Gauge - Gravity Duality at Finite N

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Abstract

Recently it has been shown that $\mathcal{N} = 4$ super Yang-Mills theory is integrable in the planar limit. Past arguments suggest the integrability is only present in the planar limit. However, this conclusion was shown to be incorrect. Two specific classes of operators were studied in the $\Delta \sim \mathcal{O}(N)$ sector. The first were labelled by Young diagrams having two long columns. The second were labelled by Young diagrams having two long rows. This result was then generalized to p long rows or columns with p fixed to be $\mathcal{O}(1)$ as $N \to \infty$. For this case, the non-planar limit was found to be integrable. In this dissertation, we extend this work by considering p to be $\mathcal{O}(N)$. We have calculated the dilation operator for the case with two impurities.

Contents

Declaration

The work described by this thesis was carried out at the School of Physics, University of the Witwatersrand, from January 2012 until June 2013, under the supervision of Professor Robert de Mello Koch.

This thesis is entirely, unless specifically contradicted in the text, the work of the candidate, Justine Alecia Tarrant, and has not been previously submitted, in whole or in part, to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Signed: _____ Name: _____

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Chapter 1

Introduction

With the publication of 'A Dynamical Theory of the Electromagnetic Field' in 1865, J.C. Maxwell had produced the unified theory of electromagnetism. He had shown that the previously unrelated theories of electricity, magnetism and optics were manifestations of the same phenomenon. This was the first example of unification, and it turned out to be an extremely prolific idea. The electromagnetic force is now considered to be a fundamental force of nature. There are three more fundamental forces: the strong nuclear, weak nuclear and gravitational forces. The strong and weak nuclear forces were not yet known when Maxwell discovered electromagnetism. Today, a subject of considerable interest is to produce a unified theory which unifies the four fundamental forces. At present, we have managed to unite three of the fundamental forces into a unified framework known as the standard model of particle physics. Gravity is not part of this description. It has proven to be exceptionally difficult to include gravity. For it to be included, we know that this requires the quantization of gravity. All attempts to find a quantum field theory (QFT) including gravity have failed. Promising unified theories are string theories which are the only well defined perturbative quantum gravity theories known at present.

A new approach, which is providing significant insight into the problem of quantum gravity, came in late 1997. Juan Maldacena conjectured an equivalence between a quantum theory of gravity and a quantum field theory without gravity. The quantum theory of gravity lives in a d dimensional spacetime. The QFT is a conformal field theory living in a $d-1$ dimensional spacetime [1]. This $d-1$ dimensional spacetime is the boundary of the d dimensional spacetime of the quantum gravity. This conjecture is called the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence and it realizes the Holographic principle, which was first described by 't Hooft and then later by Susskind [2, 3]. The Holographic principle says that studying the gravity (or bulk) theory in d dimensions is

equivalent to studying a quantum field theory on the $d-1$ dimensional boundary. The equivalence refers to the dynamical content of the two theories. Concretely, any question that can be asked and answered in the first theory, can be asked and answered in the second, and the two give identical predictions. The virtue of this correspondence is that it allows us to map difficult problems in the QFT into simpler problems in the quantum gravity. Similarly, problems that are difficult to think about in the quantum gravity can be mapped into a simpler QFT setting.

The quantum field theory that we study is called $\mathcal{N}=4$ super Yang-Mills theory. It is conjectured to be dual (or equivalent) to type IIB-string theory on an asymptotically $AdS_5 \times S^5$ geometry. It is a theory of closed strings so it includes gravity. The QFT is a super-symmetric conformal field theory in four dimensions with gauge group $U(N)$. The quantum gravity is a theory in 5 non-compact dimensions, the AdS_5 space. AdS spacetime has constant negative curvature. The sphere S^5 is a compact space with constant positive curvature. The $AdS_5 \times S^5$ spacetime is a solution of the Einstein field equations with negative (attractive) cosmological constant.

We will be working mainly on the QFT side of the duality. Our goal is to construct a complete set of operators from the fields present in the Lagrangian density of the theory. We will achieve this for the so-called SU(2) sector of the theory. These operators are then used to build the main objects of interest, they are called correlation functions. Correlation functions are the objects which correspond to observables in a quantum field theory, they encode all of the dynamical content. In QFT we can turn correlators into S-matrix elements. In a conformal field theory we read off the scaling behaviour of the 2-point function.[1]

The fields in $\mathcal{N}=4$ super Yang-Mills theory that we are interested in are called the Higgs fields ϕ_i where $i=1,...,6$. They are matrix valued fields transforming as Lorentz scalars. They transform in the adjoint of the U(N) gauge group, that is, they are $N \times N$ matrices. We combine these six Higgs fields to build three complex fields Z, Y and X. The operators whose correlation functions we study may contain any number or combination of the fields Z, Y and X. For the SU(2) sector we only use the Z and Y fields. AdS/CFT relates this choice of operator to different physical objects in the quantum gravity, based on the number of fields present in the operator. These physical objects (states) correspond to gravitons, strings, giant gravitons and new background geometries where the number of fields present are of $\mathcal{O}(1)$, $\mathcal{O}(\sqrt{N})$, $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$ respectively.

[1] For example

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

where Δ is the observable and is called the dimension.

1.1 Overview

Recently it has been shown that $\mathcal{N} = 4$ super Yang-Mills theory is integrable in the planar limit [4]. In the past arguments have been given suggesting that integrability is spoiled by non-planar corrections [5]. In 2011 this conclusion was shown to be incorrect [6]. Operators labelled by Young diagrams having a conformal dimension $\Delta \sim \mathcal{O}(N)$ were considered. Two specific classes of operators were studied. The first were labelled by Young diagrams having two long columns. The second were labelled by Young diagrams having two long rows. This result was generalized to p long rows or columns in [7] with p held fixed to be $\mathcal{O}(1)$ as $N \to \infty$. For p long rows or columns the non-planar limit was found to be integrable. In both [6] and [7] integrability was proven by showing that the dilatation operator reduces to a decoupled set of harmonic oscillators.

Integrability is a very powerful tool. In this masters research we try to determine whether integrability holds in the non-planar large N limit. We will focus on the sector with conformal dimension $\Delta \sim \mathcal{O}(N^2)$. Operators with a bare dimension of $\mathcal{O}(N^2)$ are dual to new geometries. Thus an optimistic view is that the linearized Einstein equations will be recovered from the action of the dilatation operator in this sector of the theory. We take a small step towards realizing this very exciting idea. We will try to answer two important questions. First, we would like to know whether the sector of super Yang-Mills described by operators with $\mathcal{R}$-charge of $\mathcal{O}(N^2)$ is integrable. Second, do the results match something we already know about in type-IIB string theory to which this sector is conjectured to be dual by the AdS/CFT correspondence?

We concentrate on operators with large $\mathcal{R}$-charge[2] in the large N but non-planar limit. The latter statement and in particular this limit will become transparent in Chapter 2 when we discuss a toy model. This toy model is a theory of matrices and is thus QCD-like. It is useful to study matrix models because $\mathcal{N} = 4$ super Yang-Mills theory is also a theory of matrices. In Chapter 3 we deal with the components that go into building the AdS/CFT correspondence. Here, we briefly discuss D-branes. Chapter 4 is concerned with supersymmetric gauge theories. This is the type of theory we deal with in this dissertation. In Chapter 5 we study the mathematical objects used to build the basic operators in our theory, these are the Schur and restricted Schur polynomials. Finally, Chapter 6 considers the idea at the heart of this research area, the AdS/CFT correspondence. Here, we discuss some tests for the validity of the correspondence. Chapter 7 contains the results, that is, the calculation performed during this project. It discusses the dilatation operator. We conclude

[2]The $\mathcal{R}$-charge is a conserved charge associated with supersymmetry. The $\mathcal{R}$-charges commute with all the bosonic generators of the Poincarè group. However they do not commute with the supercharges in the super Poincarè group.

this work in Chapter 8.

1.2 Background

The main tools that are required for this research are mathematical. Group representation theory for the symmetric and unitary groups is used to build projectors and then restricted Schur polynomials. The restricted Schur polynomials play a central role since they form a complete set of operators and are used to build a basis for the local operators of the field theory. These operators are dual to the new geometries of interest to us [8]. We briefly discuss the group representation theory used and introduce Young diagrams. Our restricted Schur polynomials are labelled by these Young diagrams. We will discuss restricted Schur polynomials in Chapter 5. We end this section with a brief discussion of string theory.

1.2.1 Group Representation Theory and Young Diagrams

Group representation theory is used to construct and describe Schur polynomials. It is also used to construct projectors, which are used to build the Schur polynomials. In particular, we focus on unitary ($U(N)$) and symmetric (S_n) group theory. We use Young diagrams to implement group theoretic concepts, as they are a strong algebraic tool. The Young diagrams label all the irreducible representations for the groups that we choose to study.

The group theories of the symmetric and unitary groups are intimately linked via *Schur-Weyl duality*. Consider the vector space[3] $V^{\otimes n}$. The action on this space by the unitary group is given by

$$U : T^{i_1,i_2\ldots,i_n} \to \Gamma(U)^{i_1}_{j_1}\Gamma(U)^{i_2}_{j_2}\ldots\Gamma(U)^{i_n}_{j_n}T^{j_1,j_2\ldots,j_n}$$

where $\Gamma(U)$ is a matrix representing $U \in U(N)$ in the fundamental representation.[4] Similarly the action of the symmetric group on this vector space may also be defined. It is given by

$$\sigma : T^{i_1,i_2\ldots,i_n} \to T^{i_{\sigma(1)},i_{\sigma(2)}\ldots,i_{\sigma(n)}}$$

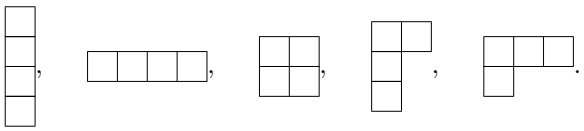
where σ permutes the indices of T. These actions commute with each other.[5] As a result, we may label representations of $U(N)$ with precisely the same labels used to label representations of S_n. This is similar to when we label hydrogen energy eigenstates by angular momentum because angular momentum commutes with the Hamiltonian. For S_n different

[3]The definition of this vector space is given in Appendix A.

[4]Our convention here is that an upstairs index belongs to the fundamental representation whilst a downstairs index belongs to the anti-fundamental representation.

[5]Proof given in Appendix A.

representations of the same group are found by arranging n boxes in all possible legal ways to form a Young diagram. For S_4 all possible Young diagrams are



Each Young diagram labels a different representation of S_4. The dimension of the representation is given by

$$d_R(S_n) = \frac{n!}{\text{hooks}_R}$$

where R is the label corresponding to the particular Young diagram. R labels an irreducible representation of S_n. To explain the factor $hooks_R$ consider any box in R. Start at the bottom of the column containing that box. Move vertically upward, imagine drawing a line as you go along, up toward the box in mind. At this point, turn right and exit the right most box of the row the box is in. The number of boxes this line moves through during this process is called the *hook length*. Thus, each box in R will have its own hook length. In the equation above, $hooks_R$ is given by the product of all hook lengths. For example

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

has three hook lengths, one for each box. To move through the first box in the first row, starting from the bottom and exiting on the right we had to move through 3 boxes. The product of the hook lengths is $3 \times 1 \times 1 = 3$.

Next consider the unitary group U(N). The rule is that any Young diagram with at most N rows of boxes labels an irreducible representation of $U(N)$. The number of boxes in any row is unrestricted. The dimension of an irreducible representation of U(N) is given by

$$D_R(U(N)) = \frac{\text{factors}_R}{\text{hooks}_R}$$

where the hooks are calculated in the same way as for S_n. A factor of $N - i + j$ is given to the box in the $i - th$ row and $j - th$ column. The product of the factors from each box is $factors_R$. As an example consider the representation

of S_3 and $U(3)$. It has dimensions

$$d_{\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = \frac{3!}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = 2$$

and

$$D_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}} = \frac{\begin{smallmatrix}3&4\\2&\end{smallmatrix}}{\begin{smallmatrix}3&1\\1&\end{smallmatrix}} = \frac{N(N+1)(N-1)}{3} = 8$$

under the symmetric and unitary group respectively. The irreducible representation of U(N) with a single column of N boxes has unit dimension[6]. Similarly, the irreducible representation for S_n with a single column of n boxes has unit dimension.

1.2.2 Superstring theory

Superstring theory is the generalization of bosonic string theory. It extends the theory to include fermions and there are five such theories. These theories are all supersymmetric. In this theory, fermions are introduced by saying that each boson has a supersymmetric fermion partner. Type I superstring theory describes open and closed strings which are unoriented. This theory has $SO(32)$ gauge symmetry. Type IIA string theory describes closed oriented strings with a $U(1)$ gauge symmetry. Type IIB string theory is the theory that we focus on in this dissertation. It also describes closed oriented strings. The final two theories are the heterotic string theories, both describing closed, oriented strings. The modern point of view claims that these perturbative theories are all different limits of a single underlying theory.

[6]For $SU(N)$ these columns are dropped. What this means is that

$$\begin{smallmatrix}\square&\square&\square\\ \square&\square&\\ \square&&\end{smallmatrix} \sim \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}$$

as an irreducible representation of SU(3).

Chapter 2

Matrix Models

We begin by studying a free theory in zero dimensions[1]. By free theory we mean that our action will be quadratic, there being no cubic, quartic or other interaction terms. We work in 0-dimensions. A consequence of this is that we only need to deal with the combinatorics of the gauge group indices (matrix indices). Our matrix valued fields M, are $N \times N$ hermitian matrices transforming in the adjoint representation of the Unitary group $U(N)$ as

$$M \to M' = U^{\dagger} M U.$$

Strictly speaking, a 0-dimensional universe has only a single point, so there is no notion of a local symmetry. However, we declare that physical observables are invariant under the above transformation. In this sense, we say that our observables are 'gauge' invariant. The aforementioned action is given by

$$S = \omega Tr\left(M^2\right),$$

where

$$Tr\left(M^2\right) = \sum_{i,j=1}^{N} M_{ij} M_{ji} = \sum_{i=1}^{N} M_{ii}^2 + 2\sum_{i>j} M_{ij} M_{ji}.$$

The trace in our action ensures gauge invariance. We can build operators in a natural way by taking traces of products of the field M

$$\mathcal{O}_n = Tr\left(M^n\right).$$

[1]It may be noted that this toy model is not merely pedagogical. This was shown by T. Eguchi and H. Kawai in [9]. In this paper the authors showed that the Schwinger-Dyson equations of the reduced model are equivalent to the Schwinger-Dyson equations of quantum chromodynamics in 3+1 dimensions.

We use a path integral to calculate correlation functions of the general operators given above. These are the objects which have physical significance. The path integral is given by

$$Z[J] = \int [dM] e^{-S+Tr(JM)} \tag{2.1}$$

where the integral is over all possible matrix configurations. This is accomplished by an integral over the N^2 independent parameters since M is hermitian[2]. Thus $M_{ii} = M_{ii}^*$ and $M_{ij} = M_{ji}^*$. Thus

$$[dM] = \prod_{i=1}^{N} dM_{ii} \prod_{i>j}^{N} dM_{ij}^R dM_{ij}^I$$

where dM_{ij}^R is the real part and dM_{ij}^I is the imaginary part. The first product encompasses the diagonal elements and the second accounts for the rest. We couple in a source J which is used to calculate correlation functions. It is chosen to be hermitian so that the action remains hermitian. We can now write down the correlation functions of the traced observables

$$\langle \mathcal{O}_n \rangle = \langle Tr\,(M^n) \rangle = \frac{d^n}{dJ_{ab} dJ_{bc} ... dJ_{da}} Z[J]|_{J=0}, \tag{2.2}$$

where

$$Z[J=0] = \left(\sqrt{\frac{\pi}{\omega}} \right)^{N^2}.$$

By absorbing a factor of $\left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2}$ in the measure $[dM]$, we can set $Z[J=0] = 1$. This convention will be assumed hereafter. After completing the square, we arrive at the following simple expression for the generating function of correlation functions

$$Z[J] = \exp\left(\frac{Tr(J^2)}{4\omega} \right).$$

The only non-vanishing correlation functions are those with an even number of M's. The

[2]An $N \times N$ real matrix has N^2 elements. For a Hermitian matrix there are $2N^2$ elements since each entry has two degrees of freedom. However for the diagonal elements $M_{ii} = M_{ii}^*$ which removes N degrees of freedom. Thus we are left with $2(N^2 - N) + N$ degrees of freedom. Hermiticity means that $M_{ij} = M_{ji}^*$, which relates elements below the diagonal to those above. This removes half the off-diagonal degrees of freedom. So we have $(N^2 - N) + N = N^2$ degrees of freedom.

basic correlation function is the two-point function or propagator. It is given by[3]

$$\begin{aligned}\langle M_{ij}M_{kl}\rangle =& \int [dM]e^{-S}M_{ij}M_{kl}\\ =&\frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z[J]|_{J=0}\\ =&\frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}\exp\left(\frac{J_{mn}J_{nm}}{4\omega}\right)|_{J=0}\\ =&\frac{d}{dJ_{ji}}\left[\frac{J_{kl}}{2\omega}\exp\left(\frac{Tr(J^2)}{4\omega}\right)\right]|_{J=0}\\ =&\frac{\delta_{il}\delta_{jk}}{2\omega}.\end{aligned} \tag{2.3}$$

Thus for the propagator we take the outer two labels and form a delta, then take the inner two and form another delta, including a factor $\frac{1}{2\omega}$ each time. Using the latter rule, the 2-point function of the traced operator is

$$\begin{aligned}\langle\mathcal{O}_2\rangle =&\langle Tr(M^2)\rangle\\ =&\langle M_{ij}M_{ji}\rangle\\ =&\frac{\delta_{ii}\delta_{jj}}{2\omega}\\ =&\frac{N^2}{2\omega},\end{aligned}$$

where $\delta_{ii}=N$. The trace operators $\langle\mathcal{O}_n\rangle$ have no free indices and their expectation values are polynomials of N. For $Tr(M)^2$ we get

$$\langle M_{ii}M_{jj}\rangle = \frac{\delta_{ij}\delta_{ji}}{2\omega} = \frac{N}{2\omega}.$$

For what follows we will normalize our operators so that they have an expectation value that is order 1 in the large N limit. This amounts to setting $\mathcal{O}_n$ to

$$\mathcal{O}_n = \frac{Tr\left(M^n\right)}{N^{(n+2)/2}}.$$

Thus $\langle\mathcal{O}_2\rangle = \frac{1}{2\omega}$. As a final example consider

$$\langle Tr\left(M^4\right)\rangle = \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle,$$

[3]We take derivatives as follows:

$$\begin{aligned}\frac{d}{dJ_{ij}}Tr(JM) =&\frac{d}{dJ_{ij}}\left(J_{kl}M_{lk}\right)\\ =&\delta_{ik}\delta_{jl}M_{lk}\\ =&M_{ji}.\end{aligned}$$

Thus notice that differentiation with respect to J_{ij} produces an M_{ji}, i.e. it switches the labels on the M. This is a consequence of the U(N) covariance of the theory.

there are three Wick contractions that can be performed here to give the following three terms

$$\begin{aligned}
\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle &= \langle M_{ij}M_{jk}\rangle\langle M_{kl}M_{li}\rangle \;+\; \langle M_{ij}M_{li}\rangle\langle M_{jk}M_{kl}\rangle \\
&\quad + \langle M_{ij}M_{kl}\rangle\langle M_{jk}M_{li}\rangle \\
&= \frac{\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll}}{(2\omega)^2} + \frac{\delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}}{(2\omega)^2} + \frac{\delta_{il}\delta_{jk}\delta_{ji}\delta_{kl}}{(2\omega)^2} \\
&= \frac{1}{(2\omega)^2}\left[2N^3 + N\right] \qquad (2.4)
\end{aligned}$$

therefore

$$\langle \mathcal{O}_4\rangle = \frac{1}{(2\omega)^2}\left[2 + \frac{1}{N^2}\right].$$

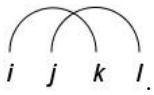
From the examples we have just studied, it is simple to read off the following Feynman rules:

1. For each matrix element M_{ij} draw a pair of dots labelled by these matrix indices,
2. Put all of the dots on a dashed line,
3. Join pairs of dots by ribbons such that;
   - ribbons stay above the dashed line,
   - ribbon ends join ribbon ends,
   - ribbons are not twisted,
4. Each of the two lines composing the ribbon join pairs of dots, giving a Kronecker delta labelled by these indices,
5. For each trace join matching indices by a line,
6. For each ribbon (propagator) there is a factor of $\frac{1}{2\omega}$.

For example, the first ribbon graph below is valid. The second, is not allowed by the rules above, since it constitutes a twisted ribbon

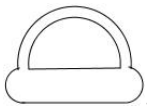
$\langle M_{ij}M_{kl}\rangle =$ 

,

$\langle M_{ij} M_{kl} \rangle \neq$ 

.

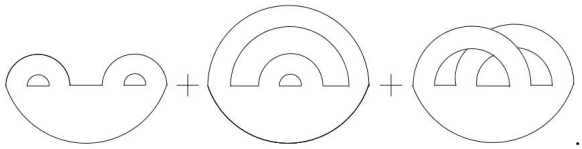
These Feynman diagrams are called *ribbon graphs*. For the traced operators we get

$\langle M_{ij} M_{ji} \rangle =$ 
, $\langle M_{ii} M_{jj} \rangle =$ 
.

Recall that, for $\langle M_{ij} M_{ji} \rangle$ we had an N^2 dependence. Notice that the corresponding ribbon graph has two closed loops. Similarly for $\langle M_{ii} M_{jj} \rangle$ we had an N dependence and the ribbon graph has only one closed loop. Thus we associate a factor of N with each closed loop. So we add another rule

- For each closed loop there is a factor of N.

The lines joining the traced indices are not ribbons and do not constitute a propagator. Thus there is only one propagator in the diagrams for $\langle M_{ij} M_{ji} \rangle$ and $\langle M_{ii} M_{jj} \rangle$, yielding a single factor of $\frac{1}{2\omega}$ in agreement with what we found earlier. Next, consider the last example given above, $\langle Tr\left(M^4\right) \rangle$. This example is more interesting, as it yields a ribbon graph with ribbons that cross, which is different to twisting. This graph is given by

$\langle Tr\left(M^4\right) \rangle =$ 
.

These terms correspond to the terms in the second line of (2.4) respectively. The diagrams which can be drawn in the plane without any of the ribbons crossing (the first two above) are called *planar* diagrams. When this can't be done (as in the last diagram) we call them *non-planar*. Using our previous rules we get a factor of $\frac{1}{(2\omega)^2}$ for the two propagators. Secondly, the first two diagrams have three closed loops and thus produce factors of N^3 each. The last non-planar diagram only has one loop and thus produces a factor of N. This reproduces the last line of (2.4) exactly. Notice that the non-planar diagram is suppressed by a factor of $\frac{1}{N^2}$.

2.1 The Large N Limit and Factorization

Suppose that we decide to let $N \to \infty$. Then

$$\langle \mathcal{O}_2 \rangle = \frac{1}{2\omega}$$

and

$$\langle \mathcal{O}_2 \mathcal{O}_2 \rangle = \frac{1}{(2\omega)^2} \left[1 + \mathcal{O}\left(\frac{1}{N^2} \right) \right] .$$

Then we can write

$$\begin{aligned} \langle \mathcal{O}_2 \mathcal{O}_2 \rangle =& \langle \mathcal{O}_2 \rangle^2 \left[1 + \mathcal{O}\left(\frac{1}{N^2} \right) \right] \\ \to& \langle \mathcal{O}_2 \rangle^2 \end{aligned}$$

for $N \to \infty$. Notice that the contributions from the non-planar diagram fell away in this limit. This process is called *Factorization*. In the large N limit, correlation functions of operators built from a fixed number of fields factorize. That is, the number of fields in each operator is fixed when taking the large N limit. Concretely, factorization implies

$$\langle \prod_i \mathcal{O}_i \rangle = \prod_i \langle \mathcal{O}_i \rangle \left[1 + \mathcal{O}\left(\frac{1}{N^2} \right) \right] . \tag{2.5}$$

This is a useful property providing a powerful simplification, meaning we only need to draw the planar ribbon graphs and then sum them to get the correct correlation function in this limit. This is called the large N, *planar* limit. The planar approximation refers to the suppression of the non-planar diagrams. It is important to stress that the number of fields in the correlation functions were kept constant when we took $N \to \infty$ and that there were $\mathcal{O}(1)$ fields. This will not always be the case. For $\mathcal{O}(\sqrt{N})$ or more fields, letting $N \to \infty$ results in an increase in the number of fields present in the correlation functions. Increasing the number of fields also increases the number of Wick contractions that need to be performed, producing large combinatoric factors. As a result, the planar approximation fails in this large N limit. This point will be treated in greater detail later on.

We will now argue that factorization implies that the theory is described by classical dynamics, i.e there is a classical theory we can use to study the large N limit in the QFT. To make this argument, return to the problem of computing expectation values given some probability density describing our system. Suppose that μ_i is the probability to be in the *i-th* state, then $\sum_i \mu_i = 1$. If our observables are labelled by $\mathcal{O}_n$, the expectation value is

$$\langle \mathcal{O}_n \rangle = \sum_i \mu_i \mathcal{O}_n(i)$$

where $\mathcal{O}_n(i)$ is the value of $\mathcal{O}_n$ in the i-th state. Then (2.5) gives

$$\sum_i \mu_i \mathcal{O}_{n_1}(i)\mathcal{O}_{n_2}(i)\mathcal{O}_{n_3}(i)... = \sum_{i_1} \mu_{i_1}\mathcal{O}_{n_1}(i_1) \sum_{i_2} \mu_{i_2}\mathcal{O}_{n_2}(i_2) \sum_{i_3} \mu_{i_3}\mathcal{O}_{n_3}(i_3)...$$

which can only be satisfied by $\mu_i = 1$ for $i = i^*$ and $\mu_i = 0$ for $i \neq i^*$. Here the i^*-th state is the single configuration that is contributing. Thus the system is in a definite state with no uncertainty. The expectation value is given by an integral over all possible configurations

$$\langle \mathcal{O} \rangle = \int [dM] e^{-S} \mathcal{O}.$$

When only one configuration contributes, we are in a classical limit of the theory. This single configuration being the classical configuration. Clearly then, factorization which implies that a single configuration dominates the computation of the correlation functions at large N, implies that the large N limit of the theory is captured by the classical limit of some theory.

2.2 The Interacting Matrix Model

We have demonstrated that factorization is a property of the large N limit for a class of correlators of the free theory. One might then wonder if this conclusion holds even after interactions are included. This question will form an important part of this section. As an initial modification, one might introduce a quartic term, with coupling constant g, into the action of the free theory; the action then becomes:

$$S = \omega Tr\left(M^2\right) + gTr\left(M^4\right)$$

and

$$Z[J, g] = \int [dM] e^{-S + Tr(JM)}. \tag{2.6}$$

The normalization of the measure is chosen so that $Z[J = 0, g = 0] = 1$. This is exactly the same normalization as we used in the previous section. Then to first order in g

$$Z[J = 0, g] = \int [dM] e^{-\omega Tr\left(M^2\right)} \left(1 - gTr\left(M^4\right)\right).$$

Using the results of the last section we can write

$$\begin{aligned} Z[J = 0, g] =& 1 - g\langle Tr\left(M^4\right)\rangle \\ =& 1 - \frac{g}{(2\omega)^2}\left[2N^3 + N\right]. \end{aligned} \tag{2.7}$$

The expectation value of $Tr(M^2)$ is

$$\begin{aligned} \langle Tr\left(M^2\right)\rangle_{int} =& \int [dM] e^{-\omega Tr\left(M^2\right)} Tr\left(M^2\right)\left(1 - gTr\left(M^4\right)\right) \\ =& \langle Tr\left(M^2\right)\rangle - g\langle Tr\left(M^2\right) Tr\left(M^4\right)\rangle \\ =& \frac{N^2}{2\omega} - \frac{g}{(2\omega)^3}\left[2N^5 + 9N^3 + 4N\right]. \end{aligned} \tag{2.8}$$

From these results it is clear that we need to add a Feynman rule for the interaction vertex. The vertex is given by

$= -g.$

To first order we have

$Z[J=0,g] = 1 +$

.

Once again, using the rules stated above we can reproduce the correct powers of N from the diagrams, to arrive at (2.7). For example, consider the first diagram. There are two ribbons (propagators) and a single vertex giving the factor of $-\frac{g}{(2\omega)^2}$. There are three closed loops giving the N^3. For (2.8) there are 15 Wick contractions for the second term and thus 15 diagrams in total with the g vertex. To first order in g and keeping only the connected diagrams we get

$\langle Tr\left(M^2\right)\rangle_{con} =$

.

The disconnected diagrams are dropped because they can be absorbed into the normalization. This corresponds to choosing the normalization of the measure so that $Z[J=0,g]=1$. In what follows we will assume this normalization. From the connected diagrams we can write down

$$\langle Tr\left(M^2\right)\rangle_{con} = \frac{N^2}{2\omega} - \frac{g}{(2\omega)^3}\left[8N^3 + 4N\right].$$

We have expanded to first order in g in perturbation theory. The terms of order g are the first perturbative corrections as a result of the fact that there is an interaction. Even if g is small, these corrections need not be small due to their N dependence. In fact each correction becomes larger and larger with increasing N. It is the highest order correction that was added that dominates each time. Clearly then, the attempt at a perturbative expansion in the parameter g does not lead to a well defined expansion.

2.3 The 't Hooft Limit

The failure of perturbation theory when using the coupling constant g was fixed by 't Hooft. He introduced a new parameter λ such that

$$\lambda = gN.$$

We keep λ fixed as we send $N \to \infty$. That is, we consider a double scaling limit in which we scale $N \to \infty$, together with $g \to 0$ holding $\lambda = gN$ fixed. Then

$$\begin{aligned} Z[J=0,g] =& 1 - \frac{1}{(2\omega)}\left[2\lambda N^2 + \lambda\right] \\ =& 1 + N^2\left[-\frac{2\lambda}{2\omega} + \mathcal{O}\left(\lambda^2\right)\right] + N^0\left[-\frac{\lambda}{2\omega} + \mathcal{O}\left(\lambda^2\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \langle Tr\left(M^2\right)\rangle_{con} =& \frac{N^2}{2\omega} - \frac{1}{(2\omega)^3}\left[8\lambda N^2 + 4\lambda\right] \\ =& N^2\left[\frac{1}{2\omega} - \frac{8\lambda}{(2\omega)^3} + \mathcal{O}\left(\lambda^2\right)\right] + N^0\left[-\frac{4\lambda}{(2\omega)^3} + \mathcal{O}\left(\lambda^2\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \tag{2.10}$$

The terms belonging to N^0 came from the non-planar diagrams and similarly for $\mathcal{O}\left(\frac{1}{N^2}\right)$. Again we see that these contributions are suppressed by $\frac{1}{N^2}$. The series in (2.9) and (2.10) are double expansions, having two expansion parameters $\frac{1}{N^2}$ and λ. Expanding to first order in λ we see that the correction is no longer N^3. In fact the power of N drops by $\frac{1}{N^2}$ for each correction. The 't Hooft limit provides a way to use perturbation theory sensibly. Although we have only discussed a few examples here, it is a general conclusion that factorization holds for any matrix QFT, and in particular, also in the interacting theory. For example, calculate $\langle Tr(M^2)^2\rangle_{con}$ to first order in g. We expect to find

$$\begin{aligned} \langle Tr(M^2)^2\rangle_{con} =& \langle Tr(M^2)\rangle^2 - 2g\langle Tr(M^2)\rangle\langle Tr(M^2)Tr(M^4)\rangle + \mathcal{O}(g^2) \\ \to& \frac{1}{(2\omega)^2} + \mathcal{O}(\frac{1}{N^2}) - g\frac{1}{(2\omega)^4}(4 + \mathcal{O}(\frac{1}{N^2})) \end{aligned}$$

in the large N limit, where we have divided by the largest power of N in the first term and subsequent corrections separately. Doing the calculation we arrive at

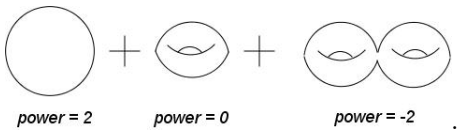
$$\begin{aligned} \langle Tr(M^2)^2\rangle_{con} =& \int [dM] e^{-\omega Tr\left(M^2\right)} Tr\left(M^2\right) Tr\left(M^2\right)\left(1 - gTr\left(M^4\right)\right) \\ =& \langle Tr\left(M^2\right)^2\rangle - g\langle Tr\left(M^2\right)^2 Tr(M^4)\rangle \\ =& \frac{1}{(2\omega)^2}(N^4 + 2N^2) - g\frac{1}{(2\omega)^4}(4N^7 + 18N^5 + ...) \\ \to& \frac{1}{(2\omega)^2} + \mathcal{O}(\frac{1}{N^2}) - g\frac{1}{(2\omega)^4}(4 + \mathcal{O}(\frac{1}{N^2})) \end{aligned}$$

which matches what we expected from the above. Thus we have shown that factorization does indeed hold up to order g in the interacting theory.

2.4 Powers of N Correspond to Different Topologies

Ribbon graphs can be drawn on surfaces with different topologies in such a way that the ribbons do not intersect. In such case, we say that the graph triangulates a particular topological surface. The topology of the surface it triangulates gives the N dependence of each ribbon diagram in the 't Hooft limit. Planar diagrams can be drawn in the plane without any of the ribbons crossing. They may then be said to triangulate a sphere. Consider the first two diagrams in $Z[J = 0, g]$. These are planar and hence can be drawn on the sphere. They come with factors of N^2 according to (2.9). Thus the sphere is associated with N^2. The last diagram has a factor of N^0, it is non-planar and can be drawn on a torus. Thus a torus is associated with N^0. Consider $\langle Tr(M^2)\rangle_{con}$, here the act of taking the trace adds a new feature to the ribbon graph. This is the bubble-like feature at the bottom of each ribbon graph in $\langle Tr(M^2)\rangle_{con}$ given in Section 2.2. It appears after tracing over the outer and inner indices in $\langle M_{ij}M_{ji}\rangle_{con}$. Again these will triangulate topological surfaces, except these will contain holes dues to these traces. As before terms having N^2 will correspond to the sphere, but it will be a sphere with holes.

For each power of N we can find a topological surface. Therefore, instead of adding up all the ribbons graphs for the 0-point function $Z[J = 0, g]$, we can sum over the topological surfaces corresponding to each power of N. This is convenient since there are fewer topologies than there are ribbon graphs, as each surface captures all diagrams associated with a given power of N. Thus the 0-point function is

$Z[J = 0, g] =$ 

.

The 'power' in the diagram refers to the power of N. For example, the tree level term has N^2. The power of N for the surface is given by the Euler characteristic χ which is given by[4]

$$\chi = 2 - 2g - b.$$

Here g is called the genus of the surface and is related to the number of handles the surface has and b is the number of boundaries. For example

- Sphere: $g = 0 = b$ therefore we get N^2
- Torus: $g = 1$, $b = 0$ therefore we get N^0

[4]For a proof of this result, consult Appendix B.

- Cylinder: $g = 0$, $b = 2$ therefore we get N^0
- Pretzel (torus with two handles): $g = 2$, $b = 0$ therefore we get $\frac{1}{N^2}$.

This matches (2.10). Summing all the ribbon graphs is thus equivalent to adding up all the topologies triangulated. Surfaces triangulated by ribbon graphs are identified with world sheets traced out by propagating strings. Consider the 2-point function. The relevant topologies are shown in Fig. 2.1. Here we get a cylinder, a cylinder with one handle (loop)

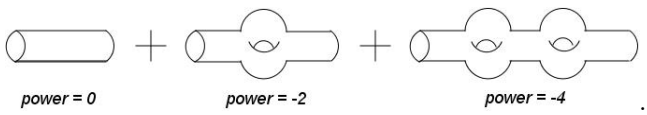


Figure 2.1: Loop expansion for propagating string.

and a cylinder with two handles. The powers of N are as follows

- Cylinder: $g = 0$, $b = 2$ therefore we get N^0
- Cylinder with a handle: $g = 1$, $b = 2$ therefore we get N^{-2}
- Cylinder with two handles: $g = 2$, $b = 2$ therefore we get N^{-4}

2.4.1 Loop expansion

A loop expansion organises the theory according to powers of $\hbar$. Consider ϕ^4 theory, for which the action is given by

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda^4\phi^4 \right). \tag{2.11}$$

When we study ϕ^4 theory we usually set $\hbar = c = 1$ and study $\int \mathcal{D}\phi \exp(iS)$. Putting $\hbar$ back into the theory gives

$$\int \mathcal{D}\phi e^{i\frac{S}{\hbar}}.$$

The propagator, which is the inverse of the quadratic term in the Lagrangian, goes like $\hbar$. The vertex, on the other hand, goes like $\frac{1}{\hbar}$. We thus make the following correspondence

$$\text{propagator} \leftrightarrow \hbar \tag{2.12}$$

$$\text{vertex} \leftrightarrow \frac{1}{\hbar}. \tag{2.13}$$

Then given some loop expansion we are able to classify each Feynman diagram in the loop expansion in terms of some power of $\hbar$. For every loop added, we add a power of $\hbar$. In

Figure 2.1, the first diagram is just a propagator. Here, every time a loop is added we gain a power of $\frac{1}{N^2}$. Thus we make the association

$$\hbar_{string} \longleftrightarrow \frac{1}{N^2}.$$

The classical theory we use to study the large N limit of our QFT is therefore a classical theory of strings.

2.5 Failure of the Planar Approximation

The large N limit has to be treated carefully depending on the number of fields out of which our operators are built. For $\mathcal{O}(1)$ fields as we send $N \to \infty$ the number of fields remain constant. There are only a few Wick contractions and therefore only a few diagrams that need to be summed. Thus the $\frac{1}{N^2}$ suppression is not overpowered by the small combinatoric factors generated by adding a small number of diagrams. Additionally, for $\mathcal{O}(1)$ fields the planar and large N limit coincide. However for $\mathcal{O}(\sqrt{N})$ or more fields, the $\frac{1}{N^2}$ suppression is overpowered by the large combinatoric factors arising from performing many Wick contractions. Although the non-planar diagrams are suppressed in the usual way, the sheer number of non-planar diagrams that can be drawn outweighs the number of planar diagrams. As a result, the non-planar contribution can not be neglected, yielding a large N limit where the planar approximation fails.

Chapter 3

AdS Space, Conformal Field Theory and D-branes

This chapter deals briefly with Anti de Sitter spacetime and conformal field theory. It is by no means comprehensive, but gathers the information needed to explain the main ideas in this dissertation. This chapter is based largely on chapter two in [10].

3.1 Basics of AdS space

Anti de Sitter space is a space with constant negative curvature with Lorentzian signature [11]. It is a solution to the Einstein field equations with a constant energy-momentum tensor, known as a cosmological constant [11]. This cosmological constant is negative (attractive). Anti de Sitter space enjoys an $SO(2, d-1)$ invariance in d dimensions. In d dimensions, AdS_d space may be represented by a hyperboloid

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = R^2$$

embedded in flat $d+1$-dimensional space, where R is constant. The metric of the flat $d+1$-dimensional space is given by

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2.$$

The metric of the AdS space is obtained by the induced metric on the hyperboloid.

3.2 Definition of a CFT

A conformal transformation is a coordinate tranformation that preserves 'angles' but not lengths.[1] More precisely

$$\frac{x^\mu y^\nu g_{\mu\nu}}{\sqrt{x^\sigma x^\tau g_{\sigma\tau} y^\gamma y^\beta g_{\gamma\beta}}}$$

is invariant under coordinate transformations. That is, it is a transformation that preserves the metric up to some scale[2]

$$g_{\mu\nu}(x) \to (\lambda(x))^2 g_{\mu\nu}(x).$$

A conformal field theory is a quantum field theory that has conformal symmetry. This extra symmetry implies that there are extra conservation laws present. For CFT's the laws of physics are scale invariant, that means that the same laws that govern the quantum scale also govern the cosmological scale. All scale and Poincarè invariant theories are believed to be CFT's [10]. At this juncture one may ask how it is then possible to use CFT's to discuss physics. These questions will, hopefully, be answered in the next section.

3.3 Reasons for studying CFT's

Conformal field theories look as though they should not describe nature since we know nature behaves differently at different scales. Yet we study them all the same. In this section, we try to give a motivation for why CFT's are so useful in trying to understand QFT's and their importance in physics.

CFT's are important for understanding the space of all QFT's. QFT's come equipped with a scale, i.e. some cut off Λ, above which we can't do physics. QFT's are thus effective field theories at some scale. There may be a hierarchy of scales. According to the Wilson renormalization group (RG), the low energy effective action (LEEA) is largely determined by the set of low energy degrees of freedom and the low energy symmetries. All the detailed dynamics of the microscopic (UV) theory are summarized into the couplings of the LEEA. At low energy only relevant and marginal operators contribute. Close to a free field fixed point (FFFP), it is easy to tell which operators are marginal and relevant. Marginal operators have couplings with length dimension zero, whilst relevant operators have length dimensions which are negative. The beta function β describes how the coupling changes as we flow from the UV to low energy (IR). At a fixed point the beta function vanishes. Thus each fixed

[1] We have placed 'angles' in inverted commas because this definition is not restricted to the geometrical sense of an angle, which we are used to.

[2] The scale factor is squared to emphasize that we cannot change the overall sign on the metric.

point corresponds to a scale invariant QFT. It is generally true that a scale invariant QFT enjoys full conformal symmetry. Thus, the endpoints of the RG trajectories are CFT's. Further, we can imagine any QFT is a CFT perturbed by relevant or irrelevant operators.

The renormalization group also describes another phenomenon known as *universality*. Universality says that systems with different microscopic behaviours share the same long distance behaviour at the phase transition point (where the degrees of freedom are rearranged). This is because there are very few relevant and marginal operators. Thus, many different UV theories share the same IR dynamics.

Next we discuss the role of CFT's in physics. Two important applications are worldsheet string theory and the AdS/CFT correspondence. For worldsheet string theory we work in two dimensions with parameters τ and σ. These are scale invariant under

$$\tau \to \lambda\tau, \qquad \sigma \to \lambda\sigma.$$

So we have a two dimensional CFT describing the worldsheet. This worldsheet is embedded in spacetime and we have the fields $X^\mu(\sigma, \tau)$ living on the worldsheet.

Of great importance to this dissertation is the AdS/CFT correspondence. This is like studying gravity in a "box" where the CFT is located on the boundary of the box provided by the AdS background. The isometry of *AdS* space matches the conformal group of the CFT.

3.4 Conformal Group and Lie Algebra

We said that conformal transformations are coordinate transformations leaving the metric invariant up to a rescaling

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}. \tag{3.1}$$

In flat space this becomes

$$\eta_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} = (\Omega(x))^2 \eta_{\mu\nu}. \tag{3.2}$$

To find the generators of the conformal group we must consider an infinitesimal conformal transformation of the type

$$\tilde{x}^\mu = x^\mu + \zeta^\mu, \qquad \Omega(x) = 1 + \omega(x),$$

where ζ^μ is called the *conformal Killing vector*. We substitute this into (3.2) and make a first order Taylor expansion to get

$$\eta_{\alpha\beta}(\delta^\alpha_\mu + \partial_\mu \zeta^\alpha)(\delta^\beta_\nu + \partial_\nu \zeta^\beta) = (1 + \omega(x))^2 \eta_{\mu\nu} = (1 + 2\omega(x))\eta_{\mu\nu}.$$

To $\mathcal{O}(1)$ in ζ we get back, trivially, that $\eta_{\mu\nu} = \eta_{\mu\nu}$. To $\mathcal{O}(\zeta)$ we arrive at the *conformal Killing vector equation*

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = 2\omega \eta_{\mu\nu}.$$

Contract both sides with $\eta^{\mu\nu}$ to obtain $\omega = \frac{1}{d}(\partial \cdot \zeta)$ with d the dimension of space. Thus, we finally get

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = \frac{2}{d}(\partial \cdot \zeta)\eta_{\mu\nu}. \tag{3.3}$$

We need to solve this last equation. We consider the cases of $d = 2$ and $d > 2$ Minkowski space.

Consider the first case. In 2 dimensional Minkowski space there are two independent choices for the indices. We can either set $\mu = 0$, $\nu = 1$ or we can set $\mu = \nu = 0$[3]. This choice then yields the following two equations

$$\partial_0 \zeta_1 + \partial_1 \zeta_1 = 0, \qquad \partial_0 \zeta_0 = -\partial_1 \zeta_1$$

The solution to these two differential equations is

$$\zeta_0 = f_1(x^0 + x^1) + f_2(x^0 - x^1), \qquad \zeta_1 = g_1(x^0 + x^1) + g_2(x^0 - x^1)$$

where

$$g_1(x^0 + x^1) - g_2(x^0 - x^1) = -f_1(x^0 + x^1) + f_2(x^0 - x^1) + constant.$$

There are thus an infinite number of solutions and therefore an infinite number of generators; one for each conformal Killing vector.

Next consider the case of $d > 2$. Act with $\partial^\mu \partial^\nu$ on both sides of (3.3) to get

$$2\partial^\mu \partial_\mu \partial_\nu \zeta^\nu = \frac{2}{D}\partial^\mu \partial_\mu (\partial \cdot \zeta)$$

which gives

$$\partial^\mu \partial_\mu (\partial \cdot \zeta) = \Box(\partial \cdot \zeta) = 0.$$

From this last equation it follows that the parameter ζ must be at most quadratic in x, since it is cubic in derivatives. In particular we have that[4]

$$\partial_\alpha \partial_\beta \partial_\nu \zeta_\lambda = 0. \tag{3.4}$$

[3]We could also set $\mu = \nu = 1$ but this will give the same information as setting $\mu = \nu = 0$. Hence there are only two independent choices

[4]Proof in Appendix C.

The following are possible for the ζ parameter

$$\zeta^\mu = a^\mu, \quad \text{translations}, \tag{3.5}$$

$$\zeta^\mu = b^\mu x^2 - 2x^\mu(b\cdot x), \quad \text{special conformal transformations}, \tag{3.6}$$

$$\zeta^\mu = \lambda x^\mu, \quad \text{scaling transformations (dilatation)}, \tag{3.7}$$

$$\zeta^\mu = \omega^\mu_\nu x^\nu, \quad \text{Lorentz transformations}. \tag{3.8}$$

The special conformal transformation is a position dependent rescaling and a position dependent translation. The Lorentz transformations contain the boosts and rotations. There are $1 + d + d + d(d-1)/2 = \frac{(d+1)(d+2)}{2}$ components for the parameters of conformal transformations. The generators are given by

$$\begin{aligned} P_\mu &= -\,i\partial_\mu, \\ K_\mu &= -\,i\left[2x_\mu\partial\cdot x + x^2\partial_\mu\right], \\ D &= \quad ix\cdot\partial, \\ M_{\mu\nu} &= -\,i\left[x_\mu\partial_\nu - x_\nu\partial_\mu\right]. \end{aligned}$$

The new generators are the dilatation generator D for λ and K_μ for b_μ. We are able to assemble the generators into an antisymmetric $(d+2)\times(d+2)$ matrix

$$J_{MN} = \begin{pmatrix} M_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & \frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ -\frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned} J_{\mu,d+1} &= \frac{K_\mu - P_\mu}{2}, \\ J_{\mu,d+2} &= \frac{K_\mu + P_\mu}{2}, \\ J_{d+1,d+2} &= D. \end{aligned}$$

The symmetry group for the conformal transformations is $SO(2,d)$. The Lie algebra is given by

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= -\,i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ [M_{\mu\nu}, K_\rho] &= -\,i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -\,i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \\ [P_\mu, K_\nu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D, \\ [D, K_\mu] &= iK_\mu, \\ [D, P_\mu] &= -\,iP_\mu, \\ [M_{\mu\nu}, D] &= 0. \end{aligned}$$

This is the same group as $d+1$-dimensional Anti de Sitter space.

3.5 D-branes

Closed strings are able to move freely through space [11]. On the other hand, open strings cannot move freely through space. Open strings have boundary conditions satisfied by their endpoints [11]. D-branes are extended objects defined by the property that open strings can end on them [12]. They give the means in which we may deal with these boundary conditions. D-branes may be considered as topological defects in string theory [12]. These 'walls' are in fact dynamical and respond to external excitations [11]. They also have degrees of freedom living on them [11]. We label the endpoints of the open string with "Chan-Paton factors". These correspond to the D-branes on which the string ends [11]. Each open string state may then be labelled by $|ij\rangle\lambda^a_{ij}$, where the λ^a_{ij} are $N \times N$ matrices if there are N D-branes [11]. It can be shown that these matrices belong to $U(N)$ and that these open string states live in the adjoint representation of $U(N)$. Since we also have $\mathcal{N}=4$ supersymmetry in 4 dimensions, the theory on the 4 [5] dimensional world volume of these N D3-branes is $\mathcal{N}=4$ super Yang-Mills theory with gauge group $U(N)$.

[5] A Dp brane has a $p+1$ dimensional world volume and here $p=3$.

Chapter 4

Supersymmetric Gauge Theories

In this Chapter we discuss supersymmetry and supersymmetric gauge theories. We shall present the supersymmetry algebra and then go on to discuss super Yang-Mills gauge theories.

4.1 Supersymmetry

During the 1960's physicists were wondering about the kinds of symmetries possible in particle physics [11]. They wanted to know whether spacetime symmetries and internal symmetries could be combined in a non-trivial manner. They found that it could not be done [11]. This was proved by Coleman and Mandula in their no-go theorem in 1967 [13]. Further investigation gave rise to supersymmetry which was able to bypass the Coleman-Mandula theorem. Supersymmetry is a symmetry relating bosons (having integer value spin) and fermions (having half-integer spin). Each fermion and boson has a superpartner sharing the same mass and quantum numbers, save spin. The spin differs from their superpartners spin by one-half. Supersymmetry is the unique extension of Poincarè symmetry in $d+1$ - dimensional QFT with $d > 1$ [14]. It enlarges the Poincarè algebra by including spinor *supercharges* Q^a_α (left Weyl spinor) and $(Q^a_\alpha)^\dagger$ (right Weyl spinor) with $a = 1, ..., \mathcal{N}$ and $\alpha = 1, 2$ [15]. $\mathcal{N}$ is the number of independent supersymmetries of the algebra [15]. These Weyl spinors are related to the 4-component Dirac spinor as follows

$$Q^a = \begin{pmatrix} Q^a_\alpha \\ (Q^a_\alpha)^\dagger \end{pmatrix}$$

The supercharges transform as Weyl spinors of $SO(1,3)$ [15]. They also commute with translations. The algebra of these charges is given in the following section.

4.1.1 Graded Lie algebras

One of the assumptions made in the Coleman-Mandula theorem was that the algebra of conserved charges is a Lie algebra [11]. This assumption is too restrictive. To amend this, the notion of a Lie algebra was then generalized to that of a graded Lie algebra. This allows one to evade the Coleman-Mandula theorem [11]. A graded Lie algebra is one that has generators which satisfy an anticommuting law [11]

$$\{Q_\alpha, Q_\beta\} = \text{other generators.}$$

4.1.2 Supersymmetry algebra in $3+1$ dimensions

The supersymmetry algebra is given by

$$\begin{aligned}\{Q_\alpha, Q^\dagger_{\dot\beta}\} =& P_\mu \sigma^\mu_{\alpha\dot\beta} \\ \{Q_\alpha, Q_\beta\} = 0 =& \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\}.\end{aligned}$$

The supersymmetry algebra is invariant under a global phase rotation of all supercharges [15]. This forms the group $U(1)_R$. Further, for $\mathcal{N} > 1$, different supercharges may be rotated into one another under the group $SU(\mathcal{N})_R$. These automorphism symmetries of the supersymmetry algebra are called *R-symmetries*. These symmetries may be broken in quantum field theory by anomaly effects [15].

4.2 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory is an important quantum field theory. The theory was first studied in the framework of string theory toroidal compactification in [16, 17]. Historically there was interest in this model as a result of its finiteness [18]. Later in 1997, Maldacena published his conjecture and the small field opened up once again. $\mathcal{N} = 4$ super Yang-Mills is a maximally supersymmetric, non-abelian gauge theory in four dimensions with four supercharges (sixteen real supercharges) [18]. We say maximal because in four dimensions the maximal number of supercharges is four. We will focus on gauge group U(N)

in this dissertation. The Lagrangian for this theory is [15]

$$\mathcal{L} = tr\left[-\frac{1}{2g^2}F^{\mu\nu}F_{\mu\nu} + \frac{\theta_I}{8\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu\phi^i D^\mu\phi^i \right.$$
$$\left. + \sum_{a,b,i} gC_i^{ab}\lambda_a[\phi^i,\lambda_b] + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a[\phi^i,\bar{\lambda}^b] + \frac{g^2}{2}\sum_{i,j}[\phi^i,\phi^j]^2\right]$$

where g is the gauge coupling and θ_I is the instanton angle. The constants C_i^{ab} and C_{iab} are related to the Clifford Dirac matrices for $SU(4)_R$. The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

is the Poincarè dual of F. Finally, we have

$$D_\mu = \partial_\mu\lambda + i[A_\mu, \lambda].$$

The theory has 6 real fields, called the Higgs fields, which transform as Lorentz scalars. They can be used to construct more general operators in the quantum field theory. There are other fields present in this theory, any of which may also be used to build general operators. For example, there are four fermions and gauge bosons. However we are only concerned with the scalars in this dissertation. The Higgs fields transform in the adjoint of the gauge group $U(N)$. We assemble the scalar fields into the following 3 complex combinations [7]

$$Z = \phi_1 + i\phi_2,$$
$$Y = \phi_3 + i\phi_4,$$
$$X = \phi_5 + i\phi_6.$$

The next task is to construct gauge invariant operators built out of these complex scalar fields. We require gauge invariant operators to ensure we are describing physical observables. According to AdS/CFT these operators have a dual description in type-IIB string theory on asymptotically $AdS_5 \times S^5$ backgrounds [7]. Therefore symmetries in the gauge theory should be reflected in the string theory. We'll explain the correspondence in Chapter 6. At present we will briefly discuss some of the interpretations of AdS/CFT relevant to this chapter. Single trace operators with $\mathcal{O}(1)$ fields were identified to be dual to gravitons whilst operators with $\mathcal{O}(\sqrt{N})$ fields were found to be dual to strings [7]. Multi-matrix trace operators with $\mathcal{O}(N)$ fields were identified with giant gravitons and multi-matrix trace operators with $\mathcal{O}(N^2)$ fields were identified with half-BPS geometries [19, 20, 21, 22]. To

recapitulate, the duality here refers to the equivalence of dynamics described in the two theories, i.e. the gauge theory dynamics is equivalent to the gravity dynamics.

We work in the SU(2) sector of super Yang-Mills theory, which means we only consider operators built using two of the complex fields above. We will be considering operators with $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$ fields. In particular, we use operators called restricted Schur polynomials, which are built using n Z's and m Y's. Restricted Schur polynomials are discussed in the next chapter.

4.3 BPS states

BPS states are massive representations of an extended supersymmetry algebra, with the mass being equal to the supersymmetry central charge[1]. These states play a vital role in discussions of non-perturbative duality symmetries. Theories with extended supersymmetry have been shown to have a rich dynamical structure [23]. Occasionally, supersymmetry representations may be smaller than usual owing to the fact that some supersymmetry operators are null, so that they cannot create new states. Thus the action of some supercharges may vanish [24]. This depends on the relation between the mass of a multiplet and some central charge appearing in the algebra. These central charges depend on electric and magnetic charges of the theory. They also depend on the expectation values of scalars (moduli) or coupling constants in rigid supersymmetry theories [24]. In a sector with given charges, BPS states are the lowest-lying states and they saturate the so-called BPS bound [24]. This bound has the following form for point-like states [24]

$$2M \geq \text{maximal eigenvalue of Z}$$

where M is the mass and Z is the central charge matrix.

There is a special behaviour related to BPS states. First, at generic points in the moduli space, they are completely stable [24]. This is true for theories with more than eight conserved supercharges like $\mathcal{N} = 4$ supersymmetry [24]. Second, their mass formula is expected to be exact if ones uses the renormalized values for the mass and moduli (couplings).

It is also possible to find BPS states that only have a certain number of supersymmetries preserved. Half-BPS states have half the supersymmetries preserved. Similarly, Quarter-BPS states only have a quarter of the supersymmetries preserved, and so on. For $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions there are sixteen supersymmetries. Thus half-BPS states break eight supersymmetries and quarter-BPS states break twelve supersymmetries.

[1] In a CFT like $\mathcal{N} = 4$ super Yang-Mills theory, it is the scaling dimension of the state that is equal to the central charge.

4.3.1 The half-BPS sector

The so called half-BPS sector, is one in which our operators are built out of a single complex matrix field and for which half of the supersymmetries of the gauge theory are preserved. This will be the sector of choice for all the work carried out in this dissertation.

The dictionary which allows us to move across the duality is, for the half-BPS sector, well established [19, 20, 21, 22]. To make use of this, one must first define the scaling or conformal dimension Δ, such that

$$\Delta = \Delta_0 + \gamma$$

where Δ_0 is the bare (or classical) dimension and γ is the anomalous dimension. The number of fields present in the Schur polynomial is given by Δ. In the half-BPS sector Δ is equal to the number of units of $\mathcal{R}$-charge[2]. Each field Z gives a single unit of $\mathcal{R}$-charge. There exists a mapping from the $\mathcal{R}$-charge in the QFT to the angular momentum in the string theory. In addition, the scaling dimension spectrum matches the energy spectrum in the dual string theory [8]. This provides a way to move between the gauge theory and the string theory.

The identification of operators with $\mathcal{R}$-charge of $\mathcal{O}(1)$ and gravitons has been checked using the AdS/CFT correspondence [8]. The computation performed in [25] reproduces field theory correlation functions using supergravity graviton calculations. For operators with $\mathcal{R}$-charge of $\mathcal{O}(\sqrt{N})$ it was found [8], [26] that the eigenvalues[3] of the Dilatation operator corresponded to the expected string energies. Thus these operators do indeed correspond to strings [8]. For the details see reference [29] in [8]. Since the non-planar diagrams can not be dropped for $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$, we have to use the Schur polynomials [8]. A big problem with the large $\mathcal{R}$-charge operators is that we are not explicitly able to build these restricted Schur polynomials [7].

[2]The $\mathcal{R}$-charge is a conserved charge associated with supersymmetry.

[3]The eigenvalues of the Dilatation operator are the scaling dimensions, Δ.

Chapter 5

Schur and Restricted Schur Polynomials

We will focus on studying Schur and restricted Schur polynomials using symmetric and unitary group theory, S_n and $U(N)$ respectively.

5.1 Schur polynomials

A Schur polynomial is defined for a single type of arbitrary matrix Z as follows

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} ... Z^{i_{n-1}}_{i_{\sigma(n-1)}} Z^{i_n}_{i_{\sigma(n)}}.$$

The label R is a group theory label. Young diagrams with n boxes are in a one-to-one correspondence with the irreducible representations of the symmetric group S_n [27]. Young diagrams with any number of boxes and having less than or equal to N rows are in one-to-one correspondence with the irreducible representations of the Unitary group $U(N)$. Schur polynomials are thus associated with a particular irreducible representation R of S_n [27]. The factor $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in the irreducible representation R. For $Z \in U(N)$, $\chi_R(Z)$ is the character of the matrix Z in irreducible representation R of $U(N)$. The indices i_1,...,i_n range from 1,...,m where Z is an $m \times m$ matrix. The lower indices in each term of the sum are a particular permutation of the indices i_1,...,i_n acted on by σ.

The two point correlation function for Schur polynomials in the free field theory limit is given by [28]

$$\langle \chi_R(Z) \chi_S(Z^\dagger) \rangle = \delta_{RS} f_R$$

where f_R is a product with one factor for each box in R

$$f_R = \prod_{i,j}(N - i + j).$$

We see that this correlator is diagonal in the Young diagram labels. The factor $(N - i + j)$ is called a weight and is given to the box in the i-th row and j-th column.

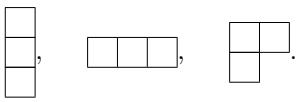
5.2 Restricted Schur polynomials

We concentrate on operators with $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$ fields. For such cases, we use *restricted Schur polynomials*. They are characters of irreducible representations of $U(N)$. Restricted Schur polynomials are the multi-matrix generalization of the Schur polynomial [29]. We will study restricted Schur Polynomials composed of n Z's and m Y's. They are given by

$$\chi_{R,(r,s),jk}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s),jk}(\sigma) Y^{i_1}_{i_{\sigma(1)}} ... Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} ... Z^{i_{m+n}}_{i_{\sigma(m+n)}}.$$

The labels R and (r,s) are group theory labels. R corresponds to a Young diagram with $n+m$ boxes, being an irreducible representation (irrep) of S_{n+m}. Similarly r and s correspond to Young diagrams with n and m boxes being irreps of S_n and S_m respectively. Together (r,s) label an irreducible representation of $S_n \times S_m$ [7]. The group $S_n \times S_m$ is a subgroup of S_{n+m}. The labels j and k are multiplicity labels appearing since (r,s) may be subduced more than once from R. $\chi_{R,(r,s),jk}(\sigma) = Tr_{(r,s)\alpha\beta}(\Gamma_R(\sigma))$ is called the restricted character [30]. It is not a trace over the whole space, but only over the (r,s) subspace. More precisely it is a trace over the subspace whose column index belongs to the $S_n \times S_m$ irreducible representation $(r,s)\alpha$ and whose row index belongs to the $S_n \times S_m$ irreducible representation $(r,s)\beta$ [30]. Note that this is only a trace in the usual sense when $\alpha = \beta$. When this is not the case we are not summing over diagonal matrix elements [30].

There is a nice agreement between the number of restricted Schur polynomials and the number of gauge invariant operators, which we briefly discuss here. We said that R corresponds to a Young diagram with $n+m$ boxes, but n and m also tell us how many fields of type Z and type Y we have, respectively. Suppose $n = 3$ and $m = 0$. Then the following are the possible Young diagrams we have labelling representations of R



From these we can build three restricted Schur polynomials. The gauge invariant operators are

$$Tr(Z^3), \qquad (Tr(Z))^3, \qquad Tr(Z^2)Tr(Z).$$

The number of gauge invariant operators is equal to the number of restricted Schur polynomials.

For Z and Y, the free theory two point correlation function for restricted Schur polynomials is [29]

$$\langle \chi_{R,(r,s)jk}(Z,Y)\chi_{T,(t,u)lm}(Z,Y)^\dagger\rangle = \delta_{R,(r,s)T,(t,u)}\delta_{kl}\delta_{jm}f_R\frac{\text{hooks}_R}{\text{hooks}_r\text{hooks}_s}$$

where f_R is

$$f_R = \prod_{i,j}(N-i+j),$$

a product of weights given to each box in the Young diagram. We see that the two point function is diagonal in all Young diagram labels as well as the multiplicity labels.

The Schur polynomials with $\mathcal{O}(N)$ fields are dual to giant gravitons. The restricted Schur polynomials with $\mathcal{O}(N)$ fields are dual to excited giant gravitons [31, 30, 32]. As mentioned earlier, Schur polynomials comprising $\mathcal{O}(N^2)$ fields are dual to new background geometries.

Chapter 6

The AdS/CFT Correspondence

In this chapter we briefly review the AdS/CFT correspondence [1], also known as the Maldacena conjecture. We shall present a summarized version of the calculation that motivated Maldacena. We then go on to explain the conjecture in as simple a way as possible. The role of Schur and restricted Schur polynomials is discussed thereafter. This chapter then concludes with a discussion relating the variables in the two theories.

6.1 Motivation

The Maldacena conjecture describes the conjectured equivalence between a theory with gravity and a theory without gravity. More specifically, it relates type-IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions[1]. The simple idea behind the motivation for this equivalence comes from considering N coincident, parallel $D3$ branes, where N is the rank of the gauge group. This system can be described in two different ways. The first indication that the two descriptions were equivalent in some appropriate limit came from the calculations of the absorption cross-sections for the low energy waves incident on the stack of $D3$ branes from a transverse direction [33, 34, 35]. The first description of the $D3$ brane stack is in terms of $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory on its world volume [35]. At low energies it interacts with the bulk closed string excitations [35]. The second description is in terms of the Ramond-Ramond charged 3-brane background of type-IIB closed superstring theory [35]. If N is large then the stack acts as a heavy object embedded in the theory of closed strings, containing gravity. The stack is charged and acts like a source for supergravity fields [27]. This description is only trustworthy at large N and large ‘t Hooft coupling, where the background is approximately

[1]This is not the most general form of the conjecture.

flat. The first description is in terms of open and closed strings, where the $D3$ branes are treated as boundary conditions for the open strings (recall that open strings end on D branes) [27]. The second description is only in terms of closed strings. We will discuss the calculation of these cross sections next. It must be noted that the calculation is very different on either side of the correspondence, even though we calculate the same object in the quantum field theory as in the gravity theory. In the field theory we are interested in calculating a cross section by summing Feynman diagrams as described below. However, in the gravity theory we are concerned with solving a wave equation coupled to some potential (brane). We are then interested in the amount of flux lost to the throat region of the brane. For further details see the original paper by Klebanov in [35].

The low energy limit for the D branes coupled to massless bulk fields is considered [35]. For example, a graviton (a closed string) is scattered off the N coincident $D3$ brane stack. The action is given by

$$S = S_{\text{brane}} + S_{\text{int}} + S_{\text{bulk}}$$

where [35]

$$S_{int} = \frac{\sqrt{\pi}}{\kappa} \int d^4x \left[tr \left(\frac{1}{4} \Phi F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{4} C F^*_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{2} h^{\alpha\beta} T_{\alpha\beta} \right]$$

and $T_{\alpha\beta}$ is the stress energy tensor of $\mathcal{N} = 4$ SYM theory. The above action tells us that the branes are coupled to the dilaton Φ, the Ramond-Ramond scalar C and the graviton $h_{\alpha\beta}$ [33, 34, 35]. The cross section for the graviton scattering off the brane stack was calculated to leading order and weak coupling. The computation involves summing Feynman diagrams.

The cross section for the graviton scattering off the massive Ramond-Ramond charged 3-brane background with geometry [35],

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2\right)$$

was also calculated. The computation entails solving a Scroedinger-like wave equation with potential sourced by the 3-brane geometry. Both were found to be in agreement. In the low energy limit we get long wavelength propagating modes decoupled from the heavy charged background. This gives free IIB-supergravity. In the near horizon region, i.e. $r \to 0$, excited modes are red shifted. Thus, this gives IIB-superstring theory on $AdS_5 \times S^5$.

6.2 The conjecture

Consider a non-empty box. This box has a boundary, given by the walls of the box. They contain everything that is known as 'inside' of the box. Stated plainly, the Maldacena

conjecture proposes an equivalence or duality between a theory of gravity living inside the box and a quantum field theory living on the boundary of the box. This quantum field theory has no gravity. After work carried out in [34, 33], Maldacena noticed the comparison given in the table below.

Description	Everywhere	Near Brane
D brane stack	free IIB supergravity	$\mathcal{N} = 4$ U(N) SYM
Ramond-Ramond charged source	free IIB supergravity	IIB string theory on $AdS_5 \times S^5$

Table 6.1: Comparison between the D brane stack and the Ramond-Ramond charged source.

In the low energy limit, S_{int} vanishes and we find that

$$S_{bulk} \to S_{IIB\ supergravity}$$
$$S_{brane} \to S^{3+1}_{\mathcal{N}=4SYM}.$$

It was after this comparison that Maldacena made the conjecture that type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ U(N) super Yang-Mills theory in 3+1 dimensions. The conjecture extends to other quantum gravities in Anti de Sitter space and supersymmetric conformal field theories on the boundaries of these Anti de Sitter spaces. The correspondence is a strong/weak coupling duality [10]. For the large N 't Hooft limit, AdS/CFT relates the region of weak field strength coupling (small g^2_{YM}) $\lambda = g^2_{YM}N$ in the super Yang-Mills theory to the region of high curvature in the string theory, and vice versa [10].

6.3 The role of Schur polynomials in AdS/CFT

The two theories described above are two different languages in which to describe the same dynamics. The power of this observation lies in the fact that, in general, calculations that are difficult in the field theory are easy in the string theory and vice versa [27]. Thus the next important step is to develop a dictionary that can be used to make the correspondence into a precise statement. Schur and restricted Schur polynomials are available to do this for us.

6.4 Measurable quantities

In quantum field theory our observables are given by the S-matrix. For a conformal field theory this is no longer a sensible object to pick as our observable. This is because there is

no notion of 'far' past and 'far' future. The two point function for a CFT is given by

$$\langle \mathcal{O}_a(\vec{x})\mathcal{O}_b(\vec{y})\rangle = \frac{\delta_{ab}}{|\vec{x}-\vec{y}|^{2\Delta}}.$$

Here Δ is the conformal dimension of the operators involved. These operators have a well defined scaling dimension. This quantity replaces the S-matrix as our observable in the CFT. At the classical level, Δ counts the number of fields in the operator. At the full quantum level, the conformal dimensions are the eigenvalues of the Dilatation operator.

These conformal dimensions are related via the AdS/CFT correspondence to energies (eigenvalues of the Hamiltonian) in the string theory [36]. The Hamiltonian of the string theory is related to the Dilatation operator of the CFT on the boundary of the AdS space. To see how this is done consider the following example. Working in $1+1$ dimensional Minkowski space we have the following metric

$$ds^2 = -dt^2 + dx^2.$$

Put $y = it$. This Wick rotates us to Euclidean space. Then we get

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

after changing to spherical coordinates. Now do the following coordinate transformation, set $r = e^t$ to get

$$ds^2 = e^{2t}dt^2 + e^{2t}d\theta^2.$$

Which, after a conformal rescaling, is just the metric of the $\mathbb{R}\times S^1$ boundary of the AdS_3 space,

$$ds^2 = dt^2 + d\theta^2.$$

If we consider a time translation $t \to t + a$, of which the Hamiltonian is the generator, we find that $r \to e^a r$. Thus time translations in the string theory result in scale (rescaling) transformations in the CFT. Hence we see the connection between energy and the conformal dimension.

6.5 Relating Parameters

The relation between string theory parameters and those of the gauge theory are as follows [1]

$$g_s = g_{YM}^2$$
$$\left(\frac{R}{l_s}\right)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda.$$

Here g_s is the string coupling and l_s is the string length. R is the radius of curvature of the AdS_5 and S^5 spaces of $AdS_5 \times S^5$. The parameters of super Yang-Mills theory with gauge group $U(N)$ are N and g_{YM}^2, the coupling constant.

6.6 Tests of the AdS/CFT correspondence

The conjecture is well tested. For an excellent review and more references see [10]. Here we briefly discuss the tests mentioned in chapter 3 in [10], which deals with the case of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and type IIB string theory on $AdS_5 \times S^5$. A direct comparison of correlators in the field theory to the string theory correlators is generally not possible. This is because presently we can only compute most correlators perturbatively in λ on the field theory side and perturbatively in $1/\sqrt{\lambda}$ on the string theory side [10]. Thus comparing properties which depend on the coupling is not possible[2]. It was eventually realised that there are several properties of these theories that do not depend on the coupling. These may be compared to test the duality. These properties are [10]:

1. Global symmetries. These do not in general change as we change the coupling[3].
2. The moduli space of the theory does not depend on the coupling.
3. Qualitative behaviour of the theory upon deformations by relevant or marginal operators also does not depend on the coupling.
4. Other tests include: existence of confinement for the finite temperature theory [37].

[2]For recent dramatic progress, in the planar limit, see the review [4] and references therein.
[3]Although changes may occur for extreme values of the coupling [10].

Chapter 7

The Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory

7.1 The scaling dimension

The dilatation operator provides a means to investigate the scaling dimensions in a conformal field theory [38]. There are a couple of ways to calculate the scaling dimensions for local operators in a conformal field theory. We shall discuss these shortly. First it is important to understand how the scaling dimension affects correlation functions in the conformal field theory. These correlation functions obey certain relations due to conformal symmetry [38]. This restricts, greatly, the structure of correlation functions of the theory.

7.2 Action of the dilatation operator

The Dilatation operator for $\mathcal{N} = 4$ super Yang-Mills theory was studied extensively in the planar limit by Beisert in [38, 39]. In this dissertation and in this section in particular, we study the dilatation operator in a large N but non-planar limit. The dilatation operator in the $SU(2)$ sector of the theory is given by [39]

$$\mathcal{D} = \sum_{j=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2} \right)^j \mathcal{D}_{2j},$$

where $\mathcal{D}_{2j}$ is the j-th loop contribution to the dilatation operator. It acts on the restricted Schur polynomial $\chi_{R,(r,s)jk}(Z, Y)$ as studied in [40, 41, 6]. The action of the one loop ($j = 1$) dilatation operator is then given by [39]

$$\mathcal{D}_2 = -g_{YM}^2 Tr \left([Y, Z] \left[\frac{d}{dY}, \frac{d}{dZ} \right] \right).$$

Henceforth we refer to the one loop dilatation operator as $\mathcal{D}$ for notational ease. It is convenient to work with normalized operators so that we arrive at a unit two point function [8]. The normalized operators $O_{R,(r,s)(Z,Y)}$ can be found from

$$\chi_{R,(r,s)jk}(Z,Y) = \sqrt{\frac{f_R \text{hooks}_R}{\text{hooks}_r \text{hooks}_s}} O_{R,(r,s)jk}(Z,Y).$$

where $\chi_{R,(r,s)jk}(Z,Y)$ is the restricted Schur polynomial discussed in Chapter 5. The action of the dilatation operator on these normalised operators is given by [41]

$$DO_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} N_{R,(r,s)jk;T,(t,u)lq} O_{T,(t,u)lq}(Z,Y)$$

where[1]

$$N_{R,(r,s)jk;T,(t,u)lq} = -\, g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \times$$
$$\times\, Tr\left(\left[\Gamma_R((1,m+1)), P_{R\to(r,s)jk}\right] I_{R'T'} \left[\Gamma_T((1,m+1)), P_{T\to(t,u)lq}\right] I_{T'R'}\right).$$

The factor $c_{RR'}$ is the factor of the corner box removed from Young diagram R to obtain diagram R'. Similarly T' is obtained by removing a box from T. The intertwiner $I_{R'T'}$ is a map from the carrier space of irreducible representation R' to the carrier space of irreducible representation T'. Schur's Lemma imposes the condition that the Young diagrams for R' and T' be of the same shape in order to obtain a non-zero intertwiner. For the case of only two impurities, there are no multiplicity labels. Thus, the action of the dilatation operator becomes

$$DO_{R,(r,s)}(Z,Y) = \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} O_{T,(t,u)}(Z,Y).$$

It is the trace appearing in $N_{R,(r,s);T,(t,u)}$ in this last equation that we wish to calculate. First, however, we need to find the individual components making up the matrix that is to be traced, that is, the projectors, the swaps and the intertwiners.

7.3 Operators with dimension of $\mathcal{O}(N^2)$

For this dissertation we have focused on the sector of the theory consisting of operators with scaling dimension $\mathcal{O}(N^2)$. In this instance, Young diagrams have $\mathcal{O}(N)$ rows and $\mathcal{O}(N)$ columns. We are only considering the case of two impurities in which case two boxes are to be removed from the Young diagrams to subduce the representation (r,s) from R. Let r_i and r_j be the lengths of row i and row j respectively. Then for any two rows for which

[1] For a derivation of this result see [8], [41].

$|i-j| \sim \mathcal{O}(1)$ we have that $|r_i - r_j| \sim \mathcal{O}(1)$. Row r_1 corresponds to the first row which has the longest length. The lengths of the following rows decrease as we move down the Young diagram. We are only allowed to remove boxes such that a valid Young diagram remains. There are then $\mathcal{O}(N)$ ways to remove the first box and $\mathcal{O}(N)$ ways to remove the second box. The location of the boxes to be removed is illustrated in the diagram below.

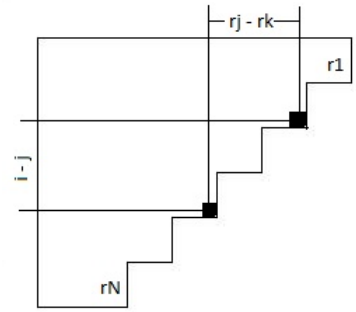


Figure 7.1: Diagram displaying the rows and columns of the boxes to be removed from the Young diagram.

The content of box i is $c_i = r_i - i$. The difference between the content of the two boxes is

$$c_{ij} = r_j - r_i + i - j$$

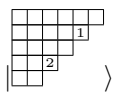
where we let $i > j$. This is called the *axial distance* between boxes i and j.

7.4 States

In this section we discuss the labelling of our states. The irreducible representation R of S_{n+2} may be thought of as a partition of $n+2$ boxes. We label states as follows

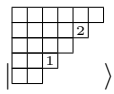
$$|\{R\}, \text{row of first box removed}, \text{row of second box removed}; a\rangle$$

where $\{R\}$ is the collection of row lengths of R and a labels a Young-Yammonouchi state for the remaining n boxes. An example of how these labels work follows below



$$\rangle = |\{6,5,4,3,2\},2,4;a\rangle$$

and



$$\rangle = |\{6,5,4,3,2\},4,2;a\rangle$$

where $a = 1, ..., d_r$ and d_r is the dimension of the representation r of S_n, r is obtained by removing the labeled boxes from R. The impurity irreducible representations are then labelled [42] as

$$|\{R\}, \square\square, i, j; a\rangle = \sqrt{\frac{c_i - c_j + 1}{2(c_i - c_j)}}|\{R\}, i, j; a\rangle + \sqrt{\frac{c_i - c_j - 1}{2(c_i - c_j)}}|\{R\}, j, i; a\rangle$$

where we have included the label for the s representation concerned. In this case we have the symmetric representation. Similarly for the antisymmetric representation

$$|\{R\}, \begin{smallmatrix}\square\\\square\end{smallmatrix}, i, j; a\rangle = \sqrt{\frac{c_i - c_j - 1}{2(c_i - c_j)}}|\{R\}, i, j; a\rangle - \sqrt{\frac{c_i - c_j + 1}{2(c_i - c_j)}}|\{R\}, j, i; a\rangle.$$

Note that the same index a is used on both sides of these equations. This is because it is the same Young-Yammonouchi states for the Z-boxes. Here r is a partition of n boxes. In terms of our diagram above

$$c_i - c_j = r_i - r_j - i + j.$$

The row lengths of the irreducible representation r is related to the row lengths of R by

$$r_k = R_k - \delta_{ik} - \delta_{jk}$$

where the two boxes to be removed are removed from the *i-th* and *j-th* rows.

7.5 Projectors

The action of the projectors on the Z boxes is trivial, simply being given by an identity operator. However, their action on the Y boxes remains non-trivial with the projectors being given by

$$P_{R,s_1,i,j} = \sum_{a=1}^{d_r} |\{R\}, \square\square, i, j; a\rangle\langle\{R\}, \square\square, i, j; a|$$

and

$$P_{R,s_2,i,j} = \sum_{a=1}^{d_r} |\{R\}, \begin{smallmatrix}\square\\\square\end{smallmatrix}, i, j; a\rangle\langle\{R\}, \begin{smallmatrix}\square\\\square\end{smallmatrix}, i, j; a|$$

7.6 The swap $\Gamma(n, n+1)$

There is a simple rule for matrices representing an adjacent permutation. The permutation we need is not an adjacent permutation. Fortunately any permutation may be written in terms of adjacent permutations. For our case, $(1, 3) = (1, 2)(2, 3)(1, 2)$. To evaluate this action we need to label a third box. As a first step we rewrite our states to incorporate

the placement of a third box which is to be moved around when acted upon by the swap $\Gamma(n, n+1)$. The states may then be written as (this is an identity between subspaces)

$$\{|\{R\}, i, j; a\rangle\} = \{\oplus_k \theta(r_k - r_{k-1} - 1)|\{R\} i, j, k; \tilde{a}\rangle\}$$

where k labels the row of the third box to be removed and the direct sum runs over all possible ways in which a third box may be removed to produce a valid Young diagram. The Heaviside function ensures this for us. It is defined by

$$\begin{aligned} \theta(x) =& 1 \qquad \text{for } x \geq 0 \\ \theta(x) =& 0 \qquad \text{for } x < 0. \end{aligned}$$

The action of the adjacent permutations on a state is

$$\Gamma_R(1,2)|\{R\}, i, j, k; a\rangle = \frac{1}{c_i - c_j}|\{R\}, i, j, k; a\rangle + \sqrt{1 - \frac{1}{(c_i - c_j)^2}}|\{R\}, j, i, k; a\rangle$$

and

$$\Gamma_R(2,3)|\{R\}, i, j, k; a\rangle = \frac{1}{c_j - c_k}|\{R\}, i, j, k; a\rangle + \sqrt{1 - \frac{1}{(c_j - c_k)^2}}|\{R\}, i, k, j; a\rangle.$$

We will deal with the final swap $\Gamma_R(1,3)$ in the section discussing the trace. Using results from that section we will find that we are afforded a large simplification that allows us to avoid writing out every term in the calculation of the swap. There are many terms.

7.7 Intertwiners

For a general derivation of the intertwiners see Appendix D. In particular, the examples given describe how the intertwiners are constructed for this case. They are given by

$$I_{R'T'} = \sum_A |\{R\}, j; A\rangle\langle\{T\}, k; A|$$

and

$$I_{T'R'} = \sum_A |\{T\}, k; A\rangle\langle\{R\}, j; A|$$

where A labels a Young-Yammonouchi state in R'_j or T'_k where $R'_j = T'_k$. R'_j is the Young diagram obtained by the removal of a single box from row j of R. T'_k is the Young diagram obtained by the removal of a single box from row k of T.

7.8 The mixing of states

Here we discuss which states mix with each other. Further we identify the leading contribution to the dilatation operator. This will simplify the number of terms that need to be

dealt with in the large N limit. The leading terms each make an $\mathcal{O}(N)$ contribution. In the first case $R = T$, $r = t$ but s need not be equal to u. For this case the third box may be removed from any row, contributing $\mathcal{O}(N)$ terms. Each term will have a different coefficient depending upon the row from which the box is removed. For the second case we have $R \neq T$, $r = t$ and again s need not be equal to u. Again we find that there are $\mathcal{O}(N)$ terms, each with a different coefficient.

There are two more cases for which $r \neq t$. In both of these cases we are forced to remove the third box from a single definite location. This means that only one term contributes. So the first two terms yield the main contribution to the dilatation operator. These two cases also tell us that the second labels must match.

7.9 The trace

In this section we discuss the calculation of

$$\mathrm{Tr}\left(\left[\Gamma_R((1,m+1)), P_{R\to(r,s)}\right] I_{R'T'} \left[\Gamma_T((1,m+1)), P_{T\to(t,u)}\right] I_{T'R'}\right).$$

We labeled our projectors differently to employ a more transparent notation. For this computation $m = 2$ for two impurities. The trace can be written as

$$\mathrm{Tr}\left([\Gamma_R((1,3)), P_{R,s,i,j}]\, I_{R'T'}\, [\Gamma_T((1,3)), P_{T,u,k,i}]\, I_{T'R'}\right).$$

Consider the case of $s = u = s_1$. There are four terms that need to be calculated. They are

$$\begin{aligned}
T_1 =& \mathrm{Tr}\left(P_{R,s_1,i,j}\Gamma_R((1,3))I_{R'T'}\Gamma_T((1,3))P_{T,s_1,k,i}I_{T'R'}\right)\\
T_2 =& \mathrm{Tr}\left(P_{R,s_1,i,j}\Gamma_R((1,3))I_{R'T'}P_{T,s_1,k,i}\Gamma_T((1,3))I_{T'R'}\right)\\
T_3 =& \mathrm{Tr}\left(\Gamma_R((1,3))P_{R,s_1,i,j}I_{R'T'}\Gamma_T((1,3))P_{T,s_1,k,i}I_{T'R'}\right)\\
T_4 =& \mathrm{Tr}\left(\Gamma_R((1,3))P_{R,s_1,i,j}I_{R'T'}P_{T,s_1,k,i}\Gamma_T((1,3))I_{T'R'}\right).
\end{aligned}$$

For the full calculation, see Appendix E. We found $T_1 = T_4$ and $T_2 = T_3$. The trace is given by

$$T_1 = \sum_l \sqrt{\frac{c_i - c_j - 1}{2(c_i - c_j)}}\sqrt{\frac{c_k - c_j + 1}{2(c_k - c_j)}}(\alpha_1(l) + \alpha_2(l))(\beta_1(l) + \beta_2(l))d_{R''_{jil}}$$

and

$$T_2 = \sum_l \sqrt{\frac{c_k - c_i + 1}{2(c_k - c_i)}}\sqrt{\frac{c_i - c_j - 1}{2(c_i - c_j)}}(\beta_1(l) + \beta_2(l))(\alpha_1(l) + \alpha_2(l))d_{R''_{jil}}.$$

Where

$$\alpha_1(l) = \sqrt{\frac{c_k - c_i + 1}{2(c_k - c_i)}} \frac{1}{c_i - c_l} \frac{1}{c_k - c_i}$$
$$\alpha_2(l) = \sqrt{\frac{c_k - c_i - 1}{2(c_k - c_i)}} \frac{1}{c_k - c_l} \sqrt{1 - \frac{1}{(c_i - c_k)^2}}$$

and

$$\beta_1(l) = \sqrt{\frac{c_i - c_j + 1}{2(c_i - c_j)}} \frac{1}{c_j - c_l} \sqrt{1 - \frac{1}{(c_i - c_j)^2}}$$
$$\beta_2(l) = \sqrt{\frac{c_i - c_l - 1}{2(c_i - c_j)}} \frac{1}{c_i - c_l} \frac{1}{c_j - c_i}.$$

7.10 Dilatation operator coefficient

In this section we explain how to evaluate the coefficient

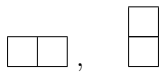
$$g_{YM}^2 \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}}$$

in the large N limit. The Young diagrams R, T, r, t, s and u each have p rows. Let R_i, T_i, r_i, t_i, s_i and u_i denote the row lengths of each row respectively for $i = 1, 2, ..., p$. The row length of a row is given by the number of boxes in that particular row. We assume that p is fixed to be $\mathcal{O}(N)$. The top row, which is always the longest row, has the value $i = 1$. Similarly, the bottom row, which is the shortest row, has $i = p$.

As explained in Section 7.8, the second labels r and t of the dilatation operator must be the same, that is $r = t$. Thus

$$\text{hooks}_r = \text{hooks}_t.$$

For the case of two impurities, we have two possible representations



both of which have hooks = 2. Thus

$$\text{hooks}_s = \text{hooks}_u.$$

Next, recall that f_R is the product of the factors in Young diagram R. We also know that $R' = T'$. Hence we learn that

$$c_{RR'} \sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'} c_{TT'}}$$

where $c_{RR'}$ is the factor associated to the box that must be removed from R to obtain R' and $c_{TT'}$ is the factor associated to the box that must be removed from T to obtain T'. Finally, we are left with evaluating

$$\sqrt{\frac{\text{hooks}_T}{\text{hooks}_R}}.$$

For operators with dimension $\mathcal{O}(N)$, the difference between row lengths in the Young diagrams with $p \sim \mathcal{O}(1)$ are of $\mathcal{O}(N)$. In this case we may employ a useful approximation, for details on this calculation see Appendix F. We state the result below. For the diagrams R and T, the row lengths R_i are of order N. Further, R and T may differ at most by the placement of a single box. This implies that $R_i = T_i$ for all but two values of i, suppose these values are $i = a, b$. For these values of i we the following relation between the row lengths in R and T

$$R_a = T_a - 1, \qquad R_b = T_b + 1.$$

We find that

$$\frac{\text{hooks}_T}{\text{hooks}_R} = \frac{R_b}{R_a}(1 + \mathcal{O}(N^{-1})).$$

For the case where our operators have dimension $\mathcal{O}(N^2)$ we do not have such an approximation. Here, the difference between row lengths is only $\mathcal{O}(1)$. So that

$$\frac{\text{hooks}_T}{\text{hooks}_R} \neq \frac{R_b}{R_a}\left(1 + O(N^{-1})\right).$$

We may write

$$\frac{\text{hooks}_T}{\text{hooks}_R} = \frac{\text{hooks}_T}{\text{hooks}_{T'}}\frac{\text{hooks}_{R'}}{\text{hooks}_R}\frac{\text{hooks}_{T'}}{\text{hooks}_{T''}}\frac{\text{hooks}_{R''}}{\text{hooks}_{R'}}$$

where we recall that $\frac{\text{hooks}_{R''}}{\text{hooks}_{T''}} = 1$. R'' is the diagram obtained from R' by removing a single box. Similarly for T''. These factors may be found exactly using the following

$$\frac{\text{hooks}_R}{\text{hooks}_{R'_j}} = R_j \frac{\prod\limits_{\substack{k \\ k\neq 0}} (R_1 - R_k + k)}{\prod\limits_{\substack{k \\ k\neq 0}} (R_1 - R_k + k - 1)}.$$

Chapter 8

Conclusion

This dissertation has investigated a new limit of the AdS/CFT correspondence by employing restricted Schur polynomials. The AdS/CFT correspondence plays a central part in modern string theory, revolutionizing the field by providing a means to study quantum field theories without any inclusion of gravity. We are then able to relate them to theories of gravity. This uses the holographic principle. The quantum field theory studied in this dissertation is $\mathcal{N} = 4$ super Yang-Mills theory. This theory is conjectured, via the aforementioned correspondence, to be dual to type IIB string theory on asymptotically $AdS_5 \times S^5$ geometry.

The purpose of the current study was to evaluate the action of the one loop dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory in the sector with conformal dimension $\Delta \sim \mathcal{O}(N^2)$. Further we aimed to diagonalize this action and then treat the corresponding operator as a Hamiltonian for some dynamical system. The energy spectrum of the Hamiltonian was then to be calculated. The main goal was to derive a linearised Einstein equation.

The action of the one loop dilatation operator was evaluated. Future work aims to extend the computation to include many impurities, to diagonalise the dilatation operator and then calculate the energy eigenvalues. The next exciting step will be to find a linearized Einstein equation, as proposed.

Previous work [4] has shown that $\mathcal{N} = 4$ super Yang-Mills theory is integrable in the planar limit. Thus, an interesting question is whether or not integrability holds in other large N limits of the theory [7]. Past arguments have shown that integrability is spoiled by non-planar corrections [5]. In 2011 this conclusion was shown to be incorrect in [6]. Operators labelled by Young diagrams having a conformal dimension of $\Delta \sim \mathcal{O}(N)$ were considered. Two specific classes of operators were studied. The first class had operators labelled by Young diagrams having two long columns. The second comprised operators labelled by Young diagrams containing two long rows. This result was generalized to p long

rows or columns in [7]. For the case of p long rows or columns, the non-planar limit was found to be integrable. In both cases [6, 7] integrability was proven by showing that the dilatation operator reduces to a decoupled set of harmonic oscillators. In this dissertation we explore this idea further by extending the study to a Young diagram with $\mathcal{O}(N)$ long rows and $\mathcal{O}(N)$ long columns.

The significance of this work will be to provide a further proof of integrability in a new large N non-planar limit, contributing to the work already mentioned above.

It is important to note that this calculation of the dilatation operator is free from any approximations, it is the exact large N result. The main obstacle in finding restricted Schur polynomials is that it is not easy to construct the projectors for the case of more than two impurities. The Schur-Weyl duality that was discovered in the displaced corners approximation makes finding these projectors rather simple.

Further work could look toward finding similar approximations that would make the problem more tractable for the case of more impurities. It would be interesting to note whether there is a type of Schur-Weyl duality present in the large N limit considered in this study.

Appendix A

Schur-Weyl Duality

In this appendix we show that the actions of the unitary and symmetric groups commute.

A.1 Definition of the action of the Unitary and Symmetric groups on $V^{\otimes n}$

Define a vector space $V^{\otimes n}$ given by

$$V^{\otimes n} \equiv \underbrace{V \otimes V \otimes ... \otimes V}_{n \text{ times}}$$

where V is an N dimensional vector space. The action of U(N) on this space is given by

$$U : T^{i_1, i_2 \ldots, i_n} \to \Gamma(U)^{i_1}_{j_1} \Gamma(U)^{i_2}_{j_2} ... \Gamma(U)^{i_n}_{j_n} T^{j_1, j_2 \ldots, j_n}$$

and the action of the symmetric group is

$$\sigma : T^{i_1, i_2 \ldots, i_n} \to T^{i_{\sigma(1)}, i_{\sigma(2)} \ldots, i_{\sigma(n)}}.$$

In the above $\Gamma(U)$ is a matrix representing $U \in U(N)$ in the fundamental representation. σ is a permutation belonging to S_n.

A.2 Commuting Actions

First act with the unitary group then act with the symmetric group to get

$$\begin{aligned} \sigma U : T^{i_1, i_2 \ldots, i_n} =& \sigma \left(\Gamma(U)^{i_1}_{j_1} \Gamma(U)^{i_2}_{j_2} ... \Gamma(U)^{i_n}_{j_n} T^{j_1, j_2 \ldots, j_n} \right) \\ =& \Gamma(U)^{i_\sigma(1)}_{j_1} \Gamma(U)^{i_\sigma(2)}_{j_2} ... \Gamma(U)^{i_\sigma(n)}_{j_n} T^{j_1, j_2 \ldots, j_n}. \end{aligned}$$

Now act with the symmetric group and then the unitary group

$$\begin{aligned} U\sigma : T^{i_1,i_2\ldots,i_n} =& U\left(T^{i_{\sigma(1)},i_{\sigma(2)}\ldots,i_{\sigma(n)}}\right) \\ =& \Gamma(U)^{i_\sigma(1)}_{j_1}\Gamma(U)^{i_\sigma(2)}_{j_2}\ldots\Gamma(U)^{i_\sigma(n)}_{j_n}T^{j_1,j_2\ldots,j_n}. \end{aligned}$$

Thus U and σ are commuting actions. Consequences of this are that symmetric states remain symmetric after a unitary operation. Similarly, unitarity is also preserved after a symmetric operation.

Appendix B

The Euler Characteristic

In this Appendix we give a proof showing that Euler's formula is a topological invariant. We also show, in a simple way, that Euler's characteristic χ is related the genus of a given surface. The genus tells us about the topology of a surface.

B.1 $V - E + F$ is a topological invariant

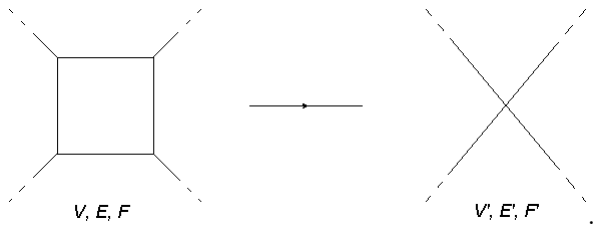


Figure B.1: Diagram of the shrinking of a face to a point, resulting in the loss of four edges.

In the above diagram V is the number of vertices, E is the number of edges and F is the number of faces. Considering this diagram we see that if we were to shrink the face down to a point in a smooth uniform way, the following would happen

$$V' = V - 3, \qquad E' = E - 4, \qquad F' = F - 1$$

that is, we'd lose three vertices, four edges and one face. Euler's invariant is given by

$$V - E + F.$$

After our changes to the above surface we get

$$V' - E' + F' = V - 3 - (E - 4) + F - 1 = V - E + F.$$

So $V - E + F$ is invariant.

For another example consider

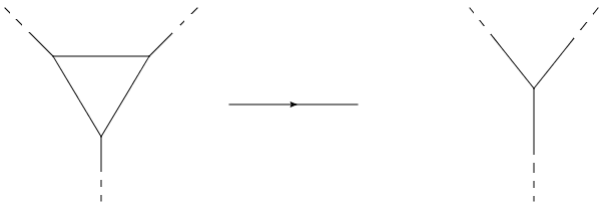


Figure B.2: Diagram of the shrinking of a face to a point, resulting in the loss of three edges.

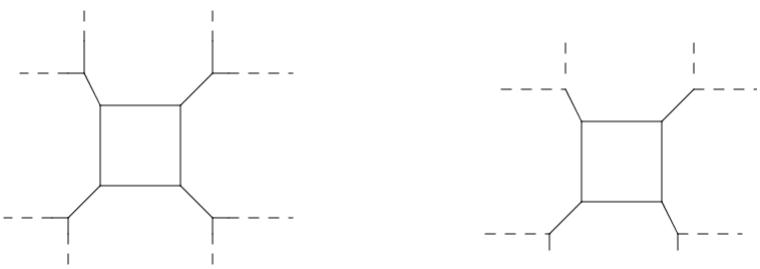
where we have once again shrunk a face down to a point in a smooth uniform way. Here we lose two vertices, three edges and one face. Again, we find

$$V' - E' + F' = V - E + F.$$

These two surfaces, although they look slightly different have the same topology. They also have no handles.

B.2 Adding a handle

Next we consider adding a handle to some surface to see whether this changes $V - E + F$ and thus the topology. To do so look at the following diagram of some shape. We only concentrate on two of the many faces present



To get a handle we need to remove two faces and glue the holes together, this is not a smooth deformation. Thus, we expect Eulers formula to change by some factor. Removing two faces together means we lose two faces in the process. So

$$V' = V, \qquad E' = E, \qquad F' = F - 2.$$

Next, glue the edges together, this results in a loss of four edges and four vertices

$$V'' = V' - 4, \qquad E'' = E' - 4, \qquad F'' = F'.$$

Then, one finds

$$V'' - E'' + F'' = V - E + F - 2$$

where a factor of -2 has been gained. Thus, adding a handle is the same as subtracting two off of the value for $V - E + F$. Every time a handle is added we lose an additive factor of two, that is we subtract $2g$ from $V - E + F$, where g is the number of handles. It is also called the genus. To add a boundary to the surface we remove a face to get

$$V' = V, \qquad E' = E, \qquad F' = F - 1.$$

Thus we get $-b$ for b boundaries. The master formula then becomes

$$V - E + F = \chi = 2 - 2g - b.$$

Appendix C

Proof of Equation (3.4)

Start off by multiplying with $\partial_\alpha \partial_\beta$ on both sides of the conformal Killing equation

$$\partial_\alpha \partial_\beta \partial_\lambda \zeta_\nu + \partial_\alpha \partial_\beta \partial_\nu \zeta_\lambda = \partial_\alpha \partial_\beta \frac{2}{D} (\partial \cdot \zeta) \eta_{\nu\lambda} \tag{C.1}$$

then

$$\partial_\alpha \left[\partial_\beta \partial_\lambda \zeta_\nu + \partial_\beta \partial_\nu \zeta_\lambda \right] = 0 \tag{C.2}$$

$$\partial_\alpha \left[\partial_\lambda \partial_\beta \zeta_\nu + \partial_\lambda \partial_\nu \zeta_\beta \right] = 0 \tag{C.3}$$

$$\partial_\alpha \left[\partial_\nu \partial_\lambda \zeta_\beta + \partial_\nu \partial_\beta \zeta_\lambda \right] = 0. \tag{C.4}$$

Then compute (C.2) + (C.4) - (C.3) to obtain

$$\partial_\alpha \partial_\beta \partial_\nu \zeta_\lambda = 0. \tag{C.5}$$

This means that ζ_λ must be at most a polynomial of degree two.

Appendix D

Intertwiners

In this appendix we take a look at how the intertwiners arise. We consider the sum over S_{n+m-1} considered in [7], performed to obtain equation (D.2) in [7]. We consider a specific example. Consider the following representation of S_6 labelled by R

$$R = \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; .$$

The representations of R' that can be subduced from R by removing a single box are given by

$$R' = \begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \square & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; , \quad \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; , \quad \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \; .$$

For any $\sigma \in S_5$, we have a matrix representation of σ in the S_6 representation R, given by $\Gamma_R(\sigma)$. Thus we may write

$$\Gamma_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}}(\sigma) = \begin{bmatrix} \Gamma_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}}(\sigma) & & \\ & \Gamma_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}}(\sigma) & \\ & & \Gamma_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}}(\sigma) \end{bmatrix} .$$

The block diagonal form is expected since the matrix representation of $\Gamma_R(\sigma)$ is reducible when restricted to the S_5 subgroup. The dimension of this matrix is given by the dimension of the representation R

$$d_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} = d_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \square & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} + d_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \square & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} + d_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}} = 5 + 6 + 5 = 16.$$

Next consider another representation of S_6, for example

$$T = \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} .$$

The irreducible representation of S_6 resulting from the removal of a single box is given by

$$T' = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}\ .$$

Then the matrix representation of $\sigma \in S_5$ in the S_6 representation T, given by $\Gamma_T(\sigma)$. That is

$$\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline\end{array}}(\sigma) = \left[\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}(\sigma)\right]$$

which has dimension

$$d_{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline\end{array}} = d_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}} = 5.$$

Now recall that the fundamental orthogonality relation gives

$$\sum_{\sigma \in S_5} \Gamma_R(\sigma)_{ij}\Gamma_S(\sigma^{-1})_{kl} = \delta_{RS}\delta_{il}\delta_{jk}\frac{|S_5|}{d_R}$$

where $|S_5| = 5!$ is the order (size) of the group S_5. We see that there are only non-zero contributions to the equation before equation (D.2) in [7] when $R = S$. Then

$$\begin{aligned}\sum_{\sigma \in S_5} \Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}}(\sigma)_{ij}\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline\end{array}}(\sigma^{-1})_{kl} &= \sum_{\sigma \in S_5}\left[\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}}(\sigma) \oplus \Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}}(\sigma) \oplus \Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}(\sigma)\right]_{ij}\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}(\sigma^{-1})_{kl} \\ &= \Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}(\sigma)_{i-11,j-11}\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}(\sigma^{-1})_{kl} \\ &= \delta_{k,j-11}\delta_{i-11,l}\frac{5!}{d_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}}}\end{aligned}$$

where $i, j = 12, ..., 16$ and $k, l = 1, ..., 5$ for non-zero contributions. These deltas are what we relabel as our intertwiners.

D.1 Constructing intertwiners

D.1.1 Example 1

Suppose we have the representation R where * indicates the box to be removed to form the representation R'.

$$R = \begin{array}{|c|c|c|}\hline & & * \\ \hline & \\ \cline{1-2}\end{array}$$

To form the intertwiner we pick another representation T such that when we form T' we have that $R' = T'$. Thus we may choose our T to be

$$T = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline * \\ \cline{1-1}\end{array}\ .$$

For this case the intertwiner is

$$I_{R'T'} = |\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}\rangle\langle\begin{array}{|c|c|}\hline 5 & 3 \\ \hline 4 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}| + |\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}\rangle\langle\begin{array}{|c|c|}\hline 5 & 4 \\ \hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}|$$

$$= \sum_{i=1}^{2} |\begin{array}{|c|c|c|}\cline{1-3} & & 1 \\ \cline{1-3} & \\ \cline{1-2}\end{array}, i\rangle\langle\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline 1 \\ \cline{1-1}\end{array}, i|$$

where i runs over all possible ways to complete the Young labelling of $R' = T'$. Note that we complete the labelling identically for each term in the intertwiner whilst keeping the first box to be removed fixed.

A general intertwiner may then be written, for $R' = T'$, as

$$I_{R'T'} = \sum_{i=1}^{d_{R'}} |S_{R'}, i\rangle\langle S_{T'}, i|$$

D.1.2 Example 2

Here we consider a more complex example by considering a representation R of S_6. Let

$$R = \begin{array}{|c|c|c|}\cline{1-3} & & * \\ \cline{1-3} & \\ \cline{1-2} \\ \cline{1-1}\end{array}$$

and

$$R' = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1}\end{array}$$

There are three possible intertwiners here corresponding to T_1, T_2 and T_3, all of which give $R' = T'_k$ where $k = 1, 2, 3$. The T representations are given by

$$T_1 = \begin{array}{|c|c|c|}\cline{1-3} & & * \\ \cline{1-3} & \\ \cline{1-2} \\ \cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & * \\ \hline\end{array}, \quad T_3 = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1} * \\ \cline{1-1}\end{array}.$$

We will end up with

$$I_{R'T'_k} = \sum_{i=1}^{d_{R'}} |S_{R'}, i\rangle\langle S_{T'_k}, i|.$$

Appendix E

The Trace

We aim to calculate the following trace

$$\mathrm{Tr}\left([\Gamma_R((1,3)), P_{R,s,i,j}]\, I_{R'T'}\, \left[\Gamma_T((1,3)), P_{T,u,k,i)}\right] I_{T'R'}\right)$$

for the case $s = u = s_1$. Consider the first term, given by

$$T_1 = \mathrm{Tr}\left(P_{R,s_1,i,j}\Gamma_R((1,3))I_{R'T'}\Gamma_T((1,3))P_{T,s_1,k,i}I_{T'R'}\right).$$

Using the cyclic properties of the trace we may express this as follows

$$T_1 = \mathrm{Tr}\left(\Gamma_T((1,3))P_{T,s_1,k,i}I_{T'R'}P_{R,s_1,i,j}\Gamma_R((1,3))I_{R'T'}\right).$$

Notice that this looks almost identical to T_4

$$T_4 = \mathrm{Tr}\left(\Gamma_R((1,3))P_{R,s_1,i,j}I_{R'T'}P_{T,s_1,k,i}\Gamma_T((1,3))I_{T'R'}\right)$$

with R and T swapped. Intuitively these two terms should be equal to each other. This is explained by considering how the dilatation operator acts on our normalized operators. We must also remember, importantly, that the dilatation operator D is Hermitian. Then

$$DO_{R,(r,s)} = \sum_{\tilde{T},(t',u')} N_{R,(r,s),\tilde{T},(t',u')} O_{\tilde{T},(t',u')},$$

where the two point function is

$$\langle O^{\dagger}_{T,(t,u)} DO_{R,(r,s)}\rangle = N_{R,(r,s),T,(t,u)}$$

since our operators are normalized to one. Now take the complex conjugate

$$\langle O^{\dagger}_{R,(r,s)} D^{\dagger} O_{T,(t,u)}\rangle = N_{T,(t,u),R,(r,s)}$$

where $D^{\dagger} = D$. So this looks like we are just swapping the rows and columns of a Hermitian matrix. All we have done is swapped R and T, but we arrive at the same answer.

To calculate T_1 and T_4 we first need to calculate a term of the following type

$$\text{projector} \times \text{intertwiner} \times \text{projector}.$$

The first term is

$$\sum_a^{d_r}\sum_A |\{T\},\square\square,k,i;a\rangle\langle\{T\},\square\square,k,i;a|\{T\},k;A\rangle\langle\{R\},j;A|\{R\},\square\square,i,j;a\rangle\langle\{R\},\square\square,i,j;a|$$
$$=\sum_a \sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}|\{T\},\square\square,k,i;a\rangle\langle\{R\},\square\square,i,j;a|.$$

Similarly

$$P_{R,s_1,i,j}I_{R'T'}P_{T,s_1,k,i}=\sum_a \sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}|\{R\},\square\square,i,j;a\rangle\langle\{T\},\square\square,k,i;a|.$$

E.1 Action of the swap

The action of the swap on $|\{R\},\square\square,i,j;a\rangle$ is given by

$$\begin{aligned}\Gamma_R(1,3)|\{R\},\square\square,i,j;a\rangle =&(12)(23)(12)|\{R\},\square\square,i,j;a\rangle\\ =&(12)(23)|\{R\},\square\square,i,j;a\rangle\end{aligned}$$

where the last line follows because $|\{R\},\square\square,i,j;a\rangle$ is already in the symmetric representation. Thus

$$\begin{aligned}(12)(23)|\{R\},\square\square,i,j;a\rangle =&(12)(23)\sum_l|\{R\},\square\square,i,j,l;\tilde{a}\rangle\\ =&(12)(23)\sum_l\left[\sqrt{\frac{c_i-c_j+1}{2(c_i-c_j)}}|\{R\},i,j,l;\tilde{a}\rangle\right.\\ &\left.+\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}|\{R\},j,i,l;\tilde{a}\rangle\right]\\ =&(12)\sum_l\left[\sqrt{\frac{c_i-c_j+1}{2(c_i-c_j)}}\left(\frac{1}{c_j-c_l}|\{R\},i,j,l;\tilde{a}+...\right)\right.\\ &\left.+\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\left(\frac{1}{c_i-c_l}|\{R\},j,i,l;\tilde{a}\rangle+\sqrt{1-\frac{1}{(c_i-c_l)^2}}|\{R\},j,l,i;\tilde{a}\rangle\right)\right]\\ =&\sum_l\left[\sqrt{\frac{c_i-c_j+1}{2(c_i-c_j)}}\frac{1}{c_j-c_l}\left(\sqrt{1-\frac{1}{(c_i-c_j)^2}}|\{R\}j,i,l;\tilde{a}\rangle+...\right)\right.\\ &+\sqrt{\frac{c_i-c_l-1}{2(c_i-c_j)}}\frac{1}{c_i-c_l}\left(\frac{1}{c_j-c_i}|\{R\},j,i,l;\tilde{a}\rangle+...\right)\\ &\left.+\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{1-\frac{1}{(c_i-c_l)^2}}\left(\frac{1}{c_j-c_l}|\{R\},j,l,i;\tilde{a}\rangle+...\right)\right.\end{aligned}$$

The ellipses indicate all the extra terms which vanish once the overlap with $\langle\{R\}, j; A|$ is taken. Similarly,

$$\begin{aligned}\Gamma_T(1,3)|\{T\},\square\square,k,i;a\rangle = \sum_l \Bigg[&\sqrt{\frac{c_k-c_i+1}{2(c_k-c_i)}}\frac{1}{c_i-c_l}\left(\frac{1}{c_k-c_i}|\{T\}k,i,l;\tilde{a}\rangle+...\right)\\ &+\sqrt{\frac{c_k-c_i+1}{2(c_k-c_i)}}\sqrt{1-\frac{1}{(c_i-c_l)^2}}\left(\frac{1}{c_k-c_l}|\{T\},k,l,i;\tilde{a}\rangle+...\right)\\ &+\sqrt{\frac{c_k-c_i-1}{2(c_k-c_i)}}\frac{1}{c_k-c_l}\left(\sqrt{1-\frac{1}{(c_i-c_k)^2}}|\{T\},k,i,l;\tilde{a}+...\right).\end{aligned}$$

Note that we may check these results by preforming the alternate but equivalent calculation using (23)(12)(23). Though this calculation is more lengthy it must give the same result.

E.2 Putting it all together

We are now in a position to calculate the trace. It is given by

$$\begin{aligned}T_1 =& \sum_{a,A}\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}\langle\{T\},k;A|\Gamma_T|\{T\},\square\square,k,i;a\rangle\langle\{R\},\square\square,i,j;a|\Gamma_R|\{R\},j;A\rangle\\ =& \sum_{a,a'}\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}\langle\{T\},k,i;a'|\Gamma_T|\{T\},\square\square,k,i;a\rangle\langle\{R\},\square\square,i,j;a|\Gamma_R|\{R\},j,i;a'\rangle\\ =& \sum_{\tilde{a},b}\sum_l\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}\langle\{T\},k,i,l;b|\Gamma_T|\{T\},\square\square,k,i,l;\tilde{a}\rangle\langle\{R\},\square\square,i,j,l;\tilde{a}|\Gamma_R|\{R\},j,i,l;b\rangle\\ =& \sum_{\tilde{a},b}\sum_l\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}(\alpha_1(l)+\alpha_2(l))\delta_{\tilde{a}b}(\beta_1(l)+\beta_2(l))\delta_{\tilde{a}b}\\ =& \sum_l\sqrt{\frac{c_i-c_j-1}{2(c_i-c_j)}}\sqrt{\frac{c_k-c_j+1}{2(c_k-c_j)}}(\alpha_1(l)+\alpha_2(l))(\beta_1(l)+\beta_2(l))d_{R''_{jil}}\end{aligned}$$

where we see that only the some of the terms survived corresponding to the following constants

$$\begin{aligned}\alpha_1(l) =& \sqrt{\frac{c_k-c_i+1}{2(c_k-c_i)}}\frac{1}{c_i-c_l}\frac{1}{c_k-c_i}\\ \alpha_2(l) =& \sqrt{\frac{c_k-c_i-1}{2(c_k-c_i)}}\frac{1}{c_k-c_l}\sqrt{1-\frac{1}{(c_i-c_k)^2}}\end{aligned}$$

and

$$\begin{aligned}\beta_1(l) =& \sqrt{\frac{c_i-c_j+1}{2(c_i-c_j)}}\frac{1}{c_j-c_l}\sqrt{1-\frac{1}{(c_i-c_j)^2}}\\ \beta_2(l) =& \sqrt{\frac{c_i-c_l-1}{2(c_i-c_j)}}\frac{1}{c_i-c_l}\frac{1}{c_j-c_i}.\end{aligned}$$

We found T_1 to be equal to T_4.

E.3 T_2 and T_3

We found that T_2 was equal to T_3. For this case we had to calculate the following term

$$I_{R'T'}P_{T,s_1,k,i} = \sum_{a}^{d_r} \sqrt{\frac{c_k - c_i + 1}{2(c_k - c_i)}} |\{R\}, j, i; a\rangle\langle\{T\}, \square\square, k, i; a|$$

The result is

$$T_2 = \sum_{l} \sqrt{\frac{c_k - c_i + 1}{2(c_k - c_i)}} \sqrt{\frac{c_i - c_j - 1}{2(c_i - c_j)}} (\beta_1(l) + \beta_2(l))(\alpha_1(l) + \alpha_2(l)) d_{R''_{jil}}.$$

Appendix F

Large N value of a factor in the dilatation operator coefficient

We calculate the following factor in the dilatation operator coefficient

$$\sqrt{\frac{\text{hooks}_T}{\text{hooks}_R}}.$$

First we show how this is done for $\mathcal{O}(N)$ with $p \sim \mathcal{O}(1)$ rows or columns. We want a ratio of the product of hook lengths for two different Young diagrams R and T. Let us first consider the Young diagrams R and R', where R' is the usual Young diagram obtained from R by the removal of a single box from some row or column. Let us suppose we remove the box from row a, we represent this Young diagram by R'_a. This ratio is easy to calculate, we illustrate this with some examples below.

F.1 Example 1: Two rows

Consider a Young diagram having two rows of boxes. Let the first row have $R_1 = 8$ boxes and the second row have $R_2 = 4$ boxes. The Young diagram with its hooks lengths entered is given below

9	8	7	6	4	3	2	1
4	3	2	1				

.

For the diagram R'_2 we remove a single box from row two of R

9	8	7	5	4	3	2	1
3	2	1					

.

Then the ratio of these hook lengths is given by

$$\frac{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 9&8&7&6&4&3&2&1\\\hline 4&3&2&1\\\cline{1-4}\end{array}}{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 9&8&7&5&4&3&2&1\\\hline 3&2&1\\\cline{1-3}\end{array}} = \frac{9\times 8\times 7\times 6\times 4\times 3\times 2\times 1\times 4\times 3\times 2\times 1}{9\times 8\times 7\times 5\times 4\times 3\times 2\times 1\times 3\times 2\times 1}$$

where we can see that the following factors cancel against each other: $9, 8, 7, 4, 3, 2, 1$ from the first rows in each. And $3, 2, 1$ from the second rows of each. Thus from the numerator we are left with the factor 6 from the first row and 4 from the second, whilst from the denominator we are left with 5. This yields

$$\frac{4\times 6}{5}.$$

There is a pattern here, there will always be cancellations such as these. Notice that the numbers $6, 5$ and 4 can be written in terms of the row lengths. Also note that $R_1 - R_2 = 4$. Thus the previous equation may be written as

$$\frac{R_2(R_1 - R_2 + 2)}{(R_1 - R_2 + 1)}.$$

In the large N limit, for operators with dimension $\mathcal{O}(N)$, the difference in row lengths is $\mathcal{O}(N)$. Thus $R_1 - R_2 \sim \mathcal{O}(N)$. And we may approximate the last equation as follows

$$\frac{R_2(R_1 - R_2 + 2)}{(R_1 - R_2 + 1)} \sim \frac{R_2(N + 2)}{(N + 1)} \sim R_2.$$

F.2 Example 2: Three rows

Next we consider three rows. Let $R_1 = 6$, $R_2 = 4$ and $R_3 = 2$. Then

$$R_1 - R_2 = 2, \qquad R_1 - R_3 = 4.$$

The ratio of the product of hook lengths is given by

$$\frac{\begin{array}{|c|c|c|c|c|c|}\hline 8&7&5&4&2&1\\\hline 5&4&2&1\\\cline{1-4}2&1\\\cline{1-2}\end{array}}{\begin{array}{|c|c|c|c|c|c|}\hline 8&6&5&4&2&1\\\hline 5&3&2&1\\\cline{1-4}1\\\cline{1-1}\end{array}}$$

after removing a box from the third row. The factors we are left with after cancellation are

$$\frac{7\times 4\times 2}{6\times 3}$$

which can again be written in terms of the row lengths as

$$\frac{R_3(R_1 - R_2 + 2)(R_1 - R_3 + 3)}{(R_1 - R_2 + 1)(R_1 - R_3 + 2)}.$$

F.3 Simplifying ratios of products of hook lengths

In general we find that

$$\frac{\text{hooks}_R}{\text{hooks}_{R'_j}} = R_j \frac{\prod\limits_{\substack{k \\ k\neq 0}} (R_1 - R_k + k)}{\prod\limits_{\substack{k \\ k\neq 0}} (R_1 - R_k + k - 1)}$$

which goes like R_j in the large N limit. From [7] we know that

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{(T_a - 1 + p - a)!(T_b + 1 + p - b)!}{((T_a + p - a)!(T_b + p - b)!} \prod_{\substack{k\neq a \\ k\neq b}} \frac{|T_a - T_k| + |k - a|}{|T_a - 1 - T_k| + |k - a|} \times$$

$$\times \prod_{\substack{k\neq a \\ k\neq b}} \frac{|T_b - T_k| + |k - b|}{|T_b + 1 - T_k| + |k - b|} \frac{|T_b - T_a| + |a - b|}{|T_a - T_b - 2| + |a - b|} \,.$$

In the large N limit this becomes

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{R_b}{R_a} \left(1 + O(N^{-1})\right).$$

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