Journal of Physics: Conference Series **1468** (2020) 012186 doi:10.1088/1742-6596/1468/1/012186

Sum rules and asymptotic behaviors of neutrino mixing and oscillations in matter

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Abstract. Similar to the case in vacuum, it is straightforward to describe neutrino oscillations in matter with the effective lepton flavor mixing matrix $\widetilde{U}$ and neutrino mass-squared differences $\widetilde{\Delta}_{ji} \equiv \widetilde{m}_j^2 - \widetilde{m}_i^2$ $(i, j = 1, 2, 3)$. By calculating two sets of sum rules of $\widetilde{U}$ and $\widetilde{\Delta}_{ji}$, we have derived exact expressions of $|\widetilde{U}_{\alpha i}|^2$ and $\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}$ (for $\alpha, \beta = e, \mu, \tau$ and $i = 1, 2, 3$) and discussed the asymptotic behaviors of $|\widetilde{U}_{\alpha i}|^2$ and $\widetilde{\Delta}_{ji}$ in very dense matter (i.e., in the limit of the matter parameter $A = 2\sqrt{2}G_{\rm F}N_e E$ approaching infinity).

In 1978, L. Wolfenstein pointed out that the coherent forward scattering of neutrinos with electrons and nucleons through charged- and neutral-current interactions must be taken into account when considering the neutrino oscillations in matter [1]. This can cause big (resonance) amplification of the effective flavor mixing angles in matter even when their counterparts in vacuum are small [2, 3]. Such matter effects were subsequently demonstrated to play an important role in solving the long-standing solar neutrino problem and in understanding the data of accelerator neutrino oscillation experiments [4]. With the advent of the era of accurate measurements of neutrino oscillation parameters, a lot of efforts have been made to formulate matter effects on lepton flavor mixing and neutrino oscillations recently [5, 6, 7, 8, 9, 10, 11, 12, 13].

With the help of two sets of sum rules of the effective lepton flavor mixing matrix $\widetilde{U}$ and effective neutrino mass-squared differences $\widetilde{\Delta}_{ji} \equiv \widetilde{m}_j^2 - \widetilde{m}_i^2$ in matter, we are going to derive the exact expressions for the nine moduli of $\widetilde{U}$ and nine sides of the effective Dirac unitarity triangles in the complex plane. Compared with the previous formulas of this kind [14], our results are independent of the less intuitive terms $\widetilde{m}_j^2 - m_i^2$ with m_i and $\widetilde{m}_i$ denoting respectively for neutrino masses in vacuum and their effective counterparts in matter (for $i, j = 1, 2, 3$). Furthermore, a comprehensive and analytical analysis of the asymptotic behaviors of $|\widetilde{U}_{\alpha i}|^2$ and $\widetilde{\Delta}_{ji}$ (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) in very dense matter (i.e., in the limit of the matter parameter $A = 2\sqrt{2}G_{\rm F}N_e E$ approaching infinity) is performed, which is at least conceptually interesting [13].

In the standard three-flavor scenario, the Hamiltonian responsible for the propagation of neutrinos in matter can be expressed as

$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31}\end{pmatrix}U^{\dagger} + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\right]$$
$$\equiv \frac{1}{2E}\left[\widetilde{U}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \widetilde{\Delta}_{21} & 0 \\ 0 & 0 & \widetilde{\Delta}_{31}\end{pmatrix}\widetilde{U}^{\dagger} + BI\right] , \quad (1)$$

where I stands for a 3×3 identity matrix, $A = 2EV_{\rm cc}$ and $B = \widetilde{m}_1^2 - m_1^2 - 2EV_{\rm nc}$ with $V_{\rm cc} = \sqrt{2}G_{\rm F}N_e$ and $V_{\rm nc} = -G_{\rm F}N_n/\sqrt{2}$ being matter potential terms arising respectively form weak charged- and neutral-current interactions of neutrinos with electrons and neutrons in matter. By taking the trace of $\mathcal{H}_{\rm m}$, we get $B = (\Delta_{21}+\Delta_{31}+A-\widetilde{\Delta}_{21}-\widetilde{\Delta}_{31})/3$. According to the analytical expressions of $\widetilde{m}_i^2$ (for $i=1,2,3$) given in the literature [15, 16], we have

$$\widetilde{\Delta}_{21} = \frac{2}{3}\sqrt{x^2-3y}\sqrt{3\left(1-z^2\right)} \ ,$$
$$\widetilde{\Delta}_{31} = \frac{1}{3}\sqrt{x^2-3y}\left[3z+\sqrt{3\left(1-z^2\right)}\right] \ , \quad (2)$$
$$B = \frac{1}{3}x - \frac{1}{3}\sqrt{x^2-3y}\left[z+\sqrt{3\left(1-z^2\right)}\right]$$

in the case of normal mass ordering (NMO) with $m_1 < m_2 < m_3$, where

$$x = \Delta_{21}+\Delta_{31}+A \ ,$$
$$y = \Delta_{21}\Delta_{31} + A\left[\Delta_{21}\left(1-|U_{e2}|^2\right)+\Delta_{31}\left(1-|U_{e3}|^2\right)\right] \ , \quad (3)$$
$$z = \cos\left[\frac{1}{3}\arccos\frac{2x^3-9xy+27A\Delta_{21}\Delta_{31}|U_{e1}|^2}{2\sqrt{\left(x^2-3y\right)^3}}\right] \ .$$

Given Eq. (1), a direct calculation of the nine elements of $\mathcal{H}_{\rm m}$ and $\mathcal{H}_{\rm m}^2$ leads to two sets of sum rules. These sum rules, together with the unitarity conditions of U and $\widetilde{U}$, constitute a full set of linear equations of $\widetilde{U}_{\alpha 1}\widetilde{U}^*_{\beta 1}$, $\widetilde{U}_{\alpha 2}\widetilde{U}^*_{\beta 2}$ and $\widetilde{U}_{\alpha 3}\widetilde{U}^*_{\beta 3}$:

$$\sum_{i=1}^{3}\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i} = \sum_{i=1}^{3}U_{\alpha i}U^*_{\beta i} = \delta_{\alpha\beta}$$
$$\sum_{i=1}^{3}\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}\widetilde{\Delta}_{i1} = \sum_{i=1}^{3}U_{\alpha i}U^*_{\beta i}\Delta_{i1} + A\delta_{e\alpha}\delta_{e\beta} - B\delta_{\alpha\beta} \ , \quad (4)$$
$$\sum_{i=1}^{3}\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}\widetilde{\Delta}_{i1}(\widetilde{\Delta}_{i1}+2B) = \sum_{i=1}^{3}U_{\alpha i}U^*_{\beta i}\Delta_{i1}\left[\Delta_{i1}+A(\delta_{e\alpha}+\delta_{e\beta})\right] + A^2\delta_{e\alpha}\delta_{e\beta} - B^2\delta_{\alpha\beta} \ ,$$

where the Greek and Latin subscripts run over (e,μ,τ) and $(1,2,3)$, respectively. Solving Eq. (4) in the case $\alpha=\beta$ and $\alpha\neq\beta$, we can get the exact expressions of the nine moduli $|\widetilde{U}_{\alpha i}|^2$ and nine sides of the Dirac unitarity triangles $\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}$ in terms of $U_{\alpha i}U^*_{\beta i}$, Δ_{ji}, $\widetilde{\Delta}_{ji}$ and A. Thus we get rid of the less intuitive terms $\widetilde{m}_j^2 - m_i^2$ in which the effective neutrino masses $\widetilde{m}_j$ do not have a definite physical meaning. Note that the above formulas only apply to the neutrino beam with normal mass ordering. For the inverted mass ordering (IMO) case with $m_3 < m_1 < m_2$, we need

Table 1. The analytical expressions of $\widetilde{\Delta}_{ji}$ (for $ji = 21, 31$) and $\widetilde{U}$ in the $A \to \infty$ limit.

	(NMO, ν)	(NMO, $\overline{\nu}$)
$\widetilde{\Delta}_{21}$	$\Delta_{31}(1 - \lvert U_{e3}\rvert^2) - \Delta_{21}\lvert U_{e1}\rvert^2$	A
$\widetilde{\Delta}_{31}$	A	A
$\widetilde{U}$	$\begin{pmatrix} 0 & 0 & 1 \\ \sqrt{1-\lvert U_{\mu3}\rvert^2} & \lvert U_{\mu3}\rvert & 0 \\ -\lvert U_{\mu3}\rvert & \sqrt{1-\lvert U_{\mu3}\rvert^2} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-\lvert U_{\mu3}\rvert^2} & \lvert U_{\mu3}\rvert \\ 0 & -\lvert U_{\mu3}\rvert & \sqrt{1-\lvert U_{\mu3}\rvert^2} \end{pmatrix}$

to change $\widetilde{\Delta}_{21}$, $\widetilde{\Delta}_{31}$ and B in Eq. (2) to their counterparts in this case [13]. When it comes to an antineutrino beam, one should make the replacements $U \to U^*$, $V_{\rm cc} \to -V_{\rm cc}$ ($A \to -A$) and $V_{\rm nc} \to -V_{\rm nc}$.

Now let us discuss the the asymptotic behaviors of $|\widetilde{U}_{\alpha i}|^2$ in the $A \to \infty$ limit using their exact formulas derived above. In Table 1, we show the analytical expressions of $\widetilde{\Delta}_{ji}$ (for $ji = 21, 31$) and $\widetilde{U}$ in this extreme case, where a neutrino beam with the normal mass ordering (NMO, ν) and an antineutrino beam with the normal mass ordering (NMO, $\overline{\nu}$) are considered, respectively. We find that only one degree of freedom is needed to describe $\widetilde{U}$ in the $A \to \infty$ limit. Considering the standard parametrization of $\widetilde{U}$ in terms of three effective mixing angles $(\widetilde{\theta}_{12}, \widetilde{\theta}_{13}, \widetilde{\theta}_{23})$ and the effective Dirac CP phase $(\widetilde{\delta})$, it is always possible to remove $\widetilde{\delta}$ from $\widetilde{U}$ if the $A \to \infty$ limit is taken, and then we are left with a trivial flavor mixing angle (e.g., $\widetilde{\theta}_{13} = \pi/2$ or 0) and a nontrivial flavor mixing angle which is neither $\widetilde{\theta}_{12}$ nor $\widetilde{\theta}_{23}$. This kind of subtle parameter redundancy was not noticed in the previous papers (see, e.g., Refs. [10, 12]), where specific but misleading values of $\widetilde{\delta}$, $\widetilde{\theta}_{12}$ and $\widetilde{\theta}_{23}$ have been obtained in the $A \to \infty$ limit. The other two cases — a neutrino beam with the inverted mass ordering (IMO, ν) and an antineutrino beam with the inverted mass ordering (IMO, $\overline{\nu}$) can similarly be discussed [13]. In Fig. 1, we illustrated how each of the nine effective quantities $|\widetilde{U}_{\alpha i}|^2$ evolves with the matter parameter A in the two case (NMO, ν) and (NMO, $\overline{\nu}$). One can see that $\widetilde{U}$ asymptotically approaches a constant matrix in the $A \to \infty$ limit when taking the value of A larger than 10^{-2} eV^2. This means that we can use one degree of freedom to approximately describe the flavor mixing effects in this range of A.

In summary, we have established some more intuitive formulas for both nine $|\widetilde{U}_{\alpha i}|^2$ and nine $\widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}$, by which we have analytically unravelled the asymptotic behaviors of $|\widetilde{U}_{\alpha i}|^2$ and $\widetilde{\Delta}_{ji}$ in very dense matter for the first time. We conclude that $\widetilde{U}$ contains only a single degree of freedom in the $A \to \infty$ limit. In this extreme case, we get trivial $\widetilde{\theta}_{13}$ and no CP violation in neutrino oscillations; and thus it is meaningless to discuss the individual values of $\widetilde{\theta}_{12}$ and $\widetilde{\theta}_{23}$.

This work was supported in part by the National Natural Science Foundation of China under grant No. 11775231 and grant No. 11835013.

References

[1] Wolfenstein L 1978 *Phys. Rev.* D **17** 2369
[2] Mikheev S P and Smirnov A Y 1985 *Sov. J. Nucl. Phys.* **42** 913 [Yad. Fiz. **42**, 1441 (1985)]
[3] Mikheev S P and Smirnov A Y 1986 *Nuovo Cim.* C **9** 17

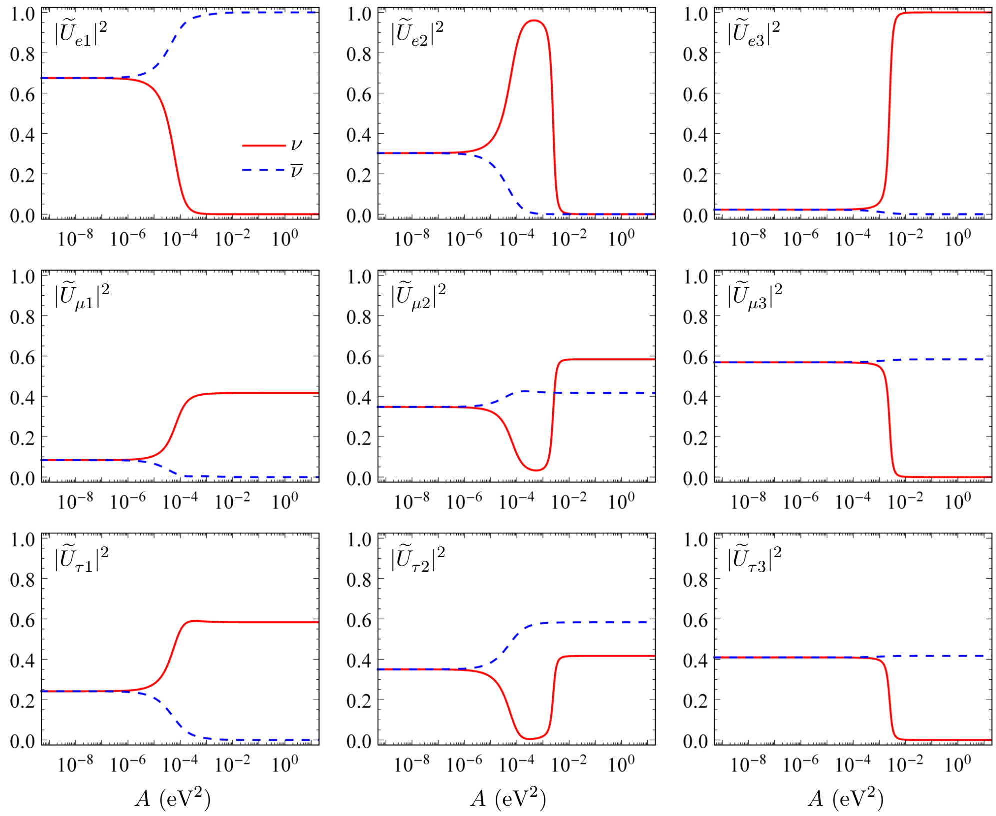

Figure 1. The evolution of $|\widetilde{U}_{\alpha i}|^2$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) with the matter parameter A in the normal neutrino mass ordering case, where the best-fit values of six neutrino oscillation parameters have been input [17].

[4] Tanabashi M *et al* [Particle Data Group] 2018 *Phys. Rev.* D **98** 030001
[5] Li Y F, Wang Y and Xing Z Z 2016 *Chin. Phys.* C **40** 091001
[6] Xing Z z and Zhu J y 2016 *JHEP* **1607** 011
[7] Li Y F, Zhang J, Zhou S and Zhu J y 2016 *JHEP* **1612** 109
[8] Chiu S H and Kuo T K 2018 *Phys. Rev.* D **97** 055026
[9] Li Y F, Xing Z z and Zhu J y 2018 *Phys. Lett.* B **782** 578
[10] Xing Z z, Zhou S and Zhou Y L 2018 *JHEP* **1805** 015
[11] Huang G y, Liu J H and Zhou S 2018 *Nucl. Phys.* B **931** 324
[12] Wang X and Zhou S 2019 *JHEP* **1905** 035
[13] Xing Z Z and Zhu J Y 2019 *Nucl. Phys.* B **949** 114803
[14] Xing Z z 2000 *Phys. Lett.* B **487** 327
[15] Barger V D, Whisnant K, Pakvasa S and Phillips R J N 1980 *Phys. Rev.* D **22**, 2718
[16] Zaglauer H W and Schwarzer K H 1988 *Z. Phys. C* **40** 273
[17] Esteban I, Gonzalez-Garcia M C, Hernandez-Cabezudo A, Maltoni M and Schwetz T 2019 *JHEP* **1901** 106