Black Holes or Gray Stars? That's the Question: Pseudo-Complex General Relativity

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Abstract After a short review on attempts to extend General Relativity, pseudo-complex variables are introduced. We restate the main properties of these variables. The variational principle has to be modified in order to obtain a new theory. An additional contribution appears, whose origin is a repulsive, dark energy. The general formalism is presented. As examples, the Schwarzschild and the Kerr solutions are discussed. It is shown that a collapsing mass inceasingly accumulates dark energy until the collapse is stopped. Rather than a black hole, a gray star is formed. We discuss a possible experimental verification, investigating the orbital frequency of a particle in a circular orbit.

1 Introduction

General Relativity (GR) is a well accepted theory which has been verified by many experimental measurements. One prediction of this theory is the existence of *black holes*, which are formed once a very large mass suffers a gravitational collapse. Astronomical observations seem to confirm this prediction, finding large mass concentrations in the center of most galaxies. These masses vary from several million solar masses to up to several billion solar masses. However, a black hole implies the appearance of an event horizon, below which an external observer cannot penetrate, thus, excluding a part of space from observation. A black hole also implies a singularity at its center. Both consequences from GR may be, from a philosophical point of view, unacceptable and one would like to find a possibility to avoid them. A black hole is an extreme object and one would not be surprised that GR has to be modified

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for such situations. For example, the singularity could be avoided, considering a quantized GR, not yet available.

There have been several attempts to generalize GR. Einstein [1, 2] introduced complex variables in order to unify GR with Electrodynamics. Later on, other groups continued this research (see for example [3, 4] and references therein) calling the new theory *complexified GR*. The real component of the complex variable is given by x^μ while the imaginary component is given by $l\frac{p^\mu}{m}$, where p^μ is the momentum of a particle and m is its mass. As a by-product a minimal length parameter l appears, for dimensional reasons. One of the motivations to continue the investigation of the complex GR is the *Born's equivalence principle*. Born noted [5, 6] that in GR there is an asymmetry between the coordinates and momenta, while in Quantum Mechanics they occur in a symmetric manner. In order to recuperate the symmetry he proposed a modified length element, adding to ds^2, the length square element, an additional term $l^2 g_{\mu\nu} du^\mu du^\nu$, with u^μ as the four velocity and $g_{\mu\nu}$ the metric (Born used instead of u^μ the p^μ/m). Again the minimal length parameter appears due to dimensional reasons. In [7] it was recognized that the new length element is related to a *maximal acceleration*, $a \leq 1/l$. Many other groups joined in this investigation [8–14] and we will show that it is automatically contained in the proposed pseudo-complex extension of GR (which we will call from here on pc-GR). In [15, 16] a non-symmetric metric is considered and we will also show that it is contained within a pseudo-complex (pc) description.

In Sect. 2 we will introduce the pc-variables and mention some important properties. In the same section the formulation of the pc-GR is resumed. In Sect. 3 we present the results of the pc-Schwarzschild and pc-Kerr solution. It will be shown that in the pc-GR dark energy accumulates around a large mass concentration, which will finally stop the gravitational collapse, forming rather a gray star than a black hole. There will be no event horizon, thus allowing an external observer to access all region of space. Also in this section, the circular motion of a particle around a gray star is considered, with possible experimental verification. In Sect. 4 the conclusions will be drawn.

2 Formulation of the Pseudo-Complex General Relativity

First we resume some basic properties of pc-variables: A pseudo-complex variable is given by $X = X_R + IX_I$, with X_R as the pseudo-real and X_I the pseudo-imaginary component. It is of great advantage to write it in terms of the *zero divisor basis* (the notation becomes obvious further below) $X = X_+\sigma_+ + X_-\sigma_-$, with $\sigma_\pm = \frac{1}{2}(1 \pm I)$. The $\sigma_\pm$ obey the relations $\sigma_\pm^2 = \sigma_\pm$ and $\sigma_+\sigma_- = 0$. The last property is the definition of a zero divisor. When one defines as the complex conjugate $X^* = X_R - IX_I$, which implies $\sigma_\pm^* = \sigma_\mp$, then for elements in the zero divisor basis ($X = \lambda\sigma_\pm$) the norm squared $\mid X \mid^2 = XX^*$ is zero. One can look at it as a "generalized" zero. Calculations in the zero divisor basis are particularly simple. For example, products and division of functions can be done independently in each zero divisor component. Also differentiation and integration can be defined, similar to complex analysis (with some slight changes). For more details, please consult [17, 18]. In the literature there exist several names for the pc-variables. Sometimes they are called hyper-complex, hyperbolic or semi-complex.

The consequences of using pc-variables for the Lorentz transformation are as follows: A finite Lorentz transformation is given by

$$
\begin{aligned}
e^{i\omega_{\mu\nu}\Lambda_{\mu\nu}} &= e^{i\omega^{+}_{\mu\nu}\Lambda^{+}_{\mu\nu}}\sigma_{+} + e^{i\omega^{-}_{\mu\nu}\Lambda^{-}_{\mu\nu}}\sigma_{-} \\
\Lambda_{\mu\nu} &= X_{\mu}P_{\nu} - X_{\nu}P_{\mu} \\
\Lambda^{\pm}_{\mu\nu} &= X^{\pm}_{\mu}P^{\pm}_{\nu} - X^{\pm}_{\nu}P^{\pm}_{\mu} \\
\omega_{\mu\nu} &= \omega^{+}_{\mu\nu}\sigma_{+} + \omega^{-}_{\mu\nu}\sigma_{-} \quad .
\end{aligned} \tag{1}
$$

It divides into a Lorentz transformation in each zero-divisor component. The generators look the same, except now the variables are pseudo-complex. In the zero-divisor component the coordinates are given by $X^{\pm}_{\mu}$ and the momenta by $P^{\pm}_{\nu}$. Because $\sigma_{+}\sigma_{-} = 0$, the two Lorentz transformations commute, thus we have

$$
SO_{+}(3,1) \otimes SO_{-}(3,1) \supset SO(3,1). \tag{2}
$$

The standard Lorentz group is contained in the direct product and is reached by projecting the pseudo-complex parameters, coordinates and momenta to their real parts, i.e.,

$$
\begin{aligned}
\omega_{\mu\nu} \rightarrow \omega^{R}_{\mu\nu} &= \frac{1}{2}\left(\omega^{+}_{\mu\nu} + \omega^{-}_{\mu\nu}\right) \\
X_{\mu} &\rightarrow x_{\mu} \\
P_{\nu} &\rightarrow p_{\nu} \quad .
\end{aligned} \tag{3}
$$

This projection method has to be applied also to the metric components.

That pseudo-complex variables also proved to be very useful was demonstrated in [19]: As shown in [19], the field equation for a scalar boson field is obtained from the Lagrangian density $\frac{1}{2}\left(D_{\mu}\Phi D^{\mu}\Phi - M^{2}\Phi^{2}\right)$, where Φ is the pc-boson field, $M = M_{+}\sigma_{+} + M_{-}\sigma_{-}$ is a pc-mass and D_{μ} a pc-derivative. The propagators of this theory are the ones of Pauli-Villars, which already are regularized. One obtains the same propagator in the standard theory, with a non-pc scalar field, using the Lagrange density $-\frac{1}{(M_{+}^{2}-M_{-}^{2})}\phi\left(\partial_{\mu}\partial^{\mu} + M_{+}^{2}\right)\left(\partial_{\mu}\partial^{\mu} + M_{-}^{2}\right)\phi$, where ϕ is now a real valued function, M_{+} is identified with the physical mass m and $M_{-} >> M_{+}$ with the regularizing mass. Note, that this theory is highly non-linear while the pc-description is linear. This indicates that a pc-description can substantially simplify the structure of the theory and we can expect something similar in the pc-formulation of GR.

Let us now return to the pc-GR: The pc-extension of GR is quite direct within the zero divisor components. The first attempts are published in [20, 21] and in a more recent article [22] which includes modifications. Here we will present a short review. We introduce the pc-metric via

$$g_{\mu\nu}(X, A) = g^{+}_{\mu\nu}(X_+, A_+)\sigma_+ + g^{-}_{\mu\nu}(X_-, A_-)\sigma_-, \tag{4}$$

were the metric is assumed to be symmetric (in Moffat's theory of a non-symmetric metric [15, 16], the σ_+ component is the metric $g_{\mu\nu}$, while the σ_- component is its transposed, so in principle Moffat's theory is contained in our theory, if we skip the restriction to a symmetric metric). The metric components depend on the variables $X^{\mu}_{\pm}$ and parameters, denoted shortly as $A_{\pm}$. In each zero-divisor component a GR is constructed in the same manner as in standard GR. The pc-coordinates have the structure

$$X^{\mu} = x^{\mu} + Ilu^{\mu} \quad . \tag{5}$$

Again, due to dimensional reasons, a minimal length parameter has to be introduced. Because it is just a parameter, it is not affected by any relativistic transformation, contrary to the believe that a minimal length is related to the breaking of Lorentz symmetry. The error made is to relate a minimal length to a physical length, which is affected by a Lorentz transformation. Here, the minimal length is a parameter and thus cannot be affected by such a transformation. The consequences are very important. For example, in [19] a pc-Field Theory was developed, demonstrating that a minimal length parameter does not affect the known symmetries, thus the calculations of Feynman diagrams remain very simple and that the propagators of the theory are automatically regularized.

In mathematical terms we can explain the pc-extension of GR in terms of the following chain

$$G_+ \otimes G_+ \supset G. \tag{6}$$

In each component a standard GR is formulated. The base manifold is given by $X^{\mu}_{\pm}$ and the tangent spaces are given by $U^{\mu}_{\pm}$. Note, that U^{μ} includes the acceleration. Excluding the acceleration leads to G.

The pc-length square element is given by

$$d\omega^2 = g_{\mu\nu}(X, A)DX^{\mu}DX^{\nu} \quad , \tag{7}$$

where D refers to a pc-differential [19, 20].

One may ask, what are the corrections due to the minimal length l? This will lead to the conclusion that all other theories, mentioned in the introduction, are a consequence of a pc-description. An expansion up to lu^{μ} is given by

$$g_{\mu\nu}(X) \approx g_{\mu\nu}(x) + lu_{\lambda}F^{\lambda}_{\mu\nu}(x). \tag{8}$$

The norm of the four-velocity can not be larger than 1. Considering that the minimal length is probably very small (Planck length), one can safely take into account only the first term. Thus, in the metric tensor $g_{\mu\nu}(X)$ the pc-coordinates X^{μ} are substituted by x^μ. With this and expressing the pc-coordinates explicitly in terms of their pseudo-real and pseudo-imaginary components, the $d\omega^2$ acquires the form

$$d\omega^2 \approx g_{\mu\nu}(x)\left(dx^\mu dx^\nu + l^2 du^\mu du^\nu\right) + 2Ilg_{\mu\nu}(x)dx^\mu du^\nu \quad . \tag{9}$$

The terms in du^μ can not be neglected when effects near the maximal accelerations are considered. The du^μ are *differentials* of velocities, thus accelerations, and can reach values of the order of $1/l$. When the motion of a particle is considered, the $d\omega^2$ has to be real. This provides the condition

$$g_{\mu\nu}(x)dx^\mu du^\nu = 0 \quad , \tag{10}$$

which is nothing but the dispersion relation. With (10) the length square element acquires the form as used in the theories mentioned in the introduction. There, the dispersion relation is introduced by hand while here it appears as a logical consequence.

When maximal acceleration effects are of no importance, one can also neglect the terms proportional to l and l^2 in (9).

All properties of tensors, four derivatives, Christoffel symbols, etc. can be directly extended from standard GR, defining them in each zero-divisor component as done in standard GR [20, 22, 23]. The only concept which has to be modified is the variational principle. If one uses (S denotes the action) $\delta S = \delta S_+\sigma_+ + \delta S_-\sigma_- = 0$, then we would obtain $\delta S_\pm = 0$, which correspond to two separated theories. In order to get a new theory, in [24, 25] a modified variational principle was proposed, namely that the variation has to be within the zero divisor (it can be interpreted as a "generalized zero"). This leads to field equations which on their right hand side are not zero but proportional to an element in the zero divisor. Our convention is to set it proportional to σ_-. Thus the Einstein equations read ($c = 1$)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; -8\pi\kappa T_{\mu\nu}\sigma_- \quad . \tag{11}$$

The $R_{\mu\nu}$ are the components of the pc-Ricci tensor, while R is the Ricci scalar. On the right hand side appears an energy-momentum tensor which describes the presence of an additional field which is always there in a pc-description. This field will turn out to have the properties of a dark energy and it will introduce a repulsion against gravitational collapse.

3 pc-Schwarzschild and pc-Kerr solutions

In [22] we presented the pc-Schwarzschild and pc-Kerr solutions. Of interest here is the g_{00} component, namely

$$g_{00} = \left(1 - \frac{2m}{r} + \frac{\Omega(r)}{r}\right) \quad . \tag{12}$$

(Here, we neglect for the moment a possible factor e^f [22], which we set to 1.) We already restricted to the first term in the expansion in lu^μ. The $\Omega(r)$ is a not yet known function in the radial distance r. We model it by $\Omega = \frac{B}{2r^2}$. This leads to a correction in the metric of $\frac{B}{2r^3}$. The correction to the metric components have to depend at least on $1/r^3$, because a dependence on $1/r^2$ with a large B is excluded by experiments in the solar system [26].

One may speculate about the origin of the dark energy. One possibility are the vaccum fluctuations (Casimir effect): In [27] the Casimir effect in a gravitational backgound is investigated, within the Hartle-Hawking vacuum. No recoupling of the vacuum fluctuations with the gravitational field is considered. Thus, there is still the Schwarzschild metric present with an event horizon at the Schwarzschild radius. As a result, the expectation value of the trace of the energy-momentum tensor, due to the vacuum fluctuations, falls off proportional to $1/r^6$. This would mean that the mass, represented by the energy density, falls of proportional to $1/r^3$. Because no recoupling with the gravitational field is considered, the calculation has to stop at the Schwarzschild radius. Below that, no time can be defined in the same way as outside. In the pc-GR the recoupling of the dark energy energy-momentum tensor with the gravitational field is automatically included in (11). This leads to the correction in (12). Using the result in [27] literally, would imply a correction to the metric proportional to $1/r^4$. We will assume that the correction to the metric falls off like $1/r^3$ instead. This is the minimal correction which can be implemented not yet in conflict with current astronomical observations [26]. We expect to change the r-dependence, when the recoupling to the gravitational field is included in the calculation of the Casimir effect. Therefore, the model assumption that the corrections to the metric behave as $1/r^3$ is a rather good one. Proposing $1/r^4$ does not change our results significantly!

After this consideration, we return to the discussion of the pc-GR: In order to have the same interpretation of time in all regions of space, the g_{00} component has to be larger than zero. This introduces a minimal value of B.

Note, that the $\sqrt{g_{00}}$ component is proportional to an effective potential, with angular momentum zero [28]. With this, the effective potential is proportional to

$$\sqrt{1 - \frac{2m}{r} + \frac{B}{2r^3}} \quad . \tag{13}$$

For large distance, the potential is similar to the standard Schwarzschild solution. The differences start to appear near the Schwarzschild radius. The event horizon vanishes, because g_{00} never becomes zero. At smaller radial distances, the potential becomes repulsive, which is the consequence of the accumulation of dark energy. This changes the picture of a gravitational collapse: When a large mass is contracted due its gravitational influence, dark energy starts to accumulate and increases when the collapse advances. The collapse is finally stopped when enough dark energy accumulates and acts against the gravitational attraction. Thus, instead of a black hole the result is rather a *gray star*, though the gray star resembles pretty much a black hole seen from far apart. Therefore, from now on we will always refer to a gray star.

Today we know that the gray stars in the centers of galaxies rotate nearly at maximum speed. Thus, instead of the Schwarzschild solution one has to take the Kerr solution, which describes stars in rotation. The pc-Kerr solution was obtained in [22, 29]. Please look there for details.

In order to relate the theory to experiment, we investigated the motion of a particle in a circular geodesic orbit around a gray star. This may be related to the possible observation of a plasma cloud orbiting such a star [30]. In Fig. 1 the orbital frequency is plotted versus the radial distance. As can be seen, the orbital frequency differs little from the standard Kerr solution until r is of the order of the Schwarzschild radius. Towards smaller radial distances, the orbital frequency is smaller in the pc-description, showing a maximum value, after which it diminishes. The maximum is a result of the structure of g_{00} which has a global minimum at about two-thirds of the Schwarzschild radius. For radii below that value the expression for the orbital frequency gets imaginary and we do not expect to observe circular geodesic orbits

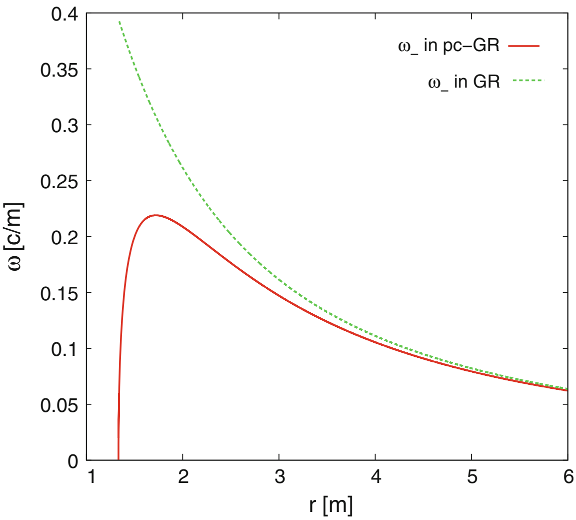

Fig. 1 The orbital frequency of a particle in a circular orbit around a gray star, as a function on the radial distance r. The units of ω are in $\frac{m}{c}$ while the radial distance is in units of half the Schwarzschild radius. $r = 2$ corresponds to the Schwarzschild radius and $\omega = 0.22$ is equivalent to about 0.11/min (For this computation we took the mass of Sagittarius A, the center of our galaxy, which is of the order $3 \times 10^7\ M_{\rm sun}$). The standard Kerr solution is given by the upper line, while the pc-solution is given by the lower line

anymore. The curve for the pc-Kerr solution stops at this value. The curve for the standard GR stops at the point of the last stable orbit.

The result was obtained assuming that $\Omega = \frac{B}{2r^2}$. If it goes with a larger power in r, the pc-solution approaches the standard Kerr solution, but will always show a maximum and the last stable orbit will be further out, i.e., the basic results will be the same.

This result has important consequences in the experimental verification of pc-GR and we refer to the talk given by T. Boller [30].

4 Conclusions

In this contribution we reviewed the pseudo-complex General Relativity. The extension of the standard GR to pc-variables is direct due to the property that the zero-divisor components commute. In each component a standard GR is constructed. In order to obtain a new theory, the variational principle has to be changed. The variation of the action has now to be within the zero-divisor, i.e., it has to be a "generalized zero". This introduces a new energy-momentum tensor in the Einstein equations, describing a dark energy field.

As a consequence of this dark energy-field, the gravitational collapse of a large mass is halted as soon as enough dark energy has accumulated. Due to this, no event horizon is formed and no singularity either. Instead of a black hole rather a *gray star* is formed. This answers the question in the title!

A possible experimental verification is proposed, determining the orbital frequency of a particle around a gray star.

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