Journal of Physics: Conference Series



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IOP Conf. Series: Journal of Physics: Conf. Series **839** (2017) 012024 doi:10.1088/1742-6596/839/1/012024

# A Note on the Off-Axis Gaussian Beams Propagation in Parabolic Media

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**Abstract.** The form of the off-axis Gaussian beam is constructed by solving the paraxial wave equation for a parabolic index medium. This construction is accomplished by reducing the problem to the solution of an Ermakov equation. It is shown that the dynamics of the system is comprised in, both, the beam width and the position of the center of the wavepacket, that are defined by the focusing properties of the material. The cases of on-axis behavior and propagation in a homogeneous medium are obtained as special cases.

## 1. Introduction

The description of propagation processes of paraxial beams has been a problem of main interest since old times. For instance, it is well known that Hermite-Gaussian and Laguerre-Gaussian modes are complete sets of solutions to the paraxial wave equation that have a number of applications in the description of laser resonators and waveguides modes [1] and in many other areas of physics [2]. These modes are characterized to be shape-invariant under propagation along the optical axis having transversal optical intensity distributions consistent of rectangular or axial-symmetric patterns with Gaussian envelops. In this context, the description of propagation of Gaussian beams in homogeneous, as well as in graded index media becomes relevant for all applications involving these kind of modes. The Gaussian-type solution of the paraxial wave equation in generic inhomogeneous media has been considered from different approaches [3, 4]. The particular case of a transverse quadratic refractive index is interesting because of its focusing, re-directing and collimation properties [3–6]. In the present work, we deal with the construction of off-axis Gaussian wavepacket-type solutions to the paraxial wave equation for a parabolic medium. It is shown that the problem is reduced to the solution of an Ermakov equation for the beam width together with a classical dynamical law of motion for the center of the wavepacket. Some examples are presented in order to illustrate our results.

## 2. Gaussian beams in quadratic-index media

Consider an inhomogeneous medium whose refractive index profile is given by

$$n(r) = n_0 \left(1 - \frac{\Omega^2}{2} r^2\right), \tag{1}$$

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where $r$ is the transversal radial coordinate. In the weakly-guiding regime we have $\Omega^2 r^2 \ll 1$ and the paraxial approximation is valid. Here $n_0$ is the refractive index along the optical axis, which in this case is chosen as the $z-$axis, and $\Omega \in \mathbb{R}$ is a constant parameter defining the focusing properties of the medium. In the paraxial regime the electric field amplitude of a monochromatic, TE wave has the form $E(\mathbf{r}, z) = U(\mathbf{r}, z)e^{ik_0 n_0 z}$, where $\mathbf{r} = (x, y)$ is the transversal position vector, $k_0$ is the wave number in free space and the function $U(\mathbf{r})$ is such that its second derivative with respect to the longitudinal coordinate can be neglected. Under this conditions $U$ satisfies the paraxial wave equation

$$\frac{i}{k_0}\frac{\partial U}{\partial z} = \left\{ -\frac{1}{2k_0^2 n_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{n_0}{2}\Omega^2 r^2 \right\} U. \tag{2}$$

We are interested in finding Gaussian wavepacket-type solutions to the equation (2) which represent physical beams carrying finite transverse optical power as they propagate along the optical axis. In the off-axis general case

$$U(\mathbf{r}, z) = N(z)\exp\left\{ i\left[ S(z)(\mathbf{r} - \mathbf{r}_0(z))^2 + k_0 \mathbf{p}_0(z)\cdot\mathbf{r} + g(z)\right]\right\}, \tag{3}$$

with $\mathbf{r}_0(z) = (x_0(z), y_0(z))$ the (variable) position of the center of the wavepacket. The normalization factor $N(z)$ and the coefficient $S(z)$ are, in general, assumed to be complex-valued functions of $z$, while the functions $\mathbf{p}_0(z)$ and $g(z)$ are, without loss of generality, supposed to be real-valued. Substituting (3) into (2) we find that $S(z)$ and $N(z)$ satisfy the set of equations

$$\frac{d}{dz}\left(\frac{2S}{k_0 n_0}\right) + \left(\frac{2S}{k_0 n_0}\right)^2 + \Omega^2 = 0, \tag{4}$$

$$\frac{d}{dz}\ln N(z) = -\frac{2}{k_0 n_0}S(z). \tag{5}$$

It also follows from (2) that $\mathbf{p}_0 = n_0 \dot{\mathbf{r}}_0$, meaning that the center of the wavepacket obeys the dynamical law of a symmetric two-dimensional classical harmonic oscillator of frequency $\Omega$

$$\ddot{\mathbf{r}}_0 + \Omega^2 \mathbf{r}_0 = 0, \tag{6}$$

and $g(z) = \frac{k_0 n_0}{2}(\mathbf{r}_0 \cdot \dot{\mathbf{r}}_0)$ (here the dot stands for derivation with respect to $z$).

Note that equation (4) is a complex Riccati equation. Observe also that the imaginary part of $S$ is related to the width $w(z)$ of the wavepacket. Hence, it is convenient to write $S$ in the form [7, 8]

$$\frac{2}{k_0 n_0}S(z) = \frac{1}{R(z)} + \frac{2i}{k_0 n_0 w^2(z)}. \tag{7}$$

Actually, the function $R(z)$ turn out to be the radius of curvature of the wavefront [1, 9]. As this expression is introduced in (4), the imaginary part of this equation leads us to $\frac{1}{R(z)} = \frac{d}{dz}\ln w(z)$, while the real part is transformed into the Ermakov equation

$$\frac{d^2 w}{dz^2} + \Omega^2 w = \frac{4}{k_0^2 n_0^2 w^3}. \tag{8}$$

On the other hand, the expression (7) also provide the form of the normalization factor $N(z)$ through (5). Indeed

$$N(z) = \frac{N_0}{w(z)}e^{-i\chi(z)}, \quad \dot{\chi}(z) = \frac{2}{k_0 n_0 w^2(z)}, \tag{9}$$

where $N_0$ is a constant to be fixed and the function $\chi$, known as the Guoy phase, is a cumulative on-axis phase shift experienced by the wavefront in passing through a focal plane [1] which is determinant in a number of optical phenomena [10] and can be interpreted as a geometric phase [11, 12].

It is well known that the solution of the Ermakov equation $\ddot{w} = M(z)w + \alpha/w^3$, with $\alpha$ a constant, can be explicitly constructed provided a solution of the corresponding linear equation $\ddot{w} = M(z)w$ is known [13, 14]. Thus, in this case, after some algebra we get [8]

$$w(z) = w_0 \left[\cos^2(\Omega z) + \frac{1}{(\Omega z_R)^2}\sin^2(\Omega z)\right]^{1/2}, \qquad z_R = \frac{k_0 n_0 w_0^2}{2}, \tag{10}$$

where $w_0 = w(0)$ is the initial width and $z_R$, known as the Rayleigh range or collimation distance, is a parameter characterizing the divergent nature of the optical beam. On the other hand, from (6), the trajectory of the center of the wave-packet is given by

$$\mathbf{r}_0(z) = \mathbf{a}\cos(\Omega z) + \mathbf{b}\sin(\Omega z)/\Omega, \tag{11}$$

where $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ are, respectively, its initial position and velocity. In this way, the functions $w(z)$ and $\mathbf{r}_0(z)$ encode all the information about the propagation processes of the light beam.

The expression for the Gaussian mode is then (compare to [7])

$$U(\mathbf{r}) = \frac{N_0}{w(z)} e^{-i(\chi(z)+g(z))} e^{\frac{ik_0 n_0 (\mathbf{r}-\mathbf{r}_0(z))^2}{2R(z)}} e^{-\frac{(\mathbf{r}-\mathbf{r}_0(z))^2}{w^2(z)}} e^{ik_0 n_0 \dot{\mathbf{r}}_0(z)\cdot\mathbf{r}}, \tag{12}$$

with

$$R(z) = \frac{\tan(\Omega z)}{\Omega z_R\left(1-\Omega^2 z_R^2\right)}\left[1 + \left(\frac{\Omega z_R}{\tan(\Omega z)}\right)^2\right], \quad \chi(z) = \frac{2}{k_0 n_0}\int^z \frac{dt}{w^2(t)}. \tag{13}$$

Observe the oscillating behavior of these functions inherited from the harmonic nature of $w$ and $\mathbf{r}_0$. This fact is due to the focalization properties of the medium: the beam experiences, at the same time, the spreading effects of wave propagation and the medium confinement, resulting in a periodically self-focusing wave-packet with period $\pi/\Omega$ whose center follows the trajectory of a two-dimensional harmonic oscillator of frequency $\Omega$ in the transversal plane. If the center of the wavepacket is fixed on the optical axis, then $\mathbf{r}_0 = 0$ and we get the on-axis Gaussian mode

$$U(\mathbf{r}) = \frac{N_0}{w(z)} e^{-i\chi(z)} e^{\frac{ik_0 n_0 r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}. \tag{14}$$

In the case that $\Omega = 0$ we recover the well known width, radius of curvature and Guoy phase shift for a Gaussian beam in a homogeneous medium [1, 9]

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2}, \quad R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right], \tag{15}$$

and $\chi(z) = \arctan(z/z_R)$. Notice that in this particular case $w_0$ is the minimum width and so it is called the waist radius. This means that the transversal plane $z = 0$ is actually the focal plane of the system. The beam doubles its cross section at a distance $z = z_R$ from the focus ($w(z_R) = \sqrt{2}w_0$). Within this distance the width remains nearly constant but out of this region it increases as a linear function of $z$ with a total divergence angle given by $2\theta_0 = 2w_0/z_R$ [1]. On the other hand, in the case that $\Omega = 1/z_R$, the divergent nature of the beam is perfectly

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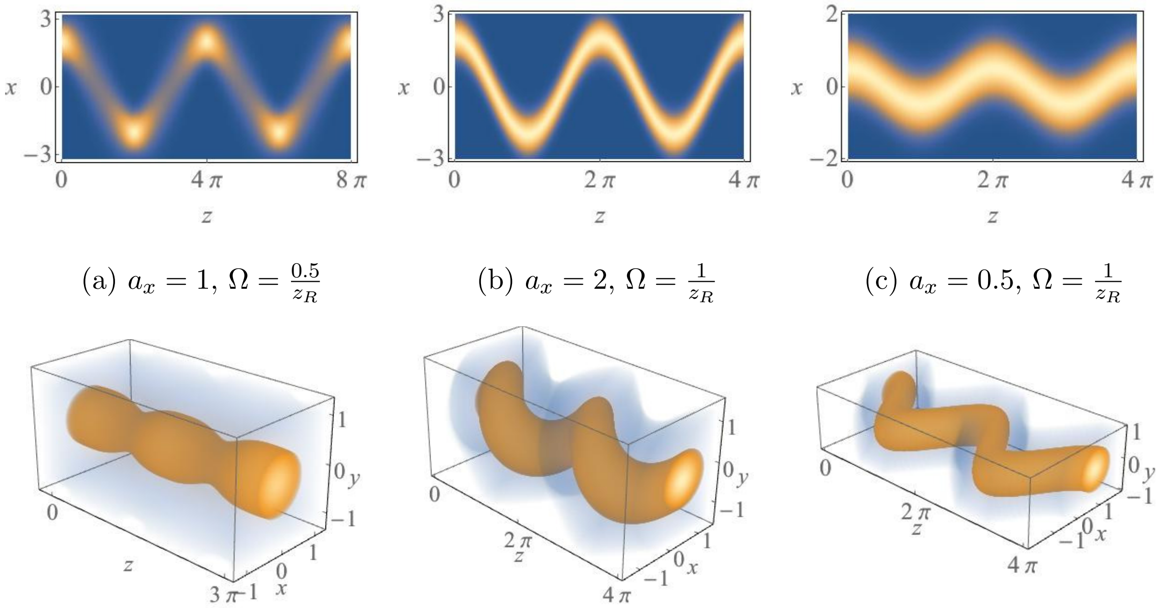


(a) $a_x = 1$, $\Omega = \frac{0.5}{z_R}$ (b) $a_x = 2$, $\Omega = \frac{1}{z_R}$ (c) $a_x = 0.5$, $\Omega = \frac{1}{z_R}$

(d) $a_x = b_y = 0$, $\Omega = \frac{0.7}{z_R}$ (e) $a_x = 0.5, b_y = \frac{0.5}{\Omega}$, $\Omega = \frac{1}{z_R}$ (f) $a_x = 1$, $b_y = 0$, $\Omega = \frac{1}{z_R}$

**Figure 1.** Upper row: Optical intensity distributions $|U(\mathbf{r}, z)|^2$ of the Gaussian modes as functions of $x$ and $z$ in the longitudinal plane $y = 0$ for the parameters indicated in each case. In (a) it can be observed the combination of the off-axis and re-focusing dynamics. Lower row: Optical intensity distributions as functions of all coordinates. In (d) it is shown the on-axis behavior for the case of $\Omega = 0.7/z_R$. In (e) and (f) the maximum of the beam evolves along rotating and oscillating trajectories respectively, according to the values of $a_x$ and $b_y$. The transverse and longitudinal coordinates are expressed, respectively, in units of $w_0$ and $z_R$.

balanced with the focalization properties of the medium and the beam propagates with constant width $w(z) = w_0$.

In Figure 1 we present the optical intensity distribution $|U(\mathbf{r}, z)|^2$ of the Gaussian beam for different values of the parameters. In all cases we have chosen $a_y = b_x = 0$ in order to present clear plots. This is the case in which the center of the beam departs from a point on the $x$-axis with a velocity oriented in the $y$-direction. In the upper row (Figures 1 (a),(b),(c)) it is shown the longitudinal plane $y = 0$ of the distribution for $b_y = 0$ and different values of $a_x$ and $\Omega$. Notice, in (a), that the period of the trajectory of the maximum of intensity is twice the period at which the beam recovers its initial width. In (b) and (c), the parameter $\Omega$ was chosen to match $1/z_R$, and so, the beam propagates with a constant width, following the oscillating dynamics of its maximum. In Figure 1(d) we have set $a_x = b_y = 0$ in order to show the on-axis self-focusing behavior of the optical intensity as a function of coordinates. In (e) and (f), as $\Omega = 1/z_R$, the beam propagates with constant width while it is redirected following, respectively, helical and sinusoidal (plane) trajectories according to the values of $a_x$ and $b_y$.

## 3. Summary

The construction of the general off-axis Gaussian mode, in the paraxial regime, was presented for the case of a medium with a quadratic refractive index profile. This construction was accomplished by reducing the problem to the solution of an Ermakov-type equation for the width of the Gaussian beam. It was shown that the field amplitude is completely defined by the width and the trajectory of the center of the wavepacket, meaning that these two functions are determinant in the description of the propagation processes of the light beam. As special cases, the on-axis Gaussian beam in a quadratic index profile, as well as the parameters for the propagation in a homogeneous medium were obtained for $\mathbf{r}_0 = 0$ and $\Omega \to 0$ respectively. Some examples were presented in order to illustrate our results.

**Acknowledgments**
The financial support from CONACyT (PhD scholarship for Z. Gress, grant 257292) and from Instituto Politécnico Nacional, Mexico through the project SIP20170233 is acknowledged. The authors are indebted to Marco Enríquez for enlightening comments.

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