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MONOPOLES AND CONFINEMENT IN THE $3D$ $SU(2)$ LATTICE GAUGE THEORY

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Using a plaquette formulation for lattice gauge models we describe monopoles of the $3D$ $SU(2)$ theory which appear as configurations in the complete axial gauge and violate the continuum Bianchi identity. Furthemore we derive a dual representation for the Wilson loop in arbitrary representation and calculate the form of the interaction between generated electric flux and monopoles in the region of weak coupling relevant for the continuum limit. The effective theory which controls the interaction is a generalized version of the sine-Gordon model. The string tension is calculated within the semiclassical approximation.

1 Introduction

The problem of the permanent confinement of quarks inside hadrons attracts attention of the theoretical physicists for the last three decades (see [1] and refs. therein for a recent review of the problem). Two of the most popular and the most elaborated mechanisms of confinement are based on the condensation of certain topologically nontrivial configurations - the so-called center vortices or monopoles. In this paper we are interested in the second of these configurations. It was proposed in [2] in the context of continuum compact three dimensional ($3D$) electrodynamics that the string tension is nonvanishing in this theory at any positive coupling constant, and the contribution of monopoles to the Wilson loop was estimated in the semiclassical approximation. Later this consideration was extended to $U(1)$ lattice gauge theory (LGT) in $3D$ [3]. It turns out that these are precisely monopole configurations which make the string tension nonvanishing at all couplings. A rigorous proof of this property was done in [4]. While the monopoles of abelian gauge models can be given by a gauge invariant definition it is not the case for nonabelian models. The most popular approach consists of a partial gauge fixing such that some abelian subgroup of the full nonabelian group remains unbroken. Then, one can define monopoles in a nonabelian theory as monopoles of the unbroken abelian subgroup. Here we propose a different way to define monopoles in nonabelian models. Its main feature is complete gauge fixing. Monopoles appear as defects of smooth gauge fields which violate the Bianchi identity in the continuum limit, in the full analogy with abelian models. Our principal approach is to rewrite the compact LGT in the plaquette (continuum field-strength) representation and to find a dual form of the nonabelian theory. The Bianchi identity appears in such formulation as a condition on the admissible configurations. This allows to reveal the relevant field configurations contributing to the partition function and various observables. Such a program was accomplished for the abelian LGT in [3]. Here we are going to work out the corresponding approach for nonabelian models on the example of $3D$ $SU(2)$ LGT.

2 Plaquette formulation and monopoles

LGT was formulated by K. Wilson in terms of group valued matrices on links of the lattice as fundamental degrees of freedom [5]. The plaquette representation was invented originally in the continuum theory by M. Halpern and extended to lattice models by G. Batrouni [6]. In this representation the plaquette matrices play the role of the dynamical degrees of freedom and satisfy certain constraints expressed by Bianchi identities in every cube of the lattice. In papers [7], [8], [9] we have developed a different plaquette formulation which we outline below.

We start from the partition function that can be written on the dual lattice as [10]

$$Z = \int \prod_{l,k} d\omega_k(l) e^{-\frac{2\beta}{8}\omega_k^2(l)} \prod_x \frac{|W_x|}{\sin W_x} \sum_{m(x)=-\infty}^{\infty} \int \prod_k d\alpha_k(x) \exp\left[-i\sum_k \alpha_k(x)\frac{\omega_k(x)}{2} + 2\pi i m(x)\alpha(x)\right] , \qquad (1)$$

where $\alpha(x) = (\sum_k \alpha_k^2(x))^{1/2}$, $W_x = \frac{1}{2}(\sum_k \omega_k^2(x))^{1/2}$ and similarly for W_l. $m(x)$ are arbitrary integers. Auxiliary

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fields $\alpha_k(x)$ have been introduced for integral representation of the Bianchi identity (see [9] for details)

$$\left[\sum_k \frac{\omega_k(x)}{2}^2\right]^{\frac{1}{2}} = 2\pi m(x) \ , \ \omega_k(x) = \sum_{n=1}^{3} (\omega_k(x,n) - \omega_k(x-e_n,n)) + \mathcal{O}(\omega_k^2(l)) \ .$$

Here, six links $l=(x,n)$ are attached to a site x and $\omega_k(l)$ are link variables dual to original plaquettes. In the continuum limit the last constraint reduces to the familiar Bianchi identity if one takes $m(x)=0$ for all x. However, when $m(x)$ differs from zero one gets violation of the continuum Bianchi identity at the point x. Clearly, $m(x)\neq 0$ configuration corresponds to the monopole configuration of nonabelian gauge field. Therefore, we may interpret the summation over $m(x)$ as a summation over monopole charges which exist due to the periodicity of $SU(2)$ delta-function (in close analogy with $U(1)$ model).

3 Effective monopole model for $SU(2)$ LGT at large β

Here we would like to calculate the contribution of monopole configurations to the partition function and to the Wilson loop. We make expansion of the action, invariant measure and Jacobian around nontrivial monopole configuration. To do this one should get exact classical equation and find its solution on the nontrivial monopole configuration. We consider that fluctuations around this configuration should be small and restrict ourselves only to the classical solution. We also have to assume that connectors (see its definition in [9]) are not important in generating the string tension and produce only smooth perturbative corrections to the dual gluons.

Starting from (1) we can obtain the following saddle-point equations for fields $\omega_k(l)$ and $\alpha_k(x)$

$$\omega_k(x) = \sum_{l\in x}\omega_k(l) + \epsilon^{kmn}\sum_{l<l'\in x}\omega_m(l)\omega_n(l') + ... = 4\pi m(x)\frac{\alpha_k(x)}{\alpha(x)} \ ,$$
$$-\frac{2\beta}{4}\omega_k(l) - i\frac{1}{2}[(\alpha_k(x)-\alpha_k(x+e_n)) + ... = 0 \ . \qquad (2)$$

To find the solution of (2) we use an anzatz $\omega_k(l)=\tau_k\omega^s(x)$, $\alpha_k(x)=\tau_k\alpha^s(x)$ where τ_k have properties $\tau_k\cdot\tau_n=0$, $k\neq n$, $\sum_k\tau_k^2=1$. After these substitutions we obtain a solution for functions $\omega^s(l)$ and $\alpha^s(x)$

$$\omega^s(l) = -2\pi D_l(y)m(y) \ , \ \alpha^s(x) = i\pi\beta G_{xx'}m(x')$$

We use this solution to compute the Wilson loop of the size $R\times T$ in the representation j. Let S be some surface dual to the surface S_{xy} which is bounded by the loop C and consisting of links dual to plaquettes of the original lattice. The expectation value of $W_j(C)$ at $\beta\to\infty$ we present in the form

$$\langle W_j(C)\rangle \ = \ \frac{1}{2j+1}\ \left\langle\ \chi_j\left(\frac{1}{2}\Omega_C\right)\right\rangle \ , \ \Omega_C \ = \ \left(\sum_k\Omega_k^2(C)\right)^{\frac{1}{2}} \ , \ \Omega_k(C) = \sum_{l\in S}\omega_k(l) + \mathcal{O}(\omega_k^2(l)) \ . \qquad (3)$$

Then, by a substitution

$$\omega_k(l)\to \ \tau_k\omega^s(l) + \frac{1}{\sqrt{2\beta}}\omega_k(l) \ , \ \alpha_k(x)\to \ \tau_k\alpha^s(x) + \sqrt{2\beta}\alpha_k(x)$$

the expectation value of $W(C)$ in (3), averaged by the ensemble in (1), is presented at $\beta\to\infty$ in the form

$$\langle W_j(C)\rangle \ = \frac{1}{Z}\frac{1}{2j+1}\prod_x\sum_{m(x)\ =\ -\infty}^{\infty} e^{S_{mon}} \int_{-\infty}^{\infty}\prod_{l,k}d\omega_k(l)\int_{-\infty}^{\infty}\prod_{x,k}d\alpha_k(x)\ \chi_j\left(\frac{1}{2}\Omega_C\right)$$
$$\times\prod_{l,k}\exp[-\frac{1}{8}\omega_k^2(l) - i\frac{1}{2}\omega_k(l)\left(\alpha_k(x+e_n)-\alpha_k(x)\right)] \ , \qquad (4)$$

where $\chi_j(x)$ is a $SU(2)$ character and the effective action S_{mon} is of the form

$$S_{mon} = -2\beta\pi^2 m(x)G_{xx'}m(x') \ . \qquad (5)$$

To perform the summation over monopole configurations $m_x=0,\pm 1$ we follow the strategy of Refs.[4], [11]. Using the decomposition $G_{xx'}=B_{xx'}+G_{xx'}(M)$, where $B_{xx'}=G_{xx'}-G_{xx'}(M)$ we rewrite the effective action (5) in the form

$$S_{mon} = -2\beta\pi^2 m(x)B_{xx'}m(x') - 2\beta\pi^2 G_0(M)\sum_x m_x^2 - 2\beta\pi^2\sum_{x\neq x'}m(x)G_{xx'}(M)m(x') \ . \qquad (6)$$

The first term in (5) is presented as

$$e^{-2\beta\pi^2 m(x)B_{xx'}m(x')} = (\det B^{-1}_{xx'})^{3/2}\int_{-\infty}^{\infty}\prod_x d\phi_x \exp[-\frac{1}{2\beta}\phi_x B^{-1}_{xx'}\phi_{x'} + i2\pi\phi_x m(x)] \ .$$

The behaviour of $G_{xx'}(M)$ in the thermodynamic and continuum limits is well known

$$G_{xx'}(M) = \frac{2}{\pi R}e^{-\frac{1}{2}MR}, \ R = \left[\sum_k (x_k - x'_k)^2\right]^{\frac{1}{2}} \ .$$

This behaviour allows us to keep only self-energy of the monopoles $S^{SE}_{mon} = -\pi^2 G_0(M)\sum_x m_x^2$ if $MR \gg 1$.

After algebraic manipulations we keep in the sums over monopoles only configurations $m = 0, \pm 1$. This gives the effective model which appears to be of the sine-Gordon type. To write down the final expression we make use of the fact that $B^{-1}_{xx'}(M) \approx G^{-1}_{xx'}$ for M sufficiently large and integrate over fluctuations $\omega_k(x)$ and $\alpha_k(x)$ when $m(x) = 0$. Since we are interested in recovering PT (perturbation theory) result at large β we consider small fluctuations $\omega_k(l) \approx 0$ and use the following asymptotics of $SU(2)$-character uniformly valid in j

$$\langle W_j(C)\rangle \ \simeq \ \int_{S^2} d\sigma_k \left\langle e^{-i\sqrt{j(j+1)}\frac{\Omega_k(C)}{2}\sigma_k}\right\rangle \ \simeq \ \int_{S^2} d\sigma_k \exp[-ij_k(l)\omega_k(l)] \ , \qquad (7)$$

where expression for $\frac{\Omega_k(C)}{2}$ is given in (3) and $j_k(l) \ = \ \frac{1}{2}\sqrt{j(j+1)}\sigma_k$ for $l \in S$.

Substituting representation (7) in the expression (4) we integrate out the fluctuations to find

$$\langle W_j(C)\rangle \ = \ \frac{1}{Z}H^{gl}_j H^{mon}_j \ ,$$

where H^{gl}_j is the conventional PT result

$$H^{gl}_j = e^{-\frac{j(j+1)}{2\beta}\sum_{b,b'\in S} G_{bb'}}$$

and for H^{mon}_j one arrives at the following effective model

$$H^{mon}_j = \int_{-\infty}^{\infty}\prod_{x,k} d\phi_x \exp(-S^{mon}_{eff}[\phi_x]),$$

where

$$S^{mon}_{eff}[\phi_x] = \frac{1}{4\beta}\sum_{x,n}(\phi_x - \phi_{x+n})^2 - \gamma\sum_x \frac{1}{2j+1}\chi_j\left(\frac{\Omega_C}{2}\right)\cos 2\pi\phi_x \ .$$

Here $\gamma = 2\exp[-2\pi^2\beta G_0(M)]$ and $\Omega_C = 2\pi\sum_{b\in S} D_b(x)$.

This model possesses an important feature. We see surface independence of the Wilson loop due to the sources which enter through $SU(2)$ group character (see properties of link Green function in our paper [9]).

To perform semiclassical calculations we take the continuum limit. In this limit we get the following saddle-point equation of sine-Gordon type

$$\Delta\alpha(x) = -m^2\frac{1}{2j+1}\chi_j\left(\frac{\Omega_C}{2}\right)\sin\alpha(x) \ . \qquad (8)$$

Here we have introduced the Debye mass

$$m^2 = 32\pi^2\beta e^{-2\pi^2\beta G_0(M)} \ . \qquad (9)$$

After calculations we find

$$\frac{\Omega_C}{2} \ = \ \pi\varphi(z) + \psi(z,x,y;R,T)$$

where $\varphi(z) \ = \ sign(z)$ and $\psi(x,y;R,T)$ is a correction which decays with R,T growing. If one neglects this correction for large Wilson loop one arrives to

$$\frac{1}{2j+1}\chi_j\left(\frac{\Omega_C}{2}\right) = \frac{1}{2j+1}\frac{\sin(2j+1)\frac{\Omega_C}{2}}{\sin\frac{\Omega_C}{2}} = \cos\left(\pi\varphi(z)\right)$$

for all half-integer j. In this approximation for all half-integer j we write down the saddle-point equation (8) as

$$\Delta\phi(x) \ = \ 2\pi\delta'(x)\theta(x,y;R,T) - m^2\sin\phi(x) \ , \qquad (10)$$

where $\theta(x,y;R,T)$ is nonzero only for $x,y \in S$. Far from the boundaries of the contour C the saddle-point equation (10) is essentially one dimensional and has the solution

$$\phi(z) \ = \ 4sign(z)\arctan(e^{-sign(z)mz}) \ . \tag{11}$$

This solution have an essential property

$$\phi(+0) - \phi(-0) \ = \ 2\pi \ . \tag{12}$$

We want to stress that there is no have such a nontrivial solution (with important property (12)) for all integer j. The solution (12) leads to the desirable area law for all half-integer representations

$$\langle W_j(C)\rangle \approx \ e^{-\sigma(j=n+1/2)A_C - \frac{j(j+1)}{2\beta}\sum_{b,b'\in S} G_{bb'}} \ ,$$

where A_C is the area of Wilson loop C and the string tension reads

$$\sigma = \frac{1}{\pi^2\beta}m \ .$$

The second term in the exponent is the leading term of the PT [9]. String rension for all integer j is vanishing. The mass of dual photons is given in (9).

4 Conclusion

In this paper we calculated effective model at large values of β for the expectation value of the Wilson loop in $3D$ $SU(2)$ LGT. This model appears to be a sine-Gordon type and it is valid for all values of representations j of $SU(2)$ group. This model takes into account both the dual photons and the monopole contributions. For all half-integer representation in the semiclassical approximation we have found that the Wilson loop obeys the area law and string tension $\sigma \sim m$. Therefore, proposed mechanism of confinement is able to reproduce this essential feature of the model. It remains unclear if this contribution is also necessary condition of confinement. Our calculations also support the result by Polyakov that the string tension is nonzero only for odd representations j in $U(1)$ LGT [12].

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References

[1] J. Greensite, Prog.Part.Nucl.Phys. **51**, 1 (2007) .

[2] A.M. Polyakov, Nucl.Phys. B **120**, 429 (1977).

[3] T. Banks, J. Kogut, R. Myerson, Nucl.Phys. B **121**, 493 (1977).

[4] M. Göpfert, G. Mack, Commun.Math.Phys. **82** 545 (1982) .

[5] K. G. Wilson, Phys.Rev. D **10** 2445 (1974).

[6] M.B. Halpern, Phys.Rev. D **19** 517 (1979); Phys.Lett. B **81** 245 (1979); G. Batrouni, Nucl.Phys. B **208** 467 (1982).

[7] O. Borisenko, S. Voloshin, M. Faber, Analytical study of low temperature phase of $3D$ LGT in the plaquette formulation, Proc. of NATO Workshop "Confinement, Topology and Other Non-perturbative Aspects of QCD", Ed. by J. Greensite, and S. Olejnik, Kluwer Academic Publishers, 33, 2002; [hep-lat/0204028].

[8] O. Borisenko, S. Voloshin, M. Faber, "Perturbation Theory for Non-Abelian Gauge Models in The Plaquette Formulation", Preprint of University of Technology of Vienna, IK-TUW-Preprint 0312401, 2003.

[9] O. Borisenko, S. Voloshin, M. Faber, Field strength formulation, lattice Bianchi identities and perturbation theory for non-Abelian models, Nuclear Physics B **816** [FS], 399 (2009); [hep-lat/0508003].

[10] O. Borisenko, V. Kushnir, A. Velytsky, Phys.Rev. D **62**, 025013 (2000) .

[11] F. Conrady, Analytic derivation of dual gluons and monopoles from $SU(2)$ lattice Yang-Mills theory. III. Plaquette representation, 2006, [hep-th/0610238].

[12] A.M. Polyakov, *Gauge Fields and Strings (Contemporary Concepts of Physics: V.3)*, (Harwood Academic Publishers, Chur and London, 1987).