Asymptotic Symmetries and Faddeev-Kulish states in QED and Gravity

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Abstract

When calculating scattering amplitudes in gauge and gravitational theories one encounters infrared (IR) divergences associated with massless fields. These are known to be artifacts of constructing a quantum field theory starting with free fields, and the assumption that in the asymptotic limit (i.e. well before and after a scattering event) the incoming and outgoing states are non-interacting. In 1937, Bloch and Nordsieck provided a technical procedure eliminating the IR divergences in the cross-sections. However, this did not address the source of the problem: A detailed analysis reveals that, in quantum electrodynamics (QED) and in perturbative quantum gravity (PQG), the interactions cannot be ignored even in the asymptotic limit. This is due to the infinite range of the massless force-carrying bosons. By taking these asymptotic interactions into account, one can find a picture changing operator that transforms the free Fock states into asymptotically interacting Faddeev-Kulish (FK) states.

These FK states are charged (massive) particles surrounded by a “cloud” of soft photons (gravitons) and will render all scattering processes infrared finite already at an S-matrix level. Recently it has been found that the FK states are closely related to asymptotic symmetries. In the case of QED the FK states are eigenstates of the large gauge transformations – U(1) transformations with a non-vanishing transformation parameter at infinity. For PQG the FK states are eigenstates of the Bondi-Metzner-Sachs (BMS) transformations – the asymptotic symmetry group of an asymptotically flat spacetime. It also appears that the FK states are related the Wilson lines in the Mandelstam quantization scheme. This would allow one to obtain the physical FK states through geometrical or symmetry arguments. We attempt to clarify this relation and present a derivation of the FK states in PQG from the gravitational Wilson line in the eikonal approximation, a result that is novel to this thesis.
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<tr>
<td>BCH</td>
<td>Baker-Campbell-Hausdorff</td>
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<tr>
<td>BMS</td>
<td>Bondi-Metzner-Sachs</td>
</tr>
<tr>
<td>BN</td>
<td>Bloch-Nordsieck</td>
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<tr>
<td>EH</td>
<td>Einstein-Hilbert</td>
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<td>FK</td>
<td>Faddeev-Kulish</td>
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<tr>
<td>GCT</td>
<td>General Coordinate Transformation</td>
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<tr>
<td>GR</td>
<td>General Relativity</td>
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<td>IR</td>
<td>Infrared</td>
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<td>LGT</td>
<td>Large Gauge Transformation</td>
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<tr>
<td>PQG</td>
<td>Perturbative Quantum Gravity</td>
</tr>
<tr>
<td>QED</td>
<td>Quantum Electrodynamics</td>
</tr>
<tr>
<td>QFT</td>
<td>Quantum Field Theory</td>
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<td>UV</td>
<td>Ultraviolet</td>
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Aim for the stars,
If you fail,
You'll go to future timelike infinity.
1. Introduction

1.1. A Modern Approach to Field Theory

Quantum field theory strives to explain the interaction between matter and the fundamental forces of the universe. The basic building blocks of the theory are free particle fields. Subsequently, we enforce interaction between these fields in order to calculate the amplitudes of different processes. However, this construction is faulty. One often encounters infinities in both the infrared (IR) low energy limit and the ultraviolet (UV) high energy limit. To obtain finite and physical results one needs to apply the entire machinery of renormalization and other techniques. Meanwhile, the question as to why these infinities arose in the first place remains.

This thesis will focus on the infrared limit of quantum electrodynamics (QED) and perturbative quantum gravity (PQG). The claim is that the reason behind the infrared divergence is that our starting point – the free fields – is unphysical. An accelerated electron will undergo bremsstrahlung – the emission and absorption of low energy photons, the so called soft photons. In order to take this into account in a perturbative approach within QED one needs to sum over an infinite amount of divergent Feynman diagrams. Combined with the sum over all loop diagrams these infinities will cancel in a non-trivial way. The method was developed by Bloch and Nordsieck [1] and gives a finite result that agrees with experiments, but the procedure itself is not theoretically satisfying.

From the special theory of relativity we know that fields do not vanish at infinity. Massless fields go to (come from) future (past) null infinity, while massive fields go to (come from) future (past) timelike infinity. This is inconsistent with the assumption that the fields are asymptotically free. By doing a detailed analysis of the asymptotics of QED we will in fact find that it does not have the asymptotics of a free theory. The operator that describes the asymptotic behaviour will create a cloud of soft photons around each charged particle, which is the analogue of the classical Coulomb field. We can therefore create a new starting point of our theory, one of physical electrons and positrons which always interact with a cloud of soft photons. These states are referred to Faddeev-Kulish (FK) states, or dressed matter states [2–5]. The virtue of the Faddeev-Kulish approach is that the infrared divergence is cancelled at the S-matrix level, in contrast to the Bloch-Nordsieck method where it is cancelled at the cross-section level.

In quantum field theory we are heavily dependent on the existence of symmetries in order to distinguish the physical processes from the unphysical ones. In fact a
large class of processes that would have been divergent are discarded from the start due to symmetry considerations.\(^1\) Are there any additional symmetries, that are not immediately evident, but are responsible for the cancellation of the divergence? This question remained unanswered until the discovery of a new global symmetry of QED - large gauge transformations (LGT) \([6–10]\). Remarkably one can construct eigenstates of LGT charge using the Faddeev-Kulish states. By so doing the reason for the infrared divergence becomes apparent – our starting point of unphysical states has an ill-defined LGT charge, resulting in divergent amplitudes between unphysical processes \([11]\).

The IR limit of perturbative quantum gravity is qualitatively similar to the IR limits of QED, and the same type of divergence appears. Fortunately the Faddeev-Kulish formalism can be extended to gravity, where massive particles are dressed with clouds of soft gravitons \([12]\). Similar to the LGT there exists an asymptotic symmetry group of gravitational scattering – the Bondi-Metzner-Sachs (BMS) group, which is the symmetry group of an asymptotically flat spacetime \([13–17]\). Similarly, the FK states of gravity can be used to create eigenstates of the BMS charge \([18,19]\).

Using the formalism of Mandelstam quantization \([20–23]\) we are able to make a geometric derivation of the QED Faddeev-Kulish operator in terms of Wilson lines from infinity in flat space \([24,25]\). This approach has been used in Rindler and Schwarzschild backgrounds in order to analyse FK states around black hole horizons and their relation to soft hair \([26]\). In this thesis we extend the formalism to gravitational FK states using the eikonal method \([27–33]\) and further analyse the relation between the Wilson line approach and the asymptotic analysis made by Kulish and Faddeev.

### 1.2. Motivation and Structure

The overall aim of this thesis is to present and investigate the physical starting point of the Faddeev-Kulish states in quantum field theory. Since we do not have a satisfactory theory of quantum gravity yet, it is important to investigate the many parallels between quantum electrodynamics and perturbative quantum gravity in the infrared regime. The relation of FK states with asymptotic symmetries is at the heart of the matter in both theories. This thesis culminates in a geometrical approach that ties all threads together.

Further, this thesis aims to give a holistic presentation of both the traditional methods to solve the IR divergence, as well as the recent developments in the research field. Some novel results are presented in the end.

The reader is expected to have familiarity with quantum field theory and an introduction to the basic concepts of general relativity. The thesis is written on such a level that a master student with the prerequisite knowledge should be able to follow it in its entirety. Thus, the thesis begins with a thorough introduction to some fundamental mathematical concepts of differential geometry and group theory used throughout the

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\(^1\)The most known example are processes that do not conserve the four-momentum or electric charge.
text. This constitutes chapter 2. “Mathematical Preliminaries”. Supplementary theorems and lemmas used, that did not fit into the narrative of chapter 2, nor the main body of the thesis can be found in appendix A. “Additional Theorems and Lemmas”. chapter 3. “Prelude: The Infrared Divergence” introduces the reader to the original problem of the IR divergence in QED together with the traditional Bloch-Nordsieck method of obtaining a finite the cross-section.

The main body of the thesis is divided into two overarching parts. The first, Part I. “Asymptotic Symmetries”, will familiarize the reader with the symmetries of spacetime and gauge theories. It begins with chapter 4. “Spacetime Geometry”, which presents the field theoretic formulation of general relativity, the Poincaré group and QFT in curved spacetimes. The following chapter, chapter 5. “Large Gauge Symmetries”, will introduce the reader to the group of large gauge transformations, which are U(1) transformations that do not vanish at infinity. Following on the same thread, chapter 6. “The Bondi-Metzner-Sachs Group”, will present the BMS group, which is composed of the transformations that are asymptotic isometries of an asymptotically flat spacetime.

Part II. “Asymptotic States” treats the construction of the physical Faddeev-Kulish states and their relation to the asymptotic symmetries. It opens with chapter 7. “Faddeev-Kulish States in QED”, which incorporates a detailed review of the construction of the FK states in QED and the properties of the coherent state space the FK states live in. The mechanism that cancels the IR divergence is presented and it is shown that these states are in fact eigenstates of the large gauge transformations. Chapter 8. “Perturbative Quantum Gravity” introduces the reader to perturbative quantum gravity and proceeds to construct FK states of scalar particles surrounded to clouds of gravitons. These states are shown to be eigenstates of the BMS transformations. We proceed with chapter 9. “The Wilson Line Perspective”, where the reader is familiarized with the concept of Wilson lines, which are used to construct a gauge invariant formulation of QFT, in which the states are proven to be equivalent to FK states. Here a geometric interpretation of the relation between FK states and asymptotic symmetries is presented.
2. Mathematical Preliminaries

The purpose of this chapter is to make the thesis self-contained and should be regarded as reference material or a dictionary, rather than mandatory reading. The reader should be able to turn to this chapter in order to study underlying mathematical definitions and theorems used throughout the thesis. For the sake of clarity, most definitions are almost identical to the source material cited, with some minor notational alterations. A reader well-versed in mathematics can simply skim through this chapter and upon requirement go back to refresh their memory of the subject at hand. It is however heavily recommended to read the section on conventions and notation.

2.1. Conventions and Notation

Definitions are denoted by \( \equiv \), equalities by \( = \), isomorphisms by \( \cong \) and representations by \( \overline{=} \). Subsets are denoted by \( \subseteq \), elements in sets by \( \in \). Approximations are denoted by \( \approx \) and implications by \( \Rightarrow \). The symbol \( \rightarrow \) is used to denote mappings, limits and transformations. We denote direct sums with \( \oplus \) and direct products with \( \otimes \).

Throughout the thesis we will use index notation whenever we are dealing with (pseudo-)tensorial or vector quantities, i.e. work with components of (pseudo-)tensors. The Einstein summation convention is used throughout the thesis unless otherwise stated, meaning that terms with the same index repeated exactly twice, with one index upstairs and the other downstairs, are summed over the range of the index.

\[
A^\mu B_\mu \equiv \sum_{\mu=0}^{n} A^\mu B_\mu = A^0 B_0 + \cdots + A^n B_n. \tag{2.1}
\]

When an index is summed over it is said that it is contracted. Formally, the use of index notation implies the choice of some coordinates. Spacetime indices will be denoted by Greek letters and range from 0 to 3. The most commonly used letters will be \( \alpha, \beta, \gamma \) or \( \mu, \nu, \lambda, \kappa \). By convention, in Cartesian coordinates the index 0 will correspond to the time component, while indices 1, 2 and 3 corresponds to the three spatial components. Lower-case Latin letters are used for quantities that range over the three spatial components, 1, 2 and 3. The most commonly used letters will be \( i, j, k \). Additionally, we will often encounter quantities that only depend on the angular components, thus we will use capital Latin letters that range over 2 and 3. The most commonly used letters will be \( A \) and \( B \). Further, when convenient and will not be the
cause of confusion we will suppress the index entirely for sake of notational brevity. In those cases, italic letters as \( x \) or \( p \) will denote the four-vectors, while bold letters as \( \mathbf{x} \) or \( \mathbf{p} \) denote three-vectors. Notably, this will often be used when we contract two indices, for example

\[
pk = p \cdot k \equiv p^\mu k_\mu, \quad |\mathbf{x}|^2 = x^i x_i. \tag{2.2}
\]

Concerning the metric signature, we will use the sign convention of \( \eta_{\mu \nu} \equiv \text{diag}(-+++), \) which is commonly referred to as the east coast convention or the general relativity convention. To illustrate, this results in the summation

\[
A^\mu B_\mu = A^i B_i + A^0 B_0. \tag{2.3}
\]

If we contract the momentum \( p^\mu \) of a massive particle with itself, we will obtain

\[
p^2 \equiv p^\mu p_\mu = p^i p_i - p_0^2 = -m^2, \tag{2.4}
\]

where \( m \) is the particle's invariant mass, derived from special relativity. Note that in the case of a flat metric the spatial components can be written with both indices down, \( p^i p_i = p_\mu p_\mu. \) A slashed quantity is defined as a vector contracted with the gamma matrix,

\[
\not{p} \equiv p_\mu \gamma^\mu. \tag{2.5}
\]

Further, the partial derivative is often denoted by

\[
\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{2.6}
\]

Regarding other kinds of derivatives, a generic spacetime-covariant derivative will be denoted by \( \nabla_\mu, \) while \( D_A \) is reserved for the spacetime covariant derivative on the two-sphere, while the Lie derivative is denoted by \( \mathcal{L}_\xi. \) A gauge covariant derivative will instead be denoted by \( D_\mu. \) Other notational shorthands include the determinant of the metric, \( g \equiv \det(g_{\mu \nu}) \) and \( d^3p \equiv \frac{d^3\mathbf{p}}{12\pi^2 2\epsilon_0}, \) for the Lorentz invariant measure, where we have embedded all convention dependent scalars obtained from Fourier transformations. Further, normal ordering of creation and annihilation operators is implied and suppressed in the notation. As for units, we will be utilizing the system of natural units, where we set \( c = \hbar = \epsilon_0 = k_B = 1. \) Lastly, it should be observed that the Feynman diagrams are written with the initial states entering from the left and final states exiting to the right.

### 2.2. Differential Geometry

Differential geometry is the branch of mathematics devoted to the study of differentiable manifolds and as such it is immensely important in physics. The aim of this section is not to give the reader an all-encompassing introduction to differentiable manifolds as a whole, but rather to define and discuss the fundamental concepts needed in order to grasp the general theory of relativity and gauge theories. The contents are therefore presented in a such a way that their connection to physics will be as clear as possible.
2.2. Differential Geometry

2.2.1. Manifolds and Tensors

A differentiable manifold is collection of points, which locally looks like Euclidean space. It can be helpful to think of manifolds as a set of continuous points, where the “transition” between to neighbouring points is always differentiable. One of the most fundamental notions in differential geometry is the notion of mappings between manifolds.

**Definition 2.1** (Mapping). Let $M$ and $N$ be two sets. A mapping $\phi : M \to N$ maps each element $m \in M$ to a unique element $n \in N$.

**Definition 2.2** (Homeomorphism). Let $M$ and $N$ be two topological spaces. A mapping $\phi : M \to N$ is called a homeomorphism if it is one-to-one and if both $\phi$ and its inverse $\phi^{-1} : N \to M$ are continuous. The two topological spaces $M$ and $N$ are then said to be homeomorphic, $M \cong N$.

A mapping can also be called a map or a function. We can now introduce some terminology to describe the relation between two manifolds.

**Definition 2.3** (Diffeomorphism). Let $M$ and $N$ be two manifolds. A map $\phi : M \to N$ that is one-to-one and such that both $\phi$ and $\phi^{-1}$ are differentiable is called a diffeomorphism. The two manifolds $M$ and $N$ are then said to be diffeomorphic, $M \cong N$.

A diffeomorphism is a special case of a homeomorphism for manifolds. Since a diffeomorphism must be one-to-one, the dimension of $M$ and $N$ must agree. It also follows that a diffeomorphism cannot change the underlying structure or topology of the manifold, it is therefore equivalent to an active coordinate transformation. We are now in a position to state a more precise definition of differentiable manifolds.

**Definition 2.4** (Differentiable manifold). A set $M$ together with a collection of open subsets $\{O_\alpha\}$ is called a differentiable manifold if the following properties hold:

1. Each point $p \in M$ lies in at least one subset $O_\alpha$.
2. For each $\alpha$, there exists a diffeomorphism $\phi_\alpha : O_\alpha \to U_\alpha$, where $U_\alpha \subset \mathbb{R}^n$.
3. If any two sets $O_\alpha$ and $O_\beta$ overlap, $O_\alpha \cap O_\beta \neq \emptyset$, the mapping $\phi_\beta \circ \phi_\alpha^{-1}$, which maps points in $\phi_\alpha[O_\alpha \cap O_\beta] \subset U_\alpha$ to $\phi_\beta[O_\alpha \cap O_\beta] \subset U_\beta$ must be continuous and differentiable.

We will denote the manifold with $M$.

The first property requires that the subsets $\{O_\alpha\}$ cover $M$, the second property ensures that the manifold is locally diffeomorphic to an open subset of Euclidean space and
the third property ensures that the transition map between two open subsets is itself differentiable.

In order to assign values to specific points on the manifolds we introduce a set of coordinates \( x^\mu \). A coordinate system is a set function that attributes each point on the manifold a \textbf{unique} set of values. We can further define curves \( x^\mu(\tau) \), parametrized by \( \tau \), along the manifold. This allows us to grasp the concept of directional derivatives.

\textbf{Definition 2.5 (Directional derivative).} For some point \( p \in M \), a curve \( x^\mu(\tau) \) and a scalar function \( f(x^\nu) \) we can construct
\[
\left. \frac{d}{d \tau} f(x^\nu) \right|_p = \left. \frac{dx^\mu}{d \tau} \right|_p \frac{\partial}{\partial x^\mu} f(x^\nu) \left|_p \equiv \xi^\mu \partial_\mu f(x^\nu) \left|_p \right. ,
\]
(2.7)
where \( \xi^\mu \) is vector tangent to the curve \( x^\mu(\tau) \) and \( \partial_\mu = \frac{\partial}{\partial x^\mu} \) is the partial derivative along the curve \( x^\mu = \text{const.} \) for some set of coordinates \( x^\mu \). The sum \( \xi^\mu \partial_\mu \) is called the \textbf{directional derivative} along \( x^\mu(\tau) \).

Note that due to the summation \( \xi^\mu \partial_\mu \) is a coordinate independent quantity. The quantity \( \xi^\mu \) will specify the directional derivative and give us a notion of a direction. This means that directional derivatives are in fact tangent vectors. Often we will refer to the quantity \( \xi^\mu \) as a \textbf{contravariant} vector, when in fact it is a contravariant vector \textbf{component}, however the terms can be used interchangeably provided that we have a specified coordinate basis. The direction of the partial derivative will be along lines \( x^\mu = \text{const.} \) for some given coordinate system, meaning that they will specify the coordinate basis, \( \partial_\mu = \hat{e}_\mu \).

Often we are interested in knowing how objects behave under general coordinate transformations (GCT) or diffeomorphisms. The trivial example is a scalar.

\textbf{Definition 2.6 (Scalar).} A scalar is an object that is invariant under all general coordinate transformations.

Since a quantity with contracted indices, \( A^\mu B_\mu \), is a scalar, we can find the properties of objects with indices under general coordinate transformations (GCT).
\[
\xi^\mu \partial_\mu = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \equiv \xi^\nu \frac{\partial}{\partial x^\nu} \Rightarrow \xi^\nu = \frac{\partial x'^\nu}{\partial x^\mu} \xi^\mu \cdot
\]
(2.8)
Likewise, a quantity with indices downstairs transforms as \( \xi'^\mu = \frac{\partial x^\mu}{\partial x'^\nu} \xi^\nu \). We can now define vectors solely by their transformation laws under GCT. Consider the following transformations,
\[
dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \tag{2.9a}
\]
\[
\partial_\mu = \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}. \tag{2.9b}
\]
Definition 2.7 (Contravariant vector). A contravariant vector \( x^\mu \) is an object that transforms as a differential \( dx^\mu \) under general coordinate transformations,

\[
x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu.
\] (2.10)

From eq. (2.7) we notice that the notion of vectors is defined at a specific point \( p \), in fact vectors live in the tangent space of \( M \) at \( p \), and two vectors at different points live in different vector spaces.

Definition 2.8 (Tangent space). For a manifold \( M \), the set of all vectors \( \{X_p\} \) that are tangent to \( M \) at point \( p \) form the vector space \( T_p \), called the tangent space of \( M \) at \( p \) [34].

The dual of a contravariant vector is called a **covariant vector**, or a 1-form. A covariant vector \( \omega_p \) at point \( p \) is a function that maps the contravariant vector \( X_p \) at point \( p \) to a real scalar value, \( \omega_p(X_p) \in \mathbb{R} \). Covariant vectors live in the cotangent space.

Definition 2.9 (Covariant vector). A covariant vector \( x_\mu \) is an object that transforms as a partial derivative \( \partial_\mu \) under general coordinate transformations,

\[
x'_\mu = \frac{\partial x'^\nu}{\partial x_\mu} x^\nu.
\] (2.11)

Definition 2.10 (Cotangent space). For a \( n \)-dimensional manifold \( M \), the space dual to the tangent space \( T_p \) at point \( p \), is called the cotangent space \( T^*_p \) [34].

These definitions allows us to extend the notion of vectors to the more general **tensors**, which are objects that may contain both covariant and contravariant indices.\(^1\)

Definition 2.11 (Tensor). A tensor \( T_{\mu\nu\cdots}^{\alpha\beta\cdots} \) is an object that transforms as a differential \( dx^\mu \) for each contravariant index, and as a partial derivative \( \partial_\nu \) for each covariant index,

\[
T_{\mu\nu\cdots}^{\alpha\beta\cdots} \rightarrow T_{\mu'\nu'\cdots}^{\alpha'\beta'\cdots} = \frac{\partial x'^\alpha}{\partial x^\alpha} \frac{\partial x'^\beta}{\partial x^\beta} \cdots \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \cdots T_{\mu\nu\cdots}^{\alpha\beta\cdots}.
\] (2.12)

The rank of a tensor is the number of its free (uncontracted) indices. For example, \( T^{\mu\nu} \) is a rank 2 contravariant tensor, \( T_{\mu\nu} \) is a rank 2 covariant tensor and \( T_\nu^\mu \) is a rank 1+1 mixed tensor. Generically the order of the indices is important, since nothing requires a tensor to be symmetric under the permutation of indices. Scalars are tensors of rank 1\(^1\)

\(^1\)Even though we have not written out any coordinate dependence in our definitions, all tensorial properties hold for tensor fields, \( T_{\mu\nu\cdots}^{\alpha\beta\cdots}(x) \). Unless otherwise states the reader can assume that the objects at hand always are tensor fields.
0, contravariant vectors are tensors of rank $1 + 0$, while covariant vectors are tensors of rank $0 + 1$.

Any equation containing only tensorial objects is called **covariant**. In this context the word covariant refers to the fact that the equation holds true for any coordinate system. For example given the tensorial equality $A_{\alpha\beta\gamma} = B_{\alpha\beta\gamma}$, after a GCT we obtain,

$$
\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\gamma} A_{\mu\nu\rho} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\gamma} B_{\mu\nu\rho} \Rightarrow A_{\mu\nu\rho} = B_{\mu\nu\rho}.
$$

(2.13)

Which demonstrates the power of tensorial equations - they allow us to make coordinate independent statements and give us the freedom to chose the coordinates best suited for the problem at hand. It is important to note that any given quantity where all the indices are summed over is a scalar and thus is invariant under GCT.

Some care needs to be taken since not all objects with indices are tensors, an important example is the Christoffel symbol $\Gamma_{\mu\nu}^\lambda$, which we shall discuss in further detail in section 2.2.3. "Spacetime Curvature and the Covariant Derivative". Further, one will often encounter **densities**, which have special transformation rules. We refer to objects that transforms as scalars, but with additional factors of the Jacobian as **scalar densities**. Generically we are interested in tensor densities.

**Definition 2.12** (Tensor density). An object $D_{\mu_1...\mu_n}^{\alpha_1...\alpha_m}$ that transforms as a rank $m + n$ tensor except for extra factors of the determinant Jacobian,

$$
D'_{\mu_1...\mu_n}^{\alpha_1...\alpha_m} = \left| \frac{\partial x}{\partial x'} \right|^w D_{\mu_1...\mu_n}^{\alpha_1...\alpha_m}
$$

is called a tensor density with weight $w$ [36].

### 2.2.2. Metric Spaces

To introduce further structure on our manifold we can want to have a well-defined notion of distances. To this end we need to introduce a distance function or metric on our manifold, which will define the scalar product between vectors. The metric must therefore be a bilinear map, i.e. a covariant tensor of rank 2.

**Definition 2.13** (Metric). Let $M$ be a vector space, $X,Y \in M$. The bilinear map $g : M \times M \rightarrow \mathbb{R}$ is called a **metric** if it satisfies the following two properties: (i) Symmetry: $g(X,Y) = g(Y,X)$, (ii) Nondegeneracy: $g(X,Y) = 0$, $\forall X$ if and only if $Y=0$. We call the combination $(M,g)$ a metric space [35].

Suppose that we are working in a coordinate basis given by the basis $\{\hat{e}_\mu\}$. We are then able to express the metric in terms of its components in the given coordinate basis,

---

2It is unfortunate that this word coincides with the word for objects with index down. The terms meanings are distinct and not directly related.
Lemma 2.14. The metric $g(X, Y)$ can be written in terms of its components as follows,

$$g(X, Y) \equiv X \cdot Y = x^\mu \hat{e}_\mu y^\nu \hat{e}_\nu = x^\mu y^\nu g_{\mu \nu},$$

(2.15)

where $\hat{e}_\mu \hat{e}_\nu \equiv g_{\mu \nu}$, is the metric written in its components.

Say that we are interested in knowing how covariant vectors and contravariant vectors are related to each other. The metric mapped two contravariant vectors to a scalar, meaning that $x^\mu y^\nu g_{\mu \nu} \in \mathbb{R}$. We can relate the quantities as

$$g(X, Y) = x^\mu y^\nu g_{\mu \nu} = x^\mu y_\mu,$$

(2.16)

where we have defined $y_\mu \equiv y^\nu g_{\mu \nu}$. The metric allows us to lower and raise indices - it allows us to map contravariant vectors in the tangent space to their dual covariant vectors in the cotangent space.

The metric determinant $g \equiv \det g_{\mu \nu}$ transforms as

$$g' = \det g'_{\mu \nu} = \det \left( \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha \beta} \right) = \det \left( \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \right) g = \left| \frac{\partial x}{\partial x'} \right|^2 g, \quad (2.17)$$

where $\left| \frac{\partial x}{\partial x'} \right|$ is the Jacobian corresponding to the coordinate transformation - the metric determinant is a scalar density of weight 2. By demanding that $\det g_{\mu \nu}$ does not vanish we can define the inverse metric.

**Definition 2.15** (Inverse metric). For a given metric $g_{\mu \nu}$ on a manifold $M$, we can define the inverse metric $g^{\mu \nu}$ by the following relation

$$g_{\mu \nu} g^{\nu \sigma} = g^{\sigma \mu} g_{\nu \mu} = \delta^\sigma_\mu, \quad (2.18)$$

where $\delta^\sigma_\mu$ is the Kronecker delta.

The scalar product of a vector with itself gives us its magnitude, or length. The metric thus allows us to measure distances on our manifold. Consider the length of an infinitesimal vector $dx^\mu$,

$$ds^2 \equiv g_{\mu \nu} dx^\mu dx^\nu, \quad (2.19)$$

where the quantity $ds^2$ is called the line element.\(^3\) The proper time is defined as $d\tau^2 \equiv -ds^2$. The signature of a metric is determined by the number of positive and negative eigenvalues. This allows us to distinguish two important classes of metrics:

(i) Riemannian or Euclidean metrics, where all eigenvalues are positive, (ii) pseudo-Riemannian metrics where $n$ eigenvalues are negative and $m$ eigenvalues are positive.

\(^3\)Since the line element must be a scalar and thus invariant under GCT, we can show that the metric $g_{\mu \nu}$ is in fact a tensor since its transformation must cancel that of the tensors $dx^\mu dx^\nu$. 
where $n + m = d$ are the number of dimensions of the manifold. An important special case of pseudo-Riemannian metrics are the Lorentzian metrics, where one of the eigenvalues differs by a sign from the rest.

A common example of a Euclidean metric is the metric of $\mathbb{R}^3$. We can represent this metric with a three dimensional matrix in Cartesian coordinates,

$$
\eta_{ij} = \text{diag}(1,1,1),
$$

or likewise by the line element

$$
ds^2 = dx^2 + dy^2 + dz^2.
$$

The most important example of a Lorentzian metric is the flat Minkowski metric of special relativity. Written out on matrix form in Cartesian coordinates,$^4$

$$
\eta_{\mu\nu} = \text{diag}(-1,1,1,1),
$$

or likewise by the line element

$$
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.
$$

This is the metric that characterizes Minkowski space $\mathbb{R}^{1,3}$. The above two metrics were both flat, in general we will work with curved metrics, such as the metric of the unit two-sphere $S^2$, which has the line element

$$
d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2,
$$

in spherical polar coordinates. In relativistic physics one exclusively works with Lorentzian metrics. This compels us to extend our definition of manifolds.

**Definition 2.16 (Lorentzian manifold).** Let $U$ be an open subset $U \subset M$ of the $n$-dimensional manifold $M$. If there exists a map $\phi : U \to \mathbb{R}^{1,n-1}$ such that $\phi$ is differentiable, then $M$ is a differentiable Lorentzian manifold.

The metric serves a multitude of purposes; it allows us to raise and lower indices, to define a scalar product on our manifold, which in turn gives us the notion of lengths and distances. In fact, all information about the geometry of a manifold can be extracted from the metric.

**Definition 2.17 (Spacetime).** The combination of a differentiable manifold $M$ together with a Lorentzian metric $g_{\mu\nu}$ is called a spacetime and is denoted by $(M, g_{\mu\nu})$.

---

$^4$We are using natural units and have chosen the signature where the time component is negative.
2.2.3. Spacetime Curvature and the Covariant Derivative

In the previous section we mentioned that the Euclidean and Minkowski metrics are flat, but we never specified what that means. In general a manifold will be curved, meaning that while it is locally diffeomorphic to $\mathbb{R}^{1,3}$, it will have a different global structure. Recall that vectors live in the (co)tangent space at a specific point. This result in some difficulties when we want to compare two vectors at different points - they do not live in the same space! A general diffeomorphism does not need to affect all points equivalently, rather the mapping can dependent on spacetime coordinates. Thus two vectors at different points that have identical components in one frame will generically have different components in another frame. We can find the extremal (timelike) path between two points $p_1$ and $p_2$ as the integral of the proper time,

$$\tau(p_1, p_2) = \int_{p_1}^{p_2} (-g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}} = \int_0^1 d\sigma \left( -g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right)^{\frac{1}{2}}. \tag{2.25}$$

Since we are interested in the extremal path we can treat the above integral with the principle of least action for the Lagrangian $L = (-g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma})^{\frac{1}{2}}$. The Euler-Lagrange equations for the above Lagrangian will be

$$\frac{d^2 x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \tag{2.26}$$

This is known as the geodesic equation, where

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{g^{\lambda\alpha}}{2} \left( \partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right) \tag{2.27}$$

is the Christoffel symbol.\(^6\) The solutions eq. (2.26) are geodesics - a generalization of straight lines to curved spacetimes. We must note that even though the Christoffel symbol is an object with indices it is not a tensor. This can be seen from its transformation law derived from the definition eq. (2.27),

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x'\beta}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}. \tag{2.28}$$

The first term is what one expects for a tensor, however we obtain another, inhomogeneous term which prevents it from being a tensor. Likewise, consider the transformation of $\partial_\mu V^\nu$,

$$\partial'_\mu V'^\nu = \frac{\partial x'^\alpha}{\partial x^\alpha} \partial_\mu \left[ \frac{\partial x'^\nu}{\partial x'^\beta} V^\beta \right] = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial V^\beta}{\partial x'^\alpha} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x'^\alpha \partial x^\beta} V^\beta. \tag{2.29}$$

\(^5\)Were we to find the extremal path of $ds$ we would obtain a spacelike path.

\(^6\)Technically, the Christoffel symbol is the components of the Christoffel connection, also known as Levi-Civita connection, Riemann connection, metric connection or simply connection. We assume throughout this thesis that the connection is torsion-free.
once again we see that the first term is what we expect from the transformation of a tensor, while the second term prevents the partial derivative acting on a vector (tensor) from being a tensor. Consider the combination

\[ \nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha. \] (2.30)

By combining eqs. (2.28) and (2.29) we see that it follows the transformation

\[ \partial'_{\mu} V'^{\nu} + \Gamma'_{\mu\alpha} V'^{\alpha} = \partial_{\nu} \partial'_{\mu} V^\gamma + \Gamma'_{\beta\gamma} V^\beta, \] (2.31)

which tells us that \( \nabla_\mu V^\nu \) is in fact a tensor! The operator \( \nabla_\mu \) is called a covariant derivative since it gives us a way to differentiate vectors in such a way that the covariance of the equations is preserved. In the case of vector fields, the covariant derivative parallel transports a vector field to the point \( p \) from an infinitesimally separated point \( p + \epsilon \) along a geodesic, where it is subtracted from the same vector field at point \( p \). The Christoffel symbol serves precisely the role of parallel transport. In general, for a tensor field we will define the covariant derivative as follows.

**Theorem 2.18 (Covariant derivative).** The covariant derivative \( \nabla_\rho \) with respect to \( x^\rho \) of a tensor \( T_{\alpha\beta\cdots}^{\alpha\beta\cdots} \) is defined as

\[ \nabla_\rho T_{\mu
u\cdots}^{\alpha\beta\cdots} \equiv \partial_\rho T_{\mu\nu\cdots}^{\alpha\beta\cdots} + \Gamma^\alpha_{\rho\sigma} T_{\mu\nu\cdots}^{\alpha\beta\cdots} + \cdots - \Gamma^\sigma_{\mu\rho} T_{\alpha\nu\cdots}^{\alpha\beta\cdots} - \cdots, \] (2.32)

where a positive term is added for every contravariant index and a negative term is added for every covariant index of the tensor \( T_{\mu\nu\cdots}^{\alpha\beta\cdots} \).

We see that this definition reduces to eq. (2.30) for a contravariant vector. Note that in the case of a scalar quantity, \( \phi(x) \), the covariant derivative coincides with the partial derivative since \( \partial'_\mu \phi(x) = \frac{\partial \phi}{\partial x^\nu} \partial'_\nu \phi(x) \) transforms like a tensor. Thus we see that the Christoffel symbol acts as the connection of our manifold. It allows us to compare objects located at different points and gives us a well-defined way to compare tangent vectors living in different tangent spaces. It is important to note that neither the metric, nor the connection, is a property of the manifold itself, but rather an additional structure one introduced by the definition of a scalar product.

**Lemma 2.19.** The covariant derivative of the metric vanishes,

\[ \nabla_\lambda g_{\mu\nu} = 0. \] (2.33)

**Proof.** We see that

\[ \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} g_{\mu\alpha}, \] (2.34)

However, one can always choose a local inertial frame, or Riemann normal coordinates, where both the first derivative of the metric and the Christoffel symbols vanishes. Since statement eq. (2.34) is a tensorial statement is must hold for all frames, thus we conclude that \( \nabla_\lambda g_{\mu\nu} = 0 \) [36].
We now ask ourselves the following question: what is the simplest tensor that we can construct using only the metric $g^{\mu \nu}$ and its derivatives. Remembering that the Christoffel symbol $\Gamma^\lambda_{\mu \nu}$, which solely consists of the metric and its first derivatives, is not a tensor, it is suggested that the structure will of the sought tensor be more involved. Further, since we always have the freedom to chose a local inertial frame, the sought tensor cannot depend solely on the metric’s first derivatives, or solely be a combination of the Christoffel symbols, since it would imply that it vanishes in all frames. Thus, we present the following tensor,

$$ R^\lambda_{\mu \nu \kappa} \equiv \partial_\kappa \Gamma^\lambda_{\mu \nu} - \partial_\nu \Gamma^\lambda_{\mu \kappa} + \Gamma^\alpha_{\mu \nu} \Gamma^\lambda_{\kappa \alpha} - \Gamma^\alpha_{\mu \kappa} \Gamma^\lambda_{\nu \alpha}, $$

which we will refer to as the **Riemann curvature tensor**. A detailed derivation can be found in [36]. It can be shown that it does in fact transform as a tensor, which is suggested by the antisymmetry in $\nu$ and $\kappa$, which causes cancellation of the inhomogeneous terms. Further, the Riemann tensor contains derivatives of the Christoffel symbol, which in turn results in second derivatives of the metric that do not vanish in a local inertial frame. All things considered, the Riemann tensor contains terms with both the metric, together with its first and second derivatives. Qualitatively this means that it is a tensor that gives us a measure of how much the metric deviates from flatness, i.e. the curvature of spacetime. In a local inertial frame the metric will be Minkowskian at a point $p$, where its first derivatives vanishes. However, the second derivatives of the metric will generically be non-vanishing and give us terms that are related to the extent that the metric will deviate from the Minkowski metric in the neighbourhood of $p$. We can express the Riemann tensor in terms of the covariant derivative.

**Lemma 2.20.** The commutator of two covariant derivatives acting on a vector $B^\lambda$ is the Riemann tensor contracted with the vector.

$$ [\nabla_\mu, \nabla_\nu] B^\lambda = R^\lambda_{\kappa \mu \nu} B^\kappa. $$

Further, we will define two additional quantities from the Riemann tensor, both of which are measures of the curvature.

$$ R_{\mu \nu} \equiv R^\alpha_{\mu \alpha \nu}, $$

is called the **Ricci curvature tensor** and is the Riemann tensor contracted with itself.

$$ R \equiv R^\beta_\beta = g^{\beta \mu} R^\alpha_{\mu \alpha \beta}, $$

is called the **Ricci scalar**, or scalar curvature, and is the Ricci tensor contracted with itself, alternatively the Riemann tensor with all indices contracted. Since the curvature tensors and the curvature scalar are measures of a spacetime’s geometry, they are profoundly important in general relativity and quantum theories of gravity.
2.2. Differential Geometry

2.2.4. The Lie Derivative

Let \( \xi^\mu \) be a smooth vector field on a manifold \( M \), which generates the diffeomorphisms \( \sigma_\xi \). Since a diffeomorphism can be interpreted as an active coordinate transformation we can express it in terms of the infinitesimal transformation

\[
\sigma_\xi : x^\mu \rightarrow x'^\mu = x^\mu + \epsilon \xi^\mu. \tag{2.39}
\]

For example, consider the rotation of \( S^2 \) in the \( \phi \)-direction, then eq. (2.39) corresponds to the diffeomorphism

\[
\sigma_\xi(\theta, \phi) = (\theta, \phi + \epsilon \xi). \tag{2.40}
\]

By differentiating eq. (2.39) we obtain

\[
\frac{\partial x'^\mu}{\partial x^\nu} = \delta_{\nu}^\mu + \epsilon \partial_\nu \xi^\mu. \tag{2.41}
\]

Now consider the general coordinate transformation of a rank 2 contravariant tensor,

\[
T^{\mu\nu}(x) \rightarrow T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}(x)
= (\delta_{\alpha}^\mu + \epsilon \partial_\alpha \xi^\mu)(\delta_{\beta}^\nu + \epsilon \partial_\nu \xi^\nu) T^{\alpha\beta}(x). \tag{2.42}
\]

\[
= T^{\mu\nu}(x) + \epsilon T^{\alpha\nu}(x) \partial_\alpha \xi^\mu + \epsilon T^{\mu\beta}(x) \partial_\beta \xi^\nu + O(\epsilon^2).
\]

Consider the Taylor expansion of the same tensor \( T^{\mu\nu}(x) \), but shifted infinitesimally in the \( \xi \) direction,

\[
T^{\mu\nu}(x') = T^{\mu\nu}(x + \epsilon \xi) = T^{\mu\nu}(x) + \epsilon \xi^{\alpha} \partial_\alpha T^{\mu\nu} + O(\epsilon^2). \tag{2.43}
\]

**Definition 2.21** (Lie derivative). Let \( T^{\mu\nu}(x) \) be a tensor and \( \xi^\mu \) be a smooth vector field on a manifold \( M \). The quantity

\[
\mathcal{L}_\xi T^{\mu\nu} \equiv \lim_{\epsilon \to 0} \frac{T^{\mu\nu}(x') - T'^{\mu\nu}(x')}{\epsilon}, \tag{2.44}
\]

where \( x'^\mu = x^\mu + \epsilon \xi^\mu \), is called the **Lie derivative** of \( T^{\mu\nu} \) along \( \xi \) [37].

Using eqs. (2.43) and (2.42) we can express the Lie derivative of a rank 2 contravariant tensor as follows

\[
\mathcal{L}_\xi T^{\mu\nu}(x) = \xi^{\alpha} \partial_\alpha T^{\mu\nu}(x) = T^{\alpha\nu}(x) \partial_\alpha \xi^\mu - T^{\mu\alpha}(x) \partial_\alpha \xi^\nu. \tag{2.45}
\]

Note that covariant tensors would contribute with terms with positive sign due to their transformation laws. Following a similar procedure we can express the Lie derivative of a general tensor.
Lemma 2.22. The Lie derivative of any general tensor, \( T^{\alpha_3 \cdots}_{\mu_3 \cdots} \), along a smooth vector field \( \xi^\mu \) can be expressed as follows

\[
L_\xi T^{\alpha_3 \cdots}_{\mu_3 \cdots} = \xi^\rho \partial_\rho T^{\alpha_3 \cdots}_{\mu_3 \cdots} - T^{\rho \alpha_3 \cdots}_{\mu_3 \cdots} \partial_\rho \xi^\alpha - \cdots + T^{\alpha_3 \cdots}_{\rho \mu_3 \cdots} \partial_\mu \xi^\rho + \cdots,
\]  
(2.46)

where a negative term is added for every contravariant index and a positive term is added for every covariant index of the tensor \( T^{\alpha_3 \cdots}_{\mu_3 \cdots} \) [37].

The Lie derivative follows the regular rules for derivatives - it is a linear operation and it satisfies the Leibniz rule. Further, it is type-preserving since the Lie derivative of a tensor of rank \( m + n \) is a tensor of rank \( m + n \). We can now see that the effect of an infinitesimal coordinate transformation on any tensor is the old tensor at the same coordinate point plus the Lie derivative \( L_\xi T^{\alpha_3 \cdots}_{\mu_3 \cdots} \), where \( \xi \) parametrizes the transformation [36].

By setting the Lie derivative of the metric \( g_{\mu \nu} \) to zero along some vector field we can find its isometries, that is diffeomorphisms along which it is invariant. A field along which the metric is invariant is called a Killing vector field, and by Noether’s theorem A.1 it will have a corresponding conserved quantity.

Theorem 2.23. Let \( g_{\mu \nu} \) be the metric tensor and \( \xi^\mu \) be a smooth vector field. If the Lie derivative of the metric along \( \xi^\mu \) equals to zero,

\[
L_\xi g_{\mu \nu} = 0,
\]  
(2.47)

then \( \xi^\mu \) is a Killing vector field.

Proof. Consider the Lie derivative of the metric,

\[
L_\xi g_{\mu \nu} = \xi^\alpha \partial_\alpha g_{\alpha \beta} + g_{\mu \alpha} \partial_\nu \xi^\alpha + g_{\alpha \nu} \partial_\mu \xi^\alpha.
\]  
(2.48)

Since the covariant derivative of a metric is equal to zero we can subtract it from the above expression,

\[
L_\xi g_{\mu \nu} = \xi^\alpha \partial_\alpha g_{\alpha \beta} + g_{\mu \alpha} \partial_\nu \xi^\alpha + g_{\alpha \nu} \partial_\mu \xi^\alpha - \xi^\alpha \nabla_\alpha g_{\mu \nu}
\]

\[
= \xi^\alpha \partial_\alpha g_{\alpha \beta} + g_{\mu \alpha} \partial_\nu \xi^\alpha + g_{\alpha \nu} \partial_\mu \xi^\alpha - \xi^\alpha (\partial_\alpha g_{\mu \nu} - \Gamma^\beta_{\alpha \mu} g_{\beta \nu} - \Gamma^\beta_{\alpha \nu} g_{\mu \beta})
\]

\[
= g_{\mu \alpha} \partial_\nu \xi^\alpha + g_{\alpha \nu} \partial_\mu \xi^\alpha + \xi^\alpha \Gamma^\beta_{\alpha \mu} g_{\beta \nu} + \xi^\alpha \Gamma^\beta_{\alpha \nu} g_{\mu \beta}
\]

\[
= \partial_\nu \xi_\mu + \partial_\mu \xi_\nu + g_{\beta \nu} \Gamma^\beta_{\alpha \mu} \xi^\alpha + g_{\mu \beta} \Gamma^\beta_{\alpha \nu} \xi^\alpha
\]

\[
= g_{\mu \beta} (\partial_\nu \xi^\beta + \Gamma^\beta_{\alpha \nu} \xi^\alpha) + g_{\beta \nu} (\partial_\mu \xi^\beta + \Gamma^\beta_{\alpha \mu} \xi^\alpha)
\]

\[
= g_{\mu \beta} \nabla_\nu \xi^\beta + g_{\beta \nu} \nabla_\mu \xi^\beta
\]

\[
= \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu.
\]  
(2.49)
By setting the Lie derivative equal to zero, we can see from the last line that we obtain Killing’s equation,

\[ \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0. \tag{2.50} \]

\[ \square \]

2.3. Group Theory

In this section we will present the definitions and theorems for the basic concepts in group theory. The concept of groups and group theory is used as a powerful mathematical tool in almost all fields of physics.

**Definition 2.24 (Group).** Let \( G \) denote a nonempty set and * denote a law of composition (binary operation). We will call \((G,*)\) a **group** under the law of composition * if the following properties hold:

(i) **Closure:** For all \( a, b \in G \), the product \( a \ast b \) is an element of \( G \).

(ii) **Associativity:** For all \( a, b, c \in G \), we have \( a \ast (b \ast c) = (a \ast b) \ast c \).

(iii) **Identity:** There exists an identity element \( e \in G \), such that \( e \ast a = a \ast e = a \) for all \( a \in G \).

(iv) **Inverse:** For each \( a \in G \) there exists an inverse element \( a^{-1} \in G \) such that \( a \ast a^{-1} = a^{-1} \ast a = e \) [38].

Henceforth for the sake of brevity we will denote the group with simply \( G \) and the product \( a \ast b \equiv ab \), given that the notation does not cause any misunderstanding. For the special case where the group operation is commutative the group is called **Abelian**.

**Definition 2.25 (Abelian group).** A group \( G \) is called Abelian if \( ab = ba \) for all \( a, b \in G \) [38].

A group that is not Abelian is called **non-Abelian**. It is often of interest to study a subset \( H \subseteq G \) that itself forms a group, such a subset is called a **subgroup**.

**Definition 2.26 (Subgroup).** A subset \( H \subseteq G \) is called a subgroup if the following properties hold:

(i) **Closure:** For all \( h, k \in H \), the product \( hk \in H \).

(ii) **Identity:** The identity element \( e \in G \) is an element \( e \in H \).

(iii) **Inverse:** For each \( h \in H \) there exists an inverse element \( h^{-1} \in H \).
This is denoted by $H \leq G$ [34].

Note that the property of associativity is automatically inherited to $H$ from the larger group $G$. All groups have two trivial subgroups, namely $\{e\}$ and the whole group itself.

**Definition 2.27** (Normal subgroup). A subgroup $H$ of the group $G$ is called a **normal subgroup** of $G$ if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$. This is denoted by $H \triangleleft G$ [38].

**Definition 2.28** (Cosets). For $H \leq G$ and $a \in G$, the set

$$aH = \{x \in G \mid x = ah \text{ for some } h \in H\}$$

(2.51a)

is called the **left coset** of $H$ in $G$ determined by $a$. Similarly, the **right coset** is defined by

$$Ha = \{x \in G \mid x = ha \text{ for some } h \in H\}.$$  

(2.51b)

The number of left cosets of $H$ in $G$ is called the **index** of $H$ in $G$ and is denoted by $[G : H]$ [38].

**Theorem 2.29.** The left and right cosets for a normal subgroup are identical.

**Proof.** Let $N \triangleleft G$ and $g \in G$. It follows from definition 2.27 that

$$gN = gNg^{-1} = g^{-1}Ng = Ng, \quad \forall g \in G.$$ 

(2.52)

Note that this does not imply that every element of $N$ commutes with every element in $G$, but rather this means that for every element $g \in G$ and $n \in N$ there exists another element $n' \in N$ such that $gn = n'g$. Now consider the product of two different cosets of the normal subgroup $N \triangleleft G$, with $g_1, g_2 \in G$,

$$g_1Ng_2N = g_1(g_2g_2^{-1})Ng_2N = g_1g_2(g_2^{-1}Ng_2)N = g_1g_2NN = g_1g_2N.$$ 

(2.53)

Since $g_1g_2 \in G$, this leads to the conclusion that the product of two cosets of a normal subgroup is always another coset. In fact, the set of all cosets of a normal subgroup, itself forms a group. This can be verified by the fact that $eN$ constitutes the identity element and $g^{-1}N = (gN)^{-1}$ forms the inverse, thus the group axioms of definition 2.24 are satisfied.

**Definition 2.30** (Factor group). The set of all cosets of a normal subgroup $N \triangleleft G$ itself forms a group, which we call the **factor (quotient) group** and denote by $G/N$. 

There are several ways to define products between groups, one of them is the semidirect product.

**Definition 2.31 (Semidirect product).** Let $G$ be a group with the normal subgroup $N$ and another subgroup $H$. If there exists a unique $n \in N$ and a unique $h \in H$ such that $g = nh$ for every element $g \in G$, we will say that $G = N \rtimes H$ is the **semidirect product** of $N$ and $H$.

### 2.3.1. Homomorphisms

Often it is interesting to study how a group transforms under different mappings. Mappings where the group structure is preserved are of considerable importance and thus we will study the definitions of the central concepts.

**Definition 2.32 (Homomorphism).** Let $G$ and $G'$ be two groups. A map $\phi : G \to G'$ is called a **homomorphism** if $\phi$ preserves the products, $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$ [34].

**Definition 2.33 (Isomorphism).** The special case when a homomorphism is one-to-one and onto is called an **isomorphism**. If an isomorphism between two groups exists then the two groups are said to be **isomorphic**, which is denoted by $G \cong G'$ [34].

If two groups are isomorphic they are virtually identical in all their group properties from the perspective of abstract algebra. In physics, we often use **representations** of groups, for example in terms of matrices. Some representations of a group can be more suitable than other representations of the same group depending on the context.
3. Prelude: The Infrared Divergence

Summary

- The original occurrence of the infrared divergence is presented.
- The Bloch-Nordsieck theorem is presented schematically.
- We find that elastic scattering is prohibited in QED.
- At $S$-matrix level we find that the IR divergence amounts to a diverging complex phase.

Any scattering process involving charged particles will always result in the emission of radiation, which we refer to as \textbf{bremsstrahlung}. Consider the scattering of a charged particle by the Coulomb field of a heavy nucleus, as seen in fig. 1. From the Feynman diagram we can write down the corresponding Feynman amplitude of this process,

\[
\mathcal{M} = -e^2 \bar{u}(p') \left[ \gamma(k) iS_F(p' + k) A_e(q) + A_e(q) iS_F(p - k) \gamma(k) \right] u(p)
\]

\[
= -ie^2 \bar{u}(p') \left[ \gamma(k) \frac{-(p' + k) + m}{2p'k} A_e(q) + A_e(q) \frac{-(p - k) + m}{-2pk} \gamma(k) \right] u(p), \quad (3.1)
\]

where $u(p)$ and $\bar{u}(p')$ are the constant four-spinors one obtains from the Dirac equation, $\epsilon_\mu(k)$ is the photon polarization tensor and slash denoted a vector contracted with the gamma matrix $\slashed{B} \equiv B_\mu \gamma^\mu$. The momentum space potential of a heavy nucleus in the Coulomb gauge is $A_e^\mu(q) = (\frac{Ze}{|q|}, 0, 0, 0)$, where $Z$ is the atomic number and $q = p' + k - p$. Since the nucleus is several orders of magnitude heavier than the electron and $Z \gg 1$ we are justified to treat its Coulomb field as an external classical field. Note that the denominator in the fermion propagator, $iS_F(p) = -\frac{p + m}{p^2 + m^2}$, is calculated using $(p \pm k)^2 + m^2 = p^2 \pm 2pk + k^2 + m^2 = \pm 2pk$, where we used $p^2 = -m^2$ and $k^2 = 0$. 
Say that we are interested in the case when the emitted radiation consists of a photon with very low energy, \( \omega \approx 0 \), a so called soft photon. We will work in an approximation called the eikonal approximation, where we disregard all factors of \( k \) in the numerator, which is justified by the low energy assumption. It follows that \( q = p' - p \) and \( |p| = |p'| \).

From the Dirac equation we find that constant spinor \( u(p) \) satisfies \( (\gamma^\mu p + m)u(p) = 0 \), see appendix B.2. “The Dirac Field”, meaning that

\[
(-\gamma^\mu + m)\psi(k)u(p) = (-2\gamma^\mu + [p + m]\psi(k)u(p) = -2p^\mu\epsilon_\mu(k)u(p),
\]

thus we can reduce eq. (3.1) to

\[
\mathcal{M} = -ie^2\bar{u}(p')\hat{A}_e(q)u(p)\left[\frac{p\epsilon}{pk} - \frac{p'\epsilon}{p'k}\right],
\]

where \( pe \equiv p^\mu\epsilon_\mu(k) \). We note that \( ie\bar{u}(p')\hat{A}_e(q)u(p) = \mathcal{M}_0 \) is the Feynman amplitude for the elastic scattering of a charged particle by a heavy nucleus without any emission of radiation, that is an elastic scattering. The differential cross-section for soft photon bremsstrahlung becomes

\[
\left(\frac{d\sigma}{d\Omega}\right)_B = \left(\frac{d\sigma}{d\Omega}\right)_0\frac{\alpha}{(2\pi)^2}\left[\frac{p'\epsilon}{p'k} - \frac{p\epsilon}{pk}\right]^2 \frac{d^3k}{\omega},
\]

where \( \left(\frac{d\sigma}{d\Omega}\right)_0 \) is the differential cross-section of the elastic scattering process and \( \alpha = \frac{e^2}{4\pi} \) is the fine structure constant. We are now in a position to conclude that both the amplitude and the cross-section will diverge as \( \omega \to 0 \). This occurrence is called the infrared divergence. Clearly this poses a problem; either the theory is incorrect or we are missing a piece of the puzzle. Not only is this a problem in quantum electrodynamics, but an almost identical divergence occurs in the case of perturbative quantum gravity. The study and solution of this conundrum is the main objective of this thesis.
3. Prelude: The Infrared Divergence

More generally we can consider the processes depicted in fig. 2, where the blob $\mathcal{B}$ can stand for any IR finite part of the graph, including both tree-level processes and any number of internal loops.\footnote{There exists additional diagrams, where the soft photon is emitted from the incoming particle, or where the self-energy loop of fig. 2d is on the external leg. These diagrams are not depicted due to redundancy, but are taken into consideration in the calculations. Soft photon emission from internal legs are IR finite, since the momentum is off-shell.} The more general type of amplitude for a soft photon emission process, as seen in fig. 2b, will have the form

$$\frac{d\sigma}{d\Omega'} = \frac{\alpha}{(2\pi)^2} |\mathcal{B}|^2 \frac{pp'}{(pk)(p'k)} \frac{d^3 k}{\omega} \sum_{\text{pol}}$$

where we have summed over the polarizations. Any observer in a physical process will have some maximum energy resolution, $\Delta E$, within which energy levels cannot be distinguished. Thus, the soft photon energy should be integrated over a range with $|k| \leq \Delta E$ as the upper limit. In order to avoid the divergence at $|k| = 0$ we will introduce a lower cut-off at $|k| = \lambda$. This is equivalent to giving the photon a fictitious, non-physical mass $\lambda$. The cross-section becomes

\begin{align*}
\frac{d\sigma}{d\Omega'} = \frac{\alpha}{(2\pi)^2} |\mathcal{B}|^2 \frac{pp'}{(pk)(p'k)} \frac{d^3 k}{\omega} \sum_{\text{pol}}
\end{align*}
$$\sigma_b = \frac{\alpha}{(2\pi)^2} |\mathcal{B}|^2 \ln \left( \frac{\Delta E}{\lambda} \right) \int d\Omega \frac{pp'}{(E - p\cos\theta)(E' - p'\cos\theta')}.$$  \hspace{1cm} (3.6)

where \(E\) and \(E'\) are the energies of the external fermions, while \(\theta\) and \(\theta'\) are the angles between \(p\) or \(p'\) and \(k\).

The graphs of the type fig. 2c are experimentally indistinguishable from a graph with soft photon emission and must be added to the total cross-section.

$$\sigma_c = -\frac{\alpha}{(2\pi)^2} |\mathcal{B}|^2 \ln \left( \frac{E}{\Delta E} \right) \int d\Omega \frac{pp'}{(E - p\cos\theta)(E' - p'\cos\theta')}.$$  \hspace{1cm} (3.7)

When summed with eq. (3.6) this graph will give us a contribution proportional to \(\ln(\Delta E/E)\). The fermion self-energy graph, fig. 2d, will cancel against the soft photon emission from the incoming external leg. By summing all graphs we obtain the total experimental cross-section

$$\sigma_{exp} = -\frac{\alpha}{(2\pi)^2} |\mathcal{B}|^2 \ln \left( \frac{E}{\Delta E} \right) \mathcal{K} + \mathcal{O}(\alpha^2),$$  \hspace{1cm} (3.8)

where

$$\mathcal{K} = \int d\Omega \frac{pp'}{(E - p\cos\theta)(E' - p'\cos\theta')} - 1.$$  \hspace{1cm} (3.9)

Incidentally we see that in the total experimental cross-section has no cut-off dependence. The factors of \(\lambda\) obtained from the different graphs cancelled each other and experimental cross-section will be IR finite. The Bloch-Nordsieck theorem states that the infrared divergences cancel in each order of perturbation theory \([1]\). By summing all graphs to all orders we will obtain a total inelastic cross-section proportional to

$$\sigma_{exp} = |\mathcal{B}|^2 \left( \frac{\Delta E}{E} \right)^{\alpha \mathcal{K}/(2\pi)^2}.$$  \hspace{1cm} (3.10)

It is interesting to note that the cross-section vanishes as \(\Delta E \to 0\), since the quantity \(\mathcal{K}\) is strictly positive. This means that strict elastic scattering is prohibited given that the observer has an infinite energy resolution. The reader is referenced to \([39,40]\) for a more detailed calculation and discussion of the IR divergence and the Bloch-Nordsieck formalism.

Using this formalism we obtained a cross-section that is finite and agrees well with experiments. However we still have the problem of a IR divergent \(S\)-matrix. In addition to the real diverges part there also exists a complex phase \(e^{i\Phi(x)}\) that diverges in the IR limit. In \([41–43]\) Weinberg showed that any scattering process in QED will introduce a phase factor between incoming and outgoing particle pairs. Each different particle pair \((n,m)\) will contribute with the following phase factor,
\[
\Phi(t) = \frac{e_n e_m}{4\pi} \frac{p_m p_m'}{\sqrt{\left(p_m p_m'\right)^2 + (m_m m_m')^2}} \text{sgn} \ln \frac{|q|}{r_0},
\]  

(3.11)

where the index \(n\) and \(m\) denotes the incoming and outgoing particles, \(e_m\) and \(p_m\) is the charge and momentum of the \(m\)th particle, respectively. Thus the phase factor \(\Phi(t)\) diverges for soft processes since \(p_m \to p_m'\) as the soft photon momentum \(k \to 0\). This result can be generalized to gravity, giving a very similar phase factor [44].

The study of the infrared divergences is the central theme of this thesis. Even though the original problem was formally solved by Bloch & Nordsieck in 1937 [1] several fundamental questions were still left unanswered. Is it possible to construct a method that does not only make the cross-section finite, but also makes the \(S\)-matrix finite and well-defined? Is the cancellation of the IR divergence in the Bloch-Nordsieck formalism a happy coincidence, or is there some thitherto unknown symmetry of the theory that is responsible for the cancellation? Is there any relation to the gauge independence of the theory?

All the above questions have a common answer – yes. The problem turns out to be in the foundation of quantum field theory. The states that we build our theory from, free Fock states, are unphysical by construction. Our starting point are free particles, between which we subsequently impose interactions. However, an electron is always charged, which means that it must \textbf{always} interact with the electromagnetic field. Part II. “Asymptotic States” will be devoted to the construction of physical matter states. We will however see that these states are intimately connected to asymptotic symmetries of the theory, which is the topic of Part I. “Asymptotic Symmetries”.
Part I
Asymptotic Symmetries
4. Spacetime Geometry

Summary

- The Lagrangian formulation of the general theory of relativity is presented and the Einstein field equations are derived as the equations of motion for the Einstein-Hilbert action.
- The Minkowski spacetime is presented and discussed in further detail.
- Conformal transformations and Penrose diagrams are introduced.
- The Poincaré group is found to be the symmetry group of the Minkowski spacetime.
- The basic concepts of QFT in curved spacetimes are presented.

4.1. The Einstein-Hilbert Action

The theory of general relativity can be formulated as a classical field theory in terms of an action principle and a Lagrangian, to which the Einstein field equations will be the corresponding equations of motion [35, 36]. The matter-free (vacuum) theory will be described by the Einstein-Hilbert action,

\[ S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \ R, \tag{4.1} \]

where \( g_{\mu\nu} \) is the metric tensor, \( R \) is the Ricci scalar, \( G \) is Newton’s constant and \( \sqrt{-g} \equiv \sqrt{-\det(g)} \). The combination \( d^4x \sqrt{-g} \) is called the invariant volume element since it is left invariant under general coordinate transformations, \( d^4x \sqrt{-g} = d^4x' \sqrt{-g'} \). The Ricci scalar is defined in terms of the Ricci tensor

\[ R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{4.2} \]

\(^1\text{We have excluded the Gibbons-Hawking-York term, since it will not affect the analysis at hand.}\)
The equations of motion for the metric field can be found by extremizing the action (4.1), which is done by setting the variation of the action with respect to the metric field \( g_{\mu\nu} \) to zero. We find that

\[
\delta \left( \sqrt{-g} R \right) = \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} + \sqrt{-g g^{\mu\nu} \delta R_{\mu\nu}},
\]

(4.3)

where the variation is with respect to the metric field \( g_{\mu\nu} \), \( \delta f \equiv \frac{\delta f}{\delta g_{\mu\nu}} \). Using \( \delta R_{\mu\nu} = \nabla_\nu (\delta \Gamma^\lambda_{\mu\lambda}) - \nabla_\lambda (\delta \Gamma^\lambda_{\mu\nu}) \), where \( \Gamma^\lambda_{\mu\nu} \) is the Christoffel symbol and \( \nabla_\mu \) is the covariant derivative compatible with the metric \( g_{\mu\nu} \), we can rewrite the third term as a total derivative. This term will therefore amount to zero as we integrate over all space. Using lemma A.4 we find that the variation of the metric determinant is equal to

\[
\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}.
\]

(4.4)

Using \( \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \) the variation of the action takes the form

\[
\delta S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \delta g_{\mu\nu}.
\]

(4.5)

By requiring that the action is extremized for any small variation \( \delta g_{\mu\nu} \) we find the extremal action and thus obtain the **Einstein field equations** in vacuum,

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0.
\]

(4.6)

The solutions of eq. (4.6) are called **vacuum solutions**. Some exact vacuum solutions are the Minkowski, Schwarzschild and Kerr spacetimes. The former will be discussed in further detail in the following section.

### 4.2. The Minkowski Spacetime

When studying the Einstein field equations in vacuum (4.6) we can find one trivial, but very important solution - the flat Minkowski spacetime. This Lorentzian flat metric is the geometry of special relativity and from it one can find the causal structure of flat spacetime. In Cartesian coordinates \((t, x, y, z)\) it can be represented by a \(4 \times 4\) matrix,

\[
\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1).
\]

(4.7)

The line element then becomes

\[
d s^2 \equiv \eta_{\mu\nu} d x^\mu d x^\nu = -d t^2 + d x^2 + d y^2 + d z^2.
\]

(4.8)

\(^2\text{Note that we have chosen the convention with the time coordinate being negative. One could choose the convention where } \eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1) \text{ instead, the only change would be that equations containing one instance of the metric would gain a minus sign.}\)
4.2. The Minkowski Spacetime

Since there is a one-to-one relation between the metric and the line element, the terms will often be used interchangeably. No single coordinate system can be used to describe all physical systems in a satisfactory way, one often needs to express the Minkowski metric in different coordinates. For example, the Minkowski line element in spherical coordinates \( (t, r, \theta, \phi) \) takes the form

\[
ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2,
\]

where \( d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)d\phi^2 \) is the line element of the two-sphere \( S^2 \) in polar coordinates. We can make the coordinate transformation \( u = t - r \), where \( u \) is called the retarded time and rewrite the angular terms using the unit metric on the \( S^2, \gamma_{AB} = \frac{g_{AB}}{r^2} \), where the capital Latin indices \( A, B... \) range over 2 and 3. We obtain the following form for the Minkowski line element,

\[
ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^Adx^B.
\]

These coordinates are called the retarded Bondi coordinates and are especially suitable when we are interested in studying outgoing light at infinity. If we are interested in incoming light from infinity, we can use the advanced Bondi coordinates instead,

\[
ds^2 = -dv^2 + 2dvdv + r^2\gamma_{AB}dx^Adx^B,
\]

where \( v = t + r \) is the advanced time coordinate. It is important to note that the new coordinates \( u \) and \( v \) are lightlike (null).

4.2.1. Conformal Transformations

When we are interested in studying the large-scale structure of spacetime we need a well-defined method of analysing the behaviour at the boundary of spacetime. To this end we introduce conformal transformations, that map the coordinate values at infinity in the original metric, to some finite coordinate value for an unphysical metric.

**Definition 4.1** (Weyl scaling). Let \( M \) be a manifold with the metric \( g_{\mu\nu} \). The following scaling of the metric,

\[
g_{\mu\nu}(x) \rightarrow \hat{g}_{\mu\nu}(x) = K^2g_{\mu\nu}(x),
\]

is called a Weyl scaling, given that \( K \) is a smooth, strictly positive function. \( K \) is called the conformal factor [45].

It is said that \( \hat{g}_{\mu\nu}(x) \) arises from \( g_{\mu\nu}(x) \) via a Weyl scaling.

**Definition 4.2** (Conformal isometry). Let \( M \) be a manifold with the metric \( g_{\mu\nu} \). The diffeomorphism \( \phi : M \rightarrow M \) defines a map \( \phi^* : T_p \rightarrow T_{\phi(p)} \) for which \( (\phi^*g(x))_{\mu\nu} = K^2g_{\mu\nu}(x') \) is called a conformal isometry [35,45].
In contrast to the Weyl scaling, a conformal isometry involves a change of coordinates, \( x^\mu \rightarrow x'^\mu \), in which the resulting metric is written. We note that for \( K = 1 \) the conformal isometry reduces to an ordinary isometry of the spacetime \((M, g_{\mu\nu})\).

**Definition 4.3** (Conformal transformation). The specific combination of a Weyl scaling and a conformal isometry, such that

\[
g_{\mu\nu}(x) \xrightarrow{W} K^2 g_{\mu\nu}(x) \xrightarrow{GCT} g_{\mu\nu}(x'), \tag{4.13}
\]

is called a **conformal transformation** [45].

The conformal factor of the conformal isometry cancelled the conformal factor of the Weyl scaling and we have recovered the original metric, but written in the new coordinate system.

Consider the metric of the unit two-sphere \( S^2 \) in spherical coordinates,

\[
d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2. \tag{4.14}
\]

By introducing the complex **stereographic coordinates**, we can express the coordinates \( \theta \) and \( \phi \) in terms of the complex coordinates \( z \) and \( \bar{z} \) by the relation

\[
z = \cot\left(\frac{\theta}{2}\right) e^{i\phi}. \tag{4.15}
\]

The metric of the unit two-sphere then takes the form

\[
d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2 = \frac{4}{(1 + z\bar{z})^2} dzd\bar{z}. \tag{4.16}
\]

The coordinate \( z \) corresponds to a point in the complex plane. Note that for this choice of coordinates the \( S^2 \) metric \( \gamma_{AB} \) becomes anti-diagonal

\[
\gamma_{AB} dx^A dx^B = \frac{4}{(1 + z\bar{z})^2} dzd\bar{z} \Rightarrow \gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}. \tag{4.17}
\]

Further, since the metric \( \gamma_{z\bar{z}} \) is anti-diagonal we have \( \sqrt{\det(-\gamma_{AB})} = \gamma_{z\bar{z}} \) and we can use \( \gamma_{z\bar{z}} \) to lower the indices,

\[
D_z = \gamma_{zB} D^B = \gamma_{z\bar{z}} D^z + \gamma_{z\bar{z}} D^\bar{z} = \gamma_{z\bar{z}} D^\bar{z}. \tag{4.18}
\]

By performing a Weyl scaling with \( K = \frac{1 + z\bar{z}}{2} \) we have made a conformal transformation of the \( S^2 \) metric. The conformal transformation of \((\theta, \phi)\) induces a transformation of the complex plane onto the sphere, namely the **Möbius transformation**,

\[
z \rightarrow z' = \frac{az + b}{cz + d}, \quad ad - bc = 1, \tag{4.19}
\]

where \( a, b, c \) and \( d \in \mathbb{C} \). Together with the constraint \( ad - bc = 1 \), the constants form a six parameter group [46].
Theorem 4.4. The Möbius group is homomorphic to $\text{SL}(2, \mathbb{C})$.

Proof. Let $V$ be a two-dimensional complex vector space. Consider the two-vector $v \in V$, with components $v_1$ and $v_2$. Then to each transformation in eq. (4.19) there exists a corresponding linear transformation of the group $\text{SL}(2, \mathbb{C})$ that has the form

\[ v'_1 = av_1 + bv_2, \]
\[ v'_2 = cv_1 + dv_2, \] (4.20)

which can be seen by identifying $z = v_1/v_2$. Note that the transformation of eq. (4.20) can be obtained by

\[ A v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}, \] (4.21)

where the matrix $A \in \text{SL}(2, \mathbb{C})$. The constraint $ad - bc = 1$ of the parameters exactly corresponds to the requirement that $\det(A) = 1, \forall A \in \text{SL}(2, \mathbb{C})$.

By applying the Möbius transformation of eq. (4.19) on the metric of eq. (4.16) we obtain a conformally related metric,

\[ d\Omega^2 = K^2 \frac{4}{(1 + z'\bar{z})^2} dz'\bar{z}', \] (4.22)

where the conformal factor $K$ is given by

\[ K = \frac{(az + b)(\bar{a}\bar{z} + \bar{b}) + (cz + d)(\bar{c}\bar{z} + \bar{d})}{1 + z\bar{z}}. \] (4.23)

Since the conformal factor is left unchanged by setting $a \rightarrow -a, b \rightarrow -b, c \rightarrow -c,$ and $d \rightarrow -d,$ it follows that $\text{SL}(2, \mathbb{C})$ furnishes a double cover of the Möbius group. Note that $\text{SL}(2, \mathbb{C})$ also furnishes a double cover of the Lorentz group $\text{SO}(1,3)$, implying an isomorphism between the Möbius group and the Lorentz group, see theorem A.6.

Returning to the retarded Bondi coordinates in flat spacetime, eq. (4.10), we can introduce a new coordinate system $(u, R, \theta, \phi)$ where $R = r^{-1}$, the Minkowski line element becomes,

\[ ds^2 = R^{-2} \left( -R^2 du^2 - 2dudR + d\theta^2 + \sin^2 \theta d\phi^2 \right). \] (4.24)

Note that points at infinity are now at a finite coordinate value, namely $R = 0$. Due to the factor of $R^{-2}$ the physical metric blows up near $R = 0$ and has a coordinate singularity at infinity. This singularity is not a physical one, it is due to the choice of coordinates. We can see that after Weyl scaling with $K^2 = R^2$, the conformally related line element,

\[ d\hat{s}^2 = K^2 ds^2 = -R^2 du^2 - 2dudR + d\theta^2 + \sin^2 \theta d\phi^2, \] (4.25)

does not have any coordinate singularity at $R = 0$ and allows a smooth extension to infinity.
4.2. The Minkowski Spacetime

\[ r = 0 \]

\[ i^+ \]

\[ i^- \]

\[ u = \text{const.} \]

\[ 0 \]

\[ i_0 \]

\[ I^+ \]

\[ I^- \]

\[ I^0 \]

\[ u = \text{const.} \]

\[ r = \text{const.} \]

The infinite range of \( u \) and \( v \) have now been mapped to the range of \(-\frac{\pi}{2} \leq v', u' \leq \frac{\pi}{2}\). This allows us to study the large scale structure of Minkowski spacetime in a Penrose diagram, see fig. 3.

We are now in a position to study the different kinds of radial infinities present in flat spacetime. Since the Penrose diagrams are a conformally mapped from the regular spacetime diagrams they preserve angles. All null world lines will have a 45° angle in relation to the \( r = 0 \) line. We can conclude that all ingoing radial light rays begin in \( I^- \) and all outgoing radial light rays end at \( I^+ \), which we refer to as past and future null infinity, respectively. All massive particles move along timelike trajectories and begin at \( r = 0 \) and \( t = -\infty \), thus they begin at \( i^- \) and end in \( i^+ \), which we refer to as past and future timelike infinity. Similarly, spacelike world lines end at spacelike infinity, \( i_0 \). At the endpoints of \( I^+ \) and \( I^- \) there will be boundary region to \( i^- \), \( i_0 \) and \( i^+ \). We will denote the boundary of \( i^+ \) and \( I^+ \) as \( I^+_+ \), while the boundary of \( i_0 \) and \( I^+ \) will be denoted as \( I^+_-. \) Similarly, \( I^-_- \) is the boundary of \( i_0 \) and \( I^- \), while \( I^-_+ \) is the boundary of \( i^- \) and \( I^- \). Since each point in fig. 3 is a two-sphere, these boundary regions will be two-spheres.
### 4.2.3. The Poincaré Group

Say that we are interested in finding the symmetry group of Minkowski spacetime, that is the group of transformations under which the Minkowski metric is invariant. These symmetries will be the isometries of our spacetime and we can find them using Killing’s equation (2.50),

\[ \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0, \]  

(4.27)

where the covariant derivative has been reduced to a partial derivative, since the Christoffel symbol of flat spacetime vanishes in Cartesian coordinates. By taking a further derivative and permuting the indices, one can show that

\[ 2 \partial_\alpha \partial_\mu \xi_\nu = 0. \]  

(4.28)

The fact that the second derivatives of the Killing vectors vanishes implies that they are linear in the coordinates,

\[ \xi_\nu(x) = a_\nu + b_{\nu\lambda} x^\lambda, \]  

(4.29)

where \( a_\nu \) is an arbitrary covariant vector and \( b_{\nu\lambda} \) is an antisymmetric tensor, since substituting eq. (4.29) into Killing’s equation gives \( b_{\nu\lambda} + b_{\lambda\nu} = 0 \). Note that \( a_\nu \) and \( b_{\nu\lambda} \) are constant tensors and not tensor fields. Thus we have a total of 10 unknown independent choices of the components of \( a_\nu \) and \( b_{\nu\lambda} \), yielding 10 independent Killing vector fields.

These 10 Killing vector fields correspond to the transformations of the four translations, three spatial rotations and three Lorentz boosts. The translations form the four parameter Abelian translation group \( \mathbb{R}^{1,3} \), while the rotations and boosts form the six parameter non-Abelian Lorentz group \( \text{SO}(1,3) \). The semidirect product of the translations and the Lorentz group,

\[ \mathbb{R}^{1,3} \rtimes \text{SO}(1,3), \]  

(4.30)

forms the Poincaré group, which is the symmetry group of the Minkowski spacetime. From Noether’s theorem A.1 one can derive seven physical conserved charges associated with the symmetries - energy, momentum and angular momentum.

### 4.3. QFT in Curved Spacetime

Suppose that we add a matter term \( \mathcal{L}_M \) to the Einstein-Hilbert action, eq. (4.1),

\[ S = S_{EH} + S_M = \int d^4x \left( \mathcal{L}_G + \mathcal{L}_M \right), \]  

(4.31)
where the subscript $G$ denotes the gravitational Lagrangian density from eq. (4.1) and $M$ denotes the matter Lagrangian density. For the sake of simplicity we will only consider scalar fields when working in curved spacetimes.

Varying the action with respect to each matter field separately will yield their dynamics on a curved spacetime background. By background, it is meant that we set the metric to a specific solution of the Einstein vacuum equations and that it will not change due to the matter distribution. Since $\mathcal{L}_G$ is independent of the matter fields, the equations of motion for a matter field in curved spacetime will be similar to the field equations in flat spacetime, with the modification that all instances of the Minkowski metric $\eta_{\mu\nu}$ is replaced by the curved background metric $g_{\mu\nu}$ and the partial derivatives will be replaced by the corresponding covariant derivatives. For example, the equations of motion of a scalar field $\phi(x)$ in curved spacetime are

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi - m^2 \phi = 0. \quad (4.32)$$

The equations of motion for the (source-free) Maxwell field in a curved spacetime are

$$\nabla_\mu F^{\mu\nu} = 0, \quad (4.33a)$$
$$\nabla_{[\mu} F_{\nu\lambda]} = 0, \quad (4.33b)$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor and the square brackets denote antisymmetrization of all indices. The reader should note that even though the covariant derivative of a scalar field is the partial derivative, $\nabla_\nu \phi = \partial_\nu \phi$, the second covariant derivative of a scalar field becomes $\nabla_\mu \nabla_\nu \phi = \nabla_\mu (\partial_\nu \phi) = \partial_\mu \partial_\nu \phi - \Gamma^\alpha_{\mu\nu} \partial_\alpha \phi$.

We are now free to expand our matter fields in harmonic modes and impose commutation relations between the creation and annihilation operators. The resulting theory will be a quantum field theory in curved spacetime, where the metric field $g_{\mu\nu}$ is fixed to be a specific background. Several complications arise with this procedure, we will briefly mention some of them. Firstly, in order to make a decomposition in harmonic modes of the scalar field, one needs to be able to have a clear distinction between positive and negative frequency solutions to the Klein-Gordon equation (4.32). This requires that there exists a timelike Killing vector field that provides us with a preferred time coordinate with respect to which we can make the Fourier transformation. This results in difficulties of interpreting the quantized fields in terms of particles for non-stationary spacetimes. In the standard approach to field theory the particles are assumed to be well-defined and non-interacting asymptotically, whilst the notion of particles is not well defined during the interaction that happens within a microscopic region of spacetime. Consequently, we do not have a well-defined notion of "in" and "out" Fock states in spacetimes that are not asymptotically flat, or spacetimes in macroscopic regions that are curved, since the interaction would take place over a large spacetime region [35]. The Faddeev-Kulish states introduced in 7. “Faddeev-Kulish States in QED” do in fact take interactions over macroscopic distances into account, thus it is interesting to
ask whether one can construct a quantum field theory on a macroscopically curved spacetime using the Faddeev-Kulish states as the starting point.

In the path integral formulation of QFT in Minkowski spacetime, one often performs a Wick rotation by setting $t = -i\tau$, so that the metric becomes Euclidean. This is an analytic continuation of the regular $\mathbb{R}^{1,3}$ Minkowski spacetime to a complexified $\mathbb{C}^4$ Minkowski spacetime, which is divided into the original Lorentzian sector and the Euclidean sector in which calculations are done after Wick rotation. No similar decomposition in Lorentzian and Euclidean sectors can be done for generic curved spacetimes, thus no clear analogue of Wick rotations exists. The approach is only applicable for static (time independent) spacetimes [35].

4.4. The Einstein Field Equations with Matter Sources

Returning to the Einstein-Hilbert action with matter sources eq. (4.31), we must also vary it with respect to the metric field $g_{\mu\nu}$ in order to obtain the full dynamics of the system including the gravitational field itself. The gravitational Lagrangian density $\delta \mathcal{L}_G$ is unchanged with regards to the vacuum scenario in eq. (4.1) and thus will provide the terms of eq. (4.5). The matter Lagrangian density $\mathcal{L}_M$ will in fact contain terms with the metric field, thus the variation has the form

$$\delta S_M = \frac{1}{2} \int d^4x \sqrt{-g} \, T^{\mu\nu} \delta g_{\mu\nu},$$

where the stress-energy tensor $T^{\mu\nu}$ is defined as the variation of the matter action with respect to the metric field,

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}}. \quad (4.35)$$

Thus the Einstein field equations with matter sources are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}. \quad (4.36)$$

In total there are six independent field equations, 10 from the fact that $g_{\mu\nu}$ is a symmetric tensor, but four degrees of freedom are removed by the virtue of the Bianchi identity. The matter equations eq. (4.32) and eq. (4.33) coupled with the field equations (4.36) for the corresponding stress-energy tensor are called the Einstein-Klein-Gordon equations and Einstein-Maxwell equations, respectively. These coupled equations will describe the full dynamics of the system in question.

It should be stated that the stress-energy tensor of eq. (4.35) acts as the source of spacetime curvature. However, this definition is not equivalent to the canonical stress-energy tensor in quantum field theory $\tilde{T}^{\mu\nu}$, defined from Noether’s theorem A.1,

$$\tilde{T}^{\mu\nu} \equiv g^{\mu\nu} \mathcal{L}_M - \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi_a)} \partial^{\nu} \phi_a. \quad (4.37)$$
where the index $a$ denotes the field components. The definitions coincide for the Klein-Gordon field, but differ for higher spins. For example, the canonical stress-energy tensor for the Maxwell field is neither symmetric nor invariant under $U(1)$ gauge transformations, nor does it carry information about the intrinsic angular momentum of field. Thus, it cannot be a consistent source for the spacetime curvature in the Einstein field equations (4.36), which is necessarily symmetric and cannot be $U(1)$ gauge dependent. It is possible to derive a symmetrized stress-energy tensor from the canonical stress-energy tensor,

$$\Theta^{\mu\nu} \equiv \tilde{T}^{\mu\nu} + \frac{1}{2} \partial_\lambda [S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}],$$

where $S^{\mu\nu\lambda}$ is the spin current. The tensor $\Theta^{\mu\nu}$ is called the Belinfante-Rosenfeld tensor, it is conserved, symmetric and equivalent to the Einstein-Hilbert stress-energy tensor in the field equations [44, 47, 48].
5. Large Gauge Symmetries

Summary

- Large gauge transformations (LGT) are defined by imposing relaxed fall-off conditions on the Maxwell field.
- These transformations are U(1) gauge transformation for a specific set of transformation parameters and constitute the asymptotic symmetry group of QED.
- The expression for the generator of LGT is derived.
- We find that conservation of LGT charge must be satisfied asymptotically in all QED processes.

In electrodynamics we always have the freedom of choosing a U(1) gauge. After fixing the gauge, say for example the Lorenz gauge $\partial^{\mu}A_{\mu} = 0$, we still have the freedom of a residual gauge transformation,

$$A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\theta,$$

where $\partial_{\mu}\partial^{\mu}\theta = 0$,

$$\text{(5.1)}$$

for some function $\theta(x)$. For the special case of $\theta = \text{const}.$ we call it a global phase transformation, otherwise for any function $\theta(x) \neq \text{const.}$ we call it a local phase transformation, since it depends on the coordinate $x$. The usual boundary condition is that we let our fields vanish at infinity, however this is problematic when we want to analyse the asymptotics of photons, since we know that light reaches future null infinity $\mathcal{I}^+$. Thus, we will relax this condition and instead impose the following, looser boundary conditions at future null infinity$^1$,
5.1. The Generator of LGT

\[
\lim_{r \to \infty} A_r = O(r^{-2}) \quad \lim_{r \to \infty} A_u = O(r^{-1}) \quad \lim_{r \to \infty} A_{z/\bar{z}} = O(1),
\]

where we have used the retarded Bondi coordinates of eq. (4.10). Due to this looser restriction the resulting gauge transformation will be called a large gauge transformation (LGT) \([6, 7, 11, 49]\), it is a gauge transformation that is non-vanishing at infinity. The boundary conditions eq. (5.2) will lead to the following restriction on \(\theta(x)\),

\[
\lim_{r \to \infty} \theta(u, r, z, \bar{z}) = \varepsilon(z, \bar{z}) + O(r^{-1}),
\]

where \(\varepsilon(z, \bar{z})\) is an arbitrary function on the two-sphere. Defining the asymptotic value of the photon field, \(A_{z/\bar{z}}^{(\infty)}\),

\[
\lim_{r \to \infty} A_{z/\bar{z}} = A_{z/\bar{z}}^{(\infty)} + O(r^{-1}),
\]

we find that it transforms according to

\[
A_{z/\bar{z}}^{(\infty)} \rightarrow A_{z/\bar{z}}^{(\infty)\prime} = A_{z/\bar{z}}^{(\infty)} + \partial_{z/\bar{z}} \varepsilon(z, \bar{z}).
\]

We must note that this gauge transformation is not a gauge transformation in an exact sense - it is only defined at null infinity in contrast to the ”small” gauge transformations which are defined over all spacetime.

5.1. The Generator of LGT

From Noether’s theorem A.1 we know that any continuous transformation that keeps the Lagrangian invariant is a symmetry of the theory and therefore has a corresponding conserved current. The associated conserved charge will be the generator of the transformation. One can show that the commutator between a field \(\phi_a(x)\) and a conserved charge, \(Q_\varepsilon\) parametrized by a \(\varepsilon(x)\), is proportional to the variation of the field,

\[
\left[\phi_a(x), Q_\varepsilon\right] = -i\delta_\varepsilon \phi_a(x).
\]

In our case this relation reduces to

\[
\left[A_{z/\bar{z}}^{(\infty)}, Q_\varepsilon^+\right] = -i\partial_{z/\bar{z}} \varepsilon(z, \bar{z}),
\]

where the + denotes that we are at future null infinity. The Noether current for a local U(1) transformation can be calculated using eq. (A.2),

\[
J_\theta^\mu = -F^{\mu\nu} \partial_\nu \theta + J_M^\mu \theta = -F^{\mu\nu} \partial_\nu \theta + (\partial_\nu F^{\nu\mu}) \theta = \partial_\nu (F^{\nu\mu} \theta),
\]

where \(J_M^\mu(x)\) is the matter current, \(F^{\mu\nu}\) is the field strength tensor and we have used Maxwell’s equations, \(J_M^\mu = \partial_\nu F^{\nu\mu}\). The conserved LGT charge is measured at null
5.1. The Generator of LGT

infinity and is found by a spatial integration of usual Noether current. However we need to do a general coordinate transformation from the Cartesian coordinates in eq. (A.2) to the retarded Bondi coordinates,

\[ J_0^0(x) = \partial x^0 \frac{\partial}{\partial \tilde{x}^\mu} \tilde{J}_\theta^\mu(\tilde{x}) = \tilde{J}_0^0(\tilde{x}) + \tilde{J}_1^0(\tilde{x}), \]  

(5.9)

where the terms with tildes are in Bondi coordinates. By lowering the index on the current we find that

\[ \tilde{J}_\theta^0(\tilde{x}) = \tilde{g}_{0\mu} \tilde{J}_\theta^\mu(\tilde{x}) = -\tilde{J}_0^0(\tilde{x}) - \tilde{J}_1^0(\tilde{x}). \]  

(5.10)

This means that \( J_0^0(x) = -\tilde{J}_0^0(\tilde{x}) \). Since we will henceforth work solely in the Bondi coordinates we can drop the tildes and suppress the argument. The conserved LGT charge is

\[ Q_\varepsilon = -\lim_{r \to \infty} \int d^3x \sqrt{-g} J_0^0 = -\lim_{r \to \infty} \int d^3x \sqrt{-g} \nabla^\nu (F_{\nu 0}^0) \]

\[ = -\lim_{r \to \infty} \int d^3x \sqrt{-g} \nabla^i (F_{i 0}^0) = -\lim_{r \to \infty} \int d^3x \partial^i (\sqrt{-g} F_{i 0}^0) \]

\[ = -\lim_{r \to \infty} \int d^2z dr \partial^r (\sqrt{-g} F_{r 0}^0) - \lim_{r \to \infty} \int dr d^2z \partial^A (\sqrt{-g} F_{A 0}^0). \]  

(5.11)

The last term is an integral over the angular coordinates of a total derivative and thus will vanish due to periodicity. The \( r \) integral vanishes at the lower limit since we require the LGT to vanish at small \( r \), \( \lim_{r \to 0} (r^2 \varepsilon) = 0 \), giving us only one boundary term. Using that the square-root determinant \( \sqrt{-g} = r^2 \gamma_{zz} \), we find that

\[ Q_\varepsilon^+ = -\lim_{r' \to \infty} \int d^2z r'^2 \gamma_{zz} \theta(r', z, \bar{z}) F_{r u} = -\lim_{r' \to \infty} \int d^2z r'^2 \gamma_{zz} \varepsilon(z, \bar{z}) F_{r u}. \]  

(5.12)

Writing the \( \mu = u \) component of Maxwell’s inhomogeneous equations out explicitly we see that

\[ r^{-2} \partial_r (r^2 F_{r u}) - \partial_u F_{r u} + r^{-2} (D^z F_{z u} + D^z F_{\bar{z} u}) = J_u^M, \]  

(5.13)

where \( D^z \) denotes the covariant derivative on the two-sphere. Rewriting eq. (5.12) in terms of a \( u \)-integral and inserting the above expression we obtain

\[ Q_\varepsilon^+ = -\lim_{r' \to \infty} \int d^2z du r'^2 \gamma_{zz} \varepsilon(z, \bar{z}) \partial_u F_{r u} \]

\[ = -\lim_{r' \to \infty} \int d^2z du \gamma_{zz} \varepsilon(z, \bar{z}) \left[ \partial_r (r^2 F_{r u}) + (D^z F_{z u} + D^z F_{\bar{z} u}) - r^2 J_u^M \right]. \]  

(5.14)
5.1. The Generator of LGT

From the fall-off conditions eq. (5.2) we see that \( F_{ru} = O(r^{-2}) \), meaning that the \( \partial_t (r^2 F_{ru}) \)-term vanishes in the \( r' \to \infty \) limit. For the second term we have that

\[
F_{zu} = \partial_z A_u - \partial_u A_z = -\partial_u A_z + O(r^{-1}).
\]  

(5.15)

Remembering that \( \gamma_{zz} = \sqrt{-\gamma_{AB}} \) and using the relation

\[
\sqrt{-g} \nabla_\mu A^\mu = \partial_\mu (\sqrt{-g} A^\mu)
\]

(5.16)

we obtain

\[
\gamma_{zz} D^z F_{zu} = \partial_z (\gamma_{zz} F_{zu}) = -\partial_z \partial_u A_z.
\]

(5.17)

Likewise, we have that the matter current \( J_u^M = O(r^{-2}) \), meaning that the rest of the terms do not vanish in the limit. Defining \( j_u \) as the \( u \)-component of the leading term of the matter current at infinity,

\[
j_u(u, z, \bar{z}) \equiv \lim_{r \to \infty} r^2 J_u^M(u, r, z, \bar{z}).
\]

(5.18)

Thus our expression for the generator of large gauge transformations will have the form

\[
Q^+ = \int_{\mathcal{I}^+} d^2z \ v \ [\varepsilon(z, \bar{z}) \ \partial_v (\partial_z A_z + \partial_{\bar{z}} A_{\bar{z}})] + \int_{\mathcal{I}^+} d^2z \ v \ \gamma_{zz} \ | \ v | \ j_u.
\]

(5.19)

The first term is linear in the fields and will be referred to as the soft part of the LGT charge, due to the fact that it can be written in terms of soft photon operators \([11]\). The second term is quadratic in the fields and will be referred to as the hard part, it will encode the radial electric field of the charged particles in the final state at \( i^+ \).

An analogous derivation at past null infinity using the advanced Bondi coordinates will give us the following expression for the LGT charge at \( \mathcal{I}^- \),

\[
Q^- = \int_{\mathcal{I}^-} d^2z \ v \ [\varepsilon(z, \bar{z}) \ \partial_v (\partial_z A_z + \partial_{\bar{z}} A_{\bar{z}})] + \int_{\mathcal{I}^-} d^2z \ v \ \gamma_{zz} \ | \ v | \ j_v.
\]

(5.20)

Since the action is invariant under the LGT transformation, i.e. it is a symmetry of the theory, it follows that the LGT charge commutes with the S-matrix,

\[
\langle \text{out} | (Q^+ S - SQ^-) | \text{in} \rangle = 0,
\]

(5.21)

implying conservation of the LGT charge \([7]\).

The group structure of the large gauge transformations is the structure of U(1), with a different set of transformation parameters. The LGT have the structure of U(1) transformations, with the distinct feature of only being defined at past/future null infinity and for a different set of transformation parameters than the “small” gauge transformations. Returning to the form eq. (5.12) of the LGT charge, we can expand both the transformation parameter \( \varepsilon(z, \bar{z}) \) and \( F_{ru} \) is spherical harmonics. The \( Y^0_0 \)-term of \( \varepsilon(z, \bar{z}) \) will be the constant piece of the transformation parameter, exactly corresponding to the electrical charge given by a global U(1) transformation. Due to the orthogonality of spherical harmonics the integral will vanish for all components of \( F_{ru} \) for which \( l \neq 0 \), meaning that only the constant piece of the field strength tensor at infinity contains information of the electric charge.
In this section we will focus on the effect of operating on the outgoing QED vacuum \( \langle 0 | \) with the generator of LGT, \( Q_+ \). Later, in chapter 7. “Faddeev-Kulish States in QED”, we will show that the action of the large gauge transformation will create a new state that is orthogonal to the original vacuum and that can in fact not be reproduced by any physical scattering process.

When acting on the vacuum, only the soft part of eq. (5.19) will survive since the hard part of the operator vanishes when the current operator \( j_u \) acts on the vacuum \( \langle 0 | \). After integration by parts with respect to \( z \) and \( \bar{z} \) the soft part of \( Q_+ \) becomes

\[
\langle 0 | Q_+^\epsilon = - \langle 0 | \int_{\mathbb{R}^+} d^2z du \partial_u \left[ A_z(u, z, \bar{z}) \partial_z \epsilon(z, \bar{z}) + A_{\bar{z}}(u, z, \bar{z}) \partial_{\bar{z}} \epsilon(z, \bar{z}) \right].
\] (5.22)

Notice that the boundary term vanishes due to periodicity. Following Gabai & Sever [11] we write the \( z/\bar{z} \) components gauge field at null infinity in terms of creation and annihilation operators in the \( z/\bar{z} \) directions only,

\[
\int du \partial_u A_z(u, z, \bar{z}) = - \lim_{\omega \to 0} \frac{\omega \gamma_{zz}}{8\pi} \left[ a_+(\omega \hat{x}_z) + a_+^\dagger(\omega \hat{x}_{\bar{z}}) \right],
\] (5.23a)

\[
\int du \partial_u A_{\bar{z}}(u, z, \bar{z}) = - \lim_{\omega \to 0} \frac{\omega \gamma_{\bar{z}z}}{8\pi} \left[ a_-(-\omega \hat{x}_z) + a_-^\dagger(-\omega \hat{x}_{\bar{z}}) \right],
\] (5.23b)

where \( \hat{x}_z \) is the unit vector in the \( z \)-direction and the subscript \( \pm \) denotes the two orthogonal polarizations of the photon. The limit \( \omega \to 0 \) makes sure that the operators create and annihilate soft photons, hence the name for the soft part of the operator. After inserting this result in eq. (5.22) and remembering that the creation operator annihilates the vacuum, the following expression remains

\[
\langle 0 | Q_+^\epsilon = \lim_{\omega \to 0} \frac{\omega}{8\pi} \int d^2z \gamma_{zz} \langle 0 | \left[ a_-(-\omega \hat{x}_z) \partial_z \epsilon(z, \bar{z}) + a_+(\omega \hat{x}_z) \partial_{\bar{z}} \epsilon(z, \bar{z}) \right].
\] (5.24)

Thus we see that the action of a LGT on vacuum adds a soft photon at null infinity, moderated by the transformation parameter \( \epsilon(z, \bar{z}) \). A corresponding expression can be derived at past null infinity for the incoming ket vacuum \( |0 \rangle \). This expression will be important when we analyse the Faddeev-Kulish states in 7.5. “Eigenstates of LGT”.
6. The Bondi-Metzner-Sachs Group

Summary

- The notions of asymptotic flatness and asymptotically flat spacetimes are defined.
- The Bondi metric is defined and expanded as a series in $\frac{1}{r}$.
- An explicit expression for the Bondi mass is calculated and it is shown that one can only have mass loss if there is news.
- The BMS group is presented as the set of transformations that are isometries of an asymptotically flat spacetime. The group structure is investigated in detail.
- BMS supertranslations are defined and an expression for the generator of supertranslations is calculated.

6.1. Asymptotically Flat Spacetimes

When we are interested in studying the gravitational field of an isolated system, say a star, it is useful to have a notion of asymptotic flatness. As we move away from the body in question, its gravitational field should decrease and thus we expect it to vanish asymptotically. Henceforth, when we refer to a spacetime as asymptotically flat, we mean that it goes towards the Minkowski geometry as we go towards infinity in some radial coordinate. This working definition will suffice within the scope of this thesis, for a more rigorous definition see Hawking & Ellis [50] or Stewart [51].

Recall the flat line element in retarded Bondi coordinates, introduced in eq. (4.10),

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B.$$  \hspace{1cm} (6.1)

We can see that the metric satisfies the so called Bondi gauge conditions,
6.1. Asymptotically Flat Spacetimes

\[ g_{rr} = g_{rA} = 0, \quad \partial_r \left( \frac{\det(g_{AB})}{r^2} \right) = 0. \] \hspace{1cm} (6.2)

We will now consider the finer details of the coordinates \((u, r, \theta, \phi)\) and propose the form of the most general metric that can be regarded as asymptotically flat. The crucial properties imposed on the coordinates are:

(i) the hypersurfaces \(u = \text{const.}\) are everywhere tangent to the local light cone,
(ii) \(r\) is the luminosity distance,
(iii) the scalars \(\theta\) and \(\phi\) are constant along each ray.

**Proposition** (Bondi metric). The most general four dimensional metric that can be written in coordinates which satisfy properties (i) – (iii) above, must have a line element of the form

\[ ds^2 = \frac{Vr}{r} e^{2\beta} du^2 - 2e^{2\beta} dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du), \] \hspace{1cm} (6.3)

where \(A, B = 2, 3\), \(h_{AB}\) is a metric for the radial coordinates and

\[ \det(h_{AB}) = b(u, \theta, \phi). \] \hspace{1cm} (6.4)

Here \(V, \beta, U^A\) and \(h_{AB}[\det(h_{AB})]^{-1/2}\) are any six scalar functions of the coordinates and \(b(u, \theta, \phi)\) of its arguments [15].

We will refer to eq. (6.3) as the **Bondi metric**. The proof can be found in the original papers by Bondi, van der Burg, Metzner and Sachs [13–15]. Recall that there exists six independent Einstein field equations and note that the Bondi metric contains six undefined functions, thus the metric has sufficient degrees of freedom to be the most general asymptotically flat metric. Writing the metric out in matrix form,

\[
\begin{pmatrix}
\frac{V}{r} e^{2\beta} + r^2 \gamma_{AB} U^A U^B & -e^{2\beta} & -r^2 h_{2B} U^B & -r^2 h_{3B} U^B \\
-e^{2\beta} & 0 & 0 & 0 \\
-r^2 h_{2A} U^A & 0 & r^2 h_{22} & r^2 h_{23} \\
-r^2 h_{3A} U^A & 0 & r^2 h_{23} & r^2 h_{33}
\end{pmatrix}
\] \hspace{1cm} (6.5)

we can study it more easily. By demanding that the Bondi metric approaches the Minkowski metric for large \(r\) we impose that

\[
\lim_{r \to \infty} \beta = \lim_{r \to \infty} U^A = 0, \quad \lim_{r \to \infty} \frac{V}{r} = -1, \quad \lim_{r \to \infty} h_{AB} = \gamma_{AB}. \] \hspace{1cm} (6.6)
where $\gamma_{AB}$ is the unit metric of the two-sphere from eq. (4.10). By solving the Einstein field equations with the ansatz of eq. (6.3), the detailed asymptotic behaviour was found to be the following \cite{14,15}

\begin{align}
V &= -r + 2m_B(u, \theta, \phi) + O(r^{-1}), \\
\beta &= -c(u, \theta, \phi)c^*(u, \theta, \phi)(2r)^{-2} + O(r^{-4}), \\
h_{AB}dx^A dx^B &= (d\theta^2 + \sin^2 \theta d\phi^2) + O(r^{-1}), \\
U^A &= O(r^{-2}).
\end{align}

(6.7)

Changing to the stereographic coordinates of eqs. (4.15) and (4.16) and expanding the Bondi metric eq. (6.3) as a series in $\frac{1}{r}$, we find

\begin{equation}
ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{\bar{z}z} dz d\bar{z} + \frac{2m_B}{r} du^2 + rC_{zz} dz^2 + rC_{\bar{z}\bar{z}} d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}} dud\bar{z} + \cdots
\end{equation}

(6.8)

where $\gamma_{\bar{z}z} = \frac{2}{(1+z\bar{z})^2}$ is the anti-diagonal metric on the two-sphere. Note that even though some of the terms are constants with respects to $r$ or even contain an $r$-dependent factor, they are all suppressed with a factor of $r^{-1}$ with respect to the corresponding terms in the Minkowski metric. The function $m_B(u, z, \bar{z})$ is called the Bondi mass aspect, from which we can define the Bondi mass \cite{52} as follows,

\begin{equation}
m(u) \equiv \frac{1}{2\pi} \int d^2 z \gamma_{\bar{z}z} m_B(u, z, \bar{z}),
\end{equation}

(6.9)

where the integration is over the two-sphere. The Bondi mass is measured at future null infinity, in contrast to the ADM mass measured at spatial infinity. Further we have

\begin{equation}
U_z = -\frac{1}{2} D^z C_{zz}, \quad U_{\bar{z}} = -\frac{1}{2} D^\bar{z} C_{\bar{z}\bar{z}},
\end{equation}

(6.10)

here $D^A$ denotes the covariant derivative on the unit two-sphere. $C_{zz}$ and $C_{\bar{z}\bar{z}}$ are called the radiative data functions, from which we can define the news functions,

\begin{equation}
N_{zz} = \partial_u C_{zz}, \quad N_{\bar{z}\bar{z}} = \partial_u C_{\bar{z}\bar{z}}.
\end{equation}

(6.11)

In problems involving gravitational radiation it is interesting to study the mass loss of gravitating bodies, which is related to the energy of the emitted gravitational waves. From the $uu$-component of the field equations we can impose the constraint.
\[ \partial_u m_B = -\frac{1}{2} \partial_u \left[ D^z U_z + D^\bar{z} U_{\bar{z}} \right] - T_{uu}, \quad (6.12) \]

where the \( T_{uu} \) is the total outgoing radiational flux, defined by

\[ T_{uu} = \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r \to \infty} [r^2 T^M_{uu}]. \quad (6.13) \]

Here the first term of eq. (6.13) is the gravitational contribution, while \( T^M \) is the stress-energy contribution of the matter [18]. The matter term will vanish for large \( r \), since we do not have any matter at future null infinity. By combining eqs. (6.9), (6.12) and (6.13) we obtain

\[ \partial_u m = -\frac{1}{8\pi} \sqrt{\det g} \partial_u m_B \]

(6.14)

We begin with analysing the first of the two above integrals. Using eq. (4.18) the covariant derivative of the first term will become

\[ \gamma_{\bar{z}z} D_z U_z = \gamma_{\bar{z}z} U_{\bar{z}} - \gamma_{\bar{z}z} \nabla_{\bar{z}} \nabla_z \gamma_{\bar{z}z}. \quad (6.15) \]

However, from the definition of the Christoffel symbol, eq. (2.27), we see that they will vanish,

\[ \Gamma^A_{z\bar{z}} = \frac{\gamma^{AB}}{2} \left( \partial_z \gamma_{\bar{z}B} + \partial_{\bar{z}} \gamma_{zB} - \partial_B \gamma_{z\bar{z}} \right) \]

\[ = \frac{1}{2} \left( \gamma^A_{\bar{z}z} \partial_z \gamma_{z\bar{z}} + \gamma^A_z \partial_{\bar{z}} \gamma_{z\bar{z}} - \gamma^A \partial_z \gamma_{z\bar{z}} - \gamma^A \partial_{\bar{z}} \gamma_{z\bar{z}} \right) = 0. \]

Thus the first terms of eq. (6.14) becomes

\[ -\frac{1}{4\pi} \int d^2 z \ \partial_\bar{z} \left[ \partial_z U_z + \partial_{\bar{z}} U_{z} \right]. \quad (6.17) \]

For both terms we are integrating over a derivative and will be left with some boundary term, but since our integral is over the two-sphere this boundary term must vanish. Thus the entire integral of eq. (6.17) will be equal to zero. We are left with

\[ \partial_u m = -\frac{1}{8\pi} \int d^2 z \ \gamma_{z\bar{z}} N_{zz} N^{zz}. \quad (6.18) \]

Since \( \gamma_{z\bar{z}} = \gamma_{\bar{z}z}^{-1} \) we have that,

\[ N_{zz} N^{zz} = \gamma_{z\bar{z}} \gamma_{\bar{z}z} N^{zz} \gamma_{z\bar{z}} \gamma_{\bar{z}z} N_{zz} = N^{zz} N_{zz} \equiv |N|^2. \]

Thus, our final expression for the mass loss is

\[ \partial_u m = -\frac{1}{8\pi} \int d^2 z \ \gamma_{z\bar{z}} |N|^2, \quad (6.20) \]

and we can draw the following conclusion.
**Lemma 6.1.** There is mass loss if and only if there is news.

The recall that the news tensor is related the $u$-derivative of the radiative data functions in the metric. The physical statement of the above lemma is that if a a gravitating system loses mass, then gravitational radiation will always be emitted. Conversely, if a system emits gravitational radiation, then there will always be a mass loss. The news tensor brings the news of the mass loss in a gravitating system.

### 6.2. The BMS Group

The asymptotic expansion of the Bondi line element, eq. (6.8), must asymptotically go towards the Minkowski line element as $r \to \infty$. Thus, we conclude that the metric components must obey the following fall-off conditions,

$$
\begin{align}
\delta g_{uu} &= \mathcal{O}(r^{-1}), & \delta g_{ur} &= \mathcal{O}(r^{-2}), \\
\delta g_{uA} &= \mathcal{O}(1), & \delta g_{AB} &= \mathcal{O}(r), \\
\delta g_{rr} &= 0, & \delta g_{rA} &= 0, & \delta g_{A}^A &= 0,
\end{align}
$$

(6.21a, b, c)

where the first two rows are seen from the line element, while eq. (6.21c) can be seen from the Bondi gauge conditions eq. (6.2). An infinitesimal change of any tensor, $\delta T^{\alpha\beta\cdots}_{\mu\nu\cdots}$, along some vector field $\xi^\mu$ is given by the Lie derivative, $\mathcal{L}_\xi T^{\alpha\beta\cdots}_{\mu\nu\cdots}$, see 2.2.4. The Lie Derivative. The Lie derivative of the metric tensor can be expressed in terms of covariant derivatives,

$$
\mathcal{L}_\xi g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu,
$$

(6.22)

which is shown in the proof of theorem 2.23. Further, according to theorem 2.23, if the Lie derivative of the metric vanishes along a field $\xi$, then that field is a Killing vector field. This means that by constructing asymptotic Killing equations, using the fall-off conditions of eq. (6.21), we can find the asymptotic isometries of the Bondi metric. Similar to the case of the isometries of Minkowski spacetime, we will find that the transformations along the asymptotic Killing vector fields will constitute a group. Somewhat surprisingly, the asymptotic symmetry group of an asymptotically flat spacetime is not identical to the symmetry group of flat spacetime (the Poincaré group), but it is rather a much larger group, namely the Bondi-Metzner-Sachs (BMS) group.

**Definition 6.2** (BMS transformation). Let $u$, $r$, $\theta$ and $\phi$ be the standard coordinates for an asymptotically flat space-time, where $u$ is the retarded time and $r$ is the luminosity distance. The BMS transformation is defined by the following transformation of the coordinates,
\[ u \rightarrow u' = K(\theta, \phi)[u - \alpha(\theta, \phi)] \quad (6.23a) \]
\[ \theta \rightarrow \theta' = \theta'(\theta, \phi) \quad (6.23b) \]
\[ \phi \rightarrow \phi' = \phi'(\theta, \phi), \quad (6.23c) \]

where the transformation \((\theta, \phi) \rightarrow (\theta', \phi')\) is a conformal transformation of the two-sphere \(S^2\) onto itself, with \(K\) being the corresponding conformal factor and \(\alpha(\theta, \phi)\) being any smooth, twice differentiable real function defined on the two-sphere \(S^2\) [53].

**Theorem 6.3.** The BMS transformations form a group.

**Proof.** We know that the conformal transformations form a group, so the transformation \((\theta, \phi) \rightarrow (\theta', \phi')\) and function \(K\) have all the necessary properties. What is left to prove is that if one carries out two consecutive BMS transformations, the resulting \(\tilde{\alpha}\) will indeed be a smooth, twice differentiable function. Consider the transformation

\[ u' \rightarrow u'' = K'[u' - \alpha'] \]
\[ = K'[K(u - \alpha) - \alpha'] \]
\[ = K'K[u - (\alpha + \frac{\alpha'}{K})] \]
\[ = \tilde{K}[u - \tilde{\alpha}]. \quad (6.24) \]

Since both \(\alpha\) and \(\alpha'\) are smooth, twice differentiable functions then \(\tilde{\alpha} = \alpha + \frac{\alpha'}{K}\) will be both smooth and twice differentiable as well [15]. \qed

**Definition 6.4** (Supertranslations). The transformations

\[ \theta' = \theta, \quad \phi' = \phi, \quad u' = u - \alpha(\theta, \phi), \quad (6.25) \]

are called supertranslations, \(\mathcal{S}\) [15].

**Theorem 6.5.** The supertranslations form an Abelian normal subgroup of the BMS group.

**Proof.** The supertranslations are a normal subgroup of the BMS group if \(bsb^{-1} \in \mathcal{S}\) for all \(b \in \text{BMS}\) and \(s \in \mathcal{S}\). Since the transformation \(s\) leaves the angles invariant the total transformation \(bsb^{-1}\) will as well, \((bsb^{-1})_{\theta,\phi} = (bb^{-1})_{\theta,\phi} = 1\). In general the \(u\)-coordinate will be transformed non-trivially,
\[ u \to \tilde{u} = K^{-1} \left[ \bar{u} + K\alpha \right] = K^{-1} \left[ u' - \alpha' + K\alpha \right] = K^{-1} \left[ K(u - \alpha) - \alpha' + K\alpha \right] = u - K^{-1}\alpha'. \] (6.26)

The resulting transformation is a supertranslation, thus \( \mathcal{S} \vartriangleleft \text{BMS} \). That any two supertranslations commute follows from eq. (6.25).

**Corollary 6.6.** The factor group \( \text{BMS}/\mathcal{S} \) is isomorphic to \( \text{SO}(1,3) \).

**Proof.** The factor group will be characterized by transformations where \( \alpha(\theta, \phi) = 0 \), while \( \theta \) and \( \phi \) transform conformally. These transformation will exactly be the ones of the Möbius group, which we have shown is isomorphic to the Lorentz group, see 4.2.1. “Conformal Transformations”.

Under a supertranslation the set of null hypersurfaces \( u = \text{const} \) transform into a different set of null hypersurfaces \( u' = \text{const} \) without there being any Lorentz rotation involved. It is sometimes useful to write the function \( \alpha(\theta, \phi) \) in terms of the spherical harmonics,

\[ \alpha(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta, \phi), \] (6.27)

where \( \alpha_{lm} \in \mathbb{R} \) are constants. The special case when \( \alpha_{lm} = 0 \) for \( l \geq 2 \), the function \( \alpha(\theta, \phi) \) becomes

\[ \alpha(\theta, \phi) = \varepsilon_0 + \varepsilon_1 \sin \theta \cos \phi + \varepsilon_2 \sin \theta \sin \phi + \varepsilon_3 \cos \theta, \] (6.28)

where \( \varepsilon_0, \ldots, \varepsilon_3 \in \mathbb{R} \). Thus, we reduce the supertranslation group to a four parameter group - namely the **translations** of the Poincaré group [15]. Likewise in stereographic coordinates we can express the function \( \alpha(z, \bar{z}) \) as

\[ \alpha(z, \bar{z}) = \frac{A + B z + \bar{B} \bar{z} + C z \bar{z}}{1 + z \bar{z}}, \] (6.29)

where \( A, C \in \mathbb{R} \) and \( B \in \mathbb{C} \). Since the supertranslations constitute a normal subgroup of BMS, we can conclude that the translations forms a four-parameter normal subgroup of the BMS group [51].
In general a Poincaré transformation may be regarded as the composition of a translation and a Lorentz transformation. Similarly, a BMS transformation is a composition of a supertranslation and a Lorentz transformation and the BMS group is semidirect product of the supertranslation group and the Lorentz group,\(^1\)

\[
\text{BMS} \cong \mathcal{S} \rtimes \text{SO}(1,3).
\]

(6.30)

Thus, the BMS group contains many "copies" of the Poincaré group, one for each supertranslation that is not a translation.

### 6.3. Generator of Supertranslations

We can define the BMS group as the set of vector fields along which the fall-off conditions eq. (6.21) are preserved. The extended BMS group is generated by

\[
\lambda^u \partial_u = f \partial_u + V^A \partial_A + \frac{1}{2} (D^A V_A)(u \partial_u - r \partial_r) + \cdots,
\]

(6.31)

where \( f(z, \bar{z}) \) is an arbitrary function on the two-sphere which acts as the transformation parameter of supertranslations and \( V^A(z, \bar{z}) \) is a vector field on the two-sphere that parametrizes the superrotations [57]. In this thesis we will solely focus on the supertranslations. In particular we are interested in the so called radiative modes, that is modes of non-zero Bondi news at \( \mathcal{I}^+ \). The generator of the BMS supertranslations on these modes has the form [16,57,58]

\[
T(f) = \frac{1}{4\pi G} \int_{\mathcal{I}^+} \, d^2z \, \gamma_{zz} f(z, \bar{z}) m_B,
\]

(6.32)

where \( m_B \) is the Bondi mass aspect from eq. (6.8). Integrating the constraint equation eq. (6.12), we obtain

\[
m_B = -\frac{1}{2} \left[ \partial^z U_z + \partial^\bar{z} U_{\bar{z}} \right] - \int du \, T_{uu}.
\]

(6.33)

We can now divide the generator of supertranslations into two parts,

\[
T(f) = T_S(f) + T_H(f),
\]

(6.34)

where \( T_S(f) \) is called the soft part and \( T_H(f) \) is called the hard part,

\[
T_S(f) = \frac{1}{8\pi G} \int \, d^2z \, \left[ \partial_z U_z + \partial_{\bar{z}} U_{\bar{z}} \right] f(z, \bar{z}),
\]

(6.35a)

\[
T_H(f) = \frac{1}{4\pi G} \int du \, d^2z \, \gamma_{zz} f(z, \bar{z}) T_{uu}.
\]

(6.35b)

\(^1\)Note that the expansion in eq. (6.8) is up to the first non-trivial order. Higher order terms would yield terms containing the angular momentum aspect and would extend SO(1,3) to the group of superrotations. The extended BMS group is the semidirect product of supertranslations and superrotations [54–56].
6.3. Generator of Supertranslations

From eq. (6.10) we have that \( U_{z/\bar{z}} = -\frac{1}{2} \frac{1}{2} D_{z/\bar{z}} C_{zz/\bar{z}\bar{z}} \), so integrating the above expression by parts twice and moving the derivatives to \( f(z, \bar{z}) \) will give us

\[
T_S(f) = -\frac{1}{16\pi G} \int d^2z \gamma_{z\bar{z}} \left[ C_{zz} D_z f + C_{\bar{z}\bar{z}} D_{\bar{z}} f \right].
\] (6.36)

Since we know from eq. (6.11) that \( N_{zz/\bar{z}\bar{z}} = \partial_u C_{zz/\bar{z}\bar{z}} \) we can express the generator in terms of the news tensor by integrating over \( u \),

\[
T_S(f) = -\frac{1}{16\pi G} \int d^2z \int du \gamma_{z\bar{z}} \left[ N_{zz} D_z f + N_{\bar{z}\bar{z}} D_{\bar{z}} f \right].
\] (6.37)

Let us now define the operator

\[
N(z, \bar{z}) \equiv \gamma_{z\bar{z}} \int du \gamma_{z\bar{z}} = \gamma_{z\bar{z}} \int du \gamma_{z\bar{z}}.
\] (6.38)

The operator \( N(z, \bar{z}) \) can be written in terms of creation and annihilation operators of soft gravitons,

\[
N(z, \bar{z}) = -\frac{\kappa}{8\pi} \lim_{\omega \to 0} \left[ \omega a_+ (\omega \hat{x}_z) + \omega a_-^\dagger (\omega \hat{x}_z) \right],
\] (6.39)

where \( \kappa = \sqrt{32\pi G} \) and \( \omega \) is the energy of the created graviton [17]. The expression for the soft part of the supertranslations becomes.

\[
T_S(f) = \lim_{\omega \to 0} \frac{\omega}{4\pi \kappa} \int d^2z \left[ a_+ (\omega \hat{x}_z) + a_-^\dagger (\omega \hat{x}_z) \right] D_{\bar{z}} f + \text{h.c.}.
\] (6.40)

Now this relation is true for all functions \( f(z, \bar{z}) \) on the sphere, meaning that we can choose a specific transformation parameter that would be particularly convenient. Following Choi & Akhoury [19] we set \( f'(z', \bar{z}') = \frac{(1+z'\bar{z}')^2(\bar{z}'-\bar{z})}{(1-zz')(\bar{z}'-\bar{z})} \), so that \( D_{\bar{z}} f'(z', \bar{z}') = 2\pi \delta(z' - z) \), which gives us

\[
T_S(f) = -\frac{1}{4G} N(z, \bar{z}).
\] (6.41)

As we shall see in chapter 8. “Perturbative Quantum Gravity”, the supertranslations are important for the asymptotic states in perturbative quantum gravity.
Part II

Asymptotic States
7. Faddeev-Kulish States in QED

Summary

- By analysing the asymptotic dynamics of the QED current we find a non-vanishing interaction.
- We introduce the asymptotic interaction picture of QED. The picture changing operator will consist of two parts – the radiation operator \( W(t) \) and the phase operator \( e^{i\Phi(t)} \).
- The radiation operator creates a cloud of soft photons around each charged particle, creating dressed Faddeev-Kulish (FK) states.
- The space of the FK states \( \mathcal{H}_{FK} \) is investigated.
- We find that the IR divergence is cancelled at an \( S \)-matrix level.
- The FK states are found to be the eigenstates of the LGT.
- The reason to why the infrared divergence originally occurred was that we used Fock states with an ill-defined LGT charge.

7.1. The Asymptotic Hamiltonian

Consider the Lagrangian density of quantum electrodynamics,

\[
\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

where we have introduced the gauge covariant derivative \( \mathcal{D}_\mu \equiv \partial_\mu - ieA_\mu \) and the slash denotes contraction with the gamma matrix \( \gamma^\mu \). By writing out the covariant derivative explicitly we obtain,

\[
\mathcal{L}_{QED} = \bar{\psi}(i\partial - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e\bar{\psi}A\psi,
\]
7.1. The Asymptotic Hamiltonian

Here the first two terms describe the free dynamics of the Dirac field $\psi$ and the gauge field $A_\mu$. The last term describes the interaction between the two fields.

In order to find the different scattering processes allowed in the theory we can make an $S$-matrix expansion. The probability amplitude for the transition to some final state $\langle f |$ from some initial state $| i \rangle$ is calculated by

$$\langle f | S | i \rangle \equiv S_{fi}. \quad (7.3)$$

The initial state is defined at a time $t_i = -\infty$ long before the scattering and the final state is defined at a time $t_f = +\infty$ long after the scattering. The underlying assumption in this formalism is that the interaction dies off at infinity, and thus the asymptotics of the initial and final states are those of free particles. At first glance this assumption sounds justified, but as discussed in chapter 4. “Spacetime Geometry” the gauge field does not vanish at infinity - since it is null it goes towards $\mathcal{I}^+$. Likewise, a massive field does not vanish either, but goes towards $i^+$, see fig. 3 for reference. In fact by studying the interaction Lagrangian more closely we will find that the asymptotics of QED is not a free theory and we will assert that it is precisely this assumption that lead to the infrared divergence.

We will follow the derivation of Kulish & Faddeev [2] closely, but similar constructions were made earlier by Dollard [3], Chung [4] and Kibble [5]. We can find the asymptotic potential by analysing the asymptotics of the interaction potential,

$$V = -\int d^3x \, \mathcal{L}_I = -\int d^3x \, e\bar{\psi} A \psi = -\int d^3x \, J^\mu A_\mu, \quad (7.4)$$

where we have defined the standard QED current $J^\mu(x) \equiv e\bar{\psi}(x)\gamma^\mu \psi(x)$. We begin with expanding the fields in terms of creation and annihilation operators, appendix see B. “Canonical Commutation Relations”.

By expanding the interaction potential, eq. (7.4), in terms of the creation and annihilation operators we will obtain eight terms. Sorting them by the matter operators we will have two cases 1) terms containing only two creation operators or only two annihilation operators, or 2) terms containing a mix of creation and annihilation operators. By studying the exponentials of the different terms we can ascertain which terms vanish at $|t| \to \infty$ and which will remain finite and thus govern the asymptotics of QED.

Terms of the first kind, with the operators $c^\dagger_s(p) d^\dagger_i(p') a^j_{\mu}(k)$, will have exponentials of the form

$$\sim \int d^3 x \, \tilde{d}^3 k \, \tilde{d}^3 p \, \tilde{d}^3 p' \, \exp(-i(p + p' \pm k)x) \quad (7.5)$$

$$= \int d^3 x \, \tilde{d}^3 k \, \tilde{d}^3 p \, \tilde{d}^3 p' \, \exp\left(-i(p^0 + p'^0 \pm k^0)t\right) \exp(-i(p + p' - k)x)$$

Since we integrate over $k$ we can send $k \to -k$ in the terms with photon creation operators. Remembering that $k^0 = \omega_k = \omega_{-k}$, this results in sending the operator $a^\dagger_{\mu}(k) \to a^\dagger_{\mu}(-k)$. Integrating over $d^3x$ will then result in a delta function,
7.1. The Asymptotic Hamiltonian

\[
\begin{align*}
\mathcal{H}_{\text{as}}(t) &= -e \int d^3k \, \bar{\psi}(k) \gamma^\mu \frac{\partial \phi}{\partial \phi} \\
&= -e \int d^3k \, \bar{\psi}(k) \gamma^\mu \phi(k) e^{ikp_0 t} + a^\dagger_\mu(k) e^{-i\frac{pk}{m} t},
\end{align*}
\]
}

where \( \rho(p) \equiv d^4(p) d_\nu(p) - c_\nu(p) c_\nu(p) \) is the charged matter density operator. It is worth mentioning that this expression is entirely free of spinorial terms, meaning that soft photons couple to electrons with no regard to their spin.
7.2. The Asymptotic Interaction Picture

In the ordinary interaction (Dirac) picture of quantum field theory we have two central equations,

\[ i \frac{d}{dt} \phi_I = [\phi_I, H_0], \]  
\[ i \frac{d}{dt} |s\rangle_I = V_I |s\rangle_I. \]

The first equation is the Heisenberg-like equation, which governs the time evolution of the field operators \( \phi(x) \) and gives us the equations of motions for the corresponding field. The solutions are the free fields (non-interacting), which after a Fourier transformation give us the annihilation and creation operators. The second equation is the Schrödinger-like equation, which gives us the time evolution of the states. The solutions are

\[ |\tilde{s}(t = +\infty)\rangle = S_D(V_I) |s(t = -\infty)\rangle. \]

A specific initial state \( |s(t = -\infty)\rangle \) is evolved into some general final state \( |\tilde{s}(t = +\infty)\rangle \) by the action of the Dyson S-matrix,

\[ S_D(V_I) = T \left\{ e^{-i \int_{-\infty}^{\infty} dt \, V_I} \right\}. \]

The states \( |\tilde{s}(t = +\infty)\rangle \) and \( |s(t = -\infty)\rangle \) are assumed to be the free Fock states constructed from the free field operators. This assumption is faulty for interaction potentials \( V_I \) that are non-vanishing at infinite times, \( V_I(t = \pm \infty) = V_I^{as}(t) \neq 0 \). This is generically the case for interactions that are mediated by massless bosons - precisely the interactions where one encounters infrared divergences.

This calls for the formulation of a new picture – the asymptotic interaction picture. Operators and states in this picture will be denoted by the subscript \( A \) and are related to the interaction picture by

\[ O_A \equiv U^{-1} O_I U \]  
\[ |s\rangle_A \equiv U^{-1} |s\rangle_I \]

where \( U \) is the picture changing operator that describes the asymptotic dynamics of our theory. This operator \( U \) must satisfy the Schrödinger-like equation for the asymptotic interaction picture,

\[ i \frac{d}{dt} U(t) = U(t) V_I^{as}. \]
The time evolution equations in the asymptotic interaction picture will be

\[
\frac{id}{dt} \phi_A = [\phi_A, H_0 + V_A^{as}], \quad (7.16a)
\]

\[
\frac{id}{dt} |s\rangle_A = (V_A - V_A^{as}) |s\rangle_A. \quad (7.16b)
\]

Note that the term \( V_A - V_A^{as} \) vanishes at infinity by construction, meaning that the solution for the states will be expressed in terms of the free Fock states,

\[
|\bar{s}(t = +\infty)\rangle = S_A(V_A - V_A^{as}) |s(t = -\infty)\rangle, \quad (7.17)
\]

for

\[
S_A(V_A - V_A^{as}) = T \left\{ e^{-i \int_{-\infty}^{\infty} dt (V_A - V_A^{as})} \right\}. \quad (7.18)
\]

Even though the states can now be expressed in terms of the free Fock states, this \( S \)-matrix is no longer the ordinary Dyson \( S \)-matrix \( S_D \). It is a more complicated operator constructed from the asymptotically interacting field operators \( \phi_A \). Remember that the \( S \)-matrix itself is an operator, this gives us the relation

\[
S_A(V_A - V_A^{as}) = U^{-1} S_D(V_I - V_I^{as}) U. \quad (7.19)
\]

This allows us to express the \( S \)-matrix element of a scattering to some final state \( \langle f | \) from some initial state \( |i\rangle \) as

\[
\langle f | S_A(V_A - V_A^{as}) |i\rangle_F = \langle f | U^{-1} S_D(V_I - V_I^{as}) U |i\rangle_F = \langle f | S_D(V_I - V_I^{as}) |i\rangle_{FK}, \quad (7.20)
\]

where we have defined the Faddeev-Kulish states, \( |s\rangle_{FK} \), in terms of the asymptotically free Fock states \( |s\rangle_F \) as

\[
|s\rangle_{FK} \equiv U(t) |s\rangle_F. \quad (7.21)
\]

The solution eq. (7.15) is the time-ordered exponential of the asymptotic potential, \( U(t) = T \left\{ \exp \left( -i \int_{t'}^t dt' V_I^{as} \right) \right\} \). Taylor expanding both terms after time-ordering will give us

\[
U(t) = \exp \left( -i \int_{t'}^t dt' V_I^{as} (t') - \frac{1}{2} \int_{t'}^t dt' \int_{t'}^{t''} ds Q(t', s) \right). \quad (7.22)
\]

where \( Q(t_1, t_2) \) is the commutator of the asymptotic potential with itself.

\[
Q(t_1, t_2) \equiv [V_I^{as}(t_1), V_I^{as}(t_2)]. \quad (7.23)
\]
7.3. Faddeev-Kulish States

Since \( Q(t_1, t_2) \) commutes with \( V_{\mathcal{F}}^{\text{as}} \) we can use the Baker-Campbell-Hausdorff (BCH) formula, lemma A.3, to write the asymptotic operator \( U(t) \) on the form

\[
U(t) = e^{R(t)} e^{i\Phi(t)},
\]

(7.24)

where

\[
R(t) = -i \int dt' V_{\mathcal{F}}^{\text{as}} = e \int \tilde{d}^3k \, \tilde{d}^3p \, \frac{p^\mu}{p k} \left( a^\dagger_\mu(k)e^{-i\frac{p k}{2p_0}t} - a_\mu(k)e^{i\frac{p k}{2p_0}t} \right) \rho(p),
\]

(7.25a)

\[
\Phi(t) = \frac{i}{2} \int dt' \int t' ds \, Q(t', s)
\]

\[
= -\frac{e^2}{4\pi} \int \tilde{d}^3p \, \tilde{d}^3q \,: \rho(p) \rho(q) : \frac{pq}{\sqrt{(pq)^2 + m^2}} \text{sgn } t \ln \frac{|t|}{t_0}.
\]

(7.25b)

The infinite range of the Coulomb potential make the asymptotic fermion states dependent on all other electrons in the system. The operator \( e^{i\Phi(t)} \) does not contain any net creation or annihilation operators and will therefore only contribute with a phase. When this phase operator acts on a particle pair the charge density operators will "extract" the momenta of the particle pair and contributes with a relative phase. The resulting phase factor will be identical to the phase calculated by Weinberg, eq. (3.11), but with opposite sign and consequently they will cancel each other.

In contrast, the radiation operator \( W(t) \equiv e^{R(t)} \) does contain photon operators and is responsible for creating a coherent state of soft photons associated to each charged fermion. We will refer to this process as the dressing of fermions with soft photon clouds. It is important to remember that these two operators commute.

### 7.3. Faddeev-Kulish States

Recall that we have two equivalent ways of studying the same amplitude \( \mathcal{M} \),

\[
\mathcal{M} = \langle \bar{\Psi}_F S_A \Psi \rangle_F = \langle \bar{\Psi}_F U^{-1}(t) S_D U(t) \Psi \rangle_F,
\]

(7.26)

We can view it as the matrix element of the asymptotic \( S \)-matrix \( S_A \) between two free Fock states,
\[ \mathcal{M} = \langle \tilde{\Psi} | S_A | \Psi \rangle_F, \]  
(7.27)

where \( |\Psi\rangle \in \mathcal{F} \) are the Fock states\(^1\). Alternatively we can view it as the matrix element of the standard Dyson S-matrix between two Faddeev-Kulish states,

\[ \mathcal{M} = \langle \tilde{\Psi} | S_D | \Psi \rangle_{FK}, \]  
(7.28)

where \( |\Psi\rangle_{FK} \equiv U(t) |\Psi\rangle_F \) are the asymptotic Faddeev-Kulish states. These Faddeev-Kulish (FK) states do not live in Fock space \( \mathcal{F} \), but rather in a coherent state space \( \mathcal{H}_{FK} \), which will be treated in section 7.3.1. "The Coherent State Space \( \mathcal{H}_{FK} \)."

From now on we will mainly focus on the radiation operator \( W(t) = e^{R(t)} \). Unlike the phase operator, it does produce a qualitative change of the states. Consider an arbitrary Fock state with the spin indices suppressed,

\[ |\Psi\rangle = c_1^{\dagger}(p_1) \cdots c_n^{\dagger}(p_n) d_1^{\dagger}(q_1) \cdots d_m^{\dagger}(q_m) |0\rangle \otimes |\Phi\rangle_\gamma, \]  
(7.29)

where \( |\Psi\rangle \) contains \( n \) electron and \( m \) positrons together with some arbitrary photon state \( |\Phi\rangle_\gamma \). The charge density operator \( \rho(p') \) in \( W(t) \) will commute with any photonic state and will vanish upon acting on the vacuum. The effect of \( \rho(p') \) on fermionic states produces a qualitatively different effect due to the anticommutation relations, eq. (B.5), imposed on the fermions,

\[ \rho(p') c_1^{\dagger}(p) = c_1^{\dagger}(p) \rho(p') + c_1^{\dagger}(p') \delta^3(p - p'). \]  
(7.30)

When \( W(t) \) acts on some generic Fock state it will be reduces to the reduced radiation operator \( W_{n,m}(t) \),

\[ W(t) |\Psi\rangle = W_{n,m}(t) |\Psi\rangle, \]  
(7.31a)

\[ W_{n,m}(t) = \exp \left( e \int d^3k \ f_{n,m}^\mu(k, t) a_{\mu}^\dagger(k) - f_{n,m}^{\mu\ast}(k, t) a_\mu(k) \right), \]  
(7.31b)

\[ f_{n,m}^\mu(k, t) = \sum_{i=1}^n \frac{p_i^\mu}{p_i^2} e^{i p_i k t} - \sum_{j=1}^m \frac{q_j^\mu}{q_j^2} e^{i q_j k t}, \]  
(7.31c)

where \( i \) and \( j \) are summed over each positron and electron respectively. Conceptually, \( \rho(p') \) in the radiation operator "probes" all of momentum space for charged particles and upon finding these particles the photonic operators will create a "cloud" of soft photons around each charged particle, modulated by the functions \( f_{n,m}^\mu \).\(^2\)

---

1. When there is no possibility of confusion ordinary Fock states will be written as bras and kets without any subscript, otherwise Fock states will be written as \( |\Psi\rangle_F \).

2. Strictly speaking, due to the fact that photons travel towards \( \mathcal{J}^+ \) and massive charged particles travel towards \( i^+ \), the photonic clouds are not "around" the individual electrons, but rather associated to the electron in question.
7.3. Faddeev-Kulish States

7.3.1. The Coherent State Space $\mathcal{H}_{FK}$

The expression for the reduced radiation operator $W_{n,m}(t)$ in eq. (7.31), is an exponential that is linear in $a_\mu$ and $a_\mu^\dagger$. It has the form

$$ W_{n,m}(t) \sim \exp \left( \sum_i \alpha_i a_i^\dagger - \alpha_i^* a_i \right), \quad (7.32) $$

which is clearly reminiscent of a bosonic coherent state operator. To find the space of the Faddeev-Kulish states, let us first study the definition of Fock space $[35, 59]$.

**Definition 7.1 (Fock space).** The Hilbert space of many-particle states in quantum field theory is defined as

$$ \mathcal{F} \equiv \bigoplus_{N=0}^\infty \mathcal{H}^N, \quad (7.33) $$

where $\mathcal{H}^N \equiv \bigotimes_{i=1}^N \mathcal{H}_i$ is the N-particle Hilbert space for $N > 0$, $\mathcal{H}_i$ is a single particle Hilbert space and $\mathcal{H}^0 = \mathbb{C}$ is the space of the vacuum. We refer to $\mathcal{F}$ as **Fock space**.

It should be noted that in the case of spinors the tensor product is antisymmetric, while in the case of integer spin fields the tensor product is symmetric. Within Fock space we can construct an arbitrary particle state as a direct sum of single and many-particle states, including superpositions of states,

$$ |\Psi\rangle = \alpha_0 |0\rangle \oplus \alpha_1 |\psi_1\rangle \oplus \sum_{ij} \alpha_{2,ij} |\psi_{2i}, \psi_{2j}\rangle \oplus \cdots, \quad (7.34) $$

where $\alpha_n \in \mathbb{C}$ are the amplitudes for each $N$-particle state. In order for $\mathcal{F}$ to be a Hilbert space the inner product between states need to be convergent and positive-definite,

$$ \langle \Psi | \Psi \rangle = \sum_{n=0}^\infty |\alpha_n|^2 < \infty. \quad (7.35) $$

In momentum space we can formally construct arbitrary particle states by acting on the vacuum with creation operators, as demonstrated by eq. (7.29). A coherent state in form of $\exp \left( \sum_i \alpha_i a_i^\dagger \right) |0\rangle$ for $\alpha_i \in \mathbb{C}$ can be expanded in Fock states $\mathcal{F}$ as long as the convergence criterion is fulfilled, and lives in a coherent state space that is isomorphic to $\mathcal{F}$. However, if the convergence criterion is not met the state is not defined in Fock space. For the case of the reduced radiation operator the role of the numbers $\alpha_i$ is replaced by the functions $f_{n,m}^\mu$. The sum in the convergence criterion is replaced by the integral

$$ \int \tilde{d}^3k \; f_{n,m}^\mu f_{n,m}^{\mu*}, \quad (7.36) $$

note that there is no summation over $n$ or $m$. This integral is divergent in both the lower limit and the upper limit.$^3$ The conclusion is that the radiation operator $W(t)$ does

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$^3$Formally, the upper divergence can be dealt with by the theory of renormalization.
not define a unitary transformation in Fock space. Conceptually this is reasonable due to \( W(t) \) creating an unbounded amount of soft photons around each charged particle. Instead, the operator maps Fock space to a larger space of coherent states,

\[
e^{R(t)} : \mathcal{F} \to \mathcal{H}_{FK},
\]

where we have denoted the coherent state space with \( \mathcal{H}_{FK} \) to emphasize that it is the space of the Faddeev-Kulish states.

However, one problem arises – the radiation operator \( W(t) = e^{R(t)} \) does not satisfy the Gupta-Bleuler condition derived from the Lorenz gauge,

\[
(a_3 - a_0) |\Psi\rangle_F = 0,
\]

\[
(a_3 - a_0) |\Psi\rangle_{FK} \neq 0,
\]

since it does not commute with the photon creation and annihilation operators. This condition is crucial if we want a ghost-free formulation of the theory, where the unphysical degrees of freedom of the gauge field in terms of scalar and longitudinal photons, \( a_0 \) and \( a_3 \), are eliminated. Fortunately, the exact details of the radiation operator does not itself characterize the low-energy behaviour of the theory, but rather the asymptotic space \( \mathcal{H}_{FK} \). Kulish and Faddeev showed that the coherent state space of the dressed matter states is independent of \( t \), Lorentz and gauge invariant [2]. Thus we would like to construct an operator, \( R_f \), that maps Fock space to \( \mathcal{H}_{FK} \), but also commutes with the Gupta-Bleuler condition.

\[
e^{R_f} : \mathcal{F} \to \mathcal{H}_{FK},
\]

\[
(a_3 - a_0) |\Psi\rangle_{FK} = (a_3 - a_0) e^{R_f} |\Psi\rangle_F = e^{R_f} (a_3 - a_0) |\Psi\rangle_F = 0.
\]

The operator \( R_f \) has the form

\[
R_f \equiv \int d^3k \; d^3p \left( f^\mu(p,k) a_\mu^\dagger(k) - f^{\mu*}(p,k) a_\mu(k) \right) \rho(p).
\]

This solution is “put in by hand” in order to obtain an operator that satisfies the Gupta-Bleuler condition. The justification for this is that the new operator creates the same Hilbert space as the original operator, but does not have the problem of creating photons with unphysical degrees of freedom. The operator \( R_f \) is parametrized by the infrared functions, \( f_\mu(k,p) \), that take the following form [2,11]

\[
f_\mu(p,k) \equiv \left[ \frac{p_\mu}{p_0} + \frac{c_\mu(k)}{\omega_k} \right] \phi(p,k).
\]

Where \( \phi(p,k) \) no longer needs to be \( \exp\left(i \frac{p\cdot k}{p_0 t} \right) \), but rather an arbitrary function fulfilling the condition \( \lim_{k \to 0} \phi = 1 \). The similarity to \( f_{n,m}^\mu(k,t) \) in eq. (7.31c) should be
7.4. Cancellation of the Divergence

noted, but the functions should not be confused with each other. The four-vector $c_\mu(k)$ is a $k$-dependent null vector, meaning that $c_\mu c^\mu = 0$, which satisfies $c \cdot k = -\omega_k$. This implies that $f^\mu(p,k) k_\mu = 0$.

Similar to $e^{R(t)}$ we can interpret $e^{R_f}$ as an operator that dresses each charged particle with a cloud soft photons. From the anticommutation relations of spinors, eq. (B.5), we can show that $\{c(p), \rho(p')\} = (2\pi)^3(2\omega_p)\delta^3(p - p')c(p)$. Thus, we can obtain the following relation

$$e^{R_f} c^\dagger(p_1) c^\dagger(p_2) |0\rangle = e^{R_{f(p_1)}} c^\dagger(p_1) e^{R_{f(p_2)}} c^\dagger(p_2) |0\rangle,$$

which makes the dressing of each individual particle apparent. Here we defined

$$R_f(p) \equiv e \int d^3k \left( f^\mu(p,k) a^\dagger_\mu(k) - f^{\mu*}(p,k) a_\mu(k) \right),$$

which we will refer to as the dressing operator. In principle, we can choose to have different $c_\mu$-vectors for each dressed particle. In order to avoid introducing new divergences the $c_\mu$-vectors must we satisfy the following convergence constraint

$$\sum_{j \in \text{out}} e_j c_{j,\mu}(k) = \sum_{i \in \text{in}} e_i c_{i,\mu}(k),$$

where $e_i$ is the $i$th particle’s electric charge [11]. We will mainly consider the case when all $c_\mu$-vectors are equal, for which eq. (7.44) is satisfied due to charge conservation.

7.4. Cancellation of the Divergence

This section is devoted to showing how the infrared divergence cancels on an $S$-matrix level in the presence of FK states. As a proof of concept we will principally investigate the cancellation in processes with one-to-one scatterings. It can be generalized for processes with an arbitrary amount of incoming and outgoing particles, see [2, 4, 11].

In addition to the diagrams of fig. 2 we will obtain graphs that related to the interaction of the soft photon cloud. We can distinguish the four new kinds of interaction depicted in fig. 5:

(a) Emission of real soft photons from the clouds.

(b) An internal photon line connecting one of the clouds to one of the external legs. We refer to these diagrams as containing interacting photons, since they depict an interaction between the cloud and the external particles.

(c) Cloud-to-cloud photons that connect an incoming cloud with an outgoing cloud.

(d) Cloud-to-cloud photons that connect an incoming (outgoing) cloud with an incoming (outgoing) cloud. Note that photons can be emitted and reabsorbed by the same cloud.
Let us introduce the following shorthand notation:

\[ S_\mu(p, k) \equiv e f_\mu(p, k), \]  
\[ P_\mu(p, k) \equiv e \frac{p_\mu}{p \cdot k}, \]  
\[ C_\mu(k) \equiv e \frac{c_\mu(k)}{\omega_k}, \]

and \( S_\mu^i \equiv S_\mu(p_i, k) \). Without any loss of generality we can set \( \phi(p, k) = 1 \), so that

\[ S_\mu(p, k) = P_\mu(p, k) + C_\mu(k). \]  

In this shorthand the dressing operator of eq. (7.43) is expressed as

\[ R_f(p) \equiv \int \frac{d^3k}{(2\pi)^3} \left[ S_\mu a_\mu^i(k) - S_\mu(p, k)a_\mu(k) \right]. \]
Note that the above operator commutes with the electron operators. We can now define dressed creation and annihilation operators, that are the charged matter operators together with their individual dressing,

\[
e^{R_f(p)} c^\dagger(p) = \exp \left( \int \tilde{d}^3k \left[ S^\mu(p,k) a^\dagger_\mu(k) - S^\mu(p,k) a_\mu(k) \right] \right) c^\dagger(p), \quad (7.48a)
\]

\[
e^{-R_f(p)} c(p) = \exp \left( - \int \tilde{d}^3k \left[ S^\mu(p,k) a^\dagger_\mu(k) - S^\mu(p,k) a_\mu(k) \right] \right) c(p). \quad (7.48b)
\]

An incoming dressed electron state with momentum \( p_i \) can be expressed as

\[
|i\rangle = e^{R_f(p_i)} c^\dagger(p_i) |0\rangle = \exp \left( \int \tilde{d}^3k \left[ S^\mu_i a^\dagger_\mu(k) - S^\mu_i a_\mu(k) \right] \right) c^\dagger(p_i). \quad (7.49)
\]

Using the canonical commutation relation of the photon field, eq. (B.9), we find that

\[
\left[ \left( \int \tilde{d}^3k \ S^\mu_i a^\dagger_\mu(k) \right), \left( - \int \tilde{d}^3k \ S^\nu_i a_\nu(k) \right) \right] = \int \tilde{d}^3k \ S^\mu_i \eta_{\mu\nu} S^\nu_i. \quad (7.50)
\]

Thus we can use the BCH formula of lemma A.3 to obtain

\[
\exp \left( \int \tilde{d}^3k \left[ S^\mu_i a^\dagger_\mu(k) - S^\mu_i a_\mu(k) \right] \right)
= \exp \left( \int \tilde{d}^3k \ S^\mu_i a^\dagger_\mu(k) \right) \exp \left( - \int \tilde{d}^3k \ S^\nu_i a_\nu(k) \right) \exp \left( - \frac{1}{2} \int \tilde{d}^3k \ S^\mu_i \eta_{\mu\nu} S^\nu_i \right). \quad (7.51)
\]

Remembering that \( a_\mu(k) |0\rangle = 0 \), we find the expression for the initial state to be

\[
|i\rangle = \exp \left( \int \tilde{d}^3k \ S^\mu_i a^\dagger_\mu(k) \right) \exp \left( - \frac{1}{2} \int \tilde{d}^3k \ S^\nu_i \eta_{\mu\nu} S^\nu_i \right) c^\dagger(p_i) |0\rangle. \quad (7.52)
\]

To the lowest order in the photon creation operators this is

\[
|i\rangle = \left( 1 - \frac{1}{2} \int \tilde{d}^3k \ S^\nu_i \eta_{\mu\nu} S^\nu_i + \int \tilde{d}^3k \ S^\mu_i a^\dagger_\mu(k) \right) c^\dagger(p_i) |0\rangle. \quad (7.53)
\]

Similarly, the final state may be written as

\[
\langle f | = \langle 0 | c(p_f) e^{-R_f(p_f)}
= \langle 0 | c(p_f) \left( 1 - \frac{1}{2} \int \tilde{d}^3k \ S^\nu_f \eta_{\mu\nu} S^\nu_f + \int \tilde{d}^3k \ S^\mu_f a^\dagger_\mu(k) \right). \quad (7.54)
\]

Now we are ready to phrase the precise statement that we will show:

\footnote{There is no summation over \( i \), since it is merely shorthand for denoting the momentum \( p_i \).}
7.4. Cancellation of the Divergence

Proposition. The amplitude $\mathcal{M} = \langle f | S_D | i \rangle$ will be free of any infrared divergences.

Let us begin with the case (a), real soft photon emission. Consider the scattering amplitude of $i$ incoming hard particles into $j$ outgoing hard particles and a soft photon,

$$\mathcal{M}_{\text{soft}} \equiv \langle k, r | \Psi_{\text{out}}^{\mu_{i}} e^{-R_f} S e^{R_f} \Psi_{\text{in}}^{\mu_{i}} | 0 \rangle. \quad (7.55)$$

where $\langle k, r \rangle = \langle 0 | a_{\tau}(k) \rangle = \langle 0 | \epsilon_{\tau}^{\mu_{i}}(k) a_{\mu}(k) \rangle$ and $\Psi_{\text{out/in}}^{\mu_{i}}$ are the outgoing/incoming undressed Fock states. An outgoing soft photon can be connected to the remainder of the diagram in three ways:

(a) Attached to an external fermion leg, this is depicted in fig. 6a.

(b) Attached to a photon cloud, this is depicted in fig. 6b.

(c) Attached to an internal leg in the blob. These diagrams are always finite and will therefore not be of interest.

For each external fermion leg there will be an associated cloud. The contribution of the emitted soft photons will be given by

$$\langle k, r | e^{-R_f(p)} \rangle = \langle 0 | \epsilon_{\tau}^{\mu_{i}} a_{\mu}(k) e^{-R_f(p)} \rangle \quad (7.56)$$

$$= \langle 0 | \epsilon_{\tau}^{\mu_{i}} \left( e^{-R_f(p)} a_{\mu}(k) + [a_{\mu}(k), e^{-R_f(p)}] \right) \rangle.$$

The first of these terms will give the ordinary contribution from the soft theorem for the emission of a soft photon from an external leg,

$$\eta e^{\frac{p}{p} \cdot \epsilon_{\tau}} \frac{p}{pk}, \quad (7.57)$$

Figure 6: The diagrams involving the emission of a real soft photon.
7.4. Cancellation of the Divergence

where \( \eta = +1(−1) \) for outgoing (incoming) particles \([43]\). The second term with the commutator, \([a_\mu(k), e^{-R_f(p)}]}\), be given by

\[
\left[ a_\mu(k), e^{-R_f(p)} \right] = -e^{-R_f(p)} \left[ a_\mu(k), R_f(p) \right] = -e^{-R_f(p)} \int \tilde{d}^3k' \left( S'^\nu(p, k') \left[ a_\mu(k), a_\nu^\dagger(k') \right] - S'^\nu(p, k') \left[ a_\mu(k), a_\nu(k') \right] \right)
\]

(7.58)

\[
= -e^{-R_f(p)} S'^\nu(p, k).
\]

This means that the contribution of the soft photon emission from one of the clouds to the amplitude will be

\[
- \eta S_\mu(p, k) \epsilon_\mu^r(p, k) = -\eta e \left[ \frac{p \cdot \epsilon}{pk} + \frac{c \cdot \epsilon}{\omega_k} \right] \phi(p, k).
\]

(7.59)

For a scattering process involving \( i \) incoming particles and \( j \) outgoing particles the emitted soft photon can be attached to \( i + j \) legs and clouds. The total amplitude for the process will be

\[
\mathcal{M}_{\text{soft}} = \sum_{j \in \text{out}} e_j \left( \frac{p'_j \cdot \epsilon_r}{p'_j k} \right) - \sum_{i \in \text{in}} e_i \frac{p_i \cdot \epsilon_r}{p_i k} - \sum_{j \in \text{out}} e_j \left( \frac{p'_j \cdot \epsilon}{p'_j k} + \frac{c_j \cdot \epsilon}{\omega_k} \right) \phi(p'_j, k) + \sum_{i \in \text{in}} e_i \left( \frac{p_i \cdot \epsilon}{p_i k} + \frac{c_i \cdot \epsilon}{\omega_k} \right) \phi(p_i, k) \cdot \mathcal{M}_0
\]

(7.60)

\[
= \left[ \sum_{j \in \text{out}} e_j \frac{p'_j \cdot \epsilon_r}{p'_j k} \left( 1 - \phi(p'_j, k) \right) - \sum_{i \in \text{in}} e_i \frac{p_i \cdot \epsilon_r}{p_i k} \left( 1 - \phi(p_i, k) \right) \right] \cdot \mathcal{M}_0
\]

where \( \mathcal{M}_0 \equiv \langle 0 | \Psi^\text{out}_F e^{-R_f} S e^{R_f} \Psi^\text{in}_F | 0 \rangle \) is the amplitude for the scattering process without soft photon emission. In the soft limit the function \( \lim_{\omega_k \to 0} \phi(p, k) \to 1 \). This means that the first two terms vanish and the amplitude reduces to

\[
\mathcal{M}_{\text{soft}} = -\frac{1}{\omega_k} \left( \sum_{j \in \text{out}} e_j c_j - \sum_{i \in \text{in}} e_i c_i \right) \cdot \epsilon_r \mathcal{M}_0,
\]

(7.61)

which will vanish by the virtue of the convergence constraint on the \( c_\mu \)-vectors, eq. (7.44), meaning that \( \mathcal{M}_{\text{soft}} = 0 \). Consequently, we see that the soft photon emission from the external legs (bremsstrahlung) will exactly cancel against the soft photon emission from...
the Faddeev-Kulish clouds, eliminating the first kind of IR divergence. Qualitatively one can think of all the soft photons that would be emitted by a hard particle as already existing in the cloud. Note that soft photon emissions from internal legs in the blob can still occur, since these never where divergent to begin with.

Moving on to the case of interacting and cloud-to-cloud photons. For the sake of brevity we will only show the cancellation to one loop between the virtual photons of diagrams figs. 2c and 2d and the interacting photons of fig. 5b together with the cloud-to-cloud photons of figs. 5c and 5d. In [43] Weinberg showed that the virtual photon contribution amounts to

\[
\mathcal{M}_{\text{virt}} = \exp \left( -\frac{1}{2} \sum_{n,m} \int d^3k \frac{e_n e_m \eta_n \eta_m (p_n \cdot p_n)(p_m \cdot k)}{(p_n \cdot k)(p_m \cdot k)} \right),
\]

(7.62)

where \(n\) and \(m\) are summed over the external particles and \(\eta = +1\) for incoming particles and \(\eta = -1\) for outgoing particles. To the lowest order for a one-to-one scattering process this contribution becomes
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\[ M^{(1)}_{\text{virt}} = 1 - \frac{e^2}{2} \sum_{n,m} \int \frac{\eta_n \eta_m (p_n \cdot p_m)}{(p_n \cdot k) (p_m \cdot k)}, \]  

(7.63)

\[ = 1 - \frac{e^2}{2} \sum_{n,m} \int \frac{p_i^2}{(p_i \cdot k)^2} + \frac{p_f^2}{(p_f \cdot k)^2} - 2 \frac{p_i \cdot p_f}{(p_i \cdot k)(p_f \cdot k)}, \]

where the label \( i \) stands for initial and \( f \) for final. In our shorthand notation this is expressed as

\[ M^{(1)}_{\text{virt}} = 1 - \frac{1}{2} \int \frac{p_i^2 + p_f^2}{(p_i \cdot k)^2} - 2 \frac{p_i \cdot p_f}{(p_i \cdot k)(p_f \cdot k)}, \]  

(7.64)

where \( P_i \cdot P_f \equiv P^\mu(p_i, k) P^\nu(p_f, k) \). The four different diagrams involving interacting photons are shown in fig. 7. The contribution from the loop in fig. 7a is

\[ \int \frac{d^3k}{\sim k} S^\mu(p_i, k) \eta_{\mu\nu} i S_F(p_i - k)(ie\gamma^\nu), \]  

(7.65)

which is derived from the contraction of the soft photon cloud and the photon loop, \( i S_F \) is the fermion propagator and \( (ie\gamma^\nu) \) is the ordinary vertex factor of QED. Due to the eikonal approximation we can discard factors of \( k \) in the denominator and again using the trick of eq. (3.2) we can rewrite \( \gamma^\nu \to 2p^\nu \) to obtain

\[ M^a_{\text{int}} = \int \frac{d^3k}{\sim k} S^\mu(p_i, k) \eta_{\mu\nu} \frac{i}{-2p_i \cdot k} (2iep_i^\nu) = \int \frac{d^3k}{\sim k} S^\mu(p_i, k) \eta_{\mu\nu} e \frac{p_i^\nu}{p_i \cdot k} \]

\[ = \int \frac{d^3k}{\sim k} S^\mu(p_i, k) \eta_{\mu\nu} P^\nu(p_i, k). \]  

(7.66a)

Similarly, the remaining graphs contribute with

\[ M^b_{\text{int}} = -\int \frac{d^3k}{\sim k} S^\mu(p_i, k) \eta_{\mu\nu} P^\nu(p_f, k), \]  

(7.66b)

\[ M^c_{\text{int}} = +\int \frac{d^3k}{\sim k} S^\mu(p_f, k) \eta_{\mu\nu} P^\nu(p_f, k), \]  

(7.66c)

\[ M^d_{\text{int}} = -\int \frac{d^3k}{\sim k} S^\mu(p_f, k) \eta_{\mu\nu} P^\nu(p_i, k). \]  

(7.66d)

Summing the amplitudes and using \( S^\mu_{\text{tot}} \equiv S^\mu_f - S^\mu_i \), we find that the total contribution of the interacting photons up to first order is

\[ M^{(1)}_{\text{int}} = \int \frac{d^3k}{\sim k} \left[ S^\mu(p_f, k) - S^\mu(p_i, k) \right] \eta_{\mu\nu} \left[ P^\nu(p_f, k) - P^\nu(p_i, k) \right] \]

\[ = \int \frac{d^3k}{\sim k} S_{\text{tot}} \cdot P_{\text{tot}}. \]  

(7.67)
The cloud-to-cloud photons will have three different diagrams, the disconnected one of fig. 5c together with two diagrams where the photon is emitted and reabsorbed by the same cloud, as seen in fig. 5d. The contraction of the disconnected clod-to-cloud photon is determined by the last terms of eqs. (7.53) and (7.54),

\[
\int \tilde{d}^3k \tilde{d}^3k' S^\mu(p_f, k') S^\nu(p_i, k) \langle 0 | a^\mu(k)a^\dagger_\nu(k') | 0 \rangle .
\] (7.68)

Using the commutation relations of eq. (B.7) we can rewrite this as

\[
\mathcal{M}_{\text{ctc}}^{\text{dis}} = \int \tilde{d}^3k \ S^\mu(p_f, k) \eta_{\mu\nu} S^\nu(p_i, k) = \int \tilde{d}^3k \ S_i \cdot S_f .
\] (7.69a)

Similarly, the contribution of the connected cloud-to-cloud photons comes from the middlemost term in eqs. (7.53) and (7.54),

\[
\mathcal{M}_{\text{ctc}}^i = -\frac{1}{2} \int \tilde{d}^3k \ S^\mu(p_i, k) S^\nu(p_i, k) \langle 0 | a^\mu(k)a^\dagger_\nu(k) | 0 \rangle = \int \tilde{d}^3k \ S_i \cdot S_i .
\] (7.69b)

\[
\mathcal{M}_{\text{ctc}}^f = -\frac{1}{2} \int \tilde{d}^3k \ S^\mu(p_f, k) S^\nu(p_f, k) \langle 0 | a^\mu(k)a^\dagger_\nu(k) | 0 \rangle = \int \tilde{d}^3k \ S_f \cdot S_f .
\] (7.69c)

By summing the above amplitudes we find that the total cloud-to-cloud contribution up to first order is

\[
\mathcal{M}^{(\text{1})}_{\text{ctc}} = \int \tilde{d}^3k \ S_i \cdot S_f - \frac{1}{2} S_i \cdot S_i - \frac{1}{2} S_f \cdot S_f
\]

\[
= \frac{1}{2} \int \tilde{d}^3k \ S_i \cdot S_f + S_f \cdot S_i - S_i \cdot S_i - S_f \cdot S_f
\]

\[
= -\frac{1}{2} \int \tilde{d}^3k \ S_{\text{tot}} \cdot S_{\text{tot}} .
\] (7.70)

where we used that \( S_i \cdot S_f = \frac{1}{2} S_i \cdot S_f + \frac{1}{2} S_f \cdot S_i \). We can now calculate the total contribution of the soft photon cloud by summing eqs. (7.67) and (7.70).

\[
\mathcal{M}^{(\text{1})}_{\text{cloud}} = \mathcal{M}^{(\text{1})}_{\text{int}} + \mathcal{M}^{(\text{1})}_{\text{ctc}}
\]

\[
= \int \tilde{d}^3k \ S_{\text{tot}} \cdot P_{\text{tot}} - \frac{1}{2} \int \tilde{d}^3k \ S_{\text{tot}} \cdot S_{\text{tot}} .
\] (7.71)

Remembering that \( S^\mu(p, k) = P^\mu(p, k) + C^\mu(k) \),

\[
\mathcal{M}^{(\text{1})}_{\text{cloud}} = \int \tilde{d}^3k \ [ (P_{\text{tot}} + C_{\text{tot}}) \cdot P_{\text{tot}} - \frac{1}{2} (P_{\text{tot}} + C_{\text{tot}}) \cdot (P_{\text{tot}} + C_{\text{tot}}) ]
\]

\[
= \frac{1}{2} \int \tilde{d}^3k \ [ P_{\text{tot}} \cdot P_{\text{tot}} - C_{\text{tot}} \cdot C_{\text{tot}} ] .
\] (7.72)
7.5. Eigenstates of LGT

Note that from definition $C_\mu(k) = e^{i c_\mu(k) \frac{\omega}{2k}}$. As a consequence of eq. (7.44), $C_{\text{tot}}$ must vanish. Further, by definition $c_\mu$ is null, meaning factors of $c_\mu e^{i \omega} = 0$, and as a consequence $C_{\text{tot}} \cdot C_{\text{tot}} = 0$. We conclude that the contribution of the cloud amounts to

$$M_{\text{cloud}}^{(1)} = \frac{1}{2} \int \frac{d^3k}{2} P_{\text{tot}} \cdot P_{\text{tot}}$$

which exactly cancels the IR divergent term of $M_{\text{virt}}^{(1)}$ in eq. (7.64). This result was a proof of concept to the lowest order. It can be generalized for many-particle states, including states with different $c_\mu$-vectors. The reader is referred to Kulish & Faddeev [2], Chung [4], Gabai & Sever [11] and Jauch & Rohrlich [60] for further details.

There exists several qualitative differences between the FK method and the Bloch-Nordsieck (BN) method. Firstly, in the BN formalism the IR divergence from the bremsstrahlung cancels against the IR divergent parts of the virtual photons. This happens at a cross-section level and gives us an experimental cross-section dependent on the energy resolution. This cross-section vanishes in the limit of elastic scattering. In the FK formalism we see that the divergence is cancelled on an $S$-matrix level, where the IR divergence of the bremsstrahlung is cancelled against soft photon emission from the clouds. Meanwhile, the IR divergence from the virtual photons is cancelled against the interacting photons and cloud-to-cloud photons. The advantage of this formalism is that both the cross-section and the $S$-matrix are finite and well-defined and do not have any dependence on a energy resolution. We also obtain an explanation as to why elastic scattering is prohibited - each charged particle will always interact with its cloud by continuously emitting and reabsorbing soft photons.

7.5. Eigenstates of LGT

Recall the action of the generator of LGT $Q^+_\xi$ on the bra vacuum of eq. (5.24),

$$\langle 0| Q^+_\xi = \lim_{\omega \to 0} \frac{\omega}{8\pi} \int d^2z \gamma_{zz} \langle 0| \left[ a_- (\omega \hat{x}_z) \partial_z \xi (z, \bar{z}) + a_+ (\omega \hat{x}_{\bar{z}}) \partial_{\bar{z}} \xi (z, \bar{z}) \right]$$

This corresponds of adding a soft photon of polarization $\pm$ at future null infinity. Consider the scattering process of any physical state built from the Fock vacuum $|0\rangle$ into a state built from the a vacuum acted by $Q^+_\xi$,

$$\mathcal{M} = \langle 0| Q^+_\xi \Psi_{\text{out}}^{\text{out}} e^{-R_f} S D e^{R_f} \Psi_{\text{in}}^{\text{in}} |0\rangle,$$  

where $\Psi_{\text{out/in}}^{\text{out/in}}$ are the outgoing/incoming undressed Fock states and $S$ is the $S$-matrix. Due to the factor of $\lim_{\omega \to 0} \omega$ in the amplitude, the scattering process needs to produce a
7.5. Eigenstates of LGT

divergent factor of $\frac{1}{\omega}$ if this amplitude is to be non-vanishing. However, we have already shown that all scattering processes involving physical states are free of IR divergences, including the ones with soft photon emission. As a consequence, the amplitude of eq. (7.75) will always vanish.

$$\mathcal{M} = \langle 0 | Q_{\epsilon}^+ \psi^\text{out}_F e^{-R_f} S_D e^{R_f} \psi^\text{in}_F | 0 \rangle = 0. \quad (7.76)$$

A large gauge transformation of the original vacuum will create a state that is orthogonal to it and cannot by any means be reproduced by any physical scattering process. Let us now consider the more general case of a large gauge transformation of a dressed matter state

$$\langle 0 | c(p) e^{-R_f(p)} Q_{\epsilon}^+ = \langle 0 | Q_{\epsilon}^+ c(p) e^{-R_f(p)}$$

$$+ \langle 0 | c(p) \left[ e^{-R_f(p)}, Q_{\epsilon}^+ \right] + \langle 0 | \left[ c(p), Q_{\epsilon}^+ \right] e^{-R_f(p)}. \quad (7.77)$$

The first term is the LGT transformation of vacuum, will always vanish when contracted with an incoming ket state and will therefore be ignored. The second term is the commutator between the fermion annihilation operator and the LGT generator, $[c(p), Q_{\epsilon}^+]$. In this case the generator of LGT is acting to a non-vacuum state, meaning that we need to take both the soft and the hard part of the operator into account. Formally, before one can calculate the action of the LGT generator on a massive particle one needs change to rescaled radial coordinates\textsuperscript{5}, which are more suitable to study objects at future timelike infinity $i^+$ than the retarded Bondi coordinates. By explicit calculation the commutator becomes [8, 11]

$$\left[ c(p), Q_{\epsilon}^+ \right] = c(p) \int \frac{d^2 z}{4\pi} \frac{\gamma_{zz} \varepsilon(z, \bar{z}) m^2}{(\hat{x}_z \cdot p - \sqrt{m^2 + p^2})^2}. \quad (7.78)$$

Next we will treat the commutator between the dressing operator, $e^{-R_f(p)}$, and $Q_{\epsilon}^+$. Remember that $[e^{-R_f(p)}, Q_{\epsilon}^+] = -e^{-R_f} \left[ R_f(p), Q_{\epsilon}^+ \right]$. The resulting commutator can be calculated using the canonical commutation relation for photons,

$$\left[ R_f(p), Q_{\epsilon}^+ \right] = e \int \tilde{d}^3 k \left( f^\mu(p, k) e^\nu(k) \left[ a^\dagger_\nu(k), Q_{\epsilon}^+ \right] - f^{\mu*}(p, k) e^{\nu*}(k) \left[ a_\nu(k), Q_{\epsilon}^+ \right] \right)$$

$$= -e \int \frac{d^2 z}{4\pi} \gamma_{zz} f^\mu(\hat{k}_z) \left[ \epsilon_\mu^{-}(\hat{k}_z) \partial_z \varepsilon(z, \bar{z}) + \epsilon_\mu^{+}(\hat{k}_z) \partial_{\bar{z}} \varepsilon(z, \bar{z}) \right], \quad (7.79)$$

where $\hat{k}_z = \hat{x}_z$ is the unit vector in the $z$-direction. Note that as a result of the commutation we obtained a delta function on the form $\delta^{(3)}(k - \hat{q}_L) \delta^*(\omega_k - \omega_q)$, where

\textsuperscript{5}Rescaled radial coordinates are defined by $\tau \equiv \sqrt{r^2 - r^2}$, $\rho \equiv \frac{r}{\sqrt{r^2 - r^2}}$, meaning that the line element will be $ds^2 = -d\tau^2 + d\tau^2(\frac{d\rho^2}{1 + \rho^2} + \rho^2 \gamma_{zz} dz d\bar{z})$. 
7.5. Eigenstates of LGT

$q$ is the momentum of the creation and annihilation operators from $Q^+_\varepsilon$. This delta function must however be normalized in such a way that it introduces an additional factor of $\frac{1}{\sqrt{2}}$. This is due to that in the soft limit it is impossible to distinguish between photons of different helicities and it would otherwise result in a double counting of the states [11]. We can integrate each term by parts with respect to $z$ and $\bar{z}$ respectively, obtaining

$$
\left[R_f(p), Q^+_\varepsilon\right] = -e \int \frac{d^2 z}{4\pi} \varepsilon(z, \bar{z}) \left( \gamma_{z\bar{z}} \frac{p^2}{(p \cdot k_z)^2} + \partial_z (c^\mu \epsilon^-_\mu) + \partial_{\bar{z}} (c^\mu \epsilon^+_\mu) \right)
$$

$$
= -e \int \frac{d^2 z}{4\pi} \varepsilon(z, \bar{z}) \left( \gamma_{z\bar{z}} \frac{p^2}{(p \cdot k_z)^2} + C(z, \bar{z}) \right),
$$

(7.80)

where we have defined $C(z, \bar{z}) \equiv \partial_z (c^\mu \epsilon^-_\mu) + \partial_{\bar{z}} (c^\mu \epsilon^+_\mu)$. We note that

$$
\frac{p^2}{(p \cdot k_z)^2} = -\frac{m^2}{\left(\hat{k}_z \cdot p - \sqrt{m^2 + p^2}\right) \bar{z}}.
$$

(7.81)

Since the unit vectors $\hat{k}_z = \hat{x}_z$ are identical the contribution of $[c(p), Q^+_\varepsilon]$ in eq. (7.78) will be cancelled exactly. Thus, the LGT of a dressed matter state becomes

$$
\langle 0 | c(p) e^{-R_f(p)} Q^+_\varepsilon = \langle 0 | c(p) e^{R_f(p)} e \int \frac{d^2 z}{4\pi} \varepsilon(z, \bar{z}) C(z, \bar{z}),
$$

(7.82)

and we see that the Faddeev-Kulish states indeed are eigenstates of the large gauge transformations. For a generic outgoing state the total LGT charge will be

$$
\frac{\langle 0 | \Psi_{out FK}^+ Q^+_\varepsilon S_D \Psi_{in FK}^+ | 0 \rangle}{\langle 0 | \Psi_{out FK}^+ S_D \Psi_{in FK}^+ | 0 \rangle} = \sum_{j \in out} e_j \int \frac{d^2 z}{4\pi} \varepsilon(z, \bar{z}) C_j(z, \bar{z}),
$$

(7.83)

where $e_j$ is the electric charge of particle $j$. We see that it not dependent on any of the outgoing particles’ momenta and only depend on the different $c^\mu$-vectors that parametrize the dressings.

For further clarification, we can consider a generic eigenstates of LGT $\langle \Omega_\Lambda |$,

$$
\langle \Omega_\Lambda | Q^+_\varepsilon \equiv \int \frac{d^2 z}{4\pi} \Lambda(z, \bar{z}) \varepsilon(z, \bar{z}) \langle \Omega_\Lambda |,
$$

(7.84)

where $\Lambda(z, zb)$ is a function on $S^2$ and parametrizes which LGT sector the state belongs to. We can treat these eigenstates as new vacuum states and use them to construct our physical states [11]. The LGT of a dressed state created on this new vacuum becomes

$$
\langle \Omega_\Lambda | c(p) e^{-R_f(p)} Q^+_\varepsilon = \langle \Omega_\Lambda | c(p) e^{R_f(p)} e \int \frac{d^2 z}{4\pi} \varepsilon(z, \bar{z}) \left[ \Lambda(z, \bar{z}) + C(z, \bar{z}) \right],
$$

(7.85)
7.5. Eigenstates of LGT

Since the LGT are a symmetry of the theory it follows that $Q_\varepsilon$ commutes with the $S$-matrix and the below expression must vanish

$$\langle \Omega_{\Lambda_1} | \Psi_{FK}^{\text{out}} [Q_\varepsilon, S_D] \Psi_{FK}^{\text{in}} | \Omega_{\Lambda_2} \rangle = \int \frac{d^2z}{4\pi} \varepsilon(z, \bar{z}) \langle \Omega_{\Lambda_1} | \Psi_{FK}^{\text{out}} S_D \Psi_{FK}^{\text{in}} | \Omega_{\Lambda_2} \rangle$$  \hspace{1cm} (7.86)

$$\times \left[ \sum_{j \in \text{out}} e_j C_j - \sum_{i \in \text{in}} e_i C_i + \Lambda_2 - \Lambda_1 \right],$$

as a consequence of eq. (7.44) the $C(z, \bar{z})$-factors from the outgoing and incoming states will cancel each other. What remains is

$$\langle \Omega_{\Lambda_1} | \Psi_{FK}^{\text{out}} [Q_\varepsilon, S_D] \Psi_{FK}^{\text{in}} | \Omega_{\Lambda_2} \rangle = \langle \Omega_{\Lambda_1} | \Psi_{FK}^{\text{out}} S_D \Psi_{FK}^{\text{in}} | \Omega_{\Lambda_2} \rangle \int \frac{d^2z}{4\pi} \varepsilon(z, \bar{z}) \left[ \Lambda_2 - \Lambda_1 \right] = 0.$$  \hspace{1cm} (7.87)

Since this must be valid for all $\varepsilon(z, \bar{z})$ the expression can vanish in only two ways: (i) If the amplitude for the process vanishes, $\langle \Omega_{\Lambda_1} | \Psi_{FK}^{\text{out}} S_D \Psi_{FK}^{\text{in}} | \Omega_{\Lambda_2} \rangle = 0,$ (ii) if the bra vacuum and the ket vacuum have the same large gauge charge, $\Lambda_{\text{in}}(z, \bar{z}) = \Lambda_{\text{out}}(z, \bar{z}).$ This means that when using physical FK states, all scattering processes between states built from different LGT vacuums will always vanish. As a consequence, transitions between different LGT sectors are forbidden.

For regular Fock states without the Faddeev-Kulish dressing the $S$-matrix would have been divergent in the IR limit, meaning that the final expression of

$$\langle 0 | \Psi_{F}^{\text{out}} [Q_\varepsilon, S_D] \Psi_{F}^{\text{in}} | 0 \rangle$$  \hspace{1cm} (7.88)

would have additional terms due since the terms containing the LGT of the vacuum would no longer be vanishing, since the factor of $\lim_{\omega \to 0} \omega \langle 0 | a_\pm(\omega \hat{x}) \rangle$ would cancel against the IR divergence of $S_D.$ This would in turn result in the large gauge charge not being conserved. This means that in IR divergent processes the LGT charge cannot be conserved and conversely – if the LGT is conserved then the amplitude must be IR finite. Conservation of the large gauge charge does in fact imply the existence of Faddeev-Kulish amplitudes [19].
8. Perturbative Quantum Gravity

Summary

• The theory of perturbative quantum gravity (PQG) is presented.
• Faddev-Kulish states of scalar particles surrounded by soft graviton clouds are constructed.
• It is shown that these FK states will be eigenstates of the BMS supertranslations as well as the Bondi news tensor.

8.1. Second Quantization of the Gravitational Field

Consider the Lagrangian density for a single scalar field coupled to the metric field,

$$\mathcal{L} = \sqrt{-g} \left( -R - \frac{1}{2} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} m^2 \phi^2 \right),$$  \hspace{1cm} (8.1)

where $g \equiv \text{det}(g_{\mu \nu})$, $R$ is the Ricci scalar, $g^{\mu \nu}$ is the inverse metric tensor, $\phi$ is the scalar field, $m$ is the mass. Let the metric $g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu}$ be a small perturbation of the flat Minkowski metric, $\eta_{\mu \nu} = \text{diag}(-1,1,1,1)$, where $\kappa^2 = 32\pi G$. Note that we write partial derivatives in the Lagrangian, since the covariant derivative of a scalar field reduces to a partial derivative, $\nabla_{\mu} \phi = \partial_{\mu} \phi$, by virtue of theorem 2.18.

We will henceforth treat the Minkowski metric as a purely classical object, as is the standard procedure in QFT, but we will impose commutation relations on the metric perturbation. From the point of view of general relativity this break-up of the metric into two parts, one classical and one quantum mechanical is unappealing, it is however straight-forward from a field theoretical point of view. It should also be mentioned, that since we will assume the perturbation to be small, we will neglect higher-order terms of the perturbation and thus also neglect the rich nonlinear nature of gravity. This method of quantizing the gravitational field is called perturbative quantum gravity (PQG) or the covariant perturbation method. It is non-renormalizable in the UV regime, but as we will show by constructing Faddeev-Kulish states, it is finite in the IR regime. The
main reason to studying PQG is its similarity to QED in the IR limit as well as its relation to asymptotic symmetries of spacetime.

The inverse metric will take the form \( g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} \). Since we are only interested in terms that are linear in the perturbation we can lower and raise indices using only \( \eta^{\mu\nu} \). Using lemma A.4 with \( M = \eta^{\mu\nu} \) and \( \delta M = \kappa h^{\mu\nu} \) we can find the expression for \( \det(\eta^{\mu\nu} + \kappa h^{\mu\nu}) \),

\[
\det(g_{\mu\nu}) = \det(\eta_{\mu\nu} + \kappa h_{\mu\nu}) \\
= \det(\eta_{\mu\nu})[1 + \kappa \eta^{\mu\nu} h_{\mu\nu}] = -1 - \kappa h^\mu. \tag{8.2}
\]

After Taylor expanding the square root up to first order in \( \kappa \) we obtain

\[
\sqrt{-g} = \sqrt{1 + \kappa h^\mu} \approx 1 + \frac{\kappa}{2} h, \tag{8.3}
\]

where we have written \( h \equiv h^\mu = \eta^{\mu\nu} h_{\mu\nu} \) for the sake of notational brevity. We can now expand the Lagrangian density around the Minkowski metric,

\[
\mathcal{L} = (1 + \frac{\kappa}{2}) \left[ -R - \frac{1}{2} \partial_\mu \phi (\eta^{\mu\nu} + \kappa h^{\mu\nu}) \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \\
= \mathcal{L}_{EH} = \mathcal{L}_{KG} \\
= -R \left( 1 + \frac{\kappa}{2} \right) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \\
- \frac{\kappa}{2} \left[ h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - h^\mu \left( \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) \right], \tag{8.4}
\]

where \( \mathcal{L}_{EH} \) is the free linearized gravitational Lagrangian, \( \mathcal{L}_{KG} \) is the free Klein-Gordon Lagrangian in Minkowski spacetime and \( \mathcal{L}_I \) is the interaction Lagrangian. The stress-energy tensor for the scalar field is\(^1\)

\[
T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \frac{1}{2} \left( \partial_\rho \phi \partial^\rho \phi + m^2 \phi^2 \right), \tag{8.5}
\]

meaning that the interaction Lagrangian can be expressed in terms of the stress-energy tensor contracted with the metric perturbation,

\[
\mathcal{L}_I = -\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}. \tag{8.6}
\]

This is in close analogy to the interaction Lagrangian of QED being \( \mathcal{L}_{I}^{QED} = J_{\mu} A^\mu \). The resemblance is clear when one considers that the electromagnetic current is the source term in the Maxwell equations in the same way that the stress-energy is the source term in the Einstein field equations.

\(^1\)Note that the EH stress-tensor and the canonical stress-tensor coincide for the scalar field.
In order to impose canonical commutation relations we must first expand the fields $\phi(x)$ and $h_{\mu\nu}(x)$ in terms of creation and annihilation operators, see Appendix B.4. “The Gravitational Field”. Further, we have the freedom of choosing gauge, in the literature it is common to work in the harmonic gauge (de Donder gauge), which is the gravitational analogue of the Lorenz gauge in electrodynamics,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu_{\mu} = 0. \quad (8.7)$$

Defining

$$h_{\mu\nu}(x)^+ \equiv \int \tilde{d}^3 p \ a_{\mu\nu}(k)e^{ikx}, \quad (8.8a)$$
$$h_{\mu\nu}(x)^- \equiv \int \tilde{d}^3 k \ a^\dagger_{\mu\nu}(k)e^{-ikx}, \quad (8.8b)$$

we can impose the Gupta-Bleuler condition derived from eq. (8.7), in order for the theory to be ghost-free,

$$\langle \Phi | \partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu_{\mu} \rangle^+ = 0, \quad (8.9a)$$
$$\langle \Phi | (\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu_{\mu})^- = 0, \quad (8.9b)$$

where $|\Phi\rangle_F$ is an arbitrary Fock state. From eq. (8.8a) we can calculate

$$(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu_{\mu})^+ = \int \tilde{d}^3 k \ (\#) \left[ k^\mu a_{\mu\nu}(k) - \frac{1}{2} k_\nu a^\mu_{\mu}(k) \right] e^{ikx}, \quad (8.10)$$

where $(\#)$ is some irrelevant scalar function. Thus the Gupta-Bleuler eq. (8.9a) condition becomes

$$\int \tilde{d}^3 k \ (\#)e^{ikx} \left[ k^\mu a_{\mu\nu}(k) - \frac{1}{2} k_\nu a^\mu_{\mu}(k) \right] |\Phi\rangle_F = 0, \quad (8.11)$$

however, since the above expression is a Fourier decomposition the equality must hold for each $k$, we can thus throw away the integral and the numerical factor. The Gupta-Bleuler condition becomes

$$\left[ k^\mu a_{\mu\nu}(k) - \frac{1}{2} k_\nu a^\mu_{\mu}(k) \right] |\Phi\rangle_F = 0. \quad (8.12)$$

This condition must be satisfied by all ghost-free Fock states $|\Phi\rangle_F$, meaning all scalar and longitudinal gravitons will cancel each other and thus not be observed.
8.2. Faddeev-Kulish States in PQG

The procedure of finding the asymptotic dynamics of PQG is analogous to QED and will not be written out in detail, instead we will comment some key steps. For a detailed derivation of the Faddeev-Kulish states in PQG the reader is referred to Ware et al. [12]. Starting from the interaction Lagrangian eq. (8.4) we can repeat the analysis of the asymptotic interaction potential for QED, but keeping in mind that the source of gravity is in fact the stress-energy. We obtain

\[ V^a_\text{as}(t) = -\frac{\kappa}{2} \int d^3p \, d^3k \, p^\mu p'^\nu \left( a^\dagger_{\mu\nu}(k)e^{i\frac{p^\mu}{p^0}t} + a_{\mu\nu}(k)e^{-i\frac{p^\mu}{p^0}t} \right) \rho(p), \quad (8.13) \]

where \( \rho(p) \equiv b^\dagger(p)b(p) \) is the scalar density operator. Similar to eq. (7.25a) the radiation operator in PQG, \( W(t) = e^{R(t)} \), has the form

\[ R(t) = -i\frac{\kappa}{2} \int dt' \, V^a_\text{as} = \frac{\kappa}{2} \int d^3k \, d^3p \, \frac{p^\mu p'^\nu}{pk} \left( a^\dagger_{\mu\nu}(k)e^{-i\frac{p^\mu}{p^0}t} - a_{\mu\nu}(k)e^{i\frac{p^\mu}{p^0}t} \right) \rho(p), \quad (8.14) \]

meanwhile, the phase operator \( e^{i\Phi(t)} \) will cancel the phase calculated by Weinberg [12, 43]. Similar to the case of QED, the FK states that are dressed with the operator \( R(t) \) will not satisfy the Gupta-Bleuler condition, eq. (8.12), meaning that we need to construct a new operator that creates an equivalent space,

\[ R_f = \frac{\kappa}{2} \int d^3k \, d^3p \, \frac{p^\mu p'^\nu}{pk} \left( f^\mu\nu(p,k)a^\dagger_{\mu\nu}(k) - f^\mu\nu(p,k)a_{\mu\nu}(k) \right) \rho(p). \quad (8.15) \]

The functions \( f^\mu\nu(p,k) \) are the infrared functions, which have the form

\[ f^\mu\nu(p,k) = \left[ \frac{p_\mu p_\nu}{pk} + \frac{c^\mu\nu(p,k)}{\omega_k} \right] \phi(p,k), \quad (8.16) \]

where \( \phi(p,k) \) is a smooth function such that \( \phi \to 1 \) as \( k \to 0 \) and \( c^\mu\nu(p,k) \) is some function such that

\[ \frac{k^\mu c^\mu\nu}{\omega_k} - p_\nu. \quad (8.17) \]

The \( c_{\mu\nu} \)-tensors associated with each particle will satisfy

\[ \sum_{j \in \text{out}} c_{\mu\nu}(p_j, k) = \sum_{i \in \text{in}} c_{\mu\nu}(p_i, k). \quad (8.18) \]

For equal \( c_{\mu\nu} \) the relation follows from eq. (8.17) as a result of momentum conservation, but can be generalized for different \( c_{\mu\nu} \)-tensors [18].

\(^2\)Note that the radiation operators \( R(t) \) and \( R_f \) are not in any way related to the Ricci scalar \( R \).
8.2. Faddeev-Kulish States in PQG

8.2.1. Cancellation of the Divergence

The procedure to show that the IR divergence is cancelled by the Faddeev-Kulish formalism for PQG is analogous to the case of QED. We will briefly mention the differences compared to the calculation in 7.4. “Cancellation of the Divergence”. Firstly, the shorthand must be adjusted so that it matches the gravitational clouds,

\[ S_{\mu\nu}(p, k) \equiv \frac{\kappa}{2} f_{\mu\nu}(p, k) \],

\[ P_{\mu\nu}(p, k) \equiv \frac{\kappa p_{\mu} p_{\nu}}{2 p \cdot k} \],

\[ C_{\mu\nu}(k) \equiv \frac{\kappa}{2} c_{\mu\nu}(p, k) \frac{\omega_k}{\omega_k} \].

Consequently, the soft graviton dressing operator becomes

\[ R_f(p) \equiv \int \frac{d^3k}{4\pi} \left[ S^{\mu\nu}_{\mu\nu}(k) - S^{\mu\nu}(p, k) S^{\mu\nu}(k) \right] \].

To lowest order in the initial and final states will be

\[ |i\rangle = \left( 1 - \frac{1}{4} \int \frac{d^3k}{4\pi} S_i^{\mu\nu} I^{\mu\nu\rho\sigma} S_i^{\rho\sigma} + \int \frac{d^3k}{4\pi} S_i^{\mu\nu} a^{\mu\nu}(k) \right) b^\dagger(p_i) |0\rangle \],

\[ \langle f | = \langle 0 | b(p_f) \left( 1 - \frac{1}{4} \int \frac{d^3k}{4\pi} S_f^{\mu\nu} I^{\mu\nu\rho\sigma} S_f^{\rho\sigma} + \int \frac{d^3k}{4\pi} S_f^{\mu\nu} a^{\mu\nu}(k) \right) \].

Each soft graviton emitted from an external leg will contribute with

\[ \eta \frac{\kappa p^\mu p'^\nu}{2 p \cdot k} \epsilon^r_{\mu\nu} \],

where \( \eta = +1(-1) \) for outgoing(incoming) legs, in accordance to the soft theorem [43]. Similarly, each soft photon emission from a cloud contributes with

\[ \eta S^{\mu\nu}(p, k) \epsilon^r_{\mu\nu} \].

The first order virtual graviton contribution is

\[ \mathcal{M}^{(1)}_{\text{virt}} = \frac{\kappa^2}{16} \int \frac{d^3k}{4\pi} \left[ \frac{p_f^4}{2(p_f \cdot k)^2} + \frac{p_i^4}{2(p_i \cdot k)^2} - 2 \left( \frac{(p_f \cdot p_i)^2 - \frac{1}{2} p_f^2 p_i^2}{(p_f \cdot k)(p_i \cdot k)} \right) \right] \].

With these differences in mind, the divergence is cancelled in the same manner as in QED. For a detailed calculation the reader is directed to Ware et al. [12] and Choi et al. [18, 19].
8.3. Eigenstates of the BMS Supertranslations

Recall the generator of supertranslations,

\[ T(f) = \lim_{\omega \to 0} \frac{\omega}{4\pi \kappa} \int d^2 \bar{z} \left( [a_+ (\omega \hat{x}_z) + a_\dagger (\omega \hat{x}_{\bar{z}})] \bar{D}_z f + \text{h.c.} \right) + T_H(f), \quad (8.25) \]

where the first inside the integral is the soft part of eq. (6.40) and the second term is the hard part of eq. (6.35b)

\[ T_H(f) = \frac{1}{4\pi G} \int du \, d^2 z \, \gamma_{z \bar{z}} f(z, \bar{z}) T_{uu}. \quad (8.26) \]

Similar to the case of the large gauge transformations, we want to calculate the effect of a BMS supertranslation on a dressed matter state,

\[ \langle 0 | b(p) e^{-R_f(p)} T(f) = \langle 0 | T_S(f) b(p) e^{-R_f(p)} \]

\[ + \langle 0 | b(p) \left[ e^{-R_f(p)}, T(f) \right] + \langle 0 | \left[ b(p), T(f) \right] e^{-R_f(p)}. \quad (8.27) \]

The first term is the BMS supertranslation of the vacuum. The hard part of the operator will immediately annihilate the vacuum, giving of the soft graviton contribution,

\[ \langle 0 | T_S(f) = \lim_{\omega \to 0} \frac{\omega}{4\pi \kappa} \int d^2 \bar{z} \, \gamma_{z \bar{z}} \langle 0 | \left[ a_- (\omega \hat{x}_z) \bar{D}_z f + a_+ (\omega \hat{x}_{\bar{z}}) D_\bar{z} f \right]. \quad (8.28) \]

Completely analogous to the case of QED, this supertranslated vacuum state will be orthogonal to all states that can be created by any physical scattering,

\[ \mathcal{M} = \langle 0 | T_S(f) \Phi^\text{out}_F e^{-R_f} S_D e^{R_f} \Phi^\text{in}_F | 0 \rangle = 0. \quad (8.29) \]

The action of the BMS supertranslation of a scalar field operator will be given by the commutator \[18\]

\[ \left[ b(p), T(f) \right] = b(p) \int d^2 \bar{z} \frac{\gamma_{z \bar{z}} f(z, \bar{z}) m^4}{4\pi} \left( \sqrt{m^2 + p^2 - \hat{x}_z \cdot \hat{p}} \right)^3. \quad (8.30) \]

Similarly the commutator between the dressing operator and \( T(f) \) becomes \[18\]

\[ \left[ R_f(p), T(f) \right] = -\int d^2 \bar{z} f(z, \bar{z}) \left( \frac{p^4}{(p \cdot \hat{k}_z)^3} + C(p, z, \bar{z}) \right), \quad (8.31) \]

where we have defined \( C(p, z, \bar{z}) \equiv \partial_{\bar{z}} \partial^\bar{z} (\gamma_{z \bar{z}} c^{\mu \nu} \epsilon^-_{\mu \nu}) + \partial_z \partial^z (\gamma_{z \bar{z}} c^{\mu \nu} \epsilon^+_{\mu \nu}). \) The first term of the commutator is equal to
which will exactly cancel the contribution from the scalar field operator. A supertranslation of a dressed matter state will therefore result in

\[ \langle 0 | b(p) e^{-R_f(p)} T(f) = \int \frac{d^2 z}{4\pi} f(z, \bar{z}) C(p, z, \bar{z}). \quad (8.33) \]

The total BMS charge of a generic outgoing state in a scattering process will be

\[ \frac{\langle 0 | \Phi^\text{out}_{FK} T(f) S_D \Phi^\text{in}_{FK} | 0 \rangle}{\langle 0 | \Phi^\text{out}_{FK} S_D \Phi^\text{in}_{FK} | 0 \rangle} = \sum_{j \in \text{out}} \int \frac{d^2 z}{4\pi} (z, \bar{z}) C(p_j, z, \bar{z}). \quad (8.34) \]

Similar to the case of QED transitions between different supertranslation sectors is forbidden. We find that using FK states ensured the conservation of the BMS charge in a scattering process and by so doing it prevents the occurrence of the IR divergence [18, 19].

\[ \langle \Omega_{\Lambda_1} | \Phi^\text{out}_{FK}[T(f), S_D] \Phi^\text{in}_{FK} | \Omega_{\Lambda_2} \rangle = \langle \Omega_{\Lambda_1} | \Phi^\text{out}_{FK} S_D \Phi^\text{in}_{FK} | \Omega_{\Lambda_2} \rangle \int \frac{d^2 z}{4\pi} f(z, \bar{z}) [\Lambda_1 - \Lambda_2] = 0. \quad (8.35) \]

### 8.3.1. Bondi News Eigenstates

Another approach is to explicitly construct eigenstates of the Bondi news tensor and then see that these states are equivalent to the FK states. Recall the form of soft part of the generator of BMS supertranslations that we derived in eq. (6.41),

\[ T_S(f) = -\frac{1}{4G} N(z, \bar{z}), \quad (8.36) \]

where we have set \( f(z', \bar{z}') = \frac{(1+zz')(z'-\bar{z})}{(1-\bar{z}z)(z'-\bar{z})} \). The news tensor can be expressed in terms of creation and annihilation operators, which suggests that the eigenstate will have the form of a coherent state. Consider the state

\[ |N\rangle = \exp \left\{ \int \frac{\sqrt{4\pi}}{\tilde{k}} N^{\mu\nu}(k) \left[ a^\dagger_{\mu\nu}(k) - a_{\mu\nu}(k) \right] \right\} |0\rangle, \quad (8.37) \]

where \( a^\dagger_{\mu\nu}(k) = \epsilon^{\mu\nu}_{\lambda\sigma}(k) a^\dagger_{\lambda\sigma}(k) \) and \( a_{\mu\nu}(k) = \epsilon_{\mu\nu}^{\lambda\sigma}(k) a_{\lambda\sigma}(k) \). Due to the fact that eq. (6.39) has an overall factor of \( \omega \) that goes towards zero this state can only be an eigenstate if \( N^{\mu\nu}(k) \) has a pole at \( \omega = 0 \). Denoting the operator in the exponential of eq. (8.37) as \( \hat{N} \) we can show
Thus, we can conclude that
\[
\lim_{\omega \to 0} \omega a_+ (\omega \hat{x}) | N \rangle = \lim_{\omega \to 0} \omega a_+ (\omega \hat{x}) e^{\hat{N}} | 0 \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left( a_+ (\omega \hat{x}) e^{\hat{N}} - e^{\hat{N}} a_+ (\omega \hat{x}) \right) | 0 \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ a_+ (\omega \hat{x}), e^{\hat{N}} \right] | 0 \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ a_+ (\omega \hat{x}), \hat{N} \right] e^{\hat{N}} | 0 \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ a_+ (\omega \hat{x}), \int d^3 k \ N^{\mu \nu} (k) \left( a^\dagger_{\mu \nu} (k) - a_{\mu \nu} (k) \right) \right] | N \rangle ,
\]
where we have used that \([A, f(B)] = [A, B] \frac{\partial f(B)}{\partial B}\). Thus, we have
\[
\lim_{\omega \to 0} \omega a_+ (\omega \hat{x}) | N \rangle = \lim_{\omega \to 0} \omega \left[ a_+ (\omega \hat{x}) \int d^3 k \ N^{\mu \nu} (k) \left( a^\dagger_{\mu \nu} (k) - a_{\mu \nu} (k) \right) \right]
\]
\[
- \int d^3 k \ N^{\mu \nu} (k) (a^\dagger_{\mu \nu} (k) - a_{\mu \nu} (k)) a_+ (\omega \hat{x}) | N \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ a_+ (\omega \hat{x}) \int d^3 k \ N^{\mu \nu} (k) (\epsilon^r_{\mu \nu} (k) a^\dagger_{\nu} (k) - \epsilon^{r^*}_{\mu \nu} (k) a_{\nu} (k)) \right]
\]
\[
- \int d^3 k \ N^{\mu \nu} (k) (\epsilon^r_{\mu \nu} (k) a^\dagger_{\nu} (k) - \epsilon^{r^*}_{\mu \nu} (k) a_{\nu} (k)) a_+ (\omega \hat{x}) | N \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ \int d^3 k \ N^{\mu \nu} (k) (\epsilon^r_{\mu \nu} (k) [ (2\pi)^3 (2\omega) \delta_{r+} \delta^3 (k - \omega \hat{x}) + a^\dagger_{r} (k) a_+ (\omega \hat{x})] ) - \epsilon^{r^*}_{\mu \nu} (k) a_{r} (k) \right]
\]
\[
- \epsilon^{r^*}_{\mu \nu} (k) a_r (k) \right) a_+ (\omega \hat{x}) | N \rangle
\]
\[
= \lim_{\omega \to 0} \omega \left[ \int d^3 k \ N^{\mu \nu} (k) (\epsilon^r_{\mu \nu} (k) \delta_{r+} \delta^3 (k - \omega \hat{x}) | N \rangle ) \right]
\]
\[
= \lim_{\omega \to 0} \omega \ N^{\mu \nu} (\omega \hat{x}) \epsilon^+_{\mu \nu} (\omega \hat{x}) | N \rangle ,
\]
meaning that we can explicitly see that \(| N \rangle\) is an eigenstate of \(a_+ (\omega \hat{x})\) if \(N^{\mu \nu}\) has a soft pole. Likewise, for the creation operator we can show
\[
\lim_{\omega \to 0} \omega a^\dagger_+ (\omega \hat{x}) | N \rangle = \lim_{\omega \to 0} \omega \ N^{\mu \nu} (\omega \hat{x}) \epsilon^+_{\mu \nu} (\omega \hat{x}) | N \rangle ,
\]
by remembering that the term \(\omega e^{\hat{N}} a^+_0 \) vanish in the \(\omega \to 0\) limit and that \(\epsilon^{-*}_{\mu \nu} = \epsilon^+_{\mu \nu}\). Thus, we can conclude that
8.3. Eigenstates of the BMS Supertranslations

\[ N(z, \bar{z}) |N\rangle = -\frac{\kappa}{4\pi} \left( \lim_{\omega \to 0} \omega N^{\mu\nu}(\omega \hat{x}) \epsilon^+_{\mu\nu}(\omega \hat{x}) \right) |N\rangle \]  \hspace{1cm} (8.41)

and \( |N\rangle \) in fact is an eigenstate of \( N(z, \bar{z}) \) given that \( N^{\mu\nu}(\omega \hat{x}) \) has a pole at \( \omega = 0 \). This is an alternative way of showing that FK states are eigenstates of the BMS supertranslations and will ensure the IR-finiteness of our scattering processes. For further details on this formalism the reader is referred to Choi & Akhoury [19].
9. The Wilson Line Perspective

Summary

- Wilson lines are introduced as the connection of QED and are used to construct a geometric formulation of the interactions.
- The Mandelstam quantization scheme is presented.
- The gauge invariant fields in Mandelstam QED $\Psi(x|\Gamma)$ are shown to be equivalent to Faddeev-Kulish states.
- The Mandelstam scalar fields with gravitational Wilson lines are shown to be equivalent to FK states. This result is novel to this thesis.

Consider the local $\text{U}(1)$ gauge transformation of a matter field $\phi(x)$,

$$
\phi(x) \rightarrow e^{i\theta(x)}\phi(x),
$$

(9.1)

where $\theta(x)$ is the transformation parameter at point $x$. Even this transformation introduces an arbitrary phase to the field, the theory as a whole should be invariant under the transformation. However in a local theory the gauge transformation at point $x$ is independent of the transformation at point $y$, thus a difficulty arises in comparing the fields at two points. From the quantity

$$
\phi(y) - \phi(x) \rightarrow e^{i\theta(y)}\phi(y) - e^{i\theta(x)}\phi(x),
$$

(9.2)

we see that the norm $|\phi(y) - \phi(x)|$ depends on our choice of local phases. As a consequence we cannot immediately construct a consistent derivative. This problem is analogous to the problem of comparing vectors at different points in a manifold, which was discussed in section 2.2. "Differential Geometry". We want to construct a way to compare fields at different point in a gauge invariant way. To this end we introduce a
new bi-local field \( W(x, y) \) that depends on both points. We want this field to have the transformation law

\[
W(x, y) \rightarrow e^{i\theta(x)}W(x, y)e^{-i\theta(y)},
\]

(9.3)
in order for

\[
W(x, y)\phi(y) - \phi(x) \rightarrow e^{i\theta(x)}W(x, y)e^{-i\theta(y)}e^{i\theta(y)}\phi(y) - e^{i\theta(x)}\phi(x)
\]

\[
= e^{i\theta(x)}[W(x, y)\phi(y) - \phi(x)].
\]

(9.4)

Now we see that the norm \(|W(x, y)\phi(y) - \phi(x)|\) is independent of our choice of gauge and can be used to compare the field values at two points. The field \( W(x, y) \) is suitably called a comparator or connector. Assuming that the \( W(x, y) \) is a continuous function of the points \( x \) and \( y \), we can expand the comparator as

\[
W(x, x + \delta x) = 1 - ie\delta x^\mu A_\mu(x) + O(\delta x^2),
\]

(9.5)

where \( e \) is some coupling constant and the field \( A_\mu(x) \) parametrizes the displacement coefficients along the direction of \( \delta x^\mu \). This field allows us to compare field values of \( \phi(x) \) at infinitesimally separated points and is therefore called a connection. In fact, it serves the same purpose as the Christoffel symbol.

We can use this fact to construct a consistent definition of a derivative between two points, called a gauge covariant derivative,

\[
\mathcal{D}_\mu \phi(x) \equiv \lim_{\delta x^\mu \to 0} \frac{W(x, x + \delta x)\phi(x + \delta x) - \phi(x)}{\delta x^\mu}.
\]

(9.6)

From eq. (9.4) we see that the gauge covariant derivative transforms as

\[
\mathcal{D}_\mu \phi(x) \rightarrow e^{i\theta(x)}\mathcal{D}_\mu \phi(x).
\]

(9.7)

Inserting the expansion eq. (9.5) into the definition of the gauge covariant derivative eq. (9.6) will give us

\[
\mathcal{D}_\mu \phi(x) = \lim_{\delta x^\mu \to 0} \left[1 - ie\delta x^\mu A_\mu(x)\right] \frac{\phi(x + \delta x) - \phi(x)}{\delta x^\mu}
\]

\[
= \lim_{\delta x^\mu \to 0} \frac{\phi(x + \delta x) - \phi(x)}{\delta x^\mu} + \lim_{\delta x^\mu \to 0} \frac{-ie \delta x^\mu A_\mu(x)\phi(x + \delta x)}{\delta x^\mu}
\]

\[
= \partial_\mu \phi(x) - ieA_\mu(x)\phi(x).
\]

(9.8)

The transformation law of eq. (9.7) infers that the connection \( A_\mu(x) \) must transform as

\[
A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x).
\]

(9.9)
9. The Wilson Line Perspective

We have derived the need of a gauge field \( A_\mu(x) \) as well as its transformation properties from purely geometrical arguments. Since the covariant derivative transforms nicely, so will the commutator between two gauge covariant derivatives acting on a field,

\[
[D_\mu, D_\nu] \phi(x) \to e^{i\theta(x)} [D_\mu, D_\nu] \phi(x).
\] (9.10)

Looking more closely on the commutator we find that it is in fact a function and not an operator,

\[
[D_\mu, D_\nu] \phi = \left[ \partial_\mu, \partial_\nu \right] \phi + ie \left( [\partial_\nu, A_\mu] - [\partial_\mu, A_\nu] \right) \phi - e^2 [A_\mu, A_\nu] \phi \\
= ie \partial_\nu (A_\mu \phi) - A_\mu \partial_\nu \phi - \partial_\mu (A_\nu \phi) + A_\mu \partial_\nu \phi
\] (9.11)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor. Thus, we obtain the relation

\[
\frac{i}{e} [D_\mu, D_\nu] = F_{\mu\nu}.
\] (9.12)

As a consequence of eq. (9.10) the field strength tensor is locally gauge invariant. Recall lemma 2.20, where we expressed the Riemann curvature tensor in terms of a commutator of two spacetime-covariant derivatives,

\[
[\nabla_\mu, \nabla_\nu] B^\lambda = R^\lambda_{\kappa\mu\nu} B^\kappa.
\] (9.13)

We can now see that the field strength tensor of gauge theories is analogous to the Riemann tensor of gravitation from a geometrical point of view. Indeed, both are measures of the interaction between matter and the field in question. The fact the Riemann tensor contains 4 indices, while the field strength tensor contains 2 indices can be viewed as a consequence of gravity describing particles of spin-2, while electromagnetism describes particles of spin-1.

Now say that we start in the other end instead, we begin by defining a connection \( A_\mu \) that transforms according to eq. (9.9) and from that definition want to construct a comparator that transforms according to eq. (9.3). This allows to express the comparator with the following expression,

\[
W_\Gamma(x, y) = \exp \left( ie \int_y^x dz^\mu A_\mu(z) \right),
\] (9.14)

where the integration is evaluated along a path \( \Gamma \) between the two points \( x \) and \( y \). The quantity \( W_\Gamma(x, y) \) is called a **Wilson line** and has the property of being path dependent. Under the gauge transformation of \( A_\mu \) the Wilson line transforms as
9.1. Mandelstam Quantization

\[
\exp\left(ie \int_y^x dz^\mu A_\mu(z)\right) \to \exp\left(ie \int_y^x dz^\mu A_\mu(z) + i \int_y^x dz^\mu \partial_\mu \theta(z)\right) = e^{i\theta(x)} W_\Gamma(x, y) e^{-i\theta(y)},
\]

meaning the Wilson line has the desired transformation property of the comparator. Note that the transformation is independent of the path taken between the endpoints.

Consider a Wilson line where \( x = y \), in that case the line integral becomes a contour integral,

\[
W_\Gamma(x, x) = \exp\left(i e \int_\Sigma d\sigma_{\mu\nu} F_{\mu\nu}\right) = 1 + ie \int_\Sigma d\sigma_{\mu\nu} F_{\mu\nu} + O(e^2),
\]

where \( \Sigma \) is the surface enclosed by the curve \( \Gamma \). We can now explicitly see that the Wilson loop solely depends on \( F_{\mu\nu} \) and the local gauge invariance is manifest.

In the more general non-Abelian case, the Wilson line has the form

\[
W_\Gamma(x, y) = \mathbb{P}\left\{\exp\left(ie \int_y^x dz^\mu A_\mu(z)\right)\right\},
\]

where \( A_\mu(z) \equiv A_\mu^a(z)t^a \) is a non-Abelian gauge field and \( \mathbb{P} \) is the path-ordering operator [39,61]. The path-ordering operator is necessary since we are considering an exponential of matrices that do not necessarily commute at different points.

9.1. Mandelstam Quantization

Suppose that one desires a formulation of quantum field theory where the fields themselves are gauge invariant. This can be done by dressing each matter field with a Wilson line. In the case of QED this takes the form,

\[
\Psi(x|\Gamma) = \mathbb{P}\left\{\exp\left(ie \int_{-\infty}^x dz^\mu A_\mu(z)\right)\right\} \psi(x), \tag{9.19a}
\]

\[
\bar{\Psi}(x|\Gamma) = \mathbb{P}\left\{\exp\left(-ie \int_{-\infty}^x dz^\mu A_\mu(z)\right)\right\} \bar{\psi}(x), \tag{9.19b}
\]

where \( \psi(x) \) is the regular Dirac field. Note that each field is dressed with its own individual “tail”. The new variable \( \Psi(x|\Gamma) \) is manifestly gauge invariant\footnote{Given the assumption that \( \theta(x) \) vanishes at infinity.}. We can
now expand the matter field in harmonics modes and impose commutation relations between the field operators. This procedure is known as the Mandelstam quantization method and promotes the field variables to non-local path dependent variables and has been developed for both electrodynamics and gravity.\footnote{In the case of gravity the theory is coordinate-independent, since that is the analogy of gauge in gravitational theories.} The local theory of QED is converted to a global theory [20–23].

Acting on the Mandelstam field with a partial derivative gives us
\[
\partial_{\mu} \Psi(x|\Gamma) = \exp \left( i e \int_{-\infty}^{x} dz \mu A_{\mu}(z) \right) \left[ \partial_{\mu} - ie A_{\mu}(x) \right] \psi(x),
\]
thus, we see that the gauge covariant derivative, \( D_{\mu} = \partial_{\mu} - ie A_{\mu}(x) \), arises naturally in the Mandelstam quantization scheme. The full QED Lagrangian becomes
\[
\mathcal{L}_{\text{QED}} = \bar{\Psi}(x|\Gamma)(i\not{\partial} - m)\Psi(x|\Gamma) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x).
\]
Comparing to eq. (7.2) we see that the interaction term is implied implicitly due to the properties of eq. (9.20). In this formulation QED has clear analogies to gravitational theories – the interaction arises from purely geometrical arguments. Note that even if the gauge-invariant fields \( \Psi \) and \( \bar{\Psi} \) are path-dependent, the resulting Lagrangian will not carry the path-dependence.

9.2. Wilson Lines as Faddeev-Kulish Operators

This section is devoted to showing that the Wilson line ”tails” attached to each particle in the Mandelstam formalism are in fact equivalent to the soft photon clouds of the Faddeev-Kulish states. The result for QED is known [25], however the extension to perturbative quantum gravity is novel to this thesis.

9.2.1. Quantum Electrodynamics

Consider the Wilson line attached to the spinor fields in Mandelstam QED,
\[
W_{\Gamma}(x) = \mathbb{P} \left\{ \exp \left( i e \int_{-\infty}^{x} dz \mu A_{\mu}(z) \right) \right\},
\]
which extends from negative infinity to some finite point \( x \). Since the spinor fields are massive we will restrict the path \( \Gamma \) to timelike paths, thus we can replace the path-ordering operator with a time-ordering operator. The equations of motion for the gauge field \( A_{\mu} \) with a source \( J_{\mu} \) are
\[
\Box A_{\mu}(x) = J_{\mu}(x).
\]
We can therefore decompose the gauge field into an incoming homogeneous part and a source dependent part expressed in terms of a retarded Green’s function,

\[ A_\mu(x) = A_\mu^{in}(x) + \int d^4y \, G_{ret}(x-y) J_\mu(y), \]  

(9.24)

where \( \Box G(x-y) = \delta^4(x-y) \) and \( \Box A_\mu^{in}(x) = 0 \) by definition, which reduces the above expression to eq. (9.23). We have made the choice to express the field in terms of the incoming homogeneous field and the retarded Green’s function, while we likewise could have chosen to express \( A_\mu \) in terms of the outgoing homogeneous field combined with the advanced Green’s function. After inserting the above expression into eq. (9.22) we obtain

\[ W_\Gamma(x) = T \left\{ \exp \left( ie \int_{-\infty}^{x} dz^\mu A_\mu^{in}(z) + ie \int_{-\infty}^{x} dy^\mu \int d^4z \, G_{ret}(x-y) J_\mu(y) \right) \right\} \]

\[ = \exp \left( ie \int_{-\infty}^{x} dz^\mu A_\mu^{in}(z) \right) \times \exp(-i\Phi(x)), \]

(9.25)

where the time-ordering operator gave us a phase factor, exactly cancelling the Coulomb phase from eq. (3.11) found by Weinberg [25, 43]. Henceforth, we will focus on the homogeneous part. We can parametrize the path in terms of the proper time \( \tau \). For a \( \tau \)-independent four-velocity \( u^\mu \). Using \( z^\mu = x^\mu + \tau u^\mu \), we obtain

\[ ie \int_{-\infty}^{x} dz^\mu A_\mu^{in}(z) = ie \int_{-\infty}^{0} dt \frac{dz^\mu}{d\tau} A_\mu^{in}(x + \tau u) \]

\[ = ie \int_{-\infty}^{0} d\tau \, u^\mu A_\mu^{in}(x + \tau u) \]

\[ = ie \int d^3k \int_{-\infty}^{0} d\tau \, u^\mu \left[ a_\mu^1(k)e^{-ik(x+\tau u)} + a_\mu^\dagger(k)e^{ik(x+\tau u)} \right], \]

(9.26)

where we have expanded in harmonic modes. Using the boundary condition from [2]

\[ \int_{-\infty}^{0} d\tau e^{iuk\tau} = \frac{1}{iu_k}, \]

we find that

\[ ie \int_{-\infty}^{x} dz^\mu A_\mu^{in}(x) = -e \int d^3k \frac{p^\mu}{pk} \left[ a_\mu^1(k)e^{-ikx} - a_\mu(k)e^{ikx} \right], \]

(9.27)

where we used that \( u_\mu = \frac{p_\mu}{pk} \). Comparing with radiation operator \( W(t) \) of the original Faddeev-Kulish states, eq. (7.25a),

\[ W(t) = \exp \left\{ e \int d^3k \, d^3p \, \frac{p^\mu}{pk} \left( a_\mu^1(k)e^{-i\frac{p\cdot x}{pt}} - a_\mu(k)e^{i\frac{p\cdot x}{pt}} \right) \rho(p) \right\}, \]

(9.28)
we can see several similarities, but also differences. First of all, note the different forms of the exponential. In the end, this difference is not important, since the Faddeev-Kulish states are modified to commute with the Gupta-Bleuler condition by replacing the exponential with the infrared function $f^\mu(p, k)$,

$$\frac{p^\mu}{p^k} e^{-i \frac{e}{m} t} \rightarrow f^\mu(p, k).$$  (9.29)

More importantly, the Wilson line becomes an operator that creates a cloud of coherent photons, but only around one individual electron, not an operator acting on general multiparticle states. This can be seen by noting the absence of a charge density operator in eq. (9.27), since the starting point was the Mandelstam quantization scheme where we attributed one “tail” for each field. This is equivalent to substituting

$$\int \tilde{d}^3 p \rho(p) \rightarrow \pm e,$$  (9.30)

in eq. (7.25a), where $\pm e$ is the charge of a specific field with a fixed momentum, which in eq. (9.27) takes the form of $-e$ and by setting the momentum to a fixed $p^\mu$. Recall the Faddeev-Kulish dressing operator, eq. (7.43), that dressed a specific particle of momentum $p^\mu$ with a cloud of soft photons,

$$R_f(p) = e \int \tilde{d}^3 k \left( f^\mu(p, k)a^\dagger_\mu(k) - f^{\mu*}(p, k)a_\mu(k) \right).$$  (9.31)

Assuming that the Wilson lines are modified with the infrared functions, we can formulate an equivalence between the two formulations,

$$e^{R_f(p)} \psi(x) = \Psi(x|\Gamma_0),$$  (9.32)

where the path $\Gamma_0$ is the one specified in above, starting at $-\infty$ and going to some finite $x$. Similarly, in the many-particle case we have

$$e^{R_f} \left[ \psi_1(x_1) \cdots \psi_n(x_n) \right] = \Psi_1(x_1|\Gamma_0) \cdots \Psi_n(x_n|\Gamma_0).$$  (9.33)

We now have an equality between the original asymptotic analysis of Faddeev-Kulish and purely geometrical approach using the Mandelstam quantization scheme and Wilson lines. An additional feature of the Wilson line approach is that it can be used to study the behaviour of Faddeev-Kulish states in the Rindler and Schwarzschild spacetimes, including the study of the soft hair of black holes [26].

### 9.2.2. Gravitational Wilson Line

We would like to extend the above formalism to construct the FK operator of perturbative quantum gravity from the gravitational Wilson line. In the eikonal approximation of gravity one can express the Wilson line as [27–33]
\[ W_\Gamma(\tau_1, \tau_2) = \mathbb{P} \left\{ \exp \left( i \frac{\kappa}{2} \int_{\tau_2}^{\tau_1} d\tau \frac{p^\mu p'^\nu}{m} h_{\mu\nu} \right) \right\} , \]  
where our expression differs from the one in the references by a factor of \( \frac{1}{2} \) due to our normalization choice. The exponent can be rewritten as

\[ i \frac{\kappa}{2} \int_{\tau_2}^{\tau_1} d\tau \frac{p^\mu p'^\nu}{m} h_{\mu\nu} = i \frac{\kappa}{2} \int d^3x' \sqrt{-g} T^{\mu\nu} h_{\mu\nu} , \]  
where \( T^{\mu\nu} \) is the stress-energy tensor of a particle following a geodesic

\[ T^{\mu\nu}(x) = \int_{\tau_2}^{\tau_1} d\tau \frac{p^\mu p'^\nu}{m} \delta^3(x - x'(\tau)) . \]  
It should be noted that the gravitational Wilson line is related to the Christoffel symbol in a non-trivial way \([29,30]\). Using the Mandelstam quantization scheme we can attach this Wilson line to a scalar particle,

\[ \Phi(x|\Gamma) = \mathbb{P} \left\{ \exp \left( i \frac{\kappa}{2} \int_{-\infty}^{0} d\tau \frac{p^\mu p'^\nu}{m} h_{\mu\nu}(x + u\tau) \right) \right\} \phi(x) . \]  
We are now in a position to interpret this Wilson line as the interaction between a scalar particle and soft gravitons as it follows a geodesic specified by \( \Gamma \). According to the Mandelstam quantization scheme we set \( \tau_1 = 0 \) and \( \tau_2 = -\infty \). Using eq. (B.8) we can expand the metric perturbation in harmonics modes, the exponent becomes

\[ i \frac{\kappa}{2} \int_{-\infty}^{0} d\tau \frac{p^\mu p'^\nu}{m} h_{\mu\nu}(x + u\tau) = i \frac{\kappa}{2} \int d^3k \int_{-\infty}^{0} d\tau \frac{p^\mu p'^\nu}{m} \left[ a_{\mu\nu}(k) e^{ik(x+u\tau)} + a^\dagger_{\mu\nu}(k) e^{-ik(x+u\tau)} \right] \]

\[ = i \frac{\kappa}{2} \int d^3k \frac{p^\mu p'^\nu}{p k} \left[ a_{\mu\nu}(k) e^{ikx} - a^\dagger_{\mu\nu}(k) e^{-ikx} \right] , \]  
where we again have used the boundary condition \( \int_{-\infty}^{0} d\tau e^{iku} = \frac{1}{iu} \) from \([2]\) and assumed constant four-velocity. This is the expression for the graviton cloud associated with scalar particle of momentum \( p \). Similar to the case of QED we note the absence of the \( c_{\mu\nu} \)-tensor, the function \( \phi(p,k) \) and the matter density operator. These factors can either be put in by hand following the reasoning above, or obtained from a generalization of the Wilson line construction \([62]\).
10. Conclusions

In this thesis we have presented an encompassing review of the relation between infrared divergences, dressed matter states and asymptotic symmetries. The argument that was presented is that the Faddeev-Kulish formalism provides a more fundamental solution of IR divergences than the Bloch-Nordsieck method. Not only is the $S$-matrix finite and well-defined in the FK approach, but the relation between the asymptotic states of QED and PQG and the asymptotic symmetries become apparent. We have computed the asymptotic charges and showed that they are conserved in all physical scattering processes. This implies that there is no transition between different LGT/BMS sectors.

The resemblance of QED and PQG in the IR limit is remarkable, given the distinct high energy behaviour of the theories. The future study of the similarities between gauge and gravitational theories is an important step towards coming closer to a consistent quantum theory of gravity. However, even more astounding are the almost identical properties of large gauge transformations and BMS supertranslations. The LGT are $U(1)$ transformation that are non-vanishing at infinity, meaning that the class of allowed transformation parameters is expanded. The BMS transformations are asymptotic symmetries of spacetime itself, and have a qualitative difference from the Poincaré group. Indeed, both are asymptotic symmetries of the corresponding theory, however, the nature of the symmetries is truly dissimilar.

Perhaps the cause of the similarity arises from the common geometrical interpretation of the Faddeev-Kulish states derived from Wilson lines. In order to study the soft hair both at infinity and near the horizon, Choi & Akhoury [26] generalized the Wilson line approach to the Rindler and Schwarzschild spacetimes. Using FK states they found support for the claim that Schwarzschild black holes carry soft hair. The authors listed the extension of the analysis to perturbative quantum gravity as a natural step forward. In this thesis we provide a derivation of the gravitational FK states derived from the Wilson lines in Minkowski spacetime. The formulation of gravitational FK states in the Rindler, Schwarzschild and other spacetimes remains.

As mentioned previously, several difficulties arise as one tries to formulate quantum field theory on background that are not asymptotically flat, or are significantly curved in a large spacetime region. It is an interesting question whether this problem can be solved using Faddeev-Kulish states. Instead of deriving the gravitational FK states from an interaction Lagrangian, is it possible to deduce the behaviour of large scale interactions starting from the FK states? Perhaps the Wilson line approach could shed
light on how a FK state in, for example, de Sitter space would look and be used to formulate QFT in said spacetime. To the author’s knowledge, no such attempts have been made.

In the original derivation of the dressed matter states by Kulish & Faddeev, they relied solely on the asymptotic analysis of the interaction Lagrangian density. They did not employ any arguments from the geometric viewpoint, nor did the analysis have any foundation in the gauge character of the theory. Consequently, the relation between the Faddeev-Kulish states and asymptotic symmetries was seemingly coincidental. The Wilson line approach allows for a geometrical analysis where the gauge independence of the theory is central, and the relation to large gauge transformations are made more apparent. However, one problem remains – the operator obtained through both the asymptotic analysis and the Wilson line approach gives rise to unphysical degrees of freedom of the bosonic field. One would like to formulate a derivation starting with gauge invariant states that directly leads to the physical radiation operator, without the requirement to make any modifications “by hand”. It seems that such a derivation is possible [62], but has not been pursued in this thesis.

It has come to the author’s attention that a very recent paper by Hirai & Sugishita [63] presents one such formulation. They argue that the free Gupta-Bleuler condition is not the appropriate gauge condition that should be imposed on dressed matter states. Instead they formulate a gauge invariance condition on the asymptotic states based on Gauss’ law. Using the BRST quantization scheme Hirai & Sugishita formulate a method that does not introduce unphysical degrees of freedom and allows for a variety of dresses, for which the (unmodified) FK dressing is one of them. However, the paper is inconclusive concerning which of these dresses that are IR finite and their exact relation to asymptotic symmetry. Further studies that compare the BRST approach to the Wilson line approach are of interest.

Lastly, an extension of the dressed matter formalism to Yang-Mills theories would be of considerable importance, as it could shed light on the asymptotic symmetries of those theories. Their relation to the IR finiteness of the corresponding Faddeev-Kulish states could further substantiate the gauge/gravity duality.
Appendices

A. Additional Theorems and Lemmas

This appendix contains theorems and lemmas that did not have a natural place in the main body of the thesis. Proofs not will be provided for all theorems.

A.1. Noether’s Theorem

Theorem A.1. Every continuous transformation under which the action is invariant will give rise to an associated conserved current,

\[ \partial_\mu J^\mu_\epsilon = 0, \]  
\[ (A.1) \]

where \( \epsilon(x) \) is the transformation parameter and \( J^\mu_\epsilon \) is the Noether current. The current is defined by

\[ J^\mu_\epsilon = \frac{\partial L}{\partial (\partial_\mu \phi_a)} \delta_\epsilon \phi_a + L \delta x^\mu. \]  
\[ (A.2) \]

Here \( \delta_\epsilon = \phi_a'(x') - \phi_a(x) \), where the subscript \( a \) denotes the field, and \( \delta x^\mu = x'^\mu - x^\mu \). The variation of the action is given by

\[ \delta S = \int d^4x \, \partial_\mu J^\mu_\epsilon = 0. \]  
\[ (A.3) \]

This implies the local conservation of having a corresponding charge,

\[ Q_\epsilon = \int d^3x \, J^0_\epsilon. \]  
\[ (A.4) \]

Such a transformation is called a symmetry [64].

A.2. The Riemann-Lebesgue Lemma

Lemma A.2.

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} dx \, f(x)e^{\pm inx} = 0, \]  
\[ (A.5) \]

for all \( f \in L^1(\mathbb{R}) \) [65].
A.3. Baker-Campbell-Hausdorff Formula

Lemma A.3. For two operators $A$ and $B$, the following holds

$$e^A e^B = e^{A + B + \frac{1}{2} [A, B]},$$

(A.6)
given that $[A, [A, B]] = [B, [A, B]] = 0$ [65].

A.4. Determinant of Variation

Lemma A.4. The determinant of any matrix $M$ together with a small variation $\delta M$ will have the form

$$\det(M + \delta M) \approx \det M \left(1 + M^{-1} \delta M\right).$$

(A.7)

Proof. For any matrix $M$ we have that $\det M = e^{\text{tr} \ln M}$, thus

$$\det(M + \delta M) = e^{\text{tr} \ln (M + \delta M)} = e^{\text{tr} \ln (M(1 + M^{-1} \delta M))}$$

$$= e^{\text{tr} \ln M} e^{\text{tr} \ln (1 + M^{-1} \delta M)} = \det M e^{\text{tr} \ln (1 + M^{-1} \delta M)}$$

$$\approx \det M e^{\text{tr} (M^{-1} \delta M)} \approx \det M \left(1 + M^{-1} \delta M\right),$$

(A.8)

which is the desired result. \qed

A.5. Special Linear Group

Definition A.5 (Special linear group). Consider the set $\{A\}$ of $(n \times n)$ invertible matrices with entries from a field $F$. Under the matrix multiplication this set forms a group, the general linear group $\text{GL}(n, F)$. The subgroup where $\det(A) = 1$ is called the special linear group, $\text{SL}(n, F)$. The factor group of $\text{SL}(n, F)$ modulo its center is called the projective special linear group, $\text{PSL}(n, F) \equiv \text{SL}(n, F)/\{\pm 1\}$ [38].

Theorem A.6. $\text{PSL}(2, \mathbb{C})$ is isomorphic to the Lorentz group, $\text{SO}(1, 3)$.

Proof. For each point in spacetime, $(x_0, x_1, x_2, x_3)$, construct the following $(2 \times 2)$ hermitian matrix,

$$X = \begin{pmatrix} -x_0 - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & -x_0 + x_1 \end{pmatrix}.$$  

(A.9)

Note that $\det(X) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_\mu x^\mu$ is the Minkowski line element in Cartesian coordinates. Now consider any matrix $A \in \text{SL}(2, \mathbb{C})$ acting on $X$, 


$X \rightarrow X' = A^\dagger X A \quad \forall \ A \in \text{SL}(2, \mathbb{C}).$ \hspace{1cm} (A.10)

However, from definition we know that $\det(A) = 1$, meaning

$$\det(X') = \det(A^\dagger X A) = \det(A^\dagger) \det(X) \det(A) = \det(X).$$ \hspace{1cm} (A.11)

Thus, the Minkowski line element is invariant under $\text{SL}(2, \mathbb{C})$ transformations. It then follows that $\text{SL}(2, \mathbb{C})$ is homomorphic to the Lorentz group. By the first isomorphism theorem \[38\] we then have that $\text{PSL}(2, \mathbb{C}) \cong \text{SO}(1, 3)$. Since the center has index 2, $\text{SL}(2, \mathbb{C})$ is called a \textbf{double covering} of $\text{SO}(1, 3)$.
Appendix B

B. Canonical Commutation Relations

This appendix contains the canonical (anti)commutation relations of scalars, spinors, photons and gravitons.

B.1. Real Scalar Field

The real scalar field can be expanded in terms of creation and annihilation operators

\[ \phi(x) = \int d^3p \left( b(p)e^{ipx} + b^\dagger(p)e^{-ipx} \right), \]  

(B.1)

The creation and annihilation operators, \( b(p) \) and \( b^\dagger(p) \) satisfy the commutation relations

\[ \left[ b(p), b^\dagger(p') \right] = (2\pi)^3(2\omega_p)\delta^3(p - p'). \]  

(B.2)

B.2. The Dirac Field

The Dirac field can be expanded in terms of creation and annihilation operators,

\[ \psi(x) = \int d^3p \left( c_r(p)u_r(p)e^{ipx} + d^\dagger_r(p)v_r(p)e^{-ipx} \right), \]  

(B.3a)

\[ \bar{\psi}(x) = \int d^3p \left( c^\dagger_r(p)\bar{u}_r(p)e^{ipx} + d_r(p)\bar{v}_r(p)e^{-ipx} \right), \]  

(B.3b)

where \( c^\dagger(p) \) is the electron creation operator and \( d^\dagger(p) \) is the positron creation operator, while \( u_r(p) \) and \( v_r(p) \) are constant spinors satisfying the equations [66]

\[ (\not{p} + m)u_r(p) = \bar{u}_r(p)(\not{p} + m) = 0 \]  

(B.4a)

\[ (-\not{p} + m)v_r(p) = \bar{v}_r(p)(\not{p} + m) = 0 \]  

(B.4b)

\[ \bar{u}_r(p)u_s(p) = -\bar{v}_r(p)v_r(p) = \delta_{rs} \]  

(B.4c)

\[ \bar{u}_r(p)v_s(p) = 0. \]  

(B.4d)

The anticommutation relations for the creation and annihilation operators are

\[ \left\{ c(p), c^\dagger(p') \right\} = (2\pi)^3(2\omega_p)\delta^3(p - p'), \]  

(B.5a)

\[ \left\{ d(p), d^\dagger(p') \right\} = (2\pi)^3(2\omega_p)\delta^3(p - p'). \]  

(B.5b)
B.3. The Maxwell Field

The Maxwell field can be expanded in terms of creation and annihilation operators

\[ A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \left( a_\mu(k)e^{ikx} + a_\mu^\dagger(k)e^{-ikx} \right). \]  

(B.6)

The photon creation and annihilation operators, \( a_\mu^\dagger(k) \) and \( a_\mu(k) \), satisfy the commutation relations

\[ \left[ a_\mu(k), a_\nu^\dagger(k') \right] = \eta_{\mu\nu} (2\pi)^3 (2\omega_k) \delta^3(k-k'), \]  

(B.7)

where \( \eta_{\mu\nu} \) is the Minkowski metric and \( a_\mu(k) \equiv \epsilon_\mu^r(k)a_r(k) \) where \( \epsilon_\mu^r(k) \) is the polarization tensor. Note that in order to avoid the presence of ghosts we must chose a gauge that is compatible with the Gupta-Blueler condition.

B.4. The Gravitational Field

The metric perturbation can be expanded in terms of creation and annihilation operators

\[ h_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3} \left( a_{\mu\nu}(k)e^{ikx} + a_{\mu\nu}^\dagger(k)e^{-ikx} \right). \]  

(B.8)

The graviton creation and annihilation operators, \( a_{\mu\nu}^\dagger(k) \) and \( a_{\mu\nu}(k) \), satisfy the commutation relations

\[ \left[ a_{\mu\nu}(k), a_{\rho\sigma}^\dagger(k') \right] = \frac{1}{2} I_{\mu\nu\rho\sigma} (2\pi)^3 (2\omega_k) \delta^3(k-k'), \]  

(B.9)

where \( a_{\mu\nu}(k) \equiv \epsilon_{\mu\nu}^r(k)a_r(k) \) where \( \epsilon_{\mu\nu}^r(k) \) is the polarization tensor and

\[ I_{\mu\nu\rho\sigma} = \eta_\mu^\rho \eta_\nu^\sigma + \eta_\mu^\sigma \eta_\nu^\rho - \eta_\mu^\nu \eta_\rho^\sigma. \]  

(B.10)

Note that in order to avoid the presence of ghosts we must chose a gauge that is compatible with the Gupta-Blueler condition.
References


