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Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra

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ABSTRACT

A considerable amount of the standard model's three-generation structure can be realised from just the $8\,\mathbb{C}$-dimensional algebra of the complex octonions. Indeed, it is a little-known fact that the complex octonions can generate on their own a $64\,\mathbb{C}$-dimensional space. Here we identify an $su(3) \oplus u(1)$ action which splits this $64\,\mathbb{C}$-dimensional space into complexified generators of $SU(3)$, together with 48 states. These 48 states exhibit the behaviour of exactly three generations of quarks and leptons under the standard model's two unbroken gauge symmetries. This article builds on a previous one, [1], by incorporating electric charge.

1. Why three generations?

Upon the 2012 discovery of a 125 GeV Higgs, the most straightforward four-generation chiral extension to the standard model was ruled out, [2–5]. Of course, the possibility of eventually finding a fourth generation is not excluded for every imaginable scenario, e.g. [6], [7]. However, given this new data, it seems increasingly likely that nature's game of replicating particle content comes to an end at three generations.

Although three-generation models can be relatively easy to justify experimentally, they are substantially more difficult to motivate theoretically. That is, few mathematical objects exhibit (efficiently) the group representations necessary to describe three full generations.

Indeed, it is no secret that the most well-known extensions of the standard model: $SU(5)$, $Spin(10)$ grand unified theories, and the Pati–Salam model are all naturally one-generation models. For the standard model and its most well-studied extensions, the existence of three families need be imposed by hand.

With this being said, a variety of proposals have materialized over the years, e.g. [8–21], in order to explain the curious pattern. This includes in particular a recent three-generation proposal put forward by Dubois-Violette and Todorov, based on the 27-dimensional (octonionic) exceptional Jordan algebra.

The intention of our article here is not to present a completed three-generation quantum field theory. Instead we will demonstrate a significant portion of an algebraic framework on which such a theory might be built.

Rather unconventionally, this framework does not begin with a larger mathematical object, which is then subsequently broken down into the known quark and lepton representations of the standard model. On the contrary, we will make use of just an $8\,\mathbb{C}$-dimensional algebra—an algebra whose degrees of freedom are far outweighed by the number of states which we aim to describe.

We will begin by introducing the algebra of the complex octonions, $\mathbb{C} \otimes \mathbb{O}$. This $8\,\mathbb{C}$-dimensional algebra will then be seen to generate the complex Clifford algebra $\mathbb{C}l(6)$, via its *left-action maps*. Within this $64\,\mathbb{C}$-dimensional Clifford algebra, we next identify a pair of complexified $su(3)_c$ Lie algebras. These $SU(3)_c$ generators will then be applied to the rest of the Clifford algebra, which consequently breaks down into exactly the $SU(3)_c$ representations one would expect for three full generations of quarks and leptons, [1].

Finally, we demonstrate how the action of these generators may be generalized so as to include a new $u(1)$. This $U(1)$ action then distributes 48 eigenvalues which are found to coincide with electric charge.

Hence, it is shown that a single eight-dimensional algebra can encode the behaviour of three full generations under nature's two unbroken gauge symmetries.

This article builds on [1] by (1) demonstrating that the generators of G_2 may be described in terms of *associators*, (2) by redefining the operation of Lie algebras on $\mathbb{C}l(6)$ in terms of a single action, (3) by further specifying the projection properties of

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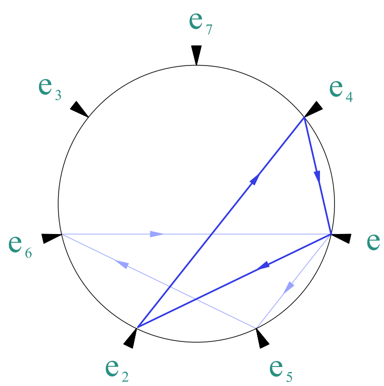

Fig. 1. Octonionic multiplication rules, [32].

quark and lepton states, and finally (4) by incorporating electric charge.

Over the years, there has been quite a number of authors who have used $\mathbb{C}l(6)$ to describe one generation of standard model fermions. These include, but are likely not limited to [22–30].

2. Introduction to $\mathbb{C}\otimes\mathbb{O}$

The complex octonions form an 8-dimensional algebra over $\mathbb{C}$, spanned by basis vectors e_j for $j=0,\dots 7$. The basis vector e_0 plays the role of the multiplicative identity, whereas the e_k for $k=1,\dots 7$ are imaginary units with $e_k^2=-1$. The remaining octonionic multiplication rules may be described succinctly by specifying $e_1e_2=e_4$, and then invoking the rules, [31],

$$\begin{aligned} &e_ie_j=-e_je_i \qquad i\neq j,\\ &e_ie_j=e_k \;\;\Rightarrow\;\; e_{i+1}e_{j+1}=e_{k+1},\\ &e_ie_j=e_k \;\;\Rightarrow\;\; e_{2i}e_{2j}=e_{2k}. \end{aligned} \tag{1}$$

Please see Fig. 1.

The octonions are perhaps best known for their property of *non-associativity*, meaning that there exists an a, b, and c in the algebra such that $a(bc)\neq(ab)c$. Hence, brackets should typically be specified whenever multiplication involves three or more elements. With this being said, readers should realise that non-associativity is by no means a foreign concept in physics. It is an under-appreciated fact that both Lie algebras and Jordan algebras likewise constitute non-associative algebras.[1]

Symmetries of the algebra: the derivations of $\mathbb{C}\otimes\mathbb{O}$ are given by the complexified 14-dimensional exceptional Lie algebra g_2. These g_2 elements can be seen to act on a generic element $f\in\mathbb{C}\otimes\mathbb{O}$ as

$$\begin{aligned}
\Lambda_1 f &= \tfrac{i}{2}\{e_1,e_5,f\}-\tfrac{i}{2}\{e_3,e_4,f\}\\
\Lambda_2 f &= \tfrac{i}{2}\{e_4,e_1,f\}-\tfrac{i}{2}\{e_3,e_5,f\}\\
\Lambda_3 f &= -\tfrac{i}{2}\{e_1,e_3,f\}+\tfrac{i}{2}\{e_4,e_5,f\}\\
\Lambda_4 f &= \tfrac{i}{2}\{e_2,e_5,f\}+\tfrac{i}{2}\{e_4,e_6,f\}\\
\Lambda_5 f &= -\tfrac{i}{2}\{e_2,e_4,f\}+\tfrac{i}{2}\{e_5,e_6,f\}\\
\Lambda_6 f &= \tfrac{i}{2}\{e_1,e_6,f\}+\tfrac{i}{2}\{e_2,e_3,f\}\\
\Lambda_7 f &= \tfrac{i}{2}\{e_1,e_2,f\}+\tfrac{i}{2}\{e_3,e_6,f\}\\
\Lambda_8 f &= \tfrac{i}{2\sqrt{3}}\{e_1,e_3,f\}+\tfrac{i}{2\sqrt{3}}\{e_4,e_5,f\}-\tfrac{i}{\sqrt{3}}\{e_2,e_6,f\}\\
g_9 f &= -\tfrac{i}{2\sqrt{3}}\{e_1,e_5,f\}-\tfrac{i}{2\sqrt{3}}\{e_3,e_4,f\}-\tfrac{i}{\sqrt{3}}\{e_2,e_7,f\}\\
g_{10} f &= -\tfrac{i}{2\sqrt{3}}\{e_4,e_1,f\}-\tfrac{i}{2\sqrt{3}}\{e_3,e_5,f\}+\tfrac{i}{\sqrt{3}}\{e_6,e_7,f\}\\
g_{11} f &= -\tfrac{i}{2\sqrt{3}}\{e_4,e_6,f\}-\tfrac{i}{2\sqrt{3}}\{e_5,e_2,f\}+\tfrac{i}{\sqrt{3}}\{e_7,e_1,f\}\\
g_{12} f &= -\tfrac{i}{2\sqrt{3}}\{e_2,e_4,f\}-\tfrac{i}{2\sqrt{3}}\{e_5,e_6,f\}+\tfrac{i}{\sqrt{3}}\{e_3,e_7,f\}\\
g_{13} f &= -\tfrac{i}{2\sqrt{3}}\{e_6,e_1,f\}-\tfrac{i}{2\sqrt{3}}\{e_2,e_3,f\}+\tfrac{i}{\sqrt{3}}\{e_7,e_4,f\}\\
g_{14} f &= -\tfrac{i}{2\sqrt{3}}\{e_1,e_2,f\}-\tfrac{i}{2\sqrt{3}}\{e_6,e_3,f\}+\tfrac{i}{\sqrt{3}}\{e_5,e_7,f\}
\end{aligned} \tag{2}$$

over $\mathbb{C}$. Here, we have made use of the *associator*, defined as $\{a,b,c\}\equiv a(bc)-(ab)c$. Readers should note that, when taken over the field of the real numbers, the first eight Λ_j generate $SU(3)$. In this case, we have chosen this $SU(3)$ so that it holds the octonionic imaginary unit e_7 constant.

3. From 8 dimensions to 64

Given the definition of the associator, it is straightforward to see that the 14 generators of equations (2) are constructed from chains of octonions acting from the left on f. In fact, the most general *left-action map*, M, may be described as

$$\begin{aligned} Mf \equiv c_0 f &+\textstyle\sum_{i=1}^6 c_i\, e_i f+\sum_{j=2}^6\sum_{i=1}^{j-1} c_{ij}\, e_i(e_jf)\\ &+\textstyle\sum_{k=3}^6\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} c_{ijk}\, e_i(e_j(e_kf))+\dots\\ &+c_{123456}\, e_1(e_2(e_3(e_4(e_5(e_6f))))), \end{aligned} \tag{3}$$

where the coefficients c_0, c_i, $\dots\in\mathbb{C}$. Readers may have noticed that the imaginary unit e_7 is not explicitly expressed in these maps. This is due to the fact that

$$e_7f=e_1(e_2(e_3(e_4(e_5(e_6f))))) \qquad \forall f\in\mathbb{C}\otimes\mathbb{O}, \tag{4}$$

thereby making e_7 redundant as a left-action map. Of course, e_7 itself holds no preferred status within the octonions, and the space of left-action maps may equivalently be generated by any six of the seven imaginary units.

The octonionic chains (3) describe all possible complex-linear maps from $\mathbb{C}\otimes\mathbb{O}$ to itself. Said more precisely, they faithfully represent the full $64\,\mathbb{C}$-dimensional space of complex endomorphisms. Indeed, even right multiplication may be re-expressed in the form of equation (3). For example,

$$fe_7=\frac{1}{2}\left(e_1(e_3f)+e_2(e_6f)+e_4(e_5f)-e_7f\right) \tag{5}$$

$\forall f\in\mathbb{C}\otimes\mathbb{O}$.

So in summary, we have shown how it is possible to build up a $64\,\mathbb{C}$-dimensional space, using only the $8\,\mathbb{C}$-dimensional $\mathbb{C}\otimes\mathbb{O}$ operating on itself from the left. For examples of earlier work which make reference to this 64-dimensional algebra, see [22], [33], [10].

[1] As an example, consider the elements $a=i\sigma_x$, $b=i\sigma_x$, $c=i\sigma_y$ in the $su(2)$ Lie algebra, where multiplication is given by the commutator.

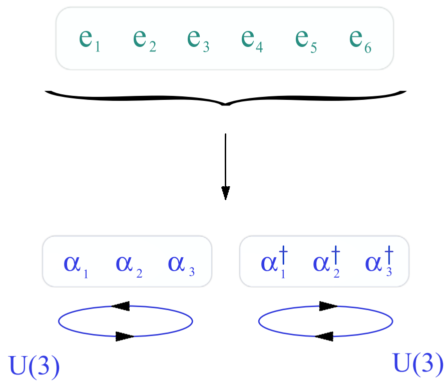

Fig. 2. Octonionic imaginary units $e_1, e_2, \ldots e_6$ generate the Clifford algebra $\mathbb{C}l(6)$. These six generators may be rewritten in terms of a basis of ladder operators, $\alpha_1, \alpha_2, \alpha_3, \alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger$, which have $U(3) = SU(3) \times U(1)/\mathbb{Z}_3$ symmetry.

4. The Clifford algebra $\mathbb{C}l(6)$

It is reasonably straightforward to show that

$$e_i(e_j f) = \begin{cases} -e_j(e_i f) & \text{when } i \neq j \\ -f & \text{when } i = j \end{cases} \tag{6}$$

$\forall f \in \mathbb{C}\otimes\mathbb{O}$, and for $i, j = 1, \ldots 6$. (This should be recognizable to the reader as Clifford algebraic structure.) In fact, the linear maps (3) give a faithful representation of the complex Clifford algebra $\mathbb{C}l(6)$.

Readers concerned about the potential conflict between the inherent associativity of a Clifford algebra, and the non-associativity of the octonions should note that *multiplication* of left-action maps is given by the *composition of maps*. Of course, the composition of maps is associative, by definition; $F\circ(G\circ H) = (F\circ G)\circ H$.

Consider now for a moment complex Clifford algebras of the form $\mathbb{C}l(2n)$ for $n \in \mathbb{Z} > 0$. It is known that the generating space of such algebras may be partitioned into an n-dimensional subspace spanned by raising operators $\{\alpha_i^\dagger\}$ and an n-dimensional subspace spanned by lowering operators $\{\alpha_i\}$. These n-dimensional subspaces are known as *maximal totally isotropic subspaces*, whose basis vectors obey

$$\{\alpha_i, \alpha_j\} = 0 = \{\alpha_i^\dagger, \alpha_j^\dagger\}, \qquad \{\alpha_i, \alpha_j^\dagger\} = \delta_{ij} \tag{7}$$

for $i, j = 1, \ldots n$, under the anticommutator: $\{a, b\} \equiv ab + ba$. For further details, please see [34].

In the case of our octonionic representation of $\mathbb{C}l(6)$, (3), the generating space is spanned by the linear maps e_i for $i = 1, \ldots 6$. These may be reorganized into a set of lowering operators

$$\alpha_1 \equiv \frac{-e_5 + ie_4}{2}, \quad \alpha_2 \equiv \frac{-e_3 + ie_1}{2}, \quad \alpha_3 \equiv \frac{-e_6 + ie_2}{2}, \tag{8}$$

and a set of raising operators,

$$\alpha_1^\dagger \equiv \frac{e_5 + ie_4}{2}, \quad \alpha_2^\dagger \equiv \frac{e_3 + ie_1}{2}, \quad \alpha_3^\dagger \equiv \frac{e_6 + ie_2}{2}, \tag{9}$$

where $\dagger$ maps the complex $i \mapsto -i$ and the octonionic $e_j \mapsto -e_j$, while reversing the order of multiplication, $(ab)^\dagger = b^\dagger a^\dagger$. Readers may confirm that these ladder operators obey equations (7), as maps acting on any $f \in \mathbb{C}\otimes\mathbb{O}$.

The structure of these ladder operators is preserved by the unitary group $U(3) = SU(3) \times U(1)/\mathbb{Z}_3$, as depicted in Figure (2). This $U(3)$ symmetry may be realised as $G\alpha_i G^{-1}$ and $G\alpha_i^\dagger G^{-1}$, for $G \equiv \exp\left(ir_j\Lambda_j + ir_0 Q\right) \in \mathbb{C}l(6)$. Here, $r_k \in \mathbb{R}$ for $k = 0, \ldots 8$, and the $SU(3)$ generators, Λ_j, are defined as in equations (2). The $U(1)$ generator is defined as

$$Qf \equiv \frac{N}{3}f = \frac{1}{3}\sum_{i=1}^{3}\alpha_i^\dagger(\alpha_i f), \tag{10}$$

acting on any $f \in \mathbb{C}\otimes\mathbb{O}$. This Q is proportional to the number operator, N, for the system, and can be seen to commute with the Λ_j. In [26], Q was identified as the generator of *electric charge*, in the context of a one-generation model.

Before moving on, we will first simplify our notation. From here forward, it will now be implicitly assumed that equations in $\mathbb{C}l(6)$ hold $\forall f \in \mathbb{C}\otimes\mathbb{O}$. That is, we will no longer write f explicitly. Furthermore, we will cease to write the nested brackets of octonionic left-action maps, (3). That is, right-to-left bracketing will also be implicitly assumed. Hence, equations such as $Qf \equiv \frac{1}{3}\sum_{i=1}^{3}\alpha_i^\dagger(\alpha_i f)$ will now simply read $Q \equiv \frac{1}{3}\sum_{i=1}^{3}\alpha_i^\dagger\alpha_i$.

5. Three generations under $SU(3)_c$

For our first result, we will now show how the $SU(3)_c$ irreducible representations corresponding to three generations of quarks and leptons may be found, using only the action of $\mathbb{C}\otimes\mathbb{O}$ on itself.

We begin by splitting $\mathbb{C}l(6)$ into two $32\,\mathbb{C}$-dimensional pieces: $\mathbb{C}l(6)s$ and $\mathbb{C}l(6)s^*$, where s is given by the linear map $s \equiv 1/2\,(1 + ie_7) \in \mathbb{C}l(6)$. Readers may confirm that both s and s^* are idempotents, and that $ss^* = s^*s = 0$.

Within the subalgebra $\mathbb{C}l(6)s$, we find a faithful representation of the Lie algebra $su(3)$, generated by eight objects of the form $\Lambda_j s$. Seeing as how $\left[\,\Lambda_j,\ s\,\right] = 0$, it may be confirmed that

$$\left[\,\frac{\Lambda_i}{2}s,\ \frac{\Lambda_j}{2}s\,\right] = ic_{ijk}\frac{\Lambda_k}{2}s \tag{11}$$

holds, where c_{ijk} are the usual $SU(3)$ structure constants.

Now, given this representation of the $SU(3)$ Lie algebra, we may subsequently apply the $\Lambda_j s$ generators to the remainder of $\mathbb{C}l(6)s$. Under the action $[\,i\Lambda_j s,\ \mathbb{C}l(6)s\,]$, the $32\,\mathbb{C}$-dimensional $\mathbb{C}l(6)s$ is found to break down as

$$\mathbb{C}l(6)s \quad\mapsto\quad \underline{\mathbf{8}} \oplus \underline{\mathbf{3}} \oplus \left(5\times\underline{\mathbf{3}}^*\right) \oplus \left(6\times\underline{\mathbf{1}}\right) \tag{12}$$

over $\mathbb{C}$. For a sample calculation, please see [1]. Invoking the complex conjugate, $i \mapsto -i$, sends particles to antiparticles, and vice versa. In other words, the commutator $[\,-i\Lambda_j^* s^*,\ \mathbb{C}l(6)s^*\,]$ induces

$$\mathbb{C}l(6)s^* \quad\mapsto\quad \underline{\mathbf{8}} \oplus \underline{\mathbf{3}}^* \oplus \left(5\times\underline{\mathbf{3}}\right) \oplus \left(6\times\underline{\mathbf{1}}\right). \tag{13}$$

Finally, these actions may be trivially combined into one single action on the full $64\,\mathbb{C}$-dimensional $\mathbb{C}l(6)$. Under $[\,i\Lambda_j s,\ \mathbb{C}l(6)s\,] + [\,-i\Lambda_j^* s^*,\ \mathbb{C}l(6)s^*\,]$, the algebra $\mathbb{C}l(6)$ breaks down into a pair of complexified $SU(3)$ Lie algebras, together with the $SU(3)$ representations

$$\left(6\times\underline{\mathbf{3}}\right) \oplus \left(6\times\underline{\mathbf{3}}^*\right) \oplus \left(6\times\underline{\mathbf{1}}\right) \oplus \left(6\times\underline{\mathbf{1}}\right). \tag{14}$$

Readers should recognize these as the $SU(3)_c$ representations necessary to describe three full generations of quarks and leptons.

6. Three generations under $SU(3)_c \times U(1)_{em}/\mathbb{Z}_3$

Now, given the ladder operator symmetry $U(3) = SU(3)_c \times U(1)_{em}/\mathbb{Z}_3$ described earlier, it is natural to wonder if these $SU(3)_c$ results may be extended so as to include $U(1)_{em}$. The most obvious electromagnetic extension to the action $[\,i\Lambda_j s,\ \mathbb{C}l(6)s\,]+[\,-i\Lambda_j^* s^*,\ \mathbb{C}l(6)s^*\,]$ is clearly $[\,iQs,\ \mathbb{C}l(6)s\,]+[\,-iQ^*s^*,\ \mathbb{C}l(6)s^*\,]$. However, we find that this action fails to assign the correct electric charges.

Hence, we are then left to ask: *Could there be a way to generalize this action so that Q produces the electric charges of standard model fermions?*

In what is to follow, we will need to introduce an idempotent, S, which is the right-multiplication analogue of s.

$$\begin{aligned} Sf &\equiv f\,\tfrac{1}{2}(1+ie_7) \\ &= \tfrac{1}{2}f-\tfrac{i}{4}e_7f+\tfrac{i}{4}e_1(e_3f)+\tfrac{i}{4}e_2(e_6f)+\tfrac{i}{4}e_4(e_5f) \end{aligned} \tag{15}$$

acting on $f\in\mathbb{C}\otimes\mathbb{O}$, or more simply,

$$S\equiv\frac{1}{2}+\frac{1}{4}\left(-ie_7+ie_{13}+ie_{26}+ie_{45}\right), \tag{16}$$

where $e_{ab}f$ is shorthand for $e_a(e_bf)$. Readers will find that equation (15) is easily confirmed given equation (5). As before, $SS^*=S^*S=0$, and furthermore, $[\,s,S\,]=[\,s,S^*\,]=0$.

Given that $S+S^*=1$, it is straightforward to see that the action $[\,i\Lambda_j s,\ \mathbb{C}l(6)s\,]+[\,-i\Lambda_j^* s^*,\ \mathbb{C}l(6)s^*\,]$ is equal to

$$\begin{aligned} &\left[\, i\Lambda_j s,\ S\,\mathbb{C}l(6)s\,\right]+\left[-i\Lambda_j^* s^*,\ S^*\mathbb{C}l(6)s^*\,\right] \\ &+\left[\, i\Lambda_j s,\ S^*\mathbb{C}l(6)s\,\right]+\left[-i\Lambda_j^* s^*,\ S\,\mathbb{C}l(6)s^*\,\right]. \end{aligned} \tag{17}$$

Furthermore, from equations (2), it is clear that $\Lambda_j^*=-\Lambda_j$, and hence the action (17) is identical to

$$\begin{aligned} &\left[-i\Lambda_j^* s,\ S\,\mathbb{C}l(6)s\,\right]+\left[\, i\Lambda_j s^*,\ S^*\mathbb{C}l(6)s^*\,\right] \\ &+\left[\, i\Lambda_j s,\ S^*\mathbb{C}l(6)s\,\right]+\left[-i\Lambda_j^* s^*,\ S\,\mathbb{C}l(6)s^*\,\right]. \end{aligned} \tag{18}$$

However, this new action is not the same as the old action when we extend these generators so as to include Q. Unlike with the Λ_j operators, $Q^*\neq -Q$.

Upon finally including Q,

$$\begin{aligned} &\left[-i(r_j\Lambda_j^*+r_0Q^*)s,\ S\,\mathbb{C}l(6)\,s\,\right] \\ &\quad+\left[\, i(r_j\Lambda_j+r_0Q)s^*,\ S^*\,\mathbb{C}l(6)\,s^*\,\right] \\ &\qquad+\left[\, i(r_j\Lambda_j+r_0Q)s,\ S^*\,\mathbb{C}l(6)\,s\,\right] \\ &\qquad\quad+\left[-i(r_j\Lambda_j^*+r_0Q^*)s^*,\ S\,\mathbb{C}l(6)\,s^*\,\right], \end{aligned} \tag{19}$$

we find that $\mathbb{C}l(6)$ breaks down as

$$\begin{aligned} &\mathbb{C}l(6)\ \mapsto \\ &\underline{\mathbf{8}}_{\mathbf{0}}\oplus(3\times\underline{\mathbf{3}}_{\frac{2}{3}})\oplus(3\times\underline{\mathbf{3}}_{-\frac{1}{3}})\oplus(3\times\underline{\mathbf{1}}_{\mathbf{0}})\oplus(3\times\underline{\mathbf{1}}_{-\mathbf{1}})\oplus \\ &\underline{\mathbf{8}}_{\mathbf{0}}\oplus(3\times\underline{\mathbf{3}}^*_{-\frac{2}{3}})\oplus(3\times\underline{\mathbf{3}}^*_{\frac{1}{3}})\oplus(3\times\underline{\mathbf{1}}_{\mathbf{0}})\oplus(3\times\underline{\mathbf{1}}_{\mathbf{1}}). \end{aligned} \tag{20}$$

To be more explicit, three generations of particles may be described by complex linear combinations of the states

$$\begin{aligned}
&\underline{\mathbf{3}}_{\frac{2}{3}}\ \to\ \begin{cases} u_1^R\equiv sS^*\left(-ie_{12}-e_{16}+e_{23}+ie_{36}\right)sS \\ u_1^G\equiv sS^*\left(-ie_{24}-e_{25}+e_{46}-ie_{56}\right)sS \\ u_1^B\equiv sS^*\left(ie_{14}+e_{15}+e_{34}-ie_{35}\right)sS \end{cases} \\
&\underline{\mathbf{3}}_{\frac{2}{3}}\ \to\ \begin{cases} u_2^R\equiv s^*S^*\left(-ie_{12}-e_{16}+e_{23}+ie_{36}\right)s^*S \\ u_2^G\equiv s^*S^*\left(-ie_{24}-e_{25}+e_{46}-ie_{56}\right)s^*S \\ u_2^B\equiv s^*S^*\left(ie_{14}+e_{15}+e_{34}-ie_{35}\right)s^*S \end{cases} \\
&\underline{\mathbf{3}}_{\frac{2}{3}}\ \to\ \begin{cases} u_3^R\equiv sS\left(-ie_4+e_5+e_{134}+ie_{135}\right)s^*S \\ u_3^G\equiv sS\left(-ie_1+e_3+e_{126}+e_{145}\right)s^*S \\ u_3^B\equiv sS\left(-ie_2+e_6-e_{123}+ie_{136}\right)s^*S \end{cases} \\
&\underline{\mathbf{3}}_{-\frac{1}{3}}\ \to\ \begin{cases} d_1^R\equiv sS^*\left(-ie_1-e_3+e_{126}-e_{145}\right)s^*S \\ d_1^G\equiv sS^*\left(ie_4+e_5+e_{134}-ie_{135}\right)s^*S \\ d_1^B\equiv sS^*\left(-ie_{124}-e_{125}-e_{146}+ie_{156}\right)s^*S \end{cases} \\
&\underline{\mathbf{3}}_{-\frac{1}{3}}\ \to\ \begin{cases} d_2^R\equiv sS^*\left(ie_2+e_6+e_{123}+ie_{136}\right)s^*S \\ d_2^G\equiv sS^*\left(-ie_{124}-e_{125}+e_{146}-ie_{156}\right)s^*S \\ d_2^B\equiv sS^*\left(-ie_4-e_5+e_{134}-ie_{135}\right)s^*S \end{cases} \\
&\underline{\mathbf{3}}_{-\frac{1}{3}}\ \to\ \begin{cases} d_3^R\equiv sS^*\left(-ie_{124}+e_{125}+e_{146}+ie_{156}\right)s^*S \\ d_3^G\equiv sS^*\left(-ie_2-e_6+e_{123}+ie_{136}\right)s^*S \\ d_3^B\equiv sS^*\left(ie_1+e_3+e_{126}-e_{145}\right)s^*S \end{cases} \\
&\underline{\mathbf{1}}_{\mathbf{0}}\ \to\ \nu_1\equiv sS\left(1+ie_{13}+ie_{26}+ie_{45}\right)sS \\
&\underline{\mathbf{1}}_{\mathbf{0}}\ \to\ \nu_2\equiv sS^*\left(3-ie_{13}-ie_{26}-ie_{45}\right)sS^* \\
&\underline{\mathbf{1}}_{\mathbf{0}}\ \to\ \nu_3\equiv s^*S^*\left(-ie_{124}-e_{125}+e_{146}-ie_{156}\right)sS \\
&\underline{\mathbf{1}}_{-\mathbf{1}}\ \to\ e_1^-\equiv sS^*\left(ie_1-e_3+e_{126}+e_{145}\right)s^*S^* \\
&\underline{\mathbf{1}}_{-\mathbf{1}}\ \to\ e_2^-\equiv sS^*\left(-ie_2+e_6+e_{123}-ie_{136}\right)s^*S^* \\
&\underline{\mathbf{1}}_{-\mathbf{1}}\ \to\ e_3^-\equiv sS^*\left(-ie_4+e_5-e_{134}-ie_{135}\right)s^*S^*.
\end{aligned} \tag{21}$$

From here, finding anti-particle states is remarkably easy. As with previous work, [32], [1], [26], [27], one simply invokes the complex conjugate, $i\mapsto -i$.

Readers should note that we are not distinguishing between the generations at this point. Hence, for example, the three $\underline{\mathbf{3}}_{\frac{2}{3}}$ representations are labelled arbitrarily as u_1, u_2, u_3, instead of u, c, t.

Finally, we mention that the electric charge assignments of equations (21) can easily be confirmed by the reader. This is facilitated by the fact that Q may be decomposed as

$$Q=\frac{1}{3}s^*S+\frac{2}{3}sS^*+s^*S^*. \tag{22}$$

For example, Q may be applied to the state u_1^R by setting $r_j=0$ and $r_0=1$ in the action (19):

$$\begin{aligned} \left[\, iQs,\ S^*u_1^Rs\,\right] &= i\left[\,\tfrac{2}{3}sS^*,\ sS^*u_1^RsS\,\right] \\ &= i\,\tfrac{2}{3}u_1^R. \end{aligned} \tag{23}$$

Hence, the action (19) assigns to u_1^R a Q charge of $\frac{2}{3}$.

7. Summary

This article demonstrates how the $SU(3)_c \times U(1)_{em}/\mathbb{Z}_3$ representations for three full generations of quarks and leptons may be generated, using just an 8 $\mathbb{C}$-dimensional algebra. In order to arrive at these 48 states, we did not simply replicate copies of $\mathbb{C}\otimes\mathbb{O}$. Instead, we considered the action of this one algebra on itself.

For those more accustomed to grand unified theories, this method should indeed seem unfamiliar. That is, it runs backwards to the usual direction of the prototypical unified theory. Standard grand unified theories begin with a sizeable Lie group, and then implement an appropriate mechanism in order to scale the symmetry group down. In contrast, this article shows how a low-dimensional algebra may act autonomously in order to scale the degrees of freedom up.

Although we have not proposed a grand unified theory here, these results do seem to point towards unification of another form. It is clear that the $SU(3)_c \times U(1)_{em}/\mathbb{Z}_3$ group elements, and also these 48 states, owe their existence to the same algebra. Ideally, *all* objects in such a model should likewise arise from the same algebra.

8. Outlook

While $\mathbb{C}\otimes\mathbb{O}$ did supply a reasonable portion of the standard model's group representation structure, we have by no means achieved a full description. For instance, nowhere in this paper have we discussed spin or chirality. And so we ask, *in what ways may these results be extended*?

In the third chapter of [27], it was shown that each of the Lorentz representations of the standard model can be identified as invariant subspaces of the algebra of the complex quaternions, $\mathbb{C}\otimes\mathbb{H}$. To be more precise, this 4$\mathbb{C}$-dimensional algebra yields: Lorentz scalars, ϕ, left- and right-handed Weyl spinors, Ψ_L, Ψ_R, Majorana spinors, Ψ_M, Dirac spinors, Ψ_D, four-vectors, p_μ, and the field strength tensor, $F_{\mu\nu}$, [35]. These Lorentz (or $SL(2,\mathbb{C})$) representations were identified as invariant subspaces of $\mathbb{C}\otimes\mathbb{H}$ under various actions of the algebra on itself. In each case, they were found to arise as a result of the outer automorphism and anti-automorphisms of the algebra, [27].

In the context of these findings, we then propose that Lorentzian degrees of freedom such as spin and chirality be described by $\mathbb{C}\otimes\mathbb{H}$, while other internal degrees of freedom such as colour and electric charge be described by $\mathbb{C}\otimes\mathbb{O}$, [29].

It is straightforward to see that $\mathbb{C}\otimes\mathbb{H}$ and $\mathbb{C}\otimes\mathbb{O}$ may be combined, via a tensor product over $\mathbb{C}$, into the algebra $(\mathbb{C}\otimes\mathbb{H})\otimes_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{O}) = \mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. The Dixon algebra $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ is the tensor product of the *only* four normed division algebras over the real numbers. (For an alternate three-generation model which makes use of tensor products of column vectors over division algebras, see [10].)

Once spin and chirality are incorporated into this model, we might then be in a position to address some obvious outstanding questions. For example,

- *How is electroweak symmetry to be described in this model? How might it incorporate Q ?*
- *How do we interpret the* $\underline{\mathbf{8}}_{\mathbf{0}}$ *representations in* $\mathbb{C}l(6)$*?*
- *What brings about the form of the action (19)?*
- *What is the connection between the recent one-generation results of [29] and this model? Or should [29] prompt the reconsideration of a 4-generation model?*
- *If the standard model's group representation structure is indeed a result of the algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$, *then what is it exactly that is so special about these algebras?*

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