SPACE-TIME ROTATIONS AND ISOBARIC SPIN

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I

INTRODUCTION

1.1. General outline

According to the principle of special relativity any theory of particles or wave fields must be invariant under rotations in four-dimensional space-time. It is well known that this requirement can be met by the introduction of wave functions having a number of components which transform linearly among themselves under the rotations in question; the transformation properties of a wave function being intimately connected with the spin of the particles it describes. In the familiar formalism for spinning particles, discussing the rotational behaviour of wave functions is essentially a matter of group theory. Conversely, the formalism reveals only those properties of rotations that are covered by the theory of groups. Now it is our idea that rotations might conceivably have some aspects which are not recognized from the study of group theory only. Spinning particles being described with the help of rotation eigenvectors, it is tempting to suggest that new aspects of rotations, should they exist, might be associated with some new spin-type property of matter.

In order to elaborate this idea, we must discuss the transformations of special relativity in terms of six variables. However, in order not to get confused in erratic calculations, it is convenient to start with a general introduction on rotations in n dimensions. This subject is treated in chapter II, where after a brief summary of some of the known properties of infinitesimal rotations it is discussed that for a close study of rotations in n dimensions one needs $\frac{1}{2}n(n-1)$ independent variables. These variables are conveniently taken to be generalizations of the three Eulerian angles known from classical mechanics. It is shown in chapter II how to represent angular-momentum operators by differential operators acting on the generalized Eulerian angles. It is the principal result of this chapter that in consequence of the introduction of Eulerian angles there appear $\frac{1}{2}n(n-1)$ new angular-momentum operators in addition to the $\frac{1}{2}n(n-1)$ familiar ones. Each of the new operators commutes with each of the familiar ones. For the sake of distinctness we note that in the present investigation the Eulerian angles are intended as spin variables. Accordingly the operators under consideration refer to the spin, rather than to the orbital motion.

Since mathematically it is advantageous to study three-dimensional rotations before proceeding to the Lorentz group, chapter III is devoted to explicit calculations on rotations in three dimensions. In this chapter we construct functions of three Eulerian angles, denoted by $Y(\omega)$, which are to replace the spinors of the usual three-dimensional theory. Contrary to what one might be led to expect, we do find appropriate functions both for integral and half-integral values of the spin. Some of the results of chapter III have already been obtained by previous authors (Reiche and Rademacher 1926, 1927, Kronig and Rabi 1927, Casimir 1931) in connection with the theory of rotations of a rigid body. As a matter of fact, the functions $Y(\omega)$ are known as angular-momentum eigenfunctions of a rigid rotator. However, with a view to the applications which we have in mind chapter III has been cast in a form rather different from what may be found elsewhere.

In the usual spinor formalism for three dimensions, any spin value j is associated with but one family of $2j+1$ spinors which under rotations transform linearly among themselves. By contrast, in chapter III we find $2j+1$ distinct families of functions for any j, each family consisting of $2j+1$ functions of the Eulerian angles. Under the usual rotations these functions transform among members of their own families only, the families as such do not mix; each of the $2j+1$ families of functions of a particular j transforms just as the corresponding family of spinors. Yet these $2j+1$ families are markedly different, and we can label them by a family-index. This family-index reveals, of course, the new property of rotations alluded to above. It is the eigenvalue of one of the new angular-momentum operators met in chapter II.

Whereas the theory of spatial rotations has no direct bearing on relativistic wave functions, it does provide us with some mathematical tools indispensable for the study of rotations in Minkowski space. Owing to the particular structure of the Lorentz group, the results of chapter III may be readily adapted to the description of rotations in four dimensions. This point is discussed in chapter IV, where the concept of Eulerian angles is extended to the transformations of special relativity. In chapter IV we express rotations in Minkowski space in terms of six angles and we show how to incorporate the spatial reflection in that scheme. By suitable choice of the generalized Eulerian angles, the function-analogues of relativistic spinors are nothing but sums of products of functions $Y(\omega)$ already encountered in chapter III. Again, we find quite a number of families of functions, several families now showing the same behaviour under all transformations of the full Lorentz group. The number of families is even larger than in the three-dimensional case. It turns out, in fact, that two family-indices are required rather than one.

The family-indices are shown to be the eigenvalues of a new three-dimensional angular-momentum operator, S', and of one component thereof, conveniently denoted by S'_3. The angular momentum S' is again of the type met in chapter II. All three components of S' commute with all transformations of the full Lorentz group. As a matter of fact, it is shown in chapter V that they commute with all operators occurring in the theory of free particles. It thus seems natural to suggest that S' might represent a new spin. A typical eigenvalue of $(S')^2$ being denoted by $s'(s'+1)$, there might conceivably exist $2s'+1$ different kinds of particles which all have the same mass, spin, parity, and s', but are distinguished from each other by some quantity closely related to S'_3. It is an essential feature of our theory that s' should be integral for bosons, half-integral for fermions.

Concerning a new spin it is well known that the description of nuclear interactions and pion-nucleon processes is greatly simplified if in addition to the ordinary spin one formally attributes a so-called isobaric spin to nucleons and π-mesons; the charge independence of nuclear forces then amounts to the conservation of the isobaric spin. However, whereas empirically the isobaric spin is considered to be a meaningful concept, its very occurrence is far from understood at present. It is customary to speak of rotations in a three-dimensional isobaric-spin space, but it is now known whether or how such a space might be interpreted. It is now suggested in chapter VI that S' might represent the isobaric spin. According to this idea, the isobaric-spin space would be embedded in Minkowski space. Charge independence would correspond to the conservation of s', and S'_3 would be closely related to the charge operator. On adopting this tentative interpretation of S' the current theory of pion-nucleon interactions is readily translated into a formalism in terms of Eulerian angles.

Whereas it is now well established that the concept of isobaric spin plays a most useful part in the description of pion-nucleon phenomena, it is not known whether the isobaric-spin formalism should be extended beyond the theory of nucleons and π-mesons. At the same time, it has been suggested by Gell-Mann (1953) and by Nakano and Nishijima (1953) that among newly discovered mesons and heavy unstable particles there might be bosons of half-integral isobaric spin as well as fermions of integral isobaric spin. The quantity s' being integral for bosons and half-integral for fermions, the classification of Gell-Mann and Nakano and Nishijima, should it be correct, would clearly rule out the possibility of identifying S' with the isobaric spin. The underlying ideas of this classification are therefore critically discussed in chapter VI, particular attention being paid to the question of charge independence. It is our conclusion that the available experimental evidence is not incompatible with the hypothesis that S' might represent the isobaric spin. However, much experimental and theoretical work will be required before the possible interpretation of S' can be established.

1.2. Related investigations

Owing to the fact that with two spherical angles in three-dimensional space one cannot construct eigenfunctions appropriate to a half-integral angular momentum, it is usually believed that for discussing spin phenomena one must inevitably introduce such abstract quantities as spinors. On the other hand, Bopp and Haag (1950) and Rosen (1951) have already pointed out that in a three-dimensional theory a spinning particle may be visualized as a rotating entity the orientation of which is described through Eulerian angles. These authors have represented spin operators by differential operators acting on three angles, and they have indicated the functions $Y(\omega)$ mentioned in section 1.1. However, it has not been suggested either by Bopp and Haag or by Rosen that Eulerian angles might be connected with a new spin. Further, their considerations are not relativistically invariant.

In a paper on the internal structure of spinning particles, Finkelstein (1955) has investigated the symmetry properties of rigid structures that rotate in Minkowski space. In his work up to six degrees of freedom are taken into account. However, Finkelstein's results are based on group-theoretical considerations, and he has not specified six variables explicitly. His paper does therefore not contain new spin operators, neither the family-indices associated therewith.

As the results of the present investigation are attained by using six variables in Minkowski space, it is interesting that as early as 1896 Klein introduced so-called Cayley-Klein parameters appropriate to what are now known as rotations in Minkowski space. At that time these quantities were presented as beautiful mathematical generalizations of the known parameters for three dimensions, with Klein's added assurance that "the non-Euclidean geometry has no metaphysical significance here or in the subsequent discussion". Klein's parameters are closely related to the Eulerian angles used in chapter IV, about in the same way as the corresponding three-dimensional quantities, which have been discussed by Whittaker (1944), secs. 10, 12. It seems that the Cayley-Klein parameters for Minkowski space have never been used for practical applications. As far as we know angular-momentum eigenvectors of six degrees of freedom have not been given previously.

Whereas in the following the spin is described in terms of six Eulerian angles, Yukawa (1950a, b, 1953a, b) has proposed a theory of so-called non-local fields which introduces spin phenomena through rectangular coordinates. Now four rectangular, i.e. three angular, variables in Minkowski space cannot describe half-integral spins, neither do they suggest a new spin. Yukawa's and our ideas are therefore not directly related, but they might mutually supplement each other in future investigations. We briefly return to this point in section 5.4.

Concerning the isobaric spin it has been suggested by Pais (1953a, b, 1954a, b) that formally there might also be an isobaric orbital angular momentum. Further Pais (1954a, b) has considered the possibility of a four-dimensional isobaric space, rather than the usual three-dimensional one, with the resultant suggestion that there might even be a third spin. In order to avoid confusion we note that whereas at first sight the present investigation might somewhat resemble Pais's work, it is not related thereto. In particular, in Pais's papers the isobaric space is introduced ad hoc, not being related to Minkowski space.

Perhaps one would ask at this point why in a four-dimensional isobaric space Pais finds two isobaric spins, whereas the usual description of spinning particles in four-dimensional space-time yields only one ordinary spin. The answer to this is quite simple. The six angular-momentum operators appropriate to a four-dimensional space, be it Euclidean or pseudo-Euclidean, can indeed be grouped so as to give two three-dimensional angular momenta. This fact is used in chapter IV, where the three-dimensional angular-momentum operators in question are denoted by P^+ and P^-. Now loosely speaking in Pais's isobaric space the isobaric spins are of types P^+ and P^-. However, if P^+ and P^- refer to operations in Minkowski space, neither of them commutes with the spatial reflection. In that case they can therefore not be used separately to characterize an elementary particle. In Minkowski space only the sum $P^+ + P^-$ commutes with the spatial reflection, and as a result there is but one ordinary spin. Since in Pais's theory the isobaric space is not related to ordinary space-time, a reflection need not be considered there. In order to fit an additional spin in Minkowski space one cannot proceed along a line similar to the one indicated by Pais.

After completion of the present work it turned out that in an attempt at developing a relativistic field theory appropriate to an extended particle Nakano (1956) has remarked: "When one quantizes motion of a rigid sphere, one obtains an internal angular momentum which is invariant under Lorentz transformation of outer space and it might be identified with the isobaric spin operator." Nakano's angular momentum is closely related to our spin S'. The truth of his idea is considerably obscured, however, by Nakano's deriving his results from a rather specialized Hamiltonian, in which the internal structure of the particle in question is represented by an angular momentum in five dimensions, μ. Through Nakano's equations (4.3) and (4.4) μ is related to an angular momentum τ, which likewise is of five dimensions. At the end of his section 6 it is suggested that τ might be related to the isobaric spin. At the same time it is remarked, however, that the equations (4.3) and (4.4) would then be incompatible with ideas on the isobaric spin presented by Gell-Mann (1953) and by Nakano himself together with Nishijima (1953). Perhaps this is the reason why the isobaric-spin idea is only casually mentioned.

Since Nakano's paper aims at constructing a new sort of wave equation, its approach has no connection with the present one. As a matter of fact, one gets the impression that according to Nakano's ideas the occurrence of the angular momentum τ results from his introducing a new Hamiltonian. In the present investigation wave equations are not modified; a proton wave function satisfies Dirac's equation just as usual. It follows from general considerations on Lorentz transformations that the proton might have a sister-particle which is likewise described by Dirac's equation.

It is a further difference between Nakano's and our work that τ is an angular momentum in five dimensions, having ten components, whereas S' is of three dimensions only. Now it is not out of the question that Nakano could also have developed a formalism with a four-dimensional τ. However, there is no experimental evidence either for a five- or for a four-dimensional isobaric spin. It will be observed from our chapter IV that as regards "isobaric-spin" operators in Minkowski space we start from an angular momentum in four dimensions, J'. This quantity is subsequently reduced to a three-dimensional one, S', by the requirement of invariance under spatial reflection. At this point it should be remarked that in Nakano's paper the behaviour of τ under spatial reflection is not considered. As a matter of fact, we feel that his work does not contain the mathematical apparatus required to this end. Even if τ were four-dimensional one could therefore not establish a satisfactory correspondence between τ and the isobaric spin from Nakano's work as it stands.

Recently it has been asserted by Rayski (1956) that the isobaric spin might be explicable within the framework of his bilocal field theory (Rayski 1955), which essentially is a version of Yukawa's (1950a, b, 1953a, b) theory of non-local fields. Writing x for the position operator divided by l, and d for the non-local anologue of $l\partial/\partial x$, l being some fundamental length, Rayski has proposed to consider the transformations $x' = \alpha x + \beta d$, $d' = \gamma x + \delta d$ satisfying the condition $\alpha\delta - \beta\gamma = 1$. These transformations are called canonical in Rayski's paper. According to Rayski the wave equations of his bilocal theory are invariant under unitary canonical transformations, and as a result the wave functions must have a number of components which transform linearly among themselves under such transformations. Now it is well known that group-theoretically there exists a correspondence between the unitary transformations of the type considered by Rayski and the rotations in three dimensions, cf. van der Waerden (1932), sec. 16. It is therefore argued by Rayski that his canonical transformations might correspond to the rotations in isobaric-spin space. However, on checking Rayski's suggestion it turns out that his equations are only invariant under the canonical transformations having coefficients α, β, γ, and δ of the special form $\cos\theta$, $\sin\theta$, $-\sin\theta$, and $\cos\theta$, in that order. As the transformations satisfying this restriction correspond to the rotations in a fixed plane, rather than to all three-dimensional rotations, Rayski's idea is therefore not rich enough to account for the full group of rotations in isobaric-spin space.

As another attempt at explaining the isobaric spin we have to discuss a paper by Schremp (1955) on wave functions for spin $\frac{1}{2}$. In Schremp's paper a four-component Dirac wave function is split into two two-component parts, and while the wave equation is left unchanged each of these parts is replaced by a quaternion. Once this transition has been carried out it is not difficult to introduce new spin-type operators which commute with the ordinary spin, and these are then interpreted as the operators representing the isobaric spin. Now it is clear that Schremp has indicated a possible notation for two commuting spins. However, such a notation has long been known, as may be seen from the conventional description of the isobaric spin. The crucial point is that in his summary Schremp asserts: "By a slight modification of the algebraic foundations of spin-$\frac{1}{2}$ theory, isotopic spin is incorporated therein, in a natural and irreducible manner. The method of approach proceeds from a consideration of the geometry of the group-space of the proper Lorentz group." However, Schremp's paper covers one page only, and it does not motivate the crucial transition from spinors to quaternions. In his references we could not find an explanation of Schremp's ideas either, so that all things considered we do not see an intrinsic relation between two spins in his work. Schremp's paper is only concerned with Dirac particles, his new spin taking the value $\frac{1}{2}$.

Schremp's work is unusual in that it tries to incorporate the isobaric spin in the framework of special relativity. This is generally believed to be impossible, and it has been suggested that in order to find a fundamental theory with two spins one would have to revise the current description of space-time. In this connection we mention a series of papers by Goto (1954a, b, c), who in the hope of coming across the isobaric spin has considered the de Sitter group and the inhomogeneous Lorentz group. It is claimed by Goto (1954b) that invariance under de Sitter transformations implies that any wave function is characterized by two sorts of spin-like quantum numbers. However, it must be noted that following an idea due to Dirac (1935) Goto has used the four additional angular-momentum operators in de Sitter space as the de Sitter analogues of the linear momenta and the energy in Minkowski space. This in itself is a very interesting possibility, but serving in that way these de Sitter operators cannot represent a new spin in addition.

Whereas in Goto's paper on the de Sitter group a particle of spin $\frac{1}{2}$ is described by a four-component wave function, in his work on the inhomogeneous Lorentz group (Goto 1954c) an eight-component wave equation for spin $\frac{1}{2}$ is proposed. Obviously this equation permits of eight independent solutions for any particular momentum; it is not difficult to see that in the absence of interactions four of these correspond to energy E, say, and four to energy $-E$, as it should be. However, in an external electric field all eight energy values are shifted by the same amount, so that one cannot say that the theory pictures protons and neutrons as two different states of one eight-component particle. There seems to be no alternative method of introducing electromagnetic interactions in Goto's wave equation in a relativistically invariant way, so that apparently the extra degree of freedom has nothing to do with the isobaric spin. It is not quite clear from Goto's work exactly how his equation should be interpreted. At the same time, it is not impossible that his or similar investigations may reveal possible new properties of elementary particles, which may be related to new spins. However, such investigations are essentially different from the work presented here, as the following ideas are all developed within the frameworks of ordinary quantum mechanics and special relativity.

II

ROTATIONS AND GENERALIZED EULERIAN ANGLES

2.1. Rotation operators

In the present chapter we shall be concerned with a metric space of n dimensions, in which coordinates x^a ($a = 1, 2, \ldots, n$) have been introduced. The metric tensor g will have covariant components

$$g_{aa} = \pm 1, \quad g_{ab} = 0 \quad (a \neq b). \tag{2.01}$$

For the squared distance between the origin and the point having coordinates x we write

$$g_{ab} x^a x^b, \tag{2.02}$$

summation being implied over indices occurring twice. As usual the contravariant components of g shall satisfy

$$g_{ab} g^{bc} = \delta_a^c, \tag{2.03}$$

where δ_a^c is the Kronecker symbol. From (2.03) it follows that g_{ab} and g^{ab} are numerically equal. With the help of the metric tensor we shall occasionally raise or lower indices according to

$$x_a = g_{ab} x^b, \quad x^a = g^{ab} x_b. \tag{2.04}$$

We shall be interested in real linear transformations

$$x'^a = \Lambda^a{}_b \, x^b \tag{2.05}$$

satisfying the condition

$$g_{ab} x^a x^b = g_{cd} x'^c x'^d. \tag{2.06}$$

From (2.05) and (2.06) it is readily seen that we must have

$$x^a = \Lambda_b{}^a \, x'^b, \tag{2.07}$$

$$\Lambda^{ca} \Lambda_{cb} = g_b^a, \quad \Lambda^{ac} \Lambda_{bc} = g_b^a. \tag{2.08}$$

Taking determinants in (2.08) yields $\det\Lambda = \pm 1$. The transformations for which $\det\Lambda = 1$ constitute the class of proper rotations, those having $\det\Lambda = -1$ involve a reflection.

Among the transformations (2.05) the infinitesimal rotations are most readily accessible to detailed investigations. These infinitesimal rotations are described by coefficients Λ of the special form

$$\Lambda_{ab} = g_{ab} + (\delta\omega)_{ab}, \tag{2.09}$$

where it is understood that the $\delta\omega$'s are infinitesimals such that products of two or more $\delta\omega$'s are negligible. From (2.08) we obtain the condition

$$(\delta\omega)_{ab} = -(\delta\omega)_{ba}, \tag{2.10}$$

which implies that any infinitesimal rotation can be expressed in terms of $\frac{1}{2}n(n-1)$ parameters $\delta\omega$.

It is often convenient to use the symbolic notation

$$x' = \Lambda x, \tag{2.11}$$

which stands for

$$x'^{a} = \Lambda x^{a} = \Lambda^{a}_{\ b}\, x^{b}. \tag{2.12}$$

Adopting that notation we shall represent infinitesimal rotations by

$$\Lambda = 1 - \tfrac{1}{2}i(\delta\omega)_{ab}P^{ab} \qquad (P^{ab} = -P^{ba}), \tag{2.13}$$

where the quantities P are operators, rather than numbers. On specializing to the particular choice for the $\delta\omega$'s which gives Λ the form $1 - i(\delta\omega)P^{ab}$ it is readily seen with (2.09) and (2.13) that

$$P^{ab}x^{c} = ig^{ca}x^{b} - ig^{cb}x^{a}. \tag{2.14}$$

It will become clear from our further work that the operators P represent the components of an angular momentum. In order to avoid the imaginary unit i occurring in the preceding formulas, several authors prefer to discuss rotations in terms of operators iP, rather than operators P. However, since in the present investigation we are mainly interested in angular momenta, it is more convenient here to use the P's from the outset. Though the P's essentially represent an angular momentum, we shall occasionally call them rotation operators all the same.

According to the foregoing formulas, $P^{ab}x$ is proportional to the change effected in x by an infinitesimal rotation in the coordinate plane containing the axes x^a and x^b. Likewise, if $f(x)$ is any reasonable function, one writes

$$\delta f(x) \equiv f([1 - \tfrac{1}{2}i(\delta\omega)_{ab}P^{ab}]x) - f(x) = -\tfrac{1}{2}i(\delta\omega)_{ab}P^{ab}f(x), \quad (2.15)$$

products of $\delta\omega$'s being neglected. To first order in the $\delta\omega$'s we also have

$$\delta f(x) = -\tfrac{1}{2}i(\delta\omega)_{ab}[\partial f(x)/\partial x^c]P^{ab}x^c, \quad (2.16)$$

from which it follows that

$$P^{ab}f(x) = [\partial f(x)/\partial x^c]P^{ab}x^c. \quad (2.17)$$

If the operators P act on functions $f(x)$ according to (2.14) and (2.17), they must satisfy the commutation relations

$$[P^{ab},P^{cd}] = ig^{ac}P^{bd} - ig^{ad}P^{bc} - ig^{bc}P^{ad} + ig^{bd}P^{ac}. \quad (2.18)$$

According to (2.18) P^{ab} and P^{cd} commute in case they have either none or both indices in common. If $a \neq c$ we may write

$$[P^{ab},P^{bc}] = -ig^{bb}P^{ac} \qquad (a \neq c). \quad (2.19)$$

In (2.14) and (2.17) P^{ab} may be represented by a differential operator according to

$$P^{ab}f(x) = -ix^a\,\partial f(x)/\partial x_b + ix^b\,\partial f(x)/\partial x_a. \quad (2.20)$$

However, in many physical applications rotation operators are used which cannot be written in the form of differential operators. For instance, in the theory of spinning particles the rotation operators pertaining to the spin are usually represented by matrices satisfying the commutation relations (2.18) and (2.19). In the following such matrices will be denoted by P as well.

The operators P transform as the components of an antisymmetric tensor. If it is understood that the P's commute with Λ, the operator P'^{ab} given by

$$P'^{ab} = \Lambda^a_{\ c}\Lambda^b_{\ d}P^{cd} \quad (2.21)$$

satisfies

$$P'^{ab}x'^c = ig^{ca}x'^b - ig^{cb}x'^a, \quad (2.22)$$

$$[P'^{ab},P'^{bc}] = -ig^{bb}P'^{ac} \qquad (a \neq c). \quad (2.23)$$

Since in the particular case of three dimensions an antisymmetric tensor transforms just as a (pseudo)vector, it is convenient to denote the three-dimensional operator P^{kl} by P_j (j,k,l = 1,2,3 cycl.). If $g^{ll} = 1$, the operators P_l satisfy the familiar commutation relations for an angular momentum in three dimensions, viz.

$$[P_j, P_k] = iP_l \qquad (j, k, l = 1,2,3 \text{ cycl.}). \tag{2.24}$$

As previously noted, an infinitesimal rotation in the x^a, x^b-plane can be described by the operator $1 - i(\delta\omega)P^{ab}$. Accordingly the operator $\exp(-i\omega P^{ab})$ represents a finite rotation. Repeated application of (2.14) shows, in fact, that

$$\begin{aligned}
&\left.\begin{aligned}\exp(-i\omega P^{ab})x^a &= x^a\cos\omega \pm x^b\sin\omega,\\ \exp(-i\omega P^{ab})x^b &= \mp x^a\sin\omega + x^b\cos\omega,\end{aligned}\right\} \quad (g^{aa} = g^{bb} = \pm 1),\\
&\left.\begin{aligned}\exp(-i\omega P^{ab})x^a &= x^a\cosh\omega + x^b\sinh\omega,\\ \exp(-i\omega P^{ab})x^b &= x^a\sinh\omega + x^b\cosh\omega,\end{aligned}\right\} \quad (g^{aa} = 1,\ g^{bb} = -1).
\end{aligned} \tag{2.25}$$

Rotation operators and their application in quantum mechanics have been discussed by van der Waerden (1932), Corson (1953), and others.

2.2. Generalized Eulerian angles

We have seen in the preceding section that any infinitesimal rotation in n dimensions can be characterized by means of $\frac{1}{2}n(n-1)$ parameters $\delta\omega$. It is not difficult to show that the most general finite rotation in n-dimensional space can also be described by $\frac{1}{2}n(n-1)$ parameters.

In order to investigate this point somewhat precisely, we consider a right-handed system of rectangular coordinate axes x fixed in space, and, in addition, a right-handed system of rectangular axes x' which may be rotated about the common origin of x and x'. Every possible orientation of the axes x' thus corresponds to a transformation $x' = \Lambda x$ having $\det\Lambda = 1$.

Our object is now to show that the most general orientation of the axes x' can be expressed in terms of $\frac{1}{2}n(n-1)$ angles. To make this clear we first characterize the orientation of the x'^n-axis by means of $n-1$ angles. Next we introduce $n-2$ more angles in order to describe the orientation of the x'^{n-1}-axis in the space perpendicular to the x'^n-axis. Proceeding in this way we must use one angle for the x'^2-axis, after which we are left with a line perpendicular to the axes $x'^n, x'^{n-1}, \ldots, x'^2$. Since the direction of the x'^1-axis along this line is determined by the condition $\det\Lambda = 1$, i.e. by the condition that we must have a proper rotation, it follows that the orientation of the axes x' can be described indeed with the help of $\frac{1}{2}n(n-1)$ angles.

The $\frac{1}{2}n(n-1)$ angles under consideration are generalizations of the three Eulerian angles known from classical mechanics. Irrespective of their precise definitions they will therefore be referred to as Eulerian angles. Introducing the symbol ω_r ($r = 1,2,\ldots,\frac{1}{2}n(n-1)$) for these Eulerian angles, we shall henceforth consider Λ as a function of the ω's, $\Lambda = \Lambda(\omega)$.

Let us now choose a particular point A' having coordinates $x'(A')$. For any given values of the Eulerian angles the coordinates $x(A')$ of that point may be computed from

$$x'(A') = \Lambda(\omega)x(A'), \qquad x(A') = \Lambda^{-1}(\omega)x'(A'). \tag{2.26}$$

In other words, the coordinates x may be considered as functions of x' and ω, the latter quantities varying independently. Changing x' then implies a transition from the point A' to a point B' the coordinates $x'(B')$ of which are likewise independent of the Eulerian angles. Varying ω amounts to a rotation of the system of axes x' as a whole with respect to the axes x, A' being kept fixed in the system x'.

2.3. The rotation operators J

In the present section we shall consider rotations of the system x' with respect to the system x. We shall particularly investigate rotations in the coordinate planes containing two axes x. It will be clear from the preceding paragraph that such rotations correspond to special transformations of the Eulerian angles. We shall now construct rotation operators which generate these transformations. In the following these operators will be denoted by J, J^{ab} pertaining to the motion of a point A' fixed in x'-space under a rotation of the system x' in the x^a,x^b-plane.

Obviously J^{ab} must transform x, i.e. the function $\Lambda^{-1}(\omega)x'$, just like P^{ab}. At the same time, there is a fundamental difference between J and P. For in deriving (2.23), for instance, it has been assumed that the operators P commute with Λ. Accordingly a rotation generated by operators P transforms x and x' while it leaves Λ unchanged. As the Eulerian angles are invariant under such a rotation, the operators P can also be used in a formalism in which moving axes x' and Eulerian angles ω are not considered. The rotations generated by operators J, on the other hand, primarily affect Λ. If x is interpreted as a function of the independent variables x' and ω, they indirectly transform x also.

They do not change x' then, however. It will be clear that the operators J occur only in a formalism with moving axes and Eulerian angles.

Being rotation operators, the J's shall satisfy

$$f([1 - \tfrac{1}{2}i(\delta\omega)_{ab}J^{ab}]\omega) - f(\omega) = -\tfrac{1}{2}i(\delta\omega)_{ab}J^{ab}f(\omega) \tag{2.27}$$

to first order in the angles $\delta\omega$. Now just as (2.17) follows from (2.15), it follows from (2.27) that we must have

$$J^{ab}f(\omega) = \sum_r [\partial f(\omega)/\partial\omega_r]J^{ab}\omega_r, \tag{2.28}$$

i.e. that the J's must be linear differential operators. Further, since J^{ab} is to describe the change undergone by $x(A')$ and $x'(A')$ if the system x' performs a rotation in the x^a, x^b-plane, we must have

$$\begin{aligned} J^{ab}x^c &= ig^{ca}x^b - ig^{cb}x^a, \\ J^{ab}x'^c &= 0, \end{aligned} \tag{2.29}$$

where it is understood that x is to be considered as a function of the independent variables x' and ω. It will be shown presently that there exists one and only one set of operators J satisfying (2.28) and (2.29).

Under the restrictions of the present section we obviously have $\partial x'/\partial\omega = 0$. In order to obtain explicit expressions for the J's we now apply the operator $\partial/\partial\omega$ to x, where $x = \Lambda^{-1}(\omega)x'$. With the help of (2.05) and (2.07) we first write

$$-i\partial x^c/\partial\omega_r = -ix'_d\,\partial\Lambda^{dc}/\partial\omega_r = -ix^a\Lambda_{da}\,\partial\Lambda^{dc}/\partial\omega_r. \tag{2.30}$$

Next we differentiate the first equation (2.08) with respect to ω to find that

$$\Lambda_{da}\,\partial\Lambda^{dc}/\partial\omega_r + \Lambda^{dc}\,\partial\Lambda_{da}/\partial\omega_r = 0. \tag{2.31}$$

On combining (2.30) and (2.31) a simple calculation now shows that

$$-i\partial x^c/\partial\omega_r = \tfrac{1}{2}(\Lambda_{da}\,\partial\Lambda^d{}_b/\partial\omega_r)\,(ig^{ca}x^b - ig^{cb}x^a), \tag{2.32}$$

which, by virtue of (2.29), may be simplified to

$$-i\partial x^c/\partial\omega_r = \tfrac{1}{2}(\Lambda_{da}\,\partial\Lambda^d{}_b/\partial\omega_r)J^{ab}x^c. \tag{2.33}$$

In connection with (2.28) we have seen that the J's must be linear differential operators if they exist at all. Hence if it is indeed possible to introduce rotation operators J, then according to (2.33) the x's must satisfy differential equations of the form

$$\sum_r B_r(\omega)\partial x^c/\partial\omega_r = 0. \tag{2.34}$$

Since (2.34) must be satisfied for any x', we must also have

$$\sum_r B_r(\omega)\partial\Lambda^{dc}(\omega)/\partial\omega_r = 0 \tag{2.35}$$

and, on multiplying by $\Lambda_d^{\ a}$,

$$\sum_r B_r(\omega)\Lambda_d^{\ a}(\omega)\partial\Lambda^{dc}(\omega)/\partial\omega_r = 0. \tag{2.36}$$

Now it follows from (2.31) that in (2.36) the coefficient of B_r is antisymmetric in a and c. Hence (2.36) can be considered as a system of $\frac{1}{2}n(n-1)$ equations for the $\frac{1}{2}n(n-1)$ unknowns B_r. In practical cases the determinant of this system does not vanish identically, and the only solution of (2.36) is $B_r \equiv 0$. By (2.33) the operators J must therefore satisfy

$$\tfrac{1}{2}(\Lambda_{da}\partial\Lambda^d{}_b/\partial\omega_r)J^{ab} = -i\partial/\partial\omega_r. \tag{2.37}$$

Conversely, if the J's satisfy (2.37) they also satisfy (2.28) and (2.29), so that it is indeed possible to introduce rotation operators J in a consistent way. In practical cases the system of linear equations (2.37) is readily solved for the J's to give

$$J^{ab} = -i\sum_r \Omega_r^{ab}(\omega)\partial/\partial\omega_r, \tag{2.38}$$

where the zeros of

$$\det[\Lambda_{da}(\omega)\partial\Lambda^d{}_b(\omega)/\partial\omega_r] \tag{2.39}$$

appear as simple poles of some functions $\Omega(\omega)$.

For future reference we note that according to (2.29) we have

$$J^{ab}\Lambda^{dc} = ig^{ca}\Lambda^{db} - ig^{cb}\Lambda^{da}, \tag{2.40}$$

which means that J^{ab} transforms Λ^{dc} just as P^{ab} transforms x^c.

2.4. The rotation operators J'

As any set of rotation operators, the J's transform as the components of an antisymmetric tensor. If keeping both x and x' fixed we temporarily introduce an auxiliary system $\bar{x}$ according to

$$\bar{x}^a = \Gamma^a{}_b x^b = \Gamma^a{}_b\Lambda_c^{\ b}x'^c, \tag{2.41}$$

Γ representing a rotation, the operator $\bar{J}^{ab}$ given by

$$\bar{J}^{ab} = \Gamma^a_{\ c}\,\Gamma^b_{\ d}\,J^{cd} \tag{2.42}$$

is the rotation operator appropriate to the motion of a point A' fixed in x'-space under a rotation in the $\bar{x}^a,\bar{x}^b$-plane. If x' and ω are taken as independent variables, $\bar{J}^{ab}$ satisfies

$$\begin{aligned} \Gamma^c_{\ d}\,\bar{J}^{ab}x^d &= ig^{ca}\bar{x}^b - ig^{cb}\bar{x}^a, \\ \bar{J}^{ab}x'^c &= 0. \end{aligned} \tag{2.43}$$

If Γ does not depend on the Eulerian angles, the first equation (2.43) may also be written in the form

$$\bar{J}^{ab}\bar{x}^c = ig^{ca}\bar{x}^b - ig^{cb}\bar{x}^a. \tag{2.44}$$

On specializing to the particular case that the axes $\bar{x}$ coincide with the axes x', we find

$$J'^{ab} = \Lambda^a_{\ c}\,\Lambda^b_{\ d}\,J^{cd} \tag{2.45}$$

for the component of J in the x'^a,x'^b-plane. It should be noted that for the operators J' the analogue of (2.44) is not valid. None the less J'^{ab} may well be used to describe the change effected in $x(A')$ if the x'-system performs a rotation in the x'^a,x'^b-plane.

For future reference we note that with (2.08) it follows that

$$J^{ab}J_{ab} = J'^{cd}J'_{cd}. \tag{2.46}$$

From (2.37) it is readily deduced with (2.31) that J' satisfies

$$-\tfrac{1}{2}(\Lambda_{ad}\,\partial\Lambda_b^{\ d}/\partial\omega_r)J'^{ab} = -i\,\partial/\partial\omega_r. \tag{2.47}$$

For the sake of distinctness we recall that thus far we have only considered the motion of a point A' fixed in x'-space, x' and ω varying independently. For discussing the motion of a point A having fixed coordinates x we must take x and ω as independent variables. Now obviously there exists an intimate relation between the motions of points A' and A in, say, the x'^a,x'^b-plane. However, it is not correct to interpret J'^{ab} as the rotation operator appropriate to the motion of a point A. For denoting the operator which generates rotations of A in the x'^a,x'^b-plane by K'^{ab}, we must have analogously to (2.29)

$$\begin{aligned} K'^{ab}x'^c &= ig^{ca}x'^b - ig^{cb}x'^a, \\ K'^{ab}x^c &= 0, \end{aligned} \tag{2.48}$$

where now x' must be considered as a function of the independent variables x and ω. A simple calculation after the pattern of the preceding section shows that the relation analogous to (2.37) reads

$$\tfrac{1}{2}(\Lambda_{ad}\,\partial\Lambda_b{}^d/\partial\omega_r)K'^{ab} = -\,i\partial/\partial\omega_r. \tag{2.49}$$

Hence it follows with (2.47) that

$$K'^{ab} = -\,J'^{ab}. \tag{2.50}$$

Accordingly we have

$$J'^{ab}\Lambda^{dc} = -\,ig^{da}\Lambda^{bc} + ig^{db}\Lambda^{ac}, \tag{2.51}$$

which means that J'^{ab} transforms Λ^{dc} just as $-P'^{ab}$ transforms x'^{d}. In order to avoid confusion we shall henceforth develop our theory in terms of operators J and J' only.

Let us illustrate the difference between J' and K' by a simple example. With the two-dimensional rotation

$$\begin{aligned} x'^1 &= x^1\cos\omega + x^2\sin\omega, \\ x'^2 &= -x^1\sin\omega + x^2\cos\omega, \end{aligned} \tag{2.52}$$

ω denotes the angle between the systems x and x' measured from x^1 towards x'^1. Accordingly we have $J^{12} = -i\partial/\partial\omega$ and likewise $J'^{12} = -i\partial/\partial\omega$. Now both J^{12} and J'^{12} refer to the rotation of a point fixed in x'-space, i.e. to the rotation of the x'-system as a whole, from the x^1-axis to the x^2-axis. Obviously ω increases under this rotation. By contrast, K'^{12} refers to the rotation of the x-system from the x'^1-axis to the x'^2-axis. The angle ω decreases under this transformation, and we have $K'^{12} = i\partial/\partial\omega$.

2.5. Commutation relations

It is easily seen from the preceding sections that the operators J and K' satisfy the same commutation relations as the rotation operators P introduced in section 2.1. Accordingly the commutation relations for operators J' are slightly different. To summarize, if $a \neq c$ we have analogously to (2.19)

$$[J^{ab}, J^{bc}] = -\,ig^{bb}J^{ac}, \qquad [J'^{ab}, J'^{bc}] = ig^{bb}J'^{ac}. \tag{2.53}$$

For our further work a crucial quantity will be the commutator between one J and one J'. By virtue of (2.45) we may write

$$[J'^{ab}, J^{ef}] = \Lambda^a_{\ c} \Lambda^b_{\ d} J^{cd} J^{ef} - J^{ef} \Lambda^a_{\ c} \Lambda^b_{\ d} J^{cd}. \qquad (2.54)$$

Now since the J's are linear differential operators, they act on functions p and q according to

$$J(pq) = pJq + qJp. \qquad (2.55)$$

Using this relation in the right-hand member of (2.54), we find with (2.40) that

$$[J'^{ab}, J^{ef}] = \Lambda^a_{\ c} \Lambda^b_{\ d} [J^{cd}, J^{ef}] - i\Lambda^{af} \Lambda^b_{\ d} J^{ed} + \\ + i\Lambda^{ae} \Lambda^b_{\ d} J^{fd} - i\Lambda^a_{\ c} \Lambda^{bf} J^{ce} + i\Lambda^a_{\ c} \Lambda^{be} J^{cf}. \qquad (2.56)$$

The first term in the right-hand member of (2.56) essentially is a sum over c and d in which c and d independently run through n values. The commutator in this term vanishes, however, unless $c = e$, or $c = f$, or $d = e$, or $d = f$. The contribution due to $c = e$ amounts to

$$- i\Lambda^{ae} \Lambda^b_{\ d} J^{fd}, \qquad (2.57)$$

as follows from (2.53). But this contribution cancels the third term in the right-hand member of (2.56), and since the contributions due to $c = f$, $d = f$, and $d = e$ cancel the remaining terms, we finally obtain

$$[J'^{ab}, J^{ef}] = 0, \qquad (2.58)$$

which means that any J commutes with any J'.

In order to avoid confusion we note two essential differences between J' and the operators P' mentioned in section 2.1: first, the commutation relations (2.53) for J' have a minus sign as compared with the commutator (2.23) between two operators P'; and secondly, J' commutes with J, but P' does not commute with P. By virtue of (2.21) we may write

$$[P'^{ab}, P^{ef}] = \Lambda^a_{\ c} \Lambda^b_{\ d} [P^{cd}, P^{ef}], \qquad (2.59)$$

and apart from the substitution (2.18) this expression cannot be simplified any further. The difference between J' and P' arises from the fact that P' commutes with the Eulerian angles while J' does not.

III

ROTATIONS IN THREE DIMENSIONS

3.1. Introduction

In the Pauli theory of spinning particles the behaviour of a particle of spin j is described by a wave function of $2j+1$ components,

$$\Psi_j(X) = \sum_{m=-j}^{j} \Phi^m(X) U_j^m, \tag{3.01}$$

X denoting the position coordinates of the particle in question. The quantities U are spinors, which under spatial rotations transform according to

$$\begin{aligned} P_3 U_j^m &= m U_j^m, \\ (P_1 \pm iP_2) U_j^m &= [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} U_j^{m\pm 1}. \end{aligned} \tag{3.02}$$

In (3.02) the P's are three-dimensional rotation operators of the type discussed in section 2.1. They are usually represented by square matrices satisfying the commutation relations (2.24). Accordingly the spinors U are written as one-column matrices.

It is well known that it takes only the commutation relations for the operators P to show that there exist rotation eigenvectors U of the behaviour (3.02), cf. Condon and Shortley (1953), secs. 2^3-3^3, Schiff (1949), sec. 24. Conversely, in the spinor analysis of rotations only the commutation relations for the operators and the transformation properties of the eigenvectors are considered. It is the purpose of the following pages to show that in a formalism with Eulerian angles the rotation eigenvectors have additional features which cannot be found in the spinor theory.

Though in the present investigation we aim at a relativistic theory, there are good reasons for starting with a formalism in three dimensions. We shall therefore first replace the operators P_j in (3.02) by the differential operators $J_j = J^{kl}$ ($j,k,l = 1,2,3$ cycl.), cf. section 2.3, where it is understood that the J's act on three Eulerian angles in spin space, not on the position coordinates X. Upon going over to operators J, the spinors U must by replaced by certain functions $Y(\omega)$. Now in order that our formalism may be physically useful, the functions Y must be transformed by the J's just as the spinors U are transformed by the P's, i.e. according to (3.02). It is not difficult to see that this is guaranteed if and only if the functions Y are eigenfunctions of $(J)^2 = J^k J_k$ and J_3 with suitable eigenvalues $j(j+1)$ and m, respectively. However, since we are working with three angles, the desired functions are not yet specified completely by two eigenvalues. Loosely speaking, we need a third characterization. At this point we observe from (2.46) that the operators J always go together with operators J'. Since any J' commutes with any J, according to (2.58), it is clear that there exist several independent functions Y for any particular combination j, m. As a matter of fact, the functions Y are conveniently taken to be simultaneous eigenfunctions of $(J)^2$, J_3, and J'_3 with eigenvalues $j(j+1)$, m, and m', say. Since, again, any J' commutes with any J, the operations J transform functions of a particular m' into functions of the same m' only, the transformation coefficients being independent of m'.

It may already be anticipated from this qualitative reasoning that the m' encountered here is the three-dimensional family-index mentioned in section 1.1. In the following pages this will be elaborated in some detail so as to provide a useful basis for the more complicated analysis in Minkowski space. From a different point of view some of the results of the present chapter have already been obtained by Reiche and Rademacher (1926, 1927), Kronig and Rabi (1927), and Casimir (1931).

3.2. Eulerian angles

In order to describe rotations in an Euclidean space of three dimensions, we introduce a right-handed system of rectangular coordinate axes x^k $(k = 1,2,3)$ fixed in space. We consider, in addition, a right-handed system of rectangular axes x' which may be rotated about the common origin of x and x'. Coinciding at first with the axes x, the system x' is carried into its most general final orientation by three successive rotations: first, a rotation through ψ from the x^1-axis to the x^2-axis; secondly, a rotation through θ from the x^3-axis to the x^1-axis; and finally, a rotation through φ from the x^1-axis to the x^2-axis. The angles φ, θ, and ψ were introduced by Euler in 1748.

It follows from the discussion of section 2.1 that the above orientation of the axes x' can be described by the transformation

$$x' = \exp(-i\varphi P_3)\exp(-i\theta P_2)\exp(-i\psi P_3)x \equiv \Lambda x, \qquad (3.03)$$

where it is understood that the operators P act on x, not on the Eulerian angles. Repeated application of (2.25) shows, in fact, that (3.03) is equivalent to

$$x'^k = \Lambda^k{}_l\, x^l, \qquad (3.04)$$

the direction cosines $\Lambda^k{}_l$ being given in table 3.1.

Table 3.1
The direction cosines $\Lambda^k{}_l$

k	$l = 1$	$l = 2$	$l = 3$
1	$\cos\varphi\cos\theta\cos\psi - \sin\varphi\sin\psi$	$\sin\varphi\cos\theta\cos\psi + \cos\varphi\sin\psi$	$-\sin\theta\cos\psi$
2	$-\cos\varphi\cos\theta\sin\psi - \sin\varphi\cos\psi$	$-\sin\varphi\cos\theta\sin\psi + \cos\varphi\cos\psi$	$\sin\theta\sin\psi$
3	$\cos\varphi\sin\theta$	$\sin\varphi\sin\theta$	$\cos\theta$

It is easily seen from table 3.1 that the angles φ and θ can be considered as spherical angles of points fixed on the x'^3-axis, the point on the unit sphere with $x'^1 = x'^2 = 0$, $x'^3 = 1$ having coordinates $x^1 = \cos\varphi\sin\theta$, $x^2 = \sin\varphi\sin\theta$, $x^3 = \cos\theta$. Since the combination φ, θ, ψ yields the same direction cosines as the combination $\varphi + \pi + 2n_1\pi$, $-\theta + 2n_2\pi$, $\psi + \pi + 2n_3\pi$ (n_1, n_2, $n_3 = 0, \pm1, \pm2, \ldots$), any orientation of the axes x' can be attained by angles satisfying

$$0 \leqq \varphi < 2\pi, \qquad 0 \leqq \theta \leqq \pi, \qquad 0 \leqq \psi < 2\pi. \qquad (3.05)$$

At the same time, it is sometimes convenient to consider Eulerian angles outside this domain of values.

3.3. Alternative introduction of Eulerian angles

Before proceeding to explicit calculations, it is advantageous to introduce Eulerian angles in an alternative way. Comparing the two modes of interpretation brings out several formulas that are very useful at various stages in our further work.

A simple geometrical consideration shows that the orientation of axis x' given in (3.03) can be realized as well by performing, first, a rotation through φ from the x^1-axis to the x^2-axis; secondly, a rotation through θ from the x^3-axis to the once rotated x'^1-axis, i.e. a rotation through θ about the once rotated x'^2-axis, which is called the line of nodes; and finally, a rotation through ψ from the twice rotated x'^1-axis to the line of nodes, i.e. a rotation through ψ about the final x'^3-axis. If we denote the line of nodes by z, and the operator P which generates rotations about this line by P_z, we may therefore write, along with (3.03),

$$x' = \exp(-i\psi P'_3)\exp(-i\theta P_z)\exp(-i\varphi P_3)x = \Lambda x. \quad (3.06)$$

Now it is well known that rotation operators in three dimensions transform as pseudovectors. On combining operators P to form pseudovectors in the directions z and x'^3, respectively, it is thus found that

$$\begin{aligned} P_z &= -(\sin\varphi)P_1 + (\cos\varphi)P_2, \\ P'_3 &= (\cos\varphi\sin\theta)P_1 + (\sin\varphi\sin\theta)P_2 + (\cos\theta)P_3. \end{aligned} \quad (3.07)$$

Substituting (3.07) into (3.06), equating the expressions for Λ (3.03) and (3.06), and putting $\psi = 0$ now yields

$$\begin{aligned} &\exp(-i\varphi P_3)\exp(-i\theta P_2) = \\ &\quad = \exp(-i\theta[-(\sin\varphi)P_1 + (\cos\varphi)P_2])\exp(-i\varphi P_3), \end{aligned} \quad (3.08)$$

which is equivalent to

$$\exp(-i\varphi P_3)P_2\exp(i\varphi P_3) = -(\sin\varphi)P_1 + (\cos\varphi)P_2. \quad (3.09)$$

By a rather lengthy, if essentially straightforward, computation equation (3.09) may be verified directly from the commutation relations (2.24) for the operators P.

In order to generalize (3.09) we now consider an auxiliary coordinate system $\bar{x}'$ which derives from x' according to

$$\bar{x}' = \exp(-i\omega P'_k)x' = \exp(-i\omega P'_k)\Lambda x. \quad (3.10)$$

Analogously to (3.03) we may also write

$$\bar{x}' = \Lambda\exp(-i\omega P_k)x, \quad (3.11)$$

from which it follows that

$$\Lambda P_k\Lambda^{-1} = P'_k. \quad (3.12)$$

The operator P' in (3.12) should be visualized as a certain linear combination of operators P. Rotation operators P_k transforming as pseudovectors, we have, in fact,

$$P'_k = \Lambda_k^{\,l} P_l . \tag{3.13}$$

It is essential in both (3.09) and (3.12) that P commutes with the angles φ, θ, ψ that appear as parameters in the exponential operators in Λ.

From (3.03) we obtain for the inverse rotation

$$x = \exp(i\psi P_3)\,\exp(i\theta P_2)\,\exp(i\varphi P_3)x' = \Lambda^{-1}x' . \tag{3.14}$$

However, since Px' is a rather complicated expression, this equation is not very suitable for practical calculations. With (3.12) we therefore transform (3.14) into

$$x = \exp(i\psi P'_3)\,\exp(i\theta P'_2)\,\exp(i\varphi P'_3)x' = \Lambda^{-1}x' . \tag{3.15}$$

According to (2.22) the quantity $P'x'$ is very simple indeed.

As noted previously, (3.09) essentially follows from the commutation relations for the operators P. The same applies to (3.12). Operators P' satisfying the same commutation relations as operators P, we thus have, along with (3.09),

$$\exp(-\,i\varphi P'_3)P'_2\,\exp(i\varphi P'_3) = -\,(\sin\varphi)P'_1 + (\cos\varphi)P'_2 . \tag{3.16}$$

Using the expression for Λ which is implicit in (3.15), we may also write, as a special analogue of (3.12),

$$\Lambda P'_3\,\Lambda^{-1} = (\cos\varphi\,\sin\theta)P'_1 + (\sin\varphi\,\sin\theta)P'_2 + (\cos\theta)P'_3 . \tag{3.17}$$

3.4. The rotation operators J and J'

Since the angles φ, θ, ψ refer to rotations of the system x' about the axes x^3, z, and x'^3, respectively, we must have

$$-\,i\partial/\partial\varphi = J_3, \quad -\,i\partial/\partial\theta = J_z, \quad -\,i\partial/\partial\psi = J'_3, \tag{3.18}$$

where J and J' stand for the differential operators defined in the preceding chapter. Using relations analogous to (3.07) we may write (3.18) in the form

$$\begin{aligned} -\,i\partial/\partial\varphi &= J_3, \\ -\,i\partial/\partial\theta &= -\,(\sin\varphi)J_1 + (\cos\varphi)J_2, \\ -\,i\partial/\partial\psi &= (\cos\varphi\,\sin\theta)J_1 + (\sin\varphi\,\sin\theta)J_2 + (\cos\theta)J_3. \end{aligned} \tag{3.19}$$

The three equations (3.19) form a system of the type (2.37) which is easily solved for the operators J. Resolving J_3 and J_z into components along the axes x', we likewise obtain three linear equations for the operators J'.

This simple geometrical construction of the operators J and J' has been essentially given by Casimir (1931), pp. 54-57. (Casimir's choice of Eulerian angles is slightly different from the one adopted here.) Treating rotation operators as pseudovectors, however, the above method is obviously restricted to three dimensions. In n-dimensional space expressions for J and J' can be derived by straightforward computation according to (2.37) and (2.47). However, this being a rather lengthy way in practical cases, we shall give another method, which - if rather formal at first sight - will prove very useful when applied to rotations in Minkowski space.

Let us then consider the motion of a point A' fixed in the system x'. In the language of chapter II, we must take the coordinates x' and the Eulerian angles ω as independent variables when describing the motion of a point A'. The coordinates x depend on x' and ω according to (3.15). Now if x' does not depend on ω, $(P')^q x'$ (q integral) does not depend on ω either, by virtue of (2.22). Hence (3.15) may be easily differentiated with respect to the Eulerian angles to give

$$\begin{aligned} -i\partial x/\partial\varphi &= \Lambda^{-1}P'_3\, x', \\ -i\partial x/\partial\theta &= \exp(i\psi P'_3)\ \exp(i\theta P'_2)P'_2\ \exp(i\varphi P'_3)x', \\ -i\partial x/\partial\psi &= P'_3\,\Lambda^{-1}x'. \end{aligned} \tag{3.20}$$

On substituting (3.16) and (3.17), (3.20) reduces to

$$-i\partial x/\partial\theta = \Lambda^{-1}[-(\sin\varphi)P'_1 + (\cos\varphi)P'_2]x', \text{ etc.} \tag{3.21}$$

Remembering that $x = \Lambda^{-1}x'$ stands for $x^k = \Lambda^{-1}x'^k$, we now write with (2.22) and (2.29)

$$\begin{aligned} J^{jk}x^l &= ig^{lj}x^k - ig^{lk}x^j = \\ &= \Lambda^{-1}(ig^{lj}x'^k - ig^{lk}x'^j) = \Lambda^{-1}P'^{jk}x'^l, \\ J_k x &= \Lambda^{-1}P'_k\, x'. \end{aligned} \tag{3.22}$$

Combined with (3.21) this expression yields

$$-i\partial x/\partial\theta = [-(\sin\varphi)J_1 + (\cos\varphi)J_2]x, \text{ etc.} \tag{3.23}$$

According to (3.23) the operators J must satisfy the system of equations (3.19) if they act on x. Now the relation (3.23) is exactly of the form (2.33), and (3.19) is of the form (2.37). Further, the determinant of (3.19) does not vanish identically. Since we have seen that (2.37) follows from (2.33) provided the determinant of (2.37) does not vanish identically, we may therefore conclude from (3.23) that the operators J are the linear differential operators that satisfy the desired system of equations (3.19).

If in (3.03) x is taken as independent variable, similar equations may be derived for the operators which describe the motion of a point A fixed in x-space as viewed from the system x'. In section 2.4 these operators have been denoted by K'. Now the above derivation of J essentially rests on the commutation relations for the operators that appear in the exponents in Λ. Further, the operators P' in (3.15) satisfy the same commutation relations as the operators P in (3.03). And finally, (3.03) derives from (3.15) by interchanging primed and unprimed quantities and substituting $-\psi$, $-\theta$, $-\varphi$ for φ, θ, ψ. Combining these facts, and denoting the present J and K' by $J(\varphi,\theta,\psi)$ and $K'(\varphi,\theta,\psi)$, respectively, we thus arrive at

$$K'_k(\varphi,\theta,\psi) = J_k(-\psi,-\theta,-\varphi). \tag{3.24}$$

By virtue of (2.50) it now follows that

$$J'_k(\varphi,\theta,\psi) = -J_k(-\psi,-\theta,-\varphi). \tag{3.25}$$

As a final result, the operators J and J' take the form

$$\begin{aligned} J_1 &= -i\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\psi}\right), \\ J_2 &= -i\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi} + \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\psi}\right), \\ J_3 &= -i\frac{\partial}{\partial\varphi}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} J'_1 &= -i\left(\sin\psi\frac{\partial}{\partial\theta} + \cos\psi\cot\theta\frac{\partial}{\partial\psi} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \\ J'_2 &= -i\left(\cos\psi\frac{\partial}{\partial\theta} - \sin\psi\cot\theta\frac{\partial}{\partial\psi} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \\ J'_3 &= -i\frac{\partial}{\partial\psi}. \end{aligned} \tag{3.27}$$

3.5. Rotation eigenfunctions

We now proceed to construct functions of three Eulerian angles which, loosely speaking, correspond to the spinors U mentioned in section 3.1. In view of the fact that the spinors U are eigenvectors of $(P)^2 = P^kP_k$ and P_3, we shall primarily consider eigenfunctions of $(J)^2$ and J_3. Since both $(J)^2$ and J_3 commute with any operator J', it is most convenient to investigate simultaneous eigenfunctions of $(J)^2$, J_3, and J'_3. The eigenvalues concerned being denoted by $j(j+1)$, m, and m', respectively, it follows from (3.26) and (3.27) that the desired functions must have the form

$$Y_j^{m,m'}(\varphi,\theta,\psi) = (1/2\pi)\Theta_j^{m,m'}(\theta)\exp(im\varphi + im'\psi). \tag{3.28}$$

The allowed values of j, m, and m' will be delimited later.

It has been discussed by Condon and Shortley (1953), secs. 2^3-3^3, by Schiff (1949), sec. 24, and others that if U_j^m is an eigenvector of $(P)^2$ and P_3 with eigenvalues $j(j+1)$ and m, respectively, it follows from the commutation relations for the operators P that

$$(P_1 \pm iP_2)U_j^m = \lambda_{m\pm1,m}U_j^{m\pm1}. \tag{3.29}$$

Now since the operators P stand for real observables, viz. for the components of spin angular momentum, they must be represented by Hermitian matrices, i.e.

$$\lambda_{m\pm1,m} = \lambda^*_{m,m\pm1}. \tag{3.30}$$

It is well known that the commutation relations for the P's combined with the condition (3.30) lead to

$$|\lambda_{m\pm1,m}|^2 = (j \mp m)(j \pm m + 1). \tag{3.31}$$

If it is understood that j and $j+1$ are taken to be non-negative, the parameters j and m in (3.31) are restricted to the values

$$j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \ldots, \qquad m = -j,\ -j+1,\ \ldots,\ j. \tag{3.32}$$

Since the operators J satisfy the same commutation relations as the operators P, and since they commute with J'_3, we must likewise have

$$(J_1 \pm iJ_2)Y_j^{m,m'} = \lambda_{m\pm1,m}Y_j^{m\pm1,m'}. \tag{3.33}$$

In order to obtain functions which correspond to the spinors U, we now impose the condition (3.30) on the coefficients λ in (3.33). Just as in the preceding paragraph, (3.30) then leads to

(3.31), j and m being restricted to the values given in (3.32). Thus, on suitable choice of the relative phases of the functions Y we shall have

$$(J_1 \pm iJ_2)Y_j^{m,m'} = [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} Y^{m\pm 1,m'}, \qquad (3.34)$$

which means that the operators J will transform the functions Y just as the operators P transform the spinors U. The index m' does not interfere with the above reasoning due to the fact that J'_3 commutes with every J. Whereas in deriving (3.34) it has been taken as a starting-point that the functions Y are eigenfunctions of $(J)^2$ and J_3, it will be clear, conversely, that any $2j+1$ eigenfunctions of J_3 which are related to one another according to (3.34) are eigenfunctions of $(J)^2$.

To show that in (3.28), (3.33), and (3.34) j and m may take on all values indicated in (3.32), and to find the allowed values of m', we now construct explicit expressions for the functions Θ. The method developed for that purpose is an extension of the method for determining the Legendre functions given by Condon and Shortley (1953), sec. 4[3]. On introducing the operator

$$J\{m';m,\pm\} = \pm \frac{d}{d\theta} - m\cot\theta + \frac{m'}{\sin\theta}, \qquad (3.35)$$

a simple calculation with (3.26) and (3.28) shows that for (3.34) to be valid the functions Θ must be connected by the recurrence formula

$$J\{m';m,\pm\}\Theta_j^{m,m'} = [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} \Theta_j^{m\pm 1,m'}. \qquad (3.36)$$

If θ now satisfies $0 < \theta < \pi$, the operator $J\{m';m,\pm\}$ may be written in the form

$$J\{m';m,\pm\} = \pm(\sin\tfrac{1}{2}\theta)^{\pm m \mp m' + 1}(\cos\tfrac{1}{2}\theta)^{\pm m \pm m' + 1} \times$$
$$\times \frac{d}{d\sin^2\frac{1}{2}\theta}(\sin\tfrac{1}{2}\theta)^{\mp m \pm m'}(\cos\tfrac{1}{2}\theta)^{\mp m \mp m'}. \qquad (3.37)$$

Hence since $J\{m';\pm j,\pm\}$ is to give zero when applied to $\Theta_j^{\pm j,m'}$, it follows that we must have

$$\Theta_j^{\pm j,m'}(\theta) = \alpha_j^{\pm j,m'}(\sin\tfrac{1}{2}\theta)^{j\mp m'}(\cos\tfrac{1}{2}\theta)^{j\pm m'}, \qquad (3.38)$$

where the α's are certain normalization coefficients which will be chosen later. On applying the operator

$$\left[\frac{(j+m)!}{(2j)!\,(j-m)!}\right]^{\frac{1}{2}} J\{m';m+1,-\}\ldots J\{m';j-1,-\}J\{m';j,-\} \qquad (3.39)$$

to the function $\Theta_j^{j,m'}$ as given by (3.38), it is now found that

$$\Theta_j^{m,m'}(\theta) = \alpha_j^{m,m'}\{+\}(\sin\tfrac{1}{2}\theta)^{-m+m'}(\cos\tfrac{1}{2}\theta)^{-m-m'} \times$$
$$\times \frac{d^{j-m}}{d(\sin^2\frac{1}{2}\theta)^{j-m}}(\sin\tfrac{1}{2}\theta)^{2j-2m'}(\cos\tfrac{1}{2}\theta)^{2j+2m'},$$
$$\alpha_j^{m,m'}\{+\} = (-1)^{j-m}\left[\frac{(j+m)!}{(2j)!\,(j-m)!}\right]^{\frac{1}{2}}\alpha_j^{j,m'}. \tag{3.40}$$

At the same time, the operator

$$\left[\frac{(j-m)!}{(2j)!\,(j+m)!}\right]^{\frac{1}{2}} J\{m';m-1,+\}\ldots J\{m';-j+1,+\}J\{m';-j,+\} \tag{3.41}$$

applied to $\Theta_j^{-j,m}$ yields

$$\Theta_j^{m,m'}(\theta) = \alpha_j^{m,m'}\{-\}(\sin\tfrac{1}{2}\theta)^{m-m'}(\cos\tfrac{1}{2}\theta)^{m+m'} \times$$
$$\times \frac{d^{j+m}}{d(\sin^2\frac{1}{2}\theta)^{j+m}}(\sin\tfrac{1}{2}\theta)^{2j+2m'}(\cos\tfrac{1}{2}\theta)^{2j-2m'},$$
$$\alpha_j^{m,m'}\{-\} = \left[\frac{(j-m)!}{(2j)!\,(j+m)!}\right]^{\frac{1}{2}}\alpha_j^{-j,m'}. \tag{3.42}$$

Explicit expressions for $\alpha\{\pm\}$ will be given in (3.54).

The relation (3.40) contains, among others, an expression for $\Theta_j^{-j,m'}$. However, this function has already been given in (3.38). In order that (3.40) be compatible with (3.38), it is necessary that

$$d^{2j}[t^{j-m'}(1-t)^{j+m'}]/dt^{2j} \tag{3.43}$$

is a constant. Since this condition is fulfilled if and only if both $j+m'$ and $j-m'$ are non-negative integers, it follows that m' is restricted to the values

$$m' = -j, -j+1, \ldots, j. \tag{3.44}$$

If m' satisfies the condition (3.44), the two expressions for Θ, (3.40) and (3.42), are consistent provided

$$\alpha_j^{-j,m'} = (-1)^{j-m'}\alpha_j^{j,m'}. \tag{3.45}$$

It is not difficult to see from the foregoing formulas that the parameters j, m, and m' may take on all values compatible with (3.32) and (3.44).

Since m and m' run through the same values, the function $\Theta_j^{m',m}$ has a meaning together with $\Theta_j^{m,m'}$. As a matter of fact, it is easily verified from (3.40) that both functions may differ by a normalization factor at most. Without loss of generality the constants $\alpha_j^{j,m'}$ can be chosen so that

$$\Theta_j^{m',m} = \Theta_j^{m,m'}. \tag{3.46}$$

By virtue of (3.36) this choice implies that

$$J\{m;m',\pm\}\Theta_j^{m,m'} = [(j \mp m')(j \pm m' + 1)]^{1/2}\Theta_j^{m,m'\pm 1}, \tag{3.47}$$

which, by (3.27), (3.28), and (3.35), leads to

$$(J'_1 \pm iJ'_2)Y_j^{m,m'} = -[(j \pm m')(j \mp m' + 1)]^{1/2}Y_j^{m,m'\mp 1}. \tag{3.48}$$

Since J_3 commutes with every J', it already follows from the commutation relations for the operators J' that

$$(J'_1 \pm iJ'_2)Y_j^{m,m'} = \mu_{m'\mp 1,m'}Y_j^{m,m'\mp 1} \tag{3.49}$$

– not $Y_j^{m,m'\pm 1}$ in view of the "wrong" sign of the commutation relations for operators J' –. Now one might be led to expect that a second condition of the type (3.30) would be required to obtain a relation for μ such as (3.31). We see here, however, that on suitable choice of the normalization coefficients α the sole condition (3.30) suffices to give two recurrence formulas, (3.34) and (3.48), and to delimit the values of j, m, and m'.

On applying the operator

$$\left[\frac{(j+m')!}{(2j)!(j-m')!}\right]^{1/2} J\{j;m'+1,-\}\ldots J\{j;j-1,-\}J\{j;j,-\} \tag{3.50}$$

to the function $\Theta_j^{j,j}$ as given by (3.38), we find, by (3.47), $\Theta_j^{j,m'}$ in the form (3.38), and it follows that we must have

$$\alpha_j^{j,m'} = \left[\frac{(2j)!}{(j+m')!(j-m')!}\right]^{1/2} \alpha_j^{j,j}. \tag{3.51}$$

Collecting results we see that the normalization factors α must be related according to (3.40), (3.42), (3.45), and (3.51) if the recurrence formulas (3.34) and (3.48) are to hold true. In order to fix the α's we now choose $\alpha_j^{j,j}$ to be

$$\alpha_j^{j,j} = (j+\tfrac{1}{2})^{1/2}\exp(j\pi i), \tag{3.52}$$

which implies that

$$\int_0^{\pi}|\Theta_j^{j,j}(\theta)|^2\sin\theta d\theta = 1. \tag{3.53}$$

On this choice for $\alpha_j^{j,j}$ the coefficients $\alpha\{\pm\}$ take the form

$$\alpha_j^{m,m'}\{+\} = \left[\frac{(2j+1)(j+m)!}{2(j-m)!(j+m')!(j-m')!}\right]^{\frac{1}{2}} \exp(m\pi i),$$
$$\alpha_j^{m,m'}\{-\} = \left[\frac{(2j+1)(j-m)!}{2(j+m)!(j+m')!(j-m')!}\right]^{\frac{1}{2}} \exp(m'\pi i). \tag{3.54}$$

Since both $j+m'$ and $j-m'$ are integers, we may write without ambiguity concerning phases

$$(\sin\tfrac{1}{2}\theta)^{2j\pm 2m'} = (\sin^2\tfrac{1}{2}\theta)^{j\pm m'}. \tag{3.55}$$

With the help of relations of the form (3.55) the differentiations in (3.40) and (3.42) are readily carried out with Leibniz's rule to give

$$\Theta_j^{m,m'}(\theta) = \sum_q \left[\frac{[(2j+1)(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}}}{q!(j-m-q)!(j-m'-q)!(m+m'+q)!\sqrt{2}} \times \right.$$
$$\left. \times (\sin\tfrac{1}{2}\theta)^{2j-m-m'-2q}(\cos\tfrac{1}{2}\theta)^{m+m'+2q}\exp[(j-q)\pi i]\right], \tag{3.56}$$

where the summation is to be extended over all integers q consistent with the factorial notation, the factorial of a negative number being meaningless.

In the foregoing paragraphs, expressions for Θ have been derived under the assumption that $0 < \theta < \pi$. Now the right-hand member of (3.56) is an integral function of θ. It will therefore be clear that the restriction on θ may be dropped, and that the functions Θ given by (3.56) satisfy the recurrence formula (3.36) for any finite θ. Apart from certain normalization factors they are the only set of solutions of (3.36). Further, since (3.55) holds true for any finite θ, it will give no confusion if we also use (3.40) and (3.42) for any finite θ.

On multiplying the two expressions for Θ, (3.40) and (3.42), and integrating by parts, it is not difficult to show that

$$\left|\int_0^\pi \Theta_j^{m,m'}(\theta)\Theta_i^{m,m'}(\theta)\sin\theta d\theta\right| = \delta_{ji}. \tag{3.57}$$

Let us now define the scalar product of any two quadratically integrable functions $f(\varphi,\theta,\psi)$ and $g(\varphi,\theta,\psi)$ by

$$(f,g) = \tfrac{1}{2}\int_0^{4\pi} d\psi \int_0^{2\pi} d\varphi \int_0^\pi f^*(\varphi,\theta,\psi)g(\varphi,\theta,\psi)\sin\theta d\theta. \tag{3.58}$$

It then follows from (3.28) and (3.57) that

$$(Y_j^{m,m'}, Y_i^{n,n'}) = \delta_{ji}\delta_{mn}\delta_{m'n'}. \tag{3.59}$$

In (3.58) the integration with respect to ψ is carried out over an interval 4π to ensure that Y_j and Y_i shall be orthogonal if j is half-integral while i is integral, or vice versa. One might as well interchange the roles of φ and ψ in (3.58).

It is not difficult to see with (3.34) and (3.48) that on adopting the definition (3.58) for the scalar product we have

$$\begin{aligned} (f, J_k g) &= (J_k f, g), \\ (f, J'_k g) &= (J'_k f, g), \end{aligned} \qquad (k = 1,2,3) \tag{3.60}$$

for any two functions f and g which may be expanded into a series of functions Y. In other words, the operators J and J' are Hermitian with respect to such functions, which is, of course, essential for a physical interpretation of these operators to be possible. It is not surprising that the operators J are Hermitian with respect to the functions Y. One should rather say that this is a natural result of the condition (3.30) together with the definition (3.58) for the scalar product. The interesting point is that J' is Hermitian without any second condition being imposed.

3.6. **Miscellaneous remarks**

3.61. *Surface harmonics*

If we compare (3.40) or (3.42) with Rodrigues's formula, we see that $\Theta_j^{m,0}$ is proportional to the associated Legendre function P_j^m. Our normalization factor has been chosen so as to agree with the convention adopted by Condon and Shortley (1953), eq. $4^3 15$. As noted already in section 3.2, the angles φ and θ may be considered as spherical angles. Accordingly the functions $Y_j^{m,0}$ are surface harmonics.

It is well known that the surface harmonics may be written as polynomials in $\cos\varphi\sin\theta$, $\sin\varphi\sin\theta$, and $\cos\theta$. In the notation of section 3.2 this amounts to polynomials in Λ_{3k}. Now by (2.14) and (2.40), J transforms Λ_{3k} just as P transforms x_k. Likewise, J' transforms Λ_{3k} just as $-P'$ transforms x'_3, according to (2.22) and (2.51), so that $J'_3 \Lambda_{3k} = 0$. From this it follows that J'_3 yields zero when applied to any polynomial in Λ_{3k}, in accordance with the fact that surface harmonics have zero m'.

Since m' is restricted to the values $-j, -j+1, \ldots, j$, it cannot be zero unless j is an integer. This agrees with the known fact that surface harmonics exist only for integral j. By contrast, the functions Y derived in the preceding section have a meaning for integral as well as for half-integral j provided m and m' satisfy the conditions (3.32) and (3.44).

3.62. *Two-valuedness*

We have seen in section 3.2 that the angles $\varphi + \pi + 2n_1\pi$, $-\theta + 2n_2\pi$, $\psi + \pi + 2n_3\pi$ refer to the same orientation of the axes x' as the angles φ, θ, ψ. However, it follows from (3.40) that

$$\Theta_j^{m,m'}(\theta) = (-1)^{m-m'}\Theta_j^{m,m'}(-\theta) = (-1)^{2j}\Theta_j^{m,m'}(\theta+2\pi). \quad (3.61)$$

Combined with (3.28) this yields

$$\begin{aligned} Y(\varphi,\theta,\psi) &= Y(\varphi+\pi,-\theta,\psi-\pi) = (-1)^{2j}Y(\varphi+2\pi,\theta,\psi) \\ &= (-1)^{2j}Y(\varphi,\theta+2\pi,\psi) = (-1)^{2j}Y(\varphi,\theta,\psi+2\pi), \end{aligned} \quad (3.62)$$

which means that the functions Y of half-integral j change sign under a rotation through 2π about the x^3-axis, the line of nodes, or the x'^3-axis, respectively.

Let us now consider, along with x^3, an auxiliary axis $\bar{x}^3$ likewise fixed in x-space. Let us also introduce the operator $\bar{J}_3$ which generates rotations about the axis $\bar{x}^3$. Obviously $\bar{J}_3$ commutes with both $(J)^2$ and J'_3. Hence for any allowed combination j, m' it has eigenfunctions $\bar{Y}_j^{\bar{m},m'}$. Since these may be written as linear combinations of the functions $Y_j^{m,m'}$ having the j and m' in question, it follows from (3.62) that $\bar{Y}$ is multiplied by $(-1)^{2j}$ when a rotation through 2π is carried out about the x^3-axis. Conversely, Y is multiplied by $(-1)^{2j}$ under a rotation through 2π about the $\bar{x}^3$-axis, or, generally, under a rotation through 2π about any axis fixed in x-space. An analogous reasoning applies to rotations about axes fixed in x'-space. A similar two-valuedness exists in the spinor formalism, where we have

$$\exp(2\pi i P_k)U_j^m = (-1)^{2j}U_j^m. \quad (3.63)$$

3.63. *Supplementary conditions*

The functions Y of integral j are well known as the angular-momentum eigenfunctions of a rigid rotator (Reiche and Rademacher 1926, 1927, Kronig and Rabi 1927, Casimir 1931). They have previously been found by solving the second-order differential equation $(J^kJ_k - \eta)Y = 0$ subject to the requirements that Y shall be a one-valued function of position, and that it shall be bounded at the singular points with $\sin\theta = 0$. Whereas we have started from the recurrence formula (3.33) subject to the condition (3.30), our functions of integral j are identical with those satisfying the usual singular-point boundary conditions.

Once the differential equation for Y has been solved for one-valued functions, it is not difficult to see that it admits of reasonable two-valued functions also. At the same time, the singular-point boundary conditions are not quite unambiguous once the restriction of one-valuedness has been dropped. They do not settle clearly, for instance, whether a function with a branch-point such as $(\sin\theta)^{\frac{1}{2}}\exp(\pm\frac{1}{2}i\varphi)$, which satisfies the second-order differential equation for $Y_{\frac{1}{2}}^{\pm\frac{1}{2},0}$, is admissible. This function is certainly ruled out by our supplementary condition.

Whereas the usual singular-point boundary conditions might thus give rise to difficulties, we have still another reason to prefer the method of section 3.5: The singular case $\theta = 0$ refers to the situation that the axes x^3 and x'^3 coincide. Naturally, it would be rather disturbing to meet with a branch-point or a divergence there which does not occur, for instance, when x^2 coincides with x'^2, i.e. when $\varphi = \psi = 0$. At the same time, it is our intention to interpret the Eulerian angles as angles in spin space, rather than in ordinary space; and one might argue that a singular behaviour of functions in spin space need not have physical implications. We are interested, first of all, in functions which transform as spinors, rather than in functions without awkward branch-points, say. Now the method of section 3.5 has been adopted to ensure that we find all functions that satisfy the spinor-like transformation formula (3.34). Our method will be particularly useful for the analysis of rotations in Minkowski space.

3.64. *Connection with hypergeometric functions*

Expressions for Θ have previously been obtained by solving the second-order differential equation concerned in terms of a series in t and $1-t$, where t stands for $\sin^2\frac{1}{2}\theta$ (Reiche and Rademacher 1926, 1927, Kronig and Rabi 1927). The functions Θ then appeared as hypergeometric functions times some simple factors. For the sake of distinctness we briefly indicate the connection between that notation and the one established in section 3.5.

Let us first introduce the auxiliary parameters

$$d = |m-m'|, \qquad s = |m+m'|, \qquad p = j - \tfrac{1}{2}d - \tfrac{1}{2}s. \tag{3.64}$$

Since $\Theta_j^{m,m'} = \Theta_j^{\bar{m},\bar{m}'}$, (3.40) and (3.42) essentially give four expressions for Θ. If we go over from j, m, m' to d, s, p in each of these expressions, one of the new equations will certainly read

$$\begin{aligned}\Theta_j^{m,m'}(\theta) &= \beta_j^{m,m'}(\sin\tfrac{1}{2}\theta)^{-d}(\cos\tfrac{1}{2}\theta)^{-s} \times \\ &\times \frac{d^p}{d(\sin^2\frac{1}{2}\theta)^p}(\sin\tfrac{1}{2}\theta)^{2d+2p}(\cos\tfrac{1}{2}\theta)^{2s+2p}, \\ \beta_j^{m,m'} &= \left[\frac{(2j+1)(d+s+p)!}{2p!\,(d+p)!\,(s+p)!}\right]^{\frac{1}{2}} \exp[(\max m, m')\pi i].\end{aligned} \tag{3.65}$$

Introducing the abbreviation

$$t = \sin^2\tfrac{1}{2}\theta \tag{3.66}$$

we therefore consider

$$d^p[t^{d+p}(1-t)^{s+p}]/dt^p. \tag{3.67}$$

By the series expansion of the hypergeometric function, we have

$$(1-t)^{s+p} = F(-s-p, c; c; t), \tag{3.68}$$

$$d[t^{c-1}F(a, b; c; t)]/dt = (c-1)t^{c-2}F(a, b; c-1; t). \tag{3.69}$$

If putting $b = c = d+p+1$ and $a = -s-p$ we now apply (3.69) p times in succession, it is found that

$$d^p[t^{d+p}(1-t)^{s+p}]/dt^p = [(d+p)!/d!]\, t^dF(-s-p, d+p+1; d+1; t). \tag{3.70}$$

Hence since

$$F(a, b; c; t) = (1-t)^{c-a-b}F(c-a, c-b; c; t), \tag{3.71}$$

it follows that

$$\begin{aligned}\Theta_j^{m,m'}(\theta) &= \gamma_j^{m,m'}(\sin\tfrac{1}{2}\theta)^d(\cos\tfrac{1}{2}\theta)^s \times \\ &\times F(-p, d+s+p+1; d+1; \sin^2\tfrac{1}{2}\theta), \\ \gamma_j^{m,m'} &= \left[\frac{(2j+1)(d+s+p)!\,(d+p)!}{2p!\,(s+p)!}\right]^{\frac{1}{2}} \frac{1}{d!} \exp[(\max m, m')\pi i],\end{aligned} \tag{3.72}$$

which is the desired expression in terms of a hypergeometric series. The hypergeometric series in (3.72) is a Jacobi polynomial of degree p. The Jacobi polynomials have been extensively discussed by Szegö (1939), chap. IV.

If we directly substitute (3.70) into (3.65) without using (3.71), we find the expression for Θ which is obtained from the right-hand member of the first equation (3.72) by replacing $|m + m'|$ by $-|m + m'|$, keeping both $|m - m'|$ and j and the phase factor fixed. However, since $|m - m'|$ is integral, and since $F(a, b; c; t)$ has no meaning in case c is a negative integer or zero, (3.72) has no analogue in which $-|m - m'|$ is substituted for $|m - m'|$. This is the reason why the absolute quantities d, s, and p must be used in a notation in terms of hypergeometric functions.

3.7. Addition of angular momenta

For our analysis of rotations in Minkowski space we shall have to know how in a formalism with Eulerian angles we may add two commuting three-dimensional angular momenta. Let us therefore consider, along with the set of operators J acting on angles ω, an analogous set of operators I acting on angles υ. Just as the operators J are associated with operators J', the operators I will be associated with operators I'. In an obvious notation the relevant eigenfunctions will be denoted by $Y_j^{m,m'}(\omega)$ and $Y_i^{n,n'}(\upsilon)$, respectively.

We can now construct simultaneous eigenfunctions of the operators $(J+I)^2$, $(J)^2$, $(I)^2$, J_3+I_3, J'_3, and I'_3 characterized by the eigenvalues $k(k+1)$, $j(j+1)$, $i(i+1)$, l, m', and n', respectively. As a matter of fact, since any primed operator commutes with any unprimed one, J and I may be added just as if no operators J' and I' were present. It is therefore readily shown by standard methods that the desired new functions must have the form

$$\sum_{m,n} (j\ i\ m\ n\,|\,j\ i\ k\ l)Y_j^{m,m'}(\omega)Y_i^{n,n'}(\upsilon), \tag{3.73}$$

where the coefficients are Clebsch-Gordan coefficients in the notation adopted by Condon and Shortley (1953), sec. 14[3].

We now proceed to combine functions (3.73) with various eigenvalues m' and n' so as to obtain eigenfunctions of $(J'+I')^2$ and $J'_3+I'_3$ with eigenvalues $k'(k'+1)$ and l', respectively. It is particularly easy to add $-J'$ and $-I'$ by virtue of the fact that these operators satisfy the commutation relations for an ordinary angular momentum, according to (2.53). Moreover, $-(J'_1 \pm iJ'_2)$ and $-(I'_1 \pm iI'_2)$ have positive matrix elements with respect to the functions Y, as shown by (3.48). It therefore follows from the well-known formulas for the addition of two ordinary angular momenta that the functions

$$\sum_{m',n',m,n} (j\ i\ {-m'}\ {-n'}\,|\,j\ i\ k'\ {-l'})(j\ i\ m\ n\,|\,j\ i\ k\ l)Y_j^{m,m'}(\omega)Y_i^{n,n'}(\upsilon) \tag{3.74}$$

are eigenfunctions of $(J+I)^2$, $(J'+I')^2$, $(J)^2$, $(I)^2$, J_3+I_3, and $J'_3+I'_3$ with eigenvalues $k(k+1)$, $k'(k'+1)$, $j(j+1)$, $i(i+1)$, l, and l', respectively. It will be observed from (3.74) that primed and unprimed operators may be added quite independently.

In (3.74) both k and k' may take on the values $j+i$, $j+i-1, \ldots, |j-i|$. Whereas $(J)^2 = (J')^2$ and $(I)^2 = (I')^2$, we need not have $(J+I)^2 = (J'+I')^2$, hence k and k' need not be equal. This is essentially due to the fact that considered as pseudovectors J'_k and I'_k are rotated with respect to one another under variations of the Eulerian angles. Whereas J_k and I_k are strictly parallel, pointing in the direction of the x^k-axis fixed in space, the orientations of J'_k and I'_k are described by the angles ω and υ, respectively, which vary independently. As a result it is not possible to interpret $J'_k + I'_k$ as the component of $J+I$ along some moving axis x'^k, and there is no reason why $(J'+I')^2$ should be equal to $(J+I)^2$.

3.8. The family-index m'

Collecting some of the results of section 3.5, we see that the functions $Y_j^{m,m'}$ with fixed j and m', m taking on the values $-j, -j+1, \ldots, j$, constitute a family which under the rotations J transforms just as the family of spinors U_j^m transforms under the rotations P. Any integral or half-integral j leads to $2j+1$ distinct families, which are labelled by the index m', $m' = -j, -j+1, \ldots, j$.

Since it is a known fact that an orbital angular momentum can be described in terms of surface harmonics, it follows from section 3.61 that we have to restrict ourselves to families of zero m', and hence integral j, when considering orbital motions. By contrast, the situation becomes much more complicated once we interpret φ, θ, and ψ as angles in spin space, and accordingly represent the spin by the differential operators J.

In the latter case, the family of spinors U_j^m ($m = -j, -j+1, \ldots, j$) with a particular j can be replaced by any of the $2j+1$ families of functions $Y_j^{m,m'}$ with that same j. We have no physical criterion now which further delimits m'. The case $m' = 0$ is certainly not preferred, since it is not even allowed for half-integral spins.

Expressing quantities in spin space in terms of Eulerian angles, we can thus write down $2j+1$ distinct wave functions analogous to (3.01), viz. the functions

$$\Psi_j^{m'}(X,\omega) = \sum_{m=-j}^{j} \Phi^m(X) Y_j^{m,m'}(\omega). \tag{3.75}$$

Since in a three-dimensional theory each of these functions can be used for the description of a particle of spin j, we would suggest for the present that there might exist $2j+1$ different kinds of particles of spin j which are distinguished from each other by the family-index m'. According to this idea, the ordinary spin, represented by J, would always go together with an additional spin, represented by J'. It will be shown in the following chapters how this can be brought into line with the requirement of Lorentz invariance.

At this stage the new spin, J', is related to the ordinary spin, J, by the equation $(J')^2 = (J)^2$. Now one might say from the preceding section that this restriction could be relaxed by combining several "elementary" particles to form a "compound" one. Accordingly our formalism with Eulerian angles would suggest an interesting difference between "elementary" and "compound" particles. However, this apparent difference is not physically significant since in a relativistic theory the ordinary spin and the new one need not be equal anyway.

IV

THE LORENTZ GROUP

4.1. Representations of the proper Lorentz group

According to the theory of special relativity, any physical equation must be invariant under spatial rotations as well as under the transformations

$$\begin{aligned}\bar{x}^j &= x^j,\\ \bar{x}^k &= x^k,\\ \bar{x}^l &= (x^l - \beta ct)(1-\beta^2)^{-\frac{1}{2}},\\ c\bar{t} &= (-\beta x^l + ct)(1-\beta^2)^{-\frac{1}{2}},\end{aligned} \qquad (j,k,l = 1,2,3 \text{ cycl.}), \qquad (4.01)$$

which must be used for describing uniform translational motions of velocity βc. The spatial rotations and the transformations of the form (4.01) together form a group, which is called the proper Lorentz group. Putting

$$\beta(1-\beta^2)^{-\frac{1}{2}} = \sinh\upsilon, \qquad (1-\beta^2)^{-\frac{1}{2}} = \cosh\upsilon, \qquad (4.02)$$

we shall find it convenient to write the third and fourth lines of (4.01) in the form

$$\begin{aligned}\bar{x}^l &= x^l\cosh\upsilon - ct\sinh\upsilon,\\ c\bar{t} &= -x^l\sinh\upsilon + ct\cosh\upsilon.\end{aligned} \qquad (4.03)$$

After Minkowski we now consider x^k and $x^0 = ct$ as coordinates in a pseudo-Euclidean space of four dimensions. For the covariant components of the metric tensor associated with x we shall use the notation g_{uv} $(u,v = 1,2,3,0)$, where

$$g_{kk} = 1, \qquad g_{00} = -1, \qquad g_{uv} = 0 \quad (u \neq v). \qquad (4.04)$$

Clearly the quadratic form $g_{uv}x^u x^v$ is now invariant under the transformations of the proper Lorentz group. Further, denoting rotation operators by P, we see from (2.25) that the Lorentz transformation (4.03) may be interpreted as the space-time rotation in Minkowski space which is described by the operator

$\exp(i\upsilon P^{k0})$. Accordingly the proper Lorentz group is identical with the group of rotations in Minkowski space.

For the construction of relativistic wave equations one will certainly need families of quantities which transform linearly among themselves under rotations in Minkowski space. In the framework of the spinor theory such quantities have already been discussed by van der Waerden (1932), sec. 20, and we briefly review his results here.

On introducing the operators P^+ and P^- defined by

$$\begin{aligned} P_j^+ &= \tfrac{1}{2}(P^{kl} + iP^{j0}), \\ P_j^- &= \tfrac{1}{2}(P^{kl} - iP^{j0}), \end{aligned} \qquad (j,k,l = 1,2,3 \text{ cycl.}), \tag{4.05}$$

we see from the commutation relations (2.18) that P^+ and P^- satisfy the equations

$$\begin{aligned} [P_j^+,P_k^+] &= iP_l^+, \\ [P_j^-,P_k^-] &= iP_l^-, \end{aligned} \qquad (j,k,l = 1,2,3 \text{ cycl.}). \tag{4.06}$$

It follows at the same time that any P^+ commutes with any P^-. Since according to (4.06) both P^+ and P^- satisfy the commutation relations for an angular momentum in three dimensions, and since P^+ commutes with P^-, there exist simultaneous eigenvectors of $(P^+)^2$, P_3^+, $(P^-)^2$, and P_3^- with eigenvalues $j^+(j^++1)$, m^+, $j^-(j^-+1)$, and m^-, respectively, in such a way that

$$\begin{aligned} j^+ &= 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \dots, & m^+ &= -j^+,\ -j^++1,\ \dots,\ j^+, \\ j^- &= 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \dots, & m^- &= -j^-,\ -j^-+1,\ \dots,\ j^-. \end{aligned} \tag{4.07}$$

By suitable normalization these eigenvectors, say $U^{m^+m^-}_{j^+j^-}$, can be made to transform according to

$$PU^{m^+m^-}_{j^+j^-} = \sum_{n^+,n^-} \langle n^+n^-|P|m^+m^-\rangle U^{n^+n^-}_{j^+j^-}, \tag{4.08}$$

where the non-vanishing coefficients have the form

$$\begin{aligned} \langle m^+m^-|P_3^+|m^+m^-\rangle &= m^+, \\ \langle m^+\pm 1\ m^-|P_1^+\pm iP_2^+|m^+m^-\rangle &= [(j^+\mp m^+)(j^+\pm m^++1)]^{1/2}, \\ \langle m^+m^-|P_3^-|m^+m^-\rangle &= m^-, \\ \langle m^+\ m^-\pm 1|P_1^-\pm iP_2^-|m^+m^-\rangle &= [(j^-\mp m^-)(j^-\pm m^-+1)]^{1/2}. \end{aligned} \tag{4.09}$$

The representation of the P's which is implicit in (4.09) is usually referred to as $D(j^+,j^-)$. It is not difficult to see that $x^1 + ix^2$; $-x^3 + x^0$; $-x^3 - x^0$; $-x^1 + ix^2$ transform as the spinors $U^{m^+ m^-}_{\frac{1}{2}\,\frac{1}{2}}$ having $m^+, m^- = \frac{1}{2},\frac{1}{2};\ \frac{1}{2},-\frac{1}{2};\ -\frac{1}{2},\frac{1}{2};\ -\frac{1}{2},-\frac{1}{2}$.

In a matrix representation according to (4.09), the operators P^+ and P^- are Hermitian, which implies that the operators P^{k0} are represented by anti-Hermitian matrices. At the same time, the matrices in question have a finite number, viz. $(2j^+ +1)(2j^- +1)$, of rows and columns. It follows from a comparison with the theory of rotations in three dimensions that the representations $D(j^+,j^-)$ are the only ones which picture P^+ and P^- as Hermitian operators. Further, it is not difficult to prove that all representations of the proper Lorentz group other than $D(j^+,j^-)$ are infinite-dimensional. The reader is referred to Corson (1953), sec. 17(b), for a discussion of this point. Since all wave equations established thus far, particularly the equations for spins 0, $\frac{1}{2}$, and 1, derive from representations $D(j^+,j^-)$, we shall restrict ourselves to such ones.

4.2. Eulerian angles

In the spirit of section 2.2 we now introduce two systems of rectangular coordinate axes in Minkowski space, x and x', the mutual orientation of which is described through six Eulerian angles. It will be particularly workable to choose the Eulerian angles in such a way that the coordinates x and x' are related to one another according to

$$x' = \exp(-i\,\varphi_2\,P^{03})\exp(-i\,\varphi_1\,P^{12})\exp(-i\,\theta_2\,P^{02}) \times$$
$$\times \exp(-i\,\theta_1\,P^{31})\exp(-i\,\psi_2\,P^{03})\exp(-i\,\psi_1\,P^{12})x \equiv \Lambda x. \qquad (4.10)$$

It is readily seen from (2.25) that (4.10) is equivalent to

$$x'^u = \Lambda^u{}_v, \qquad (4.11)$$

where the coefficients $\Lambda^u{}_v$ have the form given in table 4.1.

The sixteen transformation coefficients $\Lambda^u{}_v$ will be real provided

$$\varphi_1,\ \varphi_2 + p_1\pi i,\ \theta_1,\ \theta_2 + p_2\pi i,\ \psi_1,\ \psi_2 + p_3\pi i$$
$$(p_1, p_2, p_3 = 0, \pm 1, \pm 2, \ldots) \qquad (4.12)$$

are all real. Further, the combination

Table 4.1
The coefficients $\Lambda^{u}{}_{v}$

u	$v=1$
1	$\cos\varphi_1 \cos\theta_1 \cos\psi_1 - \sin\varphi_1 \cosh\theta_2 \sin\psi_1$
2	$-\cos\varphi_1 \cos\theta_1 \sin\psi_1 - \sin\varphi_1 \cosh\theta_2 \cos\psi_1$
3	$\cos\varphi_1 \sin\theta_1 \cosh\psi_2 - \sin\varphi_1 \sinh\theta_2 \sinh\psi_2$
0	$-\cos\varphi_1 \sin\theta_1 \sinh\psi_2 + \sin\varphi_1 \sinh\theta_2 \cosh\psi_2$
u	$v=2$
1	$\sin\varphi_1 \cos\theta_1 \cos\psi_1 + \cos\varphi_1 \cosh\theta_2 \sin\psi_1$
2	$-\sin\varphi_1 \cos\theta_1 \sin\psi_1 + \cos\varphi_1 \cosh\theta_2 \cos\psi_1$
3	$\sin\varphi_1 \sin\theta_1 \cosh\psi_2 + \cos\varphi_1 \sinh\theta_2 \sinh\psi_2$
0	$-\sin\varphi_1 \sin\theta_1 \sinh\psi_2 - \cos\varphi_1 \sinh\theta_2 \cosh\psi_2$
u	$v=3$
1	$-\cosh\varphi_2 \sin\theta_1 \cos\psi_1 + \sinh\varphi_2 \sinh\theta_2 \sin\psi_1$
2	$\cosh\varphi_2 \sin\theta_1 \sin\psi_1 + \sinh\varphi_2 \sinh\theta_2 \cos\psi_1$
3	$\cosh\varphi_2 \cos\theta_1 \cosh\psi_2 + \sinh\varphi_2 \cosh\theta_2 \sinh\psi_2$
0	$-\cosh\varphi_2 \cos\theta_1 \sinh\psi_2 - \sinh\varphi_2 \cosh\theta_2 \cosh\psi_2$
u	$v=0$
1	$\sinh\varphi_2 \sin\theta_1 \cos\psi_1 - \cosh\varphi_2 \sinh\theta_2 \sin\psi_1$
2	$-\sinh\varphi_2 \sin\theta_1 \sin\psi_1 - \cosh\varphi_2 \sinh\theta_2 \cos\psi_1$
3	$-\sinh\varphi_2 \cos\theta_1 \cosh\psi_2 - \cosh\varphi_2 \cosh\theta_2 \sinh\psi_2$
0	$\sinh\varphi_2 \cos\theta_1 \sinh\psi_2 + \cosh\varphi_2 \cosh\theta_2 \cosh\psi_2$

$$\varphi_1,\ \varphi_2,\ \theta_1,\ \theta_2,\ \psi_1,\ \psi_2 \tag{4.13}$$

yields the same scheme of coefficients as any of the combinations

$$\varphi_1 + 2q_1\pi,\ \varphi_2 + 2q_2\pi i,\ \theta_1 + 2q_3\pi,\ \theta_2 + 2q_4\pi i,\ \psi_1 + 2q_5\pi,\ \psi_2 + 2q_6\pi i$$
$$(q_r = 0, \pm 1, \pm 2, \dots);$$

$$\varphi_1 + \pi,\ \varphi_2 + \pi i,\ \theta_1,\ \theta_2,\ \psi_1 + \pi,\ \psi_2 + \pi i; \tag{4.14}$$
$$\varphi_1 + \pi,\ \varphi_2 + \pi i,\ \theta_1 + \pi,\ \theta_2 + \pi i,\ \psi_1,\ \psi_2;$$
$$\varphi_1 + \pi,\ \varphi_2,\ -\theta_1,\ -\theta_2,\ \psi_1 + \pi,\ \psi_2.$$

Any real orientation of the moving axes x' can therefore be expressed in terms of six angles that satisfy the relations

$$\begin{gathered} 0 \leqq \varphi_1 < 2\pi, \qquad 0 \leqq \theta_1 \leqq \pi, \qquad 0 \leqq \psi_1 < 2\pi, \\ -\infty < \varphi_2 < \infty, \quad -\infty < \mathrm{Rl}\,\theta_2 < \infty, \ \mathrm{Im}\theta_2 = 0, \pi, \quad -\infty < \psi_2 < \infty. \end{gathered} \tag{4.15}$$

It will be observed from table 4.1 that if the Eulerian angles satisfy (4.15), the coefficient $\Lambda^0{}_0$ is positive or negative according as $\mathrm{Im}\theta_2 = 0$ or $\mathrm{Im}\theta_2 = \pi$.

It will be convenient for our further work to introduce the six alternative angles

$$\varphi^{\pm} = \varphi_1 \pm i\varphi_2, \qquad \theta^{\pm} = \theta_1 \pm i\theta_2, \qquad \psi^{\pm} = \psi_1 \pm i\psi_2. \tag{4.16}$$

It is readily seen from (4.05) together with the commutation relations for the operators P that in terms of these new angles the relation (4.10) may be written in the form

$$\begin{aligned} x' &= \exp(-i\varphi^- P_3^-)\exp(-i\theta^- P_2^-)\exp(-i\psi^- P_3^-) \times \\ &\times \exp(-i\varphi^+ P_3^+)\exp(-i\theta^+ P_2^+)\exp(-i\psi^+ P_3^+)x \equiv \Lambda^-\Lambda^+ x, \end{aligned} \tag{4.17}$$

the operators Λ^+ and Λ^- being given by

$$\Lambda^{\pm} = \exp(-i\varphi^{\pm} P_3^{\pm})\exp(-i\theta^{\pm} P_2^{\pm})\exp(-i\psi^{\pm} P_3^{\pm}). \tag{4.18}$$

4.3. The rotation operators J and J'

We now proceed to construct the six differential operators J^{uv} which generate rotations of the x'-system in the x^u, x^v-planes. Stated more precisely, we shall derive expressions for the operators

$$\begin{aligned} J_j^+ &= \tfrac{1}{2}(J^{kl} + iJ^{j0}), \\ J_j^- &= \tfrac{1}{2}(J^{kl} - iJ^{j0}), \end{aligned} \qquad (j, k, l = 1, 2, 3 \text{ cycl.}). \tag{4.19}$$

We shall also encounter operators J'^{uv}, or, equivalently, operators defined by

$$\begin{aligned} J'^+_j &= \tfrac{1}{2}(J'^{kl} + iJ'^{j0}), \\ J'^-_j &= \tfrac{1}{2}(J'^{kl} - iJ'^{j0}), \end{aligned} \qquad (j, k, l = 1, 2, 3 \text{ cycl.}). \tag{4.20}$$

The general properties of the operators J and J' have been extensively discussed in chapter II. According to the results of section 2.5, any primed operator commutes with any unprimed one.

Further, both J^+ and J^- satisfy the commutation relations for an angular momentum in three dimensions, just as P^+ and P^-. By the same token, J'^+ and J'^- formally behave as the three-dimensional operators J'. The operators J^+ commute with the operators J^-, and the operators J'^+ commute with the operators J'^-.

Let us now compare the transformation formula (4.17) with (3.03), and let us retrace the procedure by which expressions for the three-dimensional operators J_k and J'_k have been derived. We may summarize the argument of sections 3.3 and 3.4 by saying that the formulas (3.26) and (3.27) for J_k and J'_k, respectively, essentially follow from (3.03) by differentiation and proper application of the commutation relations for the operators P_k. Now both P_k^+ and P_k^- behave just as the three-dimensional operator P_k. Hence in (4.17) both Λ^+ and Λ^- may be formally compared with the operator Λ in (3.03). Further, as any operator P^+ commutes with any operator P^-, Λ^- will in no way interfere with the construction of operators J^+ and J'^+. By the same token, Λ^+ will not interfere with J^- and J'^-. We need therefore only copy the reasoning of sections 3.3 and 3.4 to see that J_k^+, J_k^- and J'^+_k, J'^-_k may be derived from J_k and J'_k, respectively, by merely supplying the Eulerian angles with indices + or −, as the case may be. In the notation of section 3.4 this result may be written in the form

$$
\begin{aligned}
J_k^+ &= J_k(\varphi^+, \theta^+, \psi^+), \qquad & J'^+_k &= J'_k(\varphi^+, \theta^+, \psi^+), \\
J_k^- &= J_k(\varphi^-, \theta^-, \psi^-), \qquad & J'^-_k &= J'_k(\varphi^-, \theta^-, \psi^-).
\end{aligned}
\tag{4.21}
$$

As we have noted in (2.46), J and J' are related to one another according to

$$J^{uv} J_{uv} = J'^{vw} J'_{vw}. \tag{4.22}$$

It follows from (4.21) that we also have

$$(J^+)^2 = (J'^+)^2, \qquad (J^-)^2 = (J'^-)^2. \tag{4.23}$$

4.4. Rotation eigenfunctions

By virtue of the formal correspondence between operators with indices + and − and three-dimensional angular momenta, it is quite easy to obtain functions of the Eulerian angles which under the rotations J transform according to the representation $D(j^+, j^-)$. If we particularly choose such functions to be eigenfunctions of J'^{+}_3 and J'^{-}_3, then it follows from what was said in section 3.5 that they must be proportional to

$$Y^{m^+, m'^+}_{j^+}(\varphi^+, \theta^+, \psi^+) Y^{m^-, m'^-}_{j^-}(\varphi^-, \theta^-, \psi^-), \tag{4.24}$$

where j^+, m^+, j^-, m^- satisfy (4.07) while m'^+ and m'^- may take on the values

$$\begin{aligned} m'^+ &= -j^+, -j^+ + 1, \ldots, j^+, \\ m'^- &= -j^-, -j^- + 1, \ldots, j^-. \end{aligned} \tag{4.25}$$

The functions (4.24), $Y(+)Y(-)$ in an abbreviated notation, are simultaneous eigenfunctions of $(J^+)^2$, J^+_3, J'^+_3, $(J^-)^2$, J^-_3, and J'^-_3 with eigenvalues $j^+(j^+ + 1)$, m^+, m'^+, $j^-(j^- + 1)$, m^-, and m'^-, respectively. They may be compared with the spinors discussed in section 4.1 in that the functions with fixed j^+, m'^+, j^-, and m'^- constitute a family which transforms just as the family of spinors with fixed j^+ and j^-.

If $j^+ + j^-$ is half-integral, the functions $Y(+)Y(-)$ are two-valued in the sense that they change sign when one of the angles φ_1, θ_1, ψ_1 is increased by 2π, or one of the angles φ_2, θ_2, ψ_2 by $2\pi i$. It is not difficult to see from (3.62) and (4.16) that they also change sign when the angles φ_1, φ_2, θ_1, θ_2, ψ_1, ψ_2 are replaced by a combination such as $\varphi_1 + \pi$, φ_2, $-\theta_1$, $-\theta_2$, $\psi_1 + \pi$, ψ_2. The orientation of the moving axes x' is invariant under any of these substitutions.

It will be observed from section 3.5 and the subsequent discussion on supplementary conditions given in section 3.63, that the functions $Y(+)Y(-)$ are uniquely determined by the requirement that they shall transform as the spinors U discussed in section 4.1. Now $Y(+)Y(-)$ contains, among others, a factor $\exp(m^- - m^+)\varphi_2$. Since m^+ and m^- take on the values (4.07), this factor is not bounded when φ_2 tends to infinity. Yet we must certainly take $m^- - m^+$ to be real and, in general, different from zero to find functions which transform according to $D(j^+, j^-)$. It follows from section 3.5 that for our functions to transform as spinors m'^+ and m'^- must also be real, which means, then, that the factor $\exp(m'^- - m'^+)\psi_2$ is not bounded when ψ_2 tends to infinity. It will be inferred from (3.72) that if the second-order differential equation for Θ is expressed in terms of $t = \sin^2\frac{1}{2}\theta$, it is a hypergeometric equation with singularities at $t = 0$, at $t = 1$, and at the point at infinity. Clearly this equation does not permit of solutions which are bounded everywhere. It is again our supplementary condition - transformation according to $D(j^+,j^-)$ - which implicitly determines that the desired functions shall not be bounded when θ_2 tends to infinity. In case φ_2, θ_2, or ψ_2 tend to infinity the system x' performs a uniform motion with respect to the system x, the velocity of which tends to the velocity of light.

In order that our formalism may be physically useful, we must give a prescription for constructing scalar products of functions $Y(+)Y(-)$. Now we are used to computing scalar products of functions in ordinary space by integration, and it is well known that in the spinor theory we have

$$\left(U^{m^+m^-}_{j^+j^-}, U^{n^+n^-}_{i^+i^-}\right) = \delta_{j^+i^+}\,\delta_{m^+n^+}\,\delta_{j^-i^-}\,\delta_{m^-n^-}. \tag{4.26}$$

However, the Eulerian angles will be interpreted as spin variables, and we have no physical principle as yet which tells how to obtain scalar products of functions in spin space. It might seem attractive to consider integrations in spin space too, but since the functions $Y(+)Y(-)$ are not quadratically integrable over the whole domain of values of Eulerian angles that correspond to real Lorentz transformations, this is not a very workable idea. At present the best thing to do is to choose the most reasonable analogue of (4.26) in the hope that this may lead to a useful formalism. Accordingly we propose the definition

$$\left(\nu Y^{m^+,m'^+}_{j^+}(+)Y^{m^-,m'^-}_{j^-}(-),\ \nu Y^{n^+,n'^+}_{i^+}(+)Y^{n^-,n'^-}_{i^-}(-)\right) =$$
$$= \delta_{j^+i^+}\delta_{m^+n^+}\delta_{m'^+n'^+}\delta_{j^-i^-}\delta_{m^-n^-}\delta_{m'^-n'^-}, \tag{4.27}$$

where ν is a normalization constant which will drop out, however, in the applications to be discussed in chapter V. It will be shown in section 5.1 that if we adopt the definition (4.27), all results on expectation values derived from the spinor theory can be taken over into a formalism with Eulerian angles. In the present state of our experimental and theoretical knowledge we see therefore no objection against our using spin functions which are not quadratically integrable. It should be emphasized that if instead of the functions $Y(+)Y(-)$ we had chosen some other set of functions of the Eulerian angles, the latter functions would not transform according to representations $D(j^+,j^-)$. They could therefore not be used to describe spinning particles in a satisfactory way.

As stated previously, the functions $Y(+)Y(-)$ are eigenfunctions of the operators J'^+_3 and J'^-_3 with eigenvalues m'^+ and m'^-, respectively. We may also combine functions $Y(+)Y(-)$ with various m'^{+} and m'^{-} to form eigenfunctions of $(J'^{+}+J'^{-})^2$ and $J'^{+}_3+J'^{-}_3$. Although $J'^{+}_j+J'^{-}_j$ is nothing but J'^{kl} $(j,k,l = 1,2,3$ cycl.), we shall henceforth denote this operator by S'_j for convenience,

$$S'_j = J'^{+}_j + J'^{-}_j = J'^{kl} \qquad (j,k,l = 1,2,3 \text{ cycl.}). \tag{4.28}$$

It is readily shown by the reasoning of section 3.7 that the functions

$$\begin{aligned} Z^{m^+m^-,m'}_{j^+j^-,s'}(+,-) &= \\ &= \sum_{m'^+,m'^-} v(j^+\,j^-\,{-m'^+}\,{-m'^-} \mid j^+\,j^-\,s'\,{-m'})\,Y^{m^+,m'^+}_{j^+}(+)\,Y^{m^-,m'^-}_{j^-}(-) \\ &\qquad (s' = j^++j^-, j^++j^--1, \ldots, |j^+-j^-|) \end{aligned} \tag{4.29}$$

are eigenfunctions of $(J^+)^2$, J^+_3, $(J^-)^2$, J^-_3, $(S')^2$, and S'_3 with eigenvalues $j^+(j^++1)$, m^+, $j^-(j^-+1)$, m^-, $s'(s'+1)$, and m', respectively. It follows from the properties of the Clebsch-Gordan coefficients that the functions $Z(+,-)$ are orthonormal once for the scalar product of the functions $Y(+)Y(-)$ the definition (4.27) has been adopted. For future reference we note that

$$\begin{aligned} (S'_1 \pm iS'_2)Z^{m^+m^-,m'}_{j^+j^-,s'}(+,-) &= \\ &= -\,[(s' \pm m')(s' \mp m' + 1)]^{\frac{1}{2}} Z^{m^+m^-,m'\mp 1}_{j^+j^-,s'}(+,-). \end{aligned} \tag{4.30}$$

Equation (4.30) may be easily verified from (3.48) combined with the recurrence formula for the Clebsch-Gordan coefficients given by Condon and Shortley (1953), eq. $14^3 3$.

4.5. Surface harmonics in Minkowski space

We have seen in chapter III that any one- or two-valued representation of the group of rotations in three dimensions is associated with families of functions Y depending on three Eulerian angles. Surface harmonics on the other hand, which may be written as functions of $x^k/(x^l x_l)^{\frac{1}{2}}$, all transform according to one-valued representations. In Minkowski space a similar distinction exists between functions of Eulerian angles and functions of x^u. We shall briefly discuss this in the following, functions of $x^u/(|x^v x_v|)^{\frac{1}{2}}$ which transform according to representations $D(j^+,j^-)$ being referred to as surface harmonics in Minkowski space.

Since for the present we are not working with Eulerian angles, we shall denote rotation operators by P, rather than by J. Further we shall imagine P^{uv} to be represented by a differential operator according to (2.20). From (2.20) it follows that

$$(P^{23}P^{10} + P^{31}P^{20} + P^{12}P^{30})f(x) = 0. \tag{4.31}$$

Now the operator in the left-hand member of (4.31) is proportional to $(P^+)^2 - (P^-)^2$. Hence if $f(x)$ is an eigenfunction of $(P^+)^2$ and $(P^-)^2$ with eigenvalues $j^+(j^++1)$ and $j^-(j^-+1)$, respectively, $(j^+, j^- \geqq 0)$, it must have $j^+ = j^-$. Surface harmonics in Minkowski space therefore all transform according to representations $D(j,j)$.

Conversely, for any representation $D(j,j)$ there exists a family of surface harmonics. To show this it is convenient to use the auxiliary variables

$$\begin{aligned} y^{++} &= (x^1 + ix^2)/r, \quad y^{+-} = (-x^3 + x^0)/r, \\ y^{-+} &= (-x^3 - x^0)/r, \quad y^{--} = (-x^1 + ix^2)/r, \end{aligned} \qquad (r = |x^u x_u|^{\frac{1}{2}} \neq 0). \tag{4.32}$$

Since the quantities y^{++}; y^{+-}; y^{-+}; y^{--} transform as the spinors $U^{m^+m^-}_{\frac{1}{2}\,\frac{1}{2}}$ having $m^+, m^- = \frac{1}{2},\frac{1}{2};\ \frac{1}{2},-\frac{1}{2};\ -\frac{1}{2},\frac{1}{2};\ -\frac{1}{2},-\frac{1}{2}$, it is readily seen that the function $\alpha_j(y^{++})^{2j}$, $(2j = 0,1,2,\ldots)$, is an eigenfunction of $(P^+)^2$, P_3^+, $(P^-)^2$, and P_3^- with eigenvalues $j(j+1)$, j, $j(j+1)$, and j, respectively, it being understood that α_j is a normalization constant. If we now denote the surface harmonic which transforms as the spinor $U^{m^+m^-}_{j^+j^-}$ by $U^{m^+m^-}_{j^+j^-}(y)$, it follows from section 4.1 that the operator

$$\left[\frac{(j+m^+)!}{(2j)!\,(j-m^+)!}\right]^{\frac{1}{2}} (P_1^+ - iP_2^+)^{j-m^+} \tag{4.33}$$

carries $U^{jj}_{jj}(y)$ into $U^{m^+j}_{j\ j}(y)$, so that

$$U^{m^+j}_{j\ j}(y) = \alpha_j \left[\frac{(2j)!}{(j+m^+)!\,(j-m^+)!}\right]^{\frac{1}{2}} (y^{++})^{j+m^+}(y^{-+})^{j-m^+}. \tag{4.34}$$

On applying the operator

$$\left[\frac{(j+m^-)!}{(2j)!\,(j-m^-)!}\right]^{\frac{1}{2}} (P_1^- - iP_2^-)^{j-m^-} \tag{4.35}$$

to $U^{m^+j}_{j\ j}(y)$ we find that

$$U^{m^+m^-}_{j\ \ j}(y) = \alpha_j \sum_q \left[\frac{[(j+m^+)!\,(j-m^+)!\,(j+m^-)!\,(j-m^-)!]^{\frac{1}{2}}}{q!\,(j-m^+-q)!\,(j-m^--q)!\,(m^++m^-+q)!} \times \right.$$
$$\left. \times\ (y^{++})^{m^++m^-+q}(y^{+-})^{j-m^--q}(y^{-+})^{j-m^+-q}(y^{--})^q\right]. \tag{4.36}$$

where the summation is to be extended over all integers q consistent with the factorial notation.

From the fact that surface harmonics in Minkowski space must all have $j^+ = j^-$ we may conclude that the representations $D(j,j)$ are the only ones that admit of eigenvectors expressible in terms of three spherical angles. A representation such as $D(j^+,0)$ is also associated with functions of three angles, but apparently these cannot be interpreted as spherical angles in Minkowski space.

In a formalism with Eulerian angles the operators J transform the coefficients Λ^{uv} just as the operators P transform x^v, according to (2.14) and (2.40). Hence if in (4.32) we substitute Λ^{uv} for x^v, the corresponding expression (4.36) yields polynomials in Λ^{uv} ($v = 1,2,3,0$) which transform as surface harmonics. Since the operators J' transform Λ^{uv} just as $-P'$ transforms x'^u, by (2.22) and (2.51), the polynomials in question are annihilated by any operator J'^{tw} having $t \neq u$ and $w \neq u$. If we particularly choose six Eulerian angles in such a way that for a certain u the four quantities Λ^{uv} ($v = 1,2,3,0$) depend on three angles only, then these angles may be interpreted as spherical angles of a point on the x'^u-axis, and the polynomials considered above are the surface harmonics appropriate to the rotational motion of such a point. A similar situation is realized in chapter III, where the direction cosines Λ^{3k} ($k = 1,2,3$) depend on two spherical angles, the corresponding surface harmonics being annihilated by the operator $J'_3 \equiv J'^{12}$. However, we have not been able to find a set of six angles in Minkowski space which makes four coefficients Λ^{uv} depend on three angles and, at the same time, yields a workable expression for the operator $J^{vw}J_{vw}$. As a matter of fact, for several seemingly simple systems of Eulerian angles this operator cannot be separated in the six variables.

4.6. The spatial reflection

We now proceed to discuss the spatial reflection, which is represented by a unitary operator, say Q, satisfying

$$Qx^k = -x^k, \qquad Qx^0 = x^0,$$
$$Qf(x) = f(Qx), \tag{4.37}$$

$$QP^{kl}Q^{-1} = P^{kl}, \qquad QP^{k0}Q^{-1} = -P^{k0}. \tag{4.38}$$

The spatial reflection and the rotations in Minkowski space together form a group, which is called the full Lorentz group. It is assumed that physical equations must be invariant under every transformation of this group.

Equation (4.38) implies that

$$QP_k^+Q^{-1} = P_k^-,\qquad QP_k^-Q^{-1} = P_k^+, \tag{4.39}$$

from which it is readily seen that spinors must certainly satisfy

$$QU^{m^+m^-}_{j^+j^-} = \langle m^-m^+|Q|m^+m^-\rangle\, U^{m^-m^+}_{j^-j^+}. \tag{4.40}$$

In order to delimit the coefficients in (4.40) we now apply the operators $Q(P_1^+ \pm iP_2^+)$ and $(P_1^- \pm iP_2^-)Q$ to $U^{m^+m^-}_{j^+j^-}$. Equating coefficients we find that

$$\langle m^-\ m^+\pm 1|Q|m^+\pm 1\ m^-\rangle = \langle m^-m^+|Q|m^+m^-\rangle. \tag{4.41}$$

It may be likewise shown that

$$\langle m^-\pm 1\ m^+|Q|m^+\ m^-\pm 1\rangle = \langle m^-m^+|Q|m^+m^-\rangle. \tag{4.42}$$

It follows from (4.41) and (4.42) that in any representation of the full Lorentz group the spatial reflection couples rotation eigenvectors according to

$$QU^{m^+m^-}_{j^+j^-} = aU^{m^-m^+}_{j^-j^+},\qquad QU^{m^-m^+}_{j^-j^+} = bU^{m^+m^-}_{j^+j^-}\qquad (j^+ \neq j^-),$$
$$QU^{m^+m^-}_{j\ j} = cU^{m^-m^+}_{j\ j}, \tag{4.43}$$

where the parameters a, b, and c are independent of m^+ and m^-.

Since Q is to be unitary, we must have $a^*a = b^*b = 1$ and $c^*c = 1$. Further, if we want the spatial reflection to be one-valued in the sense that $QQ = 1$, a and b must be related according to $ab = 1$, whereas c is restricted to the values 1 and -1. In case $j^+ + j^-$ is half-integral one also allows two-valued representations of the spatial reflection, which are characterized by $ab = -1$.

It will be clear from the foregoing discussion that the spinors $U^{m^+m^-}_{j^+j^-}$ and $U^{m^-m^+}_{j^-j^+}$ must appear together in a basis for the full Lorentz group. Hence if $j^+ \neq j^-$, an irreducible representation of the full group is of $2(2j^++1)(2j^-+1)$ dimensions. With respect to proper rotations it reduces to the direct sum $D(j^+,j^-) \oplus D(j^-,j^+)$. Since the spinors with $j^+ = j^- = j$ are carried over into one another under rotations as well as under spatial reflection, we can also construct representations of the full Lorentz group which simply derive from $D(j,j)$. The latter representations are of $(2j+1)^2$ dimensions. In the following we shall loosely denote representations of the full Lorentz group by $D(j^+,j^-) \oplus D(j^-,j^+)$ or $D(j,j)$, as the case may be.

4.7. Spatial reflection and Eulerian angles

Since moving axes x' and Eulerian angles in Minkowski space have not been used previously, we do not know as yet how they will behave under spatial reflection. Hence in order to incorporate the spatial reflection in our formalism, we shall have to extend our notion of the operator Q so that it may be applied to quantities depending on the angles, and to functions of x'. As in the previous sections, our criterion will be that the new formalism shall be as closely analogous to the spinor theory as possible.

To start with, we note that the reflection invariance of the spinor theory is due, among others, to the fact that wave functions are composed of spinors which transform among themselves under spatial reflection. Hence if a formalism with Eulerian angles makes sense indeed, and if such a formalism can be made reflection invariant in the same simple way as the spinor theory, then the spatial reflection must turn a function which corresponds to a spinor into a similar function of certain reflected Eulerian angles which again corresponds to a spinor. In other words, in order to find a useful reflection-invariant theory with Eulerian angles we shall have to assume that reflected quantities Qx' and $Q\omega$ may be introduced in such a way that for any reasonable function $f(\omega)$ we have

$$Qf(\omega) = f(Q\omega), \tag{4.44}$$

where the $Q\omega$'s are Eulerian angles which describe the orientation of the axes Qx' with respect to the axes Qx,

$$Qx'^{u} = \Lambda^{u}{}_{v}(Q\omega)Qx^{v}. \tag{4.45}$$

It is understood here that x satisfies (4.37) both as an independent variable and as a function of x' and ω.

Let us now write

$$J(\omega)f(\omega) = g(\omega), \tag{4.46}$$

where $J(\omega)$ is a rotation operator as constructed in section 4.3, while $f(\omega)$ is any differentiable function of the Eulerian angles. According to (4.44) we then have

$$QJ(\omega)Q^{-1}f(Q\omega) = QJ(\omega)f(\omega) = g(Q\omega), \tag{4.47}$$

and so the operators J and Q satisfy

$$QJ(\omega)Q^{-1} = J(Q\omega). \tag{4.48}$$

It may be likewise shown that we also have

$$QJ'(\omega)Q^{-1} = J'(Q\omega). \tag{4.49}$$

According to the foregoing formulas, the quantities Qx, Qx', $Q\omega$, and $J(Q\omega)$ are related to one another in exactly the same way as the quantities x, x', ω, and $J(\omega)$; the operators $J(Q\omega)$ are linear combinations of the operators $\partial/\partial Q\omega$. It will be assumed in the following that from ω we may compute $Q\omega$, and vice versa, in such a way that the operators $J(\omega)$ may also be written as linear combinations of operators $\partial/\partial Q\omega$.

In order to get an idea as to a proper choice for the angles $Q\omega$, we observe from (2.29) that if x is considered as a function of x' and ω - satisfying (4.37) - we have

$$QJ^{kl}Q^{-1}Qx^{v} = J^{kl}Qx^{v}, \qquad QJ^{k0}Q^{-1}Qx^{v} = -J^{k0}Qx^{v}$$
$$(k, l = 1,2,3;\ v = 1,2,3,0). \tag{4.50}$$

If we now express the operators QJQ^{-1} and J in terms of the angles $Q\omega$, and if we consider Qx as a function of Qx' and $Q\omega$, then (4.50) is a system of linear differential equations for the functions $\Lambda_u{}^{v}(Q\omega)Qx'^{u}$ ($v = 1,2,3,0$). Since this system must be satisfied for any Qx', it may be shown by the methods developed in section 2.3, equations (2.33)-(2.37), that the operator Q transforms J according to

$$QJ^{kl}Q^{-1} = J^{kl}, \qquad QJ^{k0}Q^{-1} = -J^{k0}. \tag{4.51}$$

In view of (4.38) this is a very satisfactory result.

The most obvious choice for $Q\omega$ consistent with (4.48) and (4.51) is

$$Q\varphi^{+} = \varphi^{-} + 2n_1\pi, \quad Q\theta^{+} = \theta^{-} + 2n_2\pi, \quad Q\psi^{+} = \psi^{-} + 2n_3\pi,$$
$$Q\varphi^{-} = \varphi^{+} + 2n_4\pi, \quad Q\theta^{-} = \theta^{+} + 2n_5\pi, \quad Q\psi^{-} = \psi^{+} + 2n_6\pi,$$
$$(n_r = 0, \pm 1, \pm 2, \ldots), \tag{4.52}$$

which entails, by (3.26), (3.27), and (4.21), that

$$\begin{aligned} QJ_k^+Q^{-1} &= J_k^-, & QJ'^+_kQ^{-1} &= J'^-_k, \\ QJ_k^-Q^{-1} &= J_k^+, & QJ'^-_kQ^{-1} &= J'^+_k, \end{aligned} \tag{4.53}$$

in accordance with (4.51). It is readily seen from table 4.1, with (4.16), that (4.52) leads to

$$\begin{aligned} Q\Lambda^{kl} &= \varepsilon\Lambda^{kl}, & Q\Lambda^{k0} &= -\varepsilon\Lambda^{k0}, \\ Q\Lambda^{0l} &= -\varepsilon\Lambda^{0l}, & Q\Lambda^{00} &= \varepsilon\Lambda^{00}, \end{aligned} \qquad (\varepsilon = \exp(\pi i \sum_r n_r)), \tag{4.54}$$

which implies that

$$Qx'^k = -\varepsilon x'^k, \qquad Qx'^0 = \varepsilon x'^0. \tag{4.55}$$

It is to be noted that the angles $Q\omega$ are not uniquely determined by the requirement (4.51). As a matter of fact, if we multiply Q, (4.52), by the rotation φ^+, θ^+, ψ^+, φ^-, θ^-, $\psi^- \to \varphi^+ + \pi$, $-\theta^+$, $\psi^+ + \pi$, $\varphi^- + \pi$, $-\theta^-$, $\psi^- + \pi$, then we get an alternative transformation which not only satisfies (4.51), but even (4.53). However, since this transformation does not lead to any new result as compared with (4.52), we shall not discuss it separately.

Since J commutes with any J', J is invariant under a rotation through α in a plane fixed in the x'-system, where it is understood that α does not depend on the Eulerian angles. As a result the product of Q times such a rotation satisfies (4.51). However, such a product is either not essentially different from Q, or it transforms the coefficients Λ^{uv} so intricately that it does not seem to be a proper choice for the spatial reflection.

Conversely, according to (4.37) and (4.45) $x'^u x'_u$ is invariant under spatial reflection. Hence if $\bar{Q}$ denotes a typical choice for the spatial-reflection operator, then the transformation $x' \to \bar{Q}x'$ can be expressed in terms of rotations and reflections. Furthermore, the transformation in question involves an odd number of reflections, for otherwise x could be transformed into $\bar{Q}x$ by the chain of rotations $x \to x' \to \bar{Q}x' \to \bar{Q}x$. From this it follows that $\bar{Q}x'$ can be written as $\Gamma Q x'$, where Qx' is given by (4.55), while Γ represents a rotation which transforms the four variables Qx'^u among themselves. In case Γ is a rotation through an integer times 2π, or a rotation which transforms Qx' into $-Qx'$, $\bar{Q}$ is not essentially different from Q. Further, if Γ is another type of rotation, it is either a rotation in the plane through the Qx'^0-axis and the axis having direction cosines $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$, 0 with respect to the Qx'-system, or it is not symmetrical in the three space-like coordinates Qx'^k. In the former case it makes

$\bar{Q}x'^k$ depend on x'^0, and in the latter case it yields an expression for $\bar{Q}x'$ which is not symmetrical in the three space-like coordinates. In neither case $\bar{Q}$ seems to be a proper choice for the spatial-reflection operator. As regards the spatial reflection we shall therefore restrict ourselves to the operator Q introduced above.

The operator Q transforms functions $Y(+)Y(-)$ according to

$$\begin{aligned} QY^{m^+,m'^+}_{j^+}(+)Y^{m^-,m'^-}_{j^-}(-) &= \varepsilon_{j^+j^-} Y^{m^+,m'^+}_{j^+}(-)Y^{m^-,m'^-}_{j^-}(+), \\ \varepsilon_{j^+j^-} &= \exp\left(2\pi i\left[j^+(n_1+n_2+n_3)+j^-(n_4+n_5+n_6)\right]\right). \end{aligned} \tag{4.56}$$

Loosely speaking we may say that Q turns j^+, m^+, m'^+, j^-, m^-, m'^- into j^-, m^-, m'^-, j^+, m^+, m'^+, in that order.

Since Q commutes with the operators S' introduced in (4.28), it turns an eigenfunction of $(S')^2$ and S'_3 with eigenvalues $s'(s'+1)$ and m' into an eigenfunction with the same s' and m'. As a matter of fact, we may write

$$\begin{aligned} &QZ^{m^+m^-,m'}_{j^+j^-,s'}(+,-) = \\ &= \varepsilon_{j^+j^-} \sum_{m'^+,m'^-} \nu\left(j^+j^- -m'^+ -m'^- \mid j^+j^- s' -m'\right) Y^{m^+,m'^+}_{j^+}(-)Y^{m^-,m'^-}_{j^-}(+) \end{aligned} \tag{4.57}$$

which by the relation

$$\begin{aligned} &\left(j^+j^- -m'^+ -m'^- \mid j^+j^- s' -m'\right) = \\ &\qquad = (-1)^{j^++j^--s'}\left(j^-j^+ -m'^- -m'^+ \mid j^-j^+ s' -m'\right) \end{aligned} \tag{4.58}$$

can be simplified to

$$QZ^{m^+m^-,m'}_{j^+j^-,s'}(+,-) = (-1)^{j^++j^--s'}\,\varepsilon_{j^+j^-} Z^{m^-m^+,m'}_{j^-j^+,s'}(+,-). \tag{4.59}$$

The relation (4.58) is readily verified by considering Condon and Shortley's (1953) equation 14^35 for the Clebsch-Gordan coefficients, and substituting $\varkappa = j + m - \varkappa'$ therein. The transformation (4.59) is quite analogous to the spatial reflection in the spinor theory. It is two-valued if $\Sigma_r n_r$ is odd while $j^+ + j^-$ is half-integral, one-valued otherwise.

It was discussed in the preceding section that the $(2j+1)^2$ spinors U having fixed $j^+ = j^- = j$ are carried over into one another under rotations as well as under spatial reflection. The same holds true now for the $(2j+1)^2$ functions $Z(+,-)$ of fixed s', m'; fixed $j^+ = j^- = j$; m^+, $m^- = -j, -j+1, \ldots, j$. By contrast, since Q does not turn m'^{+}, m'^{-} into itself, it does not apply to the functions $Y(+)Y(-)$ having fixed m'^{+}, m'^{-}; fixed $j^{+} = j^{-} = j$; m^{+}, $m^{-} = -j, -j+1, \ldots, j$. Whereas $Y(+)Y(-)$ and $Z(+,-)$ show the same behaviour under the rotations generated by operators J, only the functions $Z(+,-)$ allow a description of the spatial reflection analogous to the spinor representation. When expressing wave functions in terms of Eulerian angles in spin space we shall therefore replace spinors by functions $Z(+,-)$, rather than by functions $Y(+)Y(-)$. As a result we shall have to consider the family-indices s' and m' associated with the three-dimensional angular momentum S'. This point will be discussed in the next chapter.

It has been mentioned above that for half-integral $j^{+} + j^{-}$ the spatial reflection in one- or two-valued according as $\Sigma_r n_r$ is even or odd. Let us now denote the spatial-reflection operators concerned by Q_e and Q_o, respectively. Omitting irrelevant subscripts, we then have

$$(Q_e Q_o - Q_o Q_e) Z_{j^{+}j^{-}} \neq 0 \qquad (2j^{+}+2j^{-} = 1,3,5,\ldots). \tag{4.60}$$

All possible operators Q_e commute among themselves, and likewise all possible operators Q_o.

In connection with (4.60) we recall that along with the spatial reflection the time-reflection also satisfies the relations (4.38) and (4.39). Hence for half-integral $j^{+} + j^{-}$, i.e. for half-integral spin, we can see that the space- and the time-reflection do not commute by representing one transformation by Q_e, and the other by Q_o. If it would be desirable to have two non-commuting operators which are both one-valued, or both two-valued, then we might use Q_e, iQ_o, or iQ_e, Q_o, respectively. As a matter of fact, it is well known that in the theory for Dirac particles - derived from $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$ - the space- and the time-reflection must not commute, cf. Umezawa (1956), chap. III, sec. 2. Under Racah's (1937) choice both operators are two-valued, being represented by the Dirac matrices γ^0 and $\gamma^1\gamma^2\gamma^3$, respectively. Now it is well known that in Dirac's theory rotations in the x^u, x^v-plane are generated by the operator $-i(\gamma^u\gamma^v - \gamma^v\gamma^u)/4$, cf. Umezawa, loc. cit. In this connection it is worth noting that for spin $\frac{1}{2}$ the operators J^{uv}, iQ_e, and Q_o may be readily translated into $-i(\gamma^u\gamma^v - \gamma^v\gamma^u)/4$, γ^0, and $\gamma^1\gamma^2\gamma^3$, or $-i(\gamma^u\gamma^v - \gamma^v\gamma^u)/4$, $\gamma^1\gamma^2\gamma^3$, and γ^0 for that matter.

V

WAVE FUNCTIONS FOR SPINNING PARTICLES

5.1. General principles

In a relativistically invariant theory a spinning particle is usually described by a wave function of the form

$$\Psi(X) = \sum_{p} \Phi^{p}(X)U^{p}, \tag{5.01}$$

where X stands for the position coordinates of the particle in question, while the quantities U^{p} are spinors which transform among themselves according to one of the finite-dimensional irreducible representations of the full Lorentz group. The U^{p}'s are nothing but the spinors $U^{m^{+}m^{-}}_{j^{+}j^{-}}$ discussed in chapter IV, but for the sake of simplicity we shall use abridged indices in the following. It may be inferred from section 4.6 that the running index p in (5.01) takes on $2(2j^{+}+1)(2j^{-}+1)$ or $(2j+1)^{2}$ values according as $j^{+} \neq j^{-}$ or $j^{+} = j^{-} = j$.

The wave equation for Ψ is a system of simultaneous linear differential equations to be satisfied by the functions Φ. For details concerning relativistic wave equations the reader may consult Corson (1953).

In a theory with wave functions of the form (5.01) the spinors U may be acted upon by matrix operators γ according to

$$\gamma U^{p} = \sum_{q} \langle q|\gamma|p\rangle U^{q}, \tag{5.02}$$

where any U^{q} is in the same family with U^{p}, i.e. where U^{p} and the U^{q}'s transform irreducibly among themselves under Lorentz transformations. It is essential that both the rotation operators P and the spatial-reflection operator Q are of type γ.

The spinors U are orthonormal. Accordingly the expectation value of the operator which carries Ψ into

$$\sum_{p} \Upsilon^{p}(X)U^{p} \tag{5.03}$$

equals

$$\sum_{p} \int \Phi^{p*}(X)\Upsilon^{p}(X)dV, \tag{5.04}$$

where dV stands for $dX^{1}dX^{2}dX^{3}$.

In the usual spinor theory the transformation properties of the U's are so essential that if we try to replace (5.01) and (5.02) by something else, we cannot hope to attain any result unless we introduce a family of new quantities which behave the same as the U's under rotations as well as under spatial reflection. If we particularly want to develop a theory in which the spin is expressed in terms of Eulerian angles, we shall therefore have to replace the family of spinors U^p by a family of functions $Z^{p,m'}_{s'}(\omega)$ having fixed s' and m'. Conversely, if the functions Φ are such that (5.01) describes a free particle of specified mass and spin, then any of the functions

$$\Psi^{m'}_{s'}(X,\omega) = \sum_p \Phi^p(X) Z^{p,m'}_{s'}(\omega) \tag{5.05}$$

can be used for such a particle as well. For if we assume the Z's to be orthonormal just like the U's, and if we replace (5.02) by

$$\gamma Z^{p,m'}_{s'}(\omega) = \sum_q \langle q|\gamma|p\rangle Z^{q,m'}_{s'}(\omega), \tag{5.06}$$

then (5.05) yields the same expectation values for operators γ as (5.01). The same applies to operators such as $\partial/\partial X$ and, in fact, to any operator C which can be expressed entirely in terms of operators γ, $\partial/\partial X$, and X, and therefore does not affect s' and m'. As in a theory for free particles we only have operators of type C, it follows that in the absence of interactions (5.05) yields the same results as (5.01), irrespective of the particular values of s' and m'. This is not undesirable, as the established theory for free particles is quite satisfactory. By virtue of its analogy with the spinor theory, the free-particle formalism with Eulerian angles may be readily quantized by known field-theoretical methods.

Whereas for free particles the new formalism is straightforward, the situation becomes much more complicated once interactions are present. For if the family-indices s' and m' have any physical significance, they must obviously play a part in at least some interaction schemes. In other words, we cannot construct a sound interaction theory unless the interpretations of s' and m' are known. In the present investigation we shall therefore not try to extend our formalism so as to include interactions. At the same time, it is hoped that the introduction of Eulerian angles may eventually contribute to overcoming the serious difficulties still encountered in present-day interaction theory.

According to section 4.4, s' and m' are the eigenvalues of the three-dimensional angular momentum S' defined in (4.28), and of the component S'_3 thereof. Now since the operator γ in (5.06) couples only functions Z having the same s' and m', and since it does not depend on either s' or m', S' commutes with γ. As a matter of fact, it is not difficult to see that it commutes with any operator of type C. We note here that if in (5.02) γ represents the rotation operator P^{uv}, then in (5.06) it stands for $J^{uv}(\omega)$. If in (5.02) it represents the spatial reflection, then the same holds true in (5.06).

5.2. The family-indices s' and m'

Due to the occurrence of family-indices s' and m', our formalism with Eulerian angles suggests that for any particular values of mass and spin there may exist several kinds of particles. Distinguished from each other by s' and m', these particles may be expected to show the same behaviour as long as no interactions are present, according to the preceding section.

Now if (5.05) is a normalized wave function for a free particle of specified mass and spin, then the operator

$$- [(s' \pm m')(s' \mp m' + 1)]^{-1/2}(S'_1 \pm iS'_2) \qquad (5.07)$$

carries (5.05) into a similar normalized wave function for a free particle of the same mass, spin, and s'. The new particle has upper index $m' \mp 1$ instead of m', however, as may be seen from (4.30). The new upper index is $m' \mp 1$ indeed, rather than $m' \pm 1$, owing to the fact that the commutation relations for primed operators have a minus sign as compared with the usual commutators for angular momenta, cf. (2.53). Since S' commutes with the spatial-reflection operator, according to section 4.7, the new particle has the same parity as the original one. Summarizing we so arrive at the conclusion that there might exist a group of particles which all have the same mass, spin, parity, and s', are transformed into one another under operations of the form (5.07), and can be distinguished by the quantum number m', i.e. by some physical quantity closely related to S'_3. This group might contain $2s'+1$ particles at most.

When the functions Z in (5.05) transform according to the representation $D(j^+,j^-) \oplus D(j^-,j^+)$, $(j^+ \neq j^-)$, the wave function Ψ has $2(2j^++1)(2j^-+1)$ components Φ; when the Z's transform according to $D(j,j)$, the wave function Ψ has $(2j+1)^2$ components Φ. If it is now understood that in the latter case $j^+ = j^- = j$, then in either case Ψ in principle refers to an assembly of particles having spins $j^+ + j^-$, $j^+ + j^- - 1, \ldots, |j^+ - j^-|$. If Ψ is to describe only particles of a definite spin s, one has to make certain supplementary conditions so as to reduce the number of independent components Φ. However, we need not bother about the details of that procedure; at present the point is that Ψ refers to integral or half-integral spins according as $2j^+ + 2j^-$ is even or odd. Since we know from (4.29) that s' may take on the values $j^+ + j^-$, $j^+ + j^- - 1, \ldots, |j^+ - j^-|$, that suffices to see that s' must be integral for bosons, and half-integral for fermions. It should be emphasized that s' need not be equal to the spin. This is a considerable generalization as compared with the three-dimensional theory of chapter III, where we had $(J)^2 = (J')^2$.

The representation $D(0,0)$ obviously refers to zero spin and zero s'. Dirac's equation derives from $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$. As a result it describes particles of spin $\frac{1}{2}$, and according to the ideas presented here it refers to $s' = \frac{1}{2}$. The familiar vector field of spin 1 transforms according to $D(\frac{1}{2},\frac{1}{2})$. Its four components are subject to one supplementary condition to ensure that the spin shall be 1 rather than 0. For a vector field s' may be either 0 or 1.

As for particles of low spin and low s', we thus far lack a theory for spin 0 and $s' = 1$. It will be shown in the next section that a formalism for this case can be constructed without much difficulty.

5.3. Fields having spin 0 and $s' = 1$

In the present section we shall develop a quantum field theory for spin 0 and $s' = 1$. Starting from an appropriate Lagrangian function, we shall first construct a classical field theory, or, equivalently, a quantum-mechanical formalism for a single particle. After that we shall briefly indicate how to carry out the usual process of quantization so as to admit of the creation and annihilation of particles. Whereas for reasons of principle the present formalism cannot be identical with the familiar theory for spin 0, it leads to the very same results as regards the expectation values of observables not depending on s' and m'. For a review of the field-theoretical methods employed in the present section the reader is referred to Wentzel (1949), chap. I.

The simplest possible representation for $s' = 1$ is $D(\frac{1}{2},\frac{1}{2})$. It requires a wave function of four components Φ. By suitable linear combination the Φ's can be chosen so that under rotations they transform as the components of a four-vector. They will be denoted by Φ^u accordingly.

In the following we shall use the abbreviations

$$\partial^u = \partial/\partial X_u, \qquad \Phi^{u,v} = \partial\Phi^u(X)/\partial X_v. \tag{5.08}$$

Further we shall use the auxiliary parameter $\mu = mc/\hbar$, m denoting the mass of the particles under consideration. It will be assumed that $m \neq 0$.

In the absence of further restrictions, the four functions Φ^u describe particles of spins 0 and 1. It is an essential feature of the familiar theory for spin 1 that there Φ^0 is redundant. That guarantees that the theory describes only particles of spin 1, the admixture of spin 0 being eliminated. For our present purpose we shall now see that the functions Φ^k are redundant. To this end we shall choose our Lagrangian to be independent of $\Phi^{k,0}$ ($k = 1,2,3$), so that the momenta conjugate to the Φ^k's vanish identically. After some manipulations the Φ^k's can be eliminated then in favour of the momentum conjugate to Φ^0, denoted by Π, so as to give a formalism entirely in terms of Φ^0 and Π.

To make it comprehensible now that the field Φ^u can have the properties of an established field of spin 0, say φ, we recall that under rotations the momentum conjugate to φ transforms as x^0. At the same time, the Hamiltonian equations of motion are invariant under interchange of coordinates and momenta. It is therefore natural to anticipate that the familiar theory for spin 0 can be equivalently formulated in terms of a "reciprocal" wave function φ^0 and its conjugate π. It will presently be shown that this is the case indeed, φ^0 and π being nothing but the quantities Φ^0 and Π alluded to above.

Whereas our argument concerning the canonical equations of motion seems obvious, it must be noted that the reciprocal Hamiltonian density contains $\Pi^{,k}$. As a result it cannot be directly obtained from a Lagrangian function. It therefore seems not superfluous to show that starting from the Lagrangian function

$$\mathsf{L} = \mu^2\Phi^{u*}\Phi_u + \Phi^{v*}{}_{,v}\,\Phi^{w}{}_{,w} \tag{5.09}$$

we nevertheless arrive at the reciprocal formalism for free charged particles of spin 0. The simplification to neutral particles will be obvious.

The choice (5.09) being taken for L, the familiar variational problem

$$\delta\int \mathsf{L}\,dV dX^0 = 0 \tag{5.10}$$

leads to the wave equations

$$\mu^2\Phi^u - \Phi_w{}^{,uw} = 0. \tag{5.11}$$

Differentiating (5.11) with respect to X_v and remembering that $\partial^v\partial^w = \partial^w\partial^v$, we obtain the supplementary conditions

$$\Phi^{u,v} = \Phi^{v,u}. \tag{5.12}$$

On combining (5.11) and (5.12) it is now found that

$$\mu^2\Phi^u - \Phi^{u,v}{}_v = 0, \tag{5.13}$$

i.e. that Φ^u must satisfy the Klein-Gordon equation as usual.

On account of (5.09) the canonical energy-momentum tensor takes the form

$$\mathsf{T}^{uv} = -\Phi^{w*}{}_{,w}\Phi^{u,v} - \Phi^{u*,v}\Phi^w{}_{,w} + \mathsf{L}g^{uv}. \tag{5.14}$$

By virtue of (5.12) T^{uv} is symmetric with respect to u and v, which permits us to anticipate that particles described by the functions Φ^u will indeed have spin 0.

According to (5.14) the energy density is given by

$$\mathsf{H} \equiv \mathsf{T}^{00} = -\Phi^{w*}{}_{,w}\Phi^{0,0} - \Phi^{0*,0}\Phi^w{}_{,w} - \mu^2\Phi^{w*}\Phi_w - \Phi^{u*}{}_{,u}\Phi^v{}_{,v}. \tag{5.15}$$

However, since we are primarily interested in the total energy $H = \int\mathsf{H}dV$, rather than in H, it is convenient to go over to the function $\bar{\mathsf{H}}$ which derives from H according to

$$\bar{\mathsf{H}} = \mathsf{H} + \partial_k(\Phi^{u*}{}_{,u}\Phi^k) + \partial_k(\Phi^{k*}\Phi^u{}_{,u}). \tag{5.16}$$

In terms of $\bar{\mathsf{H}}$ the total energy takes the form

$$H \equiv \int\mathsf{H}dV = \int\bar{\mathsf{H}}dV, \tag{5.17}$$

where only a surface integral at infinity has been discarded. By virtue of (5.11) we have

$$\bar{\mathsf{H}} = \mathsf{H} + \Phi^{u*}{}_{,u}\Phi^k{}_{,k} + \Phi^{k*}{}_{,k}\Phi^u{}_{,u} + 2\mu^2\Phi^{k*}\Phi_k, \tag{5.18}$$

which means that

$$\bar{\mathsf{H}} = \mu^2\Phi^{k*}\Phi_k + \mu^2\Phi^{0*}\Phi^0 + \Phi^{u*}{}_{,u}\Phi^v{}_{,v}. \tag{5.19}$$

In the present formalism the momenta conjugate to the Φ's take the form

$$\Pi_u \equiv (1/c)\partial\mathsf{L}/\partial\Phi^u{}_{,0} = (1/c)g_u^0\Phi^{v*}{}_{,v}. \tag{5.20}$$

According to (5.20) all three momenta Π_k vanish identically. It is therefore convenient to use the abbreviations

$$\Phi = \Phi^0, \qquad \Pi = \Pi_0. \tag{5.21}$$

On account of the wave equations (5.11) the space-like field components Φ^k may now be eliminated by using the relations

$$\mu^2\Phi^k = c\Pi^{*,k}. \tag{5.22}$$

Accordingly $\bar{\mathrm{H}}$ may be written in the form

$$\bar{\mathrm{H}} = \mu^2\Phi\Phi^* + (c^2/\mu^2)\Pi_{,k}\Pi^{*,k} + c^2\Pi\Pi^*. \tag{5.23}$$

In terms of Φ and Π the canonical momentum density G and the charge-current density s take the form

$$\begin{aligned} \mathrm{G}^k &\equiv (1/c)\mathrm{T}^{0k} = -\Pi\Phi^{,k} - \Phi^{*,k}\Pi^*, \\ s^k &= (iec^2/\hbar\mu^2)(\Pi^{,k}\Pi^* - \Pi^{*,k}\Pi), \\ s^0 &= (iec/\hbar)(\Phi^*\Pi^* - \Phi\Pi), \end{aligned} \tag{5.24}$$

where e is the elementary charge. Besides G we shall also consider the quantity $\bar{\mathrm{G}}$ given by

$$\bar{\mathrm{G}}^k = \Pi^{,k}\Phi + \Phi^*\Pi^{*,k}. \tag{5.25}$$

As regards the total momentum G and the total angular momentum M we have

$$\begin{aligned} G^k &\equiv \int \mathrm{G}^k dV = \int \bar{\mathrm{G}}^k dV, \\ M^{kl} &\equiv \int (X^k\mathrm{G}^l - X^l\mathrm{G}^k)dV = \int (X^k\bar{\mathrm{G}}^l - X^l\bar{\mathrm{G}}^k)dV, \end{aligned} \tag{5.26}$$

surface integrals at infinity being discarded.

Let us now compare the foregoing expressions with the familiar formulas for a field of spin 0. Denoting the field variables of the latter field by lower-case symbols, we have, according to Wentzel (1949), chap. II,

$$\begin{aligned} \mathrm{H} &= \pi^*\pi + c^2\varphi^{*,k}\varphi_{,k} + \mu^2c^2\varphi^*\varphi, \\ \mathrm{G}^k &= -\pi\varphi^{,k} - \varphi^{*,k}\pi^*, \\ s^k &= (iec^2/\hbar)(\varphi^{*,k}\varphi - \varphi^{,k}\varphi^*), \\ s^0 &= -(iec/\hbar)(\pi\varphi - \pi^*\varphi^*). \end{aligned} \tag{5.27}$$

If we now make the change of variables

$$\Phi \to \pm\pi^*/\mu, \qquad \Pi \to \mp\mu\varphi^*, \tag{5.28}$$

then the expressions for $\overline{H}$, $\overline{G}$, and s given in (5.23) – (5.25) exactly go over into the expressions (5.27). From this we may conclude that except for the family-indices s' and m' our formalism in terms of Φ and Π yields the same physical results as the familiar theory for spin 0. The functions Φ^u are appropriate to what is considered as a scalar or a pseudoscalar field according as under spatial reflection they behave as the components of a vector or a pseudovector.

The foregoing remarks apply to a classical field theory or, equivalently, to a quantum-mechanical formalism for a single particle of spin 0. The field Φ may be readily quantized by postulating that Φ and Π shall be operators satisfying the commutation relations

$$[\Pi(X^k,X^0),\Phi(\bar{X}^k,X^0)] = -i\hbar\,\delta(X^1-\bar{X}^1)\delta(X^2-\bar{X}^2)\delta(X^3-\bar{X}^3). \tag{5.29}$$

The quantized theory is completely determined by the properties of the classical theory together with the commutation relations imposed on Φ and Π. Since under the substitution (5.28) the relations (5.29) exactly go over into the commutation relations for the (pseudo)scalar field, it follows that the analogy between Φ and the familiar fields of spin 0 will be completely preserved after the quantization procedure has been carried out.

In order to describe the interaction with an electromagnetic field A one may start from the Lagrangian function

$$L = \mu^2\Phi^{u*}\Phi_u + [(\partial_v + \frac{ie}{\hbar c}A_v)\Phi^{v*}][(\partial_w - \frac{ie}{\hbar c}A_w)\Phi^w] + L_{\text{free}}, \tag{5.30}$$

where L_{free} stands for the Lagrangian of the free electromagnetic field. It may be seen by straightforwardly generalizing our foregoing considerations that the Lagrangian (5.30) again leads to a formalism in entire agreement with the established theory. Likewise interactions usually associated with the interaction Lagrangian

$$L_{\text{int}} = -\varphi^*B - B^*\varphi - \varphi^{*,u}\Gamma_u - \Gamma^{u*}\varphi_{,u} \tag{5.31}$$

may be equivalently discussed in terms of Φ by starting from

$$\begin{aligned} L_{\text{int}} = {} & \pm(1/\mu c)\Phi^{u*}{}_{,u}B \pm (1/\mu c)B^*\Phi^u{}_{,u} + (1/\mu^2c^2)B^*B \\ & \pm (\mu/c)\Phi^{u*}\Gamma_u \pm (\mu/c)\Gamma^{u*}\Phi_u + (1/c^2)\Gamma^{u*}\Gamma_u. \end{aligned} \tag{5.32}$$

Here B and Γ depend on the variables of fields other than Φ. The symbol $\pm$ refers to the two possible substitutions indicated in (5.28). It is of course understood that (5.30) and (5.32) should be modified if the interactions concerned would depend on s' and m'.

The results of the present section may be summarized by the statement that a field said to be (pseudo)scalar can be successfully described in terms of a (pseudo)vector. Whereas a scalar can only have $s' = 0$, the s' of a four-vector may be either 0 or 1. According to the ideas presented here the s' of a field having spin 0 may therefore certainly take the values 0 and 1.

5.4. Note on non-local fields

In most field theories the fields are described by point functions $\Phi(X)$, and the Lagrangian density depends on the field variables at a single space-time point only, as in the preceding section. Such theories are called local. They picture bare particles as mathematical points. Now all relativistic theories with local interaction give rise to certain divergences, infinite self-energies among others. Since an infinite self-energy also occurs in the classical theory of a point electron, it has been suggested that at least some divergences could be removed if particles were attributed finite extensions in space-time. Various attempts have therefore been made to introduce a finite particle radius in a relativistically invariant way. A general discussion of this point may be found in the textbooks by Schweber, Bethe, and de Hoffmann (1955), I, secs. 11*c*, 20*b*, and Umezawa (1956), chap. I, sec. 5.

It is a common feature of all extended-particle theories that the coupling between two fields is no longer localized at a single point; it is spread over a finite region in space-time. For instance, the interaction term $\varphi^*(X)\mathrm{B}(X)$ in (5.31) may be replaced by

$$\int \varphi^*(\bar{X})\,\mathrm{B}(\bar{\bar{X}})F(X,\bar{X},\bar{\bar{X}})\,d\bar{V}d\bar{X}^0 d\bar{\bar{V}}d\bar{\bar{X}}^0, \tag{5.33}$$

where F is a form factor related to the structure of the particles under consideration.

In (5.33) and, in fact, in nearly all so-called non-local theories, the field variables themselves are still completely localized. Yukawa (1950a, b, 1953a, b), on the other hand, has proposed a theory in which the field variables are non-local too in that they depend on two sets of rectangular coordinates, X and r in Yukawa's notation. The X's represent the usual position coordinates, and it has been shown by Fierz (1950a, b) that the r's are spin variables. In other words, Yukawa's X and r have been denoted by X and x in the present investigation. His non-local fields picture particles as extended spinning entities whose linear dimensions and intrinsic rotation are described through x. Yukawa's theory has been investigated by several authors, many references being given by Yukawa (1953a) himself.

It has been pointed out by Fierz (1950a, b) that in the absence of interactions a non-local field is equivalent to an assembly of ordinary local fields of various spins. Now obviously the simplest type of non-local field is described by a scalar function $\Phi(X, x)$. The spin properties of such a field must be expressed entirely in terms of the internal variables x, i.e. in terms of rectangular coordinates. From section 4.5 it may therefore be inferred that a scalar non-local field describes only particles the wave functions of which transform according to representations $D(j, j)$. This is, in fact, exactly what has been asserted by Fierz.

Scalar non-local fields pertaining to integral spins only, Yukawa has tried to extend his theory by considering spinor fields

$$\Psi(X, x) = \sum_{p} \Phi^{p}(X, x) U^{p}, \tag{5.34}$$

the spins of which are connected with x as well as with the spinors U. At present it is still too early to say whether this idea will eventually lead to a satisfactory theory. At the same time, a more unified approach to the description of all kinds of particles seems possible if instead of (5.34) we use wave functions $\Psi(X, x', \omega)$. According to the preceding sections, the latter functions can describe integral as well as half-integral spins provided a sufficient number of Eulerian angles is introduced. Whereas spins 0 and 1 may be discussed in terms of rectangular or spherical coordinates, as in Yukawa's theory, a wave function $\Psi(X, x', \omega)$ must essentially depend on six angles if it is to describe Dirac particles. For the rotation eigenvectors pertaining to $D(\frac{1}{2}, 0)$ depend on the three angles ω^{+}, $D(0, \frac{1}{2})$ demands three angles ω^{-}, so that for $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$ six angles are required indeed. The introduction of Eulerian angles obviously leads to the occurrence of the family-indices s' and m', which are not found in Yukawa's work.

It has been shown by Yukawa (1953a) that in a non-local field theory one may possibly eliminate infinite self-energies. It is another interesting feature that in the wave equation for a non-local field the usual mass parameter may be replaced by an operator acting on x. Actual masses can be interpreted then as the solutions of an eigenvalue problem, particles corresponding to eigenstates of internal motion. This point has been examined by Yukawa (1950a, b, 1953a, b) as well as by Hara, Marumori, Ohnuki, and Shimodaira (1954). It is hoped that through this device of mass quantization one may eventually discover some method in the confusing array of elementary particles, but is not known as yet how to construct a mass operator appropriate to what is realized in nature. We note here that throughout the years many alternative attempts have been made at constructing a mass spectrum for elementary particles. References pertaining to this problem have been given by Hara et al., and in addition we mention papers by Enatsu (1954a, b, 1956) as well as Finkelstein's (1955) and Pais's (1953a, b) work cited in section 1.2.

For lack of firm guiding principles the theory of non-local fields is still in an early state of development. At the same time, it shows some promising aspects, and it seems reasonable to hope that these may be retained if instead of Yukawa's type of field we consider a non-local field $\Psi(X, x', \omega)$. As compared with other field functions, $\Psi(X, x', \omega)$ shows the novel feature of depending on relativistically invariant coordinates x', which presumably might be quite an asset when it comes to specifying particle extensions in a relativistically invariant way. To summarize, non-local functions $\Psi(X, x', \omega)$ seem quite well suited to the study of various problems concerning the possible internal structure of elementary particles.

VI

THE ISOBARIC SPIN

6.1. Nucleons and π-mesons

As early as 1932, shortly after the discovery of the neutron, it was suggested by Heisenberg that proton and neutron should be regarded as two states of one fundamental particle, the nucleon. In Heisenberg's formalism the charge state of a nucleon is represented by an operator T_3 having eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$. In analogy to Pauli's theory for the spin, T_3 is formally taken to be a component of an angular momentum in three dimensions, T, the components of which are denoted by T_1, T_2, and T_3. The quantity represented by T has later been called the isobaric - or, less correctly, the isotopic - spin. The nucleon is attributed an isobaric spin $\frac{1}{2}$, and the notation is now chosen such that proton and neutron are eigenstates of T_3 with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Accordingly the operator $T_1 + iT_2$ transforms a neutron wave function into a proton wave function, and $T_1 - iT_2$ accomplishes the inverse transformation. It will be convenient in the following to denote typical eigenvalues of $(T)^2$ and T_3 by $t(t+1)$ and t_3, respectively.

When in 1936 the first evidence was found that at low energies the proton-proton and the neutron-neutron nuclear forces are the same as the proton-neutron force, provided states with the same angular momentum and parity are compared, it was pointed out by Cassen and Condon that in the language of isobaric spin this charge independence of nuclear forces could be given an extremely simple expression. As a matter of fact, if the Pauli principle is generalized to the statement that a more-nucleon wave function shall be antisymmetric in the space, spin, and isobaric-spin coordinates of any two nucleons, the hypothesis of charge independence is equivalent to the hypothesis that in any particular state of angular momentum and parity the nuclear interaction does not depend on the charges of the nucleons under consideration, i.e., not on the resultant t_3. Under this assumption the Hamiltonian describing nuclear interactions must be a scalar in isobaric-spin space, and the total isobaric spin is a good quantum number in a nuclear system in which Coulomb effects may be neglected. Obviously the resultant t_3 is also a good quantum number, denoting the total charge. In the course of years strong evidence has been found for the validity of charge independence in nuclear systems at low energies. For a detailed account of this matter the reader is referred to a recent review article by Burcham (1955).

The hypothesis of charge independence should not be confused with the assumption of charge symmetry, which says only that the nuclear forces between two protons are the same as those between two neutrons. Obviously charge independence implies charge symmetry, but the latter principle is less far-reaching in that it makes no statement concerning proton-neutron interactions. Charge symmetry has also been discussed in Burcham's paper cited above.

As nuclear forces are now considered to be a consequence of the virtual exchange of mesons, especially π-mesons, between nucleons, it is essential that the π-meson can be incorporated in the isobaric-spin formalism in a charge-independent way. As a matter of fact, the now familiar charge-independent Hamiltonian for the interaction between nucleons and π-mesons was already proposed by Kemmer (1938) nearly ten years before the actual discovery of the π-meson. After Kemmer the π-meson is attributed isobaric spin 1, its charge states π^+, π^0, and π^- being eigenstates of T_3 with eigenvalues 1, 0, and -1, respectively.

It is well known that a direct test of charge independence in pion-nucleon phenomena is provided by a comparison of the reactions

$$\begin{aligned} p + p &\to d + \pi^+, \\ n + p &\to d + \pi^0, \end{aligned} \tag{6.01}$$

where p, n, and d stand for proton, neutron, and deuteron, respectively. Since for the deuteron $t = 0$, the final states in (6.01) both have $t = 1$. Now obviously a two-proton system also has $t = 1$. However, the proton-neutron system under consideration is an equal mixture of states with $t = 0$ and states with $t = 1$. Hence if in the processes (6.01) the isobaric spin is conserved, the differential cross-sections should be in the ratio of 2 to 1 at all angles. This has been checked experimentally at nucleon energies of about 400 MeV, and strong evidence for charge independence has thus been found, cf. Hildebrand (1953) and Schluter (1954). For a further discussion of the role of the isobaric spin in pion-nucleon processes the reader is referred to Bethe and de Hoffmann (1955), II.

6.2. Speculation on S'

In spite of the fact that the isobaric-spin formalism appropriate to nucleons and π-mesons has been known for nearly twenty years, the very occurrence of an isobaric spin is still a rather puzzling feature. Introduced ad hoc for the description of nuclear forces and pion-nucleon interactions, the isobaric spin is far from understood at present. It is not clear, for instance, whether an isobaric spin should be attributed to electrons, neutrinos, or μ-mesons. Neither is there any indication as to whether the so-called rotations in isobaric-spin space are in any way related to ordinary rotations, translations, or other established transformations. In this situation it seems natural to suggest that the operator S' discussed in the preceding chapter might represent the isobaric spin. More particularly, since the commutation relations for the components of S' have a minus sign as compared with the commutators for an ordinary angular momentum in three dimensions, we shall consider the possibility of interpreting $-S'_k$ as the operator T_k $(k = 1,2,3)$. Some implications of this tentative idea will be discussed in the following pages.

Inasmuch as nucleons and π-mesons have spins $\frac{1}{2}$ and 0, and isobaric spins $\frac{1}{2}$ and 1, respectively, the operator S' may be easily incorporated in the current theory of pion-nucleon interactions provided for the π-meson we use the formalism developed in section 5.3. According to section 5.3, a particle of spin 0 and $s' = 1$ must be described by a vector wave function having four components $\Phi^u(X)$. Since for $s' = 1$ the index m' may take the values 1, 0, and -1, the most general wave function for spin 0 and $s' = 1$ has twelve components in all. In an obvious notation it may be written in the form

$$\sum_{m'} g_{uv}\Phi^{um'}(X)Z^{v,m'}_{\ \ 1}(\omega), \tag{6.02}$$

summation being implied over the tensor indices u and v. Since we may combine functions $Z^{v,m'}_{\ \ 1}$ with the same v but different superscripts m' to form functions $Z^{v,k}_{\ \ 1}$ which under rotations generated by operators S'_l transform as the components of a vector in three dimensions, the wave function (6.02) may also be expanded according to

$$g_{uv}\Phi^u{}_k(X)Z^{v,k}_{\ \ 1}(\omega), \tag{6.03}$$

where it is understood that the summation convention applies to the tensor index k. It follows, by the way, from what was said in connection with (2.40) and (2.51) that $Z^{v,k}_{\ \ 1}$ is proportional to the coefficient Λ^{kv} given in table 4.1.

Let us now choose the functions $\Phi^u{}_k$ to be real, let us introduce a constant μ such that the mass of the π-meson is equal to $\mu h/c$, and let us write $\Pi_0{}^k$ for the momentum conjugate to $\Phi^0{}_k$. With the help of section 5.3 we may then translate the usual theory of the π-meson into a formalism with Eulerian angles provided we replace the usual wave function $\varphi^k(X)$ by $\mp\Pi_0{}^k(X)/\mu$, cf. (5.28). The usual eight-component nucleon wave function $\psi(X)$ may be simply replaced by a function of the form

$$\Psi(X, \upsilon) = \sum_{p,\, m'} \Upsilon^{pm'}(X) Z^{p,\, m'}_{\ \frac{1}{2}}(\upsilon), \tag{6.04}$$

where $-m'$ equals $\frac{1}{2}$ for a proton state and $-\frac{1}{2}$ for a neutron state. The index p runs through four values. Obviously the Eulerian angles υ in (6.04) must be independent of the ω's in (6.02) and (6.03).

It follows from section 5.3 that if $-S'_k$ is interpreted as the isobaric-spin operator T_k, the current theory of pion-nucleon interactions, which derives from the interaction Hamiltonian

$$H_{\text{int}} = f\overline{\psi}(X)\gamma_5 T_k \psi(X)\varphi^k(X) \qquad (\overline{\psi} = \psi^*\gamma^0), \tag{6.05}$$

can be expressed in terms of the wave functions (6.03) and (6.04) by going over to

$$\overline{H}_{\text{int}} = \mp(f/\mu)\left(\Psi(X,\upsilon), -\gamma^0\gamma_5 S'_k(\upsilon)\Psi(X,\upsilon)\right)_\upsilon \Pi_0{}^k(X). \tag{6.06}$$

In (6.06) the operators γ are the operators of type (5.06) that correspond to the usual Dirac matrices γ^0 and γ_5. The symbol $(\ldots,\ldots)_\upsilon$ signifies that the scalar product with respect to υ must be taken, not, however, the scalar product with respect to X. The symbol $\mp$ refers to the two possible substitutions indicated in (5.28).

Whereas it is of course very satisfactory that the operator S' may be naturally fitted into the current description of pions and nucleons, it will be observed that this description is insensitive to the nature of the isobaric spin; going over from (6.05) to (6.06) does not alter our quantitative predictions on pions and nucleons to any extent. Yet our speculation on S' might provide a useful starting-point for future investigations. In particular, as introducing Eulerian angles seems to open up new ways for studying the internal structure of elementary particles, it is hoped that it may contribute to overcoming the divergence difficulties encountered in the current theory of pions and nucleons.

If S' is interpreted as the isobaric spin, charge independence corresponds to the conservation of the quantum number s'. In particular, s' will be conserved under pion-nucleon reactions consistent with (6.06). This does not imply, however, that s' should be conserved under any reaction. As a matter of fact, in the previous chapters we have demanded that any physical equation shall be invariant under the ordinary transformations of the full Lorentz group; as regards a formalism with Eulerian angles, this requirement by itself necessitated the introduction of families of functions Z. The functions Z have not been constructed on the ground that there must be invariance under rotations generated by operators S'. Now since any operator S'_k commutes with any operator for an ordinary Lorentz transformation, invariance with respect to S' is quite independent of invariance under Lorentz transformations. It is therefore well possible that some equations are invariant under rotations generated by operators S' while others are not. This is very satisfactory since, for instance, electromagnetic interactions obviously are not charge independent.

If our formalism with Eulerian angles corresponds to physical reality, all particles should be incorporated therein; it does not seem very reasonable to assume that for some particles the spin S' would be meaningful while for others it would not. If S' would indeed represent the isobaric spin, the isobaric-spin formalism should therefore be extended to include all particles. However, for lack of experimental criteria we shall not try to assign isobaric spins to electrons, neutrinos, and μ-mesons. Neither shall we discuss the photon in this connection.

As we have shown in section 5.2, the quantum number s' must be integral for bosons and half-integral for fermions. At the same time, it has been suggested by Gell-Mann (1953) and independently by Nakano and Nishijima (1953) that among newly discovered particles there might be bosons of half-integral isobaric spin as well as fermions of integral isobaric spin. Since for the present investigation this point is very crucial indeed, we shall now summarize some relevant properties of the particles in question, and we shall examine the underlying assumptions of the classification proposed by Gell-Mann and others. Anticipating the argument of the following sections, we note that while this classification has been tentatively adopted by many workers, no conclusive evidence has yet been found either for or against it.

6.3. Hyperons and heavy mesons

In recent years several types of particles have been discovered whose behaviour is far from understood at present. Accordingly the particles in question are now being referred to as curious particles. They are called heavy mesons or hyperons according as they are intermediate in mass between the π-meson and the proton, or between the neutron and the deuteron. For a qualitative discussion of curious particles the reader is referred to Bethe and de Hoffmann (1955), II, sec. 51. Numerical data and references to original papers have been summarized by A.M.Shapiro (1956) and M.M.Shapiro (1956). Additional information may be found in the proceedings of the sixth Rochester conference (Rochester 1956).

Among the hyperons the particles denoted by Λ^0, Σ^+, Σ^-, and Ξ^- are now well established. They are fermions having lifetimes of the order of 10^{-10} sec. The Λ^0, for instance, decays into a proton and a π^--meson according to

$$\Lambda^0 \to p + \pi^- + Q \qquad (Q \approx 37 \text{ MeV}), \tag{6.07}$$

from which it follows that the mass of the Λ^0 is approximately 2181 times the electron mass m_e. The main data on hyperons are summarized in table 6.1.

Table 6.1

Particle	Decay	Q (MeV)	Mass (m_e)
Λ^0	$p + \pi^-$	37	2181
Σ^+	$p + \pi^0$	116	2327
	$n + \pi^+$	110	
Σ^-	$n + \pi^-$	118	2342
Ξ^-	$\Lambda^0 + \pi^-$	67	2586

As regards the heavy mesons - alternatively called K-particles - at least six different modes of decay have been reported. It is one of the most puzzling features of "elementary" particles that all heavy mesons observed thus far have nearly equal masses, about 965 times the electron mass. Their lifetimes are of the order of 10^{-8} sec. There is no conclusive evidence against some K-particles being fermions, but the best-established ones, decaying into three and two π-mesons, definitely are bosons. In theoretical discussions only the bosons are being considered. At present it is hoped that several types of heavy-meson events correspond to alternative decay modes of one basic particle, but at the same time some considerable evidence has been found to the effect that the K-particle which decays into three π-mesons cannot have the same spin and parity as the one that decays into two π-mesons (Dalitz 1955a, 1956, Orear, Harris, and Taylor 1956). Various speculations concerning this point have been put forward at the sixth Rochester conference.

If protons are bombarded with high-energy π^--mesons, several per cent of the collisions lead to the production of curious particles. From this it follows that curious particles are rather strongly coupled to nuclear matter. If Λ^0-particles were now produced copiously by the reaction $\pi^- + p \rightarrow \Lambda^0 + \pi^0$, one would expect that the Λ^0 should decay extremely rapidly, for instance by the chain of virtual processes $\Lambda^0 \rightarrow \pi^0 + p + \pi^- \rightarrow p + \pi^-$. Discussing this point quantitatively, Pais (1952) showed that from production data one would estimate lifetimes of about 10^{-21} sec for all known curious particles. The established curious particles have lifetimes of the order of 10^{-10} to 10^{-8} sec, however, so that apparently the interaction operating in production is much stronger than the one responsible for decay.

The most reasonable way out of this difficulty was shown by Pais (1952), who suggested that curious particles could not be readily produced singly. In recent years this has been strikingly confirmed by experiments, and it now appears well established that Λ- and Σ-particles are produced according to

$$\begin{aligned} &\pi + N \rightarrow \Lambda + K^+, \qquad \pi + N \rightarrow \Sigma + K^+, \\ &\pi + N \rightarrow \Lambda + K^0, \qquad \pi + N \rightarrow \Sigma + K^0, \end{aligned} \tag{6.08}$$

where N stands for the nucleon. There is some evidence for the production scheme

$$\pi^- + n \rightarrow \Xi^- + K^0 + K^0 \tag{6.09}$$

(Sorrels, Leighton, and Anderson 1955). Further it has been experimentally found that negative K-particles can be captured by nucleons according to

$$K^- + N \rightarrow \Lambda + \pi, \qquad K^- + N \rightarrow \Sigma + \pi. \tag{6.10}$$

The so-called associated-production reactions (6.08)-(6.10) are much faster than the decays given in table 6.1. It is noteworthy that reactions such as $n + n \rightarrow \Lambda^0 + \Lambda^0$ have not been observed.

Comparison of the reaction schemes (6.08) and (6.10) suggests that the mesons K^+ and K^- are mutual antiparticles in the same sense as the mesons π^+ and π^-. It has first been pointed out by Gell-Mann and Pais (1954) that from the fact that the process $n + n \rightarrow \Lambda^0 + \Lambda^0$ does not occur with appreciable probability it follows that a neutral meson K^0 must likewise have an antiparticle, $\tilde{K}^0$, distinct from itself. (The better-known neutral bosons, viz. the π^0-meson and the photon, are identical with their antiparticles.) Interesting ideas concerning K^0 and $\tilde{K}^0$ have been presented by Gell-Mann and Pais (1955), Pais and Piccioni (1955), and others.

6.4. Strangeness and isobaric spin

Ever since it became clear that curious particles are not produced singly, attempts have been made to express their behaviour in terms of suitable conservation laws. It has particularly been suggested to invoke the isobaric spin in this connection, but before proceeding to that point it will be convenient to discuss the concept of strangeness.

In some way or other, many theoretical papers on curious particles (Gell-Mann 1953, Nakano and Nishijima 1953, Gell-Mann and Pais 1954, Nishijima 1954, 1955, d'Espagnat and Prentki 1955, 1956, Sachs 1955, Goldhaber 1956, Cerulus 1956) contain a quantity known as strangeness, which is often denoted by the symbol S. Pions, heavy mesons, nucleons, and hyperons are attributed strangeness quantum numbers according to table 6.2.

Table 6.2

Particle	π^-, π^0, π^+	K^0, K^+	K^-	n, p	Λ^0	Σ^-, Σ^+	Ξ^-
Strangeness	0	1	-1	0	-1	-1	-2

If a particle is assigned strangeness S, then its antiparticle is assigned $-S$. The strangeness of a combination of particles is defined to be the sum of the strangenesses of the individual particles.

If the above definition of strangeness is adopted, S is conserved under pion-nucleon reactions as well as under the associated-production processes (6.08)-(6.10). The hyperon decays summarized in table 6.1 satisfy $\Delta S = 1$. It is now assumed that a reaction can only be fast provided it has $\Delta S = 0$. Reactions with $\Delta S = \pm 1$ are assumed to be slow, and $\Delta S = \pm 2$ is classified as forbidden. The latter selection rule accounts for the observed fact that Ξ^- is produced in association with two heavy mesons. It forbids processes as $n + n \rightarrow \Lambda^0 + \Lambda^0$, as well as the decay $\Xi^- \rightarrow n + \pi^-$. Electrons, neutrinos, and μ-mesons are usually not incorporated in the strangeness scheme, but recently Cerulus (1956) has attempted to extend the concept of strangeness in such a way that it applies to these particles too.

Whereas the strangeness concept is very useful in classifying curious-particle interactions, it does not account for the fact that there exist several groups of curious particles of nearly equal masses but different charges. In order to describe this feature the curious particles have been tentatively attributed isobaric spins (Gell-Mann 1953, Nakano and Nishijima 1953, Gell-Mann and Pais 1954, Nishijima 1954, 1955, d'Espagnat and Prentki 1955. 1956, Sachs 1955, Goldhaber 1956). After Gell-Mann (1953) and Nakano and Nishijima (1953) the current isobaric-spin model has been constructed in such a way that the strangeness selection rules correspond to similar selection rules for the isobaric spin. It is assumed that the quantum number t_3 is conserved under any fast reaction, and it is strongly suggested that all fast reactions are charge independent, conserving the total isobaric spin t too.

Now if Λ^0 essentially has no charged counterpart of nearly equal mass, the reaction in which Λ^0 is produced cannot be charge independent unless Λ^0 has $t = 0$. Since no Λ^+ or Λ^- seems to exist, Λ^0 is therefore attributed isobaric spin 0. It is known that Λ^0 has half-integral spin, so that at this point the novel situation is raised of a fermion of integral isobaric spin.

It is not difficult to see that if it is assumed that fast reactions are charge independent, that Λ^0 has $t = 0$, and that no doubly-charged curious particles exist, the only possible isobaric-spin classification is the one given in table 6.3.

According to the ideas summarized in table 6.3, the K-mesons are bosons of half-integral isobaric spin, the hyperons Λ and Σ are fermions of integral isobaric spin, while Ξ has half-integral spin and half-integral isobaric spin.

Under the assumptions leading to table 6.3, K^- must have a neutral counterpart, $\tilde{K}^0$, which has $t_3 = \frac{1}{2}$. The particles K^0 and $\tilde{K}^0$ are not identical, for they do not have the same t_3. It is assumed that $\tilde{K}^0$ and K^- are the antiparticles of K^0 and K^+, respectively, cf. the end of the preceding section. If the heavy mesons do not all have the same spin and parity, cf. section 6.3,

Table 6.3
The isobaric spin according to Gell-Mann (1953) and Nakano and Nishijima (1953)

Particle	K^0	K^+	K^-	$(\tilde{K}^0)$	Λ^0	Σ^-	(Σ^0)	Σ^+	Ξ^-	$((\Xi^0))$
t	$\frac{1}{2}$		$\frac{1}{2}$		0		1		$\frac{1}{2}$	
t_3	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Symbols in single brackets refer to particles the existence of which is not fully established. Symbols in double brackets refer to particles which have not been observed.

then there should be at least two doublets of type K^0, K^+, and for the antiparticles as many doublets of type K^-, $\tilde{K}^0$.

For charge independence in the sense of table 6.3 to be valid for the fast reactions (6.08)-(6.10), $\Sigma^{\pm}$ and Ξ^- must also have neutral counterparts. Concerning a Σ^0 it appears quite probable that several events of the type

$$\pi^- + p \rightarrow \Sigma^0 + K^0 \tag{6.11}$$

have been observed, where the Σ^0 decays very fast into a Λ^0 plus a photon (A.M.Shapiro 1956 and references given there, Budde et al. 1956). Evidence for a Ξ^0 has not been reported, but in the present experimental situation this cannot yet be taken as an indication against the ideas of Gell-Mann and Nakano and Nishijima.

In a model according to table 6.3, only reactions with $\Delta t_3 = 0$ are allowed to be fast, reactions having $\Delta t_3 = \pm\frac{1}{2}$ are required to be slow, and $\Delta t_3 = \pm 1$ is forbidden. It is not difficult to see that these selection rules correspond to the strangeness selection rules discussed above. For if for nucleons, π-mesons, and curious particles the charge divided by $|e|$ is denoted by q, then by tables 6.2 and 6.3 the strangeness and the third component of isobaric spin are related according to

$$q = t_3 + \tfrac{1}{2}S + \tfrac{1}{2}\varepsilon, \tag{6.12}$$

where ε equals 0 for bosons, 1 for nucleons and hyperons, and -1 for antinucleons and antihyperons. In (6.12) it is understood that for a particle and its antiparticle q, t_3, S, and ε take opposite values. For a combination of particles the resultant ε signifies the number of heavy particles, i.e. the number of nucleons and hyperons minus the number of antinucleons and antihyperons. If it is now assumed that the total charge q and the heavy-particle number ε are conserved in any reaction, it follows that the above selection rules for t_3 and S are fully equivalent.

6.5. Charge independence

It will be observed from the preceding section that much experimental evidence on curious-particle production and decay can be successfully expressed in terms of a quantity known as strangeness. It seems reasonable to anticipate that the so-called strangeness selection rules, i.e. the rules on fast, slow, and forbidden reactions, will play a fundamental role in any future theory of elementary particles. By contrast, the current isobaric-spin classification of curious particles is far from established. Since, for instance, the assignment $t = 0$ for Λ^0 is directly coupled to the hypothesis that Λ^0 essentially has no charged counterpart, and that it is produced in a charge-independent reaction, it is our impression that as regards the isobaric-spin model indicated in table 6.3 the question of charge independence is extremely crucial. In the light of our present investigation it seems appropriate to discuss this point in some detail.

The hypothesis of charge independence in the Gell-Mann and Nakano and Nishijima sense could be tested most directly by comparing the reactions

$$\begin{aligned} K^- + d &\to \Sigma^- + p, \\ K^- + d &\to \Sigma^0 + n. \end{aligned} \tag{6.13}$$

It may be seen from a reasoning analogous to the one indicated for the reactions (6.01) that if table 6.3 is valid and if charge independence applies to all fast reactions, the processes indicated in (6.13) must occur in the ratio of 2 to 1. The test (6.13) and similar more complicated tests have been proposed by Lee (1955), Case, Karplus, and Yang (1956), and Feldman (1956). To our knowledge it has not yet been possible, however, to check any of the predicted relationships between curious-particle processes.

It has been discussed by Dalitz (1955b) that the study of binding energies in hyperfragments, i.e. nuclei containing a bound hyperon, generally a Λ^0-particle, may provide a test of the proposed $t = 0$ assignment for Λ^0. However, as far as we know the available evidence on hyperfragments does not yet permit a conclusion pertaining to this point either.

Since through the associated-production interactions curious particles are rather strongly coupled to nuclear matter, a possible charge dependence of any of these interactions would somewhat destroy the presumed charge independence of pion-nucleon forces. Hence accurate measurements of pion-nucleon phenomena will in principle give information on the behaviour of curious particles as regards charge independence. Now it was pointed out in section 6.1 that the charge independence of pion-nucleon interactions at low energies has been established beyond reasonable doubt. This does not permit one, however, to conclude that all fast reactions are charge independent. Concerning the role of curious particles in low-energy phenomena, it has been remarked by Bethe and de Hoffmann (1955), II, sec. 51*h*, that even at an energy of about 1000 MeV in the center-of-mass system the cross-section for production of curious particles in pion-nucleon collisions is only a few per cent of that for pion production. Hence although virtual processes such as

$$\begin{aligned} p &\rightarrow \Lambda^0 + K^+, \\ n &\rightarrow \Lambda^0 + K^0 \end{aligned} \tag{6.14}$$

are essentially likely by the strangeness selection rules, the coupling constant for (6.14) might be considerably smaller than the coupling constant for the pion-nucleon interaction.

Even if the interaction (6.14) is stronger than anticipated by Bethe and de Hoffmann, the virtual production of curious particles according to (6.14) can only cause a short-range interaction, the range being of order $\hbar/m_K c$ or less. In this connection it is remarked by Bethe and de Hoffmann that on account of the interaction with pions the forces between nucleons at distances smaller than $0.4\hbar/m_\pi c$ appear to be strongly repulsive. As a result low-energy nuclear phenomena will not be substantially influenced by the virtual production of curious particles.

Concerning pion-nucleon scattering it is pointed out that in the S-state of total isobaric spin $\frac{3}{2}$ there is a strong repulsion between π-mesons and nucleons, the range of which is about $2\hbar/m_N c$. For P-wave scattering the interaction is attractive it is true, giving a resonance near 200 MeV ascribed to the $P_{3/2}$-state of $t = \frac{3}{2}$, but as regards P-states it is argued by Bethe and de Hoffmann that the main interaction comes from the production of an additional virtual π-meson, which requires much less energy than that of a virtual heavy meson.

Perhaps the latter reasoning is not fully convincing, particularly as regards the S-wave interaction of $t = \frac{1}{2}$, which seems to be attractive at low energy, cf. Bethe and de Hoffmann (1955), II, sec. 33i, but notwithstanding this it is generally felt that from low-energy data one cannot draw any conclusion concerning the charge-independence character of curious-particle phenomena. As a matter of fact, it was just this consideration that made the above-mentioned authors (Lee 1955, Case, Karplus, and Yang 1956, Feldman 1956) propose alternative tests. Obviously it should be possible to obtain information on charge independence from experiments in the high-energy region, but here the measurements are still very inaccurate.

6.6. Concluding remarks

If all fast reactions should be shown to be charge independent in the Gell-Mann and Nakano and Nishijima sense, the isobaric-spin model given in table 6.3 would provide an appropriate starting-point for future investigations. Since it implies that there are fermions of integral isobaric spin as well as bosons of half-integral isobaric spin, one could not reasonably maintain then that S' would represent the isobaric spin. The operator S' might have quite another interpretation, but at present we have no particular suggestion as to that point.

If, by contrast, charge independence in the sense of table 6.3, and likewise charge symmetry, should fail as a guiding principle, it would not be easy to determine the role of the isobaric spin. There would be no compelling reason, for instance, to attribute $t = 0$ to Λ^0. As a matter of fact, the concept of isobaric spin might be altogether irrelevant to curious particles. If it were assumed that it might be meaningful to incorporate such particles in an isobaric-spin scheme, it should be borne in mind that it is not known whether all members of the same isobaric multiplet must have the same strangeness.

Considering that for the present some speculation seems inevitable anyway, we would suggest that it be investigated whether under the assumption that S' were the isobaric spin one could develop a reasonable theory of elementary particles. As a first step at describing curious particles in such a theory one might examine the isobaric-spin classification proposed in table 6.4.

If this scheme should be correct, the strangeness and the third component of isobaric spin would be related according to

$$q = t_3 + S + \tfrac{1}{2}\varepsilon. \qquad (6.15)$$

Table 6.4
Tentative isobaric-spin classification

Particle	K^0	K^+	$((K^{++}))$	$((K^{--}))$	K^-	$(\tilde{K}^0)$
t		1			1	
t_3	-1	0	1	-1	0	1

Particle	$((\Lambda^-))$	Λ^0	$((\Sigma^{--}))$	Σ^-	(Σ^0)	Σ^+	$((\Xi^{--}))$	Ξ^-
t	$\frac{1}{2}$		$\frac{3}{2}$				$\frac{1}{2}$	
t_3	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Symbols in single brackets refer to particles the existence of which is not fully established. Symbols in double brackets refer to particles which have not been observed.

Hence as regards the strangeness selection rules, table 6.4 would be equally useful as the model due to Gell-Mann and Nakano and Nishijima provided the usual criterion $\Delta t_3 = 0, \pm\frac{1}{2}$, or ± 1 for fast, slow, or forbidden reactions were simply replaced by $\Delta t_3 = 0, \pm 1, \pm 2$. Table 6.4 essentially differs from the latter model, however, in that it has room for hypothetical particles K^{++}, K^{--}, Λ^-, Σ^{--}, and Ξ^{--}, none of which has thus far been observed. If table 6.4 were tentatively adopted as a starting-point for further work, one should account for the non-appearance of these particles. Naturally one would try to invoke electromagnetic self-energy effects in this connection. As a matter of fact, table 6.4 has been constructed in such a way that within each multiplet the unobserved particle is more highly charged than the observed ones. Table 6.4 also differs from the model proposed by Gell-Mann and others in that it does not require the existence of a Ξ^0.

As regards the strong associated-production coupling between curious particles and nuclear matter, we would suggest to consider a charge-independent interaction involving all particles entered in table 6.4, and, superimposed thereupon, an electromagnetic interaction which accounts for the non-appearance of the highly-charged particles. Obviously the total interaction would thus not be charge independent. Yet our scheme has the attraction of retaining the charge independence of the specific associated-production coupling, which as an ordering principle commends itself through its simplicity. It will be observed, however, that quantitatively charge independence according to table 6.4 is not identical with charge independence in the Gell-Mann and Nakano and Nishijima sense.

From a computational point of view the present suggestion is rather complicated, it is true. However, if the coupling between curious particles and nucleons should indeed be only a few per cent of that between nucleons and π-mesons (Bethe and de Hoffmann 1955, II, sec. 51*h*), then neglecting electromagnetic interactions – which are weaker than the pion-nucleon interaction by a factor of the order of 1/137 – as compared with the curious-particle interaction would not be reliable anyway. Unfortunately, if electromagnetic effects should considerably distort the associated production, it would be very difficult to check the basic hypothesis of charge independence.

If the Σ-particles should have $s' = \frac{3}{2}$, they could not be described in terms of the representation $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$; one should use a more complicated wave function than the Dirac one. Now it is not known at present whether the Σ's satisfy Dirac's equation. As a matter of fact, observations reported by Fowler et al. (1955) and by Walker and Shephard (1956) suggest that the spins both of Λ^0 and of the Σ's are at least $\frac{3}{2}$. This has not been confirmed, however, by the experiments of Budde et al. (1956). If the Σ-particles should be of high spin, using a many-component wave function which transforms according to a complicated representation of the Lorentz group would be inevitable anyway. The classification $s' = \frac{3}{2}$ could be easily taken into account then. If, on the other hand, the Σ's should be elementary particles of spin $\frac{1}{2}$, one should construct a wave equation for spin $\frac{1}{2}$ and $s' = \frac{3}{2}$ starting, for instance, from a representation such as $D(1,\frac{1}{2}) \oplus D(\frac{1}{2},1)$. At present it cannot be said whether the desired equation should be equivalent to the Dirac one.

In connection with the classification $s' = \frac{3}{2}$ it should also be examined whether the Σ's actually are elementary particles. If table 6.4 should be valid and if a Σ-particle should be regarded as a compound of a nucleon and a K^-- or a $\tilde{K}^0$-meson (Goldhaber 1956), or of a Λ-particle and a π-meson, which is possible by the strangeness selection rules, the classification $s' = \frac{3}{2}$ would give no formal complications. The angular momenta S' of the constituent particles could be coupled then to a resultant having $s' = \frac{3}{2}$ irrespective as to whether the Σ's are of high spin. In this respect a compound particle of spin $\frac{1}{2}$ can be discussed more easily than an elementary one owing to the fact that whereas for elementary particles we have $(J^+)^2 = (J'^+)^2$ and $(J^-)^2 = (J'^-)^2$, cf. (4.23), for compounds primed and unprimed operators are not so intimately related. As J^+ and J^- formally behave as angular-momentum operators in three dimensions, this may be seen directly from section 3.7.

It should be emphasized that if all fast reactions should be shown to be charge independent in the Gell-Mann and Nakano and Nishijima sense, table 6.4 would not be tenable. In the opposite case, however, the present state of our experimental and theoretical knowledge would not suffice to decide for any particular isobaric-spin classification. We feel that it would be almost impossible in the latter situation to extend the theory of the isobaric spin, and, in fact, the theory of "elementary" particles, beyond its present empirical state unless a formalism were found which intrinsically fits a new spin into the description of nature. Since such a formalism has been developed in the foregoing pages, it is hoped that the present investigation may contribute to a better understanding of "elementary" particles.

NOTE ADDED IN PROOF

After the present paper had been set up, strong evidence (Wu et al. 1957, Garwin et al. 1957) has been reported to the effect that parity is not conserved in decay interactions. It is now assumed that only strong and electromagnetic interactions are invariant under spatial reflection, rather than all interactions as has been suggested at the beginning of section 4.6. At present it appears quite possible that there are but four K-particles, K^0, K^+, K^-, $\tilde{K}^0$ in the notation of section 6.3, which at times disintegrate as if they had even parity, then again as if their parity were odd. This does not imply, however, that for K-particles parity is not a meaningful concept. According to the ideas presented by Lee and Yang (1956), who proposed the experiments that have led to the new view on parity, the parity of the Λ^0 relative to the nucleon is a matter of definition. Once it is defined, the parities of other curious particles can be determined from data on strong interactions. If this idea is correct, all particles which take part in strong interactions, i.e. nucleons, π-mesons, and curious particles, should still be described by wave functions the components of which transform among themselves under spatial reflection. Hence as regards such particles it still follows from section 4.7 that in a formalism with Eulerian angles they can only be described in a satisfactory way provided wave functions are chosen the components of which are all attributed the same s' and m'. In other words, in a formalism with Eulerian angles nucleons, π-mesons, and curious particles should still be characterized by an additional spin S'. If, on the other hand, among the light fermions there should be particles the parity of which cannot be determined, then perhaps the spatial reflection would not turn such particles into themselves. To give a simple example, one could entertain the idea of a Dirac wave function the $D(\frac{1}{2},0)$-components of which have $s' = \frac{1}{2}$, $m' = \frac{1}{2}$, while the $D(0,\frac{1}{2})$-components have $s' = \frac{1}{2}$, $m' = -\frac{1}{2}$. After spatial reflection the $D(\frac{1}{2},0)$-components of that wave function would have $m' = -\frac{1}{2}$, the $D(0,\frac{1}{2})$-components would have $m' = \frac{1}{2}$. There would thus be a formal distinction between the reflected and the unreflected wave function, which might have something to do with the observed asymmetry of weak interactions. Obviously the wave function under consideration could not be attributed an overall spin S'. Perhaps this is not undesirable, as the light fermions thus far seem to evade an isobaric-spin classification.

SUMMARY

After some preparatory work on rotations and generalized Eulerian angles, the transformations of special relativity are discussed in terms of six Eulerian angles in Minkowski space. Families of functions Z are constructed which transform linearly among themselves under rotations in space-time as well as under spatial reflection. It is found that for any finite-dimensional representation of the full Lorentz group there exist several families of functions Z, which are distinguished from each other by two family-indices. The family-indices are shown to be the eigenvalues of a new three-dimensional angular-momentum operator, S', and of one component thereof. Any component of S' commutes with any operator which generates an ordinary Lorentz transformation. It is essential that the eigenvalues of S' are integral for one-valued, half-integral for two-valued representations of the Lorentz group.

If the six Eulerian angles are interpreted as spin variables, spinors can be replaced by functions Z. In a formalism in which wave functions for spinning particles are expressed in terms of functions Z, the eigenvalues of S' should be integral for bosons, half-integral for fermions. In such a formalism the angular momentum S' commutes with any operator which occurs in the theory of free particles. It is now suggested that S' might represent a new spin. More particularly, S' is tentatively interpreted as the isobaric spin. In view of the connection between S' and the spin, this interpretation is incompatible with the isobaric-spin classification of heavy unstable particles which has been proposed by Gell-Mann (1953) and by Nakano and Nishijima (1953). However, it is pointed out that the present state of our experimental knowledge does not yet permit a decision in favour of any particular isobaric-spin classification. Much experimental and theoretical work will be required before the possible interpretation of S' can be established.

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STELLINGEN

I

Wordt de in paragraaf 2.3 van dit proefschrift genoemde determinant

$$\det[\Lambda_{da}(\omega)\partial\Lambda^{d}{}_{b}(\omega)/\partial\omega_r]$$

aangeduid met Δ, dan kan de grootheid

$$d\tau = \Delta\, d\omega_1 d\omega_2 \ldots d\omega_{n(n-1)/2}$$

worden beschouwd als het bij de kromlijnige coordinaten ω behorende „volume-element". De in paragraaf 2.3 ingevoerde operatoren J^{ab} voldoen aan de betrekking

$$J^{ab}J_{ab} = -\frac{1}{\Delta}\sum_{q,r} g_{ac}g_{bd}\frac{\partial}{\partial\omega_q}\Delta\Omega_q^{ab}\Omega_r^{cd}\frac{\partial}{\partial\omega_r}.$$

II

Is a een eindig positief getal, dan bestaat bij elke willekeurig kleine positieve ε en elke even functie $F(t)$, die gedefinieerd en continu is in het interval $-a \leqq t \leqq a$, een functie $f(x)$ zodanig dat in het genoemde interval uniform in t geldt

$$\left|F(t) - \int_0^1 f(x)J_0(tx)x\,dx\right| < \varepsilon.$$

III

Staafjes met lengte a, die elkaar niet kunnen overdekken, worden volgens het toeval verdeeld over een oneindig lange rechte lijn. Is de gemiddelde tussenruimte tussen de uiteinden van twee opeenvolgende staafjes gelijk aan λ, dan is de verwachtingswaarde van de afstand van een willekeurig gekozen oorsprong O tot het $n$de staafjesmidden rechts van O gelijk aan

$$n(a+\lambda) - a(\tfrac{1}{2}a+\lambda)/(a+\lambda).$$

IV

De argumenten, waarmee Lepore duidelijk tracht te maken dat het aanbeveling verdient om aan de derde component van de isobarische spin van proton en antiproton dezelfde waarde toe te kennen, zijn niet steekhoudend. Lepore's vergelijkingen (5') en (5'') voor het antinucleon, die ook zijn onderzocht door Györgyi, kunnen op eenvoudige wijze zo worden veranderd dat de door Györgyi gesignaleerde discrepantie tussen de eendeeltjesbeschrijving en de gequantiseerde veldentheorie van het nucleon wordt opgeheven.

Lepore, J.V. (1956), Phys. Rev. **101**, 1206.
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V

De opmerking van Umezawa dat zogenaamde contravariante en complex geconjugeerde spinoren bij tijdomkeer niet onder elkaar transformeren is onjuist.

Umezawa, H. (1956), Quantum field theory, North-Holland Publishing Company, Amsterdam, p. 53.

VI

De beschouwingen van Wentzel naar aanleiding van het impulsmoment van een vectorveld geven geen juist inzicht in het begrip spin.

Wentzel, G. (1949), Quantum theory of fields, Interscience Publishers, Inc., New York, p. 87.

VII

In tegenstelling tot hetgeen is gesuggereerd door Lee en Yang volgt uit de omstandigheid dat de pariteit niet behouden blijft bij vervalsprocessen geenszins het bestaan van „rechtse" en „linkse" protonen.

Lee, T.D., en Yang, C.N. (1956), Phys. Rev. **104**, 254.

VIII

Het zou in een behoefte voorzien, wanneer de commissie voor symbolen, eenheden en nomenclatuur van de „International Union for pure and applied Physics" zich duidelijk zou uitspreken over de vraag of zij de operatie „differentiëren naar x" aangegeven zou willen zien met d/dx dan wel met $\mathrm{d}/\mathrm{d}x$.

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C. van Winter, 26 februari 1957