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BLACK HOLES: *A WINDOW INTO STRONG (QUANTUM?) GRAVITY*

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Abstract

Black holes are regions of space-time where gravity becomes so strong to confine everything. Their classical general relativistic description however shows critical aspects when faced with the established quantum nature of matter. Alternative approaches and descriptions, like the horizon quantum mechanics and corpuscular models, have therefore been proposed in order to investigate their quantum structure, and search for new phenomenological signatures.

1 Gravitational collapse of quantum matter

The classical, general relativistic description of the gravitational collapse of a compact, massive object predicts the end-point of its evolution will be a space-time singularity, where the energy density diverges (along with tidal forces), provided a trapping surface forms at some point during the collapse and the weak energy condition is preserved all along [1]. In other words, if a black hole forms, general relativity predicts there is going to be a real singularity at its centre (see left panel in Fig. 1). However, matter is quantum, and such a singularity simply clashes with the Heisenberg uncertainty principle. One also gets a flavour of the sort of effects that the quantum nature of matter implies, for example, from the famous Hawking's discovery of black hole evaporation [2]: the space-time around the collapsing matter evolves in time and particles are produced in the vacuum state of any quantum field on such a background (see right panel in Fig. 1).

The Hawking effect has raised a number of concerning paradoxes about the possibility of building a consistent quantum description of gravity. Most notably, the prediction that information stored in the initial state of the collapsing star will go lost after the complete evaporation of the hole hinders the unitarity of the whole process. However, one should notice that the Hawing effect is derived by quantising small perturbations around the classical model of the collapse, which

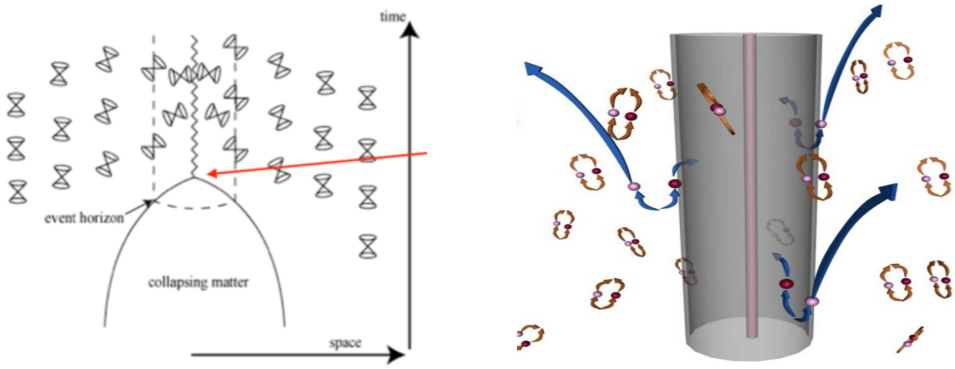

Figure 1: *Left panel: Classical Oppenheimer-Snyder model representing the collapse of a ball of dust that ends into a central singularity (red arrow) hidden inside the Schwarzschild horizon (dashed lines). Right panel: Hawking radiation as pair creation of virtual particles outside the black hole horizon.*

leaves us hope that a fully quantum treatment of the whole matter-gravity system will solve such issues. Of course, the big missing piece is now "quantum gravity".

2 Quantum gravity and black holes

It is common lore that quantum gravity should become relevant at the Planck length and mass [1],

$$\ell_{\rm p} = \sqrt{\hbar\, G_{\rm N}} \simeq 10^{-35}\,{\rm m} \qquad \text{and} \qquad m_{\rm p} = \sqrt{\hbar/G_{\rm N}} \simeq 10^{19}\,{\rm GeV}\ . \tag{1}$$

This argument is not mere numerology, but follows from the classical key concept of the gravitational radius of a static and spherically symmetric self-gravitating source, for which this quantity determines the existence of horizons.

A static spherically symmetric metric can always be written as

$$\mathrm{d}s^2 = g_{ij}(r)\,\mathrm{d}x^i\,\mathrm{d}x^j + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\ , \tag{2}$$

where $x^i = (t, r)$ and the area of a sphere parameterised by θ and ϕ is $\mathcal{A} = 4\,\pi\,r^2$. The location of a horizon is then determined by the vanishing of the null geodesic expansion,

$$g^{ij}\,\nabla_i r\,\nabla_j r = g^{rr} = 0\ . \tag{3}$$

Moreover, Einstein equations yield $g^{rr} = 1 - r_{\rm H}(r)/r$, where $r_{\rm H}(r) = 2\,\ell_{\rm p}\,m(r)/m_{\rm p}$ is the gravitational radius determined by the Misner-Sharp mass function

$$m(r) = 4\,\pi\int_0^r \rho(\bar r)\,\bar r^2\,\mathrm{d}\bar r\ , \tag{4}$$

with $\rho = \rho(r)$ the static matter density. A horizon then exists where the gravitational radius satisfies $r_{\rm H}(r) = r$, for some $r > 0$. In the vacuum far outside the region where the source is located, the Misner-Sharp mass approaches the Arnowitt-Deser-Misner (ADM) mass of the source, $m(r) \to M$, and the gravitational radius likewise becomes the Schwarzschild radius

$$R_{\rm H} = 2\,\ell_{\rm p}\,\frac{M}{m_{\rm p}} \cdot 42 \tag{5}$$

[1] I will use units with the speed of light $c = 1$.

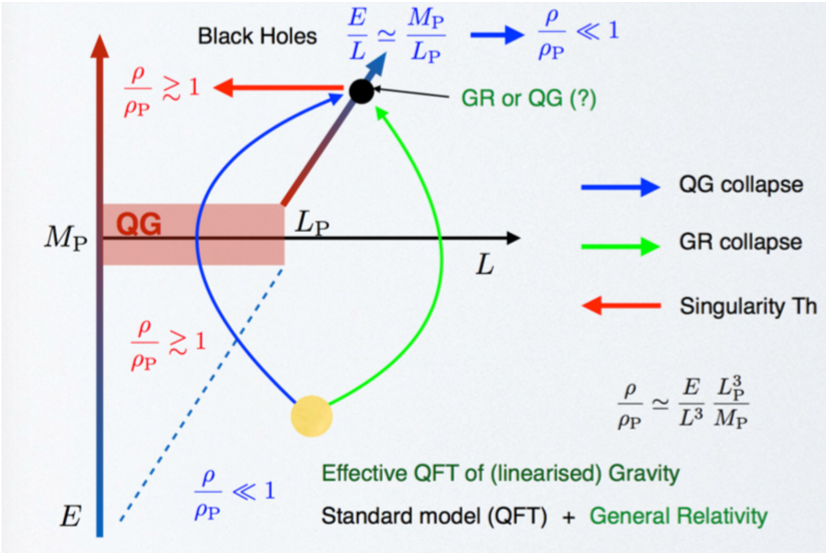

Figure 2: *"Phase space" of gravity. Energy E grows on the vertical axis, length L increases on the horizontal axis.*

The Heisenberg principle of quantum mechanics introduces an uncertainty in the particle's spatial localisation of the order of the Compton-de Broglie length, $\lambda_M \simeq \ell_p \, m_p / M$. It follows that one expects R_H only makes sense if

$$R_H \gtrsim \lambda_M \qquad \text{or} \qquad M \gtrsim m_p \ , \tag{6}$$

which immediately explains the key physical role played by the Planck scale and its relation with the existence of black holes in the quantum theory.

Our present knowledge is summarised in Fig. 2, where the relevant parameter is the energy density ρ in units of the Planck density ρ_p. We live in a region of extremely low ρ/ρ_p, in the bottom right corner, where the quantum field theoretical Standard Model of particles and classical general relativity describe very well our world. The region denoted by QG is where we meet both Planck length and mass, for which we presumably need a full quantum theory of gravity. From this region starts the line corresponding to black holes, moving up along which the energy density inside the horizon decreases and a black holes with larger mass E appear more and more classical. The yellow disk represents the starting point of, say, a collapsing star, which should produce a black hole (the black dot) according to general relativity. There are clearly two possibilities: either the star becomes a black hole by evolving through classical gravitational configurations (green line) or going through the quantum gravity region (blue line). The important point is that, according to classical general relativity, once the black hole forms, the matter in the star is forced to further contract and enter the quantum gravity regime. The conclusion is therefore that black holes are unavoidable and they make quantum gravity necessary as well.

If we look at black holes as bound states of gravity, we can draw an analogy with the (better understood) non-linear QCD theory:

1. like QCD confines quarks and gluons below the scale $\Lambda_{QCD} \simeq 220\,\text{MeV}$, Einstein gravity confines everything within a horizon R_H. Both effects occur in the non-perturbative regime

of the respective theories and there is no reason to believe that understanding black hole formation is going to be any easier than the (still open) problem of solving QCD;

2. like QCD becomes asymptotically free at energies much above $\Lambda_{\rm QCD}$, black holes should become asymptotically classical for $M \gg m_{\rm p}$.

A perhaps overlooked difference is that we have plenty of experimental data supporting the previous two points in QCD, whereas we have practically no data from the strong regime of gravity (beside the recently detected gravitational waves), which makes the above considerations about black holes purely theoretical expectations. Nonetheless, like one can consider effective descriptions of QCD around the scale of confinement, we could also envisage attempting at a quantum description of specific quantities of physical relevance for black holes, rather than insisting in deriving their properties from a candidate general theory of quantum gravity. In the following we shall describe one of such attempts.

3 Horizon Quantum Mechanics

As matter sources are described by quantum physics, the quantities that define the ADM mass M should also be considered as quantum variables, and the Horizon Quantum Mechanics (HQM) was precisely proposed in order to describe the Schwarzschild radius (5) quantum mechanically [3]. It is important to emphasise that the HQM differs from most previous attempts in which the gravitational degrees of freedom of the horizon, or of the black hole metric, are quantised independently of the state of the source. In the HQM, the gravitational radius is instead quantised together with the matter source, which is more akin to the non-linear general relativistic description of the gravitational interaction in the strong regime, and to DeWitt's mini-superspace approach [4].

We restrict our analysis to spherically symmetric sources which are both localised in space and at rest in the chosen reference frame. Let α denote the set of quantum numbers parametrising the spectral decomposition of the source, and write a matter state as

$$| \psi_{\rm S} \rangle = \sum_\alpha C_{\rm S}(E_\alpha) \, | \, E_\alpha \rangle \ , \tag{7}$$

where the sum is over the eigenstates of a given Hamiltonian H,

$$\hat{H} = \sum_\alpha E_\alpha | \, E_\alpha \rangle \langle E_\alpha \, | \ . \tag{8}$$

We can then replace the ADM mass with the expectation value of this Hamiltonian,

$$M \to \langle \psi_{\rm S} \, | \hat{H} | \, \psi_{\rm S} \rangle = \langle \psi_{\rm S} \, | \sum_\alpha E_\alpha | \, E_\alpha \rangle \langle E_\alpha \, | \, \psi_{\rm S} \rangle = \sum_\alpha |C_{\rm S}(E_\alpha)|^2 \, E_\alpha \ . \tag{9}$$

We also introduce the gravitational radius eigenstates

$$\hat{R}_{\rm H} \, | \, R_{{\rm H}\beta} \rangle = R_{{\rm H}\beta} \, | \, R_{{\rm H}\beta} \rangle \ , \tag{10}$$

so that a physical state for our system can be described by linear combinations

$$| \, \Psi \rangle = \sum_{\alpha,\beta} C(E_\alpha, R_{{\rm H}\beta}) \, | \, E_\alpha \rangle | \, R_{{\rm H}\beta} \rangle \tag{11}$$

which satisfy the algebraic (Hamiltonian) constraint (5), that is

$$0 = \left(\hat{H} - \frac{m_{\rm p}}{2\,\ell_{\rm p}} \, \hat{R}_{\rm H} \right) | \, \Psi \rangle = \sum_{\alpha,\beta} \left(E_\alpha - \frac{m_{\rm p}}{2\,\ell_{\rm p}} \, R_{{\rm H}\beta} \right) C(E_\alpha, R_{{\rm H}\beta}) \, | \, E_\alpha \rangle | \, R_{{\rm H}\beta} \rangle \ . \tag{12}$$

The solution is clearly given by

$$C(E_\alpha, R_{\mathrm{H}\beta}) = C(E_\alpha, 2\,\ell_\mathrm{p}\, E_\alpha/m_\mathrm{p})\,\delta_{\alpha\beta} \ , \tag{13}$$

where $\delta_{\alpha\beta}$ is the identity in the space of our quantum numbers.

By tracing out the gravitational radius, we must recover the matter state (7), which implies

$$C\left(E_\alpha, 2\,\ell_\mathrm{p}\, E_\alpha/m_\mathrm{p}\right) = C_\mathrm{S}(E_\alpha) \ . \tag{14}$$

Likewise, by integrating out the matter states, we obtain the horizon wave-function

$$|\, \psi_\mathrm{H}\, \rangle = \sum_\alpha C_\mathrm{S}(m_\mathrm{p}\, R_{\mathrm{H}\alpha}/2\,\ell_\mathrm{p})\,|\, R_{\mathrm{H}\alpha}\, \rangle \ , \tag{15}$$

where $m_\mathrm{p}\, R_{\mathrm{H}\alpha}/2\,\ell_\mathrm{p} = E(R_{\mathrm{H}\alpha})$. In the continuum, the normalised wave-function

$$\psi_\mathrm{H}(R_\mathrm{H}) = \langle\, R_\mathrm{H} \mid \psi_\mathrm{H}\, \rangle = \mathcal{N}_\mathrm{H}\, C_\mathrm{S}(m_\mathrm{p}\, R_\mathrm{H}/2\,\ell_\mathrm{p}) \tag{16}$$

yields the probability to detect a gravitational radius of size R_H associated with the particle in the quantum state $|\, \psi_\mathrm{S}\, \rangle$. We can further define the conditional probability density that the particle lies inside its own gravitational radius as

$$\mathcal{P}_<(R_\mathrm{H}) = P_\mathrm{S}(R_\mathrm{H})\, \mathcal{P}_\mathrm{H}(R_\mathrm{H}) \ , \tag{17}$$

where

$$P_\mathrm{S}(R_\mathrm{H}) = 4\,\pi \int_0^{R_\mathrm{H}} |\psi_\mathrm{S}(r)|^2\, r^2\, \mathrm{d}r \tag{18}$$

is the usual probability that the particle is found inside a sphere of radius $r = R_\mathrm{H}$, and

$$\mathcal{P}_\mathrm{H}(R_\mathrm{H}) = 4\,\pi\, R_\mathrm{H}^2\, |\psi_\mathrm{H}(R_\mathrm{H})|^2 \tag{19}$$

is the probability density that the value of the gravitational radius is R_H. One can view $\mathcal{P}_<(R_\mathrm{H})$ as the probability density that the sphere $r = R_\mathrm{H}$ is a trapping surface, and the probability that the particle is a black hole (regardless of the horizon size) will be obtained by integrating (17),

$$P_\mathrm{BH} = \int_0^\infty \mathcal{P}_<(R_\mathrm{H})\, \mathrm{d}R_\mathrm{H} \ , \tag{20}$$

which will depend on the observables and parameters of the specific matter state $|\, \psi_\mathrm{S}\, \rangle$.

3.1 Single particle and GUP

Let us consider a massive particle at rest in the origin of the reference frame described by the spherically symmetric Gaussian wave-function

$$\psi_\mathrm{S}(r) = \frac{e^{-\frac{r^2}{2\,\ell^2}}}{\ell^{3/2}\,\pi^{3/4}} \ , \tag{21}$$

with $\ell \simeq \lambda_m \simeq \ell_\mathrm{p}\, m_\mathrm{p}/m$. The corresponding momentum space wave-function

$$\psi_\mathrm{S}(p) = \frac{e^{-\frac{p^2}{2\,\Delta^2}}}{\Delta^{3/2}\,\pi^{3/4}} \ , \tag{22}$$

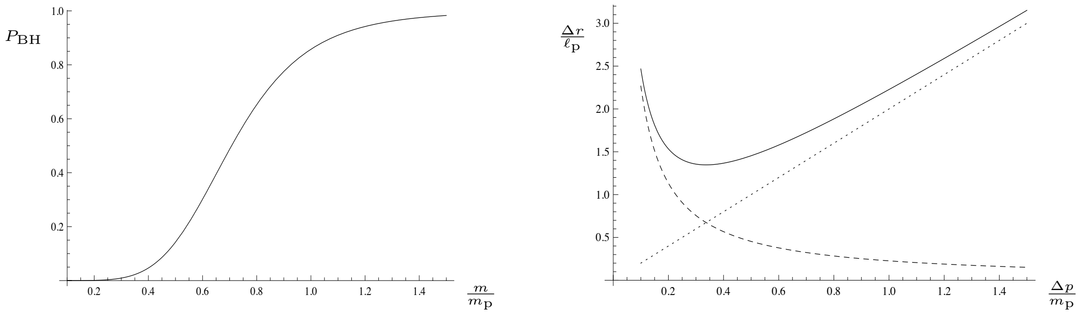

Figure 3: *Left panel: probability a Gaussian state is a black hole for increasing mass m. Right panel: generalised uncertainty relation* (26) *for* $\gamma = 1$.

has a width $\Delta = m_{\rm p}\,\ell_{\rm p}/\ell \simeq m$. We can also assume the relativistic mass-shell relation in flat space, $E^2 = p^2 + m^2$, which yields the normalized horizon wave-function

$$\psi_H(R_{\rm H}) = \frac{\ell^{3/2}\, e^{-\frac{m_{\rm p}^2\, R_{\rm H}^2}{8\, m^2\, \ell_{\rm p}^2}}}{2^{3/2}\, \pi^{3/4}\, \ell_{\rm p}^3} \ . \tag{23}$$

From the plot of the corresponding $P_{\rm BH}$ in the left panel of Fig. 3, it appears pretty obvious that the particle is most likely a black hole if $m \gtrsim m_{\rm p}$, in agreement with the qualitative result (6).

For the state (21), the uncertainty in radial size is given by

$$\Delta r^2 \simeq \ell^2 \simeq \ell_{\rm p}^2\, \frac{m_{\rm p}^2}{\Delta p^2} \ . \tag{24}$$

Analogously, the uncertainty in the horizon radius will be given by

$$\Delta R_{\rm H}^2 \simeq \frac{\ell_{\rm p}^4}{\ell^2} \simeq \ell_{\rm p}^2\, \frac{\Delta p^2}{m_{\rm p}^2} \ , \tag{25}$$

which, combined linearly with Eq. (24), yields the generalised uncertainty relation

$$\Delta r = \Delta r + \gamma\, \Delta R_{\rm H} \simeq \ell_{\rm p}\, \frac{m_{\rm p}}{\Delta p} + \gamma\, \ell_{\rm p}\, \frac{\Delta p}{m_{\rm p}} \ . \tag{26}$$

From the plot in the right panel of Fig. 3 (for $\gamma = 1$), one can see there is a minimum measurable length $\Delta r \gtrsim 1.3\, \sqrt{\gamma}\, \ell_{\rm p}$ obtained for $\Delta p \simeq m_{\rm p}$.

A crucial observation is that $\Delta R_{\rm H} \sim m \sim R_{\rm H}$, which seems to imply that the horizon of very massive sources fluctuate wildly, contrary to the expectation that astrophysical black holes should be classical objects. This leads us to consider alternative models of black holes, whose source is not localised within a very narrow wave-function (limiting to a point-like singularity).

3.2 BEC black holes

In the corpuscular model introduced by Dvali and Gomez [5], black holes are bound states of gravitons of spatial size $R_{\rm H}$, effectively forming a Bose-Einstein condensate (BEC) at a critical point. This picture emerges by considering the Newtonian potential generated by a star of mass M as made of N (virtual) gravitons of effective mass $m \simeq m_{\rm p}\, \ell_{\rm p}/\lambda_m$,

$$V_{\rm N}(r) \simeq -\frac{G_{\rm N}\, M}{r} = -\frac{\ell_{\rm p}\, N\, m}{r\, m_{\rm p}} \ . \tag{27}$$

After the star collapses to form a black hole [6], these gravitons are contained within a ball of radius $r \simeq R_{\rm H} \simeq \lambda_m$ and must be (at least) "marginally bound", that is [5]

$$E_K + U_m \simeq 0 \ . \tag{28}$$

where $E_K \simeq m$ and the average potential energy per graviton is

$$U_m \simeq m\, V_{\rm N}(\lambda_m) := -N\, \alpha\, m \ , \tag{29}$$

with the effective gravitational coupling $\alpha = \ell_{\rm p}^2/\lambda_m^2 = m^2/m_{\rm p}^2$. When the condition (28) is reached, the gravitons are "maximally packed", and their number satisfies $N\,\alpha \simeq 1$. The effective graviton mass correspondingly scales as $m \simeq m_{\rm p}/\sqrt{N}$, while the total mass of the black hole scales like

$$M = N\, m \simeq \sqrt{N}\, m_{\rm p} \ . \tag{30}$$

Moreover, the horizon area is spontaneously quantised as expected [7], that is

$$4\,\pi\, R_{\rm H}^2 \simeq \lambda_m^2 \simeq \ell_{\rm p}^2\, N \ . \tag{31}$$

This BEC black hole will emit gravitational Hawking radiation, since reciprocal $2 \to 2$ graviton scatterings inside the condensate give rise to a depletion rate

$$\dot N \sim -\frac{1}{N^2}\, N^2\, \frac{1}{\sqrt{N}\,\ell_{\rm p}} \ , \tag{32}$$

where the factor N^{-2} comes from α^2, the N^2 factor is combinatoric, and the last factor comes is the characteristic energy of the process $\Delta E \sim m$. This rate reproduces the standard decay law

$$\dot M \simeq m_{\rm p}\, \frac{\dot N}{\sqrt{N}} \sim -\frac{m_{\rm p}}{N\,\ell_{\rm p}} \sim -\frac{m_{\rm p}^3}{\ell_{\rm p}\, M^2} \ , \tag{33}$$

and allows one to read off the "effective" Hawking temperature

$$T_{\rm H} \simeq \frac{m_{\rm p}^2}{8\,\pi\, M} \sim m \sim \frac{m_{\rm p}}{\sqrt{N}} \ . \tag{34}$$

A more refined model was analysed in Refs. [8], in which we introduced candidate quantum states for both the BEC black hole and the emitted Hawking quanta. Such states were analysed using the HQM and their horizon uncertainty decreases for larger N,

$$\frac{\Delta R_{\rm H}}{R_{\rm H}} \simeq \frac{1}{N} \ , \tag{35}$$

which shows that such extended models of black holes correctly reproduce the expected behaviour in the macroscopic limit $N \simeq M/m_{\rm p} \gg 1$.

4 Summary and outlook

Given the difficulty in conceiving a full quantum theory of gravity, one can focus on a quantum description of particularly relevant quantities for specific problems. The HQM is precisely such an attempt for the gravitational radius of a matter source, which Einstein theory teaches us is a crucial quantity in black hole formation. This approach was applied to many different situations [3, 8, 9] and will be further investigated and extended in the future.

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