Gravitational self-force effect on the periastron shift in Schwarzschild spacetime

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Abstract

Recently we developed a time-domain code to calculate the gravitational self-force on a point particle moving around a Schwarzschild black hole. To know how the force affects the particle's motion, it is useful to estimate the self-force correction of some characteristic variables of the orbits (e.g. ISCO frequency, periastron shift). In this work, we focus on the self-force effect on the periastron shift and investigate how to evaluate the correction from the numerical results of the self-force with our time-domain code.

1 Introduction

The problem of the motion of a point particle in black hole spacetime is one of the fundamental issues in general relativity, which recently has been studied well, motivated by the requirement of a bank of gravitational wave templates for the gravitational wave observation. To predict the motion acculately, we need to calculate the *self-force* (or back-reaction force) exerted on the particle and incorporate it to the equation of motion correctly. A breakthrough in the self-force problem has been made by Mino, Sasaki and Tanaka [1] and Quinn and Wald [2], since then a lot of effort to devise a practical method of calculating the self-force based on their works has been done. The "mode-sum scheme" [3] is considered as a promising way to derive the self-force. This scheme is based on multipole decomposition of the retarded field, and relies on standard methods of black hole perturbation theory. This has since been implemented by various authors on a case-by case basis (See [4] for a review of the recent progress in this issue).

At the early stage, the self-force of scalar-field toy model, instead of the gravitational self-force, was mainly investigated and proved that the mode-sum scheme does work well. In extending the analysis from the scalar-field case to the gravitational case, we face the difficulty associated with the gauge dependence of the gravitational self-force. The gravitational perturbation in the vicinity of the point particle, which is required to derive the self-force, is best described using the *Lorenz* gauge, which preserves the local isotropic nature of the point singularity. Therefore, the mode-sum scheme is originally constructed under the Lorenz gauge condition. On the other hand, the field equations that govern the global evolution of the metric perturbation are more tractable in gauges which comply well with the global symmetry of the black hole background, like the Regge-Wheeler gauge [5] for the Schwarzschild geometry or the "radiation" gauges [6] for the Kerr geometry. Now, in calculating the local self-force we need, essentially, to subtract a suitable local, divergent piece of the perturbation from the full (retarded) perturbation field. In doing so, both fields (local and global) must be given in the same gauge; the "gauge problem" arises since the two fields are normally calculated in different gauges.

There are two strategies to settle the problem. One is that we derive the equation of motion in a convenient gauge for calculating the metric perturbation [7, 8]. This idea is based on the work by Detweiler and Whiting [9], in which the motion is depicted as the geodesic of a smooth perturbed spacetime. Another one, which we adopt here, is that we solve the perturbation equations directly in the Lorenz gauge. The calculation is therefore done entirely within the Lorenz gauge, the mode-sum scheme is implemented in a straightforward way. This "all-Lorenz-gauge" strategy is made possible (at least for the Schwarzschild case) following a recent work by Barack and Lousto [10]. They provided a practical formulation of the Lorenz-gauge perturbation equations in Schwarzschild spacetime and demonstrated their formulation is

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suitable for numerical calculation. In our recent works [11, 12], based on their formulation, we developed a code to calculate the gravitational self-force for bound orbits in Schwarzschild geometry.

Our next step is to consider the effect of the gravitational self-force on the particle's orbit. The self-force corrections in some characteristic variables of the orbit are good indicators to estimate the self-force effect and also to compare with the results of other approaches (e.g. post-Newtonian or numerical relativity, and so on). In our previous work [13], as the first example, we reported the self-force-induced shifts in the location and frequency of the inner most circular orbit (ISCO) in Schwarzschild spacetime. The result for the ISCO frequency shift is supported by the recent work on the Effective One Body formalism by Damour [14]. In this work, we focus on the periapsis advance of eccentric orbits, which is one of the characteristic variables, and give the formula of the correction in terms of the components of the self-force.

Throughout this work, we denote the masses of a orbiting point particle and a central Schwarzschild black hole as μ and M, respectively. Also we use standard geometrized units with $c = G = 1$ and metric signature $(-+++)$.

2 Periapsis advance: geodesic case

First, we review the periapsis advance in the geodesic case. The radial component of the geodesic equations in Schwarzschild spacetime is given

$$\left(\frac{dr_p}{d\tau}\right)^2 = R(r_p); \quad R(r) \equiv \mathcal{E}_0^2 - f(r)\left(1 + \frac{\mathcal{L}_0^2}{r^2}\right), \tag{1}$$

where $f(r) = 1 - 2M/r$, τ is the proper time along the orbit, $r_p(\tau)$ is the orbital radius. $\mathcal{E}_0$ and $\mathcal{L}_0$ are the specific energy and angular momentum parameters of the particle, which conserve along the geodesic orbit. An eccentric orbit is bounded in the range of $r_{\rm min} \le r \le r_{\rm max}$, where $r_{\rm min/max}$ satisfy $R(r_{\rm min}) = R(r_{\rm max}) = 0$ and $4M < r_{\rm min} \le r_{\rm max}$. $r_{\rm min}$ and $r_{\rm max}$ correspond to the periastron and apastron radius respectively. We can define a parametrization of eccentric orbits, the (dimensionless) semi-latus rectum, p, and the eccentricity, e, so that

$$p \equiv \frac{2r_{\rm min}r_{\rm max}}{M(r_{\rm min}+r_{\rm max})}, \quad e \equiv \frac{r_{\rm max}-r_{\rm min}}{r_{\rm max}+r_{\rm min}}. \tag{2}$$

With this parametrization, the orbital radius is given by

$$r_p(\chi) = \frac{pM}{1+e\cos\chi}, \tag{3}$$

where χ is a monotonically increasing parameter ("radial phase") along the worldline [15]. By using χ, we reexpress the t and φ components of the geodesic equations as

$$\begin{aligned}
\frac{dt_p}{d\chi} &= \frac{\mathcal{E}_0}{f(r_p)}\left[\mathcal{E}_0^2 - f(r_p)\left(1+\frac{\mathcal{L}_0^2}{r_p^2}\right)\right]^{-1/2}\left(\frac{dr_p}{d\chi}\right) \equiv W_t(r_p;\mathcal{E}_0,\mathcal{L}_0), \\
\frac{d\varphi_p}{d\chi} &= \frac{\mathcal{L}_0}{r_p^2}\left[\mathcal{E}_0^2 - f(r_p)\left(1+\frac{\mathcal{L}_0^2}{r_p^2}\right)\right]^{-1/2}\left(\frac{dr_p}{d\chi}\right) \equiv W_\varphi(r_p;\mathcal{E}_0,\mathcal{L}_0).
\end{aligned} \tag{4}$$

By integrating Eq. (4) over χ, we define the radial period and the increase of the phase for one radial period as

$$T_r \equiv \int_0^{2\pi}\frac{dt_p}{d\chi}d\chi, \quad \Delta\varphi \equiv \int_0^{2\pi}\frac{d\varphi_p}{d\chi}d\chi. \tag{5}$$

Now we can define the periapsis advance

$$\delta_0(p,e) \equiv \Delta\varphi - 2\pi, \tag{6}$$

which represents the fractional difference between two frequencies

$$\Omega_\varphi = \left(1+\frac{\delta_0}{2\pi}\right)\Omega_r,$$

where $\Omega_r \equiv 2\pi/T_r$ and $\Omega_\varphi = \Delta\varphi/T_r$ are the radial and azimuthal frequencies respectively.

3 Self-force correction in periapsis advance

Next, we consider the conservative correction in the periapsis advance caused by the self-force. In the same manner as [16], the conservative pieces of t and φ components of the force are given by

$$F_t^{\rm cons}(\chi) = \frac{1}{2}[F_t(\chi) - F_t(-\chi)], \quad F_\varphi^{\rm cons}(\chi) = \frac{1}{2}[F_\varphi(\chi) - F_\varphi(-\chi)], \tag{7}$$

where we treat F_t and F_φ as functions of χ. From the equations of motion, we find the rates of change of the specific energy and angular momentum

$$\frac{d\mathcal{E}}{d\tau} = -F_t, \quad \frac{d\mathcal{L}}{d\tau} = F_\varphi. \tag{8}$$

Integrating Eq. (8) over χ, we obtain the corrected energy and angular momentum

$$\mathcal{E}(\chi) = \mathcal{E}_0 + \Delta\mathcal{E}_0 + \delta\mathcal{E}(\chi), \quad \mathcal{L}(\chi) = \mathcal{L}_0 + \Delta\mathcal{L}_0 + \delta\mathcal{L}(\chi), \tag{9}$$

where $\Delta\mathcal{E}_0$ and $\Delta\mathcal{L}_0$ represent the conservative shifts in $\mathcal{E}$ and $\mathcal{L}$ at $\chi = 0$, and

$$\delta\mathcal{E}(\chi) = -\int_0^\chi F_t^{\rm cons}(\chi')\left(\frac{d\tau}{d\chi'}\right)d\chi', \quad \delta\mathcal{L}(\chi) = -\int_0^\chi F_\varphi^{\rm cons}(\chi')\left(\frac{d\tau}{d\chi'}\right)d\chi'. \tag{10}$$

All $\Delta\mathcal{E}_0$, $\Delta\mathcal{L}_0$, $\delta\mathcal{E}(\chi)$ and $\delta\mathcal{L}(\chi)$ are in the order of μ. In a similar manner to the geodesic case, we can define the periapsis advance

$$\delta(p,e) \equiv \int_0^{2\pi} W_\varphi(r_p,;\mathcal{E},\mathcal{L})d\chi - 2\pi, \tag{11}$$

and then taking the $O(\mu)$ terms from the above equation gives us the self-force correction as

$$\delta_{SF}(p,e) \equiv \int_0^{2\pi} \delta W_\varphi(r_p,;\mathcal{E}_0,\mathcal{L}_0)d\chi, \tag{12}$$

where

$$\begin{aligned}\delta W_\varphi(r_p,;\mathcal{E}_0,\mathcal{L}_0) &= \left.\frac{\partial W_\varphi}{\partial \mathcal{E}}\right|_0 [\Delta\mathcal{E}_0 + \delta\mathcal{E}(\chi)] + \left.\frac{\partial W_\varphi}{\partial \mathcal{L}}\right|_0 [\Delta\mathcal{L}_0 + \delta\mathcal{L}(\chi)] \\ &= \frac{p(p-3-e^2)^{1/2}[(p-2)^2-4e^2]^{1/2}}{e^2(p-6-2e\cos\chi)^{3/2}}\left[\frac{\mathcal{E}(\pi)}{4\cos^2(\chi/2)} - \frac{\mathcal{E}(\chi)}{\sin^2\chi}\right] \\ &\quad - \frac{p^{-1/2}(p-3-e^2)^{1/2}}{Me^2(p-6-2e\cos\chi)^{3/2}}\left[\frac{(1-e)^2(p-2+2e)\mathcal{L}(\pi)}{4\cos^2(\chi/2)}\right. \\ &\qquad \left. - \frac{[p(1+e^2)-2(1+3e^2)+2e(p-3-e^2)\cos\chi]\mathcal{L}(\chi)}{\sin^2\chi}\right].\end{aligned} \tag{13}$$

Equation (13) may seem singular at $\chi = 0, \pi$, but that is not the case. Local analysis around these points shows that δW_φ is regular, although the direct implementation may require special care near $\chi = 0, \pi$.

4 Summary and discussion

In this work, we considered the self-force effect on the periapsis advance of an eccentric orbit in Schwarzschild spacetime. We gave a formula of the correction induced by the conservative piece of the self-force, which can by calculated numerically by our time-domain code. In practice, however, it is not easy to implement the formula numerically because it contains double integrals of the force and then the numerical accuracy gets worse. One way to improve it is to use integration by parts. Although it makes the formula more complicated, it may reduce the loss of the numerical accuracy. To do so, again, we have to take care on the singular behavior of each term in Eq. (13) at $\chi = 0, \pi$. This reduction is left for future study.

So far we considered only the conservative piece of the self-force, and we assume that the orbital parameters (p, e) are constant. In reality, however, the dissipation also affects the particle's orbit in the secular evolution and the parameters change in time. Even in this case, we can define the periapsis shift as

$$\delta_{SF}^{\rm actual}(t_1) = \int_{t_1}^{t_2} \frac{d\varphi}{dt} dt - \delta_0(p_1, e_1), \tag{14}$$

where t_1 and t_2 are consecutive radial turning points, and (p_1, e_1) are the orbital parameters at $t = t_1$. If the orbit evolve adiabatically, the actual correction of the periapsis shift can be given approximately as the time average of the instantaneous (conservative) correction over the orbital period,

$$\delta_{SF}^{\rm actual}(t_1) \simeq \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \delta_{SF}(p, e) = \delta_{SF}\left(\frac{p_1 + p_2}{2}, \frac{e_1 + e_2}{2}\right) + O(\mu^2). \tag{15}$$

The proof and feasibility of this relation should be surely investigated.

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