\mu\nu}\,d_{\mu\nu}(k) \;=\; d-2 \;=\; 2-2\,\epsilon\,, \quad k^\mu\,d_{\mu\nu}(k) \;=\; k^\nu\,d_{\mu\nu}(k) \;=\; 0 \\ &n^\mu\,d_{\mu\nu}(k) \;=\; n^\nu\,d_{\mu\nu}(k) \;=\; 0 \end{aligned} \tag{5.122}$$

Kinematics

We need a bit kinematics for the massless photon and massive fermions

$$\begin{aligned} &p = p' + k \\ &k^2 = 0\,, \quad p'^2 \;= m^2\,; \quad p^2 \;\longrightarrow\; \mathcal{O}(m^2) \quad \text{in collinear limit} \\ &k^\mu = z\,p^\mu + k_\perp^\mu + k_r^\mu\,, \quad z \;:=\; k^0/p^0\,, \quad p\cdot k_\perp \;=\; 0\,, \quad \vec{k}_r \;=\; \vec{0} \\ &p'^\mu = p^\mu - k^\mu \;=\; (1-z)\,p^\mu - k_\perp^\mu - k_r^\mu \end{aligned} \tag{5.123}$$

From Eq. (5.123), we obtain immediately

$$\boxed{k_r^0 \;=\; -\,k_\perp^0\,, \quad \vec{k}_\perp^{\,2} \;=\; \mathcal{O}(m^2)\,, \quad k_\perp^0 \;=\; \mathcal{O}(m^2/p^0)} \tag{5.124}$$

The matrix element with photon emission now reads

$$|\mathcal{M}(p',k)|^2 = \frac{e^2\,Q_f^2}{(2\,p'\cdot k)^2}\,\overline{\mathcal{A}}(p)\,(\not{p}' + \not{k} + m)\,\gamma_\nu\,(\not{p}' + m)\,\gamma_\mu\,(\not{p}' + \not{k} + m)\,\mathcal{A}(p)\,d^{\mu\nu}(k) \tag{5.125}$$

Using the identity

$$\boxed{(\not{p}' + \not{k} + m)\,\gamma_\nu\,(\not{p}' + m) \;=\; \not{k}\,(-\not{p}' + m)\,\gamma_\nu + 2\,p'_\nu\,(\not{p}' + \not{k} + m)} \tag{5.126}$$

hence

$$|\mathcal{M}(p',k)|^2 = \frac{e^2\,Q_f^2}{(2\,p'\cdot k)^2}\,\overline{\mathcal{A}}(p)\;\left[\not{k}\,(-\not{p}' + m)\,\gamma_\nu\,\gamma_\mu + 2\,p'_\nu\,(\not{p}' + \not{k} + m)\,\gamma_\mu\right]\;(\not{p}' + \not{k} + m)\,\mathcal{A}(p)\,d^{\mu\nu}(k) \tag{5.127}$$

Using the identities

$$\boxed{\gamma_\nu\,\gamma_\mu\,d^{\mu\nu} \;=\; g_{\mu\nu}\,d^{\mu\nu} \;=\; -\,d+2} \tag{5.128}$$

and

$$\boxed{(\not{p}' + \not{k} + m)\,\gamma_\mu \;=\; 2\,p'_\mu + 2\,k_\mu + \gamma_\mu\,(-\not{p}' - \not{k} + m)} \tag{5.129}$$

Hence we have

$$\begin{aligned}
|\mathcal{M}(p',k)|^2 &= \frac{e^2\,Q_f^2}{(2\,p'\cdot k)^2}\,\overline{\mathcal{A}}(p)\,\left[-2\,p'\cdot k\,\not{k}\,g_{\mu\nu} + 4\,p'_\nu\,p'_\mu\,(\not{p}' + \not{k} + m) - 4\,p'_\nu\,\gamma_\mu\,p'\cdot k\right]\,\mathcal{A}(p)\,d^{\mu\nu}(k) \\
&= \frac{e^2\,Q_f^2}{(2\,p'\cdot k)}\,\overline{\mathcal{A}}(p)\,\left[2\,(1-\epsilon)\,\not{k} - \frac{2\,m^2}{p'\cdot k}\,(\not{p}' + \not{k} + m) + 4\,\frac{p'\cdot n}{k\cdot n}\,(\not{p}' + \not{k} + m)\right. \\
&\left. +2\,\not{p}' - 2\,\frac{\not{k}\,p'\cdot n + p'\cdot k\,\not{n}}{k\cdot n}\right]\,\mathcal{A}(p)
\end{aligned} \tag{5.130}$$

In high-energy limit, a charged fermion energy p^0 is much larger than its mass m and hence we can neglect the fermion mass. Now we consider the collinear limit: $k_r^\mu = k_\perp^\mu = 0$ and $p^2 \longrightarrow \mathcal{O}(m^2)$

$$\not{k} \approx z\,\not{p}\,, \quad \not{p}' \approx (1-z)\,\not{p}\,, \quad \frac{m^2}{2\,p'\cdot k} \;=\; \mathcal{O}(1)\,, \quad \frac{p'\cdot n}{k\cdot n} \;=\; \frac{1}{z} - 1 + \mathcal{O}(m/p^0) \tag{5.131}$$

The photon emission matrix element in the collinear limit now becomes

$$\begin{aligned}
|\mathcal{M}(p',k)|^2 \approx \frac{e^2\,Q_f^2}{2\,p'\cdot k}\,\overline{\mathcal{A}}(p)\,&\left[2\,(1-\epsilon)\,z\,\not{p} - \frac{2\,m^2}{p'\cdot k}\,\not{p} + 4\,\left(\frac{1}{z} - 1\right)\,\not{p}\right. \\
&\left. +2\,(1-z)\,\not{p} - 2\,z\,\not{p}\,\left(\frac{1}{z} - 1\right) - 2\,\frac{p'\cdot k}{k\cdot n}\,\not{n}\right]\,\mathcal{A}(p)
\end{aligned} \tag{5.132}$$

The last term in Eq. (5.132) vanishes since $p'\cdot k \approx \mathcal{O}(m^2)$. Finally we obtain

$$\begin{aligned}
|\mathcal{M}(p',k)|^2 &\approx \frac{e^2\,Q_f^2}{2\,p'\cdot k}\,\overline{\mathcal{A}}(p)\,\left[2\,(1-\epsilon)\,z\,\not{p} - \frac{2\,m^2}{p'\cdot k}\,\not{p} + 4\,\left(\frac{1-z}{z}\right)\,\not{p}\right]\,\mathcal{A}(p) \\
&= \frac{e^2\,Q_f^2}{p'\cdot k}\,\overline{\mathcal{A}}(p)\,\left[(1-\epsilon)\,z\; - \frac{m^2}{p'\cdot k}\; + 2\,\left(\frac{1-z}{z}\right)\right]\,\not{p}\,\mathcal{A}(p) \\
&= \frac{e^2\,Q_f^2}{p'\cdot k}\,\overline{\mathcal{A}}(p)\,\left[-\epsilon\,z\; - \frac{m^2}{p'\cdot k}\; + \frac{1+(1-z)^2}{z}\right]\,\not{p}\,\mathcal{A}(p) \\
&= \frac{e^2\,Q_f^2}{p'\cdot k}\,\left[P_{\gamma f}(z) - \frac{m^2}{p'\cdot k}\right]\,|\mathcal{M}_{Born}(p)|^2
\end{aligned} \tag{5.133}$$

where we have introduced the photon splitting function[6] in d dimension

$$P_{\gamma f}(z) \;:=\; \frac{1+(1-z)^2}{z} - \epsilon\,z \tag{5.134}$$

[6]For the gluon radiation in QCD, this is exactly the Altarelli-Parisi splitting function $P_{qq}(x)$ with appropriate substitution [6]

$$z \;\to\; 1 - x$$

.

5.6.2 Initial state radiation

The Born matrix element is given by (see Fig. 5.4)

$$|\mathcal{M}_{Born}(p)|^2 \;=\; \overline{\mathcal{A}}(p)\,(\not{p}+m)\,\mathcal{A}(p) \tag{5.135}$$

Kinematics

We need a bit kinematics for the photon emission in the initial state radiation (see Fig. 5.5)

$$\boxed{x \;:=\; \frac{k^0}{p^0}\,, \quad z \;=\; \frac{k^0}{p'^0}\,, \quad p \;=\; p' + k} \tag{5.136}$$

Substitutions from final state radiation, then we have

$$k \;\to\; -\,k\,, \quad p \leftrightarrow p' \tag{5.137}$$

and

$$z \;\to\; -\,z \;=\; \frac{x}{x-1}\,, \quad P_{\gamma f}(-z) \;=\; \frac{1+(1+z)^2}{-z} + \epsilon\, z \;=\; \frac{1}{x-1}\,P_{\gamma f}(x) \tag{5.138}$$

Hence the matrix element for an emission of a collinear photon with momentum k is given by

$$|\mathcal{M}(p,k)|^2 \approx \frac{e^2\,Q_f^2}{p\cdot k}\,\frac{1}{1-x}\,\left[P_{\gamma f}(x) - (1-x)\,\frac{m^2}{p\cdot k}\right]\,|\mathcal{M}_{Born}((1-x)\,p)|^2 \tag{5.139}$$

with

$$p' \;\approx\; (1-x)\,p\,, \quad k \;\approx\; -\,x\,p \tag{5.140}$$

in the collinear limit.

5.7 One-loop calculations: examples

The quark self-energy contributions

For simplicity we consider the Feynman gauge in which $\alpha = 1$; so the expression for the quark self-energy term (see Fig. 5.6) to order α_s in the dimensional regularization scheme is[7]

$$\begin{aligned} -i\,\Sigma_{ab}(p) &= \mu^\epsilon \int \frac{d^dk}{(2\,\pi)^d}\,\frac{-i\,g_{\mu\nu}}{k^2+i\epsilon}\,\left(-i\,g_s\,t^C_{ad}\,\gamma^\mu\right)\,\frac{i}{(\not{p}-\not{k})-m_q+i\epsilon}\,\left(-i\,g_s\,t^C_{db}\,\gamma^\mu\right) \\ &= -g_s^2\,\mu^\epsilon \int \frac{d^dk}{(2\,\pi)^d}\,\frac{g_{\mu\nu}}{k^2+i\epsilon}\,\gamma^\mu\,\frac{1}{(\not{p}-\not{k})-m_q+i\epsilon}\,\gamma^\nu\,t^C_{ad}\,t^C_{db} \\ &= -i\,\Sigma_{\text{QED}}\,t^C_{ad}\,t^C_{db} \end{aligned} \tag{5.141}$$

[7] Here we work in $d = 4-\epsilon$ dimensions.

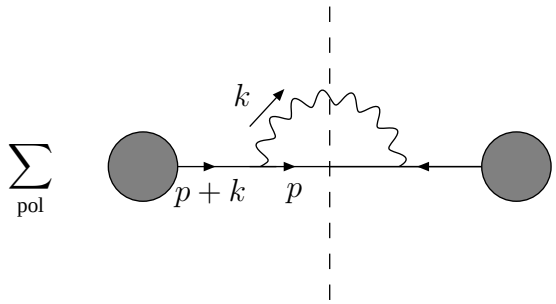


Figure 5.1: Final state radiation: soft photon emission.

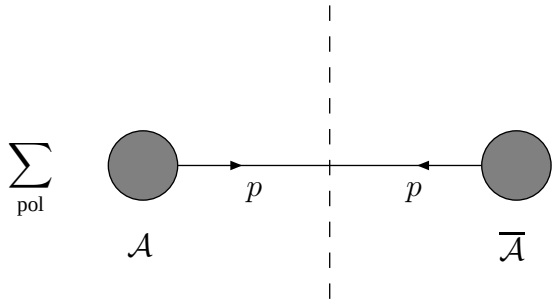


Figure 5.2: Final state radiation: Born diagram.

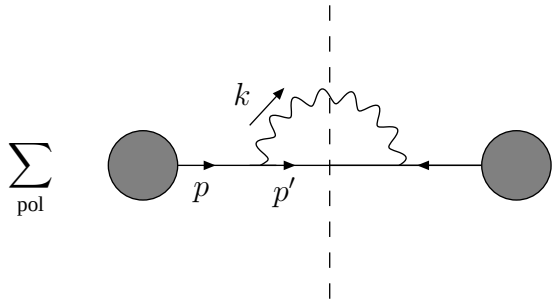


Figure 5.3: Final state radiation: collinear photon emission.

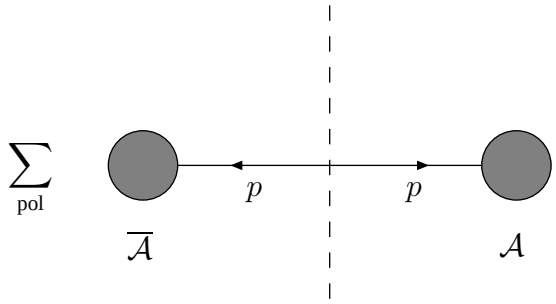


Figure 5.4: Initial state radiation: Born diagram.

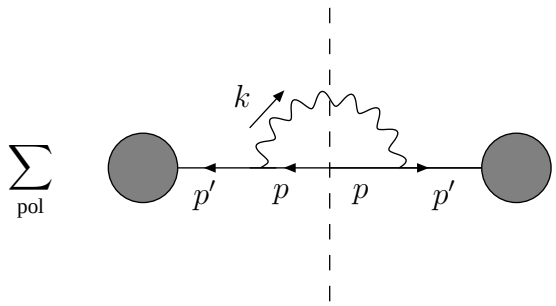


Figure 5.5: Initial state radiation: collinear photon emission.

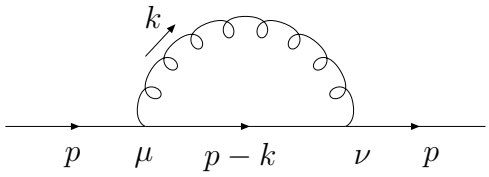


Figure 5.6: The one-loop contribution to the quark self-energy diagram in QCD.

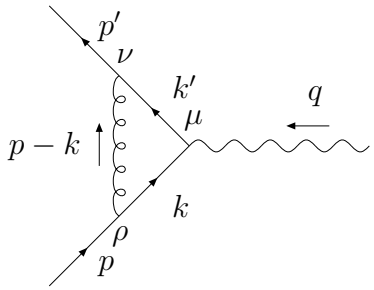


Figure 5.7: The one-loop contribution to the gluon vertex diagram in QCD.

where

$$\sum_A t^A_{ab}\, t^A_{bc} \;=\; C_F\, \delta_{ac}, \quad C_F \;=\; \frac{N^2-1}{2\,N} \tag{5.142}$$

and

$$-i\,\Sigma_{\rm QED} \;\;=\;\; -g_s^2\,\mu^\epsilon \int \frac{d^dk}{(2\,\pi)^d} \frac{\gamma_\mu\,[\not p - \not k + m_q]\,\gamma^\mu}{(k^2+i\epsilon)\,\left[(p-k)^2-m_q^2+i\epsilon\right]} \tag{5.143}$$

Here we consider the massless and on mass-shell quark (antiquark), so we have

$$-i\,\Sigma_{\rm QED} \;\;=\;\; -g_s^2\,\mu^\epsilon \int \frac{d^dk}{(2\,\pi)^d} \frac{\gamma_\mu\,[\not p - \not k]\,\gamma^\mu}{(k^2+i\epsilon)\,\left[(p-k)^2+i\epsilon\right]} \tag{5.144}$$

The infrared divergence (mass singularity) comes from the massless quark and antiquark. After some tedious calculations, we find

$$\begin{aligned} -i\,\Sigma_{\rm QED} \qquad &= \qquad -i\,g_s^2\,\mu^\epsilon\,(2-d)\,\not p \int_0^1 dz\,(1-z) \int \frac{d^dk'}{(2\,\pi)^d}\,\frac{1}{i}\,\frac{1}{(k')^4} \\ &\approx \qquad \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \;=\; 0 \\ &\uparrow \\ \epsilon_{UV} \;&=\; \epsilon_{IR} \end{aligned} \tag{5.145}$$

where $k' \;=\; k - p\,z$ and we have used the scaleless integral

$$\boxed{\int d^dk\,\frac{1}{[k^2]^n} \;\propto\; \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} = 0} \tag{5.146}$$

Here the ϵ_{UV} and ϵ_{IR} poles are used to regularize the ultraviolet and infrared divergences, respectively. So the final result is independent of any scale ($p^2 \;=\; 0$). Eq. (5.146) is very useful as long as we do not specifically care about the pole coefficients.

The vertex corrections

In this section we compute the order of α_s vertex corrections (see Fig. 5.7) to the quark and antiquark annihilation in QCD. Applying the Feynman rules, we find $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$, where

$$\delta\Gamma^\mu(p',p) = \mu^\epsilon \int \frac{d^dk}{(2\,\pi)^d} \left[\frac{-i\,g_{\nu\rho}}{(k-p)^2+i\epsilon}\right] \left[-i\,g_s\,\gamma^\nu\,t^A_{cj}\right] \frac{i}{\not k' - m_{\bar q} + i\epsilon} \times \left[\frac{1}{\sqrt{2}}\,\gamma^\mu\,\frac{1}{2}\,\left(1-\gamma^5\right)\,V_{ji}\right] \frac{i}{\not k - m_q + i\epsilon} \left[-i\,g_s\,\gamma^\rho\,t^A_{ib}\right] \tag{5.147}$$

Assume that $m_q \simeq m_{\bar q} = 0$, then $\delta\Gamma^\mu$ becomes

$$\delta\Gamma^\mu(p',p) \;=\; -i\,g_s^2\,\frac{\mu^\epsilon}{\sqrt{2}}\left(t^A_{cj}\,V_{ji}\,t^A_{ib}\right) \times I^\mu \tag{5.148}$$

where

$$I^\mu = \int \frac{d^dk}{(2\,\pi)^d}\,\frac{\gamma^\nu(\not k + \not q)\gamma^\mu\,\frac{1}{2}\,(1-\gamma^5)\,\not k\,\gamma_\nu}{(k-p)^2\,(k+q)^2\,k^2} = \int \frac{d^dk}{(2\,\pi)^d}\,\frac{[\gamma^\nu\,\not k\,\gamma^\mu\,\not k\,\gamma_\nu + \gamma^\nu\,\not q\,\gamma^\mu\,\not k\,\gamma_\nu]}{(k-p)^2\,(k+q)^2\,k^2}\,\frac{1}{2}\,\left(1-\gamma^5\right) \tag{5.149}$$

where $k' = k+q$. The gamma matrices in $d = 4-\epsilon$ dimensions satisfy

$$\gamma^\nu\,\not k\,\gamma^\mu\,\not k\,\gamma_\nu = (2-d)\,\not k\gamma^\mu\not k \quad \text{and} \quad \gamma^\nu\,\not q\,\gamma^\mu\,\not k\,\gamma_\nu = -2\,\not k\,\gamma^\mu\,\not q - (d-4)\,\not q\,\gamma^\mu\,\not k \tag{5.150}$$

So I^μ becomes

$$I^\mu \;=\; -2\int \frac{d^dk}{(2\,\pi)^d}\,\frac{\left[\frac{d-2}{2}\,\not k\,\gamma^\mu\,\not k + \not k\,\gamma^\mu\,\not q + \frac{d-4}{2}\,\not q\,\gamma^\mu\,\not k\right]}{(k-p)^2\,(k+q)^2\,k^2}\,\frac{1}{2}\,\left(1-\gamma^5\right) \tag{5.151}$$

Using the Feynman parametrization prescription, the denominator can be expressed as

$$\frac{1}{(k-p)^2\,(k+q)^2\,k^2} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{D^3} \tag{5.152}$$

where

$$D = \ell^2 - \triangle + i\epsilon\,, \quad \ell = k + y\,q - z\,p \quad \text{and} \quad \triangle \equiv -x\,y\,q^2 \tag{5.153}$$

or more generally

$$\frac{1}{A_1^{n_1}A_2^{n_2}\cdots A_k^{n_k}} = \frac{\Gamma(n_1+n_2+\cdots+n_k)}{\Gamma(n_1)\Gamma(n_2)\cdots\Gamma(n_k)}\int_0^1 d\alpha_1\cdots d\alpha_k \frac{\alpha_1^{n_1-1}\alpha_2^{n_2-1}\cdots\alpha_k^{n_k-1}\delta\left(1-\sum_i\alpha_i\right)}{(\alpha_1A_1+\cdots+\alpha_kA_k)^{n_1+n_2+\cdots+n_k}} \tag{5.154}$$

Now let us calculate the numerator of Eq. (5.151)

$$\bar v(p')\,\left[\frac{d-2}{2}\not k\gamma^\mu\not k + \not k\gamma^\mu\not q + \frac{d-4}{2}\not q\gamma^\mu\not k\right]\,\frac{1}{2}\,(1-\gamma^5)\,u(p) \tag{5.155}$$

after some tedious calculations, we obtain

$$\bar{v}(p')\,\left[-\frac{(2-d)^2}{2\,d}\ell^2+(1-x)\,\left(1-y\,\frac{d-2}{2}\right)\,q^2+\frac{d-4}{2}\,y\,q^2\right]\gamma^\mu\frac{(1-\gamma^5)}{2}\,u(p) \tag{5.156}$$

where we have used

$$\not{\ell}\,\gamma^\mu\,\not{\ell}\;=\;\frac{2-d}{d}\,\ell^2\,\gamma^\mu\quad\text{and}\quad\int\frac{d^d\ell}{(2\,\pi)^d}\,\frac{\ell^\mu}{D^3}\;=\;0 \tag{5.157}$$

Finally, we have to compute the following expression

$$\begin{aligned}\bar{v}(p')\,I^\mu\,u(p) &= -2\int\frac{d^d\ell}{(2\,\pi)^d}\int_0^1 dx\,dy\,dz\,\frac{2\,\delta(x+y+z-1)}{(\ell^2-\triangle+i\epsilon)^3}\times\left[\bar{v}(p')\,\gamma^\mu\,\frac{(1-\gamma^5)}{2}\,u(p)\right]\\ &\times\left[-\frac{(2-d)^2}{2\,d}\,\ell^2+(1-x)\,\left(1-y\,\frac{d-2}{2}\right)\,q^2+\frac{d-4}{2}\,y\,q^2\right]\\ &=\bar{v}(p')\,\gamma^\mu\,\frac{(1-\gamma^5)}{2}\,u(p)\times A(q^2)\end{aligned} \tag{5.158}$$

We can divide $A(q^2)$ into two terms $A\;=\;A^a(q^2)+A^b(q^2)$. In order to calculate $A(q^2)$, we could resort to the following integration formulae,

$$\begin{aligned}\int\frac{d^d\ell}{(2\,\pi)^d}\frac{\ell^2}{(\ell^2-\triangle+i\epsilon)^3} &= \frac{i}{4}\,\frac{d}{(4\,\pi)^{d/2}}\,\Gamma(2-d/2)\,\triangle^{d/2-2}\\ \int\frac{d^d\ell}{(2\,\pi)^d}\frac{1}{(\ell^2-\triangle+i\epsilon)^3} &= \frac{1}{2}\,\frac{-i}{(4\,\pi)^{d/2}}\,(2-d/2)\,\Gamma(2-d/2)\,\triangle^{d/2-3}\end{aligned} \tag{5.159}$$

hence we have

$$\begin{aligned}A^a(q^2) &= \frac{i}{(4\,\pi)^{d/2}}\,(-q^2)^{d/2-2}\,\frac{\Gamma(d/2-2)\,\Gamma(d/2)}{\Gamma(d-2)}\,\Gamma(2-d/2)\,(d/2-2)\\ A^b(q^2) &= -2\,\frac{i}{(4\,\pi)^{d/2}}\,(-q^2)^{d/2-2}\,(2-d/2)\,\Gamma(2-d/2)\,\frac{\Gamma(d/2-2)\,\Gamma(d/2)}{\Gamma(d-2)}\\ &\times\;\left[-\frac{2}{\epsilon}-\frac{d}{(d-2)^2}\,\frac{d-2}{2}+\frac{d-4}{d-2}\right]\end{aligned} \tag{5.160}$$

Finally we obtain the expression for $\bar{v}(p')\,\delta\Gamma^\mu\,u(p)$:

$$\begin{aligned}\bar{v}(p')\,\delta\Gamma^\mu\,u(p) &= -i\,g_s^2\,\frac{\mu^\epsilon}{\sqrt{2}}\,\left(t^A_{cj}\,V_{ji}\,t^A_{ib}\right)\times\bar{v}(p')\,\gamma^\mu\,\frac{(1-\gamma^5)}{2}\,u(p)\times\\ &(-q^2)^{d/2-2}\,\frac{i}{(4\,\pi)^{d/2}}\,\frac{1}{\Gamma(1-\epsilon)}\,\frac{\Gamma(2-\epsilon/2)}{1-\epsilon}\Gamma(\epsilon/2)\Gamma(-\epsilon/2)\times\\ &\left\{-\frac{\epsilon}{2}-2\,\frac{\epsilon}{2}\,\left[-\frac{2}{\epsilon}+\frac{\epsilon-4}{2\,(2-\epsilon)}-\frac{\epsilon}{2-\epsilon}\right]\right\}\\ &=\frac{\alpha_s}{4\,\pi}\,\left(\frac{4}{3}\right)\,\left(\frac{4\pi\mu^2}{q^2}\right)^{\frac{\epsilon}{2}}\times\bar{v}(p')\,\frac{1}{\sqrt{2}}\,\gamma^\mu\,\frac{(1-\gamma^5)}{2}\,u(p)\times\\ &\frac{\Gamma(1+\frac{\epsilon}{2})\,\Gamma^2(1-\frac{\epsilon}{2})}{\Gamma(1-\epsilon)}\,\left(-\frac{8}{\epsilon^2}-\frac{6}{\epsilon}-8+\pi^2+\mathcal{O}(\epsilon)\right)\end{aligned} \tag{5.161}$$

5.8 A review of the Standard Model (SM)

5.8.1 Abelian gauge theory: Quantum electrodynamics (QED)

We start with a Lagrangian

$$\mathcal{L}_0 = \overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x) \tag{5.162}$$

which is invariant under a global $U(1)$ symmetry:

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\,q\,\theta}\psi(x) \\ \partial_\mu\psi(x) &\rightarrow \partial_\mu\psi'(x) = e^{i\,q\,\theta}\partial_\mu\psi(x) \end{aligned} \tag{5.163}$$

with spacetime independent group parameter θ. There is a conserved current according to Noether's theorem:

$$J_\mu(x) = q\,\overline{\psi}(x)\,\gamma_\mu\,\psi(x) \qquad \Longrightarrow \qquad \partial^\mu J_\mu(x) = 0 \tag{5.164}$$

In the case of quantum electrodynamics, the phase invariance is promoted to the level of a local transformation in order to describe the gauge interactions between electrons and photons, *i.e.* the phase θ depends on the spacetime point. So we demand the global $U(1)$ symmetry to local symmetry, this means

$$\theta \rightarrow \theta(x) \tag{5.165}$$

so

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\,q\,\theta(x)}\psi(x) \\ \partial_\mu\psi(x) &\rightarrow \partial_\mu\psi'(x) = e^{i\,q\,\theta(x)}\partial_\mu\psi(x) + i\,q\,e^{i\,q\,\theta(x)}\psi(x)\partial_\mu\theta(x) \end{aligned} \tag{5.166}$$

To maintain the local gauge invariance, we introduce the covariant derivative.

$$D_\mu = \partial_\mu + i\,q\,A_\mu(x) \tag{5.167}$$

in such a way that

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\,q\,\theta(x)}\psi(x) = U(x)\psi(x) \\ D_\mu\psi(x) &\rightarrow D'_\mu\psi'(x) = e^{i\,q\,\theta(x)}D_\mu\psi(x) = U(x)D_\mu\psi(x) \end{aligned} \tag{5.168}$$

i.e. both $\psi(x)$ and $D_\mu\psi(x)$ transform the same way under $U(1)$ local symmetry. A^μ is the spin 1 gauge field (photon field) and transforms under the local gauge symmetry as

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu\theta(x) \tag{5.169}$$

Note that the transformation property ensures that

- $D_\mu\psi(x) \to D'_\mu\psi'(x) \;=\; U(x)D_\mu\psi(x)$
- gauge invariance of field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The commutator of covariant derivatives obeys

$$F_{\mu\nu} = \frac{1}{iq}\,[D_\mu, D_\nu] = \frac{1}{iq}\,[\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.170}$$

Then obviously the generalized Lagrangian

$$\begin{aligned}\mathcal{L}_\psi &= \overline{\psi}(x)\,(i\gamma^\mu D_\mu - m)\,\psi(x) \\ &= \overline{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) - q\,\overline{\psi}(x)\,\gamma_\mu\,\psi(x)\,A^\mu\end{aligned} \tag{5.171}$$

is invariant under the local gauge transformations. The complete QED Lagrangian has two contributions: matter and gauge field contributions:

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_{\text{gauge}} \tag{5.172}$$

with

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\,F_{\mu\nu}(x)\,F^{\mu\nu}(x) \tag{5.173}$$

The electron-photon coupling in Eq. (5.171) is called minimal coupling of the photon field A_μ to the electromagnetic current $J^\mu = q\,\overline{\psi}(x)\,\gamma^\mu\,\psi(x)$; and $\mathcal{L}_{\text{gauge}}$ cannot contain a term proportional to

$$A_\mu A^\mu$$

since this term is not gauge invariant under Eq. (5.169).

5.8.2 Non-abelian gauge theory

The starting point is a Lagrangian of free or self-interacting fields, *i.e.* symmetric under a global symmetry.

$$\mathcal{L}_\psi(\psi, \partial_\mu\psi) \;=\; \overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x) \tag{5.174}$$

where

$$\psi = \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_n\end{pmatrix} = \text{multiplet of a compact Lie group } G \tag{5.175}$$

The Lagrangian is symmetric under the transformation

$$\psi \to \psi' \;=\; U(\theta)\psi, \quad U(\theta) \;=\; \exp\{i\,t^A\,\theta^A\}, \quad U^\dagger U = UU^\dagger = 1 \tag{5.176}$$

If U is unitary, the t^A are hermitian matrices, called group generators. θ^A ($A = 1, \cdots, N^2 - 1 = \dim G$) are spacetime independent parameters. We can expand the unitary matrix around the unit element of the group G

$$U(\theta) = 1 + i\, t^A\, \theta^A + \mathcal{O}(\theta^2) \tag{5.177}$$

this means that the t^A generate infinitesimal transformation around the unit matrix element of the group. For $SU(N)$ matrix (unitary and $\det U = 1$), there are $N^2 - 1$ traceless, hermitian generators

$$t^A = \frac{\lambda^A}{2} \tag{5.178}$$

The generators for any representation of G satisfy the Lie Algebra relation

$$\left[t^A, t^B\right] \;=\; i\, f^{ABC}\, t^C \tag{5.179}$$

where the f^{ABC} are called the structure constants of the group G. The starting hypothesis is that Lagrangian is invariant under G

$$\mathcal{L}_\psi(\psi, \partial_\mu\psi) = \mathcal{L}_\psi(\psi', \partial_\mu\psi'), \qquad \psi' \,=\, U(\theta)\psi \tag{5.180}$$

Now we promote the global symmetry to local symmetry by gauging the theory, which means that we allow the parameters θ^A to be function of the spacetime coordinates.

$$\theta^A \to \theta^A(x) \quad \Longrightarrow \quad U \,\to\, U(x) \tag{5.181}$$

so now we have

$$U(x) = 1 + i\, t^A\, \theta^A(x) + \mathcal{O}(\theta^2) \tag{5.182}$$

We obtain a local invariant Lagrangian if we make the substitution

$$\mathcal{L}_\psi(\psi, \partial_\mu\psi) \to \mathcal{L}_\psi(\psi, D_\mu\psi) \tag{5.183}$$

with

$$D_\mu = \partial_\mu + i\, g\, A^A_\mu(x)\, t^A \equiv \partial_\mu + i\, g\, A_\mu(x), \qquad A_\mu \equiv \sum_A A^A_\mu\, t^A \,=\, A_\mu \cdot t \tag{5.184}$$

where g is the gauge coupling, t^A is the generator of the group G in ψ representation and A^A_μ are gauge fields. Here A_μ is an $N \times N$ matrix. $\mathcal{L}_\psi(\psi, D_\mu\psi)$ is local gauge invariant if

$$\begin{aligned} \psi(x) \;&\to\; \psi'(x) \,=\, U(x)\psi(x) \\ D_\mu\psi(x) \;&\to\; D'_\mu\psi'(x) \,=\, U(x)D_\mu\psi(x) \,=\, U(x)D_\mu U^{-1}(x)U(x)\psi(x) \end{aligned} \tag{5.185}$$

i.e. the covariant derivative transforms as

$$D_\mu \to U(x) D_\mu U^{-1}(x) \tag{5.186}$$

implying that

$$A^A_\mu \to A'^A_\mu \;=\; A^A_\mu - f^{ABC}\,\theta^B\,A^C_\mu - \frac{1}{g}\partial_\mu\theta^A + \mathcal{O}(\theta^2) \tag{5.187}$$

We can build the kinetic term for the A^A_μ fields from the field strength:

$$[D_\mu, D_\nu] \;=\; i\,g\,t\cdot F_{\mu\nu} \;=\; i\,g\,t^A F^A_{\mu\nu} \;=\; i\,g\,F_{\mu\nu}, \quad F_{\mu\nu} \equiv t^A F^A_{\mu\nu} \tag{5.188}$$

Here $F_{\mu\nu}$ is an $N \times N$ matrix and

$$F^A_{\mu\nu} \;=\; \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g f^{ABC} A^B_\mu A^C_\nu \tag{5.189}$$

which transforms homogeneously under a local gauge transformation

$$F_{\mu\nu} \to F'_{\mu\nu} \;=\; U F_{\mu\nu} U^{-1} \tag{5.190}$$

We also note that $\mathrm{Tr}\, F_{\mu\nu}\, F^{\mu\nu}$ is invariant:

$$F^A_{\mu\nu} F^{\mu\nu}_A \sim \mathrm{Tr}\, F_{\mu\nu}\, F^{\mu\nu} \to \mathrm{Tr}\, U F_{\mu\nu} U^{-1} U F^{\mu\nu} U^{-1} = \mathrm{Tr}\, F_{\mu\nu}\, F^{\mu\nu} \tag{5.191}$$

This is true only for finite dimensional representation of the gauge group. Now we can construct the gauge invariant Lagrangian for gauge and matter fields,

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Tr}\, F_{\mu\nu}(x) F^{\mu\nu}(x) + \overline{\psi}(x)\,(i\not{D} - m)\,\psi(x) \tag{5.192}$$

Normalizing the generators t^A as

$$\mathrm{Tr}\, t^A\, t^B \;=\; \frac{1}{2}\,\delta^{AB} \tag{5.193}$$

we have $-\frac{1}{2}\,\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \;=\; -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A$. As in the abelian case, the fermion/gauge field coupling is of the form

$$\mathcal{L}_{\text{int}} \;\sim\; g\, J^\mu_A\, A^A_\mu \tag{5.194}$$

where $J^A_\mu \;=\; \overline{\psi}\,\gamma_\mu\, t^A\,\psi$ is the fermionic part of the Noether current. There are some remarks about the non-abelian gauge theory:

- $A^A_\mu A^\mu_A$ are not gauge invariant, this means that gauge bosons of unbroken non-abelian gauge theories are massless.
- We have cubic and quartic gauge boson self-interactions due to $F^A_{\mu\nu} F^{\mu\nu}_A$

$$(\partial A) A^2, \quad A^4$$

- Gauge invariance + renormalizability (absence of higher powers of fields and covariant derivatives in Lagrangian) determine gauge boson/matter couplings and gauge boson self-interactions.

- If $G = SU(3)_C$ and the fermions are in triplets,

$$\psi = \begin{pmatrix} \psi_{\text{red}} \\ \psi_{\text{blue}} \\ \psi_{\text{green}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{5.195}$$

  we have the QCD Lagrangian with $3^2 - 1 = 8$ gluons.

5.8.3 Electroweak theory

Gauge boson interactions

The standard electroweak theory [76–78] is based on the more complicated gauge group $SU(2) \times U(1)$. Here we have suppressed the indices L and Y for $SU(2)$ and $U(1)$, which mean left-handed structure and weak hypercharge, respectively. Essentially, an $SU(2)$ gauge symmetry is applied to left-handed fermion fields only and an independent $U(1)$ gauge symmetry is present in order to incorporate the electric charge Q and unify the weak and electromagnetic interactions in a common gauge structure. Initially the Lagrangian of this model contains three massless gauge bosons A^a_μ $(a = 1, 2, 3)$ of $SU(2)$ gauge group and one massless gauge boson, B_μ, associated with the $U(1)$ gauge group. The gauge symmetry does not allow any mass term for W and Z bosons. More precisely, local gauge invariance and renormalizability completely determine the kinetic terms for the gauge bosons. The Lagrangian of the gauge bosons is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{5.196}$$

The field strength tensors of the $SU(2)$ gauge fields A^a and the $U(1)$ gauge field B are

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned} \tag{5.197}$$

g being the $SU(2)$ gauge coupling. Note that the vector bosons A^a already have self-interactions because of the non-abelian property of their symmetry group $SU(2)$. This is similar to the fact that gluons carry colour charge in QCD. The coupling of the gauge fields to matter fields is achieved via the covariant derivative

$$D_\mu \psi = \left(\partial_\mu - i\, g\, T^a\, A^a_\mu - i\, g'\, Y_\psi\, B_\mu \right) \psi \tag{5.198}$$

and the Lagrangian of the interaction is given by

$$\mathcal{L}_{\text{int}} = \overline{\psi}\, i\gamma^{\mu} D_{\mu}\, \psi \tag{5.199}$$

g' is the $U(1)$ gauge coupling. The matrices T^a are a representation of the $SU(2)$ weak isospin algebra and Y_ψ is the weak hypercharge of the $U(1)$. In order to specify the coupling to matter we have to choose the $SU(2)$ representation T^a and the $U(1)$ gauge charge Y_ψ for the matter fields. Here the value of the generator (charge) Y_ψ depends on the fermion field. Three group generators T^a correspond to three gauge bosons $A^1_\mu, A^2_\mu, A^3_\mu$; the group generators for gauge doublets are

$$T^a = \frac{\tau^a}{2}, \quad a = 1, 2, 3 \tag{5.200}$$

and for gauge singlets (e^i_R, ν^i_R)

$$T^a = 0 \tag{5.201}$$

They all satisfy the $SU(2)$ commutation relations:

$$\left[T^a, T^b\right] \;=\; i\,\epsilon^{abc}\, T^c, \qquad \epsilon^{123} = 1 \tag{5.202}$$

and the explicit expression of Pauli matrices are

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.203}$$

Defining

$$W^{\pm}_{\mu} = \frac{A^1_\mu \mp i\, A^2_\mu}{\sqrt{2}} \quad \text{and} \quad T^{\pm} = T^1 \pm i\, T^2 \tag{5.204}$$

we have

$$T^a A^a_\mu \;=\; \frac{1}{\sqrt{2}} T^{+} W^{+}_{\mu} + \frac{1}{\sqrt{2}} T^{-} W^{-}_{\mu} + T^3 A^3_\mu \tag{5.205}$$

where the matrices $T^{\pm}$ and T^3 satisfy the relations

$$\begin{aligned} \left[T^{+}, T^{-}\right] &= 2\, T^3 \\ \left[T^3, T^{\pm}\right] &= \pm T^{\pm} \end{aligned} \tag{5.206}$$

T^+ and T^- are raising and lowering operators. In the doublet representation of $SU(2)$ we have

$$T^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{5.207}$$

It is worth mentioning that A^3_μ and B_μ carry identical quantum numbers ($T^3 = 0, Y_\psi = 0$), so at the end they will combine to produce two neutral gauge bosons: A_μ and Z_μ. The neutral weak current was discovered at CERN in 1973, while the Z boson was discovered at CERN in 1983.

The Higgs mechanism: Spontaneous symmetry breaking

Experimentally, the weak gauge bosons are massive, there is only one massless gauge boson in nature, the photon. The Lagrangian Eq. (5.196) describes four massless vector bosons forming a singlet (B_μ) and a triplet ($W^\pm_\mu, A^3_\mu$). So the model cannot describe the real world. If we add the explicit mass terms for the three weak bosons, then it would violate local gauge invariance and spoil the renormalizability of the theory. Therefore it is necessary to introduce a mechanism of symmetry breaking by which the three weak bosons obtain masses. The mass generation can be implemented through the Higgs mechanism [41–43]: generate mass terms from the kinetic energy term of a scalar doublet field ϕ that undergoes spontaneous symmetry breaking (SSB).

In the standard electroweak theory, the gauge group $SU(2) \times U(1)$ is broken by the Higgs mechanism. Introducing a single complex doublet of scalar Higgs fields

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{5.208}$$

on which the matrices τ^a act. The Higgs Lagrangian is given by

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger \, D_\mu \phi - V(\phi^\dagger \phi) \tag{5.209}$$

which is gauge invariant under local $SU(2) \times U(1)$ transformations. The coupling of the gauge fields to the scalar field is achieved using the covariant derivative

$$\begin{aligned} D_\mu \phi &= \left(\partial_\mu - i\, g\, T^a\, A^a_\mu - i\, g'\, Y_\phi\, B_\mu\right) \phi, \qquad Y_\phi = \frac{1}{2} \\ &= \left(\partial_\mu - i\, g\, \frac{\tau^a}{2}\, A^a_\mu - i\, g'\, \frac{1}{2}\, B_\mu\right) \phi \end{aligned} \tag{5.210}$$

The Higgs potential is chosen to be of the form

$$V(\phi^\dagger \phi) = V_0 + \lambda \left(\phi^\dagger \phi\right)^2 - \mu^2 \phi^\dagger \phi, \qquad \mu^2, \lambda > 0 \tag{5.211}$$

Note that this potential has a wrong sign for the mass term. As a result, with the parameters $\mu^2, \lambda > 0$, this potential has a classical minimum which is not at $\phi = 0$; instead the potential has minima at

$$|\phi| \, = \, \sqrt{\frac{\mu^2}{2\,\lambda}} \, = \, \frac{v}{\sqrt{2}} \tag{5.212}$$

All these minimum configurations (ground states) are connected by gauge transformations, that change the phase of the complex field ϕ without altering its modulus. v is called the vacuum expectation value (VEV) of the neutral component of the scalar Higgs doublet. When the system chooses one of the ground states, this ground state is no longer symmetric under the gauge transformation. However the Lagrangian is still gauge invariant under the gauge transformation and all properties connected with it still hold (*e.g.* current conservation). This phenomenon is called spontaneous symmetry breaking. We will discuss the consequences of the Higgs mechanism in more detail in next subsection.

Glashow-Weinberg-Salam theory

We start with the gauge and scalar sector of the theory. The $SU(2) \times U(1)$ gauge invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)^\dagger \, D_\mu \phi - V(\phi^\dagger \phi) \tag{5.213}$$

where the field strength tensors are given by Eq. (5.197); the covariant derivative connecting gauge fields and scalar field is given by Eq. (5.210). We choose the ground state to be

$$\begin{aligned} A^a_\mu &= B_\mu = 0 \\ < \phi > &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} = \phi_0 \end{aligned} \tag{5.214}$$

Note that only a scalar field can have a VEV. The VEV of a fermion or vector field would break Lorentz invariance. The generators of the gauge group $SU(2) \times U(1)$ are matrices $T^a = \frac{\tau^a}{2}$ and $Y_\phi = \frac{1}{2}$. Now we would like to show that the ground state breaks the gauge symmetry. An arbitrary state Φ is invariant under a symmetry operation $\exp(i\, T^a\, \theta^a)$ if

$$e^{i\, T^a\, \theta^a} \Phi \; = \; \Phi \tag{5.215}$$

This means that a state is invariant if

$$\boxed{T^a \, \Phi \; = \; 0} \tag{5.216}$$

For the $SU(2) \times U(1)$ case we have

$$\begin{aligned} \tau^1 \, \phi_0 &\neq 0 \\ \tau^2 \, \phi_0 &\neq 0 \\ \tau^3 \, \phi_0 &\neq 0 \\ Y_\phi \, \phi_0 &= \frac{1}{2} \, \phi_0 \neq 0 \end{aligned} \tag{5.217}$$

Here the generators τ^a and Y_ϕ correspond to broken generators , the consequence of which is that all the gauge bosons will receive positive masses. However, it may be the case that the generators of the group leave the vacuum (ground state) invariant, in which case the corresponding gauge bosons will remain massless; and the corresponding generators are called unbroken generators. Now we examine the effect of the electric charge operator Q on the vacuum state. The generator Q satisfies

$$Q \, \phi_0 \; = \; (T^3 + Y_\phi) \, \phi_0 \; = \; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \, \phi_0 \; = \; 0 \tag{5.218}$$

the electric charge symmetry is unbroken. So there is an unbroken subgroup with a single generator Q: This is the subgroup $U(1)_Q$ of $SU(2) \times U(1)$. This subgroup corresponds to a massless

gauge field, which is the electromagnetic field (photon A_μ). The choice of the vacuum expectation value in Eq. (5.214) breaks the $SU(2) \times U(1)$ gauge symmetry, since it identifies a specific direction in the internal group space. Now we shall consider small perturbations of the fields around the vacuum,

$$\phi(x) = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}\left[v + \chi(x) + i\eta(x)\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \exp\left(\frac{iT^a\theta^a(x)}{v}\right) \begin{pmatrix} 0 \\ v + \chi(x) \end{pmatrix} \tag{5.219}$$

We still have four real degrees of freedom, (three θ^a and one χ), equivalent to the two complex fields. We can use the unitary gauge in which

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \chi(x) \end{pmatrix} \tag{5.220}$$

In this gauge, the Goldstone fields $\theta^a(x)$ have been rotated away by an $SU(2)$ gauge transformation.

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = U(x)\phi(x), \qquad U(x) = \exp\left(-\frac{iT^a\theta^a(x)}{v}\right) \\ T \cdot A_\mu &\rightarrow T \cdot A'_\mu = U\, T \cdot A_\mu\, U^{-1} - \frac{i}{g}\left(\partial_\mu U\right) U^{-1} \end{aligned} \tag{5.221}$$

The $\theta^a(x)$ degrees of freedom no longer appear in the Higgs Lagrangian, they will reappear as the longitudinal modes of the massive gauge bosons. The Higgs boson χ is the only remaining dynamical field. In order to find the mass terms for the gauge bosons and Higgs boson, we need to calculate the quadratic Lagrangian, which means the calculation of the covariant derivative of the field ϕ

$$\begin{aligned} D_\mu\phi &= \begin{pmatrix} -\frac{ig}{2\sqrt{2}}(A^1_\mu - iA^2_\mu)(v+\chi) \\ -\frac{i}{2\sqrt{2}}(g'B_\mu - gA^3_\mu)(v+\chi) + \frac{1}{\sqrt{2}}\partial_\mu\chi \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu\chi \end{pmatrix} - \frac{i}{2}\left(1 + \frac{\chi}{v}\right) \begin{pmatrix} g\,v\,W^+_\mu \\ -\sqrt{(g^2+g'^2)/2}\,v\,Z_\mu \end{pmatrix} \end{aligned} \tag{5.222}$$

The physical weak bosons are linear combinations of the gauge ones, so we have defined

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right) \\ Z_\mu &= \frac{1}{\sqrt{g^2+g'^2}}(gA^3_\mu - g'B_\mu) = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu \\ A_\mu &= \frac{1}{\sqrt{g^2+g'^2}}(gB_\mu + g'A^3_\mu) = \sin\theta_W A^3_\mu + \cos\theta_W B_\mu \end{aligned} \tag{5.223}$$

where

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \tag{5.224}$$

and it is easy to show that

$$Z_\mu^2 + (A_\mu)^2 = (A_\mu^3)^2 + B_\mu^2 \tag{5.225}$$

Hence, the covariant derivative to the quadratic part of the Lagrangian is

$$[(D_\mu\phi)^\dagger D^\mu\phi]^{(2)} = \frac{1}{2}(\partial_\mu\chi)^2 + \left[\frac{g^2v^2}{4}W_\mu^+W^{-\mu} + \frac{1}{2}\left(\frac{(g^2+g'^2)v^2}{4}\right)Z_\mu^2\right]\left(1+\frac{\chi}{v}\right)^2 \tag{5.226}$$

To quadratic order, the kinetic term of the vector fields in the quadratic Lagrangian is

$$\begin{aligned} -\frac{1}{4}\hat{F}^a_{\mu\nu}\hat{F}_a^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} &= -\frac{1}{2}W^+_{\mu\nu}W^{\mu\nu-} - \frac{1}{4}\hat{F}^3_{\mu\nu}\hat{F}_3^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &= -\frac{1}{2}W^+_{\mu\nu}W^{\mu\nu-} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{5.227}$$

where

$$\begin{aligned} \hat{F}^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \\ W^\pm_{\mu\nu} &= \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{5.228}$$

Thus, the quadratic Lagrangian contains standard kinetic terms for the complex vector field $W^\pm_\mu$ and the real vector fields A_μ and Z_μ. Collecting all together, we obtain the quadratic Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)} &= -\frac{1}{2}W^+_{\mu\nu}W^{\mu\nu-} + M_W^2W^+_\mu W^{-\mu} \\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad -\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{M_Z^2}{2}Z_\mu Z^\mu \\ &\quad +\frac{1}{2}(\partial_\mu\chi)^2 - \frac{M_\chi^2}{2}\chi^2 \end{aligned} \tag{5.229}$$

We therefore find that the W and Z gauge bosons have acquired masses, given by

$$M_W = \frac{g\,v}{2}, \quad M_Z = \frac{\sqrt{g^2+g'^2}\,v}{2} = \frac{M_W}{\cos\theta_W} \tag{5.230}$$

The photon remains massless because there are no terms quadratic in the field A_μ; the quadratic Lagrangian also describes a massive real scalar field χ (Higgs boson field) with mass given by

$$M_\chi = \sqrt{2\,\lambda}\,v \tag{5.231}$$

From the measured value of the Fermi constant

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,M_W^2}} \tag{5.232}$$

the vacuum expectation value of the Higgs field, v, is 246 GeV.

In summary, the Higgs field ϕ acquires a non-zero vacuum expectation value at a particular point on the circle of minima away from the point where $\phi = 0$ and the symmetry is spontaneously broken. Three Goldstone bosons of the four scalar field components get eaten by the gauge bosons to form massive vector bosons ($W^{\pm}_{\mu}, Z_{\mu}$), and a single physical scalar particle remains: The Higgs boson χ with mass given by $M_{\chi} = \sqrt{2\,\lambda}\,v$.

There are $WW\chi$ and $ZZ\chi$ couplings from $2\chi/v$ term in Eq. (5.226)

$$\mathcal{L}_{\chi VV} = \frac{2\,M_W^2}{v} W_{\mu}^{+} W^{-\mu}\,\chi + \frac{M_Z^2}{v} Z^{\mu} Z_{\mu}\,\chi = g\,M_W\,W_{\mu}^{+} W^{-\mu}\,\chi + \frac{1}{2}\frac{g\,M_Z}{\cos\theta_W} Z^{\mu} Z_{\mu}\,\chi \quad (5.233)$$

There are also $\chi\chi WW$ and $\chi\chi ZZ$ couplings from χ^2/v^2 term. An important fact is that Higgs coupling is proportional to mass. Finally we summarize some key points of Glashow-Weinberg-Salam theory:

- To break the symmetry spontaneously, we introduce a scalar Higgs field ϕ in the fundamental representation of $SU(2)$ with non-zero VEV:

$$< \phi >= \phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

- Broken symmetry will be applied if the original symmetry were global rather than gauge.
- Let G be a gauge group, the generators of G can be divided into unbroken generators $\{t_h\}$ and broken generators $\{t_\alpha\}$.
- If G is a global symmetry group, then the theory would contain massless Goldstone fields, with number equal to broken generators.
- If G is a gauge group, then the gauge fields corresponding to unbroken generators remain massless, while the gauge fields corresponding to broken generators become massive.
- Goldstone bosons corresponding to broken generators $\{t_\alpha\}$ disappear from the spectrum; they get eaten by the gauge particles to form massive vector bosons.

Fermion interactions

From experimental facts, charged currents couple only to left-handed fermions and neutral currents couple to a massless photon A_μ and a neutral gauge boson Z_μ, the gauge group is

$SU(2) \times U(1)$. For the case of leptons or quarks the left-handed fields are put into doublets; the right-handed fields are all $SU(2)$ singlets

$$\psi_L = \frac{1}{2}(1-\gamma^5)\,\psi, \quad \psi_R = \frac{1}{2}(1+\gamma^5)\,\psi, \quad \psi = \psi_L + \psi_R \tag{5.234}$$

It is easy to show

$$\overline{\psi}\, i\,\partial\!\!\!/\,\psi \;=\; \overline{\psi}_L\, i\,\partial\!\!\!/\,\psi_L + \overline{\psi}_R\, i\,\partial\!\!\!/\,\psi_R \tag{5.235}$$

Hence the first generation of left-handed lepton fields is

$$\ell_L = \frac{1}{2}(1-\gamma^5)\begin{pmatrix}\nu_e\\ e\end{pmatrix} = \begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix} \tag{5.236}$$

which is $SU(2)$ doublet, and the first generation of right-handed lepton fields are

$$\nu_{eR} = \frac{1}{2}(1+\gamma^5)\,\nu_e, \quad e_R = \frac{1}{2}(1+\gamma^5)\,e \tag{5.237}$$

which form $SU(2)$ singlets. Similarly the quark fields form left-handed doublets

$$q_L \;=\; \frac{1}{2}(1-\gamma^5)\begin{pmatrix}u\\ d'\end{pmatrix} = \begin{pmatrix}u_L\\ d'_L\end{pmatrix}, \quad \frac{1}{2}(1-\gamma^5)\begin{pmatrix}c\\ s'\end{pmatrix}, \quad \frac{1}{2}(1-\gamma^5)\begin{pmatrix}t\\ b'\end{pmatrix} \tag{5.238}$$

and the right-handed quark fields are singlets. The primes on the down type quarks will be explained later. All these relations hold for each family. Before symmetry breaking, the coupling of the fermions to the vector bosons is given by

$$\begin{aligned}\mathcal{L}_\psi &= \overline{\psi}\, i\,D\!\!\!/\,\psi \;=\; \overline{\psi}_L\, i\,D\!\!\!/\,\psi_L + \overline{\psi}_R\, i\,D\!\!\!/\,\psi_R \\ &= \overline{\psi}_L\, i\gamma^\mu\left(\partial_\mu - i\,g\,T^a\,A^a_\mu - i\,g'\,Y_L\,B_\mu\right)\psi_L + \overline{\psi}_R\, i\gamma^\mu\left(\partial_\mu - i\,g'\,Y_R\,B_\mu\right)\psi_R\end{aligned} \tag{5.239}$$

with

$$Y_\psi\,\psi_L \;=\; Y_L\,\psi_L, \quad Y_\psi\,\psi_R \;=\; Y_R\,\psi_R \tag{5.240}$$

Here ψ denote left- and right-handed quarks and leptons. The $U(1)$ charges, Y_L and Y_R, are chosen to satisfy the relation $Q = T^3 + Y_\psi$, so that after symmetry breaking we obtain the correct values of the electric charges Q. Table 5.2 lists all the values of weak isospin and hypercharge for quarks and leptons. Now we focus on one generation of leptons (quarks work the same way)

$$\mathcal{L}_\psi = i\bar{\ell}_L D\!\!\!/\,\ell_L + i\bar{e}_R D\!\!\!/\,e_R \tag{5.241}$$

where the covariant derivative is given by Eq. (5.198) and a right-handed neutrino would have zero coupling both to $SU(2)$ and to $U(1)$, so we have simply omitted this field. We will find that it is also useful to rewrite Eq. (5.198) in terms of the gauge boson mass eigenstates W^+_μ, W^-_μ, A_μ and Z_μ fields

$$D_\mu \;=\; \partial_\mu - i\,\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\,\frac{g}{\cos\theta_W}\,Z_\mu\,(T^3 - \sin^2\theta_W\,Q) - i\,e\,A_\mu\,Q \tag{5.242}$$

where the values of e and the weak $SU(2)$ charge g are related by

$$\boxed{g \;=\; \frac{e}{\sin\theta_W}} \tag{5.243}$$

To work out the physical consequences of the fermion-gauge boson couplings, we should write Eq. (5.241) in terms of the vector boson mass eigenstates using the form of the covariant derivative given in Eq. (5.242). Thus the fermion Lagrangian can be divided into three pieces: kinetic term, charged current and neutral current.

$$\mathcal{L}_\psi = \mathcal{L}_{\text{kin}} + g\,(\,W^+_\mu\,J^{\mu+}_W + W^-_\mu\,J^{\mu-}_W + Z_\mu\,J^\mu_Z) + e\,A_\mu\,J^\mu_{EM} \tag{5.244}$$

where

$$\mathcal{L}_{\text{kin}} \;=\; i\bar{\ell}_L\partial\!\!\!/\,\ell_L + i\overline{e}_R\partial\!\!\!/\,e_R \tag{5.245}$$

$$\begin{aligned}\mathcal{L}_{\text{CC}} \;&=\; g\,(\,W^+_\mu\,J^{\mu+}_W + W^-_\mu\,J^{\mu-}_W\,) \;=\; \frac{g}{\sqrt{2}}\,\overline{\ell}_L\,(W^+_\mu T^+ + W^-_\mu T^-)\,\gamma^\mu\,\ell_L \\ &=\; \frac{g}{\sqrt{2}}\left[W^+_\mu\,\overline{\nu}_{eL}\,\gamma^\mu\,e_L + W^-_\mu\,\overline{e}_L\,\gamma^\mu\,\nu_{eL}\right]\end{aligned} \tag{5.246}$$

The neutral current interactions involving Z_μ vector boson is

$$\begin{aligned}\mathcal{L}_{\text{NC}} \;&=\; g\,Z_\mu\,J^\mu_Z \;=\; \frac{g}{\cos\theta_W}\,Z_\mu\left[\overline{\ell}_L\,(T^3 - \sin^2\theta_W\,Q)\,\gamma^\mu\,\ell_L + \overline{e}_R\,(T^3 - \sin^2\theta_W\,Q)\,\gamma^\mu\,e_R\right] \\ &=\; \frac{g}{\cos\theta_W}\,Z_\mu\left[\overline{\nu}_{eL}\left(\frac{1}{2}\right)\gamma^\mu\,\nu_{eL} + \overline{e}_L\left(-\frac{1}{2} + \sin^2\theta_W\right)\gamma^\mu\,e_L + \overline{e}_R\,(\sin^2\theta_W)\,\gamma^\mu\,e_R\right]\end{aligned} \tag{5.247}$$

where

$$Q\,\psi_q \;=\; q\,\psi_q \tag{5.248}$$

This procedure works for leptons and also for the quarks, *e.g.* the charged current Lagrangian for quark sector is

$$\frac{g}{\sqrt{2}}\left[W^+_\mu\,\overline{u}_L\,\gamma^\mu\,d_L + W^-_\mu\,\overline{d}_L\,\gamma^\mu\,u_L\right] \tag{5.249}$$

The charged currents of lepton and quark sector for one generation is given by

$$\begin{aligned}J^{\mu+}_W \;&=\; \frac{1}{\sqrt{2}}\left[\overline{\nu}_{eL}\,\gamma^\mu\,e_L + \overline{u}_L\,\gamma^\mu\,d_L\right] \\ J^{\mu-}_W \;&=\; \frac{1}{\sqrt{2}}\left[\overline{e}_L\,\gamma^\mu\,\nu_{eL} + \overline{d}_L\,\gamma^\mu\,u_L\right]\end{aligned} \tag{5.250}$$

Finally we comment that the theoretical motivation for grouping the quarks and leptons as shown in Table 5.2 is that complete families are required for the cancellation of anomalies in the currents which couple to gauge fields. This cancellation shows that Ward identities, which are crucial for the proof of renormalizability of the gauge theory at quantum level, are still validated.

Fermion			T_L^3	Y_L	T_R^3	Y_R	Q
u	c	t	$+\frac{1}{2}$	$+\frac{1}{6}$	0	$+\frac{2}{3}$	$+\frac{2}{3}$
d	s	b	$-\frac{1}{2}$	$+\frac{1}{6}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
ν_e	ν_μ	ν_τ	$+\frac{1}{2}$	$-\frac{1}{2}$	$-$	$-$	0
e	μ	τ	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	-1

Table 5.2: Weak isospin and hypercharge assignments, they are related through $Q = T^3 + Y_\psi$.

Yukawa interactions

A direct fermion mass term

$$m_f \overline{\psi}\,\psi = m_f \left(\overline{\psi}_R\,\psi_L + \overline{\psi}_L\,\psi_R\right) \tag{5.251}$$

is not gauge invariant under $SU(2)$ or $U(1)$ gauge transformation. The Higgs field ϕ can give masses to the fermions via Yukawa interactions with the fermion fields. After spontaneous symmetry breaking, a Yukawa interaction of the form $g_f\,\overline{f}_L\,\phi\,f_R$ leads to a fermion mass, $m_f = g_f\,v/\sqrt{2}$. The Yukawa Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; & -\Gamma_d^{ij}\,\bar{q}_L'^i\,\phi\,d_R'^j - \Gamma_d^{ij\star}\,\bar{d}_R'^i\,\phi^\dagger\,q_L'^j \\ & -\Gamma_u^{ij}\,\bar{q}_L'^i\,\phi_c\,u_R'^j + \text{h.c.} \\ & -\Gamma_e^{ij}\,\bar{\ell}_L^i\,\phi\,e_R^j + \text{h.c.}\end{aligned} \tag{5.252}$$

where

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \quad \phi_c = i\,\tau^2\,\phi^\star = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix} \tag{5.253}$$

and here we have replaced the Higgs boson notation χ with H. q_L and ℓ_L are left-handed doublet fields and d_R, u_R, e_R are right-handed $SU(2)$ singlet fields. The primes for q_L', d_R' and u_R' mean that they are quark fields that are generic linear combination of the mass eigenstates u and d. Γ_d^{ij}, Γ_u^{ij} and Γ_e^{ij} are 3×3 complex matrices in generation space, spanned by the indices i and j. Note that neutrino masses can be implemented via Γ_ν^{ij} term. Since m_ν is very small so that we neglect it in the following. $\mathcal{L}_{\text{Yukawa}}$ is Lorentz invariant, gauge invariant and renormalizable, and

therefore it can be included in the Lagrangian. In the unitary gauge we have

$$\begin{aligned}
\bar{q}'^i_L \, \phi \, d'^j_R &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{u}'^i_L & \bar{d}'^i_L \end{pmatrix} \begin{pmatrix} 0 \\ v+H \end{pmatrix} d'^j_R = \frac{v+H}{\sqrt{2}} \bar{d}'^i_L d'^j_R \\
\bar{q}'^i_L \, \phi_c \, u'^j_R &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{u}'^i_L & \bar{d}'^i_L \end{pmatrix} \begin{pmatrix} v+H \\ 0 \end{pmatrix} u'^j_R = \frac{v+H}{\sqrt{2}} \bar{u}'^i_L u'^j_R
\end{aligned} \tag{5.254}$$

so Eq. (5.252) becomes

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= -\Gamma^{ij}_d \frac{v+H}{\sqrt{2}} \bar{d}'^i_L d'^j_R - \Gamma^{ij}_u \frac{v+H}{\sqrt{2}} \bar{u}'^i_L u'^j_R - \Gamma^{ij}_e \frac{v+H}{\sqrt{2}} \bar{e}^i_L e^j_R + \text{h.c.} \\
&= -\left[M^{ij}_u \bar{u}'^i_L u'^j_R + M^{ij}_d \bar{d}'^i_L d'^j_R + M^{ij}_e \bar{e}^i_L e^j_R + \text{h.c.} \right] \left(1 + \frac{H}{v} \right)
\end{aligned} \tag{5.255}$$

with mass matrices

$$M^{ij} = \Gamma^{ij} \frac{v}{\sqrt{2}} \tag{5.256}$$

Now we would like to diagonalize the mass matrices $M^{ij}_f (f = u, d, e)$, which can be achieved using a bi-unitary transformation U^f_L and U^f_R.

$$f'^i_L = \left(U^f_L \right)_{ij} f^j_L, \qquad f'^i_R = \left(U^f_R \right)_{ij} f^j_R \tag{5.257}$$

with U^f_L and U^f_R chosen such that

$$\left(U^f_L \right)^\dagger \, M_f \, \left(U^f_R \right) = \text{diagonal} \tag{5.258}$$

and $U^f_{L/R}$ must be unitary in order to preserve the form of the kinetic terms in the Lagrangian. We give two examples of diagonalized fermion mass matrices

$$(U^u_L)^\dagger \, M_u U^u_R = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \qquad \left(U^d_L \right)^\dagger M_d U^d_R = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \tag{5.259}$$

Hence Yukawa interactions Eq. (5.252) can be rewritten as

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= -\sum_{f',i,j} M^{ij}_f \, \bar{f}'^i_L \, f'^j_R \left(1 + \frac{H}{v} \right) + \text{h.c.} \\
&= -\sum_{f,i,j} \bar{f}^i_L \left[\left(U^f_L \right)^\dagger \, M_f \, \left(U^f_R \right) \right]_{ij} f^j_R \left(1 + \frac{H}{v} \right) + \text{h.c.} \\
&= -\sum_f m_f \left[\bar{f}_L f_R + \bar{f}_R f_L \right] \left(1 + \frac{H}{v} \right) \\
&= -\sum_f m_f \, \bar{f} f \left(1 + \frac{H}{v} \right)
\end{aligned} \tag{5.260}$$

We succeed in producing fermion masses and we got a fermion-antifermion-Higgs coupling which is proportional to the fermion mass. Obviously the Higgs Yukawa couplings are flavour diagonal, this means there is no flavour changing Higgs interactions. Recall that the charged current interaction is of the form

$$g\, W_\mu^+\, J_W^{\mu+} + \text{h.c.} \;=\; \frac{e}{\sqrt{2}\,\sin\theta_W}\, W_\mu^+\, \bar{u}_L'^i\, \gamma^\mu\, d_L'^i + \text{h.c.} \tag{5.261}$$

After the mass diagonalization described previously, this term becomes

$$\frac{e}{\sqrt{2}\,\sin\theta_W}\, W_\mu^+\, \bar{u}_L^i \left[(U_L^u)^\dagger\; U_L^d\right]_{ij} \gamma^\mu\, d_L^j + \text{h.c.} \tag{5.262}$$

We define the Cabibbo-Kobayashi-Maskawa matrix V_{CKM}.

$$V_{CKM} = (U_L^u)^\dagger\; U_L^d \tag{5.263}$$

The CKM matrix is not diagonal and then it mixes the flavours of the different quarks. It is a unitary matrix ($V_{CKM}^\dagger\, V_{CKM} = 1$) and the values of its entries must be determined from experiments. The CKM matrix connects the weak eigenstates (d', s', b') and the corresponding mass eigenstates (d, s, b) through

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{5.264}$$

For 3×3 CKM matrix, the matrix element can be parameterized by 3 angles and 1 phase, which gives rise to CP violation in Standard Model (SM). Now we look at the neutral current interaction, *e.g.* down type quarks is given by

$$g\, Z_\mu\, J_Z^\mu \;=\; \frac{e}{\sin\theta_W\, \cos\theta_W} \left[\left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_L'^i\, \gamma^\mu\, d_L'^i + \left(+\frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_R'^i\, \gamma^\mu\, d_R'^i\right] \tag{5.265}$$

After the mass diagonalization we have

$$\begin{aligned} &\frac{e}{\sin\theta_W\, \cos\theta_W} \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_L^i \left[(U_L^d)^\dagger\; U_L^d\right]_{ij} \gamma^\mu\, d_L^j + \\ &\frac{e}{\sin\theta_W\, \cos\theta_W} \left(+\frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_R^i \left[(U_R^d)^\dagger\; U_R^d\right]_{ij} \gamma^\mu\, d_R^j \\ &= \frac{e}{\sin\theta_W\, \cos\theta_W} \left[\left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_L^i\, \gamma^\mu\, d_L^i + \left(+\frac{1}{3}\sin^2\theta_W\right) Z_\mu\, \bar{d}_R^i\, \gamma^\mu\, d_R^i\right] \end{aligned} \tag{5.266}$$

So the unitary matrices cancel and the Z boson interaction is flavour diagonal also in the mass eigenstate. It works the same way for the other flavours. This mechanism is called the GIM mechanism (Glashow, Iliopoulos and Maiani) [79]. It predicts the existence of the charm quark;

there is then a second doublet (c_L, s'_L). Hence, the transitions, like $d \to s$ and $s \to d$, cancel precisely. This GIM mechanism generalizes for any number of quark generations.

Finally we summarize the key points of this section.

- The Higgs field ϕ can give masses to the fermions via Yukawa interactions with the fermion fields.
  $\longrightarrow$ No flavour changing Higgs interactions.

- The flavour is conserved in vertices involving neutral gauge bosons: A_μ, Z_μ and G_μ.
  $\longrightarrow$ GIM mechanism.
  $\longrightarrow$ Absence of flavour changing neutral current (FCNC) at the tree level.
  $\longrightarrow b \to s\gamma, \cdots$ are loop-induced in SM (high sensitivity to new physics effects).

- The charged current processes mediated by $W^\pm$ are flavour violating with the strength of violation given by the $SU(2)$ gauge coupling g and a unitary 3×3 CKM matrix.

5.8.4 The Standard Model of particle physics

The Standard Model (SM) of elementary particle physics is a gauge theory of strong and electroweak interactions. It is based on the following gauge group.

$$\boxed{SU(3)_C \times SU(2)_L \times U(1)_Y} \tag{5.267}$$

The $SU(3)_C$ is the colour group of QCD, while $SU(2)_L \times U(1)_Y$ is the Glashow-Weinberg-Salam electroweak symmetry group, which is spontaneously broken down to $U(1)_Q$, the phase group of the electric charge Q, different from the $U(1)_Y$ of weak hypercharge: $Q = T^3 + Y$, where T^3 is the third component of the weak isospin generator of $SU(2)_L$. The group $SU(3)_C \times U(1)_Q$ is believed to be an exact gauge symmetry of nature. The gauge group Eq. (5.267) contains 12 spin 1 gauge bosons:

- 8 massless gluons of $SU(3)_C$, which are responsible for the strong interactions (QCD).

- 4 gauge bosons of $SU(2)_L \times U(1)_Y$, which are responsible for the electroweak interactions, of which one is massless (photon field: A_μ) and three are massive ($W^\pm$ and Z gauge bosons) after spontaneous symmetry breaking (SSB).

These gauge bosons interact with matter fields (coloured quarks and colourless leptons) in a gauge invariant way. The field content is the following:

Gauge sector : Spin = 1

The gauge bosons are spin 1 vector particles belonging to the adjoint representation of the gauge group Eq. (5.267). Their quantum numbers are:

gluons	G_μ^A :	$(8,1,0)$	$SU(3)_C$	g_s
weak bosons	A_μ^a :	$(1,3,0)$	$SU(2)_L$	g
abelian boson	B_μ :	$(1,1,0)$	$U(1)_Y$	g'

(5.268)

where A runs over the eight colour degrees of freedom of the gluon field G_μ^A ($A = 1, \cdots, 8$) and a denotes the isospin space ($a = 1, 2, 3$). In order to avoid confusion we have changed the notation for the strong coupling constant by g_s, which we denoted with g in Chapter 1.

Fermion sector : Spin = $\frac{1}{2}$

In the Standard Model the matter fields are fermions belonging to the fundamental representation of the gauge group Eq. (5.267):

quarks				$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_Q : Q$
$q_L^i =$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	3	2	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
$u_R^i =$	u_R	c_R	t_R	$\bar{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$
$d_R^i =$	d_R	s_R	b_R	$\bar{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$
leptons							
$\ell_L^i =$	$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$	1	2	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$e_R^i =$	e_R	μ_R	τ_R	1	1	-1	-1
$\nu_R^i =$	ν_{eR}	$\nu_{\mu R}$	$\nu_{\tau R}$	1	1	0	0

where

$$Q = T^3 + Y \tag{5.269}$$

The Standard Model is described by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}} \tag{5.270}$$

where the $\mathcal{L}_{\text{Yukawa}}$ is given by Eq. (5.252), the $\mathcal{L}_{\text{Higgs}}$ is given by Eq. (5.209), and the gauge interaction is given by

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A - \frac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a - \frac{1}{4}B_{\mu\nu}\,B^{\mu\nu} \\ &\quad + i\bar{\ell}^i_L \not{D}\,\ell^i_L + i\bar{e}^i_R \not{D}\,e^i_R \\ &\quad + i\bar{q}^i_L \not{D}\,q^i_L + i\bar{u}^i_R \not{D}\,u^i_R + i\bar{d}^i_R \not{D}\,d^i_R \end{aligned} \tag{5.271}$$

Here $\not{D}$ is a notation for $\not{D} = \gamma^\mu\, D_\mu$ and i denotes three generations. The covariant derivative is given by

$$D_\mu\psi = \left(\partial_\mu - i\,g\,\frac{\tau^a}{2}A^a_\mu - i\,g'\,Y_\psi\,B_\mu + i\,g_s\,\frac{\lambda^A}{2}\,G^A_\mu\right)\psi \tag{5.272}$$

which acts on quark fields and lepton fields gives

$$\begin{aligned} D_\mu\ell_L &= \left(\partial_\mu - i\,g\,\frac{\tau^a}{2}A^a_\mu - i\,g'\,\left(-\frac{1}{2}\right)\,B_\mu\right)\ell_L \\ D_\mu e_R &= \left(\partial_\mu - i\,g'\,(-1)\,B_\mu\right)e_R \\ D_\mu q_L &= \left(\partial_\mu - i\,g\,\frac{\tau^a}{2}A^a_\mu - i\,g'\,\left(\frac{1}{6}\right)\,B_\mu + i\,g_s\,\frac{\lambda^A}{2}\,G^A_\mu\right)q_L \\ D_\mu u_R &= \left(\partial_\mu - i\,g'\,\left(\frac{2}{3}\right)\,B_\mu + i\,g_s\,\frac{\lambda^A}{2}\,G^A_\mu\right)u_R \\ D_\mu d_R &= \left(\partial_\mu - i\,g'\,\left(-\frac{1}{3}\right)\,B_\mu + i\,g_s\,\frac{\lambda^A}{2}\,G^A_\mu\right)d_R \end{aligned} \tag{5.273}$$

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