General Relativity and Gravitation (2011)
DOI 10.1007/s10714-007-0551-y

RESEARCH ARTICLE

Salvatore Capozziello · Mauro Francaviglia

Extended theories of gravity and their cosmological and astrophysical applications

Received: 17 July 2007 / Accepted: 17 July 2007

Abstract Astrophysical observations are pointing out huge amounts of "dark matter" and "dark energy" needed to explain the observed large scale structure and cosmic dynamics. The emerging picture is a spatially flat, homogeneous Universe undergoing the today observed accelerated phase. Despite of the good quality of astrophysical surveys, commonly addressed as *Precision Cosmology*, the nature and the nurture of dark energy and dark matter, which should constitute the bulk of cosmological matter-energy, are still unknown. Furthermore, up to now, no experimental evidence has been found, at fundamental level, to explain such mysterious components. The problem could be completely reversed considering dark matter and dark energy as "shortcomings" of General Relativity in its simplest formulation (a linear theory in the Ricci scalar R, minimally coupled to the standard perfect fluid matter) and claiming for the "correct" theory of gravity as that derived by matching the largest number of observational data, without imposing any theory a priori. As a working hypothesis, accelerating behavior of cosmic fluid, large scale structure, potential of galaxy clusters, rotation curves of spiral galaxies could be reproduced by means of *extending* the standard theory of General Relativity. In other words, gravity could acts in different ways at different scales and the above "shortcomings" could be due to incorrect extrapolations of the Einstein gravity, actually tested at short scales and low energy regimes. After a survey of what is intended for *Extended Theories of Gravity* in the so called "metric" and "Palatini" approaches, we discuss some cosmological and astrophysical applications where the issues related to the dark components are addressed by enlarging the Einstein theory to more general $f(R)$ Lagrangians, where $f(R)$ is a generic function of Ricci scalar R, not assumed simply linear. Obviously, this is not the final answer to the problem of "dark-components" but it can be consid-

S. Capozziello Dipartimento di Scienze fisiche Università di Napoli "Federico II", and INFN Sez. di Napoli Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia 80126 Napoli, Italy
capozzie@na.infn.it · M. Francaviglia Dipartimento di Matematica Università di Torino, and INFN Sez. di Torino Via Carlo Alberto 10 10123 Torino, Italy

ered as an operative scheme whose aim is to avoid the addition of unknown exotic ingredients to the cosmic pie.

Keywords Extended theories of gravity, Dark energy, Dark matter, Observations

1 Introduction

General Relativity (GR) is a comprehensive theory of spacetime, gravity and matter. Its formulation implies that space and time are not "absolute" entities, as in Classical Mechanics, but dynamical quantities strictly related to the distribution of matter and energy. As a consequence, this approach gave rise to a new conception of the Universe itself which, for the first time, was considered as a dynamical system. In other words, Cosmology has been enclosed in the realm of Science and not only of Philosophy, as before the Einstein work. On the other hand, the possibility of a scientific investigation of the Universe has led to the formulation of the Standard Cosmological Model [1] which, quite nicely, has matched with observations.

Despite of these results, in the last 30 years, several shortcomings came out in the Einstein theory and people began to investigate whether GR is the only fundamental theory capable of explaining the gravitational interaction. Such issues come, essentially, from cosmology and quantum field theory. In the first case, the presence of the Big Bang singularity, the flatness and horizon problems [2] led to the statement that Cosmological Standard Model, based on the GR and the Standard Model of Particle Physics, is inadequate to describe the Universe at extreme regimes. On the other hand, GR is a *classical* theory which does not work as a fundamental theory, when one wants to achieve a full quantum description of spacetime (and then of gravity).

Due to these facts and, first of all, to the lack of a definitive quantum gravity theory, alternative theories have been considered in order to attempt, at least, a semi-classical scheme where GR and its positive results could be recovered. One of the most fruitful approaches has been that of Extended Theories of Gravity (ETG) which have become a sort of paradigm in the study of gravitational interaction. They are based on corrections and enlargements of the Einstein theory. The paradigm consists, essentially, in adding higher-order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of quantum gravity [3].

Other motivations to modify GR come from the issue of a full recovering of the Mach principle which leads to assume a varying gravitational coupling. The principle states that the local inertial frame is determined by some average of the motion of distant astronomical objects [4]. This fact implies that the gravitational coupling can be scale-dependent and related to some scalar field. As a consequence, the concept of "inertia" and the Equivalence Principle have to be revised. For example, the Brans–Dicke theory [5] is a serious attempt to define an alternative theory to the Einstein gravity: it takes into account a variable Newton gravitational coupling, whose dynamics is governed by a scalar field non-minimally coupled to the geometry. In such a way, Mach's principle is better implemented [5; 6; 7].

Besides, every unification scheme as Superstrings, Supergravity or Grand Unified Theories, takes into account effective actions where non-minimal couplings to the geometry or higher-order terms in the curvature invariants are present. Such contributions are due to one-loop or higher-loop corrections in the high-curvature regimes near the full (not yet available) quantum gravity regime [3]. Specifically, this scheme was adopted in order to deal with the quantization on curved space-times and the result was that the interactions among quantum scalar fields and background geometry or the gravitational self-interactions yield corrective terms in the Hilbert–Einstein Lagrangian [8]. Moreover, it has been realized that such corrective terms are inescapable in order to obtain the effective action of quantum gravity at scales closed to the Planck one [9]. All these approaches are not the "full quantum gravity" but are needed as working schemes toward it.

In summary, higher-order terms in curvature invariants (such as R^2, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\Box R$, or $R\Box^k R$) or non-minimally coupled terms between scalar fields and geometry (such as $\phi^2 R$) have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. For instance, one can notice that such terms occur in the effective Lagrangian of strings or in Kaluza–Klein theories, when the mechanism of dimensional reduction is used [10].

On the other hand, from a conceptual viewpoint, there are no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar R, minimally coupled with matter [11]. Furthermore, the idea that there are no "exact" laws of physics could be taken into serious account: in such a case, the effective Lagrangians of physical interactions are "stochastic" functions. This feature means that the local gauge invariances (i.e., conservation laws) are well approximated only in the low energy limit and the fundamental physical constants can vary [12].

Besides fundamental physics motivations, all these theories have acquired a huge interest in cosmology due to the fact that they "naturally" exhibit inflationary behaviors able to overcome the shortcomings of Cosmological Standard Model (based on GR). The related cosmological models seem realistic and capable of matching with the CMBR observations [13; 14; 15]. Furthermore, it is possible to show that, via conformal transformations, the higher-order and non-minimally coupled terms always correspond to the Einstein gravity plus one or more than one minimally coupled scalar fields [16; 17; 18; 19; 20].

More precisely, higher-order terms appear always as contributions of order two in the field equations. For example, a term like R^2 gives fourth order equations [21], $R\,\Box R$ gives sixth order equations [20; 22], $R\Box^2 R$ gives eighth order equations [23] and so on. By a conformal transformation, any second-order derivative term corresponds to a scalar field:[1] for example, fourth-order gravity gives Einstein plus one scalar field, sixth-order gravity gives Einstein plus two scalar fields and so on [20; 24].

Furthermore, it is possible to show that the $f(R)$-gravity is equivalent not only to a scalar-tensor one but also to the Einstein theory plus an ideal fluid [25]. This feature results very interesting if we want to obtain multiple inflationary events since an early stage could select "very" large-scale structures (clusters of galax-

[1] The dynamics of such scalar fields is usually given by the corresponding Klein–Gordon Equation, which is second order.

ies today), while a late stage could select "small" large-scale structures (galaxies today) [22]. The philosophy is that each inflationary era is related to the dynamics of a scalar field. Finally, these extended schemes could naturally solve the problem of "graceful exit" bypassing the shortcomings of former inflationary models [15; 26].

In addition to the revision of Standard Cosmology at early epochs (leading to the Inflation), a new approach is necessary also at late epochs. ETGs could play a fundamental role also in this context. In fact, the increasing bulk of data that have been accumulated in the last few years have paved the way to the emergence of a new cosmological model usually referred to as the *Concordance Model*.

The Hubble diagram of Type Ia Supernovae (hereafter SNeIa), measured by both the Supernova Cosmology Project [27; 28] and the High-z Team [29; 30] up to redshift $z \sim 1$, has been the first evidence that the Universe is undergoing a phase of accelerated expansion. On the other hand, balloon born experiments, such as BOOMERanG [31] and MAXIMA [32], determined the location of the first and second peak in the anisotropy spectrum of the cosmic microwave background radiation (CMBR) strongly pointing out that the geometry of the Universe is spatially flat. If combined with constraints coming from galaxy clusters on the matter density parameter Ω_M, these data indicate that the Universe is dominated by a non-clustered fluid with negative pressure, generically dubbed *dark energy*, which is able to drive the accelerated expansion. This picture has been further strengthened by the recent precise measurements of the CMBR spectrum, due to the WMAP experiment [33; 34; 35], and by the extension of the SNeIa Hubble diagram to redshifts higher than 1 [36].

After these observational evidences, an overwhelming flood of papers has appeared: they present a great variety of models trying to explain this phenomenon. In any case, the simplest explanation is claiming for the well known cosmological constant Λ [37]. Although it is the best fit to most of the available astrophysical data [33], the ΛCDM model fails in explaining why the inferred value of Λ is so tiny (120 orders of magnitude lower!) if compared with the typical vacuum energy values predicted by particle physics and why its energy density is today comparable to the matter density (the so called *coincidence problem*).

As a tentative solution, many authors have replaced the cosmological constant with a scalar field rolling down its potential and giving rise to the model now referred to as *quintessence* [38; 39]. Even if successful in fitting the data, the quintessence approach to dark energy is still plagued by the coincidence problem since the dark energy and matter densities evolve differently and reach comparable values for a very limited portion of the Universe evolution coinciding at present era. To be more precise, the quintessence dark energy is tracking matter and evolves in the same way for a long time. But then, at late time, somehow it has to change its behavior into no longer tracking the dark matter but starting to dominate as a cosmological constant. This is the coincidence problem of quintessence.

Moreover, it is not clear where this scalar field originates from, thus leaving a great uncertainty on the choice of the scalar field potential. The subtle and elusive nature of dark energy has led many authors to look for completely different scenarios able to give a quintessential behavior without the need of exotic components. To this aim, it is worth stressing that the acceleration of the Universe only claims for a negative pressure dominant component, but does not tell anything about the nature and the number of cosmic fluids filling the Universe.

This consideration suggests that it could be possible to explain the accelerated expansion by introducing a single cosmic fluid with an equation of state causing it to act like dark matter at high densities and dark energy at low densities. An attractive feature of these models, usually referred to as Unified Dark Energy (UDE) or Unified Dark Matter (UDM) models, is that such an approach naturally solves, atleast phenomenologically, the coincidence problem. Some interesting examples are the generalized Chaplygin gas [40], the tachyon field [41] and the condensate cosmology [42]. A different class of UDE models has been proposed [43; 44] where a single fluid is considered: its energy density scales with the redshift in such a way that the radiation dominated era, the matter era and the accelerating phase can be naturally achieved. It is worth noticing that such class of models are extremely versatile since they can be interpreted both in the framework of UDE models and as a two-fluid scenario with dark matter and scalar field dark energy. The main ingredient of the approach is that a generalized equation of state can be always obtained and observational data can be fitted.

Actually, there is still a different way to face the problem of cosmic acceleration. As stressed in [45], it is possible that the observed acceleration is not the manifestation of another ingredient in the cosmic pie, but rather the first signal of a breakdown of our understanding of the laws of gravitation (in the infra-red limit).

From this point of view, it is thus tempting to modify the Friedmann equations to see whether it is possible to fit the astrophysical data with models comprising only the standard matter. Interesting examples of this kind are the Cardassian expansion [46] and the DGP gravity [47]. Moving in this same framework, it is possible to find alternative schemes where a quintessential behavior is obtained by taking into account effective models coming from fundamental physics giving rise to generalized or higher-order gravity actions [48; 49; 50; 51] (for a comprehensive review see [52]).

For instance, a cosmological constant term may be recovered as a consequence of a non-vanishing torsion field thus leading to a model which is consistent with both SNeIa Hubble diagram and Sunyaev-Zel'dovich data coming from clusters of galaxies [53]. SNeIa data could also be efficiently fitted including higher-order curvature invariants in the gravity Lagrangian [54; 56; 57; 58]. It is worth noticing that these alternative models provide naturally a cosmological component with negative pressure whose origin is related to the geometry of the Universe thus overcoming the problems linked to the physical significance of the scalar field.

It is evident, from this short overview, the high number of cosmological models which are viable candidates to explain the observed accelerated expansion. This abundance of models is, from one hand, the signal of the fact that we have a limited number of cosmological tests to discriminate among rival theories, and, from the other hand, that a urgent degeneracy problem has to be faced. To this aim, it is useful to remark that both the SNeIa Hubble diagram and the angular size–redshift relation of compact radio sources [59] are distance based methods to probe cosmological models so then systematic errors and biases could be iterated. From this point of view, it is interesting to search for tests based on time-dependent observables.

For example, one can take into account the *lookback time* to distant objects since this quantity can discriminate among different cosmological models. The lookback time is observationally estimated as the difference between the present day age of the Universe and the age of a given object at redshift z. Such an estimate is possible if the object is a galaxy observed in more than one photometric band since its color is determined by its age as a consequence of stellar evolution. It is thus possible to get an estimate of the galaxy age by measuring its magnitude in different bands and then using stellar evolutionary codes to choose the model that reproduces the observed colors at best.

Coming to the weak-field-limit approximation, which essentially means considering Solar System scales, ETGs are expected to reproduce GR which, in any case, is firmly tested only in this limit [61]. This fact is matter of debate since several relativistic theories *do not* reproduce exactly the Einstein results in the Newtonian approximation but, in some sense, generalize them. As it was firstly noticed by Stelle [64], a R^2-theory gives rise to Yukawa-like corrections in the Newtonian potential. Such a feature could have interesting physical consequences. For example, some authors claim to explain the flat rotation curves of galaxies by using such terms [65]. Others [66] have shown that a conformal theory of gravity is nothing else but a fourth-order theory containing such terms in the Newtonian limit. Besides, indications of an apparent, anomalous, long-range acceleration revealed from the data analysis of Pioneer 10/11, Galileo, and Ulysses spacecrafts could be framed in a general theoretical scheme by taking corrections to the Newtonian potential into account [67; 68].

In general, any relativistic theory of gravitation yields corrections to the Newton potential (see for example [69]) which, in the post-Newtonian (PPN) formalism, could be a test for the same theory [61]. Furthermore the newborn *gravitational lensing astronomy* [70] is giving rise to additional tests of gravity over small, large, and very large scales which soon will provide direct measurements for the variation of the Newton coupling [71], the potential of galaxies, clusters of galaxies and several other features of self-gravitating systems.

Such data will be, very likely, capable of confirming or ruling out the physical consistency of GR or of any ETG. In summary, the general features of ETGs are that the Einstein field equations result to be modified in two senses: (i) geometry can be non-minimally coupled to some scalar field, and/or (ii) higher than second order derivative terms in the metric come out. In the former case, we generically deal with scalar-tensor theories of gravity; in the latter, we deal with higher-order theories. However combinations of non-minimally coupled and higher-order terms can emerge as contributions in effective Lagrangians. In this case, we deal with higher-order-scalar-tensor theories of gravity.

Considering a mathematical viewpoint, the problem of reducing more general theories to Einstein standard form has been extensively treated; one can see that, through a "Legendre" transformation on the metric, higher-order theories, under suitable regularity conditions on the Lagrangian, take the form of the Einstein one in which a scalar field (or more than one) is the source of the gravitational field (see for example [11; 72; 73; 74]); on the other side, as discussed above, it has been studied the mathematical equivalence between models with variable gravitational coupling with the Einstein standard gravity through suitable conformal transformations (see [75; 76]).

In any case, the debate on the physical meaning of conformal transformations is far to be solved (see [78] and references therein for a comprehensive review). Several authors claim for a true physical difference between Jordan frame (higher-order theories and/or variable gravitational coupling) since there are experimental and observational evidences which point out that the Jordan frame could be suitable to better match solutions with data. Others state that the true physical frame is the Einstein one according to the energy theorems [74]. However, the discussion is open and no definitive statement has been formulated up to now.

The problem should be faced from a more general viewpoint and the Palatini approach to gravity could be useful to this goal. The Palatini approach in gravitational theories was firstly introduced and analyzed by Einstein himself [79]. It was, however, called the Palatini approach as a consequence of an historical misunderstanding [80; 81].

The fundamental idea of the Palatini formalism is to consider the (usually torsion-less) connection Γ, entering the definition of the Ricci tensor, to be independent of the metric g defined on the spacetime $\mathscr{M}$. The Palatini formulation for the standard Hilbert–Einstein theory results to be equivalent to the purely metric theory: this follows from the fact that the field equations for the connection Γ, firstly considered to be independent of the metric, give the Levi-Civita connection of the metric g. As a consequence, there is no reason to impose the Palatini variational principle in the standard Hilbert–Einstein theory instead of the metric variational principle.

However, the situation completely changes if we consider the ETGs, depending on functions of curvature invariants, as $f(R)$, or non-minimally coupled to some scalar field. In these cases, the Palatini and the metric variational principle provide different field equations and the theories thus derived differ [74; 82]. The relevance of Palatini approach, in this framework, has been recently proven in relation to cosmological applications [48; 49; 50; 51; 52; 83; 84; 85].

It has also been studied the crucial problem of the Newtonian potential in alternative theories of Gravity and its relations with the conformal factor [87; 88]. From a physical viewpoint, considering the metric g and the connection Γ as independent fields means to decouple the metric structure of spacetime and its geodesic structure (being, in general, the connection Γ not the Levi-Civita connection of g). The chronological structure of spacetime is governed by g while the trajectories of particles, moving in the spacetime, are governed by Γ.

This decoupling enriches the geometric structure of spacetime and generalizes the purely metric formalism. This metric-affine structure of spacetime is naturally translated, by means of the same (Palatini) field equations, into a bi-metric structure of spacetime. Beside the *physical* metric g, another metric h is involved. This new metric is related, in the case of $f(R)$-gravity, to the connection. As a matter of fact, the connection Γ results to be the Levi-Civita connection of h and thus provides the geodesic structure of spacetime.

If we consider the case of non-minimally coupled interaction in the gravitational Lagrangian (scalar-tensor theories), the new metric h is related to the non-minimal coupling. The new metric h can be thus related to a different geometric and physical aspect of the gravitational theory. Thanks to the Palatini formalism, the non-minimal coupling and the scalar field, entering the evolution of the gravitational fields, are separated from the metric structure of spacetime. The situation mixes when we consider the case of higher-order-scalar-tensor theories. Due to these features, the Palatini approach could greatly contribute to clarify the physical meaning of conformal transformation [86].

In this review paper, without claiming for completeness, we want to give a survey on the formal and physical aspects of ETGs in metric and Palatini approaches, considering the cosmological and astrophysical applications of some ETG models.

The layout is the following. Section 2 is a rapid overview of GR. We summarize what a good theory of gravity is requested to do and what the foundations of the Einstein theory are. The goal is to demonstrate that ETGs have the same theoretical bases but, in principle, could avoid some shortcomings of GR which is nothing else but a particular case of ETG, $f(R) = R$.

The field equations for generic ETGs are derived in Sect. 3. Specifically, we discuss two interesting cases: $f(R)$ and scalar-tensor theories considering their relations with GR by conformal transformations.

The Palatini approach and its intrinsic conformal structure is discussed in Sect. 4 giving some peculiar examples.

Cosmological applications are considered in Sect. 5. After a short summary of ΛCDM model, we show that dark energy and quintessence issues can be addressed as "curvature effects", if ETGs (in particular $f(R)$ theories) are considered. We work out some cosmological models comparing the solutions with data coming from observational surveys. As further result, we show that also the stochastic cosmological background of gravitational waves could be "tuned" by ETGs. This fact could open new perspective also in the issues of detection and production of gravitational waves which should be investigated not only in the standard framework of GR.

Section 6 is devoted to the galactic dynamics under the standard of ETGs. Also in this case, we show that flat rotation curves and haloes of spiral galaxies could be explained as curvature effects which give rise to corrections to the Newton potential without taking into account huge amounts of dark matter. Discussion and conclusions are drawn in Sect. 7.

2 What a good theory of gravity has to do: general relativity and its extensions

From a phenomenological point of view, there are some minimal requirements that any relativistic theory of gravity has to match. First of all, it has to explain the astrophysical observations (e.g., the orbits of planets, the potential of self-gravitating structures).

This means that it has to reproduce the Newtonian dynamics in the weak-energy limit. Besides, it has to pass the classical Solar System tests which are all experimentally well founded [61].

As second step, it should reproduce galactic dynamics considering the observed baryonic constituents (e.g., luminous components as stars, sub-luminous components as planets, dust and gas), radiation and Newtonian potential which is, by assumption, extrapolated to galactic scales.

Thirdly, it should address the problem of large scale structure (e.g., clustering of galaxies) and finally cosmological dynamics, which means to reproduce, in a self-consistent way, the cosmological parameters as the expansion rate, the Hubble constant, the density parameter and so on. Observations and experiments, essentially, probe the standard baryonic matter, the radiation and an attractive overall interaction, acting at all scales and depending on distance: the gravity.

The simplest theory which try to satisfies the above requirements was formulated by Albert Einstein in the years 1915–1916 [89] and it is known as the Theory of General Relativity. It is firstly based on the assumption that space and time have to be entangled into a single spacetime structure, which, in the limit of no gravitational forces, has to reproduce the Minkowski spacetime structure. Einstein profitted also of ideas earlier put forward by Riemann, who stated that the Universe should be a curved manifold and that its curvature should be established on the basis of astronomical observations [90].

In other words, the distribution of matter has to influence point by point the local curvature of the spacetime structure. The theory, eventually formulated by Einstein in 1915, was strongly based on three assumptions that the Physics of Gravitation has to satisfy.

The "Principle of Relativity", that amounts to require all frames to be good frames for Physics, so that no preferred inertial frame should be chosen a priori (if any exist).

The "Principle of Equivalence", that amounts to require inertial effects to be locally indistinguishable from gravitational effects (in a sense, the equivalence between the inertial and the gravitational mass).

The "Principle of General Covariance", that requires field equations to be "generally covariant" (today, we would better say to be invariant under the action of the group of all spacetime diffeomorphisms) [91].

And—on the top of these three principles—the requirement that causality has to be preserved (the "Principle of Causality", i.e., that each point of spacetime should admit a universally valid notion of past, present and future).

Let us also recall that the older Newtonian theory of spacetime and gravitation–that Einstein wanted to reproduce at least in the limit of small gravitational forces (what is called today the "post-Newtonian approximation")—required space and time to be absolute entities, particles moving in a preferred inertial frame following curved trajectories, the curvature of which (i.e., the acceleration) had to be determined as a function of the sources (i.e., the "forces").

On these bases, Einstein was led to postulate that the gravitational forces have to be expressed by the curvature of a metric tensor field $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ on a four-dimensional spacetime manifold, having the same signature of Minkowski metric, i.e., the so-called "Lorentzian signature", herewith assumed to be $(+,-,-,-)$. He also postulated that spacetime is curved in itself and that its curvature is locally determined by the distribution of the sources, i.e.,—being spacetime a continuum—by the four-dimensional generalization of what in Continuum Mechanics is called the "matter stress-energy tensor", i.e., a rank-two (symmetric) tensor $T^m_{\mu\nu}$.

Once a metric $g_{\mu\nu}$ is given, its curvature is expressed by the Riemann (curvature) tensor

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\sigma_{\beta\nu}\Gamma^\alpha_{\sigma\mu} - \Gamma^\sigma_{\beta\mu}\Gamma^\alpha_{\sigma\nu} \tag{1}$$

where the comas are partial derivatives. Its contraction

$$R^\alpha{}_{\mu\alpha\nu} = R_{\mu\nu}, \tag{2}$$

is the “Ricci tensor” and the scalar

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{3}$$

is called the “scalar curvature” of $g_{\mu\nu}$. Einstein was led to postulate the following equations for the dynamics of gravitational forces

$$R_{\mu\nu} = \frac{\kappa}{2} T^{m}_{\mu\nu} \tag{4}$$

where $\kappa = 8\pi G$, with $c = 1$ is a coupling constant. These equations turned out to be physically and mathematically unsatisfactory.

As Hilbert pointed out [91], they were not of a variational origin, i.e., there was no Lagrangian able to reproduce them exactly (this is slightly wrong, but this remark is unessential here). Einstein replied that he knew that the equations were physically unsatisfactory, since they were contrasting with the continuity equation of any reasonable kind of matter. Assuming that matter is given as a perfect fluid, that is

$$T^{m}_{\mu\nu} = (p+\rho) u_{\mu} u_{\nu} - p g_{\mu\nu} \tag{5}$$

where $u_{\mu}u_{\nu}$ is a comoving observer, p is the pressure and ρ the density of the fluid, then the continuity equation requires $T^{m}_{\mu\nu}$ to be covariantly constant, i.e., to satisfy the conservation law

$$\nabla^{\mu} T^{m}_{\mu\nu} = 0, \tag{6}$$

where ∇^{μ} denotes the covariant derivative with respect to the metric.

In fact, it is not true that $\nabla^{\mu} R_{\mu\nu}$ vanishes (unless $R = 0$). Einstein and Hilbert reached independently the conclusion that the wrong field equations (4) had to be replaced by the correct ones

$$G_{\mu\nu} = \kappa T^{m}_{\mu\nu} \tag{7}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{8}$$

that is currently called the “Einstein tensor” of $g_{\mu\nu}$. These equations are both variational and satisfy the conservation laws (6) since the following relation holds

$$\nabla^{\mu} G_{\mu\nu} = 0, \tag{9}$$

as a byproduct of the so-called “Bianchi identities” that the curvature tensor of $g_{\mu\nu}$ has to satisfy [1].

The Lagrangian that allows to obtain the field equations (7) is the sum of a “matter Lagrangian” $\mathscr{L}_m$, the variational derivative of which is exactly $T^{m}_{\mu\nu}$, i.e.,

$$T^{m}_{\mu\nu} = \frac{\delta \mathscr{L}_m}{\delta g^{\mu\nu}} \tag{10}$$

and of a "gravitational Lagrangian", currently called the Hilbert–Einstein Lagrangian

$$L_{HE} = g^{\mu\nu} R_{\mu\nu} \sqrt{-g} = R\sqrt{-g}, \tag{11}$$

where $\sqrt{-g}$ denotes the square root of the value of the determinant of the metric $g_{\mu\nu}$.

The choice of Hilbert and Einstein was completely arbitrary (as it became clear a few years later), but it was certainly the simplest one both from the mathematical and the physical viewpoint. As it was later clarified by Levi-Civita in 1919, curvature is not a "purely metric notion" but, rather, a notion related to the "linear connection" to which "parallel transport" and "covariant derivation" refer [92].

In a sense, this is the precursor idea of what in the sequel would be called a "gauge theoretical framework" [93], after the pioneering work by Cartan in 1925 [94]. But at the time of Einstein, only metric concepts were at hands and his solution was the only viable.

It was later clarified that the three principles of relativity, equivalence and covariance, together with causality, just require that the spacetime structure has to be determined by either one or both of two fields, a Lorentzian metric g and a linear connection Γ, assumed to be torsionless for the sake of simplicity.

The metric g fixes the causal structure of spacetime (the light cones) as well as its metric relations (clocks and rods); the connection Γ fixes the free-fall, i.e., the locally inertial observers. They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow null geodesics of Γ, so that Γ and g can be independent, a priori, but constrained, a posteriori, by some physical restrictions. These, however, do not impose that Γ has necessarily to be the Levi-Civita connection of g [95].

This justifies—at least on a purely theoretical basis—the fact that one can envisage the so-called "alternative theories of gravitation", that we prefer to call "Extended Theories of Gravitation" since their starting points are exactly those considered by Einstein and Hilbert: theories in which gravitation is described by either a metric (the so-called "purely metric theories"), or by a linear connection (the so-called "purely affine theories") or by both fields (the so-called "metric-affine theories", also known as "first order formalism theories"). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both g and Γ.

The choice (11) is by no means unique and it turns out that the Hilbert–Einstein Lagrangian is in fact the only choice that produces an invariant that is linear in second derivatives of the metric (or first derivatives of the connection). A Lagrangian that, unfortunately, is rather singular from the Hamiltonian viewpoint, in much than same way as Lagrangians, linear in canonical momenta, are rather singular in Classical Mechanics (see e.g., [96]).

A number of attempts to generalize GR (and unify it to Electromagnetism) along these lines were followed by Einstein himself and many others (Eddington, Weyl, Schrodinger, just to quote the main contributors; see, e.g., [97]) but they were eventually given up in the 1950s of twentieth Century, mainly because of a number of difficulties related to the definitely more complicated structure of a non-linear theory (where by "non-linear" we mean here a theory that is based on non-linear invariants of the curvature tensor), and also because of the new understanding of Physics that is currently based on four fundamental forces and requires the more general "gauge framework" to be adopted (see [98]).

Still a number of sporadic investigations about "alternative theories" continued even after 1960 (see [61] and references quoted therein for a short history). The search of a coherent quantum theory of gravitation or the belief that gravity has to be considered as a sort of low-energy limit of string theories (see, e.g., [99])—something that we are not willing to enter here in detail—has more or less recently revitalized the idea that there is no reason to follow the simple prescription of Einstein and Hilbert and to assume that gravity should be classically governed by a Lagrangian linear in the curvature.

Further curvature invariants or non-linear functions of them should be also considered, especially in view of the fact that they have to be included in both the semi-classical expansion of a quantum Lagrangian or in the low-energy limit of a string Lagrangian.

Moreover, it is clear from the recent astrophysical observations and from the current cosmological hypotheses that Einstein equations are no longer a good test for gravitation at Solar System, galactic, extra-galactic and cosmic scale, unless one does not admit that the matter side of Eqs. 7 contains some kind of exotic matter-energy which is the "dark matter" and "dark energy" side of the Universe.

The idea which we propose here is much simpler. Instead of changing the matter side of Einstein Equations (7) in order to fit the "missing matter-energy" content of the currently observed Universe (up to the 95% of the total amount!), by adding any sort of inexplicable and strangely behaving matter and energy, we claim that it is simpler and more convenient to change the gravitational side of the equations, admitting corrections coming from non-linearities in the Lagrangian. However, this is nothing else but a matter of taste and, since it is possible, such an approach should be explored. Of course, provided that the Lagrangian can be conveniently tuned up (i.e., chosen in a huge family of allowed Lagrangians) on the basis of its best fit with all possible observational tests, at all scales (solar, galactic, extragalactic and cosmic).

Something that—in spite of some commonly accepted but disguised opinion—can and should be done before rejecting a priori a non-linear theory of gravitation (based on a non-singular Lagrangian) and insisting that the Universe has to be necessarily described by a rather singular gravitational Lagrangian (one that does not allow a coherent perturbation theory from a good Hamiltonian viewpoint) accompanied by matter that does not follow the behavior that standard baryonic matter, probed in our laboratories, usually satisfies.

3 The extended theories of gravity

With the above considerations in mind, let us start with a general class of higher-order-scalar-tensor theories in four dimensions[2] given by the action

[2] For the aims of this review, we do not need more complicated invariants like $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ which are also possible.

$$\mathscr{A} = \int d^4x\sqrt{-g}\left[F(R,\Box R,\Box^2 R,..\Box^k R,\phi) - \frac{\varepsilon}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \mathscr{L}_m\right], \qquad (12)$$

where F is an unspecified function of curvature invariants and of a scalar field ϕ. The term $\mathscr{L}_m$, as above, is the minimally coupled ordinary matter contribution. We shall use physical units $8\pi G = c = \hbar = 1$; ε is a constant which specifies the theory. Actually its values can be $\varepsilon = \pm 1, 0$ fixing the nature and the dynamics of the scalar field which can be a standard scalar field, a phantom field or a field without dynamics (see [133; 134] for details).

In the metric approach, the field equations are obtained by varying (12) with respect to $g_{\mu\nu}$. We get

$$\begin{aligned} G^{\mu\nu} = \frac{1}{\mathscr{G}}\Bigg[& T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(F - \mathscr{G}R) + (g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\mathscr{G}_{;\lambda\sigma} \\ & + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma})(\Box^{j-i})_{;\sigma}\left(\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda} \\ & - g^{\mu\nu}g^{\lambda\sigma}\left((\Box^{j-1}R)_{;\sigma}\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda}\Bigg], \end{aligned} \qquad (13)$$

where $G^{\mu\nu}$ is the above Einstein tensor and

$$\mathscr{G} \equiv \sum_{j=0}^{n}\Box^j\left(\frac{\partial F}{\partial \Box^j R}\right). \qquad (14)$$

The differential Eqs. 13 are of order $(2k+4)$. The stress-energy tensor is due to the kinetic part of the scalar field and to the ordinary matter:

$$T_{\mu\nu} = T^m_{\mu\nu} + \frac{\varepsilon}{2}\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\phi_{;}^{\alpha}\phi_{;\alpha}\right]. \qquad (15)$$

The (eventual) contribution of a potential $V(\phi)$ is contained in the definition of F. From now on, we shall indicate by a capital F a Lagrangian density containing also the contribution of a potential $V(\phi)$ and by $F(\phi)$, $f(R)$, or $f(R,\Box R)$ a function of such fields without potential.

By varying with respect to the scalar field ϕ, we obtain the Klein–Gordon equation

$$\varepsilon\Box\phi = -\frac{\partial F}{\partial \phi}. \qquad (16)$$

Several approaches can be used to deal with such equations. For example, as we said, by a conformal transformation, it is possible to reduce an ETG to a (multi) scalar-tensor theory of gravity [69; 18; 19; 20; 100].

The simplest extension of GR is achieved assuming

$$F = f(R), \qquad \varepsilon = 0, \qquad (17)$$

in the action (12); $f(R)$ is an arbitrary (analytic) function of the Ricci curvature scalar R. We are considering here the simplest case of fourth-order gravity but we could construct such kind of theories also using other invariants in $R_{\mu\nu}$ or $R^{\alpha}_{\beta\mu\nu}$. The standard Hilbert–Einstein action is, of course, recovered for $f(R)=R$. Varying with respect to $g_{\alpha\beta}$, we get the field equations

$$f'(R)R_{\alpha\beta}-\frac{1}{2}f(R)g_{\alpha\beta}=f'(R)^{;\mu\nu}\left(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu}\right), \tag{18}$$

which are fourth-order equations due to the term $f'(R)^{;\mu\nu}$; the prime indicates the derivative with respect to R. Equation (18) is also the equation for $T_{\mu\nu}=0$ when the matter term is absent.

By a suitable manipulation, the above equation can be rewritten as:

$$G_{\alpha\beta}=\frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R)-Rf'(R)\right]+f'(R)_{;\alpha\beta}-g_{\alpha\beta}\Box f'(R)\right\}, \tag{19}$$

where the gravitational contribution due to higher-order terms can be simply reinterpreted as a stress-energy tensor contribution. This means that additional and higher-
order terms in the gravitational action act, in principle, as a stress-energy tensor, related to the form of $f(R)$. Considering also the standard perfect-fluid matter contribution, we have

$$\begin{aligned}G_{\alpha\beta}&=\frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R)-Rf'(R)\right]+f'(R)_{;\alpha\beta}-g_{\alpha\beta}\Box f'(R)\right\}\\&+\frac{T^{m}_{\alpha\beta}}{f'(R)}=T^{\text{curv}}_{\alpha\beta}+\frac{T^{m}_{\alpha\beta}}{f'(R)},\end{aligned} \tag{20}$$

where $T^{\text{curv}}_{\alpha\beta}$ is an effective stress-energy tensor constructed by the extra curvature terms. In the case of GR, $T^{\text{curv}}_{\alpha\beta}$ identically vanishes while the standard, minimal coupling is recovered for the matter contribution. The peculiar behavior of $f(R)=R$ is due to the particular form of the Lagrangian itself which, even though it is a second order Lagrangian, can be non-covariantly rewritten as the sum of a first order Lagrangian plus a pure divergence term. The Hilbert–Einstein Lagrangian can be in fact recast as follows:

$$L_{\text{HE}}=\mathscr{L}_{\text{HE}}\sqrt{-g}=\left[p^{\alpha\beta}(\Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\rho\beta}-\Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\alpha\beta})+\nabla_{\sigma}(p^{\alpha\beta}u^{\sigma}{}_{\alpha\beta})\right] \tag{21}$$

where:

$$p^{\alpha\beta}=\sqrt{-g}g^{\alpha\beta}=\frac{\partial\mathscr{L}}{\partial R_{\alpha\beta}} \tag{22}$$

Γ is the Levi-Civita connection of g and $u^{\sigma}_{\alpha\beta}$ is a quantity constructed out with the variation of Γ [1]. Since $u^{\sigma}_{\alpha\beta}$ is not a tensor, the above expression is not covariant; however a standard procedure has been studied to recast covariance in the first order theories [101]. This clearly shows that the field equations should consequently be second order and the Hilbert–Einstein Lagrangian is thus degenerate.

From the action (12), it is possible to obtain another interesting case by choosing

$$F=F(\phi)R-V(\phi),\quad \varepsilon=-1. \tag{23}$$

In this case, we get

$$\mathscr{A} = \int \mathrm{d}^4x\sqrt{-g}\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right] \tag{24}$$

$V(\phi)$ and $F(\phi)$ are generic functions describing respectively the potential and the coupling of a scalar field ϕ. The Brans–Dicke theory of gravity is a particular case of the action (24) for $V(\phi) = 0$ [102]. The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi)\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right] = -\frac{1}{2}T^{\phi}_{\mu\nu} - g_{\mu\nu}\Box_g F(\phi) + F(\phi)_{;\mu\nu}, \tag{25}$$

here $\Box_g$ is the d'Alembert operator with respect to the metric g. The energy-momentum tensor relative to the scalar field is

$$T^{\phi}_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi) \tag{26}$$

The variation with respect to ϕ provides the Klein–Gordon equation, i.e., the field equation for the scalar field:

$$\Box_g\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0 \tag{27}$$

where $F_{\phi} = \mathrm{d}F(\phi)/\mathrm{d}\phi$, $V_{\phi} = \mathrm{d}V(\phi)/\mathrm{d}\phi$. This last equation is equivalent to the Bianchi contracted identity [103]. Standard fluid matter can be treated as above.

3.1 Conformal transformations

Let us now introduce conformal transformations to show that any higher-order or scalar-tensor theory, in absence of ordinary matter, e.g.. a perfect fluid, is conformally equivalent to an Einstein theory plus minimally coupled scalar fields. If standard matter is present, conformal transformations allow to transfer non-minimal coupling to the matter component [74]. The conformal transformation on the metric $g_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{2\omega}g_{\mu\nu} \tag{28}$$

in which $\mathrm{e}^{2\omega}$ is the conformal factor. Under this transformation, the Lagrangian in (24) becomes

$$\begin{aligned}&\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right)\\ &\quad = \sqrt{-\tilde{g}}\mathrm{e}^{-2\omega}\left(F\tilde{R} - 6F\Box_{\tilde{g}}\omega - 6F\omega_{;\alpha}\omega^{\alpha}_{;} + \frac{1}{2}\tilde{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \mathrm{e}^{-2\omega}V\right)\end{aligned} \tag{29}$$

in which $\tilde{R}$ and $\Box_{\tilde{g}}$ are the Ricci scalar and the d'Alembert operator relative to the metric $\tilde{g}$. Requiring the theory in the metric $\tilde{g}_{\mu\nu}$ to appear as a standard Einstein theory [77], the conformal factor has to be related to F, that is

$$\mathrm{e}^{2\omega} = -2F. \tag{30}$$

where F must be negative in order to restore physical coupling. Using this relation and introducing a new scalar field $\tilde{\phi}$ and a new potential $\tilde{V}$, defined respectively by

$$\tilde{\phi}_{;\alpha} = \sqrt{\frac{3F_{\phi}{}^{2} - F}{2F^{2}}}\,\phi_{;\alpha}, \quad \tilde{V}(\tilde{\phi}(\phi)) = \frac{V(\phi)}{4F^{2}(\phi)}, \tag{31}$$

we see that the Lagrangian (29) becomes

$$\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right) = \sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{\phi}_{;\alpha}\tilde{\phi}_{;}^{\alpha} - \tilde{V}\right) \tag{32}$$

which is the usual Hilbert–Einstein Lagrangian plus the standard Lagrangian relative to the scalar field $\tilde{\phi}$. Therefore, every non-minimally coupled scalar-tensor theory, in absence of ordinary matter, e.g., perfect fluid, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential suitably defined by (30) and (31). The converse is also true: for a given $F(\phi)$, such that $3F_{\phi}{}^{2} - F > 0$, we can transform a standard Einstein theory into a non-minimally coupled scalar-tensor theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the scalar-tensor theories, assigned by the coupling $F(\phi)$, via the conformal transformation (30) with the constraints given by (31). Following the standard terminology, the “Einstein frame” is the framework of the Einstein theory with the minimal coupling and the “Jordan frame” is the framework of the non-minimally coupled theory [74].

In the context of alternative theories of gravity, as previously discussed, the gravitational contribution to the stress-energy tensor of the theory can be reinterpreted by means of a conformal transformation as the stress-energy tensor of a suitable scalar field and then as “matter” like terms. Performing the conformal transformation (28) in the field equations (19), we get:

$$\begin{aligned}\tilde{G}_{\alpha\beta} = & \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} \\ & + 2\left(\omega_{;\alpha;\beta} + g_{\alpha\beta}\Box\omega - \omega_{;\alpha}\omega_{;\beta} + \frac{1}{2}g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}\right).\end{aligned} \tag{33}$$

We can then choose the conformal factor to be

$$\omega = \frac{1}{2}\ln|f'(R)|, \tag{34}$$

which has now to be substituted into (20). Rescaling ω in such a way that

$$k\phi = \omega, \tag{35}$$

and $k = \sqrt{1/6}$, we obtain the Lagrangian equivalence

$$\sqrt{-g}f(R) = \sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{\phi}_{;\alpha}\tilde{\phi}_{;}^{\alpha} - \tilde{V}\right) \tag{36}$$

and the Einstein equations in standard form

$$\tilde{G}_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma} + \tilde{g}_{\alpha\beta}V(\phi), \tag{37}$$

with the potential

$$V(\phi) = \frac{\mathrm{e}^{-4k\phi}}{2}\left[\mathscr{P}(\phi) - \mathscr{N}\left(\mathrm{e}^{2k\phi}\right)\mathrm{e}^{2k\phi}\right] = \frac{1}{2}\frac{f(R) - Rf'(R)}{f'(R)^2}. \tag{38}$$

Here $\mathscr{N}$ is the inverse function of $\mathscr{P}'(\phi)$ and $\mathscr{P}(\phi) = \int \exp(2k\phi)\mathrm{d}\mathscr{N}$. However, the problem is completely solved if $\mathscr{P}'(\phi)$ can be analytically inverted. In summary, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field (see also [11; 72]).

This procedure can be extended to more general theories. If the theory is assumed to be higher than fourth order, we may have Lagrangian densities of the form [20; 80],

$$\mathscr{L} = \mathscr{L}(R, \Box R, \ldots, \Box^k R). \tag{39}$$

Every $\Box$ operator introduces two further terms of derivation into the field equations. For example a theory like

$$\mathscr{L} = R\Box R, \tag{40}$$

is a sixth-order theory and the above approach can be pursued by considering a conformal factor of the form

$$\omega = \frac{1}{2}\ln\left|\frac{\partial\mathscr{L}}{\partial R} + \Box\frac{\partial\mathscr{L}}{\partial\Box R}\right|. \tag{41}$$

In general, increasing two orders of derivation in the field equations (i.e., for every term $\Box R$), corresponds to adding a scalar field in the conformally transformed frame [20]. A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a $2n$-order theory can be, in principle, reduced to an Einstein theory plus $(n-1)$-scalar fields. On the other hand, these considerations can be directly generalized to higher-order-scalar-tensor theories in any number of dimensions as shown in [17].

As concluding remarks, we can say that conformal transformations work at three levels: (i) on the Lagrangian of the given theory; (ii) on the field equations; (iii) on the solutions. The table below summarizes the situation for fourth-order gravity (FOG), non-minimally coupled scalar-tensor theories (NMC) and standard Hilbert–Einstein (HE) theory. Clearly, direct and inverse transformations correlate all the steps of the table but no absolute criterion, at this point of the discussion, is able to select which is the "physical" framework since, at least from a mathematical point of view, all the frames are equivalent [74]. This point is up to now unsolved even if wide discussions are present in literature [78].

$\mathscr{L}_{\text{FOG}}$	$\longleftrightarrow$	$\mathscr{L}_{\text{NMC}}$	$\longleftrightarrow$	$\mathscr{L}_{\text{HE}}$
$\updownarrow$		$\updownarrow$		$\updownarrow$
FOG Eqs.	$\longleftrightarrow$	NMC Eqs.	$\longleftrightarrow$	Einstein Eqs.
$\updownarrow$		$\updownarrow$		$\updownarrow$
FOG Solutions	$\longleftrightarrow$	NMC Solutions	$\longleftrightarrow$	Einstein Solutions

4 The Palatini approach and the intrinsic conformal structure

As we said, the Palatini approach, considering g and Γ as independent fields, is "intrinsically" bi-metric and capable of disentangling the geodesic structure from the chronological structure of a given manifold. Starting from these considerations, conformal transformations assume a fundamental role in defining the affine connection which is merely "Levi-Civita" only for the Hilbert–Einstein theory.

In this section, we work out examples showing how conformal transformations assume a fundamental physical role in relation to the Palatini approach in ETGs [86].

Let us start from the case of fourth-order gravity where Palatini variational principle is straightforward in showing the differences with Hilbert–Einstein variational principle, involving only metric. Besides, cosmological applications of $f(R)$ gravity have shown the relevance of Palatini formalism, giving physically interesting results with singularity—free solutions [83]. This last nice feature is not present in the standard metric approach.

An important remark is in order at this point. The Ricci scalar entering in $f(R)$ is $R \equiv R(g,\Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$ that is a *generalized Ricci scalar* and $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsion-less connection Γ, which, a priori, has no relations with the metric g of spacetime. The gravitational part of the Lagrangian is controlled by a given real analytical function of one real variable $f(R)$, while $\sqrt{-g}$ denotes a related scalar density of weight 1. Field equations, deriving from the Palatini variational principle are:

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = T^m_{\mu\nu} \tag{42}$$

$$\nabla^\Gamma_\alpha(\sqrt{-g}f'(R)g^{\mu\nu}) = 0 \tag{43}$$

where ∇^Γ is the covariant derivative with respect to Γ. As above, we assume $8\pi G = 1$. We shall use the standard notation denoting by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e., $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$.

In order to get (43), one has to additionally assume that $\mathscr{L}_m$ is functionally independent of Γ; however it may contain metric covariant derivatives $\overset{g}{\nabla}$ of fields. This means that the matter stress-energy tensor $T^m_{\mu\nu} = T^m_{\mu\nu}(g,\Psi)$ depends on the metric g and some matter fields denoted here by Ψ, together with their derivatives (covariant derivatives with respect to the Levi-Civita connection of g). From (43) one sees that $\sqrt{-g}f'(R)g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1. As previously discussed in [82; 86], this naturally leads to define a new metric $h_{\mu\nu}$, such that the following relation holds:

$$\sqrt{-g}f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \tag{44}$$

This *ansatz* is suitably made in order to impose Γ to be the Levi-Civita connection of h and the only restriction is that $\sqrt{-g}f'(R)g^{\mu\nu}$ should be non-degenerate. In the case of Hilbert–Einstein Lagrangian, it is $f'(R) = 1$ and the statement is trivial.

The above Eq. 44 imposes that the two metrics h and g are conformally equivalent. The corresponding conformal factor can be easily found to be $f'(R)$ (in dim $\mathscr{M} = 4$) and the conformal transformation results to be ruled by:

$$h_{\mu\nu} = f'(R)g_{\mu\nu} \tag{45}$$

Therefore, as it is well known, Eq. 43 implies that $\Gamma = \Gamma_{\mathrm{LC}}(h)$ and $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. Field equations can be supplemented by the scalar-valued equation obtained by taking the trace of (42), (we define $\tau = \mathrm{tr}\hat{T}$)

$$f'(R)R - 2f(R) = g^{\alpha\beta}T^m_{\alpha\beta} \equiv \tau^m \tag{46}$$

which controls solutions of (43). We shall refer to this scalar-valued equation as the *structural equation* of the spacetime. In the vacuum case (and spacetimes filled with radiation, such that $\tau^m = 0$) this scalar-valued equation admits constant solutions, which are different from zero only if one add a cosmological constant. This means that the universality of Einstein field equations holds [82], corresponding to a theory with cosmological constant [104].

In the case of interaction with matter fields, the structural equation (45), if explicitly solvable, provides an expression of $R = F(\tau)$, where F is a generic function, and consequently both $f(R)$ and $f'(R)$ can be expressed in terms of τ. The matter content of spacetime thus rules the bi-metric structure of spacetime and, consequently, both the geodesic and metric structures which are intrinsically different. This behavior generalizes the vacuum case and corresponds to the case of a time-varying cosmological constant. In other words, due to these features, conformal transformations, which allow to pass from a metric structure to another one, acquire an intrinsic physical meaning since "select" metric and geodesic structures which, for a given ETG, in principle, *do not* coincide.

Let us now try to extend the above formalism to the case of non-minimally coupled scalar-tensor theories. The effort is to understand if and how the bi-metric structure of spacetime behaves in this cases and which could be its geometric and physical interpretation.

We start by considering scalar-tensor theories in the Palatini formalism, calling A_1 the action functional. After, we take into account the case of decoupled non-minimal interaction between a scalar-tensor theory and a $f(R)$ theory, calling A_2 this action functional. We finally consider the case of non-minimal-coupled interaction between the scalar field ϕ and the gravitational fields (g,Γ), calling A_3 the corresponding action functional. Particularly significant is, in this case, the limit of low curvature R. This resembles the physical relevant case of present values of curvatures of the Universe and it is important for cosmological applications.

The action (24) for scalar-tensor gravity can be generalized, in order to better develop the Palatini approach, as:

$$A_1 = \int \sqrt{-g}\ \left[F(\phi)R + \frac{\varepsilon}{2} \overset{g}{\nabla}_\mu \phi \overset{g}{\nabla}{}^\mu \phi - V(\phi) + \mathscr{L}_m(\Psi, \overset{g}{\nabla} \Psi)\right] \mathrm{d}^4x. \tag{47}$$

As above, the values of $\varepsilon = \pm 1$ selects between standard scalar field theories and quintessence (phantom) field theories. The relative "signature" can be selected by conformal transformations. Field equations for the gravitational part of the action are, respectively for the metric g and the connection Γ:

$$\begin{cases} F(\phi)[R_{(\mu\nu)} - \frac{1}{2}Rg_{\mu\nu}] = T^{\phi}_{\mu\nu} + T^{m}_{\mu\nu} \\ \nabla^{\Gamma}_{\alpha}(\sqrt{-g}F(\phi)g^{\mu\nu}) = 0 \end{cases} \tag{48}$$

$R_{(\mu\nu)}$ is the same defined in (42). For matter fields we have the following field equations:

$$\begin{cases} \varepsilon\Box\phi = -V_\phi(\phi) + F_\phi(\phi)R \\ \frac{\delta \mathscr{L}_m}{\delta \Psi} = 0 \end{cases} . \tag{49}$$

In this case, the structural equation of spacetime implies that:

$$R = -\frac{\tau^{\phi} + \tau^{\mathrm{m}}}{F(\phi)} \tag{50}$$

which expresses the value of the Ricci scalar curvature in terms of the traces of the stress-energy tensors of standard matter and scalar field (we have to require $F(\phi) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$\sqrt{-g}F(\phi)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \tag{51}$$

such that g and h result to be conformally related

$$h_{\mu\nu} = F(\phi)g_{\mu\nu}. \tag{52}$$

The conformal factor is exactly the interaction factor. From (50), it follows that in the vacuum case $\tau^\phi = 0$ and $\tau^m = 0$: this theory is equivalent to the standard Einstein one without matter. On the other hand, for $F(\phi) = F_0$ we recover the Einstein theory plus a minimally coupled scalar field: this means that the Palatini approach intrinsically gives rise to the conformal structure (52) of the theory which is trivial in the Einstein, minimally coupled case.

As a further step, let us generalize the previous results considering the case of a non-minimal coupling in the framework of $f(R)$ theories. The action functional can be written as:

$$A_2 = \int \sqrt{-g}\ \left[F(\phi)f(R) + \frac{\varepsilon}{2} \overset{g}{\nabla}_\mu \phi \overset{g}{\nabla}{}^\mu \phi - V(\phi) + \mathscr{L}_m(\Psi, \overset{g}{\nabla} \Psi)\right] \mathrm{d}^4 x \qquad (53)$$

where $f(R)$ is, as usual, any analytical function of the Ricci scalar R. Field equations (in the Palatini formalism) for the gravitational part of the action are:

$$\begin{cases} F(\phi)[f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu}] = T^\phi_{\mu\nu} + T^m_{\mu\nu} \\ \nabla^\Gamma_\alpha(\sqrt{-g}F(\phi)f'(R)g^{\mu\nu}) = 0. \end{cases} \qquad (54)$$

For scalar and matter fields we have, otherwise, the following field equations:

$$\begin{cases} \varepsilon \Box \phi = -V_\phi(\phi) + \sqrt{-g}F_\phi(\phi)f(R) \\ \frac{\delta \mathscr{L}_m}{\delta \Psi} = 0 \end{cases} \qquad (55)$$

where the non-minimal interaction term enters into the modified Klein–Gordon equations. In this case the structural equation of spacetime implies that:

$$f'(R)R - 2f(R) = \frac{\tau^\phi + \tau^m}{F(\phi)}. \qquad (56)$$

We remark again that this equation, if solved, expresses the value of the Ricci scalar curvature in terms of traces of the stress-energy tensors of standard matter and scalar field (we have to require again that $F(\phi) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$\sqrt{-g}F(\phi)f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \qquad (57)$$

such that g and h result to be conformally related by:

$$h_{\mu\nu} = F(\phi)f'(R)g_{\mu\nu}. \qquad (58)$$

Once the structural equation is solved, the conformal factor depends on the values of the matter fields (ϕ, Ψ) or, more precisely, on the traces of the stress-energy tensors and the value of ϕ. From equation (56), it follows that in the vacuum case, i.e., both $\tau^\phi = 0$ and $\tau^m = 0$, the universality of Einstein field equations still holds as in the case of minimally interacting $f(R)$ theories [82]. The validity of this property is related to the decoupling of the scalar field and the gravitational field.

Let us finally consider the case where the gravitational Lagrangian is a general function of ϕ and R. The action functional can thus be written as:

$$A_3 = \int \sqrt{-g}\; [K(\phi,R) + \frac{\varepsilon}{2} \overset{g}{\nabla}_\mu \phi \overset{g}{\nabla}^\mu \phi - V(\phi) + \mathscr{L}_m(\Psi, \overset{g}{\nabla} \Psi)] \mathrm{d}^4x \tag{59}$$

Field equations for the gravitational part of the action are:

$$\begin{cases} \left[\frac{\partial K(\phi,R)}{\partial R}\right] R_{(\mu\nu)} - \frac{1}{2} K(\phi,R) g_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{m}_{\mu\nu} \\ \nabla^{\Gamma}_{\alpha} \left(\sqrt{-g} \left[\frac{\partial K(\phi,R)}{\partial R}\right] g^{\mu\nu}\right) = 0. \end{cases} \tag{60}$$

For matter fields, we have:

$$\begin{cases} \varepsilon \Box \phi = -V_\phi(\phi) + \left[\frac{\partial K(\phi,R)}{\partial \phi}\right] \\ \frac{\delta L_{\mathrm{mat}}}{\delta \Psi} = 0. \end{cases} \tag{61}$$

The structural equation of spacetime can be expressed as:

$$\frac{\partial K(\phi,R)}{\partial R} R - 2K(\phi,R) = \tau^{\phi} + \tau^{m} \tag{62}$$

This equation, if solved, expresses again the form of the Ricci scalar curvature in terms of traces of the stress-energy tensors of matter and scalar field (we have to impose regularity conditions and, for example, $K(\phi,R) \neq 0$). The bi-metric structure of spacetime is thus defined by the ansatz:

$$\sqrt{-g} \frac{\partial K(\phi,R)}{\partial R} g^{\mu\nu} = \sqrt{-h} h^{\mu\nu} \tag{63}$$

such that g and h result to be conformally related by

$$h_{\mu\nu} = \frac{\partial K(\phi,R)}{\partial R} g_{\mu\nu} \tag{64}$$

Again, once the structural equation is solved, the conformal factor depends just on the values of the matter fields and (the trace of) their stress energy tensors. In other words, the evolution, the definition of the conformal factor and the bi-metric structure is ruled by the values of traces of the stress-energy tensors and by the value of the scalar field ϕ. In this case, the universality of Einstein field equations does not hold anymore in general. This is evident from (62) where the strong coupling between R and ϕ avoids the possibility, also in the vacuum case, to achieve simple constant solutions.

We consider, furthermore, the case of small values of R, corresponding to small curvature spacetimes. This limit represents, as a good approximation, the present epoch of the observed Universe under suitably regularity conditions. A Taylor expansion of the analytical function $K(\phi,R)$ can be performed:

$$K(\phi,R) = K_0(\phi) + K_1(\phi)R + o(R^2) \tag{65}$$

where only the first leading term in R is considered and we have defined:

$$\begin{cases} K_0(\phi) = K(\phi,R)_{R=0} \\ K_1(\phi) = \left(\frac{\partial K(\phi,R)}{\partial R}\right)_{R=0} \end{cases} . \tag{66}$$

Substituting this expression in (62) and (64) we get (neglecting higher order approximations in R) the structural equation and the bi-metric structure in this particular case. From the structural equation, we get:

$$R = \frac{1}{K_1(\phi)}[-(\tau^\phi + \tau^m) - 2K_0(\phi)] \tag{67}$$

such that the value of the Ricci scalar is always determined, in this first order approximation, in terms of $\tau^\phi, \tau^{\mathrm{m}}, \phi$. The bi-metric structure is, otherwise, simply defined by means of the first term of the Taylor expansion, which is

$$h_{\mu\nu} = K_1(\phi)g_{\mu\nu}. \tag{68}$$

It reproduces, as expected, the scalar-tensor case (52). In other words, scalar-tensor theories can be recovered in a first order approximation of a general theory where gravity and non-minimal couplings are any (compare (67) with (56)). This fact agrees with the above considerations where Lagrangians of physical interactions can be considered as stochastic functions with local gauge invariance properties [12].

Finally we have to say that there are also bi-metric theories which cannot be conformally related (see for example the summary of alternative theories given in [61]) and torsion field should be taken into account, if one wants to consider the most general viewpoint [62; 63]. We will not take into account these general theories in this review.

After this short review of ETGs in metric and Palatini approach, we are going to face some remarkable applications to cosmology and astrophysics. In particular, we deal with the straightforward generalization of GR, the $f(R)$ gravity, showing that, in principle, no further ingredient, a part a generalized gravity, could be necessary to address issues as missing matter (dark matter) and cosmic acceleration (dark energy). However what we are going to consider here are nothing else but toy models which are not able to fit the whole expansion history, the structure growth law and the CMB anisotropy and polarization. These issues require more detailed theories which, up to now, are not available but what we are discussing could be a useful working paradigm as soon as refined experimental tests to probe such theories will be proposed and pursued. In particular, we will outline an independent test, based on the stochastic background of gravitational waves, which could be extremely useful to discriminate between ETGs and GR or among the ETGs themselves. In this latter case, the data delivered from ground-based interferometers, like VIRGO and LIGO, or the forthcoming space interferometer LISA, could be of extreme relevance in such a discrimination.

Finally, we do not take into account the well known inflationary models based on ETGs (e.g., [13; 15]) since we want to show that also the last cosmological epochs, directly related to the so called *Precision Cosmology*, can be framed in such a new "economic" scheme.

5 Applications to cosmology

As discussed in the Introduction, many rival theories have been advocated to fit the observed accelerated expansion and to solve the puzzle of dark energy.

As a simple classification scheme, we may divide the different cosmological models in three wide classes. According to the models of the first class, the dark energy is a new ingredient of the cosmic Hubble flow, the simplest case being the ΛCDM scenario and its quintessential generalization (the QCDM models).

This is in sharp contrast with the assumption of UDE models (the second class) where there is a single fluid described by an equation of state comprehensive of all regimes of cosmic evolution [43; 44] (the *parametric density models* or generalized *EoS*[3] models).

Finally, according to the third class of models, accelerated expansion is the first evidence of a breakdown of the Einstein GR (and thus of the Friedmann equations) which has to be considered as a particular case of a more general theory of gravity. As an example of this kind of models, we will consider the $f(R)$-gravity [48; 49; 50; 51; 52; 54; 56].

Far from being exhaustive, considering these three classes of models allow to explore very different scenarios proposed to explain the observed cosmic acceleration [105; 106; 118]. However, from the above considerations, it is possible to show that one of the simplest extension of GR, the $f(R)$ gravity can, in principle, address the dark energy issues both in metric and Palatini approach. In this section, without claiming for completeness, we sketch some $f(R)$ models matching solutions against some sets of data. The goal is to show that the dark energy issue could be addressed as a curvature effect in ETGs.

[3] EoS for Equation of State.

5.1 The ΛCDM model: the paradigm

Cosmological constant Λ has become a textbook candidate to drive the accelerated expansion of the spatially flat Universe. Despite its *conceptual* problems, the ΛCDM model turns out to be the best fit to a combined analysis of completely different astrophysical data ranging from SNeIa to CMBR anisotropy spectrum and galaxy clustering [33; 119]. As a simple generalization, one may consider the QCDM scenario in which the barotropic factor $w \equiv p/\rho$ takes at a certain epoch a negative value with $w = -1$ corresponding to the standard cosmological constant. Testing whether such a barotropic factor deviate or not from -1 is one of the main issue of modern observational cosmology. How such a negative pressure fluid drives the cosmic acceleration may be easily understood by looking at the Friedmann equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}(\rho_m + \rho_\Lambda), \tag{69}$$

$$2\frac{\ddot{a}}{a} + H^2 = -p_\Lambda = -w\rho_\Lambda, \tag{70}$$

where $a(t)$ is the scale factor of the Universe, the dot denotes differentiation with respect to cosmic time t, H is the Hubble parameter and the Universe is assumed spatially flat as suggested by the position of the first peak in the CMBR anisotropy spectrum (see also Fig. 1) [31; 32; 33].

From the continuity equation, $\dot{\rho} + 3H(\rho + p) = 0$, we get for the ith fluid with $p_i = w_i\rho_i$:

$$\Omega_i = \Omega_{i,0}a^{-3(1+w_i)} = \Omega_{i,0}(1+z)^{3(1+w_i)}, \tag{71}$$

where $z \equiv 1/a - 1$ is the redshift, $\Omega_i = \rho_i/\rho_{\text{crit}}$ is the density parameter for the ith fluid in terms of the critical density which, defined in standard units, is $\rho_{\text{crit}} = 3H_0^2/8\pi G$ and, hereafter, we label all the quantities evaluated today with a subscript 0. It is important to stress that Eq. 71 works only for $w_i =$ constant. Inserting this result into Eq. 69, one gets:

$$H(z) = H_0\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}(1+z)^{3(1+w)}}. \tag{72}$$

The subscript M means all the matter content, inclusive of dark and baryonic components. Using Eqs. 69, 70 and the definition of the deceleration parameter $q \equiv -a\ddot{a}/\dot{a}^2$, one finds:

$$q_0 = \frac{1}{2} + \frac{3}{2}w(1 - \Omega_{M,0}). \tag{73}$$

The SNeIa Hubble diagram, the large scale galaxy clustering and the CMBR anisotropy spectrum can all be fitted by the ΛCDM model with $(\Omega_{M,0}, \Omega_\Lambda) \simeq$

Fig. 1 The CMBR anisotropy spectrum for different values of w. Data points are the WMAP measurements and the best fit is obtained for $w \simeq -1$. If $w \neq -1$ the clustering of dark energy has been considered in this plot

$(0.3, 0.7)$ thus giving $q_0 \simeq -0.55$, i.e., the Universe turns out to be in an accelerated expansion phase. The simplicity of the model and its capability of fitting the most of the data are the reasons why the ΛCDM scenario is the leading candidate to explain the dark energy cosmology. Nonetheless, its generalization, QCDM models, i.e., mechanisms allowing the evolution of Λ from the past are invoked to remove the Λ-problem and the *coincidence* problem.

Here, we want to show that assuming $f(R)$ gravity, not strictly linear in R as GR, it is possible to give rise to the evolution of the barotropic factor $w = p/\rho$, today leading to the value $w = -1$, and to obtain models capable of matching with the observations. However, also if the paradigm could result valid, it is very difficult to address, in the same comprehensive $f(R)$ model, different issues as structure formation, nucleosynthesis, Hubble diagram, radiation and matter dominated behaviors as we shall discuss below.

Before considering specific $f(R)$ theories, let us discuss methods to constrain models by samples of data.

5.2 Methods to constrain models by distance and time indicators

In principle, cosmological models can be constrained using suitable distance and/or time indicators. As a general remark, solutions coming from cosmological models have to be matched with observations by using the redshift z as the natural time variable for the Hubble parameter, i.e.,

$$H(z) = -\frac{\dot{z}}{z+1}. \tag{74}$$

Data can be obtained for various values of redshift z: for example, CMB probes recombination at $z \simeq 1,100$ and $z \simeq 1$ via the late integrated Sachs-Wolfe effect; for $10 < z < 100$ the planned 21 cm observations could give detailed information [150; 151; 152]; futuristic LSS surveys and SNe could probe the Universe up to $z \simeq 4$. The method consists in building up a reasonable patchwork of data coming from different epochs and then matching them with the same cosmological solution ranging, in principle, from inflation to present accelerated era.

In order to constrain the parameters characterizing the cosmological solution, a reasonable approach is to maximize the following likelihood function:

$$\mathscr{L} \propto \exp\left[-\frac{\chi^2(\mathbf{p})}{2}\right] \tag{75}$$

where $\mathbf{p}$ are the parameters characterizing the cosmological solution. The χ^2 merit function can be defined as:

$$\chi^2(\mathbf{p}) = \sum_{i=1}^{N}\left[\frac{y^{\mathrm{th}}(z_i,\mathbf{p}) - y_i^{\mathrm{obs}}}{\sigma_i}\right]^2 + \left[\frac{\mathscr{R}(\mathbf{p}) - 1.716}{0.062}\right]^2 + \left[\frac{\mathscr{A}(\mathbf{p}) - 0.469}{0.017}\right]^2. \tag{76}$$

Terms entering Eq. 76 can be characterized as follows. For example, the dimensionless coordinate distances y to objects at redshifts z are considered in the first term. They are defined as:

$$y(z) = \int_0^z \frac{\mathrm{d}z'}{E(z')} \tag{77}$$

where $E(z) = H(z)/H_0$ is the normalized Hubble parameter. This is the main quantity which allows to compare the theoretical results with data. The function y is related to the luminosity distance $D_L = (1+z)y(z)$.

A sample of data at $y(z)$ for the 157 SNeIa is discussed in the Riess et al. [36] Gold dataset and 20 radio-galaxies are in [121]. These authors fit with good accuracy the linear Hubble law at low redshift ($z < 0.1$) obtaining the Hubble dimensionless parameter $h = 0.664 \pm 0.008$, with h the Hubble constant in units of 100 km s^{-1} Mpc^{-1}. Such a number can be consistently taken into account at low redshift. This value is in agreement with $H_0 = 72 \pm 8$ km s^{-1} Mpc^{-1} given by the HST Key project [122] based on the local distance ladder and estimates coming from time delays in multiply imaged quasars [123] and Sunyaev-Zel'dovich effect in X-ray emitting clusters [124; 125]. The second term in Eq. 76 allows to extend the z-range to probe $y(z)$ up to the last scattering surface ($z \geq 1,000$). The *shift parameter* [126; 127] $\mathscr{R} \equiv \sqrt{\Omega_M}y(z_{ls})$ can be determined from the CMBR anisotropy spectrum, where z_{ls} is the redshift of the last scattering surface which can be approximated as $z_{ls} = 1048\left(1 + 0.00124\omega_b^{-0.738}\right)\left(1 + g_1\omega_M^{g_2}\right)$ with $\omega_i = \Omega_i h^2$ (with $i = b, M$ for baryons and total matter respectively) and (g_1, g_2) given in [128]. The parameter ω_b is constrained by the baryogenesis calculations contrasted to the observed abundances of primordial elements. Using this method, the value $\omega_b = 0.0214 \pm 0.0020$ is found [129].

In any case, it is worth noticing that the exact value of z_{ls} has a negligible impact on the results and setting $z_{ls} = 1,100$ does not change constraints and priors on the other parameters of the given model. The third term in the function χ^2 takes into account the *acoustic peak* of the large scale correlation function at 100 h^{-1} Mpc separation, detected by using 46,748 luminous red galaxies (LRG) selected from the SDSS Main Sample [130; 131]. The quantity

$$\mathscr{A} = \frac{\sqrt{\Omega_M}}{z_{\rm LRG}} \left[\frac{z_{\rm LRG}}{E(z_{\rm LRG})} y^2(z_{\rm LRG}) \right]^{1/3} \tag{78}$$

is related to the position of acoustic peak where $z_{\rm LRG} = 0.35$ is the effective redshift of the above sample. The parameter $\mathscr{A}$ depends on the dimensionless coordinate distance (and thus on the integrated expansion rate), on Ω_M and $E(z)$. This dependence removes some of the degeneracies intrinsic in distance fitting methods.

Due to this reason, it is particularly interesting to include $\mathscr{A}$ as a further constraint on the model parameters using its measured value $\mathscr{A} = 0.469 \pm 0.017$ [130]. Note that, although similar to the usual χ^2 introduced in statistics, the reduced χ^2 (i.e., the ratio between the χ^2 and the number of degrees of freedom) is not forced to be 1 for the best fit model because of the presence of the priors on $\mathscr{R}$ and $\mathscr{A}$ and since the uncertainties σ_i are not Gaussian distributed, but take care of both statistical errors and systematic uncertainties. With the definition (75) of the likelihood function, the best fit model parameters are those that maximize $\mathscr{L}(\mathbf{p})$.

In order to implement the above sketched method, much attention, has been devoted to standard candles, i.e., astrophysical objects whose absolute magnitude M is known (or may be exactly predicted) a priori so that a measurement of the apparent magnitude m immediately gives the distance modulus $\mu = m - M$. Specifically, the distance to the object, estimated in Mpc, is:

$$\mu(z) = 5 \log D_L(z)/Mpc + 25 \tag{79}$$

with $D_L(z)$ the luminosity distance and z the redshift of the object. The number 25 depends on the distance modulus calculated in Mpc. The relation between μ and z is what is referred to as Hubble diagram and it is an open window on the cosmography of the Universe. Furthermore, the Hubble diagram is a powerful cosmological test since the luminosity distance is determined by the expansion rate as:

$$D_L(z) = \frac{c}{H_0}(1+z) \int_0^z \frac{dz'}{E(z')} \tag{80}$$

with $E(z)$ defined above. Being the Hubble diagram related to the luminosity distance and being D_L determined by the expansion rate $H(z)$, it is clear why it may be used as an efficient tool to test cosmological models and constrain their parameters.

To this aim, however, it is mandatory that the relation $\mu = \mu(z)$ is measured up to high enough redshift since, for low z, D_L reduces to a linear function of the redshift (thus recovering the Hubble law) whatever the background cosmological model is. This necessity claims for standard candles that are bright enough to be visible at such high redshift that the Hubble diagram may discriminate among different rival theories. SNeIa are, up to now, the objects that best match these requirements.

It is thus not surprising that the first evidences of an accelerating Universe came from the SNeIa Hubble diagram and dedicated survey (like the SNAP satellite [132]) have been planned in order to increase the number of SNeIa observed and the redshift range probed.

A reliable compilation of SNeIa is the *Gold* dataset released by Riess et al. [36]. The authors have compiled a catalog containing 157 SNeIa with z in the range $(0.01, 1.70)$ and visual absorption $A_V < 0.5$. The distance modulus of each object has been evaluated by using a set of calibrated methods so that the sample is homogenous in the sense that all the SNeIa have been re-analyzed using the same technique in such a way that the resulting Hubble diagram is indeed reliable and accurate. Given a cosmological model assigned by a set of parameters $\mathbf{p} = (p_1, \ldots, p_n)$, the luminosity distance may be evaluated with Eq. 80 and the predicted Hubble diagram contrasted with the observed SNeIa one. Constraints on the model parameters may then be extracted by mean of a χ^2-based analysis defining, in this case, the above χ^2 as:

$$\chi^2_{\rm SNeIa} = \sum_{i=1}^{N_{\rm SNeIa}} \left[\frac{\mu(z_i, \mathbf{p}) - \mu_{\rm obs}(z_i)}{\sigma_i} \right]^2 \tag{81}$$

where σ_i is the error on the distance modulus at redshift z_i and the sum is over the $N_{\rm SNeIa}$ SNeIa observed. It is worth stressing that the uncertainty on measurements also takes into account errors on the redshifts and they are not Gaussian distributed.

As a consequence, the reduced χ^2 (i.e., $\chi^2_{\rm SNeIa}$ divided by the number of degrees of freedom) for the best fit model is not forced to be close to unity. Nonetheless, different models may still be compared on the basis of the χ^2 value: the lower is $\chi^2_{\rm SNeIa}$, the better the model fits the SNeIa Hubble diagram.

The method outlined is a simple and quite efficient way to test whether a given model is a viable candidate to describe the late time evolution of the Universe. Nonetheless, it is affected by some degeneracies that could be only partially broken by increasing the sample size and extending the probed redshift range. A straightforward example may help in elucidating this point. Let us consider the flat concordance cosmological model with matter and cosmological constant. It is:

$$E^2(z) = \Omega_M (1+z)^3 + (1 - \Omega_M)$$

so that $\chi^2_{\rm SNeIa}$ will only depend on the Hubble constant H_0 and the matter density parameter Ω_M. Actually, we could split the matter term in a baryonic and a non-baryonic part denoting with Ω_b the baryon density parameter. Since both baryons and non baryonic dark matter scales as $(1+z)^3$, $E(z)$ and thus the luminosity distance will depend only on the total matter density parameter and we could never constrain Ω_b by fitting the SNeIa Hubble diagram. Similar degeneracies could also happen with other cosmological models thus stressing the need for complementary probes to be combined with the SNeIa data. For a review, see the contribution by Bob Nichols in this volume.

To this aim, a recently proposed test, based on the gas mass fraction in galaxy clusters, can be considered. We briefly outline here the method referring the interested reader to the literature for further details [135; 136; 137; 138]. Both theoretical arguments and numerical simulations predict that the baryonic mass fraction in the largest relaxed galaxy clusters should be invariant with the redshift (see, e.g., Ref. [140]).

However, this will only appear to be the case when the reference cosmology in making the baryonic mass fraction measurements matches the true underlying cosmology. From the observational point of view, it is worth noticing that the baryonic content in galaxy clusters is dominated by the hot X-ray emitting intra-cluster gas so that what is actually measured is the gas mass fraction $f_{\rm gas}$ and it is this quantity that should be invariant with the redshift within the caveat quoted above. Moreover, it is expected that the baryonic fraction in clusters equals the universal ratio Ω_b/Ω_M so that $f_{\rm gas}$ should indeed be given by $b\times\Omega_b/\Omega_M$ where the multiplicative factor b is motivated by simulations that suggest that the gas fraction is slightly lower than the universal ratio because of processes that convert part of the gas into stars or eject it outside the cluster itself.

Following Ref. [139], we adopt the SCDM model (i.e., a flat Universe with $\Omega_M = 1$ and $h = 0.5$, being h the Hubble constant in units of 100 km s^{-1} Mpc^{-1}) as reference cosmology in making the measurements so that the theoretical expectation for the apparent variation of $f_{\rm gas}$ with the redshift is [139]:

$$f_{\rm gas}(z) = \frac{b\Omega_b}{(1+0.19\sqrt{h})\Omega_M}\left[\frac{D_A^{\rm SCDM}(z)}{D_A^{\rm mod}(z)}\right]^{1.5} \qquad (82)$$

where $D_A^{\rm SCDM}$ and $D_A^{\rm mod}$ is the angular diameter distance for the SCDM and the model to be tested respectively. $D_A(z)$ may be evaluated from the luminosity distance $D_L(z)$ as:

$$D_A(z) = (1+z)^{-2} D_L(z) \qquad (83)$$

with $D_L(z)$ given by Eq. 80 above.

In [139], it has been extensively analyzed the set of simulations in Ref. [140] to get $b = 0.824 \pm 0.089$. For values in the 1σ range quoted above, the main results are independent on b. It is worth noticing that, while the angular diameter distance depends on $E(z)$ and thus on h and Ω_M, the prefactor in Eq. 82 makes $f_{\rm gas}$ explicitly depending on Ω_b/Ω_M so that a direct estimate of Ω_b is (in principle) possible. Actually, for the models which we are going to consider, the quantity that is constrained by the data is the ratio Ω_b/Ω_M rather than Ω_b itself.

To simultaneously take into account both the fit to the SNeIa Hubble diagram and the test on the $f_{\rm gas}$ data, it is convenient to perform a likelihood analysis defining the following likelihood function:

$$\mathscr{L}(\mathbf{p}) \propto \exp\left[-\frac{\chi^2(\mathbf{p})}{2}\right] \tag{84}$$

with:

$$\chi^2 = \chi^2_{\rm SNeIa} + \chi^2_{\rm gas} + \left(\frac{h-0.72}{0.08}\right)^2 + \left(\frac{\Omega_b/\Omega_M - 0.16}{0.06}\right)^2 \tag{85}$$

where it is possible to define:

$$\chi^2_{\rm gas} = \sum_{i=1}^{N_{\rm gas}} \left[\frac{f_{\rm gas}(z_i, \mathbf{p}) - f_{\rm gas}^{\rm obs}(z_i)}{\sigma_{gi}}\right]^2 \tag{86}$$

being $f_{\rm gas}^{\rm obs}(z_i)$ the measured gas fraction in a galaxy clusters at redshift z_i with an error σ_{gi} and the sum is over the $N_{\rm gas}$ clusters considered. In order to avoid possible systematic errors in the $f_{\rm gas}$ measurement, it is desirable that the cluster is both highly luminous (so that the S/N ratio is high) and relaxed so that both merging processes and cooling flows are absent. A catalog of 26 large relaxed clusters, with a measurement of both the gas mass fraction $f_{\rm gas}$ and the redshift z is in [139]. These data can be used to perform a suitable likelihood analysis.

Note that, in Eq. 85, we have explicitly introduced two Gaussian priors to better constrain the model parameters. First, there is a prior on the Hubble constant h determined by the results of the HST Key project [141] from an accurate calibration of a set of different local distance estimators. Further, we impose a constraint on the ratio Ω_b/Ω_M by considering the estimates of $\Omega_b h^2$ and $\Omega_M h^2$ obtained by Tegmark et al. [142] from a combined fit to the SNeIa Hubble diagram, the CMBR anisotropy spectrum measured by WMAP and the matter power spectrum extracted from over 200,000 galaxies observed by the SDSS collaboration. It is worth noticing that, while our prior on h is the same as that used by many authors when applying the $f_{\rm gas}$ test [137; 138; 139], it is common to put a second prior on Ω_b rather than Ω_b/Ω_M. Actually, this choice can be motivated by the peculiar features of the models which one is going to consider.

With the definition (84) of the likelihood function, the best fit model parameters are those that maximize $\mathscr{L}(\mathbf{p})$. However, to constrain a given parameter p_i, one resorts to the marginalized likelihood function defined as:

$$\mathscr{L}_{p_i}(p_i) \propto \int \mathrm{d}p_1 \cdots \int \mathrm{d}p_{i-1} \int \mathrm{d}p_{i+1} \cdots \int \mathrm{d}p_n \mathscr{L}(\mathbf{p}) \tag{87}$$

that is normalized at unity at maximum. The 1σ confidence regions are determined by $\delta\chi^2 = \chi^2 - \chi^2_0 = 1$, while the condition $\delta\chi^2 = 4$ delimited the 2σ confidence regions. It is worth stressing that $\delta\chi^2 = 1$ for 1-dimensional likelihoods. Here, χ^2_0 is the value of the χ^2 for the best fit model. Projections of the likelihood function allow to show eventual correlations among the model parameters.

Using the method sketched above, the classes of models which we are going to study can be constrained and selected by observations. However, most of the tests recently used to constrain cosmological parameters (such as the SNeIa Hubble diagram and the angular size–redshift) are essentially distance-based methods. The proposal of Dalal et al. [143] to use the lookback time to high redshift objects is thus particularly interesting since it relies on a completely different observable.

The lookback time is defined as the difference between the present day age of the Universe and its age at redshift z and may be computed as:

$$t_L(z,\mathbf{p}) = t_H \int_0^z \frac{\mathrm{d}z'}{(1+z')E(z',\mathbf{p})} \tag{88}$$

where $t_H = 1/H_0 = 9.78\,\mathrm{h}^{-1}$ Gyr is the Hubble time, and, as above, $E(z,\mathbf{p})$ is the dimensionless Hubble parameter, where the set of parameters characterizing the cosmological model, $\{\mathbf{p}\}$, can be taken into account. It is worth noticing that, by definition, the lookback time is not sensible to the present day age of the Universe t_0 so that it could be possible that a model fits well the data on the lookback time, but nonetheless it predicts a wrong value for t_0. This latter parameter can be evaluated from Eq. 88 by changing the upper integration limit from z to infinity. This shows that it is a different quantity indeed since it depends on the full evolution of the Universe and not only on how the Universe evolves from the redshift z to now. That is why this quantity can be explicitly introduced as a further constraint. However, it is possible to show from the observations that $t_L(z)$ converges to t_0 already at low z and then the method can be considered reliable.

As an example, let us now sketch how to use the lookback time and the age of the Universe to test a given cosmological model. To this end, let us consider an object i at redshift z and denote by $t_i(z)$ its age defined as the difference between the age of the Universe when the object was born, i.e., at the formation redshift z_F, and the one at z. It is:

$$\begin{aligned} t_i(z) &= \int_z^\infty \frac{\mathrm{d}z'}{(1+z')E(z',\mathbf{p})} - \int_{z_F}^\infty \frac{\mathrm{d}z'}{(1+z')E(z',\mathbf{p})} \\ &= \int_z^{z_F} \frac{\mathrm{d}z'}{(1+z')E(z',\mathbf{p})} = t_L(z_F) - t_L(z). \end{aligned} \tag{89}$$

where, in the last row, we have used the definition (88) of the lookback time. Suppose now we have N objects and we have been able to estimate the age t_i of the object at redshift z_i for $i = 1,2,\ldots,N$. Using the previous relation, we can estimate the lookback time $t_L^{\mathrm{obs}}(z_i)$ as:

$$\begin{aligned} t_L^{\mathrm{obs}}(z_i) &= t_L(z_F) - t_i(z) \\ &= [t_0^{\mathrm{obs}} - t_i(z)] - [t_0^{\mathrm{obs}} - t_L(z_F)] \\ &= t_0^{\mathrm{obs}} - t_i(z) - \Delta f, \end{aligned} \tag{90}$$

where t_0^{obs} is the today estimated age of the Universe and a *delay factor* can be defined as:

$$\Delta f = t_0^{\mathrm{obs}} - t_L(z_F). \tag{91}$$

The delay factor is introduced to take into account our ignorance of the formation redshift z_F of the object. Actually, what can be measured is the age $t_i(z)$ of the object at redshift z. To estimate z_F, one should use Eq. 89 assuming a background cosmological model. Since our aim is to determine what is the background cosmological model, it is clear that we cannot infer z_F from the measured age so that this quantity is completely undetermined.

It is worth stressing that, in principle, Δf should be different for each object in the sample unless there is a theoretical reason to assume the same redshift at the formation of all the objects. If this is indeed the case (as we will assume later), then it is computationally convenient to consider Δf rather than z_F as the unknown parameter to be determined from the data. Again a likelihood function can be defined as:

$$\mathscr{L}_{lt}(\mathbf{p},\Delta f) \propto \exp\left[-\chi^2_{lt}(\mathbf{p},\Delta f)/2\right] \tag{92}$$

with:

$$\chi^2_{lt} = \frac{1}{N-N_p+1}\left\{\left[\frac{t_0^{\mathrm{theor}}(\mathbf{p})-t_0^{\mathrm{obs}}}{\sigma_{t_0^{\mathrm{obs}}}}\right]^2+\sum_{i=1}^{N}\left[\frac{t_L^{\mathrm{theor}}(z_i,\mathbf{p})-t_L^{\mathrm{obs}}(z_i)}{\sqrt{\sigma_i^2+\sigma_t^2}}\right]^2\right\} \tag{93}$$

where N_p is the number of parameters of the model, σ_t is the uncertainty on t_0^{obs}, σ_i the one on $t_L^{\mathrm{obs}}(z_i)$ and the superscript *theor* denotes the predicted values of a given quantity. Note that the delay factor enters the definition of χ^2_{lt} since it determines $t_L^{\mathrm{obs}}(z_i)$ from $t_i(z)$ in virtue of Eq. 90, but the theoretical lookback time does not depend on Δf.

In principle, such a method should work efficiently to discriminate among the various dark energy models. Actually, this is not exactly the case due to the paucity of the available data which leads to large uncertainties on the estimated parameters. In order to partially alleviate this problem, it is convenient to add further constraints on the models by using Gaussian priors[4] on the Hubble constant, i.e., redefining the likelihood function as:

$$\mathscr{L}(\mathbf{p}) \propto \mathscr{L}_{lt}(\mathbf{p})\exp\left[-\frac{1}{2}\left(\frac{h-h^{\mathrm{obs}}}{\sigma_h}\right)^2\right] \propto \exp\left[-\chi^2(\mathbf{p})/2\right] \tag{94}$$

where we have absorbed Δf in the set of parameters $\mathbf{p}$ and have defined:

$$\chi^2 = \chi^2_{lt} + \left(\frac{h-h^{\mathrm{obs}}}{\sigma_h}\right)^2 \tag{95}$$

with h^{obs} the estimated value of h and σ_h its uncertainty. The HST Key project results [122] can be used setting $(h,\sigma_h) = (0.72, 0.08)$. Note that this estimate is independent of the cosmological model since it has been obtained from local distance ladder methods. The best fit model parameters $\mathbf{p}$ may be obtained by maximizing $\mathscr{L}(\mathbf{p})$ which is equivalent to minimize the χ^2 defined in Eq. 95.

It is worth stressing again that such a function should not be considered as a *statistical* χ^2 in the sense that it is not forced to be of order 1 for the best fit model

[4] The need for priors to reduce the parameter uncertainties is often advocated for cosmological tests. For instance, in Ref. [144] a strong prior on Ω_M is introduced to constrain the dark energy equation of state. It is likely, that extending the dataset to higher redshifts and reducing the uncertainties on the age estimate will allow to avoid resorting to priors.

to consider a fit as successful. Actually, such an interpretation is not possible since the errors on the measured quantities (both t_i and t_0) are not Gaussian distributed and, moreover, there are uncontrolled systematic uncertainties that may also dominate the error budget.

Nonetheless, a qualitative comparison among different models may be obtained by comparing the values of this pseudo χ^2 even if this should not be considered as a definitive evidence against a given model. Having more than one parameter, one obtains the best fit value of each single parameter p_i as the value which maximizes the marginalized likelihood for that parameter defined in Eq. 87. After having normalized the marginalized likelihood to 1 at maximum, one computes the 1σ and 2σ confidence limits (CL) on that parameter by solving $\mathscr{L}_{p_i} = \exp(-0.5)$ and $\mathscr{L}_{p_i} = \exp(-2)$ respectively. In summary, taking into account the above procedures for distance and time measurements, one can reasonably constrain a given cosmological model. In any case, the main and obvious issue is to have at disposal sufficient and good quality data sets.

5.3 Samples of data to constrain models: the case of LSS for lookback time

In order to apply the method outlined above, we need a set of distant objects whose distances and ages can be somehow estimated. As an example for the lookback time method, let us consider the clusters of galaxies which seem to be ideal candidates since they can be detected up to high redshift and their redshift, at formation epoch[5] is almost the same for all the clusters. Furthermore, it is relatively easy to estimate their age from photometric data only. To this end, the color of their component galaxies, in particular the reddest ones, is needed.

Actually, the stellar populations of the reddest galaxies become redder and redder as they evolve. It is just a matter, then, to assume a stellar population synthesis model, and to look at how old the latest episode of star formation should be happened in the galaxy past to produce colors as red as the observed ones. This is what is referred to as *color age*. The main limitation of the method relies in the stellar population synthesis model, and on a few (unknown) ingredients (among which the metallicity and the star formation rate).

The choice of the evolutionary model is a key step in the estimate of the color age and the main source of uncertainty [145]. An alternative and more robust route to cluster age is to consider the color scatter (see [146] for an early application of this approach). The argument, qualitatively, goes in this way: if galaxies have an extreme similarity in their color and nothing is conspiring to make the color scatter surreptitiously small, then the latest episode of star formation should happen in the galaxy far past, otherwise the observed color scatter would be larger.

[5] It is worth stressing that, in literature, the cluster formation redshift is defined as the redshift at which the last episode of star formation happened. In this sense, we should modify our definition of Δf by adding a constant term which takes care of how long is the star formation process and what is the time elapsed from the beginning of the Universe to the birth of the first cluster of galaxies. For this reason, it is still possible to consider the delay factor to be the same for all clusters, but it is not possible to infer z_F from the fitted value of Δf because we do not know the detail of star formation history. This approach is particular useful since it allows to overcome the problem to consider lower limits of the Universe age at z rather than the actual values.

Table 1 Main properties of the cluster sample used for the analysis

Color age				Scatter age			
z	N	Age (Gyr)	References	z	N	Age (Gyr)	References
0.60	1	4.53	[147]	0.10	55	10.65	[149]
0.70	3	3.93	[147]	0.25	103	8.89	[149]
0.80	2	3.41	[147]	1.27	1	1.60	[153]

The data in the left part of the Table refers to clusters whose age has been estimated from the color of the reddest galaxies (color age), while that of clusters in the right part has been obtained by the color scatter (scatter age). For each data point, we give the redshift z, the number N of clusters used, the age estimate and the relevant reference

Quantitatively, the scatter in color should thus be equal to the derivative of color with time multiplied the scatter of star formation times. The first quantity may be predicted using population synthesis models and turns out to be almost the same for all the evolutionary models thus significantly reducing the systematic uncertainty. We will refer to the age estimated by this method as *scatter age*. The dataset we need to apply the method may be obtained using the following procedure. First, for a given redshift z_i, we collect the colors of the reddest galaxies in a cluster at that redshift and then use one of the two methods outlined above to determine the color or the scatter age of the cluster. If more than one cluster is available at that redshift, we average the results from different clusters in order to reduce systematic errors. Having thus obtained $t_i(z_i)$, we then use Eq. 90 to estimate the value of the lookback time at that redshift.

Actually, what we measure is $t_L^{\rm obs}(z_i) + d\Delta f$ that is the quantity that enters the definition (93) of χ^2_{lt} and then the likelihood function. To estimate the color age, following [147], it is possible to choose, among the various available stellar population synthesis models, the Kodama and Arimoto one [148], which, unlike other models, allows a chemical evolution neglected elsewhere. This gives us three points on the diagram z vs. $t_L^{\rm obs}$ obtained by applying the method to a set of six clusters at three different redshifts as detailed in Table 1.

Using a large sample of low redshift SDSS clusters, it is possible to evaluate the scatter age for clusters age at $z = 0.10$ and $z = 0.25$ [149]. Blakeslee et al. [153] applied the same method to a single, high redshift $(z = 1.27)$ cluster. Collecting the data using both the color age and the scatter age, we end up with a sample of $\sim$160 clusters at six redshifts (listed in Table 1) which probe the redshift range $(0.10, 1.27)$. This nicely overlaps the one probed by SNeIa Hubble diagram so that a comparison among our results and those from SNeIa is possible. We assume a $\sigma = 1$ Gyr as uncertainty on the cluster age, no matter what is the method used to get that estimate.

Note that this is a very conservative choice. Actually, if the error on the age were so large, the color-magnitude relation for reddest cluster galaxies should have a large scatter that is not observed. We have, however, chosen such a large error to take qualitatively into account the systematic uncertainties related to the choice of the evolutionary model.

Finally, we need an estimate of $t_0^{\rm obs}$ to apply the method. Following Rebolo et al. [120], one can choose $(t_0^{\rm obs}, \sigma_t) = (14.4, 1.4)$ Gyr as obtained by a combined analysis of the WMAP and VSA data on the CMBR anisotropy spectrum and

SDSS galaxy clustering. Actually, this estimate is model dependent since Rebolo et al. [120] implicitly assumes that the ΛCDM model is the correct one. However, this value is in perfect agreement with $t_0^{obs} = 12.6^{+3.4}_{-2.4}$ Gyr determined from globular clusters age [71] and $t_0^{obs} > 12.5 \pm 3.5$ Gyr from radioisotopes studies [154]. For this reason, one is confident that no systematic error is induced on the adopted method using the Rebolo et al. estimate for t_0^{obs} even when testing cosmological models other than the ΛCDM one.

5.4 Dark energy as a curvature effect

The methods outlined above allow to constrain dark energy models without considering the nature of dark constituents. In [106], it is shown that the most popular quintessence (dark energy) models can be reproduced, in principle, only considering "curvature effects" i.e., only generalizing the theory of gravity to some $f(R)$ which is not supposed to be simply linear in R. From our point of view, this approach seems "economic" and "conservative" and does not claim for unknown fundamental ingredients, up to now not detected, in the cosmic fluid.[6] As it is clear, from Eq. 20, the curvature stress-energy tensor formally plays the role of a further source term in the field equations so that its effect is the same as that of an effective fluid of purely geometric origin. Let us rewrite it here for convenience:

$$T^{curv}_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta} \left[f(R) - R f'(R) \right] + f'(R)^{;\mu\nu} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right\}. \quad (96)$$

Our aim is to show that such a quantity provides all the ingredients we need to tackle with the dark side of the Universe. In fact, depending on the scales, such a curvature fluid can play, in principle, the role of dark matter and dark energy. To be more precise, also the coupling $1/f'(R)$ in front of the matter stress energy tensor, see Eqs. 20, plays a fundamental role in the dynamics since it affects, in principle, all the physical processes (e.g., the nucleo-synthesis) and the observable (luminous, clustered, baryonic) quantities. This means that the whole problem of the dark side of the Universe could be addressed considering a comprehensive theory where the interplay between the geometry and the matter has to be reconsidered assuming non-linear contributions and non-minimal couplings in curvature invariants.

From the cosmological point of view, in the standard framework of a spatially flat homogenous and isotropic Universe, the cosmological dynamics is determined by its energy budget through the Friedmann equations. In particular, the cosmic acceleration is achieved when the r.h.s. of the acceleration equation remains positive. Specifically the Friedmann equation, in physical units, is

$$\frac{\ddot{a}}{a} = -\frac{1}{6} (\rho_{tot} + 3 p_{tot}) . \quad (97)$$

The subscript tot denotes the sum of the curvature fluid and the matter contribution to the energy density and pressure. From the above relation, the acceleration

[6] Following the Occam razor prescription: *"Entia non sunt multiplicanda praeter necessitatem."*

Fig. 2 Best fit curve to the SNeIa Hubble diagram for the power law Lagrangian model. Only data of "Gold" sample of SNeIa have been used

Fig. 3 The Hubble diagram of 20 radio galaxies together with the "Gold" sample of SNeIa, in term of the redshift as suggested in [155]. The best fit curve refers to the R^n-gravity model without dark matter (*left*), while in the *right* panel it is shown the difference between the luminosity distances calculated without dark matter and in presence of this component in term of redshift. It is evident that the two behaviors are quite indistinguishable

condition, for a dust dominated model, leads to:

$$\rho_{\rm curv} + \rho_m + 3p_{\rm curv} < 0 \rightarrow w_{\rm curv} < -\frac{\rho_{\rm tot}}{3\rho_{\rm curv}} \tag{98}$$

so that a key role is played by the effective quantities:

$$\rho_{\rm curv} = \frac{1}{f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3H\dot{R}f''(R)\right\}, \tag{99}$$

and

$$w_{\rm curv} = -1 + \frac{\ddot{R}f''(R) + \dot{R}\left[\dot{R}f'''(R) - Hf''(R)\right]}{\left[f(R) - Rf'(R)\right]/2 - 3H\dot{R}f''(R)}, \tag{100}$$

deduced from Eq. 96. As a first simple choice, one may neglect ordinary matter and assume a power-law form $f(R) = f_0 R^n$, with n a real number, which represents a straightforward generalization of Einstein GR in the limit $n = 1$. One can find power-law solutions for $a(t)$ providing a satisfactory fit to the SNeIa data and a good agreement with the estimated age of the Universe in the range $1.366 < n < 1.376$ [54; 105]. On the other side, one can develop the same analysis in presence of the ordinary matter component, although in such a case, one has to solve numerically the field equations. Then, it is still possible to confront the Hubble flow described by such a model with the Hubble diagram of SNeIa using the above mentioned methods (Fig. 2). The data fit turns out to be significant (see Fig. 3) improving the χ^2 value and it fixes the best fit value at $n = 3.46$ when it is accounted only the baryon contribute $\Omega_b \approx 0.04$ (according with BBN prescriptions). It has to be remarked that considering dark matter does not modify the result of the fit, as it is evident from Fig. 3, in some sense positively supporting the assumption of no need for dark matter in this model. A part the simplicity of the power law model, the theoretical implications of the best fit values found for n are telling us that dynamics related to cosmological constant (whose theoretical shortcomings are well known) could be seriously addressed by finding a reliable $f(R)$ gravity model (see also [55]).

From the evolution of the Hubble parameter in term of redshift, one can even calculate the age of Universe. In Fig. 4, it is sketched the age of the Universe as a function of the correlation between the deceleration parameter q_0 and the model parameter n. The best fit value $n = 3.46$ provides $t_{\rm univ} \approx 12.41$ Gyr.

It is worth noticing that considering $f(R) = f_0 R^n$ gravity is only the simplest generalization of the Einstein theory. In other words, it has to be considered that R^n-gravity represents just a working hypothesis as there is no overconfidence that such a model is the correct final gravity theory. In a sense, we want only to suggest

Fig. 4 Contour plot in the plane (q_0, n) describing the Universe age as induced by R^n-gravity model without dark matter. The contours refer to age ranging from 11 to 16 Gyr from up to down. The *dashed curves* define the $1-\sigma$ region relative to the best fit Universe age suggested by the last WMAP release ($13.73^{+0.13}_{-0.17}$ Gyr) in the case of Λ-CDM model [156]. At the best fit $n \simeq 3.5$ for SNeIa, the measured $q_0 \simeq -0.5$ gives a rather short age (about 11.5 Gyr) with respect to the WMAP constraint. This is an indication that the $f(R)$ model has to be further improved

that several cosmological and astrophysical results can be well interpreted in the realm of a power law extended gravity model.

As matter of fact, this approach gives no rigidity about the value of the power n, although it would be preferable to determine a model capable of working at different scales. Furthermore, we do not expect to be able to reproduce the whole cosmological phenomenology by means of a simple power law model, which has been demonstrated to be not sufficiently versatile [157; 158; 159].

For example, we can easily demonstrate that this model fails when it is analyzed with respect to its capability of providing the correct evolutionary conditions for the perturbation spectra of matter overdensity. This point is typically addressed as one of the most important issues which suggest the need for dark matter. In fact, if one wants to discard this component, it is crucial to match the experimental results related to the Large Scale Structure of the Universe and the CMBR which show, respectively at late time and at early time, the signature of the initial matter spectrum.

As important remark, we notice that the quantum spectrum of primordial perturbations, which provides the seeds of matter perturbations, can be positively recovered in the framework of R^n-gravity. In fact, $f(R) \propto R^2$ can represent a viable model with respect to CMBR data and it is a good candidate for cosmological Inflation (see [162; 163] and references therein).

In order to develop the matter power spectrum suggested by this model, we resort to the equation for the matter contrast obtained in [164] in the case of fourth order gravity (see even [165] for a review on cosmological perturbations in $f(R)$-theories). This equation can be deduced considering the conformal Newtonian gauge for the perturbed metric [164]:

$$\mathrm{d}s^2 = (1+2\psi)\mathrm{d}t^2 - a^2(1+2\phi)\Sigma_{i=1}^3(\mathrm{d}x^i). \tag{101}$$

where ψ and ϕ are now gravitational perturbation potentials. In GR, it is $\phi = -\psi$, since there is no anisotropic stress; in ETGs, this relation breaks, in general, and the $i \neq j$ components of field equations give new relations between ϕ and ψ.

In particular, for $f(R)$ gravity, due to the non-vanishing derivatives $f_{R;i;j}$ (with $i \neq j$), the $\phi - \psi$ relation becomes scale dependent. Instead of the perturbation equation for the matter contrast δ, we provide here its evolution in term of the growth index $\mathscr{F} = \mathrm{d}\ln\delta/\mathrm{d}\ln a$, which is the directly measured quantity at $z \sim 0.15$:

$$\mathscr{F}'(a) - \frac{\mathscr{F}(a)^2}{a} + \left[\frac{2}{a} + \frac{1}{a}E'(a)\right]\mathscr{F}(a) - \frac{1-2Q}{2-3Q}\cdot\frac{3\Omega_m a^{-4}}{nE(a)^2\tilde{R}^{n-1}} = 0, \tag{102}$$

(the prime, in this case, means the derivative with respect to a, n is the model parameter, being $f(R) \propto R^n$), $E(a) = H(a)/H_0$, $\tilde{R}$ is the dimensionless Ricci

Fig. 5 Scale factor evolution of the growth index: (*left*) modified gravity, in the case $\Omega_m = \Omega_{\rm bar} \sim 0.04$, for the SNeIa best fit model with $n = 3.46$, (*right*) the same evolution in the case of a ΛCDM model. In the case of R^n-gravity it is shown also the dependence on the scale k. The three cases $k = 0.01,\ 0.001,\ 0.0002$ have been checked. Only the latter case shows a very small deviation from the leading behavior. Clearly, the trend is that the growth law saturates to $\mathscr{F} = 1$ for higher redshifts (i.e., $a \sim 0.001$ to 0.01). This behavior agrees with observations since we know that comparing CMB anisotropies and LSS, we need roughly $\delta \propto a$ between recombination and $z \sim 5$ to generate the present LSS from the small fluctuations at recombination seen in the CMB

scalar, and

$$Q = -\frac{2 f_{RR}\, k^2}{f_R\, a^2}. \tag{103}$$

For $n = 1$ the previous expression gives the ordinary growth index relation for the Cosmological Standard Model. It is clear, from Eq. 102, that such a model suggests a scale dependence of the growth index which is contained into the corrective term Q so that, when $Q \to 0$, this dependence can be reasonably neglected.

In the most general case, one can resort to the limit $aH < k < 10^{-2}\,\mathrm{h\,Mpc}^{-1}$, where Eq. 102 is a good approximation, and non-linear effects on the matter power spectrum can be neglected. Studying numerically Eq. 102, one obtains the growth index evolution in term of the scale factor; for the sake of simplicity, we assume the initial condition $\mathscr{F}(a_{ls}) = 1$ at the last scattering surface as in the case of matter-like domination. The results are summarized in Figs. 5 and 6, where we have displayed, in parallel, the growth index evolution in R^n-gravity and in the ΛCDM model. In the case of $\Omega_m = \Omega_{\rm bar} \sim 0.04$, one can observe a strong disagreement between the expected rate of the growth index and the behavior induced by power law fourth order gravity models.

This negative result is evidenced by the predicted value of $\mathscr{F}(a_{z=0.15})$, which has been observationally estimated by the analysis of the correlation function for 220,000 galaxies in 2dFGRS dataset sample at the survey effective depth $z = 0.15$. The observational result suggests $\mathscr{F} = 0.58 \pm 0.11$ [166], while our model gives $\mathscr{F}(a_{z=0.15}) \sim 0.117\ (k = 0.01),\ 0.117\ (k = 0.001),\ 0.122\ (k = 0.0002)$.

Although this result seems frustrating with respect to the underlying idea to discard the dark matter component from the cosmological dynamics, it does not give substantial improvement in the case of R^n-gravity model plus dark matter. In fact, as it is possible to observe from Fig. 6, even in this case the growth index prediction is far to be in agreement with the ΛCDM model and again, at the observational scale $z = 0.15$, there is not enough growth of perturbations to match the observed Large Scale Structure. In such a case one obtains: $\mathscr{F}(a_{z=0.15}) \sim 0.29\ (k = 0.01),\ 0.29\ (k = 0.001),\ 0.31\ (k = 0.0002)$, which are quite increased with respect to the previous case but still very far from the experimental estimate.

It is worth noticing that no significant different results are obtained if one varies the power n. Of course in the case of $n \to 1$, one recovers the standard behavior if a cosmological constant contribution is added. These results seem to suggest that an ETG model which considers a simple power law of Ricci scalar, although cosmologically relevant at late times, is not viable to describe the evolution of Universe at all scales.

In other words such a scheme seems too simple to give account of the whole cosmological phenomenology. In fact, in [164] a gravity Lagrangian considering

Fig. 6 The evolution of the growth index in terms of the scale factor when dark matter is included in the whole energy budget. Again, the *left* plot shows the modified gravity evolution for the SNeIa best fit model with $n = 3.46$, while the *right* one refers to ΛCDM model

an exponential correction to the Ricci scalar, $f(R) = R + A\exp(-BR)$ (with A, B two constants), gives a grow factor rate which is in agreement with the observational results at least in the dark matter case. To corroborate this point of view, one has to consider that when the choice of $f(R)$ is performed starting from observational data (pursuing an inverse approach) as in [106], the reconstructed Lagrangian is a non-trivial polynomial in term of the Ricci scalar, as we shall see below.

A result which directly suggests that the whole cosmological phenomenology can be accounted only by a suitable non-trivial function of the Ricci scalar rather than a simple power law function. In this case, cosmological equations, coming from an $f(R)$ action, can be reduced to a linear third order differential equation for the function $f(R(z))$, where z is the redshift. The Hubble parameter $H(z)$ inferred from the data and the relation between z and R can be used to finally work out $f(R)$.

This scheme provides even another interesting result. Indeed, one may consider the expression for $H(z)$ in a given dark energy model as the input for the reconstruction of $f(R)$ and thus work out a $f(R)$ theory giving rise to the same dynamics as the input model.

This suggests the intriguing possibility to consider observationally viable dark energy models (such as ΛCDM and quintessence) only as effective parameterizations of the curvature fluid [106; 157]. As matter of fact, the results obtained with respect to the study of the matter power spectra in the case of R^n-gravity do not invalidate the whole approach, since they can be referred to the too simple form of the model. Similar considerations can be developed for cosmological solutions derived in Palatini approach (see [167] for details).

An important remark is in order at this point. If the power n is not a natural number, R^n models could be not analytic for $R \to 0$. In this case, the Minkowski space is not a solution and, in general, the post-Minkowskian limit of the theory could be bad defined. Actually this is not a true shortcoming if we consider R^n-gravity as a toy model for a (still unknown) self-consistent and comprehensive theory working at all scales.

However, the discussion is not definitely closed since some authors support the point of view that no $f(R)$ theories with $f = R + \alpha R^n$, $n \neq 1$ can evolve from a matter-dominated epoch $a(t) \propto t^{2/3}$ to an accelerated phase [159]. This result

Fig. 7 Comparison between predicted and observed values of $\tau = t_L(z) + \Delta f$ for the best fit ΛCDM model. Data in Table 1 have been used

Fig. 8 Comparison between predicted and observed values of $\tau = t_L(z) + \Delta f$ for the best fit $f(R)$ power-law model as in Fig. 3. Data in Table 1 have been used. Also for this test, it is evident the strict concordance with ΛCDM model in Fig. 7

could be the end of such theories, if the phase space analysis of cosmological solutions is not correctly faced (Figs. 7, 8, 9, 10, 11).

In [160], and recently in [161], it is shown that transient matter-dominated evolutions evolving toward accelerated phases are actually possible and the lack of such solutions in [159] depends on an incomplete parameterization of the phase space.

In general, by performing a conformal transformation on a generic $f(R)$ gravity theory, it is possible to achieve, in the Einstein frame, dust matter behaviors which are compatible with observational prescriptions. In addition, by exploiting the analogy between the two frames and between modified gravity and scalar-tensor gravity, one can realize that physical results, in the two conformally related frames, could be completely different. In other words one can pass from a non-phantom phase behavior (Einstein frame) to a phantom regime (Jordan frame) [25].

Now, we can suppose to change completely the point of view. In fact, we can rely directly with the Jordan frame and we can verify if a dust matter regime is intrinsically compatible with modified gravity.

As a first example, one can cite the exact solution provided in [54], which has been deduced working only in the Jordan Frame (FRW Universe). In particular, one is able to find a power law regime for the scale factor whose rate is connected with the power n of the Lagrangian $f(R) = f_0 R^n$.

In other words, one has $a(t) = a_0 t^\alpha$ with $\alpha = \frac{2n^2-3n+1}{2-n}$. Such an exact solution is found out when only baryonic matter is considered [175; 176]. It is evident that such a solution allows to obtain an ordinary matter behavior ($\alpha = 2/3$) for given values of the parameter n (i.e., $n \sim -0.13,\ n \sim 1.29$).

Such solutions are nevertheless stable and no transition to acceleration phase then occurs. In general, it is possible to show that solutions of the type

$$a = a_0 (t - t_0)^{\frac{2n}{3(1+w)}}, \tag{104}$$

where w is the barotropic index of standard perfect fluid, arises as a transient phase, and this phase evolves into an accelerated solution representing an attractor for the system [160]. In any case, a single solution exactly matching, in sequence, radiation, matter and accelerated phases is unrealistic to be found out in the framework of simple $f(R)$-power law theories. The discussion can be further extended as follows.

Modified gravity can span a wide range of analytic functions of the Ricci scalar where $f(R) = f_0 R^n$ only represents the simplest choice. In general, one can reverse the perspective and try to derive the form of gravity Lagrangian directly from the data or mimicking other cosmological models.

Such an approach has been developed in [106], and allows to recover modified gravity Lagrangians by the Hubble flow dynamics $H(z)$: in particular, it is possible to show that wide classes of dark energy models worked out in the Einstein frame can be consistently reproduced by $f(R)$-gravity as quintessence models with exponential potential [107].

Clearly the approach works also for the case of coupled quintessence scalar fields. In other words, the dynamics of $H(z)$, considered in the Jordan frame, is reconstructed by observational data considered in the Einstein frame then assuming one of the two frames as the “physical frame” could be misleading. Here we further develop this approach with the aim to show, in general, the viability of $f(R)$ gravity to recover a matter-dominated phase capable of evolving in a late accelerating phase.

From a formal point of view, the reconstruction of the gravity Lagrangian from data is based on the relation which expresses the Ricci scalar in terms of the Hubble parameter:

$$R = -6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right). \tag{105}$$

Now, starting from the above the $f(R)$ field equations (20) one can reconstruct the form of $f(R)$ from the Hubble parameter as a function of the redshift z exploiting the relation (105) after this expression has been rewritten in term of the redshift itself.

A key role in this discussion is played by the conservation equation for the curvature and the matter fluids which, in the case of dust matter, (i.e., $p_m = 0$) gives:

$$\dot{\rho}_{\mathrm{curv}} + 3H(1 + w_{\mathrm{curv}})\rho_{\mathrm{curv}} = -\frac{1}{f'(R)}(\dot{\rho}_m + 3H\rho_m) - \rho_m \frac{\mathrm{d}f'(R)}{\mathrm{d}t}. \tag{106}$$

In particular, one may assume that the matter energy density is conserved:

$$\rho_m = \Omega_M \rho_{\mathrm{crit}} a^{-3} = 3H_0^2 \Omega_M (1+z)^3 \tag{107}$$

with $z = 1/a - 1$ the redshift (having set $a(t_0) = 1$), Ω_M the matter density parameter (also here, quantities labelled with the subscript 0 refers to present day ($z = 0$) values). Equation (107) inserted into Eq. 106, allows to write a conservation equation for the effective curvature fluid:

$$\dot{\rho}_{\mathrm{curv}} + 3H(1 + w_{\mathrm{curv}})\rho_{\mathrm{curv}} = 3H_0^2 \Omega_M (1+z)^3 \frac{\dot{R} f''(R)}{[f'(R)]^2}. \tag{108}$$

Actually, since the continuity equation and the field equations are not independent [106], one can reduce to the following single equation

$$\dot{H} = -\frac{1}{2f'(R)}\left\{3H_0^2 \Omega_M (1+z)^3 + \ddot{R} f''(R) + \dot{R}\left[\dot{R} f'''(R) - H f''(R)\right]\right\}, \tag{109}$$

where all quantities can be expressed in term of redshift by means of the relation $\frac{\mathrm{d}}{\mathrm{d}t} = -(1+z)H\frac{\mathrm{d}}{\mathrm{d}z}$. In particular, for a flat FRW metric, one has:

$$R = -6\left[2H^2 - (1+z)H\frac{\mathrm{d}H}{\mathrm{d}z}\right], \tag{110}$$

$$f'(R) = \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^{-1} \frac{\mathrm{d}f}{\mathrm{d}z}, \tag{111}$$

$$f''(R) = \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^{-2} \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^{-3} \frac{\mathrm{d}^2 R}{\mathrm{d}z^2} \frac{\mathrm{d}f}{\mathrm{d}z}, \tag{112}$$

$$f'''(R) = \left(\frac{dR}{dz}\right)^{-3}\frac{d^3 f}{dz^3} + 3\left(\frac{dR}{dz}\right)^{-5}\left(\frac{d^2R}{dz^2}\right)^2\frac{df}{dz} - \left(\frac{dR}{dz}\right)^{-4}\left(3\frac{d^2R}{dz^2}\frac{d^2f}{dz^2} + \frac{d^3R}{dz^3}\frac{df}{dz}\right). \quad (113)$$

Now, we have all the ingredients to reconstruct the shape of $f(R)$ by data or, in general, by the definition of a suitable $H(z)$ viable with respect to observational results. In particular, we can show that a standard matter regime (necessary to cluster large scale structure) can arise, in this scheme, before the accelerating phase arises as, for example, in the so called *quiessence* model.

A quiessence model is based on an ordinary matter fluid plus a cosmological component whose equation of state w is constant but can scatter from $w = -1$. This approach represents the easiest generalization of the cosmological constant model, and it has been successfully tested against the SNeIa Hubble diagram and the CMBR anisotropy spectrum so that it allows to severely constraint the barotropic index w [108; 109; 110].

It is worth noticing that these constraints extend into the region $w < -1$, therefore models (phantom models) violating the weak energy condition are allowed. From the cosmological dynamics viewpoint, such a model, by definition, has to display an evolutionary rate of expansion which moves from the standard matter regime to the accelerated behavior in relation to the value of w. In particular, this quantity parameterizes the transition point to the accelerated epoch.

Actually, if it is possible to find out a $f(R)$-gravity model compatible with the evolution of the Hubble parameter of the quiessence model, this result suggests that modified gravity is compatible with a phase of standard matter domination. To be precise, let us consider the Hubble flow defined by this model, where, as above:

$$H(z) = H_0\sqrt{\Omega_M(1+z)^3 + \Omega_X(1+z)^{3(1+w)}} \quad (114)$$

with $\Omega_X = (1-\Omega_M)$ and w the constant parameter defining the dark energy barotropic index. This definition of the Hubble parameter implies:

$$R = -3H_0^2\left[\Omega_M(1+z)^3 + \Omega_X(1-3w)(1+z)^{3(1+w)}\right]. \quad (115)$$

The ansatz in Eq. 114 allows to obtain from Eq. 109 a differential relation for $f(R(z))$ which can be solved numerically by choosing suitable boundary conditions.

In particular we choose:

$$\left(\frac{df}{dz}\right)_{z=0} = \left(\frac{dR}{dz}\right)_{z=0}, \quad (116)$$

$$\left(\frac{d^2f}{dz^2}\right)_{z=0} = \left(\frac{d^2R}{dz^2}\right)_{z=0}. \quad (117)$$

$$f(z=0) = f(R_0) = 6H_0^2(1-\Omega_M) + R_0. \quad (118)$$

A comment is in order here. We have derived the present day values of df/dz and d^2f/dz^2 by imposing the consistency of the reconstructed $f(R)$ theory with *local*

Solar System tests. One could wonder whether tests on local scales could be used to set the boundary conditions for a cosmological problem. It is easy to see that this is indeed meaningful.

Actually, the isotropy and homogeneity of the Universe ensure that the present day value of a whatever cosmological quantity does not depend on where the observer is. As a consequence, hypothetical observers living in the Andromeda galaxy and testing gravity in his planetary system should get the same results. As such, the present day values of $\mathrm{d}f/\mathrm{d}z$ and $\mathrm{d}^2 f/\mathrm{d}z^2$ adopted by these hypothetical observers are the same as those we have used, based on our Solar System experiments. Therefore, there is no systematic error induced by our method of setting the boundary conditions.

Once one has obtained the numerical solution for $f(z)$, inverting again numerically Eq. 115, we may obtain $z = z(R)$ and finally get $f(R)$ for several values of w.

It turns out that $f(R)$ is the same for different models for low values of R and hence of z. This is a consequence of the well known degeneracy among different quiessence models at low z that, in the standard analysis, leads to large uncertainties on w. This is reflected in the shape of the reconstructed $f(R)$ that is almost w-independent in this redshift range.

An analytic representation of the reconstructed fourth order gravity model, can be obtained considering that the following empirical function

$$\ln(-f) = l_1 \, [\ln(-R)]^{l_2} \, [1 + \ln(-R)]^{l_3} + l_4 \tag{119}$$

approximates very well the numerical solution, provided that the parameters (l_1, l_2, l_3, l_4) are suitably chosen for a given value of w. For instance, for $w = -1$ (the cosmological constant) it is:

$$(l_1, l_2, l_3, l_4) = (2.6693, 0.5950, 0.0719, -3.0099).$$

At this point, one can wonder if it is possible to improve such a result considering even the radiation, although energetically negligible. Rather than inserting radiation in the (114), a more general approach in this sense is to consider the Hubble parameter descending from a unified model like those discussed in [43; 44]. In such a scheme one takes into account energy density which scales as:

$$\rho(z) = A \, \left(1 + \frac{1+z}{1+z_s}\right)^{\beta-\alpha} \, \left[1 + \left(\frac{1+z}{1+z_b}\right)^{\alpha}\right] \tag{120}$$

having defined:

$$z_s = 1/s - 1, \quad z_b = 1/b - 1. \tag{121}$$

This model, with the choice $(\alpha, \beta) = (3, 4)$, is able to mimic a Universe undergoing first a radiation dominated era (for $z \gg z_s$), then a matter dominated phase (for $z_b \ll z \ll z_s$) and finally approaching a de Sitter phase with constant energy.

In other words, it works in the way we are asking for. In such a case, the Hubble parameter can be written, in natural units, as $H = \sqrt{\frac{\rho(z)}{3}}$ and one can perform the same calculation as in the quiessence case.

Fig. 9 Evolution of the GW amplitude for some power-law behaviors of $a(t) \sim t^s$, $\phi \sim t^m$ and $f(R) \sim R^n$. The scales of time and amplitude strictly depend on the cosmological background giving a "signature" for the model

As a final result, it is again possible to find out a suitable $f(R)$-gravity model which, for numerical reasons, it is preferable to interpolate as $f(R)/R$:

$$\frac{f(R)}{R} = 1.02 \times \frac{R}{R_0}\left[1+\left(-0.04\times\left(\frac{R}{R_0}\right)^{0.31} +0.69\times\left(\frac{R}{R_0}\right)^{-0.53}\right)\times\ln\left(\frac{R}{R_0}\right)\right], \tag{122}$$

where R_0 is a normalization constant. This result once more confutes issues addressing modified gravity as incompatible with structure formation prescriptions. In fact, also in this case, it is straightforward to show that a phase of ordinary matter (radiation and dust) domination can be obtained and it is followed by an accelerated phase.

Furthermore, several recent studies are pointing out that large scale structure and CMBR anisotropy spectrum are compatible with $f(R)$ gravity as discussed in details in [111; 112] for the metric approach and in [113] for the Palatini approach.

In particular, in [111], it is shown that several classes of $f(R)$ theories can tune the large-angle CMB anisotropy, the shape of the linear matter power spectrum, and qualitatively change the correlations between the CMB and galaxy surveys. All these phenomena are accessible with current and future data and will soon provide stringent tests for such theories at cosmological scales [114; 115; 116; 117].

5.5 The stochastic background of gravitational waves "tuned" by $f(R)$ gravity

As we have seen, a pragmatic point of view could be to "reconstruct" the suitable theory of gravity starting from data. The main issues of this "inverse " approach is matching consistently observations at different scales and taking into account wide classes of gravitational theories where "ad hoc" hypotheses are avoided. In principle, as discussed in the previous section, the most popular dark energy cosmological models can be achieved by considering $f(R)$ gravity without considering unknown ingredients. The main issue to achieve such a goal is to have at disposal suitable datasets at every redshift. In particular, this philosophy can be taken into account also for the cosmological stochastic background of gravitational waves (GW) which, together with CMBR, would carry, if detected, a huge amount of information on the early stages of the Universe evolution [168; 169; 170]. Here, we want to show that cosmological information coming from cosmological stochastic background of GWs could constitute a benchmark for cosmological models coming from ETGs, in particular for $f(R)$.

As well known, GWs are perturbations $h_{\mu\nu}$ of the metric $g_{\mu\nu}$ which transform as 3-tensors. The GW-equations in the transverse-traceless gauge are

$$\Box h_i^j = 0. \tag{123}$$

Latin indexes run from 1 to 3. Our task is now to derive the analog of Eqs. 123 for a generic $f(R)$. As we have seen from conformal transformation, the extra degrees of freedom related to higher order gravity can be recast into a scalar field being

$$\widetilde{g}_{\mu\nu} = \mathrm{e}^{2\phi} g_{\mu\nu} \quad \text{with} \quad \mathrm{e}^{2\phi} = f'(R). \tag{124}$$

and

$$\widetilde{R} = \mathrm{e}^{-2\phi}\left(R - 6\Box\phi - 6\phi_{;\delta}\phi^{;\delta}\right). \tag{125}$$

The GW-equation is now

$$\widetilde{\Box}\tilde{h}_i^j = 0 \tag{126}$$

where

$$\widetilde{\Box} = \mathrm{e}^{-2\phi}\left(\Box + 2\phi^{;\lambda}\nabla_{;\lambda}\right). \tag{127}$$

Since no scalar perturbation couples to the tensor part of gravitational waves, we have

$$\widetilde{h}_i^j = \widetilde{g}^{lj}\delta\widetilde{g}_{il} = \mathrm{e}^{-2\phi} g^{lj}\mathrm{e}^{2\phi}\delta g_{il} = h_i^j \tag{128}$$

which means that h_i^j is a conformal invariant. As a consequence, the plane-wave amplitudes $h_i^j(t) = h(t)e_i^j \exp(ik_m x^m)$, where e_i^j is the polarization tensor, are the same in both metrics. This fact will assume a key role in the following discussion.

In a FRW background, Eq. 126 becomes

$$\ddot{h} + \left(3H + 2\dot{\phi}\right)\dot{h} + k^2 a^{-2} h = 0 \tag{129}$$

being $a(t)$ the scale factor, k the wave number and h the GW amplitude. Solutions are combinations of Bessel's functions. Several mechanisms can be considered for the production of cosmological GWs. In principle, we could seek for contributions due to every high-energy process in the early phases of the Universe.

In the case of inflation, GW-stochastic background is strictly related to dynamics of cosmological model. This is the case we are considering here. In particular, one can assume that the main contribution to the stochastic background comes from the amplification of vacuum fluctuations at the transition between the inflationary phase and the radiation era. However, we can assume that the GWs generated as zero-point fluctuations during the inflation undergo adiabatically damped oscillations ($\sim 1/a$) until they reach the Hubble radius H^{-1}. This is the particle horizon for the growth of perturbations. Besides, any previous fluctuation is smoothed away by the inflationary expansion. The GWs freeze out for $a/k \gg H^{-1}$ and reenter the H^{-1} radius after the reheating. The reenter in the Friedmann era depends on the scale of the GW. After the reenter, GWs can be constrained by the Sachs-Wolfe effect on the temperature anisotropy $\triangle T/T$ at the decoupling. More precisely, such fluctuations are degenerated with scalar fluctuations, but GWs can, in principle, be measured via B-polarization of the CMB. The measurement is very hard to be performed, but many experiments in this direction are presently planned. In any case, $\triangle T/T$ can always be used to derive constraints.

If ϕ acts as the inflaton, we have $\dot{\phi} \ll H$ during the inflation. Adopting the conformal time $d\eta = dt/a$, Eq. 129 reads

$$h'' + 2\frac{\chi'}{\chi}h' + k^2 h = 0 \tag{130}$$

where $\chi = a e^{\phi}$. The derivation is now with respect to η. Inside the radius H^{-1}, we have $k\eta \gg 1$. Considering the absence of gravitons in the initial vacuum state, we have only negative-frequency modes and then the solution of (130) is

$$h = k^{1/2}\sqrt{2/\pi}\frac{1}{aH}C\exp(-ik\eta). \tag{131}$$

C is the amplitude parameter. At the first horizon crossing $(aH = k)$ the averaged amplitude $A_h^k = (k/2\pi)^{3/2}\,|h|$ of the perturbation is

$$A_h^k = \frac{1}{2\pi^2}C. \tag{132}$$

When the scale a/k becomes larger than the Hubble radius H^{-1}, the growing mode of evolution is constant, i.e., it is frozen. It can be shown that $\triangle T/T \lesssim A_h^k$, as an upper limit to A_h^k, since other effects can contribute to the background anisotropy. From this consideration, it is clear that the only relevant quantity is the initial amplitude C in Eq. 131, which is conserved until the reenter. Such an amplitude depends on the fundamental mechanism generating perturbations. Inflation gives rise to processes capable of producing perturbations as zero-point energy fluctuations. Such a mechanism depends on the gravitational interaction and then $(\triangle T/T)$ could constitute a further constraint to select a suitable theory of gravity. Considering a single graviton in the form of a monochromatic wave, its zero-point amplitude is derived through the commutation relations:

$$[h(t,x),\,\pi_h(t,y)] = i\delta^3(x-y) \tag{133}$$

calculated at a fixed time t, where the amplitude h is the field and π_h is the conjugate momentum operator. Writing the Lagrangian for h

$$\widetilde{\mathscr{L}} = \frac{1}{2}\sqrt{-\widetilde{g}}\widetilde{g}^{\mu\nu}h_{;\mu}h_{;\nu} \tag{134}$$

in the conformal FRW metric $\widetilde{g}_{\mu\nu}$, where the amplitude h is conformally invariant, we obtain

$$\pi_h = \frac{\partial\widetilde{\mathscr{L}}}{\partial\dot{h}} = e^{2\phi}a^3\dot{h} \tag{135}$$

Equation (133) becomes

$$\left[h(t,x),\,\dot{h}(y,y)\right] = i\frac{\delta^3(x-y)}{a^3 e^{2\phi}} \tag{136}$$

and the fields h and $\dot{h}$ can be expanded in terms of creation and annihilation operators

$$h(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left[h(t)e^{-ikx} + h^*(t)e^{+ikx}\right], \tag{137}$$

$$\dot{h}(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left[\dot{h}(t)e^{-ikx} + \dot{h}^*(t)e^{+ikx}\right]. \tag{138}$$

The commutation relations in conformal time are

$$\left[hh'^{*}-h^{*}h'\right]=\frac{i(2\pi)^{3}}{a^{3}\mathrm{e}^{2\phi}}. \tag{139}$$

From (131) and (132), we obtain $C=\sqrt{2}\pi^{2}H\mathrm{e}^{-\phi}$, where H and ϕ are calculated at the first horizon-crossing and, being $\mathrm{e}^{2\phi}=f'(R)$, the remarkable relation

$$A_{h}^{k}=\frac{H}{\sqrt{2f'(R)}}, \tag{140}$$

holds for a generic $f(R)$ theory at a given k. Clearly the amplitude of GWs produced during inflation depends on the theory of gravity which, if different from GR, gives extra degrees of freedom. On the other hand, the Sachs-Wolfe effect could constitute a test for gravity at early epochs. This probe could give further constraints on the GW-stochastic background, if ETGs are independently probed at other scales.

In summary, the amplitudes of tensor GWs are conformally invariant and their evolution depends on the cosmological background. Such a background is tuned by a conformal scalar field which is not present in the standard GR. Assuming that primordial vacuum fluctuations produce stochastic GWs, beside scalar perturbations, kinematical distortions and so on, the initial amplitude of these ones is a function of the $f(R)$-theory of gravity and then the stochastic background can be, in a certain sense "tuned" by the theory. Viceversa, data coming from the Sachs-Wolfe effect could contribute to select a suitable $f(R)$ theory which can be consistently matched with other observations. However, further and accurate studies are needed in order to test the relation between Sachs-Wolfe effect and $f(R)$ gravity. This goal could be achieved very soon through the forthcoming space (LISA) and ground-based (VIRGO, LIGO) interferometers.

6 Applications to galactic dynamics

The results obtained at cosmological scales motivates further applications of ETGs, in particular of $f(R)$ theories. In general, one is wondering whether ETG models, working as dark energy models, can also play a role to explain the dark matter phenomenology at scales of galaxies and clusters of galaxies.

Several studies have been pursued in this direction [171; 172] but the main goal remains that to seek a unified model capable of explain dynamics at every scale without introducing ad hoc components.

6.1 Dark matter as a curvature effect: the case of flat rotation curves of LSB galaxies

It is well known that, in the low energy limit, higher order gravity implies modified gravitational potentials [69; 173]. By considering the case of a pointlike mass m and solving the vacuum field equations for a Schwarzschild-like metric [174; 176], one gets as exact solution from a theory $f(R) = f_0 R^n$, the modified gravitational potential:

$$\Phi(r) = -\frac{Gm}{2r}\left[1 + \left(\frac{r}{r_c}\right)^{\beta}\right] \tag{141}$$

where

$$\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2} \tag{142}$$

which corrects the ordinary Newtonian potential by a power-law term. As we will see, it has to be $\beta > 0$ and then $n > 0$. In particular, the best fit value will be $\beta \simeq 0.8$ and then $n = 3.2$. Standard units have been considered here. In particular, this correction sets in on scales larger than r_c which value depends essentially on the mass of the system [176]. This quantity deserves some discussion. As shown in [176], it is derived from the initial conditions of the models and it correlates with the core masses of the LSB galaxies which we have taken into account. In some sense, it is a sort of further *gravitational radius*, beside the standard Schwarzschild radius, which rules the central mass of the galaxies, and then it is different for different systems. It is interesting to note that, given a generic $2n$-order theory of gravity, it is possible to find out n characteristic radii [69] and it is intriguing to suspect that they could likely rule the structure and the stability of the astrophysical self-gravitating structures (S. Capozziello, E. De Filippis, V. Salzano, in preparation). But this is a working hypothesis which has to be firmly demonstrated.

The corrected potential (141) reduces to the standard $\Phi \propto 1/r$ for $n = 1$ as it can be seen from the relation (142). The generalization of Eq. 141 to extended systems is straightforward. We simply divide the system in infinitesimal mass elements and sum up the potentials generated by each single element. In the continuum limit, we replace the sum with an integral over the mass density of the system

Table 2 Properties of sample galaxies

Id	D	μ_0	r_d	r_{HI}	M_{HI}	Type
UGC 1230	51	22.6	4.5	101	58.0	Sm
UGC 1281	5.5	22.7	1.7	206	3.2	Sdm
UGC 3137	18.4	23.2	2.0	297	43.6	Sbc
UGC 3371	12.8	23.3	3.1	188	12.2	Im
UGC 4173	16.8	24.3	4.5	178	21.2	Im
UGC 4325	10.1	21.6	1.6	142	7.5	SAm
NGC 2366	3.4	22.6	1.5	439	7.3	IB(s)m
IC 2233	10.5	22.5	2.3	193	13.6	SBd
NGC 3274	6.7	20.2	0.5	225	6.6	SABd
NGC 4395	3.5	22.2	2.3	527	9.7	SAm
NGC 4455	6.8	20.8	0.7	192	5.4	SBd
NGC 5023	4.8	20.9	0.8	256	3.5	Scd
DDO 185	5.1	23.2	1.2	136	1.6	IBm
DDO 189	12.6	22.6	1.2	167	10.5	Im
UGC 10310	15.6	22.0	1.9	130	12.6	SBm

Explanation of the columns: name of the galaxy, distance in Mpc; disk central surface brightness in the $\mathscr{R}$ band (corrected for galactic extinction); disk scalelength in kpc; radius at which the gas surface density equals 1 $M_\odot/pc^2$ in arcsec; total HI gas mass in 10^8 $M_\odot$; Hubble type as reported in the NED database taking care of eventual symmetries of the mass distribution [176]. Once the gravitational potential has been computed, one may evaluate the rotation curve $v_c^2(r)$ and compare it with the data. For extended systems, one has typically to resort to numerical techniques, but the main effect may be illustrated by the rotation curve for the pointlike case:

$$v_c^2(r) = \frac{Gm}{2r}\left[1 + (1-\beta)\left(\frac{r}{r_c}\right)^{\beta}\right]. \tag{143}$$

Compared with the Newtonian result $v_c^2 = Gm/r$, the corrected rotation curve is modified by the addition of the second term in the r.h.s. of Eq. 143. For $0 < \beta < 1$, the corrected rotation curve is higher than the Newtonian one. Since measurements of spiral galaxies rotation curves signal a circular velocity higher than those which are predicted on the basis of the observed luminous mass and the Newtonian potential, the above result suggests the possibility that such a modified gravitational potential may fill the gap between theory and observations without the need of additional dark matter. It is worth noticing that the corrected rotation curve is asymptotically vanishing as in the Newtonian case, while it is usually claimed that observed rotation curves are flat (i.e., asymptotically constant). Actually, observations do not probe v_c up to infinity, but only show that the rotation curve is flat within the measurement uncertainties up to the last measured point. This fact by no way excludes the possibility that v_c goes to zero at infinity.

In order to observationally check the above result, one can take into account samples of low surface brightness (LSB) galaxies with well measured HI + Hα rotation curves extending far beyond the visible edge of the system. LSB galaxies are known to be ideal candidates to test dark matter models since, because of their high gas content, the rotation curves can be well measured and corrected for possible systematic errors by comparing 21-cm HI line emission with optical Hα and [NII] data. Moreover, they are supposed to be dark matter dominated so that fitting their rotation curves without this elusive component could be a strong evidence in favor of any successful alternative theory of gravity. The considered sample (Table 2) contains 15 LSB galaxies with data on the rotation curve, the surface mass density of the gas component and $\mathscr{R}$- photometric band, disk photometry extracted from a larger sample selected by de Blok and Bosma [177]. We assume the stars are distributed in a thin and circularly symmetric disk with surface density $\Sigma(r) = \Upsilon_\star I_0 \exp(-r/r_d)$ where the central surface luminosity I_0 and the disk scalelength r_d are obtained from fitting to the stellar photometry. The gas surface density has been obtained by interpolating the data over the range probed by HI measurements and extrapolated outside this range.

When fitting to the theoretical rotation curve, there are three quantities to be determined, namely the stellar mass-to-light (M/L) ratio, $\Upsilon_\star$ and the theory param-

Fig. 10 Best fit theoretical rotation curve superimposed to the data for the LSB galaxy NGC 4455 (*left*) and NGC 5023 (*right*). These two cases are considered to better show the effect of the correction to the Newtonian gravitational potential. We report the total rotation curve $v_c(r)$ (*solid line*), the Newtonian one (*short dashed*) and the corrected term (*long dashed*)

Fig. 11 Best fit curves superimposed to the data for the total sample of 15 LSB galaxies considered

eters (β, r_c). It is worth stressing that, while fit results for different galaxies should give the same β, r_c is related to one of the integration constants of the field equations. As such, it is not a universal quantity and its value must be set on a galaxy-by-galaxy basis. However, it is expected that galaxies having similar properties in terms of mass distribution have similar values of r_c so that the scatter in r_c must reflect somewhat the scatter in the terminal circular velocities. In order to match the model with the data, we perform a likelihood analysis for each galaxy, using, as fitting parameters β, $\log r_c$ (with r_c in kpc) and the gas mass fraction[7] f_g. As it is evident considering the results from the different fits summarized in Table 3, the experimental data are successfully fitted by the model. In particular, the best fit range of β ($\beta = 0.80 \pm 0.08$), corresponding to R^n gravity with $2.3 < n < 5.3$ (best fit value $n = 3.2$), seems well overlaps the above mentioned range of n fitting SNeIa Hubble diagram.

However, these are only preliminary results which do not completely solve the problem of dark matter in galaxies by models coming from ETGs and do not fit the growth of structures. In any case, further evidences on the same line of thinking are coming from other samples of galaxies (where also high surface brightness galaxies are considered) [178], or from galaxy clusters, where the dark matter range is completely different (S. Capozziello, E. De Filippis, V. Salzano, in preparation).

[7] This is related to the M/L ratio as $\Upsilon_\star = [(1 - f_g)M_g]/(f_g L_d)$ with $M_g = 1.4 M_{\rm HI}$ the gas (HI + He) mass, $M_d = \Upsilon_\star L_d$ and $L_d = 2\pi I_0 r_{\rm d}^2$ the disk total mass and luminosity.

Table 3 Best fit values of the model parameters from minimizing $\chi^2(\beta, \log r_c, f_g)$

Id	β	$\log r_c$	f_g	$\Upsilon_\star$	χ^2/dof	$\sigma_{\rm rms}$
UGC 1230	0.83 ± 0.02	-0.39 ± 0.09	0.15 ± 0.01	15.9 ± 0.5	2.97/8	0.96
UGC 1281	0.38 ± 0.01	-3.93 ± 0.80	0.65 ± 0.08	0.64 ± 0.33	3.48/21	1.05
UGC 3137	0.72 ± 0.03	-1.86 ± 0.06	0.65 ± 0.02	9.8 ± 0.9	48.1/26	1.81
UGC 3371	0.78 ± 0.05	-1.85 ± 0.01	0.41 ± 0.01	3.3 ± 0.2	0.48/15	1.30
UGC 4173	0.94 ± 0.02	-0.97 ± 0.22	0.34 ± 0.01	9.37 ± 0.04	0.12/10	0.52
UGC 4325	0.79 ± 0.07	-2.85 ± 0.44	0.70 ± 0.02	0.50 ± 0.05	0.09/13	1.19
NGC 2366	0.96 ± 0.14	-0.58 ± 0.42	0.64 ± 0.01	14.5 ± 0.9	28.6/25	1.10
IC 2233	0.42 ± 0.01	-3.50 ± 0.05	0.64 ± 0.01	1.29 ± 0.06	6.1/22	2.10
NGC 3274	0.71 ± 0.03	-2.30 ± 0.19	0.55 ± 0.03	2.3 ± 0.3	17.6/20	2.7
NGC 4395	0.13 ± 0.02	-3.68 ± 0.31	0.14 ± 0.01	7.6 ± 0.3	37.7/52	1.40
NGC 4455	0.87 ± 0.05	-2.32 ± 0.07	0.83 ± 0.01	0.42 ± 0.04	3.3/17	1.12
NGC 5023	0.81 ± 0.02	-2.54 ± 0.05	0.53 ± 0.02	0.91 ± 0.06	8.9/30	2.50
DDO 185	0.92 ± 0.10	-2.75 ± 0.35	0.90 ± 0.03	0.21 ± 0.07	5.03/5	0.81
DDO 189	0.54 ± 0.08	-2.40 ± 0.61	0.63 ± 0.04	4.2 ± 0.7	0.44/8	1.08
UGC 10310	0.72 ± 0.04	-1.87 ± 0.04	0.59 ± 0.02	1.39 ± 0.04	2.90/13	1.02

The values of $\Upsilon_\star$, the χ^2/dof are reported for the best fit parameters (with $dof = N - 3$ and N the number of datapoints) and the root mean square $\sigma_{\rm rms}$ of the fit residuals. Errors on the fitting parameters and the M/L ratio are estimated through the jacknife method hence do not take into account parameter degeneracies [176]

6.2 Dark matter haloes inspired by $f(R)$-gravity

At this point, it is worth wondering whether a link may be found between $f(R)$ gravity and the standard approach based on dark matter haloes since both theories fit equally well the same data. The *trait-de-union* between these two different schemes can be found in the modified gravitational potential which induces a correction to the rotation curve in a similar manner as a dark matter halo does. As a matter of fact, it is possible to define an *effective dark matter halo* by imposing that its rotation curve equals the correction term to the Newtonian curve induced by $f(R)$ gravity. Mathematically, one has to split the total rotation curve derived from $f(R)$ gravity as $v_c^2(r) = v_{c,N}^2(r) + v_{c,\mathrm{corr}}^2(r)$ where the second term is the correction. Considering, for simplicity a spherical halo where a thin exponential disk is embedded, one can write the total rotation curve as $v_c^2(r) = v_{c,\mathrm{disk}}^2(r) + v_{c,\mathrm{DM}}^2(r)$ with $v_{c,\mathrm{disk}}^2(r)$ the Newtonian disk rotation curve and $v_{c,\mathrm{DM}}^2(r) = GM_{\mathrm{DM}}(r)/r$ the dark matter one, $M_{\mathrm{DM}}(r)$ being its mass distribution. Equating the two expressions, we get:

$$M_{\mathrm{DM}}(\eta) = M_{\mathrm{vir}} \left(\frac{\eta}{\eta_{\mathrm{vir}}}\right) \frac{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}\mathscr{I}_0(\eta) - \mathscr{V}_d(\eta)}{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}\mathscr{I}_0(\eta_{\mathrm{vir}}) - \mathscr{V}_d(\eta_{\mathrm{vir}})}. \tag{144}$$

with $\eta = r/r_d$, $\Sigma_0 = \Upsilon_\star i_0$, $\mathscr{V}_d(\eta) = I_0(\eta/2)K_0(\eta/2) \times I_1(\eta/2)K_1(\eta/2)$ and:[8]

$$\mathscr{I}_0(\eta,\beta) = \int_0^\infty \mathscr{F}_0(\eta,\eta',\beta)k^{3-\beta}\eta'^{\frac{\beta-1}{2}}\mathrm{e}^{-\eta'}\mathrm{d}\eta' \tag{145}$$

with $\mathscr{F}_0$ only depending on the geometry of the system and "*vir*" indicating virial quantities. Equation (144) defines the mass profile of an effective spherically symmetric dark matter halo whose ordinary rotation curve provides the part of the corrected disk rotation curve due to the addition of the curvature corrective term to the gravitational potential. It is evident that, from an observational viewpoint, there is no way to discriminate between this dark halo model and a $f(R)$ power-law gravity model. Having assumed spherical symmetry for the mass distribution, it is straightforward to compute the mass density for the effective dark halo as $\rho_{\mathrm{DM}}(r) = (1/4\pi r^2)\mathrm{d}M_{\mathrm{DM}}/\mathrm{d}r$. The most interesting features of the density profile are its asymptotic behaviors that may be quantified by the logarithmic slope $\alpha_{\mathrm{DM}} = \mathrm{d}\ln\rho_{\mathrm{DM}}/\mathrm{d}\ln r$ which can be numerically computed as function of η for fixed values of β (or n). As expected, α_{DM} depends explicitly on β, while (r_c, Σ_0, r_d) enter indirectly through η_{vir}. The asymptotic values at the center and at infinity denoted as α_0 and α_∞ result particularly interesting. It turns out that α_0 almost vanishes so that in the innermost regions the density is approximately constant. Indeed, $\alpha_0 = 0$ is the value corresponding to models having an inner core such as the cored isothermal sphere [179] and the Burkert model [180; 181; 182]. Moreover, it is well known that galactic rotation curves are typically best fitted by cored dark halo models (see, e.g., [183] and references therein). On the other hand, the outer asymptotic slope is between -3 and -2, that are values typical of most dark halo models in literature. In particular, for $\beta = 0.80$ one finds

[8] Here I_l and K_l, with $l = 1,2$ are the Bessel functions of first and second type.

$(\alpha_0, \alpha_\infty) = (-0.002, -2.41)$, which are quite similar to the value for the Burkert model $(0, -3)$. It is worth noticing that the Burkert model has been *empirically* proposed to provide a good fit to the LSB and dwarf galaxies rotation curves. The values of $(\alpha_0, \alpha_\infty)$ we find for our best fit effective dark halo therefore suggest a possible theoretical motivation for the Burkert-like models. Due to the construction, the properties of the effective dark matter halo are closely related to the disk one. As such, we do expect some correlation between the dark halo and the disk parameters. To this aim, exploiting the relation between the virial mass and the disk parameters, one can obtain a relation for the Newtonian virial velocity $V_{\rm vir} = GM_{\rm vir}/R_{\rm vir}$:

$$M_d \propto \frac{(3/4\pi\delta_{\rm th}\Omega_m\rho_{\rm crit})^{\frac{1-\beta}{4}} r_{\rm d}^{\frac{1+\beta}{2}} \eta_c^{\beta}}{2^{\beta-6}(1-\beta)G^{\frac{5-\beta}{4}}} \frac{V_{\rm vir}^{\frac{5-\beta}{2}}}{\mathscr{I}_0(V_{\rm vir}, \beta)}. \tag{146}$$

One can numerically check that Eq. 146 may be well approximated as $M_d \propto V_{\rm vir}^a$ which has the same formal structure as the baryonic Tully–Fisher (BTF) relation $M_b \propto V_{\rm flat}^a$ with M_b the total (gas + stars) baryonic mass and $V_{\rm flat}$ the circular velocity on the flat part of the observed rotation curve. In order to test whether the BTF can be explained thanks to the effective dark matter halo we are proposing, we should look for a relation between $V_{\rm vir}$ and $V_{\rm flat}$. This is not analytically possible since the estimate of $V_{\rm flat}$ depends on the peculiarities of the observed rotation curve such as how far it extends and the uncertainties on the outermost points. Therefore, for given values of the disk parameters, it is possible to simulate theoretical rotation curves for some values of r_c and measure $V_{\rm flat}$ finally choosing the fiducial value for r_c which gives a value of $V_{\rm flat}$ as similar as possible to the measured one. Inserting the relation thus found between $V_{\rm flat}$ and $V_{\rm vir}$ into Eq. 146 and averaging over different simulations, one finally gets:

$$\log M_b = (2.88 \pm 0.04) \log V_{\rm flat} + (4.14 \pm 0.09) \tag{147}$$

while a direct fit to the observed data gives [184]:

$$\log M_b = (2.98 \pm 0.29) \log V_{\rm flat} + (3.37 \pm 0.13). \tag{148}$$

The slope of the predicted and observed BTF are in good agreement leading further support to the $f(R)$ gravity model. The zeropoint is markedly different with the predicted one being significantly larger than the observed one, but it is worth stressing, however, that both relations fit the data with similar scatter. A discrepancy in the zeropoint may be due to the approximate treatment of the effective halo which does not take into account the gas component. Neglecting this term, one should increase the effective halo mass and hence $V_{\rm vir}$ which affects the relation with $V_{\rm flat}$ leading to a higher than observed zeropoint. Indeed, the larger is M_g/M_d, the more the point deviate from our predicted BTF thus confirming our hypothesis. Given this caveat, we may therefore conclude with confidence that $f(R)$ gravity offers a theoretical foundation even for the empirically found BTF relation.

All these results converge toward the picture that data coming from observations at galactic, extragalactic and cosmological scales could be seriously framed in ETGs without considering huge amounts of dark energy and dark matter.

7 Discussion and conclusions

Extended Theories of Gravity can be considered as the natural extension of General Relativity. Also if they are not the final theory of gravity at fundamental level (i.e., quantum gravity), they could be a useful approach to address several shortcomings of GR. In fact, also at Solar System scales, where GR has been strongly confirmed, some conundrums come out as the indications of an apparent, anomalous, long-range acceleration revealed from the data analysis of Pioneer 10/11, Galileo, and Ulysses spacecrafts. Such results are difficult to be framed in the standard theory of GR and in its low energy limit [67].

Furthermore, at galactic scales, huge bulks of dark matter are needed to provide realistic models matching with observations. In this case, retaining GR and its low energy limit, implies the introduction of an actually unknown ingredient (a huge amount of missing matter).

We face a similar situation even at larger scales: clusters of galaxies are gravitationally stable and bound only if large amounts of dark matter are supposed in their potential wells.

Finally, an unknown form of dark energy is required to explain the observed accelerated expansion of cosmic fluid. Summarizing, almost 95% of matter-energy content of the Universe is unknown while we can experimentally probe only gravity and ordinary (baryonic and radiation) matter.

Considering another point of view, anomalous acceleration (Solar System), dark matter (galaxies, galaxy clusters and clustered structures in general), dark energy (cosmology) could be nothing else but the indications that gravity is an interaction depending on the scale and the assumption of a linear Lagrangian density in the Ricci scalar R, as the Hilbert–Einstein action, could be too simple for a comprehensive picture at any scale.

Due to these facts, several motivations suggest to generalize GR by considering gravitational actions where generic functions of curvature invariants and scalar fields are present. This viewpoint is physically motivated by several unification schemes and by field quantization on curved spacetime [8]. Furthermore, it is well known that revisions of GR can solve shortcomings at early cosmological epochs (giving rise to suitable inflationary behaviors [13; 15]) and explain the today observed accelerated behavior [48; 49; 50; 51; 52]. These results can be achieved in metric and Palatini approaches [185; 186; 187; 188; 189; 190; 191; 192; 193].

In addition, reversing the problem, one can reconstruct the form of the gravity Lagrangian by observational data of cosmological relevance through a “back scattering” procedure [106].

All these facts suggest that the theory should be more general than the linear Hilbert–Einstein one implying that extended gravity could be a suitable approach to solve GR shortcomings without introducing mysterious ingredients as dark energy and dark matter which seem without explanation at fundamental level. However, changing gravitational side could be nothing else but a matter of taste since final probes for dark energy and dark matter could come out from the forthcoming experiments as LHC.

Furthermore, in recent papers, some authors have confronted this kind of theories even with the PPN prescriptions considering both metric and Palatini approaches. The results seem controversial since in some cases [194; 195] it is argued that

Table 4 A schematic resume of recent experimental constraints on the PPN-parameters

Mercury Perihelion shift	$\lvert 2\gamma-\beta-1\rvert < 3\times10^{-3}$
Lunar laser ranging	$4\beta-\gamma-3 = -(0.7\pm1)\times10^{-3}$
Very long baseline interf.	$\lvert\gamma-1\rvert = 4\times10^{-4}$
Cassini spacecraft	$\gamma-1 = (2.1\pm2.3)\times10^{-5}$

They are the perihelion shift of Mercury [200], the Lunar Laser Ranging [202], the upper limit coming from the Very Long Baseline Interferometry [201] and the results obtained by the estimate of the Cassini spacecraft delay into the radio waves transmission near the Solar conjunction [203]

GR is always valid and there is no room for other theories while other studies [196; 197; 198] find that recent experiments as Cassini and Lunar Laser Ranging allow the possibility that ETGs could be taken into account. In particular, it is possible to define generalized PPN-parameters and several ETGs could result compatible with experiments in Solar System [61; 197; 100].

In principle, any analytic ETGs can be compared with the Hilbert–Einstein Lagrangian provided suitable values of the coefficients. This consideration suggests to take into account, as physical theories, functions of the Ricci scalar which slightly deviates from GR, i.e., $f(R) = f_0 R^{(1+\varepsilon)}$ with ε a small parameter which indicates how much the theory deviates from GR and then approximate as

$$f_0|R|^{(1+\varepsilon)} \simeq f_0|R|\left(1+\varepsilon\ln|R|+\frac{\varepsilon^2\ln^2|R|}{2}+\cdots\right). \qquad (149)$$

Actually, the PPN-Eddington parameters β and γ may represent the key parameters to discriminate among relativistic theories of gravity. In particular, these quantities should be significatively tested at Solar System scales by forthcoming experiments like LATOR [199] while the today available releases are far, in our opinion, to be conclusive in this sense, as a rapid inspection of Table 4 suggests. In other words, ETGs cannot be a priori excluded also at Solar System scales.

In this paper, we have outlined what one should intend for ETGs in the metric and in the Palatini approach. In particular, we have discussed the higher-order and the scalar-tensor theories of gravity showing the relations between them and their connection to GR via the conformal transformations.

In the so called Einstein frame, any ETG can be reduced to the Hilbert–Einstein action plus one or more than one scalar field(s). The physical meaning of conformal transformations can be particularly devised in the Palatini approach, as discussed in Sect. 4. After, we have discussed some cosmological and astrophysical applications of ETGs.

Although the results outlined are referred to the simplest class of ETGs, power law $f(R)$, they could represent an interesting paradigm. Assuming both metric and Palatini approach, it is possible to investigate the viability of $f(R)$ cosmological models. The expansion rate $H = \dot{a}/a$ may be analytically expressed as a function of the redshift z, so that it is possible to contrast the model predictions against the observations. In particular, the SNeIa Hubble diagram, the gas mass fraction in relaxed galaxy clusters, the lookback time to galaxy clusters, and radio galaxies can be used to constrain cosmological parameters by distance and time-based methods.

Also if such models are, up to now, not completely satisfactory to match all the observations, they allow to recover accelerated behavior of Hubble fluid without any unknown form of dark energy. However, the issue of structure formation has to be seriously faced in order to understand if such toy models could give rise to a self-consistent alternative theory to GR.

Furthermore, it is possible to "tune" the stochastic background of GWs and this occurrence could constitute a further cosmological test capable of confirming or ruling out ETGs once data from interferometers, like VIRGO, LIGO and LISA, will be available.

In addition, the modification of the gravitational potential arising as a natural effect in the framework of ETGs can represent a fundamental tool to interpret the rotation curves of spiral galaxies. Besides, if one considers the model parameters settled by the fit over the observational data on rotation curves, it is possible to construct a phenomenological analogous of dark matter halo whose shape is similar to the one of the so called Burkert model. Since Burkert's model has been empirically introduced to give account for the dark matter distribution in the case of LSB and dwarf galaxies, this result could represent an interesting achievement since it provides a theoretical foundation to such a model.

By investigating the relation between dark halo and the galaxy disk parameters, a relation between M_d and $V_{\rm flat}$, reproducing the baryonic Tully–Fisher, can be deduced. In fact, exploiting the relation between the virial mass and the disk parameters, one obtains a relation for the virial velocity which can be satisfactory approximated as $M_d \propto V_{\rm vir}^a$. Even such a result seems intriguing since it provides a theoretical interpretation for a phenomenological relation.

As a matter of fact, although not definitive, these phenomenological issues can represent a viable approach for future, more exhaustive investigations of ETGs. In particular, they support the quest for a unified view of the dark side of the Universe. In summary, these results seem to motivate a careful search for a fundamental theory of gravity capable of explaining the full cosmic dynamics by the only "ingredients" which we can directly and firmly experience, namely the background gravity, the baryonic matter, the radiation and also the neutrinos [204].

Acknowledgments The authors wish to thank G. Allemandi, A. Borowiec, V.F. Cardone, S. Carloni, S. Nojiri, S.D. Odintsov, A. Stabile and A. Troisi for discussions, suggestions and results presented in this paper.

References

1. S. Weinberg (1972) Gravitation and Cosmology Wiley New York
2. A. Guth (1981) *Phys. Rev. D* **23** 347
3. I.L. Buchbinder S.D. Odintsov I.L. Shapiro (1992) Effective Action in Quantum Gravity IOP Publishing Bristol
4. H. Bondi (1952) Cosmology Cambridge University Press Cambridge
5. C. Brans R.H. Dicke (1961) *Phys. Rev.* **124** 925
6. S. Capozziello R. de Ritis C. Rubano P. Scudellaro (1996) *La Rivista del Nuovo Cimento* **4** 1

7. D.W. Sciama (1953) *Mon. Not. R. Aston. Soc.* **113** 34
8. N.D. Birrell P.C.W. Davies (1982) Quantum Fields in Curved Space Cambridge University Press Cambridge
9. G. Vilkovisky (1992) *Class. Quantum Grav.* **9** 895
10. M. Gasperini G. Veneziano (1992) *Phys. Lett.* **277** 256
11. G. Magnano M. Ferraris M. Francaviglia (1987) *Gen. Relativ. Gravit.* **19** 465
12. J. Barrow A.C. Ottewill (1983) *J. Phys. A: Math. Gen.* **16** 2757
13. A.A. Starobinsky (1980) *Phys. Lett.* **91** 99
14. J.P. Duruisseau R. Kerner (1983) *Gen. Rel. Grav.* **15** 797 – 807
15. D. La P.J. Steinhardt (1989) *Phys. Rev. Lett.* **62** 376
16. P. Teyssandier Ph. Tourrenc (1983) *J. Math. Phys.* **24** 2793
17. K. Maeda (1989) *Phys. Rev. D* **39** 3159
18. D. Wands (1994) *Class. Quantum Grav.* **11** 269
19. S. Capozziello R. de Ritis A.A. Marino (1998) *Gen. Relativ. Gravit.* **30** 1247
20. S. Gottlöber H.-J. Schmidt A.A. Starobinsky (1990) *Class. Quantum Grav.* **7** 893
21. T.V. Ruzmaikina A.A. Ruzmaikin (1970) *JETP* **30** 372
22. L. Amendola A. Battaglia-Mayer S. Capozziello S. Gottlöber V. Müller F. Occhionero H.-J. Schmidt (1993) *Class. Quantum Grav.* **10** L43
23. A. Battaglia-Mayer H.-J. Schmidt (1993) *Class. Quantum Grav.* **10** 2441
24. H.-J. Schmidt (1990) *Class. Quantum Grav.* **7** 1023
25. S. Capozziello S. Nojiri S.D. Odintsov (2006) *Phys. Lett. B* **634** 93
26. L. Amendola S. Capozziello M. Litterio F. Occhionero (1992) *Phys. Rev. D* **45** 417
27. S. Perlmutter (1999) *Astrophys. J.* **517** 565
28. R.A. Knop (2003) *Astrophys. J.* **598** 102
29. A.G. Riess (1998) *Astrophys. J.* **116** 1009
30. J.L. Tonry (2003) *Astrophys. J.* **594** 1
31. P. de Bernardis (2000) *Nature* **404** 955
32. R. Stompor (2001) *Astrophys. J.* **561** L7
33. D.N. Spergel (2003) *Astrophys. J.* **148** 175
34. G. Hinshaw (2003) *Astrophys. J.* **148** 135
35. Spergel, D.N., et al.: astro-ph/0603449 (2006)
36. A.G. Riess (2004) *Astrophys. J.* **607** 665
37. V. Sahni A. Starobinski (2000) *Int. J. Mod. Phys. D* **9** 373
38. T. Padmanabhan (2003) *Phys. Rept.* **380** 235
39. E.J. Copeland M. Sami S. Tsujikawa (2006) *Int. J. Mod. Phys. D* **15** 1753
40. A. Kamenshchik U. Moschella V. Pasquier (2001) *Phys. Lett. B* **511** 265
41. T. Padmanabhan (2002) *Phys. Rev. D* **66** 021301
42. B.A. Bassett M. Kunz D. Parkinson C. Ungarelli (2003) *Phys. Rev. D* **68** 043504
43. V.F. Cardone A. Troisi S. Capozziello (2004) *Phys. Rev. D* **69** 083517
44. S. Capozziello V.F. Cardone E. Elizalde S. Nojiri S.D. Odintsov (2006) *Phys. Rev. D* **73** 043512
45. A. Lue R. Scoccimarro G. Starkman (2004) *Phys. Rev. D* **69** 044005

46. K. Freese M. Lewis (2002) *Phys. Lett. B* **540** 1
47. G.R. Dvali G. Gabadadze M. Porrati (2000) *Phys. Lett. B* **485** 208
48. S. Capozziello (2002) *Int. J. Mod. Phys. D* **11** 483
49. S. Nojiri S.D. Odintsov (2003) *Phys. Rev. D* **68** 123512
50. S.M. Carroll V. Duvvuri M. Trodden M.S. Turner (2004) *Phys. Rev. D* **70** 043528
51. G. Allemandi A. Borowiec M. Francaviglia (2004) *Phys. Rev. D* **70** 103503
52. S. Nojiri S.D. Odintsov (2007) *Int. J. Meth. Mod. Phys.* **4** 115
53. S. Capozziello V.F. Cardone E. Piedipalumbo M. Sereno A. Troisi (2003) *Int. J. Mod. Phys. D* **12** 381
54. S. Capozziello V.F. Cardone S. Carloni A. Troisi (2003) *Int. J. Mod. Phys. D* **12** 1969
55. Starobinsky, A.A.: astro-ph/0706.2041 (2007)
56. B. Li J.D. Barrow (2007) *Phys. Rev. D* **75** 084010
57. Nojiri, S., Odintsov, S.D.: hep-th/06012113 (2006)
58. Li, B., Barrow, J.D., Mota, D.F.: Phys. Rev. D (submitted), gr-qc/0705.3795
59. G. Chen B. Rathra (2003) *Astrophys. J.* **582** 586
60. S. Podariu R.A. Daly M.P. Mory B. Rathra (2003) *Astrophys. J.* **584** 577
61. C.M. Will (1993) Theory and Experiments in Gravitational Physics Cambridge University Press Cambridge
62. F.W. Hehl (1976) *Rev. Mod. Phys.* **48** 393
63. S. Capozziello G. Lambiase C. Stornaiolo (2001) *Ann. Phys. (Leipzig)* **10** 713
64. K. Stelle (1978) *Gen. Relativ. Gravit.* **9** 353
65. R.H. Sanders (1990) *Ann. Rev. Astron Astrophys.* **2** 1
66. P.D. Mannheim D. Kazanas (1989) *Astrophys. J.* **342** 635
67. J.D. Anderson (2002) *Phys. Rev. D* **65** 082004
68. Bertolami, O., Böhmer, Ch.G., Lobo, F.S.N.: gr-qc/0704.1733 (2007)
69. I. Quant H.-J. Schmidt (1991) *Astron. Nachr.* **312** 97
70. P. Schneider J. Ehlers E.E. Falco (1992) Gravitational Lenses Springer Berlin
71. L.M. Krauss M. White (1992) *Astrophys. J.* **397** 357
72. M. Ferraris M. Francaviglia G. Magnano (1988) *Class. Quantum Grav.* **5** L95
73. L.M. Sokolowski (1989) *Class. Quantum Grav.* **6** 2045
74. G. Magnano L.M. SokoLowski (1994) *Phys. Rev. D* **50** 5039
75. R.H. Dicke (1962) *Phys. Rev.* **125** 2163
76. T. Damour G. Esposito-Farèse (1992) *Class. Quantum Grav.* **9** 2093
77. S. Capozziello R. de Ritis A.A. Marino (1997) *Class. Quantum Grav.* **14** 3243
78. V. Faraoni (2004) Cosmology in Scalar-Tensor Gravity Kluwer Dordrecht
79. Einstein, A.: Sitzung-ber. Preuss. Akad. Wiss. 414 (1925)
80. H.A. Buchdahl (1979) *J. Phys. A* **12** 8 1229
81. M. Ferraris M. Francaviglia C. Reina (1982) *Gen. Relativ. Gravit.* **14** 243
82. M. Ferraris M. Francaviglia I. Volovich (1994) *Class. Quantum Grav.* **11** 1505
83. D.N. Vollick (2003) *Phys. Rev.* **D68** 063510

84. B. Li M.C. Chu (2006) *Phys. Rev. D* **74** 104010
85. Li, B., Chan, K.C., Chu, M.C.: Phys. Rev. D astro-ph/0610794 (2006) (in press)
86. G. Allemandi M. Capone S. Capozziello M. Francaviglia (2006) *Gen. Relativ. Gravit.* **38** 33
87. X. Meng P. Wang (2004) *Gen. Relativ. Gravit.* **36** 8 1947
88. X. Meng P. Wang (2004) *Gen. Relativ. Gravit.* **36** 12 2673
89. A. Einstein (1916) *Ann. der Physik* **49** 769
90. L.P. Eisenhart (1955) Riemannian Geometry Princeton University Press Princeton
91. E. Schrödinger (1960) Space-Time Structure Cambridge University Press Cambridge
92. T. Levi-Civita (1929) The Absolute Differential Calculus Blackie and Son London
93. Klein, O.: New Theories in Physics 77, International Institute of Intellectual Co-operation, League of Nations (1938)
94. E. Cartan (1925) *Ann. Econ. Norm.* **42** 17
95. A. Palatini (1919) *Rend. Circ. Mat. Palermo* **43** 203
96. V.I. Arnold (1978) Mathematical Methods of Classical Mechanics Springer Berlin
97. T. Appelquist A. Chodos P.G.O. Freund (1987) Modern Kaluza–Klein Theories Addison-Wesley Reading
98. M. Kaku (1993) Quantum Field Theory Oxford University Press Oxford
99. M.B. Green J.H. Schwarz E. Witten (1987) Superstring Theory Cambridge University Press Cambridge
100. T. Damour G. Esposito-Farese (1992) *Class. Quant. Grav.* **9** 2093
101. Ferraris, M., Francaviglia, M.: In: Francaviglia, M., (ed.) Mechanics, Analysis and Geometry: 200 Years after Lagrange. Elsevier BV, Amsterdam (1991)
102. S. Capozziello R. de Ritis C. Rubano P. Scudellaro (1996) *Int. J. Mod. Phys. D* **5** 85
103. S. Capozziello R. de Ritis (1994) *Class. Quantum Grav.* **11** 107
104. V. Sahni A. Starobinsky (2000) *Int. J. Mod. Phys. D* **9** 373
105. S. Capozziello V.F. Cardone M. Funaro S. Andreon (2004) *Phys. Rev. D* **70** 123501
106. S. Capozziello V.F. Cardone A. Troisi (2005) *Phys. Rev. D* **71** 043503
107. S. Capozziello V.F. Cardone E. Piedipalumbo C. Rubano (2006) *Class. Quant. Grav.* **23** 1205
108. S. Hannestad E. Mortsell (2002) *Phys. Rev. D* **66** 0635088
109. A. Melchiorri L. Mersini C.J. Odman M. Trodden (2003) *Phys. Rev. D* **68** 043509
110. S. Hannestad E. Mortsell (2004) *JCAP* **0409** 001
111. Y. Song W. Hu I. Sawicki (2007) *Phys. Rev. D* **75** 044004
112. Tsujikawa, S.: astro-ph/0705.1032 (2007)
113. T. Koivisto (2006) *Phys. Rev. D* **73** 083517
114. S. Boughn R. Crittenden (2004) *Nature (Lond)* **427** 45
115. P. Fasalba E. Gaztanaga (2004) *Mon. Not. R. Aston. Soc.* **350** L37

116. M.R. Nolta (2004) *Astrophys. J.* **608** 10
117. Zhang, P., Liguori, M., Bean, R., Dodelson, S.: astro-ph/0704.1932 (2007)
118. S. Capozziello (2007) *Int. J. Geom. Methods Mod. Phys.* **4** 53
119. M. Tegmark (2004) *Phys. Rev. D* **69** 103501
120. Rebolo, R., et al.: astro-ph/0402466 (2004)
121. Daly, R.A., Djorgovski, S.G.: astro-ph/0403664 (2004)
122. W.L. Freedman (2001) *Astrophys. J.* **553** 47
123. V.F. Cardone S. Capozziello V. Re E. Piedipalumbo (2002) *Astron. Astrophys.* **382** 792
124. R. Saunders (2003) *Mon. Not. R. Aston. Soc.* **341** 937
125. R.W. Schmidt S.W. Allen A.C. Fabian (2004) *Mon. Not. R. Aston. Soc.* **352** 1413
126. Y. Wang P. Mukherjee (2004) *Astrophys. J.* **606** 654
127. Y. Wang M. Tegmark (2004) *Phys. Rev. Lett.* **92** 241302
128. W. Hu N. Sugiyama (1996) *Astrophys. J.* **471** 542
129. D. Kirkman D. Tyler N. Suzuki J.M. O'Meara D. Lubin (2003) *Astrophys. J.* **149** 1
130. D. Eisenstein (2005) *Astrophys. J.* **633** 560
131. M.A. Strauss (2005) *Astrophys. J.* **124** 1810
132. SNAP web page: `http://snap.lbl.gov`
133. V. Faraoni (2005) *Class. Quantum Grav.* **22** 32352
134. C. Rubano P. Scudellaro (2005) *Gen. Relativ. Gravit.* **37** 521
135. S. Sasaki (1996) *PASJ* **48** L119
136. U. Pen (1997) *New Aston.* **2** 309
137. S.W. Allen R.W. Schmidt A.C. Fabian (2002) *Mon. Not. R. Aston. Soc.* **334** L11
138. S.W. Allen R.W. Schmidt S. Bridle (2003) *Mon. Not. R. Aston. Soc.* **346** 593
139. Allen, S.W., Schmidt, R.W., Ebeling, H., Fabian, A.C., van Speybrock, L.: astro-ph/0405340 (2004)
140. V. Eke J.F. Navarro C.S. Frenk (1998) *Astrophys. J.* **503** 569
141. W.L. Freedman (2001) *Astrophys. J.* **553** 47
142. Tegmark, M., et al.: astro-ph/0310723
143. N. Dalal K. Abazajian E. Jenkins A.V. Manohar (2001) *Phys. Rev. Lett.* **87** 141302
144. J.A.S. Lima J.S. Alcaniz (2000) *Mon. Not. R. Aston. Soc.* **317** 893
145. Worthey, G.: Ph.D. Thesis. California University, Santa Cruz, 1992
146. R.G. Bower T. Kodama A. Terlevich (1998) *Mon. Not. R. Aston. Soc.* **299** 1193
147. S. Andreon C. Lobo A. Iovino (2004) *Mon. Not. R. Aston. Soc.* **349** 889
148. T. Kodama N. Arimoto (1997) *Astron. Astrophys.* **320** 41
149. S. Andreon (2003) *Astron. Astrophys.* **409** 37
150. J.B. Wyithe A. Loeb D.G. Barnes (2005) *Astrophys. J.* **634** 715
151. M. McQuinn (2006) *Astrophys. J.* **653** 815
152. K. Sigurdson A. Cooray (2005) *Phys. Rev. Lett.* **95** 211303
153. J.P. Blakeslee (2003) *Astrophys. J.* **596** L143
154. R. Cayrel (2001) *Nature* **409** 691
155. R.A. Daly S.G. Djorgovsky (2004) *Astrophys. J.* **612** 652
156. Spergel D.N., et al.: arXiv: astro-ph/0603449 (2006)

157. Capozziello, S., Nojiri, S., Odintsov, S.D., Troisi, A.: Phys. Lett. B 135 (2006)
158. L. Amendola D. Polarski S. Tsujikawa (2007) *Phys. Rev. Lett.* **98** 131302
159. L. Amendola R. Gannouji D. Polarski S. Tsujikawa (2007) *Phys. Rev.* **D75** 083504
160. S. Carloni P. Dunsby S. Capozziello A. Troisi (2005) *Class. Quant. Grav.* **22** 4839
161. Carloni, S., Troisi, A., Dunsby, P.K.S.: gr-qc/0706.0452 (2007)
162. J.C. Hwang H. Noh (1996) *Phys. Rev. D* **54** 1460
163. J.C. Hwang H. Noh (2001) *Phys. Lett. B* **506** 13
164. P. Zhang (2006) *Phys. Rev. D* **73** 123504
165. V.F. Mukhanov H.A. Feldman R.H. Brandenberger (1992) *Phys. Rept.* **215** 203
166. O. Lahav (2002) *Mon. Not. R. Aston. Soc.* **333** 961
167. S. Capozziello V.F. Cardone M. Francaviglia (2006) *Gen. Relativ. Gravit.* **38** 711
168. S. Capozziello Ch. Corda (2006) *Int. J. Mod. Phys. D* **15** 1119
169. S. Capozziello Ch. Corda M. De Laurentis (2007) *Mod. Phys. Lett. A* **22** 1097
170. Capozziello, S., De Laurentis, M., Francaviglia, M.: Int. J. Mod. Phys. D (2007)(submitted)
171. M. Milgrom (1983) *Astroph. J.* **270** 365
172. J. Bekenstein (2004) *Phys. Rev. D* **70** 083509
173. K. Stelle (1978) *Gen. Rel. Grav.* **9** 353
174. S. Capozziello V.F. Cardone S. Carloni A. Troisi (2004) *Phys. Lett. A* **326** 292
175. S. Capozziello V.F. Cardone A. Troisi (2006) *JCAP* **0608** 1423
176. S. Capozziello V.F. Cardone A. Troisi (2007) *Mon. Not. R. Aston. Soc.* **375** 1423
177. W.J.G. de Blok A. Bosma (2002) *Astron. Astroph.* **385** 816
178. Frigerio, C., Martins, P.: Salucci, astro-ph/07032443
179. J. Binney S. Tremaine (1987) Galactic Dynamics Princeton University Books Princeton
180. A. Burkert (1995) *Astrophys. J.* **447** L25
181. A. Burkert J. Silk (1997) *Astrophys. J.* **488** L55
182. A. Borriello P. Salucci (2001) *Mont. Not. Roy. Astron. Soc.* **323** 285
183. G. Gentile P. Salucci (2004) *Mon. Not. Roy. Astron. Soc.* **351** 953
184. S.S. McGaugh (2005) *Astrophys. J.* **632** 859
185. Capozziello, S., Carloni, S., Troisi, A.: Rec. Res. Devel. Astron. Astrophys. **1**, 625 (2003), astro-ph/0303041
186. G. Allemandi A. Borowiec M. Francaviglia (2004) *Phys. Rev. D* **70** 043524
187. D.N. Vollick (2003) *Phys. Rev. D* **68** 063510
188. X.H. Meng P. Wang (2003) *Class. Quant. Grav.* **20** 4949
189. E.E. Flanagan (2004) *Phys. Rev. Lett.* **92** 071101
190. E.E. Flanagan (2004) *Class. Quant. Grav.* **21** 417
191. X.H. Meng P. Wang (2004) *Class. Quant. Grav.* **21** 951
192. G.M. Kremer D.S.M. Alves (2004) *Phys. Rev. D* **70** 023503
193. T. Multamaki I. Vilja (2006) *Phys. Rev. D* **73** 024018
194. G.J. Olmo (2005) *Phys. Rev. Lett.* **95** 261102

195. G.J. Olmo (2005) *Phys. Rev. D* **72** 083505
196. Capozziello, S., Troisi, A.: Phys. Rev. D **72** (2005)
197. S. Capozziello A. Stabile A. Troisi (2006) *Mod. Phys. Lett. A* **21** 2291
198. G. Allemandi M. Francaviglia M. Ruggiero A. Tartaglia (2005) *Gen. Rel. Grav.* **37** 1891
199. Turyshev, S.G., Shao, M., Nordtvedt, K.L.: pre-print: gr-qc/0601035
200. Shapiro, I.I.: In: Ashby, N., et al. (eds.) General Relativity and Gravitation 12. Cambridge University Press, Cambridge (1993)
201. S.S. Shapiro (2004) *Phys. Rev. Lett. D* **92** 121101
202. J.G. Williams (1996) *Phys. Rev. D* **53** 6730
203. B. Bertotti L. Iess P. Tortora (2003) *Nature* **425** 374
204. A. Capolupo S. Capozziello G. Vitiello (2007) *Phys. Lett. A* **363** 53