Boundary Terms and the 3+1 Decomposition of the Holst Action

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Abstract. Starting from the Holst action with surface terms and the fall-off conditions that make the variational principle and the covariant phase space formulation well-defined for asymptotically flat spacetimes, we rewrite the surface terms in the 3+1 decomposition of the action. We explore their relation with the surface terms in the Hamiltonian formulation in terms of Ashtekar variables. Just as for the Einstein-Hilbert action, if variations respecting asymptotic flatness are allowed, the energy and momentum in the Hamiltonian framework are not directly recovered from the 3+1 decomposition and gauge fixing of these terms.

1. Introduction

The Holst action [1] is the classical starting point in Loop Quantum Gravity. It is a firstorder covariant action based on orthonormal tetrads and Lorentz connections, whose 3+1decomposition plus partial gauge fixing gives a Hamiltonian formulation for General Relativity in terms of Ashtekar variables.

In [2], the form of the surface terms of the action, necessary for a proper treatment of asymptotically flat space-times, was given. These surface terms give a manifestly finite action even off-shell⁵, and a well-defined variational principle, reproducing Einstein's equations under all asymptotically flat variations. Furthermore, the amended action leads naturally to a well-defined covariant phase space in which the Hamiltonians generating asymptotic symmetries provide the total energy-momentum and angular momentum of the space-time.

Here we wish to analize the form of these surface terms in the canonical or 3+1 formulation of the theory and explore its relation to the canonical treatment in [5], where surface terms and Poincaré charges are derived directly from the ADM framework.

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 $^{^{5}}$ This is true for so called cylindrical temporal cut-offs or asymptotically time-translated spatial boundaries [3, 4]

2. Covariant and 3+1 actions for General Relativity

In the conventional treatment of the Einstein-Hilbert action for a space-time with a boundary, two additional terms are necessary in order to have a well-defined variational principle that yields Einstein's equations from the stationary points of the action:

$$S_{EHGH} = \frac{1}{2\kappa} \left(\int_{\mathcal{M}} d^4x \sqrt{-g} R + 2 \int_{\partial \mathcal{M}} d^3y \sqrt{|h|} (K - K_0) \right).$$
(1)

The first so called Gibbons-Hawking surface term [7] is inserted in the action so that its variation exactly cancels variations of the first derivatives of the metric, so only the metric is required to be fixed at the boundary. Here $\kappa = 8\pi G$, \mathcal{M} is an appropriate portion of space-time and $\partial \mathcal{M}$ its boundary, R the Ricci scalar of the 4-metric q, h the induced metric on $\partial \mathcal{M}$, and K the trace of its extrinsic curvature. Since this term is generally divergent for asymptotically flat solutions the last 'non-dynamical' counter term is required to make the action finite on-shell.

In a more careful treatment though, for asymptotically flat space-times, the action should be such that asymptotically flat solutions are stationary points under all variations preserving asymptotic flatness, not just under variations of compact support. The Einstein-Hilbert action with Gibbons-Hawking term does not satisfy this requirement. Under all asymptotically flat variations, its variation gives a non-vanishing surface term when Einstein's equations are satisfied. Furthermore, the counter term becomes dynamical and since it requires an embedding of $\partial \mathcal{M}$ in a reference background, which is not always guaranteed, its variation is not even well defined [4]. Several proposals or generalizations of this term exist in the literature which aim to correct this problem [6, 4].

In contrast, in the 3+1 decomposition of space-time $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, the stationarity requirement takes a different form [8, 9]. In order to have a well-defined Hamiltonian formulation and compute Poisson brackets of the constraints with various functions on the phase space, the constraints have to be finite, i.e. the integrals have to converge, in all of phase space, and they have to be functionally differentiable. So starting from the ADM action:

$$S_{ADM} = \frac{1}{2\kappa} \int dt \int_{\Sigma} d^3x (\dot{q}_{ab} p^{ab} - NH - N^a D_a), \qquad (2)$$

the Hamiltonian $H[N] = \frac{1}{2\kappa} \int d^3x NH$ and vector constraints $D[N] = \frac{1}{2\kappa} \int d^3x N^a D_a$ are supplemented with boundary terms so that they are finite on the whole of phase space and

$$\delta H[N] = \int d^3x \; \frac{\delta H[N]}{\delta q_{ab}} \delta q_{ab} + \frac{\delta H[N]}{\delta p^{ab}} \delta p^{ab}$$

for all asymptotically flat variations $(\delta q_{ab}, \delta p^{ab})$. And similarly for $D[N^a]$.

Convergence of the integrals depends of course on the fall-off conditions for the lapse Nand shift vector N^a . For a 3+1 decomposition, a cylindrical representation of spatial infinity i^{o} is most convenient so that for cartesian coordinates x^{a} in a neighborhood of i^{o} the fall-off conditions for the canonical pairs are

$$q_{ab}(x^c) = \delta_{ab} + \frac{1}{r} f_{ab}(x^c/r) + o(1/r^2), \qquad p^{ab}(x^c) = \frac{1}{r^2} h^{ab}(x^c/r) + o(1/r^3)$$
(3)

with $f_{ab}(-x^c/r) = f_{ab}(x^c/r)$ even, and $h^{ab}(-x^c/r) = -h^{ab}(x^c/r)$ odd functions on the sphere.

For lapse and shift of order 1/r or odd functions of order 1, H[N] and $D[N^a]$ are already finite and differentiable and generate proper gauge transformations or so called supertranslations at infinity respectively. If one allows N and N^a to asymptote to constant or linear functions then surface terms are needed to subtract divergences and to cancel unwanted variation terms so that

H[N] and $D[N^a]$ are differentiable. Furthermore, with these fall-off conditions, on the constraint surface H[N] and $D[N^a]$ become the generators of asymptotic Poincaré transformations at infinity, and the non-vanishing surface terms are the corresponding charges. Indeed, if (\bar{t}, \bar{y}^a) is a Lorentzian frame at infinity such that Σ_t , the leaves of the foliation, coincide with $\bar{t} = const$, then their normals $n^{\alpha} \to \partial x^{\alpha}/\partial \bar{t}$, and the Hamiltonian evolution vector field goes as

$$t^{\alpha} \to N\left(\frac{\partial x^{\alpha}}{\partial \bar{t}}\right) + N^{a}\left(\frac{\partial x^{\alpha}}{\partial \bar{y}^{a}}\right)$$

So fall-off conditions of N and N^a determine a (10 parameter) Killing vector field of asymptotic space time metric.

We emphasize here the different roles that H[N] and $D[N^a]$ play in the Hamiltonian formulation depending on the asymptotic fall-off conditions of N and N^a : as constraints and generators of gauge transformations in the bulk or as charges and generators of symmetries at infinity.

Also we note that the surface terms for energy and momentum in the 3+1 ADM action (2) can be recovered from those in the covariant action (1) only if one fixes the variations at the boundary [10]. Finally we contrast the finiteness in all of phase space of the corresponding ADM action (2) with surface terms, at least for asymptotically time-translated spatial boundaries, as opposed to finiteness on-shell for (1).

3. Actions for Loop Quantum Gravity

The Holst action with surface term is given by

$$S_{Holst} = -\frac{1}{2\kappa} \Big(\int_{\mathcal{M}} \Sigma^{IJ} \wedge (F_{IJ} + \frac{1}{\gamma} \star F_{IJ}) + \frac{1}{2\kappa} \int_{\partial \mathcal{M}} \Sigma^{IJ} \wedge (\omega_{IJ} + \frac{1}{\gamma} \star \omega_{IJ}).$$
(4)

Where F_{IJ} is the curvature of the spin connection ω_{IJ} , and $\Sigma^{IJ} = \frac{1}{2} \epsilon_{IJKL} e^K \wedge e^L$ is constructed from the co-tetrads. As already mentioned the action is not only finite on-shell but even off-shell for cylindrical temporal cut-offs, and it gives a well-defined covariant Hamiltonian framework leading to asymptotic symmetries and conserved charges identical to the ADM framework. It is natural to ask then what the 3+1 decomposition and temporal gauge fixing of these surface terms is and how they relate to the canonical treatment derived from the ADM action.

The transcription of the framework from the ADM action (2) in its manifestly finite and functionally differentiable form to Ashtekar variables was done in [5]. Boundary conditions are derived from those for the metric (3):

$$E^a_i = \bar{E}^a_i + \frac{F^a_i}{r} + o(1/r^2), \quad A^i_a = \frac{G^i_a}{r^2} + o(1/r^3),$$

with F_i^a and G_a^i even and odd functions on the sphere respectively, and similar conditions for Γ_a^i and K_a^i . Expressed in these variables the constraints become divergent due to the extra gauge degrees of freedom, so additional terms proportional to the Gauss constraint \mathcal{G}_i have to be added to cancel the divergences. The generating functional of the canonical transformation $(q_{ab}, p^{ab}) \rightarrow (A_a^i, E_i^a)$ gives rise to these additional terms in the generators, both in the volume and the surface part:

$$D[N^{a}] = \frac{1}{\kappa\gamma} \int_{\Sigma} d^{3}x \left[N^{a} (F^{i}_{ab} E^{b}_{i} - A^{i}_{a} \mathcal{G}_{i}) + \frac{\epsilon_{abc}}{2 \det E} E^{a}_{i} E^{b}_{j} \mathcal{L}_{\bar{N}} E^{c}_{j} \mathcal{G}_{i} \right] + \frac{2}{\kappa\gamma} \int_{\partial\Sigma} dS_{a} (A^{i}_{b} - \Gamma^{i}_{b}) N^{[b} E^{a]}_{i})$$
(5)

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$$H[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x N \left[\frac{E_i^a E_j^o}{\sqrt{\det E}} [\epsilon_k^{ij} F_{ab}^k - 2(1+\gamma^2) K_{[a}^i K_{b]}^j] + 2D_a(E_i^a \mathcal{G}_i) \right] - \frac{1}{\kappa} \int_{\partial \Sigma} dS_a \left[\frac{N}{\sqrt{\det E}} E_i^a \partial_b E_i^b - D_b(N/\sqrt{\det E}) E_i^b(E_i^a - \bar{E}_i^a) \right].$$
(6)

On the other hand, the 3 + 1 decomposition of (4) with the temporal gauge fixing gives

$$S = \int dt \int_{\Sigma} d^3x \left(\frac{1}{\kappa \gamma} (E^a_i \dot{A}^i_a - \Lambda^i \mathcal{G}_i) - N^a C_a - NC \right) + S_{surface}$$

with the well known constraint functionals

$$\mathcal{G}_{i} = \partial_{a}E_{i}^{a} + \epsilon_{ij}^{k}A_{a}^{j}E_{k}^{a}, \quad C_{a} = \frac{1}{\kappa\gamma}F_{ab}^{i}E_{i}^{b} - \frac{1+\gamma^{2}}{\kappa\gamma^{2}}K_{a}^{i}\mathcal{G}_{i},$$

$$C = \frac{1}{2\kappa\sqrt{\det E}}E_{i}^{a}E_{j}^{b}[\epsilon_{k}^{ij}F_{ab}^{k} - 2(1+\gamma^{2})K_{[a}^{i}K_{b]}^{j}] + \frac{(1+\gamma^{2})}{\kappa\gamma^{2}}\frac{E_{i}^{a}\partial_{a}\mathcal{G}_{i}}{\sqrt{\det E}}$$

$$\tag{7}$$

and the surface terms [11]

$$S_{surface} = \frac{1}{\kappa\gamma} \int dt \int_{\partial\Sigma_t} dS_b \left(\frac{N\gamma}{\sqrt{\det E}} E_i^a E_j^b \epsilon_k^{ij} A_a^k + N^a E_i^b A_a^i\right)$$

$$+ \frac{1}{\kappa\gamma} \int_{\Sigma_1} E_i^a A_a^i - \frac{1}{\kappa\gamma} \int_{\Sigma_2} E_i^a A_a^i$$
(8)

It can be verified then that these additional surface terms proportional to the lapse and shift in (8) are not sufficient to cure the divergences in (7) for the constant and linear fall-off of N and N^a , nor can they be combined with terms proportional to the Gauss constraint in (7) to recover the corresponding energy and momentum charges in (6) and (5).

Thus, for asymptotically flat space-times, finiteness and well-posedness of the variational principle of the Holst action (4) do not directly descend to the canonical action framework derived from its 3+1 decomposition and temporal gauge fixing. The surface terms in (4) lead to the same ADM generators of asymptotic symmetries on the covariant phase space, defined as the space of asymptotically flat solutions. However, for the more stringent requirement of finiteness of the generators on the whole phase space of the canonical framework, additional terms are needed.

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