THE RELATIVISTIC STRING

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The relativistic string can be considered as a generalization of a free particle. It is well known, that the equations of motion of a free particle arise from a variational principle, taking the length of its path in space-time as the action. Enlarging the dimension of the zero dimensional particle to a one-dimensional curve which sweeps out a two-dimensional worldsheet in space-time, one gets the equations of the relativistic string, if one takes the area of the worldsheet as its action. There are different justifications for treating the relativistic string as a physical object: the dual models can be interpreted as strings; we will not pursue this point of view, however, but refer only to Ref.[2,3,4,6] and the references therein.

Instead, we will consider the relativistic string as a theoretical problem of its own and regain the results of Ref.[1]. Stress will be laid on the geometrical point of view as well as on a detailed treatment of the classical Hamiltonian formalism [5].

The worldsheet of the string is characterized by coordinate functions

$$x^\mu(\sigma, \tau)$$

where the parameters (σ,τ) vary in a domain D, which will be specified later. The parametrization of the string is to be regular, that is the natural tangent vectors

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau} \quad , \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma}$$

have to be linearly independent. Furthermore, we will restrict ourselves to timelike surfaces, i.e. surfaces which cut the lightcone at each point.

Taking the area of the two-dimensional worldsheet as the action we have as Lagrangian

$$\mathcal{L} = -\frac{1}{\pi}\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} \quad (*) \tag{1.1}$$

where the factor $-\frac{1}{\pi}$ has been put in for convenience. In neglecting a dimensional constant in $\mathcal{L}$, we have adopted $\hbar = c = 1$ and a fundamental length. Varying $x^\mu(\sigma,\tau)$ in the action

$$\int_{D} \mathcal{L}\, d\sigma\, d\tau \tag{1.2}$$

we get the Euler-Lagrange equations

$$0 = \frac{\partial}{\partial\tau}\left(\frac{x'^\mu(\dot{x}x') - \dot{x}^\mu(x'^2)}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\dot{x}^\mu(\dot{x}x') - x'^\mu(\dot{x}^2)}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}\right) \tag{1.3}$$

To obtain these equations, we had to integrate by parts. The boundary terms, which arise in this process cannot be neglected as usual, because we deal with a finitely extended string. They have to vanish separately and give rise to the boundary equations

$$0 = \left| m_0\left(\frac{x'^\mu(\dot{x}x') - \dot{x}^\mu(x'^2)}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}\right) + m_1\left(\frac{\dot{x}^\mu(\dot{x}x') - x'^\mu(\dot{x}^2)}{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}\right) \right. \tag{1.4}$$

at boundary of D

Here (m_0, m_1) denotes the outward unit normal vector of the boundary of the parameterdomain D. We realize that adding total derivatives to the Lagrangian would have altered the boundary equations and left invariant the equations (1.3), so total derivatives represent interactions of the boundary of the string.

Multiplying (1.4) with $\dot{x}_\mu$, x'_μ, we get

$$\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}\,\Big|_{\text{at boundary}} = 0$$

that is, as $(m_0, m_1) \neq 0$

$$(\dot{x}x')^2 - \dot{x}^2 x'^2 = 0 \tag{1.5}$$

Writing this in the form

(*) We use the convention $g_{\mu\nu} = \mathrm{diag}\,(-1,+1,+1,\ldots), \mu,\nu = 0,1,\ldots$

$$x'^2 \left(\dot{x}^\mu - \frac{(\dot{x}x')\, x'^\mu}{x'^2} \right)^2 = 0 \tag{1.6}$$

we see, that the boundary condition implies, that the transverse velocity of the boundary is the light velocity. In Euclidean space there is no solution to the boundary condition, a fact which is well known from everyday experience with soap bubbles. Returning now to equation (1.4), we realize, that the denominator vanishes, so we have to check, what additional conditions are implied by (1.4). Making use of (1.5) and the linear independence of $\dot{x}^\mu$, x'^μ we get

$$m_0\, x'^2 - m_1\, (\dot{x}x') \Big|_{\text{at boundary}} = 0 \tag{1.7}$$

as a necessary condition at the boundary. This condition has to be checked for consistency with the equations of motion (1.3), for (1.5) defines, where the boundary is,(1.3) where it will move to, so the direction of the boundary is fixed. One can prove, that (1.5) and (1.7) are sufficient for (1.4).

Let us now turn to the equations of motion (1.3). They do not determine uniquely the functions $x^\mu(\sigma,\tau)$ once the initial values $x^\mu(\sigma,\tau=0)$, $\dot{x}^\mu(\sigma,\tau=0)$ are specified. In fact, they allow an arbitrary reparametrization

$$\tilde{\sigma}(\sigma,\tau) \qquad \tilde{\tau}(\sigma,\tau) \qquad \tilde{x}^\mu(\tilde{\sigma},\tilde{\tau}) = x^\mu(\sigma,\tau)$$

So to get unique solutions of the initial value problem we have to impose additional gauge conditions. We will carefully choose such a parametrization, that the equations of motion can be solved.

Suppose, we are given a solution of (1.3), choose

$$\tilde{\tau} = c \cdot n_\mu x^\mu(\sigma,\tau) \tag{1.8}$$

as new parameter τ with an arbitrary constant vector n_μ, which is subject to

$$n^2 \leq 0$$

In the second step, we impose

$$\dot{x}\, x' = 0 \tag{1.9}$$

This can be done by solving the ordinary system of differential equations

$$\frac{d\tau(\lambda)}{d\lambda} = x'^2(\sigma,\tau) \qquad \frac{d\sigma(\lambda)}{d\lambda} = -\dot{x}x'(\sigma,\tau) \tag{1.10}$$

The solutions are paths $(\sigma(\lambda), \tau(\lambda))$. Now set $\tilde{\sigma}(\sigma,\tau)$ constant on these paths and let $\tilde{\tau} = \tau$. In this new parametrization, you can check, that (1.9) is fulfilled. Notice, that this condition fixes the form of the domain D. (1.7) yields

$$m_0 = 0 \quad , \quad m_1 = 1 \tag{1.11}$$

so the boundary is given by two curves

$$\sigma_{boundary} = \text{const} = \begin{cases} \sigma_{min} \\ \sigma_{max} \end{cases}$$

and we can choose $\sigma_{min} = 0$. The condition (1.9) could be met without spoiling (1.8), so we have

$$n\dot{x} = C \qquad n x' = 0$$

Multiplying the equations of motion (1.3) with n_μ, we get on account of (1.8) and (1.9)

$$\frac{\partial}{\partial\tau}\left(\sqrt{\frac{-x'^2}{\dot{x}^2}}\right) = 0$$

with the general solution

$$\dot{x}^2 + \lambda^2(\sigma)\,x'^2 = 0 \tag{1.12}$$

The first boundary condition (1.5) implies, that in a regular parametrization we must have

$$\lambda(0) = \lambda(\sigma_{max}) = 0 \tag{1.13}$$

We can however perform the singular reparametrization

$$\tilde{\sigma}(\sigma) = \int_0^\sigma \frac{1}{\lambda(\sigma')}\,d\sigma' \tag{1.14}$$

This is an integrable, finite reparametrization, if $\lambda^2(\sigma)$ has simple zeros at 0 and σ_{max}, and normalizes the function $\tilde{\lambda}(\tilde{\sigma})$ in (1.12) to unity, that is instead of (1.12) we have

$$\dot{x}^2 + x'^2 = 0 \tag{1.15}$$

and because of the singular parametrization

$$x'^\mu \Big|_{\text{at boundary}} = 0 \tag{1.16}$$

instead of (1.13). Odd numbers of partial derivatives with respect to σ have to vanish at the boundary. In a last step, we divide τ and σ by $\frac{\sigma_{max}}{\pi}$, thus normalizing the domain D to D = $\{ (\sigma,\tau): 0 \leq \sigma \leq \pi\}$. One can check, that each step was a invertible reparametrization of the timelike worldsheet of the string, so we didn't lose any solution of the equations of motion. (1.3) is now simply the wave equation in two dimensions

$$\ddot{x} - x'' = 0 \tag{1.17}$$

The gauge conditions (1.9) and (1.15) can be written

$$(\dot{x} \pm x')^2 = 0 \tag{1.18}$$

$$n x = c \cdot \tau \tag{1.18a}$$

The most general solution to (1.17) is given by

$$x^\mu(\sigma,\tau) = g^\mu(\tau+\sigma) + h^\mu(\tau-\sigma) \tag{1.19}$$

with arbitrary functions $g^\mu(t)$ and $h^\mu(t)$. Applying the boundary conditions (1.16) at $\sigma = 0$, we get

$$g^\mu(t) = h^\mu(t) \tag{1.19a}$$

and at $\sigma = \pi$ we have as result

$$g^\mu(t+2\pi) = g^\mu(t) + \pi\alpha_0^\mu \tag{1.20}$$

with an arbitrary constant α_0^μ . Making use of the periodicity of g^μ, which resulted from the finiteness of the string, we expand g^μ in discrete modes

$$g^\mu(t) = \frac{1}{2}\left(q^\mu + \alpha_0^\mu t - i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(a_n^{*\mu} e^{int} - a_n^\mu e^{-int}\right)\right) \tag{1.21}$$

with real constants q^μ, α_0^μ and complex a_n^μ. Notice, that $g^\mu(t)$ has a simple geometric interpretation. The boundary curve $x^\mu(\sigma = 0,\tau)$ is just

$$x^\mu(\sigma = 0, \tau) = 2\, g^\mu(\tau)$$

and the world sheet is the mean value of positions of its boundary

$$x^\mu(\sigma,\tau) = \frac{1}{2}\left(x^\mu(0,\tau+\sigma) + x^\mu(0,\tau-\sigma)\right) \tag{1.22}$$

Inserting the expansion (1.21) of $g^\mu(t)$, we get

$$x^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau - i\sum_{n=1}^{\infty} \frac{\cos n\sigma}{\sqrt{n}}\left(a_n^{*\mu} e^{in\tau} - a_n^\mu e^{-in\tau}\right) \tag{1.23}$$

The constants α_0^μ, a_n^μ are subject to the gauge conditions (1.18), or equivalently the condition, that the boundary curve is lightlike; this equivalence (in this gauge) is seen from

$$\begin{aligned} 0 = (\dot{x} \pm x')^2(\sigma,\tau) &= \left(2g'(\tau+\sigma) + 2g'(\tau-\sigma) \pm 2\left(g'(\tau+\sigma) - g'(\tau-\sigma)\right)\right)^2 = \\ &= 4\left(g'(\tau\pm\sigma)\right)^2 = \dot{x}^2(0,\tau\pm\sigma) \end{aligned} \tag{1.24}$$

So, using the expansion (1.21) and the more convenient coefficients

$$\alpha_n^\mu = \sqrt{n}\, a_n^\mu \quad , \quad \alpha_{-n}^\mu = (\alpha_n^\mu)^* \tag{1.25}$$

we get

$$\begin{aligned} \dot{x}^2(0,\tau) &= \left(\sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-in\tau}\right)^2 = \\ &= \sum_{n=-\infty}^{+\infty} e^{-in\tau}\left(\sum_{m=-\infty}^{+\infty} \alpha_{n-m}^\mu \alpha_{\mu m}\right) \end{aligned} \tag{1.26}$$

The gauge conditions thus imply

$$0 = L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}^\mu \alpha_{\mu m} \tag{1.27}$$

$$\overset{n>0}{=} \sum_{m=1}^{\infty}\left(\sqrt{m(m+n)}\, a_m^{*\mu} a_{\mu\, m+n} + \sqrt{n}\, \alpha_0^\mu a_{\mu m} + \frac{1}{2}\sum_{m=1}^{n-1} \sqrt{(n-m)m}\, a_{n-m}^\mu a_{\mu m}\right)$$

For n=0 we have

$$0 = \frac{1}{2}\alpha_0^\mu \alpha_{\mu 0} + \sum_{n=1}^{\infty} n\, a_n^{*\mu} a_{\mu n} \tag{1.28}$$

and n < 0 follows from

$$L_{-n} = L_n^* \tag{1.29}$$

Exploiting the Poincare invariance of the Lagrangian (1.1) we get the conserved momentum and angular momentum of the string

$$\mathbb{P}^\mu = \int\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} d\sigma + \frac{\partial \mathcal{L}}{\partial x'^\mu} d\tau\right)$$

$$M^{\mu\nu} = \int\left(x^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\nu} d\sigma + x^\mu \frac{\partial \mathcal{L}}{\partial x'^\nu} d\tau\right) - \mu \leftrightarrow \nu \tag{1.30}$$

where the integral is taken along some arbitrary curve, which intersects the domain D. P^μ and $M^{\mu\nu}$ are conserved because of the equations of motion (1.3) and the boundary conditions (1.4). Choosing as path τ = const for convenience and inserting the expansion (1.23), we get

$$\mathbb{P}^\mu = \alpha_0^\mu$$

$$M^{\mu\nu} = \left(q^\mu \alpha_0^\nu - q^\nu \alpha_0^\mu\right) - i \sum_{n=1}^{\infty} \left(a_n^{*\mu} a_n^\nu - a_n^{*\nu} a_n^\mu\right) \tag{1.31}$$

Identifying α_0^μ with the momentum of the string and $a_n^{*\mu}$ with its excitations, we see that (1.28) is a spectrum condition, fixing the mass of the string in terms of its excitations.

To quantize the string, we have to go through a Hamiltonian formalism, and then substitute Poisson brackets by commutator, interpreting the dynamical variables as operators acting on the Hilbert space of states. Computing

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{x}^\mu - \mathcal{L} \tag{2.1}$$

we see, that $\mathcal{H}$ vanishes identically, furthermore we have the identities

$$p_\mu x'^\mu = 0 \qquad\qquad p^2 + \left(\frac{x'}{\pi}\right)^2 = 0 \tag{2.2}$$

where

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{2.3}$$

So we see, that the phase space (p_μ, x^μ) is constrained and the usual Hamiltonian formalism cannot be applied. So we will gc through a generalized Hamiltonian formalism [5], and apply it to the string.

Let us be given a Lagrangian $\mathcal{L}(Q^A, Q^A_{,\mu})$ A = 1...N of fields Q^A (which for convenience are to be functions of space time coordinates x^μ and not of some parameters (σ,τ) as we will have to deal with) and its derivatives $Q^A_{,\mu} = \frac{\partial Q^A}{\partial x^\mu}$. The canonically conjugated momenta are

$$p_A = \frac{\partial \mathcal{L}}{\partial Q^A_{,0}} \tag{2.4}$$

The first step to Hamiltonian dynamics is to inverte this definition

$$Q^A_{,0} = Q^A_{,0}\left(p_A, Q^A, Q^A_{,\alpha}\right) \qquad \alpha \neq 0 \tag{2.5}$$

and substitute the time derivatives of the fields by these functions of the momenta, the fields and their space derivatives. What is the condition, for the inversion (2.5) to be possible?
By the implicit function theorem, the matrix

$$M_{AB} = \frac{\partial p_A}{\partial Q^B_{,0}} = \frac{\partial^2 \mathcal{L}}{\partial Q^A_{,0} \partial Q^B_{,0}} \tag{2.6}$$

has to be of maximal rank. If it is not, the Euler-Lagrange equations do not (in general) determine uniquely the initial value problem. This can be seen by writing them as

$$M_{AB}\, Q^B_{,00} + f_A = 0 \tag{2.7}$$

where f_A is a function of time derivatives of the fields up to first order at most. Now if M_{AB} is not of maximal rank, that is, if there exist vectors n^A_r with

$$n^A_r\, M_{AB} = 0 \tag{2.8}$$

then one cannot solve (2.7) for the second time derivatives. So given a set of initial values

$$\begin{aligned} Q^A(x^0 = 0, x^\alpha) &= u^A(x^\alpha) \\ Q^A_{,0}(x^0 = 0, x^\alpha) &= v^A(x^\alpha) \end{aligned} \tag{2.9}$$

the second time derivatives $Q^A_{,00}$ are determined only up to arbitrary combinations of n^A_r, if (2.7) admits a solution at all, more precisely the initial values are constrained by first order equations

$$n_r^A f_r^A = 0 \tag{2.10}$$

It can be, that differentiating (2.10) gives new second order equations, independent from (2.7), thus removing some arbitrariness of the initial value problem.

On the other hand, if we have a gauge symmetry, then the solutions are determined only up to some arbitrary functions and the initial value problem is necessarily underdetermined, then we will not be able, to solve (2.4) for the time derivatives of the fields, because (2.6) is not of maximal rank. As a consequence, the phase space will be constrained by identities

$$\varphi_r\left(Q^A, Q^A_{,\alpha}, p_A\right)(x) = 0 \qquad \begin{array}{l} \alpha \neq 0 \\ A = 1 \ldots N \\ r = 1 \ldots M \end{array} \tag{2.11}$$

the number M of independent constraints being the deviation of the rank of (2.6) from its maximal value

$$\text{rank}\left(\frac{\partial p_A}{\partial Q^B_{,0}}\right) + M = N \tag{2.12}$$

For convenience, we choose a functional parametrization of the constraints

$$\phi(\lambda) = \int d^3x\ \lambda^r(x)\, \varphi_r\left(Q^A, Q^A_{,\alpha}, p_A\right) = 0 \tag{2.13}$$

with arbitrary functions $\lambda^r(x)$.

Variations of Q^A, $Q^A_{,0}$ give rise to variations of Q^A, p_A which preserve the constraints, they are thus subject to

$$0 = \int dx \left(\frac{\delta\phi(\lambda)}{\delta p_A(x)}\, \delta p_A(x) + \frac{\delta\phi(\lambda)}{\delta Q^A(x)}\, \delta Q^A(x)\right) \tag{2.14}$$

The variational derivative of a functional

$$F[Q^A] = \int dx\ \mathcal{F}(Q^A, Q^A_{,\alpha})$$

is given by

$$\frac{\delta F}{\delta Q^A(x)} = \left(\frac{\partial \mathcal{F}}{\partial Q^A} - \partial_\alpha \frac{\partial \mathcal{F}}{\partial Q^A_{,\alpha}}\right)(x) \tag{2.15}$$

(α is summed over space indices only)

Though we cannot substitute $Q^A_{,0}$ by a function of (Q^A, p_A), we can express H as a function of (Q^A, p_A), because variations of $Q^A_{,0}$ leave H invariant.

$$\delta H = \int \left(Q^A_{,0}\, \delta p_A + p_A\, \delta Q^A_{,0} - \frac{\delta L}{\delta Q^A}\, \delta Q^A - \frac{\partial \mathcal{L}}{\partial Q^A_{,0}}\, \delta Q^A_{,0} \right) d^4x \qquad (2.16)$$

The second and the fourth term cancel because of (2.4).

Varying Q^A and p_A we get from the variational principle

$$0 = \delta \int (p_A Q^A_{,0} - \mathcal{H})\, d^4x = \int d^4x \left(Q^A_{,0} - \frac{\delta H}{\delta p_A} \right) \delta p_A - \left(p_{A,0} - \frac{\delta H}{\delta Q^A} \right) \delta Q^A \qquad (2.17)$$

As the variations δp_A and δQ^A are only subject to (2.14), we get the following Hamiltonian equations

$$\left. \begin{aligned} Q^A_{,0} &= \frac{\delta H}{\delta p_A} + \frac{\delta \phi(\lambda)}{\delta p_A} \\ - p_{A,0} &= \frac{\delta H}{\delta Q^A} + \frac{\delta \phi(\lambda)}{\delta Q^A} \end{aligned} \right\} \quad \text{for some } \lambda \qquad (2.18)$$

Defining Poisson brackets

$$\{A, B\} = \int d^3x\, \frac{\delta A}{\delta Q^C(x)}\, \frac{\delta B}{\delta p_C(x)} \quad - \quad A \leftrightarrow B \qquad (2.19)$$

(2.18) can be cast into the form

$$\dot{G} = \{ G\, ,\ H + \phi(\lambda) \} \qquad (2.20)$$

for an arbitrary functional $G \left[Q^A, p_A \right]$.

These equations of motion determine uniquely the development of the system, once the initial values have been specified. The constraints (2.13) therefore cannot be imposed as additional conditions but can be imposed at the initial time only. It has then to be checked, whether the evolution of the system, governed by $H + \Phi(\lambda)$ preserves the constraints. So for (2.13) to be valid, we must have

$$\dot{\phi}(\eta) = \{ \phi(\eta)\, ,\ H + \phi(\lambda) \} = 0 \qquad (2.21)$$

for any η and some fixed λ. It is sufficient that (2.21) vanishes on account of (2.13), that is

$$\{ \phi(\eta)\, ,\ H + \phi(\lambda) \} = \phi(\kappa(\eta, \lambda)) \qquad (2.21a)$$

It can be, that there is no λ, which fulfils (2.21) and that the Poisson brackets yield independent secondary constraints $\chi(\kappa)$ on the dynamical variables. Then these secondary constraints have to be imposed on the initial values of the system and its time derivatives have to vanish.

$$\dot{\chi}(\kappa) = \{\chi(\kappa), H + \phi(\lambda)\} = 0 \tag{2.22}$$

New constraints can arise, and we have to repeat the process. Let us suppose, that after a finite number of such steps, the equations (2.21), (2.22) are fulfilled, then we can look upon them as equations for the yet undetermined function λ. The equations form a linear inhomogeneous system of equations for λ, the most general solution of which is

$$\lambda = \bar{\lambda} + c^r \lambda_r \tag{2.23}$$

where $\bar{\lambda}$ is a particular solution, λ_r are solutions of the homogeneous equation (without H) and c^r are totally arbitrary functions, exhibiting the gauge symmetry of the system.

Let us apply this formalism to the string, where the coordinates x^μ take the role of fields and the parameters (σ,τ) correspond to the space-time coordinates. As $\mathcal{H}$ vanishes, the motion is generated by the constraint functions $\varphi_\pm$ alone

$$\varphi_\pm = (\pi p \pm x')^2 = 0 \tag{2.24}$$

As Hamiltonian, we can choose

$$\mathcal{H}(\lambda^+, \lambda^-) = \frac{\lambda^+}{2}\left(p + \frac{x'}{\pi}\right)^2 + \frac{\lambda^-}{2}\left(p - \frac{x'}{\pi}\right)^2 \tag{2.25}$$

The equations of motion are (corresponding to 2.18)

$$\begin{aligned} \dot{x}^\mu &= (\lambda^+ + \lambda^-) p^\mu + (\lambda^+ - \lambda^-) \frac{x'^\mu}{\pi} \\ \pi \dot{p}^\mu &= \frac{\partial}{\partial \sigma}\left[(\lambda^+ + \lambda^-)\frac{x'^\mu}{\pi} + (\lambda^+ - \lambda^-) p^\mu\right] \end{aligned} \tag{2.26}$$

and as we integrated by parts and have finite boundaries, we get the boundary conditions

$$\pi m_0 p^\mu = m_1 \left[(\lambda^+ + \lambda^-) \frac{x'^\mu}{\pi} + (\lambda^+ - \lambda^-) p^\mu \right] \tag{2.27}$$

The phase space is constrained by

$$M^\pm(\lambda) = \pm \int_0^\pi d\sigma \frac{\pi}{4} \lambda(\sigma) \left(p \pm \frac{x'}{\pi} \right)^2 = 0 \tag{2.28}$$

The constraints obey the algebra

$$\{ M^i(\ell), M^j(k) \} = \delta^{ij} M^i(\ell k' - \ell' k) \quad , \quad i,j = +,- \tag{2.29}$$

that is, they form a closed algebra, no secondary constraints arise. Defining

$$\ell(\sigma) = \begin{cases} \frac{\pi}{4} \left(p + \frac{x'}{\pi} \right)^2 (\sigma) & \sigma \geqslant 0 \\ \frac{\pi}{4} \left(p - \frac{x'}{\pi} \right)^2 (\sigma) & \sigma \leqslant 0 \end{cases} \tag{2.30}$$

and

$$L_n = \int_{-\pi}^{+\pi} d\sigma \, e^{in\sigma} \ell(\sigma) = M^+(e^{in\sigma}) - M^-(e^{-in\sigma}) \tag{2.31}$$

we get a discrete complete set of constraints. Their algebra is particularly simple

$$\{ L_n, L_m \} = i(m-n) L_{n+m} \tag{2.32}$$

Making use of the constraints (2.2), one can cast the boundary condition into the form

$$\left. \begin{aligned} &\lambda^+ + \lambda^- = 0 \\ &\pi m_0 = (\lambda^+ - \lambda^-) m_1 \end{aligned} \right\} \text{at boundary} \tag{2.33}$$

Nevertheless, we will choose

$$\lambda^+ = \lambda^- = \frac{\pi}{2} \tag{2.34}$$

which corresponds to the singular parametrization, we dealt with above. We then have

$$H = \frac{\pi}{2} \int_0^\pi d\sigma \left(p^2 + \frac{x'^2}{\pi^2} \right) = L_0$$
$$\dot{x}^\mu = \pi p^\mu \qquad , \qquad \pi \dot{p}^\mu = x^{\mu\,\prime\prime} \tag{2.35}$$

or

$$\ddot{x} - x'' = 0 \tag{2.36}$$

The constraints give

$$0 = (\pi p \pm x')^2 = (\dot{x} \pm x')^2 \tag{2.37}$$

that is we get back the equations (1.16,1.17,1.18). They are invariant however under conformal transformations of the two-dimensional parameter space. This is, as one knows from the theory of functions e.g., a gauge group restricted by the "Cauchy Riemann differential equations" (with a different minus sign on account of the different metric). To be specific, we can allow for new parameters

$$\tilde{\sigma}(\sigma,\tau) \qquad , \qquad \tilde{\tau}(\sigma,\tau)$$

subject to

$$\frac{\partial \tilde{\sigma}}{\partial \sigma} = \frac{\partial \tilde{\tau}}{\partial \tau} \quad , \quad \frac{\partial \tilde{\sigma}}{\partial \tau} = \frac{\partial \tilde{\tau}}{\partial \sigma} \quad , \quad \left.\frac{\partial \tilde{\sigma}}{\partial \tau}\right|_{\text{at boundary}} = 0 \tag{2.38}$$

or

$$\ddot{\tau} - \tau'' = 0 \qquad , \qquad \tau'\big|_{\sigma = 0,\pi} = 0 \tag{2.39}$$

So we can choose

$$n_\mu x^\mu = c \cdot \tau \tag{2.40}$$

As $n_\mu x^\mu$ fulfils (2.39), n_μ is a constant vector with $n^2 \leq 0$ as above and the constant c turns out to be

$$c = n_\mu \alpha_0^\mu \tag{2.41}$$

the momentum in the direction of n. So we have reproduced all the equations of the Lagrangian treatment of the string, we have as the most general solution (1.23) and the constraints (1.27) and (2.31), defined in a different way, turn out to coincide, justifying the notation.

Using the expansion (1.23) and $p^{\mu} = \frac{\dot{x}^{\mu}}{\pi}$, we can solve for the modes.

$$\alpha_0^{\mu} = \int_0^{\pi} p^{\mu} d\sigma$$

$$q^{\mu} = \frac{1}{\pi}\int_0^{\pi} (x^{\mu} - \pi\tau p^{\mu}) d\sigma \tag{2.42}$$

$$\alpha^{*\mu}_{n} = \int_0^{\pi} e^{-in\tau} \cos n\sigma \left(\frac{p^{\mu}}{\sqrt{n}} + i\sqrt{n}\, x^{\mu} \right) d\sigma$$

From the definition of Poisson brackets

$$\{A, B\} = \int_0^{\pi} d\sigma \frac{\delta A}{\delta x^{\mu}(\sigma)} \frac{\delta B}{\delta p_{\mu}(\sigma)} \quad - \quad A \leftrightarrow B \tag{2.43}$$

we get

$$\{q^{\mu}, \alpha_0^{\nu}\} = g^{\mu\nu}$$

$$\{a^{*\mu}_{n}, a^{\nu}_{m}\} = i g^{\mu\nu} \delta_{n,m} \tag{2.44}$$

The other Poisson brackets of the modes vanish. The algebra of the constraints has been given already (2.32).

Quantization can proceed now along well known lines. Regard the dynamical variables x^{μ}, p_{μ} as operators acting on the Hilbert space of states and let the commutators be given by i times the Poisson bracket of the classical theory. The algebra of the modes becomes the algebra of annihilation-creation operators

$$[a^{\mu}_{m}, a^{\nu +}_{n}] = \delta_{m,n} g^{\mu\nu} \qquad n, m > 0$$

$$[q^{\mu}, \alpha_0^{\nu}] = i g^{\mu\nu} \tag{3.1}$$

It is also convenient to give the algebra of the α-operators (1.25)

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = n\, \delta_{m+n,0}\, g^{\mu\nu}$$

$$[q^{\mu}, \alpha^{\nu}_{n}] = i \delta_{n,0}\, g^{\mu\nu} \tag{3.1a}$$

Given the ground state, we can construct a basis for the Fock space

$$|\lambda, k\rangle = \prod_{i,\mu} \frac{1}{\sqrt{\lambda_{i,\mu}!}} \left(a^{+\mu}_{i}\right)^{\lambda_{i,\mu}} e^{ik_\mu q^\mu} |0\rangle \tag{3.2}$$

where the ground state has the properties

$$\alpha^\mu_m |0\rangle = 0 \tag{3.3}$$

The state (3.2) has momentum k_μ, its excitation is given by the multi-index $\lambda = (..\lambda_{\mu,i}...)$. The ordering of the creation-annihilation operators in those operators, containing $a^{+\mu}_m$ and a^ν_n in higher than first order, is either fixed by hermiticity (1.31) or is irrelevant (1.27) or gives rise to a yet undetermined c-number term. So we will define L_o as the normal ordered operator corresponding to (1.28). Because of the non-arbitrariness in the ordering of L_o, the algebra of the L_n operators is changed by an important dimension* dependent c-number. The number can be computed most easily be taking the groundstate expectation value of the L_n commutator. We have

$$\langle 0|[L_n, L_{-n}]|0\rangle \overset{n>0}{=} \langle 0| L_n L_{-n} |0\rangle =$$

$$= \frac{1}{4}\langle 0|\left\{\sum_{\mu=0}^{D-1}\sum_{m=1}^{n-1}\sqrt{m(n-m)}\, a^\mu_m a_{\mu\, n-m}\right\}\left\{\text{the same}\right\}^\dagger|0\rangle = \frac{D}{12}(n^3-n)$$

So the constraints - or gauge operators obey the algebra (cf. 2.32)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\,\delta_{m,-n} \tag{3.4}$$

Because of these commutation relations, we cannot simply have $L_n = 0$ as constraint condition, but can only impose

$$\langle \psi| L_n + \delta_{n,0}\alpha |\varphi\rangle = 0 \tag{3.5}$$

thus singling out a subspace of the Hilbert space. (Here we allowed for the c-number term, arising from the arbitrariness of the ordering of the classical L_o.) Making use of $L_{-n} = L^+_n$ we get the sufficient condition for (3.5) to be valid as

$$(L_n + \delta_{n,0}\alpha)\, |\varphi\rangle = 0 \tag{3.6}$$

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* Notice, that we didn't specify the dimension of space time up to now, so we can allow the index μ to vary from 0 to (D-1).

The set of all solutions to (3.6) will be denoted as the physical space, the states $|\psi>$ as the physical states. One now has to show, that no negative norm states appear as solutions to (3.6), so that the gauge condition eliminates the negative norm states, generated by a_n^{+o}. Before we construct the physical states, some elementary remarks. We cannot expect to construct all solutions to (3.6) at once but want to make use of an inductive process rather. We notice, that we can diagonalize the level operator

$$R = \sum_{n=1}^{\infty} n\, a_n^{+\mu} a_{\mu n} \tag{3.7}$$

in the space of solutions to (3.6). This follows from the commutation relations

$$[R, L_m] = -m L_m \tag{3.8}$$

The eigenvalues of R, the level number M, are positive integers, the eigenspace at level number M will be denoted by R^M. The construction of solutions to (3.6) will make use of complete induction with respect to the level number.

To get rid of the continuum of momenta of the states, we would like to consider states only with one fixed momentum. The condition (3.6) for n = 0 however fixes the mass of the physical states

$$(L_0 + \alpha)\,|\varphi> = \left(\tfrac{1}{2}p^2 + R + \alpha\right)|\varphi> = 0 \tag{3.9}$$

so we cannot have one fixed momentum for physical states at different levels. (3.9 will be referred to as the mass-shell condition). So we will suppose the states, we are dealing with, as having the momentum

$$(p^\mu)_M = (p^\mu)_0 + M k^\mu \tag{3.10}$$

where $(p^\mu)_M$ is the momentum at level M, and k is restricted by

$$k^2 = 0\ ,\quad k^0 > 0\ ,\quad (p^\mu)_0 k_\mu = -1\ ,\quad (p)_0^2 = -2\alpha \tag{3.11}$$

We then have

$$(p_M)^2 = -2(\alpha + M)\quad ,\quad k_\mu (p^\mu)_M = -1 \tag{3.11a}$$

that is, (3.9) is fulfilled.
Before proceeding, we will introduce some notations:

The operators K_n will be of importance, they are defined by

$$K_n = k_\mu \alpha_n^\mu \quad , \quad K_{-n} = K_n^\dagger \tag{3.12}$$

and obey the commutation relations

$$[K_n, K_m] = 0 \, , \, [L_n, K_m] = -m K_{n+m} \, , \, [R, K_m] = -m K_m \tag{3.13}$$

The transverse states $|t\rangle$ are the solutions of

$$K_n |t\rangle = L_n |t\rangle = 0 \qquad n > 0 \tag{3.14}$$

The space spanned by the transverse states at level M, $|t, M, \nu\rangle$ *) will be denoted by T^M. The following operator product, applied to transverse states will be important

$$L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} |t, M, \nu\rangle = \{\lambda, \mu\}_{M'} |t, M, \nu\rangle \tag{3.15}$$

where $\{\lambda, \mu\}$ denotes the multi-index $\lambda_1 \ldots \lambda_n \, \mu_1 \ldots \mu_m$ and

$$M' = \sum_{r=1}^{n} r \lambda_r + \sum_{s=1}^{m} s \mu_s \tag{3.16}$$

Using the commutation relations (3.13) and (3.8), one easily verifies, that the level number of $\{\lambda, \mu\}_{M'} |t, M, \nu\rangle$ is M'+M.

The importance of the states $|t, M, \nu\rangle$, which form a subspace of the physical space becomes transparent by the following lemma:
If $|t, M, \nu\rangle$ is a basis for the transverse space T^M at level M, then the states

$$\{\lambda, \mu\}_{N-M} |t, M, \nu\rangle \tag{3.17}$$

give a basis for the states at level number N, and as N varies, for the whole Hilbert space.

Using this lemma, we can write an arbitrary physical state as a linear combination of

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*) (ν denotes a degeneration parameter, M the level number)

$$|\psi\rangle = K_{-1}^{\mu_1} \dots K_{-n}^{\mu_n} |t\rangle + L_{-n} |s\rangle$$

and making use repeatedly of the commutation relations (3.4) and (3.13) we will show, that if the dimension D is 26 and $\alpha = -1$ then $|\psi\rangle$ is a linear combination of the form

$$|\psi\rangle = |t\rangle + L_{-n} |s\rangle \tag{3.18}$$

where $L_{-n}|s\rangle$ is physical and null,null meaning, that its scalar product with any physical state, including itself, vanishes. Then, the physical space is positive semidefinite. The choice of a specific vector k_μ turns out to be irrelevant, because, as $|t\rangle$ is a physical vector, it can be written as a sum of a transverse vector, constructed with a different k'_μ, and a null vector. The condition

$$K_n |\psi\rangle = 0 \qquad\qquad n > 0 \tag{3.19}$$

which corresponds to (2.30, 2.41) thus does not spoil Lorentz-covariance, but fixes only the null vector part of the physical states. It cannot be imposed however if $D < 26$, to fix a null vector part of the physical states.

Let us prove now the lemma. We show first, that the states

$$\{\lambda,\mu\}_{N-M} |t, M, \nu\rangle \qquad\qquad N-M > 0 \tag{3.20}$$

are linearly independent for fixed M and ν. Let us be given a linear combination $\sum_{\lambda,\mu} c[\lambda,\mu] \{\lambda,\mu\}_{N-M} |t, M, \nu\rangle = |f\rangle$ then for $|f\rangle$ to vanish, the terms with different excitations have to vanish separately. Consider the $a_1^{\dagger\mu}$ oscillators :

$\{\lambda\ ,\mu\}_{N-M}$ contributes terms of the form (cf. 1.27, 3.12)

$$(p a_1^\dagger)^{\lambda_1} (a_1^\dagger a_1^\dagger)^{\lambda_2} (a_1^\dagger a_2^\dagger)^{\lambda_3} \dots (a_1^\dagger a_{n-1}^\dagger)^{\lambda_n} (k \cdot a_1^\dagger)^{\mu_1}$$

Consider the terms in $|f\rangle$, which maximize the number of $a_1^{\dagger\mu}$-oscillators; these terms have to cancel. But obviously, they can do so only if all these terms have the same $\lambda_1, \lambda_2, \dots \lambda_n, \mu_1$. But then, for cancellation of the remaining excitations resulting from $K_{-s}^{\mu_s}$ to be possible, we also have $\mu_2 \dots \mu_n$ equal in all terms, so there is only one term, which has to vanish itself. So we conclude that for $|f\rangle$ to vanish, the coefficients $c[\lambda,\mu]$ have to vanish and the linear independence of (3.20) is shown.

We now prove that the states $|f>$ do not contain transverse states. For this purpose, we first define an order of the multi-indices. We say $(\lambda_i) < (\lambda'_i)$ if $\lambda_1 = \lambda'_1, \dots \lambda_{m-1} = \lambda'_{m-1}$, $\lambda_m < \lambda'_m$, the same definition applying to (μ_i). Now assume $|f>$ is transverse. Take those terms with smallest (μ_i) and choose from these the one with largest (λ_i). Suppose $(\lambda_i) \neq 0$ and especially $\lambda_1, \dots \lambda_{j-1} = 0, \lambda_j \neq 0$.
Apply K_j to $|f>$. Using the commutation relation (3.13), one sees, that (μ_i) is increased and λ decreased. Because of $K_o|t,M,\nu> = -|t,M,\nu>$ (cf. 3.11a, 3.12), the only terms where (μ_i) is not increased have the structure $L_{-j}^{\lambda_j-1} \dots |t>$, they cannot cancel with other terms, arising in the commutation process, which have greater (μ_i), so for the smallest (μ_i) terms, the greatest (λ_i) is 0. Applying our consideration to the next larger (μ_i) and noting, that K_j annihilates the terms with smaller (μ_i) (no L_{-j} are present there), we get by induction, that all λ_j have to be zero. Now apply L_n to the vector, where $\lambda_j = 0$. Commuting L_n through K_{-i} operators either increases (μ_i) or gives rise to terms with $K_j, j > 0$, which are annihilated, or terms with K_o, which have a smaller (μ_i). Take the term with smallest non-zero (μ_i), let j be the minimal value, for which $\mu_j \neq 0$ and apply L_j. The smallest term which originated in commuting L_j to the right is $\mu_j-1, \dots$, it cannot be cancelled by other terms. So there is no smallest non-zero term $\{0, \mu_i\}|t>$ in the vector $|f>$. It follows then, that the states (3.20) do not contain transverse states.
We now show, that the states (3.20) and the transverse states form a basis. Let G^N denote the space spanned by the states (3.20). T^N is orthogonal to it by definition (3.14) and because of $K_n = K_n^\dagger$, $L_{-n} = L_n^\dagger$. We prove by induction, that T^N is equal to the orthogonal complement of G^N (with respect to R^N), and as we know, that G^N and $T^N = (G^N)^\perp$ are disjoint, G^N and T^N span the whole R^N. It is easy to show the necessary conditions for N = 1. Let us be given, that up to N-1, $T^M \oplus G^M$ form a basis for R^M. Let $|\psi>$ be from $(G^N)^\perp$, then it fulfils

$$
\begin{aligned}
&\langle\psi| L_{-1}\{\lambda,\mu\}_{N-M-1} |t, M, \nu\rangle = 0 \\
&\langle\psi| L_{-2}\{\lambda,\mu\}_{N-M-2} |t, M, \nu\rangle = 0 \qquad (3.21) \\
&\langle\psi| K_{-1}\{\lambda,\mu\}_{N-M-1} |t, M, \nu\rangle = 0
\end{aligned}
$$

By induction hypothesis $\{\lambda,\mu\}_{N-M-1}|t,M,\nu>$ and $\{\lambda,\mu\}_{N-M-2}|t,M\,\nu>$ form a basis (N-M-1 and N-M-2 may be zero) for R^{N-1}, R^{N-2}.
As all scalar products of the vectors $L_1|\psi>$, $L_2\,|\psi>$, $K_1|\psi>$ with a basis vanish, the states are zero themselves. From the commutation relations (3.4),(3.13) it then follows, that

$$L_n|\psi> = K_n|\psi> = 0$$

and $|\psi>$ is a transverse state. Moreover, we can easily show now, that T^N is positive definite: it is positive semidefinite because $K_n|t> = 0$ excludes timelike excitations, it is positive definite, because it is disjoint with its orthogonal complement G^N. Having established now the lemma it is easy to proceed to the main goal of determining the physical space. Any vector can be written as a linear combination of vectors $|t,M,\nu>$, $K_{-1}^{\mu_1} \dots K_{-n}^{\mu_n}\;|t,M,\nu>$, $L_{-1}|a>$, $\tilde{L}_{-2}|b>$ because of the generating algebra of the L_n (3.4). $\tilde{L}_{-2}$ is defined by

$$\tilde{L}_2 = L_2 + \frac{3}{2}L_1^2 \qquad\qquad \tilde{L}_{-2} = \tilde{L}_2^{\dagger} \tag{3.22}$$

We only have to check

$$L_1|\psi> = \tilde{L}_2\,|\psi> = 0 \tag{3.23}$$

for $|\psi>$ to be physical. Making use of the commutation relations (3.13) and especially

$$[L_1, L_{-1}] = 2L_0 \quad , [L_1, \tilde{L}_{-2}] = 6L_{-1}(L_0+1)$$

$$[\tilde{L}_2, L_{-1}] = 6L_1L_0, \tag{3.24}$$

$$[\tilde{L}_2, \tilde{L}_{-2}] = 13\left(L_0+\frac{D}{26}\right) + 18\left(L_{-1}L_1+L_0\right)\left(L_0+1\right)$$

we see that if D = 26 and the groundstate mass squared $m^2 = 2\alpha$ is $m^2 = -2$ ($\alpha = -1$), then the states $L_{-1}|a> + \tilde{L}_{-2}|b>$ are mapped on states $L_{-1}|a'> + \tilde{L}_{-2}\,|b'>$ by L_1 and $\tilde{L}_2$. So for $|\psi>$ to be physical, the states $K_{-1}^{\mu_1} \dots K_{-m}^{\mu_m}\,|t>$ and $L_{-1}|a> + \tilde{L}_{-2}|\,b>$ have to vanish separately on applying L_1 and $\tilde{L}_2$ to $|\psi>$. Applying the same considerations as in the proof of the lemma, we get, that for $L_nK_{-1}^{\mu_1}\dots K_{-m}^{\mu_m}\,|t>$ to be zero all powers of K_{-j} have to vanish. So we get the result, which we indicated earlier: If the dimension of space time is 26 and $\alpha = -1$, all solutions of

$$L_n|\psi\rangle = (L_0+\alpha)|\psi\rangle = 0 \quad , n>0, \text{ with}$$

$$\langle\psi|\psi\rangle \neq 0$$

are of the form

$$|\psi\rangle = |t\rangle + |s\rangle \tag{3.25}$$

where $|t\rangle$ is a transverse state and $|s\rangle$ is a linear combination of states of the form $L_{-n}|\chi\rangle$, which is orthogonal to any physical state, including itself.

That 26 is an upper limit to the dimension of the string model is seen from the state

$$|\phi\rangle = \left\{ \tilde{L}_{-2} + \frac{1}{4}(D-26)(K_{-2}+K_{-1}^2) \right\} |0\rangle \tag{3.26}$$

which is physical for any dimension D, but has norm

$$\langle\phi|\phi\rangle = \frac{1}{2}(26-D) \tag{3.27}$$

If D is smaller than 26, the transverse states do not span the physical space up to null vectors, as this example shows.

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