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Gauge Theory at Asymptotical Distance and Singularity

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Abstract

One of the fundamental problems of gravity to be an infinite range interaction is whether it has a gauge group via the internal structure, so that Einstein type theories couldn't explain it yet. Gauge groups in Yang - Mill type theories, i.e. weak [11] and strong [3, 8] interactions, are generally non - Abelian Lie group via the Hermitian structure induced by complex valued smooth vector fields. A group labeled by $U(1)\otimes\widetilde{U}(d)$ obtained as the reduction of unitary Lie groups is presented as candidate a gauge group for each behavior at large scale and at small scale of a gauge field, in this context, integrated Einstein - Yang - Mills formalism on a unique geometrical frame is given. Also as seen that the (anti-) self dual gauge theory of the group $G = U(1) \otimes \widetilde{U}(4)$ is indeed a $G = U(1)$ theory in real 4 - dimension, i.e the Maxwell theory for the electromagnetism to be an infinite range interaction.

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1 Introduction

According to Georgi and Glashow model, all fundamental (namely electromagnetic, weak and strong) interactions are severally representations of a unique group of unified fields [4]. Except for electromagnetic interation others are finite range interactions and all interactions in this context are a gauge theory of the unitary Lie group via internal symmetries in the quantum regime. However the theory of the gravity to be an infinite range interaction which protecting the validity is currently Einstein' s theory via the Lorentz group $SO(1,3)$ of external symmetry on a spacetime manifold of real 4 - dimension and the quantum behavior of this interaction is still suspense, because it has not been seen a gauge theory via internal symmetry, i.e. unitary Lie groups. In Einstein type theories, the gravity is viewed as coming from the Lorentz - Poincarè - affine groups of external symmetries (more detail can be found in [6] and [7]) as relating the orthogonal group. In this text one aims construction of a gauge theory which gives the behavior a gauge field each at large scale (asymptotical distance) and small scale (singularity).

Although the unitary Lie groups to be the gauge groups of non - Abelian gauge filed are via the Hermitian structure, the existing of the internal symmetry of the gravity is suspense at the present day. Considering some morphisms of the Lie algebras of the real orthogonal groups to some unitary Lie groups, for example $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, one may be imagined to create a gauge group which represent each behavior at large scale and at small scale of a gauge field. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If Θ_a are the generators of this algebra and α^a are some smooth scalars with complex value, then a vector field on Lie algebra $\mathfrak{g}$ is defined by $Z = \alpha^a \Theta_a$, where $a = 1, \cdots, \dim[G]$. As known that the Lie algebra $\mathfrak{u}(d)$ of the group $U(d)$ has the morphism $\mathfrak{u}(d) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(d)$. Hence, any element of the $U(d)$ can be written as $g = e^{\varrho} \otimes e^{Z} \in U(d)$, where $e^{\varrho} \in U(1)$, $e^{Z} \in SU(d)$.

The number of the generators of the group $SU(d)$ having non vanishing diagonal elements are $d-1$, i.e Θ_3 for the $SU(2)$, Θ_3, Θ_8 for $SU(3)$, $\Theta_3, \Theta_8, \Theta_{15}$ for $SU(4)$, etc. Since hermitian matrices are traceless, $Tr[Z] = 0$, where Z is a hermitian matrix, the parameters accompanying the generators having non - vanishing diagonal elements can be set as zero. Then, under an exponential mapping $\mathbf{exp} : \mathfrak{g} \to G$ any element $g \in G$ can be written in exponential representation such that $g = e^{Z} \in G$. Also, the number of the non - diagonal elements of an unitary Lie group $U(d)$ is $\frac{d^2-d}{2} + \frac{d^2-d}{2} = d^2 - d$, where each $\frac{d^2-d}{2}$ belongs to the hermitian conjugate. Then one can say that the set of all e^{Z} occur a group shown by $\widetilde{U}(d)$ with $\frac{d^2-d}{2}$ generators: $e^{Z} \in \widetilde{U}(d)$.

Starting point is the group $SU(4)$ because of the $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{su}(4)$. The group $SU(4)$ was presented by Wigner to study nuclear structure [2] [10]. The generators of $SU(4)$ group in Wigner' s sense serve fourfold degeneracy of nuclear energy level and six labels requiring by reduction $SU(4) \supset SU(2) \otimes SU(2)$ is served by Draayer using a another different way [2]. Consider the morphism of the group $U(4)$ such that $U(4) \cong U(1) \otimes SU(4)$. If $exp(\varrho) \in U(1)$, where $\varrho \in \mathcal{C}^{\infty}(M \to \mathbb{C})$, any element $g(\varrho, \alpha) \in U(4)$ can be written such that $\exp(\varrho) \otimes \exp(\alpha^a \Theta_a) \in U(4)$, where $\alpha^a \in \mathcal{C}^\infty(M \to \mathbb{C})$ and Θ_a are the generators of the $SU(4)$, on the other words Gell - Mann matrices for the special unitary Lie groups. The Gell - Mann matrices Θ_a of the group $SU(4)$ are given in [5].

Consider the group $\widetilde{U}(4)$. If Θ_a are the generators of this algebra belonging to this group and $\alpha^a \in \mathcal{C}^\infty(M \to \mathbb{C})$, then a vector field on Lie algebra $\mathfrak{g}$ of the group $\widetilde{U}(4)$ is defined by $Z = \alpha^a \Theta_a$, where $a = 1, \cdots, \dim[\widetilde{U}(4)]$. Then, any element $g \in \widetilde{U}(4)$ can be written in exponential representation $\exp : \mathfrak{g} \to \widetilde{U}(4)$ such that $g = e^Z \in \widetilde{U}(4)$. Thus, one gets

$$Z = \begin{pmatrix} Z_{00} & \alpha^1 - i\alpha^2 & \alpha^4 - \alpha^5 & \alpha^9 - i\alpha^{10} \\ \alpha^1 + i\alpha^2 & Z_{11} & \alpha^6 - i\alpha^7 & \alpha^{11} - i\alpha^{12} \\ \alpha^4 + \alpha^5 & \alpha^6 + i\alpha^7 & Z_{22} & \alpha^{13} - i\alpha^{14} \\ \alpha^9 + i\alpha^{10} & \alpha^{11} + i\alpha^{12} & \alpha^{13} + i\alpha^{14} & Z_{33} \end{pmatrix}, \tag{1}$$

where diagonal elements are labelled as

$$\begin{aligned} Z_{00} &= \alpha^3 + \tfrac{\alpha^8}{\sqrt{3}} + \tfrac{\alpha^{15}}{\sqrt{6}}, \quad Z_{11} = -\alpha^3 + \frac{\alpha^8}{\sqrt{3}} + \frac{\alpha^{15}}{\sqrt{6}}, \\ Z_{22} &= -\tfrac{2\alpha^8}{\sqrt{3}} + \tfrac{\alpha^{15}}{\sqrt{6}}, \qquad Z_{33} = -\frac{3\alpha^{15}}{\sqrt{6}}. \end{aligned} \tag{2}$$

Since this matrix is Hermitian, in order to $\overline{(\alpha^4 - \alpha^5)} = \alpha^4 + \alpha^5$ the term α^5 must be complex valued. Also, other terms may be real or complex. However, it must be selected some special values for α^a. Thus, for α^4 and α^5 it may be set $\alpha^4 = 0$, $\alpha^5 = i\widetilde{\alpha}^5$. Let's consider that all term α^a are real valued complex smooth functions, that is with zero imaginary part: $\alpha^1 = \alpha^6 = \alpha^9 = \alpha^{11} = \alpha^{13} = 0$. Because unitary matrices are traceless, $Tr[Z] = \sum_i Z_{ii} = 0$, $(i = 0, \cdots, 3)$, the parameters in diagonal elements can be chosen as $\alpha^3 = \alpha^8 = \alpha^{15} = 0$. Hence the Z are obtained such that

$$Z = Y_0 \begin{pmatrix} 0 & Y_2 & Y_5 & Y_{10} \\ \overline{Y}_2 & 0 & Y_7 & Y_{12} \\ \overline{Y}_5 & \overline{Y}_7 & 0 & Y_{14} \\ \overline{Y}_{10} & \overline{Y}_{12} & \overline{Y}_{14} & 0 \end{pmatrix}, \tag{3}$$

where $Y_0 = e^\varrho$, $Y_5 = -i\widetilde{\alpha}^5$ and $Y_a = -i\alpha^a$ for the indices $a = 2, 7, 10, 12, 14$ from the generators of the $SU(4)$. The generators of the $\widetilde{U}(4)$ become Θ_2, Θ_5, Θ_7, Θ_{10}, Θ_{12} and Θ_{14}, that of the group $SU(4)$. If this generator is rearranged in new indices for the group $\widetilde{U}(4)$, then the indices (2,5,7,10,12,14) belonging to the group $SU(4)$ are set for $\widetilde{U}(4)$ such that $2 \to 1$, $5 \to 2$, $7 \to 3$, $10 \to 4$, $12 \to 3$, $10 \to 4$, $12 \to 5$, $14 \to 6$. Also, the parameters Y_i replace some complex valued smooth functions, so that $Y_0 = e^\varrho$ and $Y_i = \phi^i \in \mathcal{C}^\infty(M \to \mathbb{C})$, where $i = 1, \cdots, 6$. Then, the group $\widetilde{U}$ becomes with six parameters, that is with 6 dimensions. Hence, the generators of the group $\widetilde{U}(4)$ are

$$\theta_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \theta_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \theta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\theta_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \theta_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \theta_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad (4)$$

The non vanishing structure constant are also obtained from that of the $SU(4)$ given in [5] such that $f^1_{2,3} = f^1_{4,5} = f^2_{4,6} = f^3_{5,6} = 1/2$.

1.1 Some Notations

Let M be a smooth manifold of real n dimension with Riemannian metric and boundary, also oriented. Let $TM : \{\partial_\mu\}$ and $T^*M : \{dx^\mu\}$ be tangent and cotangent bundles on this manifold with coordinates basis, where $x^\mu \cong \mathbb{R}^n$ are local coordinates and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The exterior derivative operator is $\mathrm{d} : \Lambda^r(M) \to \Lambda^{r+1}(M) = \partial_\mu(\bullet) \cdot dx^\mu$, where $\Lambda^r(M)$ are the bundle of the exterior r - forms on this manifold: $\Lambda^r(M) = C^\infty(\bigwedge^r T^*M)$. Also, if K is any geometrical object, i.e. adjoint bundle, vector and Lie algebra, the bundle of K valued exterior r - forms is observed by $\Lambda^r(M, K) = C^\infty(K \otimes \bigwedge^r T^*M)$. The Hodge duality operator is $* : \Lambda^r(M) \to \Lambda^{n-r}(M)$.

Let G be a Lie group and V a vector space associated to a principal G - bundle $[\pi : P \to M, G]$ on a smooth manifold M. The the associated vector bundle is written as $\mathrm{F} = P \times_\Pi V$ where $\Pi : G \to \mathrm{End}(V)$. Hence, the bundle of the F exterior forms is $\Lambda^r(M, \mathrm{F}) = C^\infty(\mathrm{F} \otimes \bigwedge^r T^*M)$. Let $\nabla : \Lambda^0(M, \mathrm{F}) \to \Lambda^1(M, \mathrm{F})$ ne a connection on this bundle with $A \in \Lambda^1(M, \mathrm{F})$ is connection 1 - form. Then its covariant exterior derivatives $\mathrm{d}^\nabla : \Lambda^r(M, \mathrm{F}) \to \Lambda^{r+1}(M, \mathrm{F})$ is

$$\mathrm{d}^\nabla = \mathrm{d} + \frac{1}{2}[A, \bullet]_\wedge. \quad (5)$$

The curvature $R : \Lambda^0(M, \mathrm{F}) \to \Lambda^2(M, \mathrm{F})$ of the connection ∇ is then

$$R = \mathrm{d}^\nabla A \quad (6)$$

together with Bianchi identity

$$\mathrm{d}^\nabla R = 0. \quad (7)$$

2 Behavior At Asymptotical Distance And Singularity

Let M be a smooth manifold and P be a principal G - bundle on this manifold. Consider a connection 1 - form on this manifold generated by the gauge group $G = U(1) \otimes \widetilde{U}(d)$ and together with its curvature 2 - form

$$A = g^{-1}\mathrm{d}g + \beta, \quad R = \mathrm{d}A + \frac{1}{2}[A, A]_\wedge, \tag{8}$$

where $\beta = \beta^i\theta_i \in \Lambda^1(M, \mathfrak{g})$ be an auxiliary non - Abelian 1 - form, $\beta \wedge \beta \neq 0$. Let $\varphi \in C^\infty(M)$ is any smooth scalar and $\beta = \mathrm{d}\varphi \in \Lambda^1(M)$. Then $[\beta, \beta]_\wedge = 0$ and $\mathrm{d}\beta \equiv \mathrm{d}\mathrm{d}\varphi = 0$, that is curvature 2 - form vanishes: $R = 0$. Indeed this situation comes from the $U(1)$ part of the $G = U(1) \otimes \widetilde{U}(d)$. Since $e^\varrho \in U(1)$ and the $\varrho \in C^\infty(M \to \mathbb{C})$ in is a smooth complex scalar, $\mathrm{d}\varphi$ isn't Lie algebra valued and it becomes then equivalent by $\beta = \mathrm{d}\varphi \cong \mathrm{d}\varrho$. Hence, the connection 1 - form potential of an infinite range interaction at asymptotical distances becomes like a scalar, so that its gradient of vanishes at $|x| \to \infty$, and so one can say that the connection 1 - form is flat:

$$\lim_{|x|\to\infty} \beta \to 0 \ (or \ \beta \to \mathrm{d}\varphi), \ and \ A \sim g^{-1}\mathrm{d}g, \ and \ R = 0. \tag{9}$$

As seen that the group $U(1)$ doesn't any contribution to the curvature 2 - form, then the auxiliary 1 - form β can be written as depending only the part $\widetilde{U}(d)$, so that $\beta \in \Lambda^1(M, \widetilde{\mathfrak{u}})$, where $\widetilde{\mathfrak{u}}$ labels the Lie algebra of the group$\widetilde{U}(d)$. Thus the behavior at small scale of the infinite range interactions with a gauge group $G = U(1) \otimes \widetilde{U}(d)$ is controlled by auxiliary 1 - form $\beta \in \Lambda^1(M, \widetilde{\mathfrak{u}})$, on the other hand, at asymptotical distance one by the Abelian part the $\beta = \mathrm{d}\varphi \cong \mathrm{d}\varrho$. Then, at asymptotical singularity

$$\lim_{|x|\to 0} \beta \in \Lambda^1(M, \mathfrak{g}), \ and \ A = g^{-1}\mathrm{d}g + \beta \ (or \ A = \beta), \ and \ R \neq 0. \tag{10}$$

Let's suppose

$$\beta = \frac{\eta}{|x|} + \mathrm{d}\varphi, \tag{11}$$

where $\varphi \in C^\infty(M)$ and $\eta \in \Lambda^1(M, \mathfrak{g})$, so that

$$\eta = \eta^i_\mu \theta_i dx^\mu, \qquad \lim_{|x|\to 0 \ and \ \infty} \eta^i_\mu \to 0, \tag{12}$$

where θ_i are the generators of the $\mathfrak{g}$. Then, from the L' Hospital rule one writes

$$\lim_{|x|\to 0} \frac{\eta^i_\mu}{|x|} \to \frac{d(\eta^i_\mu)/dx}{1} \equiv \mathrm{d}\eta = \partial_\nu \eta^i_\mu dx^\nu, \tag{13}$$

and so at asymptotical singularity a gauge field stress is naturally induced. Hence if

$$A = g^{-1}\mathrm{d}g + \frac{\eta}{|x|} + \mathrm{d}\varphi, \tag{14}$$

then the curvature 2 - form at the singularity and asymptotical distance, respectively, are

$$R \to \mathrm{d}\eta \quad for \lim_{|x|\to 0} A, \quad and \quad R = 0 \quad for \lim_{|x|\to\infty} A. \tag{15}$$

Indeed this approach gives an integrated model of the infinite range interactions of the behaviors at small and large scales. Thus, as seen that the connection 1 - form is Newtonian / Coulombian style ($\mathrm{d}\beta = 0$) at large scale. Otherwise, if $\mathrm{d}\beta \neq 0$, then it is a non - Abelian character, and so it has the curvature 2 - form such that

$$R = \mathrm{d}\beta + \frac{1}{2}[\beta, \beta]_\wedge. \tag{16}$$

The theories of the gravity from Newton to metric - affine contain Newton' s constant as coupling constant, so that it is within in Einstein - Hilbert action, but pure Yang - Mills type gauge approaches for the gravity over a manifold of real 4 dimension it doesn't contain any coupling constant of interaction and the Yang - Mills invariance vanishes on real 4 dimension and it doesn't any contribution to field equations [9]. Despite this, a formalism including the Yang - Mills invariance on real 4 - dimension can be investigated.

Let M be a smooth manifold of real $n = 2r$ dimension and $\Lambda^r(M)$ be the bundle of exterior r - form. If Hodge duality operator $* : \Lambda^r(M \cong \mathbb{R}^{n=2r}) \to \Lambda^r(M \cong \mathbb{R}^{n=2r})$ acts on any exterior r - forms such that $*^2 = (-1)^r$, then the exterior r - forms serve conformally invariance if the manifold M is endowed by conformal metric [1]. If the metric of the manifold M is g, the conformal structure may be written as $\widetilde{g} = \lambda^2 g$, where λ is the conformal (smooth) parameter. Then, the Hodge duality operator $*$ acts on an exterior form $B \in \Lambda^r(M \cong \mathbb{R}^{n=2r})$ such that $*B = \lambda^{-2r}B$ [1]. For $r = 2$, the dimension of the manifold is real 4 and the Hodge duality on he curvature 2 - form has a conformal invariance:

$$*R = \lambda^{-4}R. \tag{17}$$

The importance of this case can appear indeed under the the group $G = U(1) \otimes \widetilde{U}(d)$, so that the group $U(1)$ behaves like a conformal action to the $\widetilde{U}(d)$. Hence, given a principal $G = U(1) \otimes \widetilde{U}(4)$ - bundle $[\pi : P^1 \to M, G]$ on a smooth manifold M of real 4 - dimension with conformal metric, $\widetilde{g} = e^{2\varrho}g$, where $\lambda = e^{\varrho} \in U(1)$ is the conformal parameter, a tangent space T_pM at any point $p \in M$ presents an endomorphism $\mathrm{End}(T_pM) \in \widetilde{U}(4)$. Hereafter we will show by V the tangent space T_pM. Then one has a vector bundle associated to the principal bundle P^1: $\mathrm{F}^1 = P^1 \times V/G$.

Let $\nabla \in \Lambda^1(M, \mathrm{F}^1)$ be torsion free connection on the bundle F^1, so that if ϵ is a dual base 1 - form then $\mathrm{d}^\nabla \epsilon = 0$. Hence the Einstein - Hilbert Lagrangian is

$$L_{EH}(\epsilon, A, R) = \mathrm{g}_1 \mathrm{Tr}[R \wedge *(\epsilon \wedge \epsilon)]. \tag{18}$$

The Hodge duality of the curvature $R \in \Lambda^2(M, \mathrm{F}^1)$ in real 4 - dimension is $*R \in \Lambda^2(M, \mathrm{F}^1)$. Then,considering the eq. (17),

$$*R = e^{-4\varrho} R \tag{19}$$

and Yang - Mills Lagrangian is written as

$$L_{YM}(A, R) = \mathrm{g}_2 \mathrm{Tr}[R \wedge *R] = \mathrm{g}_2 e^{-4\varrho} \mathrm{Tr}[R \wedge R], \tag{20}$$

g_1, g_2 are some constants. Also let $L_{Mat}(\epsilon, A)$ be a matter Lagrangian. Neglecting surface terms, the variation with respect to the ϵ and A of the action

$$S[\epsilon, A] = \int_M L_{EH}(\epsilon, A, R) + L_{YM}(A, R) + L_{Mat}(\epsilon, A) \tag{21}$$

reads then Yang - Mills with matter and Einstein equations, respectively, are written as

$$\mathrm{g}_2 \mathrm{d}^\nabla (e^{-4\varrho} R) + \frac{1}{2}\mathfrak{S} \quad = \quad 0, \tag{22}$$

$$\mathrm{g}_1 R \wedge *(\epsilon \wedge \epsilon \wedge \epsilon) + \mathfrak{T} \quad = \quad 0. \tag{23}$$

where $\mathfrak{S} = \frac{\delta L_{Mat}}{\delta A}$ and $\mathfrak{T} = \frac{\delta L_{Mat}}{\delta \epsilon}$ are the current forms belonging the matter field.

The connection and its curvature transform under the conformal group $e^\varrho \in U(1)$ such that $A' = A + e^{-\varrho}\mathrm{d}(e^\varrho) \equiv A + \mathrm{d}\varrho$ and $R' = R$. Considering the gauge potential given in eq. (15) and the 1 - form β in (11), this conformal transformation be important. Therefore, without the conformal action $U(1)$, that is $\varrho \to 0$,it becomes $*R = \pm R$ and $\mathrm{d}^\nabla(*R) = \mathrm{d}^\nabla R = 0$. Then one can mention from (anti-) self dual gauge theory.

3 (Anti-) Self Duality In Real 4 - Dimension

Consider a principal G - bundle $[\pi : P^0 \to M, G = U(1) \otimes \widetilde{U}(4)]$. Also given a vector bundle associated to the P^0: $\mathrm{F}^0 = P^0 \times V/G$. A connection on the F^0 is $\nabla : \Lambda^0(M, \mathrm{F}^0) \to \Lambda^1(M, \mathrm{F}^0)$ together with the covariant exterior derivative

$$\mathrm{d}^\nabla = \mathrm{d} + \frac{1}{2}[A, \bullet]_\wedge, \tag{24}$$

where $A \in \Lambda^1(M, \mathrm{F}^0)$ is connection 1 - form. Then the curvature of the connection ∇ is $R = \mathrm{d}^\nabla A$ with the Bianchi identity $\mathrm{d}^\nabla R = 0$. Here, non - existing the conformal action, that is without $U(1)$, the Hodge duality of the curvature 2 - form given in eq. (17) becomes $*R = \pm R$ in real 4 dimension, on the other word the curvature has (anti-) self dual gauge field. And so the Yang - Mills equation of the (anti-) self dual gauge field (at the same time it may be corresponded by without matter) and the Einstein equation are written as

$$\mathrm{d}^\nabla(*R) = \mathrm{d}^\nabla R = 0, \tag{25}$$

$$\mathrm{g}_1 R \wedge *(\epsilon \wedge \epsilon \wedge \epsilon) + \mathfrak{T} = 0. \tag{26}$$

Here eq. (25) is also known as instanton or Yang - Mills equation at vacuum. However, eq.(26) is the Einstein equation with matter. More explicitly, the Yang - Mills equation is written as

$$\mathbf{d}(*R) + \frac{1}{2}[A, *R] = 0. \tag{27}$$

The exterior differentiation of this expression, due to $\mathbf{d}^2(*R) = 0$, gives

$$\mathbf{d}([A, *R]) = 0. \tag{28}$$

Then, locally $[A, *R]$ is a closed or exact 3 - form:

$$\frac{1}{2}[A, *R] = \mathrm{j} + \mathbf{d}\alpha, \quad \mathrm{j} \in \Lambda^3(M), \alpha \in \Lambda^2(M), \quad \mathrm{d}\mathrm{j} = 0. \tag{29}$$

Hence, from the eq. (27) one gets

$$\mathrm{j} + \mathbf{d}(*R + \alpha) = 0. \tag{30}$$

However the manifold M has a boundary, $\partial M = 0$, the Stoke's theorem gives

$$\int_{\partial M} \mathrm{j} + \mathbf{d}(*R + \alpha) = \int_M \mathbf{d}\mathrm{j} = 0. \tag{31}$$

so the curvature and its Hodge dual become an exact 2 - form: $R = \mathrm{d}A \in \Lambda^2(M)$. As seen clearly that $A \wedge A = 0$, and so structure group G is reduced to the $U(1)$. Then the (anti-) self dual theory of the group $G = U(1) \otimes \widetilde{U}(4)$ is indeed a $G = U(1)$ theory in real 4 - dimension, i.e the Maxwell theory for the electromagnetism to be an infinite range interaction.

4 Conclusion

In result, we can say that a gauge theory of the group $G = U(1) \otimes \widetilde{U}(4)$ for the infinite range interactions in real 4 - dimension gives each Yang - Mills equation of the non - Abelian case induced by the $\widetilde{U}(4)$, and Einstein field equation by the $U(1) \otimes \widetilde{U}(4)$. However the (anti-) self dual theory of the group $G = U(1) \otimes \widetilde{U}(4)$ is indeed a $G = U(1)$ theory in real 4 - dimension.

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