Dynamical Casimir effect: some theoretical aspects

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Abstract.
This is a brief description of some recent achievements in the theory of dynamical Casimir effect, mainly in connection with the experiment which is under preparation in the University of Padua. The first part of this paper is devoted to the theory of quantum damped oscillator with arbitrary time dependence of the frequency and damping coefficient. New results for the mean number of created photons, its variance and photon distribution function are given. The second part is devoted to calculations of the time-dependent shift of resonance frequency of an electromagnetic cavity due to strong variations of dielectric properties in a thin layer near an ideally conducting wall. A simple analytical formula for this shift is derived. It generalizes the known Schwinger–Bethe–Casimir result. The influence of different parameters on the photon generation rate is discussed. A brief review of recent publications on the subject is also included.

1. Introduction

The so-called *Dynamical Casimir Effect* (DCE), i.e., a generation of photons from vacuum due to the motion of uncharged boundaries, was a subject of numerous theoretical studies for almost 40 years. An extensive list of publications until 2000 can be found in the review [1], whereas the papers published in the period from 2001 to 2004 were briefly discussed in [2]. Also, the status of the research in this area by 2005 was demonstrated in several papers published in the special issue of Journal of Optics B [3]. Therefore here I shall concentrate mainly on the publications from 2005 to 2008 and main results obtained for this period.

Among the most important publications of the past years I would like to mark out the papers [4–8] containing different concrete experimental proposals. The scheme of [6, 7] is based on the suggestion (formulated for the first time in [9, 10]) to excite true surface vibrations of a cavity wall in the GHz band and to use a beam of atoms passing through the cavity as a detector. Another idea, proposed by the MIR group of the university of Padua [4, 5], is to simulate a motion of a boundary, using an effective electron-hole 'plasma mirror', created periodically on the surface of a semiconductor slab by illuminating it with a sequence of short laser pulses. If the interval between pulses exceeds the recombination time of carriers in the semiconductor, a highly conducting layer will periodically appear and disappear on the surface of the semiconductor film, thus simulating periodical displacements of the boundary. A different scheme of simulating the dynamical Casimir effect, where periodical changes of the cavity eigenfrequency can be achieved by changing the surface impedance of a *superconducting film* illuminated by laser pulses, was proposed recently in [8]. However, it seems that the scheme of [4, 5] is the most promising one from the point of view of reaching the result in the nearest future. Therefore I concentrate only on this scheme.

It is worth remembering that the history of the dynamical Casimir effect began almost 40 years ago, when Moore showed that the motion of ideal boundaries can result in creation of quanta of the electromagnetic field from the initial vacuum state [11]. But he concluded that the effect should be extremely small, if the velocities of boundaries are much less than the velocity of light. However, even earlier, a possibility of a significant amplification of *classical* electromagnetic fields inside cavities with *oscillating* boundaries under the conditions of parametric resonance was pointed out for the first time by Askar'yan in 1962 [12]. Later, a possibility of enhancement of vacuum (zero point) fluctuations under the conditions of resonance between field modes and oscillations of boundaries was discussed in [13, 14]. However, the first evaluations of the effect gave unrealistic numbers for two reasons: (i) approximate perturbation approaches, used in that papers, are invalid, as a matter of fact, under resonance conditions; (ii) the chosen amplitudes of oscillations of the cavity length were many orders of magnitude bigger than those which could be actually achieved in practice.

More precise and realistic calculations were performed only in 1990s in the frameworks of different approaches [9, 10, 15–18]. It was shown that a significant amount of photons could be created from vacuum, if boundaries of a high-Q cavity perform small oscillations at a frequency which is multiple of some cavity eigenfrequency. In particular, if a plane boundary of a *three-dimensional* cavity performs *harmonical* oscillations with an amplitude a at the frequency $\omega_w = 2\omega_0$, where ω_0 is the eigenfrequency of the lowest electromagnetic mode in the cavity with fixed geometry, then the mean number of photons created from vacuum in this mode is given by the formula [9, 10]

$$\langle n\rangle(t) = \sinh^2\left(\varepsilon\omega_0 t\eta^3\right), \tag{1}$$

where $\varepsilon = a/\lambda$ is the maximal relative displacement of the boundary (with respect to the wavelength $\lambda = 2\pi c/\omega_0$) and $\eta = \lambda/(2L_0) < 1$ is a numerical coefficient, which depends on the cavity geometry (L_0 is the average distance between vibrating walls).

Formula (1) can be derived from a general solution for a quantum harmonic oscillator with an arbitrary time-dependent frequency obtained for the first time in the seminal paper by Husimi [19] in 1953, if one remembers that field modes behave as a set of harmonic oscillators. Husimi showed that all dynamical properties of the *quantum* oscillator are determined by the fundamental system of solutions of the *classical* equation of motion

$$\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0. \tag{2}$$

In particular, if $\omega(t) = \omega_i$ for $t \to -\infty$ and initially (at $t \to -\infty$) the oscillator was in the vacuum state, then the mean energy at the moment t equals

$$\mathcal{E}(t) = \frac{1}{4}\left[|\dot{\varepsilon}(t)|^2 + \omega^2(t)|\varepsilon(t)|^2\right], \tag{3}$$

where the function $\varepsilon(t)$ satisfies equation (2) and the initial condition

$$\varepsilon_{t\to-\infty} = \omega_i^{-1/2} e^{-i\omega_i t}. \tag{4}$$

Formula (3) holds for an arbitrary function $\omega(t)$, provided this function is *real*, i.e., the quantum evolution is *unitary*.

According to (1), one of the most important parameters which determines a possible number of created photons is an achievable value of the wall displacement amplitude. For the cavity dimensions of the order of $1 \div 100$ cm, the field resonance frequencies $(\omega_0/2\pi)$ belong to the band from 30 GHz to 300 MHz. An idea of [9, 10] was not to force the wall to oscillate as a whole at such a high frequency, but to excite oscillations of the *surface* of the cavity wall. In such a case, the amplitude a of a standing acoustic wave at frequency $\omega_w = 2\omega_0$ (coinciding with the amplitude of oscillations of the free surface) is related to the relative deformation amplitude δ inside the wall as $\delta = \omega_w a/v_s$, where v_s is the sound velocity. Since usual materials cannot bear the deformations exceeding the value $\delta_{max} \sim 10^{-2}$, the maximal possible velocity of the boundary is $v_{max} \sim \delta_{max} v_s \sim 50$ m/s (independent of the frequency). The maximal relative displacement $\varepsilon = a/L_0$ is $\varepsilon_{max} \sim (v_s/2\pi c)\delta_{max} \sim 3\cdot 10^{-8}$ for the lowest mode with the frequency $\omega_0 \sim c\pi/L_0$. Then, taking $\varepsilon = 10^{-9}$, $\omega_0/(2\pi) = 10\,\mathrm{GHz}$ and $\eta^3 = 1/3$, in $t = 1$ s one could get a big number $\sinh^2(10) \sim 10^8$ photons in an empty cavity. However, in such a case one needs a cavity with the Q-factor of the order of 10^{10}. Moreover, it is necessary to maintain the resonance condition, which means that the frequency of the wall oscillations must not deviate from $2\omega_0/(2\pi)$ by more than $\delta/(2\pi) < \varepsilon\,\omega_0\eta^3/(2\pi) \sim 3\,\mathrm{Hz}$ during the time 1 s.

On the other hand, what we really need to create photons from vacuum, it is a possibility to change the resonance frequency in a periodical way. But this can be achieved not only by changing the geometry, but by changing the electric properties of the walls or some medium inside the cavity. Hence the idea of simulating DCE and other quantum effects arose about two decades ago in the article by Yablonovitch [20], who proposed to use a medium with a rapidly decreasing in time refractive index ('plasma window') to simulate the so-called Unruh effect. Also, he pointed out that fast changes of electric properties can be achieved in semiconductors illuminated by laser pulses. This idea was propagandized by Man'ko [21], who proposed to use semiconductors with time-dependent properties to produce an analogue of the *nonstationary Casimir effect* (see also [22,23]). A more developed scheme, based on the creation of an electron-hole 'plasma mirror' inside a semiconductor slab, illuminated by a femtosecond laser pulse, was proposed in [24] (in the single-pulse case). But only recently a possibility of creating an effective 'plasma mirror' in a semiconductor slab was confirmed experimentally [25].

Quantum effects caused by a time dependence of properties of thin slabs inside resonance cavities were studied by several authors [26–29]. However, only very simple models of the media were considered in that papers: lossless homogeneous dielectrics with time-dependent permeability [29], ideal dielectrics or ideal conductors suddenly removed from the cavity [26,27] or infinitely thin conducting slabs modeled by δ-potentials with time-dependent strength [28] (this model of 'plasma sheet' was introduced in [30]). Moreover, all that models, as well as the estimations of the photon generations rate based on the simple formula (1), did not take into account inevitable losses inside the semiconductor slab during the excitation-recombination process. This is the immediate consequence of the fact that the dielectric permeability $\epsilon(x)$ of the semiconductor medium is a *complex function*: $\epsilon = \epsilon_1 + i\epsilon_2$, where $\epsilon_2 = 2\sigma/f_0$, σ and f_0 being the conductivity (in the CGS units) and frequency in Hz, respectively. Good conductors have $\epsilon_2 \sim 10^8$ at microwave frequencies, which is essentially bigger than $\epsilon_1 \sim 1 \div 10$. Although ϵ_2 is negligibly small in the non-excited semiconductor at low temperatures, it rapidly and continuously grows up to the values of the order of $10^5 \div 10^6$ during the laser pulse, returning to zero after the recombination time. In order to predict possible results of experiments and to suggest optimal choices of different parameters, one has to take into account the internal dissipation in the slab, which means that one has to generalize the Husimi solution to the case of *damped* quantum nonstationary oscillator. Such a generalization is considered in sections 2 and 3. In addition to the formulas for the mean number of quanta created in the process of parametric excitation, obtained in previous papers [2, 31–36], I give several important new results. Namely, I calculate the distribution function of created quanta and the variance of their number, which characterizes fluctuations in the quantum state of the field mode. The distribution function turns out to be *larger* than in the thermal state. Such states are called sometimes *super-chaotic states*.

Section 4 is devoted to a new approximate formula for the resonance frequency shift in the cavity, which holds (in contradistinction to all previous studies) even for *big* changes of the complex dielectric permeability. The knowledge of this frequency shift is absolutely necessary for calculating the number of created quanta. Corresponding estimations are given in section 5.

But before going to the main goal and concrete calculations, it seems useful to provide a list of other publications related to the dynamical Casimir effect, which appeared during the period from 2005 to 2008. The role of different boundary conditions was studied in [37] for a scalar field in 1+3 dimensions with the Neumann boundary condition on a single moving deformed mirror. A functional approach was applied to the problem of a single moving dispersive mirror in 1+1 dimensions in [38]. A comparison of the Dirichlet and Neumann conditions for a single relativistic mirror in 1+1 dimensions was made in [39]. The spectrum of radiation from a single dynamically deforming mirror was calculated in [40]. A scalar field between two mirrors in 1+1 dimensions with different (Dirichlet or Neumann) conditions on the mirrors was considered in [41–43]. The case of a 3D cavity with conducting and permeable oscillating plates (described by means of the Dirichlet and Neumann boundary conditions) was studied in [44, 45]. The Robin boundary conditions in 1+1 dimensions were considered in [46, 47]. The one-dimensional cavity with one and two oscillating mirrors was considered within the framework of the 'optical' approach in [48]. Generalizations of Moore's approach to the one-dimensional vibrating cavities were considered in [49]. A one-dimensional *uniformly contracting* cavity was studied in [50, 51]. The authors of [41, 52] pointed out on a possible physical realization of one-dimensional models in the case of TEM modes in cylindrical waveguides. Oscillating *spherical* cavities were considered in [41, 53, 54]. A multiple scale analysis was used in studies on oscillating ideal cavities in three dimensions in [55, 56], giving the same results that were obtained during the preceding decade. The Hamiltonian approach to the dynamical Casimir effect was revised in [57–61] and applied to the calculation of the spectrum of created particles in [62, 63] and to cosmological problems in [64]. Possible cosmological manifestations of the Dynamical Casimir Effect were studied also in [65, 66]. A comparison of analytical and numerical results was made in [67–69]. The relationship between DCE, Unruh-Hawking and other effects was analyzed in [70–72]. An account of nonlinear effects in the DCE was made in [73]. Photon creation in nonstationary media was considered in [74, 75]. Different parametric processes which can be thought as analogs of the DCE were studied in [76–85]. Methods of detection of Casimir photons were considered in [86]. The dynamical *Casimir–Polder* effect was the subject of studies [87–90]. The phenomena of decoherence and entanglement in connection with DCE were studied in [91–95].

2. Quantum damped oscillator with time-dependent parameters in the Heisenberg–Langevin approach

An immediate consequence of the time variation of electromagnetic properties of the cavity walls is the time dependence of the eigenmode frequencies. Hence it follows a simple idea that one could understand the main features of the behavior of the quantum field in the cavity by considering a single selected mode and describing it as a quantum oscillator with 'instantaneous' time-dependent frequency [96, 97]. Later on, it was justified (see, e.g., [9, 10, 98, 99]) for three-dimensional cavities without accidental degeneracy of the spectrum of eigenmode frequencies and for harmonic variations of the effective frequency. Thus I *assume* that even in the presence of dissipation and non-monochromatic periodical variations, the field problem still can be reduced approximately to the dynamics of a *single selected mode* described in the classical limit as a harmonic oscillator with time-dependent *complex* frequency $\omega_c(t) = \omega(t) - i\gamma(t)$, which can be found from the solution of the classical electrodynamical problem by taking the *instantaneous* geometry and material properties (as was done in the non-dissipative case in [28, 29]).

The scheme presented below was developed in [2, 31, 32]. It is a generalization of the *quantum noise operator* approach, first proposed in [100–103] for systems with time-independent parameters, to the case of arbitrary time dependence of the frequency and damping coefficient. In this approach, dissipative quantum systems are described within the framework of the

Heisenberg–Langevin operator equations. In the case concerned these equations can be written as

$$d\hat{x}/dt = \hat{p} - \gamma_x(t)\hat{x} + \hat{F}_x(t), \qquad d\hat{p}/dt = -\gamma_p(t)\hat{p} - \omega^2(t)\hat{x} + \hat{F}_p(t). \tag{5}$$

Here $\hat{x}$ and $\hat{p}$ are the dimensionless quadrature operators of the selected mode, normalized in such a way that the mean number of photons equals

$$\mathcal{N} = \frac{1}{2}\langle \hat{p}^2 + \hat{x}^2 - 1\rangle. \tag{6}$$

In other words, in the subsequent formulas ω and γ are the frequency and damping coefficient normalized by the initial frequency ω_i. The *two noncommuting* noise operators $\hat{F}_x(t)$ and $\hat{F}_p(t)$ are necessary to preserve the canonical commutator between the Heisenberg operators $\hat{x}(t)$ and $\hat{p}(t)$ (however, it is supposed that $\hat{F}_x(t)$ and $\hat{F}_p(t)$ commute with $\hat{x}$ and $\hat{p}$). At first glance, the presence of two extra terms $-\gamma_x(t)\hat{x} + \hat{F}_x(t)$ in the first equation in (5) seems unusual (from the point of view of the classical theory of Brownian motion). However, these terms arise quite naturally in quantum optics, for example, if one rewrites the standard Heisenberg–Langevin equation of motion for the annihilation operator $d\hat{a}/dt = (-i\omega - \gamma)\hat{a} + \hat{F}_a$ [103] in terms of quadrature components.

The system of linear equations (5) can be solved explicitly for arbitrary time-dependent functions $\gamma_{x,p}(t)$, $\omega(t)$ and $\hat{F}_{x,p}(t)$. It is convenient to represent the solutions in the form

$$\hat{x}(t) = \hat{x}_s(t) + \hat{X}(t), \qquad \hat{p}(t) = \hat{p}_s(t) + \hat{P}(t), \tag{7}$$

where the first terms represent the solutions of the homogeneous parts of equations (5) (the subscript 's' stands for 'source'):

$$\hat{x}_s(t) = e^{-\Gamma(t)}\left\{\hat{x}_0 \mathrm{Re}\left[\xi(t)\right] - \hat{p}_0 \mathrm{Im}\left[\xi(t)\right]\right\}, \tag{8}$$

$$\hat{p}_s(t) = e^{-\Gamma(t)}\left\{\hat{x}_0 \mathrm{Re}\left[\eta(t)\right] - \hat{p}_0 \mathrm{Im}\left[\eta(t)\right]\right\}. \tag{9}$$

Here $\hat{x}_0$ and $\hat{p}_0$ are the initial values of operators at $t = 0$ (taken as the initial instant) and

$$\Gamma(t) = \int_0^t \gamma(\tau)d\tau, \qquad \gamma(t) = \frac{1}{2}\left[\gamma_x(t) + \gamma_p(t)\right]. \tag{10}$$

The operators $\hat{X}(t)$ and $\hat{P}(t)$ represent the influence of the stochastic forces (note that the 'source' and 'noise' parts of the solutions commute with each other):

$$\begin{pmatrix} \hat{X}(t) \\ \hat{P}(t) \end{pmatrix} = e^{-\Gamma(t)} \int_0^t d\tau e^{\Gamma(\tau)} \begin{pmatrix} a_x^x(t;\tau) & a_x^p(t;\tau) \\ a_p^x(t;\tau) & a_p^p(t;\tau) \end{pmatrix} \begin{pmatrix} \hat{F}_x(\tau) \\ \hat{F}_p(\tau) \end{pmatrix}, \tag{11}$$

where

$$a_x^x(t;\tau) = \mathrm{Im}\left[\xi(t)\eta^*(\tau)\right], \qquad a_x^p(t;\tau) = \mathrm{Im}\left[\xi^*(t)\xi(\tau)\right], \tag{12}$$

$$a_p^x(t;\tau) = \mathrm{Im}\left[\eta(t)\eta^*(\tau)\right], \qquad a_p^p(t;\tau) = \mathrm{Im}\left[\eta^*(t)\xi(\tau)\right]. \tag{13}$$

Function $\xi(t)$ is the special solution to equation (2) with $\omega^2(t)$ replaced by the effective frequency

$$\omega_{ef}^2(t) = \omega^2(t) + \dot{\delta}(t) - \delta^2(t), \qquad \delta(t) = \frac{1}{2}\left[\gamma_x(t) - \gamma_p(t)\right]. \tag{14}$$

This special solution is selected by the initial condition $\xi(t) = \exp(-it)$ for $t \to -\infty$, which is equivalent to fixing the value of the Wronskian:

$$\xi\dot{\xi}^* - \dot{\xi}\xi^* = 2i. \tag{15}$$

The function $\eta(t)$ is defined as

$$\eta(t) = \dot{\xi}(t) + \delta(t)\xi(t). \tag{16}$$

It satisfies the identity following from (15)

$$\mathrm{Im}\,[\xi(t)\eta^*(t)] \equiv 1. \tag{17}$$

It seems natural to identify the functions $\omega(t)$ and $\gamma(t)$ in equations (5), (10) and (14) with the real and imaginary parts of the instantaneous complex cavity eigenfrequency $\omega_c(t) = \omega(t) - i\gamma(t)$. An immediate consequence of equations (7)-(9), (15) and (17) is the formula

$$[\hat{x}(t), \hat{p}(t)] = i\hbar e^{-2\Gamma(t)} + \left[\hat{X}(t), \hat{P}(t)\right]. \tag{18}$$

Using equations (11) and (17), one can verify that the commutator $[\hat{x}(t), \hat{p}(t)] = i\hbar$ is preserved exactly for arbitrary functions $\omega(t)$ and $\gamma(t)$, if one assumes that the noise operators are *delta-correlated* (the Markov approximation) with the following commutation relations:

$$\left[\hat{F}_x(t), \hat{F}_p(t')\right] = 2i\hbar\gamma(t)\delta(t-t'), \qquad \left[\hat{F}_x(t), \hat{F}_x(t')\right] = \left[\hat{F}_p(t), \hat{F}_p(t')\right] = 0. \tag{19}$$

Indeed, under these conditions one obtains

$$\left[\hat{X}(t), \hat{P}(t)\right] = e^{-2\Gamma(t)} \int_0^t 2i\hbar\gamma(t) e^{2\Gamma(\tau)} d\tau = i\hbar\left[1 - e^{-2\Gamma(t)}\right].$$

In contrast to the classical Langevin equations, which contain a single stochastic force, in the quantum case one must use *two* noise operators, otherwise the canonical commutation relations cannot be saved. The Markov approximation implies the relations

$$\langle \hat{F}_x(t)\hat{F}_x(t')\rangle = \delta(t-t')\chi_{xx}(t), \qquad \langle \hat{F}_p(t)\hat{F}_p(t')\rangle = \delta(t-t')\chi_{pp}(t), \tag{20}$$

$$\langle \hat{F}_x(t)\hat{F}_p(t') + \hat{F}_p(t')\hat{F}_x(t)\rangle = 2\delta(t-t')\chi_s(t). \tag{21}$$

Strictly speaking, the 'noise coefficients' $\chi_{xx}(t)$, $\chi_{pp}(t)$ and $\chi_s(t)$ must be derived from some 'microscopical' model, which takes into account explicitly (1) the coupling of the field mode with electron–hole pairs inside the semiconductor slab and (2) the coupling of electrons and holes with phonons or other quasiparticles, responsible for the damping mechanisms. Unfortunately, it seems that no model of this kind was considered until now. Nonetheless, some conclusions on the relations between the noise coefficients can be made, if one calculates the second-order moments of the quadrature operators. Namely, using equations (11), (17), (20) and (21) one can obtain the following *exact* expressions for the mean values of the operators $\hat{x}^2(t)$ and $\hat{p}^2(t)$ (from now on we shall use dimensionless variables, putting $\hbar = 1$; besides, we replace the symbol of function $f(t)$ by a short form f_t):

$$\begin{aligned}
\langle \hat{x}^2(t)\rangle \;=\;& e^{-2\Gamma(t)}\left\{\langle \hat{x}^2\rangle_0 \left[\mathrm{Re}\,(\xi_t)\right]^2 + \langle \hat{p}^2\rangle_0 \left[\mathrm{Im}\,(\xi_t)\right]^2 - \langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_0 \mathrm{Re}\,(\xi_t)\,\mathrm{Im}\,(\xi_t)\right\} \\
&+ e^{-2\Gamma(t)} \int_0^t d\tau e^{2\Gamma(\tau)} \Big\{\chi_{xx}(\tau)\left[\mathrm{Im}\,(\xi_t\eta_\tau^*)\right]^2 + \chi_{pp}(\tau)\left[\mathrm{Im}\,(\xi_t\xi_\tau^*)\right]^2 \\
&- 2\chi_s(\tau)\mathrm{Im}\,(\xi_t\eta_\tau^*)\,\mathrm{Im}\,(\xi_t\xi_\tau^*)\Big\},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\langle \hat{p}^2(t)\rangle \;=\;& e^{-2\Gamma(t)}\left\{\langle \hat{x}^2\rangle_0 \left[\mathrm{Re}\,(\eta_t)\right]^2 + \langle \hat{p}^2\rangle_0 \left[\mathrm{Im}\,(\eta_t)\right]^2 - \langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_0 \mathrm{Re}\,(\eta_t)\,\mathrm{Im}\,(\eta_t)\right\} \\
&+ e^{-2\Gamma(t)} \int_0^t d\tau e^{2\Gamma(\tau)} \Big\{\chi_{xx}(\tau)\left[\mathrm{Im}\,(\eta_t\eta_\tau^*)\right]^2 + \chi_{pp}(\tau)\left[\mathrm{Im}\,(\eta_t\xi_\tau^*)\right]^2 \\
&- 2\chi_s(\tau)\mathrm{Im}\,(\eta_t\eta_\tau^*)\,\mathrm{Im}\,(\eta_t\xi_\tau^*)\Big\}.
\end{aligned} \tag{23}$$

Let us consider the case of time-independent frequency $\omega = \omega_i = 1$ and time-independent damping and noise coefficients. Assuming that $\gamma \ll 1$ (small damping) one can neglect the correction $\delta^2 \sim \gamma^2$ in function $\omega_{ef}(t)$ (14) and use the solution $\xi(t) = \exp(-it)$. Then integrals in equations (22) and (23) can be calculated exactly. Supposing that $\gamma_{x,p} \sim \gamma$ and $\chi_{jk} \sim \gamma$ (in accordance with the fluctuation–dissipation theorem) and neglecting terms proportional to γ^2, one can obtain the following mean values at $t \to \infty$:

$$\langle \hat{x}^2 \rangle_\infty = \frac{1}{4\gamma} \left[\chi_{xx} + \chi_{pp} + 2\gamma_p \chi_s \right], \qquad \langle \hat{p}^2 \rangle_\infty = \frac{1}{4\gamma} \left[\chi_{xx} + \chi_{pp} - 2\gamma_x \chi_s \right], \tag{24}$$

$$\langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle_\infty = \frac{1}{2\gamma} \left[\gamma_x \chi_{pp} - \gamma_p \chi_{xx} \right]. \tag{25}$$

We see that the steady-state moments of the second order coincide with the thermodynamical equilibrium values

$$\langle \hat{x}^2 \rangle_{eq} = \langle \hat{p}^2 \rangle_{eq} = 1/2 + \langle n \rangle_{th}, \qquad \langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle_{eq} = 0 \tag{26}$$

(where $\langle n \rangle_{th}$ is the mean number of quanta in the thermal state) with the accuracy of the order of γ^2 (i.e., without *linear* corrections with respect to the damping coefficients), provided the noise coefficients are chosen as follows

$$\chi_s = 0, \qquad \chi_{xx} = \gamma_x G, \qquad \chi_{pp} = \gamma_p G \tag{27}$$

where

$$G = 1 + 2\langle n \rangle_{th} = \coth\left(\frac{\hbar \omega_i}{2 k_B \Theta} \right) \tag{28}$$

and Θ is the temperature of the reservoir.

It is important that the noise and damping coefficients cannot be quite arbitrary and independent. Namely, the coefficients χ_{pp} and χ_{xx} must obey the restriction $\chi_{pp}\chi_{xx} \ge \gamma^2$, which prevents from possible violations of the positivity of the density matrix during the evolution [104, 105]. Therefore we must demand the fulfillment of the inequality

$$G^2 \gamma_x \gamma_p \ge \left(\gamma_x + \gamma_p \right)^2 / 4. \tag{29}$$

At zero temperature of the reservoir ($G = 1$) inequality (29) can be satisfied only for

$$\gamma_x = \gamma_p = \gamma. \tag{30}$$

In the case of nonzero temperature (when $G > 1$), the positivity of the density matrix can be preserved for unequal damping coefficients γ_x and γ_p. Their exact values could be calculated in the frameworks of the 'microscopical' theory, which does not exist yet. Therefore I *assume* in this paper that equation (30) holds for nonzero temperatures, as well, so that the diffusion and damping coefficients are given by the relations

$$\chi_s = 0, \qquad \chi_{xx} = \chi_{pp} = \gamma G. \tag{31}$$

In such a case, $\delta \equiv 0$, $\eta(t) \equiv \dot{\xi}(t)$ and the effective frequency in equation (14) coincides exactly with $\omega(t)$. For this special set of coefficients, the stationary asymptotical values of the second-order statistical moments coincide *exactly* with the equilibrium values (26) for an *arbitrary* (not necessarily small) function $\gamma(t)$ (if $\omega = const$). This will be shown in subsections 2.1 and 2.2.

The main arguments in favor of the assumption (31) are based on the analysis of different 'microscopical' models describing the interaction of a selected harmonic oscillator (field mode) with a *fixed* frequency with an 'environment' described by means of multidimensional *quadratic* Hamiltonians of the most general form. Namely, it was shown in [104, 106, 107] that time-independent damping and noise coefficients, satisfying all the requirements (i.e., not allowing violations of the positivity of the density matrix), arise in these models in the only case: when the coupling between the selected and the 'bath' oscillators has the so-called 'rotating wave approximation' (RWA) form $\sum_j \hat{a}\hat{b}_j^\dagger + H.c.$ (this special kind of coupling is considered in all textbooks on quantum optics, see, e.g., [108]). Under this restriction, the models of this kind result in the set of coefficients given by equations (30) and (31). It is worth remembering in this connection that the usual justification for the exclusion of 'antirotating' terms $\hat{a}\hat{b}_j + H.c.$ is that such terms give rapidly oscillating corrections to the equations of motion, whose influence becomes small after averaging over many periods of the field mode oscillations. The problem is that the duration of laser pulses in the MIR experiment is much smaller than the period of the field oscillations. Therefore, strictly speaking, one cannot exclude a possibility that the assumption (31) is not correct. These observations show the necessity of a rigorous microscopical theory of the dynamical Casimir effect in dissipative media with time-dependent parameters.

2.1. The mean number of created quanta

The formula for the time dependence of the mean number of quanta (6) can be split in two parts

$$\mathcal{N}(t) = \mathcal{N}_s(t) + \mathcal{N}_r(t) \tag{32}$$

where the first term depends on the initial state ('signal') while the second term is determined completely by the interaction with the reservoir (it is represented by the integrals containing the coefficients χ_{jk}). The contribution of noise to the mean number of quanta under the condition (31) is given by the formula

$$\mathcal{N}_r(t) = E(t)J(t) - \mathrm{Re}\left[\tilde{E}^*(t)\tilde{J}(t)\right], \tag{33}$$

where

$$J(t) = Ge^{-2\Gamma(t)}\int_0^t d\tau e^{2\Gamma(\tau)}\gamma(\tau)E(\tau), \qquad \tilde{J}(t) = Ge^{-2\Gamma(t)}\int_0^t d\tau e^{2\Gamma(\tau)}\gamma(\tau)\tilde{E}(\tau) \tag{34}$$

and

$$E(t) = \frac{1}{2}\left[|\xi(t)|^2 + |\dot{\xi}(t)|^2\right], \qquad \tilde{E}(t) = \frac{1}{2}\left[\xi^2(t) + \dot{\xi}^2(t)\right]. \tag{35}$$

In the special case of the initial *coherent* state $|\alpha\rangle$ (which corresponds to the initial 'classical' signal in the cavity), the 'signal' contribution is given by the formula

$$\mathcal{N}_s^{(coh)}(t) = e^{-2\Gamma(t)}\left\{(\mathrm{Re}[\alpha\xi(t)])^2 + \left(\mathrm{Re}[\alpha\dot{\xi}(t)]\right)^2 + \frac{1}{2}E(t)\right\} - \frac{1}{2}. \tag{36}$$

For the initial thermal state we have

$$\mathcal{N}_s^{(th)}(t) = \frac{1}{2}\left\{G_0 e^{-2\Gamma(t)}E(t) - 1\right\}. \tag{37}$$

One should remember that formulas (33), (36) and (37) hold for sufficiently big values of time t, when the time dependent normalized frequency $\omega(t)$ returns to its initial value $\omega(-\infty) = 1$.

2.2. Fluctuations of the number of quanta

Fluctuations of the number of created quanta are characterized by the variance

$$\sigma_N = \langle \hat{\mathcal{N}}^2 \rangle - \langle \hat{\mathcal{N}} \rangle^2, \qquad \hat{\mathcal{N}} = \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2 - 1\right). \tag{38}$$

Using the decomposition (7) we can write

$$\begin{aligned} 4\sigma_N \;=\; & \langle \hat{P}^4 \rangle - \langle \hat{P}^2 \rangle^2 + \langle \hat{X}^4 \rangle - \langle \hat{X}^2 \rangle^2 + \langle \hat{P}^2\hat{X}^2 + \hat{X}^2\hat{P}^2 \rangle - 2\langle \hat{P}^2 \rangle \langle \hat{X}^2 \rangle \\ & + \langle \hat{p}_s^4 \rangle - \langle \hat{p}_s^2 \rangle^2 + \langle \hat{x}_s^4 \rangle - \langle \hat{x}_s^2 \rangle^2 + \langle \hat{p}_s^2\hat{x}_s^2 + \hat{x}_s^2\hat{p}_s^2 \rangle - 2\langle \hat{p}_s^2 \rangle \langle \hat{x}_s^2 \rangle \\ & + 4\langle \hat{p}_s^2 \rangle \langle \hat{P}^2 \rangle + 4\langle \hat{x}_s^2 \rangle \langle \hat{X}^2 \rangle + 4\langle \hat{p}_s\hat{x}_s \rangle \langle \hat{P}\hat{X} \rangle + 4\langle \hat{x}_s\hat{p}_s \rangle \langle \hat{X}\hat{P} \rangle. \end{aligned} \tag{39}$$

Let us consider, for example, the quantity $\langle \hat{P}^4 \rangle$. In view of equation (11), it is given by the four-fold integral containing different average values of the stochastic force operators of the form $\langle \hat{F}_i(\tau_1)\hat{F}_j(\tau_2)\hat{F}_k(\tau_3)\hat{F}_l(\tau_4) \rangle$. We assume that stochastic forces are *Gaussian*. Then the fourth-order average values are factorized in the sum over all different products of the second-order moments, so that we can use the following formula:

$$\begin{aligned} \langle \hat{F}_i(\tau_1)\hat{F}_j(\tau_2)\hat{F}_k(\tau_3)\hat{F}_l(\tau_4) \rangle \;=\; & \chi_{ij}(\tau_1)\chi_{kl}(\tau_3)\delta(\tau_1-\tau_2)\delta(\tau_3-\tau_4) \\ & + \chi_{ik}(\tau_1)\chi_{jl}(\tau_2)\delta(\tau_1-\tau_3)\delta(\tau_2-\tau_4) \\ & + \chi_{il}(\tau_1)\chi_{jk}(\tau_3)\delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3). \end{aligned} \tag{40}$$

As a consequence, all four-fold integrals are also factorized in the sums of the products of two ordinary integrals of the following structure:

$$I^{\nu\rho}_{\mu\lambda} = e^{-2\Gamma(t)} \int_0^t d\tau e^{2\Gamma(\tau)} a^{\nu}_{\mu}(\tau) a^{\rho}_{\lambda}(\tau) \chi_{\nu\rho}(\tau), \qquad \mu,\nu,\lambda,\rho = x,p. \tag{41}$$

The functions $a^{\nu}_{\mu}(t)$ were defined in (12) and (13). Making the only assumption $\chi_s = 0$, one obtains the following noise contribution to the second-order moments of $\hat{X}$ and $\hat{P}$:

$$\langle \hat{P}^2 \rangle = I^{xx}_{pp} + I^{pp}_{pp}, \qquad \langle \hat{X}^2 \rangle = I^{xx}_{xx} + I^{pp}_{xx}. \tag{42}$$

Besides, the following relations hold due to identities $\chi_{xp} = -\chi_{px} = i\gamma$ and (17):

$$I^{px}_{ab} + I^{xp}_{ba} \equiv 0, \qquad a,b = x,p, \tag{43}$$

$$I^{px}_{xp} + I^{xp}_{xp} = -I^{xp}_{px} - I^{px}_{px} = \frac{i}{2}\theta(t), \qquad \theta(t) \equiv 1 - e^{-2\Gamma(t)}. \tag{44}$$

Taking into account (42)-(44) and the relation $I^{\nu\nu}_{ab} = I^{\nu\nu}_{ba}$, one can write the part of σ_N given by the first line in equation (39) (which depends on the noise operators only) as follows:

$$\sigma^r_N(t) = \frac{1}{2}\left[\left(I^{xx}_{pp} + I^{pp}_{pp}\right)^2 + \left(I^{xx}_{xx} + I^{pp}_{xx}\right)^2 + \left(I^{xx}_{px} + I^{pp}_{px}\right)^2 + \left(I^{xx}_{xp} + I^{pp}_{xp}\right)^2\right] - \frac{1}{4}\theta^2(t). \tag{45}$$

If $\chi_{pp} = \chi_{xx}$, then (45) can be simplified as follows,

$$\sigma^r_N(t) = (2E^2-1)J^2 + E^2|\tilde{J}|^2 + \mathrm{Re}\left(\tilde{E}^2\tilde{J}^{*2} - 4E\tilde{E}J\tilde{J}^*\right) - \frac{1}{4}\theta^2(t), \tag{46}$$

where the functions $J(t)$, $\tilde{J}(t)$, $E(t)$ and $\tilde{E}(t)$ were defined in (34) and (35).

For the initial *thermal* state characterized by parameter G_0 (which can be different from the reservoir factor G) we obtain the following expressions for the sums of terms in the second ('signal') and third ('signal–noise') lines of equation (39):

$$\sigma_N^s(t) = \frac{1}{4} e^{-4\Gamma(t)} \left(G_0^2 \left[2E^2(t) - 1) \right] - 1 \right), \tag{47}$$

$$\sigma_N^{s-r}(t) = e^{-2\Gamma(t)} \left\{ G_0 J(t) \left[2E^2(t) - 1) \right] - 2G_0 E(t) \mathrm{Re} \left[\tilde{E}^*(t) \tilde{J}(t) \right] - \frac{1}{2}\theta(t) \right\}. \tag{48}$$

The correctness of formulas (46)-(47) can be verified in the special case of relaxation from one temperature (characterized by parameter G_0) to another (with parameter G) without changing the frequency $\omega = const = 1$. In this case $E \equiv 1$, $\tilde{E} = \tilde{J} \equiv 0$, $J(t) = G\theta(t)/2$, so that we obtain for an arbitrary function $\gamma(t)$

$$\mathcal{N}(t) = \frac{1}{2} \left[G_{ef}(t) - 1 \right], \qquad G_{ef}(t) = G\theta(t) + G_0 \left[1 - \theta(t) \right] \tag{49}$$

and

$$\sigma_N(t) = \frac{1}{4} \left[G_{ef}^2(t) - 1 \right] \equiv \mathcal{N}(t)[\mathcal{N}(t) + 1], \tag{50}$$

as it must be for thermal states.

3. Periodical variations of parameters

Having in mind applications to the dynamical Casimir effect, we are interested in the special case when the functions $\omega(t)$ and $\gamma(t)$ have the form of *periodical* pulses with the periodicity T, separated by intervals of time with $\omega = 1$ and $\gamma = 0$ (we neglect the damping of the field between pulses, supposing that the quality factor of the cavity is big enough). Then the integrals in equation (34) are reduced to the sums of n (the total number of pulses) integrals taken between the initial and final time moments of each pulse t_i and t_f. In the interval between the kth and $(k+1)$th pulses the solution to equation (2) can be written as

$$\varepsilon_k(t) = a_k e^{-it} + b_k e^{it}, \qquad a_0 = 1, \quad b_0 = 0, \tag{51}$$

where a_k and b_k are constant coefficients. Consequently, during these intervals functions $E(t)$ and $\tilde{E}(t)$ assume constant values

$$E_k = |a_k|^2 + |b_k|^2, \qquad \tilde{E}_k = 2a_k b_k. \tag{52}$$

Evidently, every two sets of the nearest constant coefficients, (a_{k-1}, b_{k-1}) and (a_k, b_k), are related by means of a linear transformation

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = M_k \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}. \tag{53}$$

Four complex elements of the 2×2 matrix M_k for the single pulse (which can be treated also as an effective 'potential barrier', if one wishes to interprete (2) as an analog of the stationary Schrödinger equation) can be expressed through two complex amplitude reflection coefficients and two complex amplitude transmission coefficients, which connect the 'plane waves' coming from the 'left' and from the 'right'. Namely, if the pulse begins at $t = 0$ at terminates at $t = t_*$, then one can write two independent solutions of equation (2) for $t < 0$ and $t > t_*$ as (remember that we assume the constant initial and final value of the frequency to be equal to $\omega_i = 1$)

$$\varepsilon^{(-)}(t) = \begin{cases} e^{it} + r_- e^{-it}, & t < 0 \\ s_- e^{it}, & t > t_* \end{cases} \qquad \varepsilon^{(+)}(t) = \begin{cases} s_+ e^{-it}, & t < 0 \\ e^{-it} + r_+ e^{it}, & t > t_* \end{cases} \tag{54}$$

The reflection coefficients $r_\pm$ and transmission coefficients $s_\pm$ are not independent, because equation (2) is invariant with respect to complex conjugation (the frequency $\omega(t)$ is real). The following relations hold [109] (the simplest way to obtain them is to calculate Wronskians for suitable pairs of independent solutions):

$$s_- = s_+ \equiv s, \qquad r_- s^* + r_+^* s = 0, \qquad |r_-|^2 + |s_-|^2 = |r_+|^2 + |s_+|^2 = 1. \tag{55}$$

Comparing equations (51) and (53) with (54) and taking into account identities (55), one can express the elements of matrix M_k as follows:

$$M_k^{(0)} = \left\| \begin{array}{cc} f_k & g_k^* \\ g_k & f_k^* \end{array} \right\|, \qquad f_k \equiv s^{-1}, \quad g_k \equiv r_+/s, \tag{56}$$

where the superscript '(0)' means that the phases of reflection and transmission coefficients correspond to the pulse starting at the moment $t = 0$ (the subscript k is supressed in coefficients s and $r_\pm$). If the kth pulse begins at the moment t_{k-1} (so that $t_0 = 0$), then one should make the time shift $t \to t - t_{k-1}$ in equation (54). This means that matrix M_k in (53) can be expressed through the matrix $M_k^{(0)}$ as

$$M_k = \Phi_{k-1}^\dagger M_k^{(0)} \Phi_{k-1}, \qquad \Phi_k \equiv \left\| \begin{array}{cc} \exp(it_k) & 0 \\ 0 & \exp(-it_k) \end{array} \right\|. \tag{57}$$

An important consequence of identities (55) is the condition of *unimodularity* of matrix M_k

$$\det M_k = |f_k|^2 - |g_k|^2 \equiv 1 \tag{58}$$

which is equivalent to the Wronskian identity (15).

For n pulses shifted in time with respect to the initial instant $t = 0$ by t_k, $k = 1, \ldots, n-1$, we have the relation (here $\mathcal{M}_n$ is the total *transfer matrix*, whereas $M_n^{(0)}$, $M_{n-1}^{(0)}$, ..., $M_1^{(0)} \equiv M_1$ are the matrices describing the action of individual pulses)

$$\left(\begin{array}{c} a_n \\ b_n \end{array} \right) = \mathcal{M}_n \left(\begin{array}{c} a_0 \\ b_0 \end{array} \right), \qquad \mathcal{M}_n = \Phi_{n-1}^\dagger M_n^{(0)} \Phi_{n-1} \Phi_{n-2}^\dagger M_{n-1}^{(0)} \cdots \Phi_1^\dagger M_2^{(0)} \Phi_1 M_1. \tag{59}$$

For strictly periodic pulses all matrices $M_k^{(0)}$ coincide with M_1 and $\Phi_k \equiv \Phi_1^k$, so that

$$\mathcal{M}_n = \Phi^{\dagger n} (\Phi M_1)^n, \qquad \Phi \equiv \left\| \begin{array}{cc} \exp(iT) & 0 \\ 0 & \exp(-iT) \end{array} \right\|, \tag{60}$$

where T is the periodicity of pulses. Since $\det(\Phi M_1) = 1$, one can use the well-known formula for the powers of any two-dimensional unimodular matrix S (see, e.g., [110])

$$S^n = U_{n-1}(z) S - U_{n-2}(z) E, \qquad z \equiv \frac{1}{2} \mathrm{Tr} S, \tag{61}$$

where E means the unit matrix and $U_n(z)$ is the Tchebyshev polynomial of the second kind. In the case involved one has $z = \frac{1}{2}\mathrm{Tr}(\Phi M_1) = \mathrm{Re}[f \exp(iT)]$ (with $f \equiv f_1$). An amplification can happen if $|z| > 1$. Then it is convenient to use the parametrization

$$\frac{1}{2} \mathrm{Tr}(\Phi M_1) = \mathrm{Re}[f \exp(iT)] = \cosh(\nu). \tag{62}$$

Using (59)–(62), one can arrive at the following expressions [2]:

$$a_n = f\,\frac{\sinh(n\nu)}{\sinh(\nu)}e^{-iT(n-1)} - \frac{\sinh[(n-1)\nu]}{\sinh(\nu)}e^{-iTn}, \qquad b_n = g\,\frac{\sinh(n\nu)}{\sinh(\nu)}e^{iT(n-1)}. \tag{63}$$

If $\mathrm{Re}[f\exp(iT)] > 1$, then ν is real parameter. If $\mathrm{Re}[f\exp(iT)] < -1$, then one can write $\nu = \tilde{\nu} + i\pi$, where $\tilde{\nu}$ is real. The maximal values of $|\mathrm{Re}[f\exp(iT)]|$ correspond to the cases of strict resonance with

$$T = T_{res} = \frac{1}{2}T_0\left(m - \varphi/\pi\right), \tag{64}$$

where $f = |f|\exp(i\varphi)$, T_0 is the period of oscillations in the selected field mode and $m = 1, 2, \ldots$ (even values of m correspond to $\mathrm{Re}[f\exp(iT)] > 1$, whereas odd values of m correspond to $\mathrm{Re}[f\exp(iT)] < -1$). Introducing the parameter

$$\delta = \omega_0\left(T - T_{res}\right) \tag{65}$$

characterizing a detuning from the strict resonance, one can write

$$\cosh(\nu) = |f|(-1)^m\cos(\delta). \tag{66}$$

One can check the fulfillment of the identity $|a_n|^2 - |b_n|^2 \equiv 1$ as a consequence of the initial identity $|f|^2 - |g|^2 \equiv 1$.

Hereafter we confine ourselves to the simplest case of the *strict resonance* (a more general treatment can be found in [32]). Then $|f|^2 = \cosh^2(\nu)$ and $|g|^2 = \sinh^2(\nu)$, so that

$$a_k = \cosh(k\nu)e^{-ikT}, \qquad b_k = \sinh(k\nu)e^{iT(k-1)+i\phi}, \tag{67}$$

$$E_k = \cosh(2k\nu), \qquad \tilde{E}_k = \sinh(2k\nu)e^{i(\phi-T)}, \tag{68}$$

where ϕ is the phase of complex number g.

3.1. Approximate formulas for effective single-pulse reflection and transmission coefficients

For small variations of the frequency $\omega(t)$ one can write a solution to equation (2) in the form (generalizing the standard WKB solution and following the approach of [111])

$$\varepsilon_W(t) = \frac{1+\zeta(t)}{\sqrt{\omega(t)}}\exp[-i\Omega(t)] + \rho_1(t)\exp[i\Omega(t)], \qquad \Omega(t) = \int_0^t \omega(\tau)d\tau. \tag{69}$$

Putting function (69) with $\zeta(t) = 0$ into equation (2) one arrives at the inhomogeneous equation

$$\dot{\rho}_1 + \frac{\dot{\omega}}{2\omega}\rho_1 = \frac{i\ddot{\sigma}}{2\omega}\exp[-2i\Omega(t)] + \frac{i\ddot{\rho}_1}{2\omega}, \qquad \sigma(t) \equiv [\omega(t)]^{-1/2}. \tag{70}$$

It is assumed that $|\rho_1| \ll \sigma$ and $\rho_1 = 0$ if $\omega = const$. This means that the second term in (69) describes a weak reflection of the 'wave' going in the negative t-direction (caused by deviations of function $\omega(t)$ from a constant value), but it is not a part of another fundamental solution of the second order differential equation (2). Since function $\rho_1(t)$ is determined by the inhomogeneous term in equation (69), it is reasonable to suppose that the rates of time variations of $\rho_1(t)$ are more or less the same as for the function $\sigma(t)$. Then one can suppose also that the second term in the right-hand side of equation (70) (which contains the second derivative of ρ_1) is much smaller than the first one. Neglecting this term (as was done in [111]), one arrives at the first order differential equation, whose solution, satisfying the condition $\rho_1 \equiv 0$ for $t < 0$ (when $\dot{\omega}(t) \equiv 0$), is as follows,

$$\rho_1(t) = \frac{\rho(t)}{\sqrt{\omega(t)}}, \qquad \rho(t) = \frac{i}{2}\int_0^t \sigma(\tau)\frac{d^2\sigma}{d\tau^2}\exp[-2i\Omega(\tau)]d\tau. \tag{71}$$

If $\omega(t)$ returns to its initial value for $t \geq t_f$, then $\rho(t)$ given by (71) also does not depend on time for $t \geq t_f$. Comparing equations (54), (56) and (71), one finds that for a single pulse

$$g = \rho(t_f)\exp\left[i\Omega(t_f) - i\omega_0 t_f\right], \qquad f = [1 + \zeta(t_f)]\exp\left[i\omega_0 t_f - i\Omega(t_f)\right].$$

Writing

$$\omega(t) = \omega_0[1 + \chi(t)], \qquad |\chi| \leq \chi_0 \ll 1 \tag{72}$$

(the function $\chi(t)$ should not be confused with the noise coefficients χ_{xx} and χ_{pp}), one obtains the following expression for the phase of the inverse single-pulse transmission coefficient f:

$$\varphi = -\omega_0\int_{t_i}^{t_f}\chi(t)dt \tag{73}$$

(the sign of the phase is corrected here, compared with previous papers [2, 32, 33, 35, 36, 112]). The function $\zeta(t)$ can be found if one calculates the Wronskian for two independent solutions: $\varepsilon(t)$ (69) and its complex conjugate partner. Neglecting higher-order terms, containing squares of function $\zeta(t)$ and products of $\rho(t)$ and $\zeta(t)$ (or their derivatives), and demanding the fulfilment of identity (15), one can obtain the following result:

$$\zeta(t) = \frac{1}{2}|\rho(t)|^2 + i\zeta_I(t), \qquad \zeta_I(t) = \int_0^t \mathrm{Im}(\rho\dot{\rho}^*)d\tau - \int_0^t \frac{\ddot{\sigma}}{2\sigma}d\tau. \tag{74}$$

For $t = t_f$ the second integral in function $\zeta_I(t)$ can be written as (using the integration by parts and taking into account that $\dot{\sigma}(0) = \dot{\sigma}(t_f) = 0$)

$$\int_0^{t_f}\frac{\ddot{\sigma}}{2\sigma}d\tau = \int_0^{t_f}\frac{\dot{\sigma}^2}{2\sigma^2}d\tau.$$

Consequently, the imaginary part of $\zeta(t_f)$ is of the order of χ^2, so that it can be considered as a small (of an order of χ_0^2) correction to the phase φ (73) of the coefficient f. Formula (71) can be simplified for $t = t_f$, if one performs an integration by parts, taking into account that function $\dot{\sigma}$ equals zero for $t = 0$ and $t = t_f$. Neglecting the integral containing the term $\dot{\sigma}^2$ (since it has an order of χ_0^2), one obtains

$$\rho(t_f) \approx -\int_0^{t_f}\frac{\dot{\sigma}}{\sigma}\exp[-2i\Omega(\tau)]d\tau.$$

Making the replacement $\dot{\sigma}/\sigma \approx -\dot{\chi}/2$ (neglecting again the terms of the order of χ_0^2) and making one more integration by parts, one can arrive at the final simple formula (since $\chi(0) = \chi(t_f) = 0$)

$$\rho(t_f) \approx i\omega_0\int_0^{t_f}\chi(t)e^{-2i\omega_0 t}dt. \tag{75}$$

The phase $\Omega(t)$ is replaced here by the product $\omega_0 t$, because an account of the additional term $\int\chi(\tau)d\tau$ in the phase is equivalent to corrections of the order of χ_0^2 for $\rho(t_f)$, which were neglected in the derivation of (75). Thus we obtain the following elements of matrix $M_1^{(0)}$ (56) satisfying the unimodularity condition with an accuracy of the order of χ_0^2

$$g \approx i\omega_0\int_0^{t_f}\chi(t)e^{-2i\omega_0 t}dt. \qquad f = \left(1 + \frac{1}{2}|g|^2\right)e^{i\varphi} \tag{76}$$

The phase φ is given by (73). The validity of the approximation used can be checked if one puts function $\rho_1(t)$ given by formula (71) in the right-hand side of equation (70) and integrates this equation again with the additional inhomogeneous part. It can be shown that corrections to the solution arising due to the new term are, indeed, of an order of χ_0^2 at $t = t_f$. ¿From (66) and (76) one obtains the formula

$$\nu^2 = \tilde{\nu}^2 = |g|^2 - \delta^2. \tag{77}$$

3.2. The contribution of noise

To find the contribution of the kth pulse to the functions $J(T)$ and $\tilde{J}(t)$ one should use the solution $\varepsilon(t)$ in the form $a_k\varepsilon_W(t - t_{k-1}) + b_k\varepsilon_W^*(t - t_{k-1})$ and calculate the integrals defined in (34) from t_{k-1} to t_k. But it is clear from the explicit form of function $\varepsilon_W(t)$ (69) obtained in the preceding section that functions $E(t)$ and $\tilde{E}(t)$ are close to constant values (52) for $t_{k-1} \le t \le t_k$, with small corrections of the order of χ_0. Neglecting these corrections, one can calculate the integrals over the duration of each pulse exactly (since $\gamma(t) = d\Gamma/dt$). Consequently, after n pulses one has

$$J_n = \frac{G}{2}e^{-2\Lambda n}\left(1 - e^{-2\Lambda}\right)\sum_{k=1}^{n} e^{2\Lambda k}E_k, \qquad \tilde{J}_n = \frac{G}{2}e^{-2\Lambda n}\left(1 - e^{-2\Lambda}\right)\sum_{k=1}^{n} e^{2\Lambda k}\tilde{E}_k \tag{78}$$

where $J_n \equiv J(nT)$, $\tilde{J}_n \equiv \tilde{J}(nT)$ and

$$\Lambda = \int_{t_i}^{t_f} \gamma(\tau)d\tau. \tag{79}$$

Numerical evaluations confirmed a high accuracy of this approximation.

Using (68) one can obtain the following explicit formulas for the sums in equation (78):

$$J_n = A_n + B_n, \qquad \tilde{J}_n = e^{i(\phi - T)}\left(A_n - B_n\right), \tag{80}$$

$$A_n = \frac{G\Lambda}{4(\nu + \Lambda)}\left(e^{2n\nu} - e^{-2n\Lambda}\right), \qquad B_n = \frac{G\Lambda}{4(\nu - \Lambda)}\left(e^{-2n\Lambda} - e^{-2n\nu}\right). \tag{81}$$

I take into account that $\Lambda, \nu \ll 1$. In particular, the difference $1 - \exp(-2\Lambda)$ is replaced by 2Λ.

3.3. Mean number of created quanta and its variance

In view of equations (33), (68) and (80), the mean number of 'noise' quanta created after n pulses in the resonance case is given by a simple formula

$$\mathcal{N}_r(n) = A_n e^{-2n\nu} + B_n e^{2n\nu}, \tag{82}$$

where $\mathcal{N}_r(n)$ stands for the value $\mathcal{N}_r(t)$ of the function (33) at the moment $t = nT$ (I hope that such a replacement will not lead to a confusion). Adding to the quantity (82) the expression (37), we obtain the total number of quanta created from the initial thermal state:

$$\mathcal{N}^{(th)}(n) = e^{2n\nu}\left(B_n + \frac{G_0}{4}e^{-2\Lambda n}\right) + e^{-2n\nu}\left(A_n + \frac{G_0}{4}e^{-2\Lambda n}\right) - \frac{1}{2}. \tag{83}$$

Only the first term in the right-hand side of this expression grows with time (the number of pulses n), and for $2n\nu \gg 1$ one arrives at the following asymptotical formula:

$$\mathcal{N}^{(th)}(n) = \frac{1}{4}e^{2n(\nu - \Lambda)}\left(G_0 + \frac{G\Lambda}{\nu - \Lambda}\right) + \mathcal{O}(1). \tag{84}$$

We see that photons can be generated provided $\nu > \Lambda$.

Combining formulas (46)-(48) with (68) and (80) we obtain the variance of the number of photons in the case of resonance excitation of the initial thermal state:

$$\sigma_N^{(th)}(n) = 2e^{4n\nu}\left(B_n + \frac{G_0}{4}e^{-2\Lambda n}\right)^2 + 2e^{-4n\nu}\left(A_n + \frac{G_0}{4}e^{-2\Lambda n}\right)^2 - \frac{1}{4}. \tag{85}$$

The leading asymptotical term of this expression for $n(\nu - \Lambda) \gg 1$ is

$$\sigma_N^{(th)}(n) \approx \frac{1}{8}e^{4n(\nu-\Lambda)}\left(G_0 + \frac{G\Lambda}{\nu - \Lambda}\right)^2 \approx 2\left[\mathcal{N}^{(th)}(n)\right]^2 \tag{86}$$

This means that the fiels mode goes asymptotically to the so called 'superchaotic' [113, 114] quantum state, whose statistics is essentially different from the statistics of the initial thermal state, characterized by formula (50).

3.4. Photon statistics

Due to the linearity of the Heisenberg–Langevin equations of motion (5), any initial *Gaussian* state remains Gaussian in the process of evolution [105]. In particular, this is true for initial *thermal* states. The photon statistics in the Gaussian states was studied in [105, 115, 116]. If the mean values of the quadrature operators (or electric and magnetic fields) are equal to zero, then this statistics is determined completely by two parameters (we assume here $\hbar = 1$),

$$\tau = \sigma_{xx} + \sigma_{pp} \equiv 1 + 2\mathcal{N}, \qquad \Delta = \sigma_{xx}\sigma_{pp} - \sigma_{px}^2 \equiv 1/(4\mu^2), \tag{87}$$

where $\mu \equiv \mathrm{Tr}(\hat\rho^2)$ is the *purity* of the (Gaussian) quantum state of the field mode described by the statistical operator $\hat\rho$. The photon distribution function $f(m) \equiv \langle m|\hat\rho|m\rangle$ (i.e., the probability to detect m quanta in the state $\hat\rho$) can be expressed in terms of the Legendre polynomials [105]:

$$f(m) = \frac{2}{\sqrt{1+2\tau+4\Delta}}\left(\frac{1+4\Delta-2\tau}{1+4\Delta+2\tau}\right)^{m/2} P_m\left(\frac{4\Delta-1}{\sqrt{(4\Delta+1)^2-4\tau^2}}\right). \tag{88}$$

The variance of the photon number distribution for an arbitrary Gaussian state (with zero quadrature mean values) is given by the formula [105]

$$\sigma_N = \frac{1}{2}\tau^2 - \Delta - \frac{1}{4}. \tag{89}$$

Comparing this formula with (83) and (85) we find

$$\Delta(n) = \left(2B_n + \frac{G_0}{2}e^{-2\Lambda n}\right)\left(2A_n + \frac{G_0}{2}e^{-2\Lambda n}\right). \tag{90}$$

In the asymptotical regime $n\nu \gg 1$ we have

$$\Delta(n) \approx \frac{G\Lambda}{4(\nu+\Lambda)}\left(\frac{G\Lambda}{\nu-\Lambda} + G_0\right)e^{2n(\nu-\Lambda)}, \qquad \frac{2\Delta}{\tau} \approx \frac{\Delta}{\mathcal{N}} \approx \frac{G\Lambda}{\nu+\Lambda}. \tag{91}$$

Formula (88) is exact. However, since we are interested in the cases of large numbers of photons created due to the parametric resonance (when $m > 1000$), it is convenient to use its asymptotical forms for $m \gg 1$. Note that the argument of the Legendre polynomial in (88) is always outside the interval $(-1, 1)$, being equal to 1 only for thermal states with $\tau = 2\sqrt{\Delta}$. Therefore it is convenient to use the following asymptotical formula [117]:

$$P_m(\cosh\xi) \approx \left(\frac{\xi}{\sinh\xi}\right)^{1/2} I_0\left([m+1/2]\,\xi\right) \tag{92}$$

(where $I_0(z)$ is the modified Bessel function), because it holds even for complex values of variable ξ, provided $\mathrm{Re}\xi \geq 0$ and $|\mathrm{Im}\xi| < \pi$. The parameter ξ is given by the formula

$$\xi = \ln\left(\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{\sqrt{(4\Delta+1)^2 - 4\tau^2}}\right). \tag{93}$$

Under realistic conditions the ratio $2\Delta/\tau$, given by equation (91), is of the order of unity for $\tau \gg 1$ (see section 5). Consequently, parameter ξ is also of the order of unity. In such a case, the function $I_0(x)$ in (92) can be replaced by its asymptotical form $I_0(x) \approx (2\pi x)^{-1/2}\exp(x)$. Then, using (93) to calculate $\sinh(\xi)$, we obtain an approximate formula

$$f(m) \approx \left[\pi(m+1/2)\sqrt{\tau^2 - 4\Delta}\right]^{-1/2}\left(\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{4\Delta + 1 + 2\tau}\right)^{m+1/2}. \tag{94}$$

Obviously, one can neglect the term $4\Delta \ll \tau^2$ in the first factor. The fraction in the second factor can be simplified as follows under the same conditions:

$$\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{4\Delta + 1 + 2\tau} = \frac{4\Delta - 1 + 2\tau - 4\Delta/\tau + \mathcal{O}(1/\tau)}{4\Delta + 1 + 2\tau} = 1 - \frac{1}{\tau} + \mathcal{O}(1/\tau^2).$$

Replacing $(1-x)^m \approx \exp(-mx)$ for $x \ll 1$, we arrive finally at the following simple formula:

$$f(m) \approx \frac{\exp[-(m+1/2)/\tau]}{\sqrt{\pi\tau(m+1/2)}} \approx \frac{\exp[-(m+1/2)/(2\mathcal{N})]}{\sqrt{2\pi\mathcal{N}(m+1/2)}}. \tag{95}$$

It holds under the conditions $\tau \approx 2\mathcal{N} \gg 1$ and $m \gg 1$. Using the Euler–MacLaurin summation formula, one can verify that the distribution function (95) has the correct normalization with an accuracy $\mathcal{O}(\tau^{-1/2})$:

$$\sum_{m=0}^{\infty} f(m) \approx \int_0^\infty f(m)dm + \mathcal{O}[f(0)] \approx \int_0^\infty \frac{\exp(-x/\tau)}{\sqrt{\pi\tau x}}dx + \mathcal{O}(\tau^{-1/2}) = 1 + \mathcal{O}(\tau^{-1/2}).$$

With the same accuracy, the moments of the distribution function are given by the formula

$$\langle m^k\rangle \equiv \sum_{m=0}^{\infty} m^k f(m) \approx \int_0^\infty x^k \frac{\exp(-x/\tau)}{\sqrt{\pi\tau x}}dx = \tau^k\frac{(2k-1)!!}{2^k} \approx \mathcal{N}^k(2k-1)!!\,. \tag{96}$$

For $k=2$ equation (96) reproduces the result (86): $\sigma_N = \langle m^2\rangle - \langle m\rangle^2 \approx 2\mathcal{N}^2$.

4. Frequency shift of the cavity mode

We see that the time-dependent relative frequency shift $\chi(t)$ is the main ingredient of the theory of DCE. The problem of shift of resonance frequencies of electromagnetic cavities due to perturbations of geometry and material properties of the walls or internal parts of the cavities was considered for the first time by Müller [118]. It is remarkable that it was also studied (a few years later) by two persons whose names are inseparable forever from the Casimir effect:

Schwinger [119] and Casimir himself [120], although that papers are less known than the others, and the results waited for their use in this field for more than half a century (for different other applications see, e.g., [121–127]). In particular, the cavity perturbation techniques is one of the main methods used in studies on photoexcitation of semiconductors [128, 129].

The main formula for the resonance frequency shift in electromagnetic cavities with ideal boundaries was derived in [118–120]. We consider its special form in the case of nonmagnetic materials, assuming that the magnetic permeability $\mu \equiv 1$, so that the magnetic vectors $\mathbf{B}$ and $\mathbf{H}$ coincide everywhere (we use the Gauss system of units). Considering monochromatic electric and magnetic fields of the form $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})\exp(-i\omega_0 t)$ and $\mathbf{H}(\mathbf{r},t) = \mathbf{H}_0(\mathbf{r})\exp(-i\omega_0 t)$, we have the following set of equations for complex vector fields $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$ in an empty cavity:

$$\mathrm{rot}\,\mathbf{E}_0 = \frac{i\omega_0}{c}\mathbf{H}_0 \tag{97}$$

$$\mathrm{rot}\,\mathbf{H}_0 = -\frac{i\omega_0}{c}\mathbf{E}_0 \tag{98}$$

For a cavity filled in with a medium described by means of a complex dielectric function $\varepsilon(\mathbf{r}) = \varepsilon_1(\mathbf{r}) + i\varepsilon_2(\mathbf{r})$ we have equations without the subscript '0', where ω is the shifted resonance frequency:

$$\mathrm{rot}\,\mathbf{E} = \frac{i\omega}{c}\mathbf{H} \tag{99}$$

$$\mathrm{rot}\,\mathbf{H} = -\frac{i\omega}{c}\,\varepsilon\,\mathbf{E} \tag{100}$$

Now we multiply equation (99) by the complex conjugate function $\mathbf{H}_0^*$, equation (100) by $-\mathbf{E}_0^*$, the complex conjugate equation (97) by $\mathbf{H}$ and the complex conjugate equation (98) by $-\mathbf{E}$. Taking the sum of these four new equations we have

$$\mathbf{H}\,\mathrm{rot}\,\mathbf{E}_0^* - \mathbf{E}_0^*\,\mathrm{rot}\,\mathbf{H} + \mathbf{H}_0^*\,\mathrm{rot}\,\mathbf{E} - \mathbf{E}\,\mathrm{rot}\,\mathbf{H}_0^* = \frac{i}{c}\left[\delta\omega\left(\mathbf{E}\mathbf{E}_0^* + \mathbf{H}\mathbf{H}_0^*\right) + \omega\,\delta\varepsilon\,\mathbf{E}\mathbf{E}_0^*\right], \tag{101}$$

where

$$\delta\omega = \omega - \omega_0, \qquad \delta\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r}) - 1. \tag{102}$$

Now we integrate both sides of equation (101) over the total volume of the cavity, taking into account the identity

$$\mathrm{div}[\mathbf{a}\times\mathbf{b}] \equiv \mathbf{b}\,\mathrm{rot}\,\mathbf{a} - \mathbf{a}\,\mathrm{rot}\,\mathbf{b}. \tag{103}$$

Due to the Gauss theorem, the volume integral in the left-hand side can be transformed into the surface integral over the total surface of the cavity

$$\int_{walls}\left([\mathbf{H}\times\mathbf{E}_0^*] - [\mathbf{E}\times\mathbf{H}_0^*]\right)\,d\,\mathbf{s}. \tag{104}$$

But this integral is equal to zero, because the scalar product of the vector integrand and the vector surface differential $d\,\mathbf{s}$ only depends on tangential components of vectors $\mathbf{E}$ and $\mathbf{E}_0$, which turn into zero on the surface of an ideal cavity. Thus we arrive at the *exact* formula

$$\frac{\delta\omega}{\omega} = -\frac{\int\delta\varepsilon(\mathbf{r})\,\mathbf{E}\mathbf{E}_0^*\,dV}{\int\left(\mathbf{E}\mathbf{E}_0^* + \mathbf{H}\mathbf{H}_0^*\right)\,dV}\,. \tag{105}$$

If the magnetic permeability μ is also different from zero, one should add to the numerator of the fraction in the r.h.s. of (105) the integral $\int\delta\mu(\mathbf{r})\,\mathbf{H}\mathbf{H}_0^*\,dV$ (this general formula is called sometimes 'Bethe–Schwinger formula'). A good discussion of formula (105) and its applications can be found in [122].

It is assumed usually that the unperturbed fields $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$ are known at all points of the empty cavity. But the perturbed fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ *are not known*, as a rule (except for the simplest cases when they can be obtained by scaling the fields $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$ in *all* points [122]). If the variation $\delta\varepsilon(\mathbf{r})$ is small everywhere, then one can believe that the perturbed field $\mathbf{E}(\mathbf{r})$ is close to $\mathbf{E}_0(\mathbf{r})$ and replace $\mathbf{E}(\mathbf{r})$ by $\mathbf{E}_0(\mathbf{r})$ in the r.h.s. of (105). Besides, if $\delta\omega \ll \omega_0$, then one can replace ω by ω_0 in the denominator of the fraction in the l.h.s., arriving at a simple approximate formula given in many textbooks [130, 131],

$$\frac{\delta\omega}{\omega_0} \approx -\frac{\int \delta\varepsilon(\mathbf{r})\,\mathbf{E}_0\mathbf{E}_0^*\,dV}{\int\left(\mathbf{E}_0\mathbf{E}_0^* + \mathbf{H}_0\mathbf{H}_0^*\right)\,dV} = -\frac{\int \delta\varepsilon(\mathbf{r})\,\mathbf{E}_0\mathbf{E}_0^*\,dV}{2\int \mathbf{E}_0\mathbf{E}_0^*\,dV} \tag{106}$$

(the second equality holds due to the well known equality of energies of electric and magnetic fields in ideal resonance cavities, $\int \mathbf{E}_0\mathbf{E}_0^*\,dV = \int \mathbf{H}_0\mathbf{H}_0^*\,dV$).

If the function $\delta\varepsilon(\mathbf{r})$ is different from zero only inside some small volume $\delta V \ll V$ (where V is the total cavity volume), then it is clear that the ratio $\delta\omega/\omega_0$ is small even for big values of this function. Moreover, it seems reasonable to believe that the difference between functions $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}_0(\mathbf{r})$ remains small outside the volume δV [120, 122, 125]. Then assuming that the energy of the field concentrated in the volume δV is still much less than the total energy of the field in the cavity, one can replace in the first approximation $\mathbf{E}(\mathbf{r})$ by $\mathbf{E}_0(\mathbf{r})$ in the denominator of the r.h.s. of (105). But this certainly cannot be done in the numerator if $\delta\varepsilon(\mathbf{r})$ is big (otherwise the frequency shift can become arbitrarily big for $\delta\varepsilon \to \infty$, which is obviously not true).

4.1. An approximation for small field variations: TE mode in a cylindrical cavity

Note that the fields depend not only on the coordinate vector $\mathbf{r}$, but also on the frequency ω which enters the Maxwell equations (99) and (100), so that we should write in fact the electric field as $\mathbf{E}(\mathbf{r};\omega)$. Let us suppose that for some geometrical configurations small variations of frequency are accompanied by *small* variations of the electric field. Then using the approximation

$$\mathbf{E}(\mathbf{r};\omega) \approx \mathbf{E}_0(\mathbf{r};\omega_0) + \left.\frac{\partial \mathbf{E}_0(\mathbf{r};\omega)}{\partial\omega}\right|_{\omega=\omega_0} \delta\omega \tag{107}$$

and putting this expression in the numerator of the fraction in the r.h.s. of equation (105) (but neglecting the term with a derivative over ω in the denominator, where it is not multiplied by the possibly big quantity $\delta\varepsilon$), we arrive at the following generalization of (106):

$$\frac{\delta\omega}{\omega_0} \approx -\frac{\int \delta\varepsilon(\mathbf{r})\,\mathbf{E}_0\mathbf{E}_0^*\,dV}{\int\left[2\mathbf{E}_0\mathbf{E}_0^* + \omega_0\delta\varepsilon(\mathbf{r})\mathbf{E}_0^*\,\partial\mathbf{E}_0(\mathbf{r};\omega)/\partial\omega|_{\omega=\omega_0}\right]\,dV}\,. \tag{108}$$

Now we may expect that the frequency shift will remain finite even if $\delta\varepsilon \to \infty$.

Let us show that formula (108) gives a correct result in the case of a thin semiconductor layer on a plane surface of a cylindrical cavity in the case of TE polarization of the electromagnetic field (this case was studied in detail, using different approaches, in [2, 31, 32]). Having in mind that the laser radiation is absorbed in a very thin layer of the width $l = \alpha^{-1} \sim 10^{-4}$–$10^{-2}$ cm (where α is the absorption coefficient), which is much smaller than the characteristic scale of spatial variations of the field $\mathbf{E}_0$ in an empty cavity (the resonance frequency $f = 2.5$ GHz corresponds to the free space wavelength $\lambda = 12$ cm) and that the creation of electron-hole pairs results in a high conductivity without a noticeable change of the real part of dielectric susceptibility, we can assume that the function $\delta\varepsilon(\mathbf{r})$ inside the slab can be approximated as

$$\delta\varepsilon(\mathbf{r}) = \tilde{\varepsilon}_1 + iBg(\mathbf{r}_\perp)\delta(z), \qquad \tilde{\varepsilon}_1 = \varepsilon_1 - 1, \tag{109}$$

where ε_1 is the lattice dielectric constant of a non-excited semiconductor ($\varepsilon_1 \approx 13$ for GaAs), z is the coordinate in the direction perpendicular to the slab surface (which is assumed to be $z = 0$) and $\mathbf{r}_\perp$ is the two-dimensional vector in the surface plane. The function $g(\mathbf{r}_\perp) \leq 1$ takes into account a possible non-uniformity of the laser pulse in the transverse plane, whereas the coefficient B is equal to the maximal total concentration of electron-hole pairs in the slab per unit surface area (we ascribe the value $\mathbf{r}_\perp = \mathbf{0}$ to this point on the surface, so that $g(\mathbf{0}) = 1$):

$$B = \int_0^D \mathrm{Im}\delta\varepsilon(\mathbf{0}, z)dz = \frac{4\pi b_{tot}|e|}{\omega_0} \int_0^D n(\mathbf{0}, z)dz\,, \qquad g(\mathbf{r}_\perp) = \frac{\int_0^D n(\mathbf{r}_\perp, z)dz}{\int_0^D n(\mathbf{0}, z)dz}. \tag{110}$$

Here D is the slab thickness, n is the concentration of electron-hole pairs, $b_{tot} = b_e + b_h$ is the total mobility of carriers related to the pair and e is the electron charge. Then (108) leads to the following formula for the relative frequency shift with respect to the empty cavity:

$$\left.\frac{\delta\omega}{\omega_0}\right|_{empty} \approx -\frac{iB\mu + \varphi_1}{1 + iB\nu}, \tag{111}$$

where

$$\mu = \frac{1}{\mathcal{W}} \int_\Sigma g(\mathbf{r}_\perp)\, |\mathbf{E}_0(\mathbf{r}_\perp, 0)|^2\, dS, \tag{112}$$

$$\nu = \frac{\omega_0}{\mathcal{W}} \int_\Sigma g(\mathbf{r}_\perp)\mathbf{E}_0^*(\mathbf{r}_\perp, 0; \omega)\, \left.\frac{\partial \mathbf{E}_0(\mathbf{r}_\perp, 0; \omega)}{\partial\omega}\right|_{\omega=\omega_0} dS, \tag{113}$$

$$\mathcal{W} = 2\int_{cav} \mathbf{E}_0\mathbf{E}_0^*\, dV, \qquad \varphi_1 = \frac{1}{\mathcal{W}} \int_{slab} \tilde{\varepsilon}_1\mathbf{E}_0\mathbf{E}_0^*\, dV. \tag{114}$$

The surface integrals in (112) and (113) are taken over the area Σ of the slab (the plane $z = 0$). The term with φ_1 is neglected in the denominator in equation (111), because it is obviously much smaller than unity for limited values of ε_1.

For a non-illuminated semiconductor slab we have $B = 0$, since we consider the case of a low temperature, when the proper conductivity of the semiconductor can be neglected. In this case the resonance frequency is shifted from the empty cavity value by the quantity $-\omega_0\varphi_1$. Consequently, the relative resonance frequency shift between the cavities with illuminated and non-illuminated semiconductors is

$$\frac{\delta\omega}{\omega_0} \approx \frac{iB\nu\,(\varphi_1 - \mu/\nu)}{1 + iB\nu}. \tag{115}$$

The maximal relative frequency shift χ_{max} corresponds to the case $B \to \infty$:

$$\chi_{max} = \varphi_1 - \mu/\nu. \tag{116}$$

On the other hand, taking the same limit in equation (111) we arrive at the frequency shift corresponding to the case when an ideal boundary of an empty cavity is shifted by the distance D inside the cavity: $\chi_{id} = -\mu/\nu$. But the formula for χ_{id} is well known since the papers [118–120]:

$$\chi_{id} = \frac{1}{\mathcal{W}} \int_{slab} \left(|\mathbf{H}_0|^2 - |\mathbf{E}_0|^2\right)\, dV\,. \tag{117}$$

Consequently,

$$\chi_{max} = \frac{1}{\mathcal{W}} \int_{slab} \left(|\mathbf{H}_0|^2 + (\varepsilon_1 - 2)|\mathbf{E}_0|^2\right)\, dV \tag{118}$$

and we can exclude the coefficient ν, expressing it through μ and χ_{id}, which can be calculated more easily. Then the actual frequency shift can be written as

$$\frac{\delta\omega}{\omega_0} \approx \frac{A\chi_{max}}{A+i} = \frac{A(A-i)}{A^2+1}\,\chi_{max}, \tag{119}$$

where

$$A = B\mu/\chi_{id} = B\frac{\int_\Sigma g(\mathbf{r}_\perp)\,|\mathbf{E}_0(\mathbf{r}_\perp,0)|^2\,dS}{\int_{slab}\left(|\mathbf{H}_0|^2 - |\mathbf{E}_0|^2\right)\,dV}\,. \tag{120}$$

The structure of equation (119) is exactly the same as was obtained in [2,31,32]) through finding solutions of the Helmholtz equation in different parts of the cavity and their adjustment on the surface of the slab. Moreover, it can be shown that formulas (118) and (120) give the same values for the coefficients χ_{max} and A as in the cited papers in the case of TE-mode (when the electric field is parallel to the slab surface, so that $|\mathbf{H}_0| \gg |\mathbf{E}_0|$ inside the slab). Formula (119) was verified in an experiment [132].

5. Evaluations of the photon generation rate

The time dependence of parameters $A(t)$ (120) and $B(t)$ (110) is determined by the time dependence of the integral concentration of carriers created by the laser pulse $\int n(z,t)dz$ over the semiconductor slab. It can be found from equations which take into account, besides the photo-absorption, the effect of diffusion and different recombination processes. In the simplest case we have [33]

$$\partial n/\partial t = \nabla\cdot(Y\nabla n) + (\alpha\zeta/E_s)I(t)e^{-\alpha z} - \beta_1 n. \tag{121}$$

Here Y is the coefficient of ambipolar diffusion, α is the absorption coefficient of the laser radiation inside the layer, E_s is the energy gap of the semiconductor (which is close to the energy of laser photons), $I(t)$ is time-dependent intensity of the laser pulse which enters the slab (it can be less than the intensity of the pulse outside the slab, because the reflection coefficient from the semiconductor surface can be rather big, due to the big value of the dielectric constant $\varepsilon_1 \sim 10$; however, the reflection can be diminished if some quarter-wavelength film is put on the surface), $\zeta \le 1$ is the efficiency of the photo-electron conversion and β_1 is the trap-assisted recombination coefficient. We have disregarded nonlinear terms $-\beta_3 n^3 - \beta_2 n^2$ in the right-hand side of (121), because for modeling the DCE one needs very small recombination times, of the order of $T_r \equiv \beta_1^{-1} \sim 20 \div 30\,\text{ps}$. Under these conditions, the contribution of neglected terms is several orders of magnitude smaller than that of the term $\beta_1 n$ [33]. In the most general case, one should use the function $I(t-z/v)$ instead of $I(t)$, where v is the group velocity. But for materials with high absorption coefficients the coordinate dependence can be neglected, if the duration of each pulse is of the order of a few picoseconds. Since equation (121) is linear, it can be solved exactly. The details of calculations can be found in [33, 36].

Here we give the most important results of our previous studies. The maximal photon generation rate is expected under the following conditions.

1) An ideal surface of the semiconductor slab, when the effect of surface recombination can be neglected, so that the boundary condition to equation (121) is $\partial n/\partial z|_{z=0} = 0$.

2) A high absorption coefficient $\alpha \gg D^{-1}$, where D is the slab thickness, so that the presence of the second boundary does not affect the carrier distribution near the irradiated surface.

3) A very short duration of the laser pulse: much less than the recombination time T_r.

4) A uniform illumination of the semiconductor slab.

Under these conditions, the time-dependent function $A(t)$ in equation (119) has a simple form

$$A(\tau) = A_0 e^{-\tau/Z}, \tag{122}$$

where the following dimensionless variables and parameters are introduced:

$$\tau = \omega_0 t \qquad Z = \frac{\omega_0}{\beta_1} = \frac{2\pi T_r}{T_0} \qquad A_0 = \frac{8\pi^2|eb|\zeta W D}{c^2 T_0 E_s S}. \tag{123}$$

Here W is the total energy of the single laser pulse, S is the surface area of the semiconductor slab, T_0 is the period of oscillations in the chosen field mode, b is the effective mobility of a pair of carriers (electrons and holes) and e is the electron charge (we use the Gauss system of units).

According to equation (84), the rate of the photon generation is determined by the difference $\nu - \Lambda$. In the case of the fundamental TE-mode of a rectangular cavity with the longitudinal length L and the transverse length B, this difference can be expressed as follows [2, 33],

$$\nu - \Lambda = \eta^3 \Delta\, F, \qquad \Delta = 2D/\lambda_0, \qquad \eta = \lambda_0/(2L), \qquad F = \tilde{\nu} - \tilde{\Lambda}, \tag{124}$$

where $\lambda_0 = cT_0$ is the wave length of the excited field mode.

The coefficients $\tilde{\nu}$ and $\tilde{\Lambda}$ are given by the expressions following from (76), (79) and (119):

$$\tilde{\nu} = Z \left| \int_0^\infty \frac{dx\, e^{-2iZx} \left[A_0 \exp(-x)\right]^2}{1 + \left[A_0 \exp(-x)\right]^2} \right|, \tag{125}$$

$$\tilde{\Lambda} = Z \int_0^\infty \frac{dx\, A_0 \exp(-x)}{1 + \left[A_0 \exp(-x)\right]^2} = Z \tan^{-1}(A_0)\,. \tag{126}$$

The integral in (125) can be calculated analytically in terms of the Gauss hypergeometric function [36]. The behavior of function $F(A_0, Z)$ was studied in detail in [2]. It was shown that the generation of quanta is impossible ($F < 0$) if $Z > 0.54$ or $A_0 < 4$. For moderate values of parameter A_0, the maximal values of F are achieved for $Z \approx 0.3$, which is an optimal value. The corresponding recombination time is close to $T_r^{opt} = T_0/(2\pi^2)$. For $T_0 = 400\,$ps (or $f_0 = 2.5\,$GHz) this means that $T_r^{opt} \approx 20\,$ps, and in no case T_r should exceed 35 ps. The value $T_r^{opt} \approx 20\,$ps is quite realistic from the point of view of the available technology (and it was confirmed in recent preliminary measurements [5]).

The optimal value of the parameter A_0 can be found in the following way. According to (123), this parameter is proportional to the energy of the laser pulse. On the other hand, if we fix the number of photons which can be created after n pulses, then formula (84) shows that $n \approx const/F$. Consequently, the function $B_0(A_0, Z) = A_0/F(Z, A_0)$ is proportional to the *total energy* of all necessary laser pulses. Choosing for each value of A_0 the optimal value of parameter Z, we obtained [2] the following best choice of parameters (marked with an asterisk), corresponding to the minimal total energy of all pulses:

$$A_{0*} = 11.3, \qquad Z_* = 0.29, \qquad F_* = 0.18, \qquad \tilde{\nu}_* = 0.61, \qquad \tilde{\Lambda}_* = 0.43. \tag{127}$$

Then formula (84) with $G = G_0 = 1$ (i.e., for zero absolute temperature of the initial field state and the cavity walls) and $n\Lambda \gg 1$ can be written as

$$\mathcal{N}_n \approx \frac{\tilde{\nu}}{4F} \exp\left(2\eta^3 F \Delta\, n\right), \qquad \mathcal{N}_{*n} \approx 0.85 \exp\left(0.36\, \eta^3 \Delta\, n\right). \tag{128}$$

A specific property of highly doped semiconductors (which are necessary to achieve a small recombination time) is a low mobility of carriers at zero absolute temperature (because the scattering on charged impurities results in the contribution to the mobility $b \sim \Theta^{3/2}$ for $\Theta \to 0$). Therefore it seems that an optimal strategy could be to prepare somehow only the resonance field mode in the vacuum state, while maintaining the walls at some finite temperature Θ chosen according to the following requirements: a high mobility of carriers, an optimal recombination time and a high quality factor of the cavity. Note that for the frequency $\omega_0/(2\pi) = 2.5\,\text{GHz}$ the temperature gain factor G defined in (28) has the numerical value $G \approx 17\,\Theta$ if $\Theta > 1$ is expressed in Kelvins. Then formula (84) with $G \gg G_0 = 1$ becomes

$$\mathcal{N}_n \approx \frac{G\tilde{\Lambda}}{4F} \exp\left(2\eta^3 F \Delta\, n\right), \qquad \mathcal{N}_{*n} \approx 10\,\Theta \exp\left(0.36\,\eta^3 \Delta\, n\right) \tag{129}$$

and the ratio of the number of created photons to the mean number of thermal photons at temperature Θ is

$$\frac{\mathcal{N}_{*n}}{\langle n\rangle_{th}(\Theta)} \approx 1.2\, \exp\left(0.36\,\eta^3 \Delta\, n\right). \tag{130}$$

An idea to use the initial thermal state of the *field* to facilitate the observation of the dynamical Casimir effect was put forward in [18]. We emphasize that here we assume that the initial field state is *vacuum*, and the amplification of the photon generation rate is due to the finite temperature of the *walls*.

An optimal value of the geometrical factor η can be determined as follows [32, 36]. The resonance wavelength λ, corresponding to the mode TE_{101} with the lowest eigenfrequency of the rectangular cavity, is related to the cavity length L and the biggest transverse dimension B as $\lambda = 2LB/\sqrt{L^2 + B^2}$ (the third dimension should be taken much smaller than L and B in order to diminish the illuminated surface of the slab; this choice excludes automatically the TM mode for the lowest eigenfrequency). Consequently, $B = \lambda/(2\sqrt{1-\eta^2})$. For the fixed values of parameter A_0 and the smallest transverse dimension, the energy of the pulse is proportional to the surface area, i.e., B, whereas the necessary number of pulses depends on L as $\eta^{-3} \sim L^3$. Therefore the total energy is proportional to the product BL^3. Minimizing this product for the fixed value of λ, which is equivalent to maximization of the function $f(\eta) = \eta^3\sqrt{1-\eta^2}$, we find the optimal value $\eta_{opt} = \sqrt{3}/2 = 0.866$, which corresponds to $L = \lambda/\sqrt{3} \approx 7\,\text{cm}$ and $B = \lambda = 12\,\text{cm}$ (for $\omega_0/(2\pi) = 2.5\,\text{GHz}$). However, since it can be difficult to illuminate such a long plate, we should consider other reasonable values of η. Fortunately, the profile of function $f(\eta)$ is rather flat in the vicinity of point η_{opt}. For this reason, the main requirement is to avoid resonances with other cavity eigenfrequencies for the given modulation depth Δ (we considered $D = 2\,\text{mm}$ and $\Delta = 1/30$). These resonances can happen due to the highly anharmonical profile of the frequency shift function (119), and their presence can diminish significantly the photon generation rate in the fundamental mode under consideration [99, 133]. The analysis shows that the best choice, permitting to avoid at least 4 accidental resonances, corresponds to the values of η in the interval between 2/3 and 3/4. Taking $\eta = 3/4$ (i.e., $L = 8\,\text{cm}$ and $B = 9\,\text{cm}$) we have $\eta^3 = 0.42$ and $f(\eta_1) = 0.28$, which is not very far from the maximal value $f(\eta_{opt}) = 0.325$.

Taking $\Delta = 1/30$, $\eta = 3/4$ and using (128), we evaluate that 10^4 photons can be created from vacuum at zero temperature of the cavity walls after about 2000 laser pulses. This number can be reduced to 500 if the walls are maintained at the liquid nitrogen temperature $\Theta = 77\,\text{K}$, according to formula (129) (in this case $\langle n\rangle_{th}(\Theta) \approx 650$).

If the third dimension can be reduced to 2.5 mm [134], then taking $T_r = 20\,\text{ps}$, $b = 0.7\,\text{m}^2\text{V}^{-1}\text{s}^{-1} \approx 2.1 \times 10^6$ CGS units and $E_g = 1.4\,\text{eV}$ (GaAs) we find that the optimal value of parameter $A_0 \approx 10$ can be achieved for the energy of a single laser pulse about 0.1 mJ. In such a case, the total energy of 500 pulses must be about 50 mJ. It can be diminished if materials with a higher mobility can be found.

What can happen if conditions 1)–4) are not satisfied completely? This can be taken into account by using the function $A(t)$ in the form

$$A(\tau) = \exp(-\tau/Z)\mathcal{J}(\tau) \tag{131}$$

with some 'formfactor' $\mathcal{J}(\tau)$. In this general case integrals generalizing (125) and (126) can be calculated only numerically. For example, the general boundary condition to equation (121) in the case of non-ideal surface is $Y\partial n/\partial z|_{z=0} = Rn(0)$, where R is the surface recombination velocity. This case was analyzed in detail in [33]. The corresponding formfactor has the form

$$\mathcal{J}_g(\tau,h) = \frac{1}{g-1}\left[ge^{h\tau}\mathrm{Erfc}\left(\sqrt{h\tau}\right) - e^{h\tau g^2}\mathrm{Erfc}\left(g\sqrt{h\tau}\right)\right], \qquad g = R/(\alpha Y), \quad h = \alpha^2 Y/\omega_0,$$

where $\mathrm{Erfc}(x)$ is the complementary error function. It was shown that for nonzero surface recombination velocity an increase of parameter h (or the absorption coefficient) diminishes the value of F. Also, an increase of h diminishes the maximal possible values of the recombination time (parameter Z) for which $F > 0$ (if $h \gg 1$). Usual mechanically polished semiconductor surfaces have the surface recombination velocity $R \sim 10^6 \div 10^7\,\mathrm{cm/s}$ [135]. Using the Einstein relation $Y = k_B\Theta|b/e|$ we estimated the diffusion coefficient at $\Theta = 4\,\mathrm{K}$ as $Y \sim 3\,\mathrm{cm}^2/\mathrm{s}$ for $b \sim 1\,\mathrm{m}^2\mathrm{V}^{-1}\mathrm{s}^{-1}$. Thus for usual surfaces the parameter g is greater than unity, moreover, it can be much greater than unity for the values of absorption coefficient smaller than $10^6\,\mathrm{cm}^{-1}$. Therefore in order to have sufficiently big values of the amplification factor F, parameter h must be smaller than unity (by an order of magnitude). For the frequency 2.5 GHz this means that the absorption coefficient α should not exceed the value of an order of $10^5\,\mathrm{cm}^{-1}$. Using some special procedures (e.g., etching the surface) one can reduce the surface recombination velocity to the values of the order of $10^2 \div 10^3\,\mathrm{cm/s}$ [135]. In such a case one has $g < 1$ for $\alpha > 10^2\,\mathrm{cm}^{-1}$, so that the influence of parameters g and h becomes insignificant. One can also diminish parameter g by means of increasing the diffusion coefficient (or the mobility). But in this case parameter h *increases* in the same proportion. It was shown in [33] that one should avoid big values of α and R, trying to maintain the parameter $M = gh = R\alpha/\omega_0$ in the region $M < 1$. For instance, preferable values of α are less than $10^3\,\mathrm{cm}^{-1}$ for 'bad' surfaces with $R \sim 10^7\,\mathrm{cm/s}$. Or one should keep $R < 10^4\,\mathrm{cm/s}$ if $\alpha \sim 10^6\,\mathrm{cm}^{-1}$.

Another reason for introducing a formfactor in equation (131) is a finite duration of laser pulses or non-uniformity of illumination. This subject was studied in [36], where different functions $\mathcal{J}(t)$ were considered. It was concluded that the non-uniformity of illumination or temporal spread of laser pulses will not deteriorate significantly the rate of photon generation if the recombination time is in the interval 10-20 ps.

In the derivation of the formulas for the frequency shift in section 4 we assumed that only the imaginary part of the complex dielectric function is changed due to the generation of electron-hole pairs in the semiconductor slab. A change of the real part can be taken into account, if one replaces the factor B in equations (109)–(120) by $B(1-i\mu)$, where $\mu = \mathrm{Re}(\Delta\varepsilon)/\mathrm{Im}(\Delta\varepsilon)$ and $\Delta\varepsilon$ is the total change of the dielectric function [35, 36]. In the case of the Drude model we have

$$b = e\tau_c/m_{ef}, \qquad \mu = \omega_0\tau_c = \omega_0 b m_{ef}/|e|,$$

where τ_c is the mean time between collisions and m_{ef} an effective mass of carriers. For realistic values $b \sim 1\,\mathrm{m}^2\mathrm{V}^{-1}\mathrm{s}^{-1}$, $m_{ef} \sim m_0$ (the mass of free electron) and $\omega_0/(2\pi) = 2.5\,\mathrm{GHz}$ we obtain $\mu \sim 0.1 \ll 1$. It was shown in [35] that the amplification coefficient $F = \tilde{\nu} - \tilde{\Lambda}$ *decreases* as function of parameter μ in the interval $0 < \mu < 1$, although it can become significantly bigger than $F(0)$ for $\mu \gg 1$. The latter case corresponds to the plasma model considered in [28–30] (see also a recent paper [136]), where the role of dissipation (the imaginary part of dielectric function) was neglected. However, at low frequencies (of an order of a few GHz) this can happen only in materials with extremely high mobilities of carriers, $b \gg |e|/(m_{ef}\omega_0) \sim 10\,\mathrm{m}^2\mathrm{V}^{-1}\mathrm{s}^{-1}$, which probably do not exist.

6. Summary

I have described several theoretical achievements of the past few years, which are connected with planned experiments on the observation of the dynamical Casimir effect. One of them is the construction of a consistent model of a damped nonstationary quantum oscillator with arbitrary time-dependent frequency and damping coefficients, based on the generalization of the Senitzky–Schwinger–Haus–Lax noise operator approach. New results given in this paper include formulas for the distribution function of photons created in a resonantly excited field mode and the fluctuations of the number of photons.

Another achievement consists in a generalization of the Müller–Schwinger–Bethe–Casimir formula for the shift of the complex resonance frequency of a cavity to the case of strong variations of the complex dielectric permeability inside a thin inhomogeneous slab. The new formula can be applied to thin semiconductor slabs irradiated by laser pulses of arbitrary shapes in space and time. The theoretical analysis and concrete numerical evaluations performed in this and previous papers show that an experimental demonstration of the dynamical Casimir effect is a quite feasible task which can be achieved in the nearest future.

Acknowledgments

A partial support of the Brazilian agency CNPq is acknowledged.

References

[1] Dodonov V V 2001 Nonstationary Casimir effect and analytical solutions for quantum fields in cavities with moving boundaries *Modern Nonlinear Optics* (*Advances in Chemical Physics Series* vol 119, part 1) ed M W Evans (New York: Wiley) pp 309–94 (*Preprint* arXiv: quant-ph/0106081)
[2] Dodonov V V and Dodonov A V 2005 Quantum harmonic oscillator and nonstationary Casimir effect *J. Russ. Laser Res.* **26** 445–83
[3] Barton G, Dodonov V V and Man'ko V I (guest editors) 2005 Special issue on Nonstationary Casimir Effect and Quantum Systems with Moving Boundaries *J. Opt. B: Quantum Semiclass. Opt.* **7** S1–126
[4] Braggio C, Bressi G, Carugno G, Del Noce C, Galeazzi G, Lombardi A, Palmieri A, Ruoso G and Zanello D 2005 A novel experimental approach for the detection of the dynamic Casimir effect *Europhys. Lett.* **70** 754–60
[5] Agnesi A, Braggio C, Bressi G, Carugno G, G Galeazzi G, Pirzio F, Reali G, Ruoso G and Zanello D 2008 MIR status report: an experiment for the measurement of the dynamical Casimir effect *J. Phys. A: Math. Theor.* **41** 164024
[6] Kim W-J, Brownell J H and Onofrio R 2006 Detectability of dissipative motion in quantum vacuum via superradiance *Phys. Rev. Lett.* **96** 200402
[7] Brownell J H, Kim W J and Onofrio R 2008 Modelling superradiant amplification of Casimir photons in very low dissipation cavities *J. Phys. A: Math. Theor.* **41** 164026
[8] Segev E, Abdo B, Shtempluck O, Buks E and Yurke B 2007 Prospects of employing superconducting stripline resonators for studying the dynamical Casimir effect experimentally *Phys. Lett.* A **370** 202–6
[9] Dodonov V V 1995 Photon creation and excitation of a detector in a cavity with a resonantly vibrating wall *Phys. Lett.* A **207** 126–32
[10] Dodonov V V and Klimov A B 1996 Generation and detection of photons in a cavity with a resonantly oscillating boundary *Phys. Rev.* A **53** 2664–82
[11] Moore G T 1970 Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity *J. Math. Phys.* **11** 2679–91
[12] Askar'yan G A 1962 Interaction between laser radiation and oscillating surfaces *Sov. Phys. – JETP* **15** 1161–2
[13] Rivlin L A 1979 Note on parametric birth of photons in model problems of cosmology *Sov. J. Quant. Electron.* **9** 1320–1
[14] Sarkar S 1988 Moving mirrors and nonclassical light *Photons and Quantum Fluctuations* ed E R Pike and H Walther (Bristol: Hilger) pp 151–72
[15] Lambrecht A, Jaekel M T and Reynaud S 1996 Motion induced radiation from a vibrating cavity *Phys. Rev. Lett.* **77** 615–18
[16] Dodonov V V 1998 Resonance photon generation in a vibrating cavity *J. Phys. A: Math. Gen.* **31** 9835–54

[17] Dalvit D A R and Mazzitelli F D 1999 Creation of photons in an oscillating cavity with two moving mirrors *Phys. Rev.* A **59** 3049–59

[18] Plunien G, Schützhold R and Soff G 2000 Dynamical Casimir effect at finite temperature *Phys. Rev. Lett.* **84** 1882–5

[19] Husimi K 1953 Miscellanea in elementary quantum mechanics. II *Prog. Theor. Phys.* **9** 381–402

[20] Yablonovitch E 1989 Accelerating reference frame for electromagnetic waves in a rapidly growing plasma: Unruh–Davies–Fulling–De Witt radiation and the nonadiabatic Casimir effect *Phys. Rev. Lett.* **62** 1742–5

[21] Man'ko V I 1992 Time-dependent invariants and nonclassical light *Quantum Measurements in Optics* (*NATO ASI Series B: Physics* vol 282) ed P Tombesi and D F Walls (New York: Plenum) pp 239–46

[22] Grishchuk L P, Haus H A and Bergman K 1992 Generation of squeezed radiation from vacuum in the cosmos and the laboratory *Phys. Rev.* D **46** 1440–9

[23] Okushima T and Shimizu A 1995 Photon emission from a false vacuum of semiconductors *Japan. J. Appl. Phys.* **34** 4508–10

[24] Lozovik Y E, Tsvetus V G and Vinogradov E A 1995 Parametric excitation of vacuum by use of femtosecond laser pulses *Phys. Scripta* **52** 184–90

[25] Braggio C, Bressi G, Carugno G, Lombardi A, Palmieri A, Ruoso G and Zanello D 2004 Semiconductor microwave mirror for a measurement of the dynamical Casimir effect *Rev. Sci. Instrum.* **75** 4967–70

[26] Cirone M, Rzążewski K and Mostowski J 1997 Photon generation by time-dependent dielectric: A soluble model *Phys. Rev.* A **55** 62–6

[27] Cirone M and Rzążewski K 1999 Electromagnetic radiation in a cavity with a time-dependent mirror *Phys. Rev.* A **60** 886–92

[28] Crocce M, Dalvit D A R, Lombardo F C and Mazzitelli F D 2004 Model for resonant photon creation in a cavity with time-dependent conductivity *Phys. Rev.* A **70** 033811

[29] Uhlmann M, Plunien G, Schützhold R and Soff G 2004 Resonant cavity photon creation via the dynamical Casimir effect *Phys. Rev. Lett.* **93** 193601

[30] Barton G and Calogeracos A 1995 On the quantum electrodynamics of a dispersive mirror. I. Mass shifts, radiation, and radiative reaction *Ann. Phys.* (NY) **238** 227–67

[31] Dodonov V V 2005 Time-dependent quantum damped oscillator with 'minimal noise': application to the nonstationary Casimir effect in non-ideal cavities *J. Opt. B: Quantum Semiclassical Opt.* **7** S445–51

[32] Dodonov V V and Dodonov A V 2006 Nonstationary Casimir effect in a cavity with periodical time-dependent conductivity of a semiconductor mirror *J. Phys. A: Math. Gen.* **39** 6271–81

[33] Dodonov V V and Dodonov A V 2006 QED effects in a cavity with time-dependent thin semiconductor slab excited by laser pulses, *J. Phys. B: At. Mol. Opt. Phys.* **39** S749–66

[34] Dodonov V V and Dodonov A V 2006 The Heisenberg–Langevin model of a quantum damped harmonic oscillator with time-dependent frequency and damping coefficients *J. Russ. Laser Res.* **27** 379–88

[35] Dodonov V V and Dodonov A V 2006 Generation of photons from vacuum in a cavity with time-dependent eigenfrequency and damping coefficient *Acta Phys. Hung. B – Quantum Electronics* **26** 29–36

[36] Dodonov V V and Dodonov A V 2008 Theory of the dynamical Casimir effect in nonideal cavities with time-dependent parameters *J. Phys.: Conference Series* **99** 012006

[37] Montazeri M and Miri M 2005 Motion-induced radiation from a dynamically deforming mirror: Neumann boundary condition *Phys. Rev.* A **71** 063814

[38] Fosco C D, Lombardo F C and Mazzitelli F D 2007 Quantum dissipative effects in moving mirrors: A functional approach *Phys. Rev.* D **76** 085007

[39] Alves D T, Granhen E R and Lima M G 2008 Quantum radiation force on a moving mirror with Dirichlet and Neumann boundary conditions for a vacuum, finite temperature, and a coherent state *Phys. Rev.* D **77** 125001

[40] Montazeri M and Miri M 2008 Radiation from a dynamically deforming mirror immersed in the electromagnetic vacuum *Phys. Rev.* A **77** 053815

[41] Dalvit D A R, Mazzitelli F D and Millan X O 2006 The dynamical Casimir effect for different geometries *J. Phys. A: Math. Gen.* **39** 6261–70

[42] Alves D T, Farina C and Granhen E R 2006 Dynamical Casimir effect in a resonant cavity with mixed boundary conditions *Phys. Rev.* A **73** 063818

[43] Alves D T and Granhen E R 2008 Energy density and particle creation inside an oscillating cavity with mixed boundary conditions *Phys. Rev.* A **77** 015808

[44] Sarabadani J and Miri M 2006 Mechanical response of the quantum vacuum to dynamic deformations of a cavity *Phys. Rev.* A **74** 023801

[45] Sarabadani J and Miri M 2007 Motion-induced radiation in a cavity with conducting and permeable plates *Phys. Rev.* A **75** 055802

[46] Mintz B, Farina C, Neto P A M and Rodrigues R B 2006 Casimir forces for moving boundaries with Robin

conditions *J. Phys. A: Math. Gen.* **39** 6559–65
[47] Farina C 2006 The Casimir effect: Some aspects *Braz. J. Phys.* **36** 1137–49
[48] Wegrzyn P 2006 An optical approach to the dynamical Casimir effect *J. Phys. B: At. Mol. Opt. Phys.* **39** 4895–4903
[49] Wegrzyn P 2007 Exact closed-form analytical solutions for vibrating cavities *J. Phys. B: At. Mol. Opt. Phys.* **40** 2621–40
[50] Fedotov A M, Lozovik Y E, Narozhny N B and Petrosyan A N 2006 Dynamical Casimir effect in a one-dimensional uniformly contracting cavity *Phys. Rev.* A **74** 013806
[51] Fedotov A M, Narozhny N B, Petrosyan A N and Lozovik Y E 2007 On the dynamical Casimir effect in a one-dimensional contracting cavity *Laser Phys.* **17** 310–5
[52] Crocce M, Dalvit D A R, Lombardo F C and Mazzitelli F D 2005 Hertz potentials approach to the dynamical Casimir effect in cylindrical cavities of arbitrary section *J. Opt. B: Quantum Semiclassical Opt.* **7** S32–9
[53] Mazzitelli F D and Millan X O 2006 Photon creation in a spherical oscillating cavity *Phys. Rev.* A **73** 063829
[54] Pascoal F, Céleri L C, Mizrahi S S and Moussa M H Y 2008 Dynamical Casimir effect for a massless scalar field between two concentric spherical shells *Phys. Rev.* A **78** 032521
[55] Setare M R and Dinani H T 2008 Particle creation in oscillating cavities with cubic and cylindrical geometry *Acta Phys. Polon.* B **39** 771–80
[56] Yuce C and Ozcakmakli Z 2008 The dynamical Casimir effect for two oscillating mirrors in 3D *J. Phys. A: Math. Theor.* **41** 265401
[57] Haro J 2006 Pair production in a cavity with moving boundaries *J. Phys. A: Math. Gen.* **39** 6433–40
[58] Haro J and Elizalde E 2006 Hamiltonian approach to the dynamical Casimir effect *Phys. Rev. Lett.* **97** 130401
[59] Haro J 2007 Dynamical Casimir effect for scalar fields I (particle creation) *Int. J. Theor. Phys.* **46** 1003–19
[60] Haro J 2007 Dynamical casimir effect for scalar fields II (energy calculation) *Int. J. Theor. Phys.* **46** 951–71
[61] Haro J and Elizalde E 2007 Physically sound Hamiltonian formulation of the dynamical Casimir effect *Phys. Rev.* D **76** 065001
[62] Haro J and Elizalde E 2008 Dynamical Casimir effect and the black body spectrum *J. Phys. A: Math. Theor.* **41** 032002
[63] Haro J and Elizalde E 2008 Black hole collapse simulated by vacuum fluctuations with a moving semitransparent mirror *Phys. Rev.* D **77** 045011
[64] Elizalde E 2008 Dynamical Casimir effect with semi-transparent mirrors, and cosmology *J. Phys. A: Math. Theor.* **41** 164061
[65] Durrer R and Ruser M 2007 Dynamical Casimir effect in braneworlds *Phys. Rev. Lett.* **99** 071601
[66] Ruser M and Durrer R 2007 Dynamical Casimir effect for gravitons in bouncing braneworlds *Phys. Rev.* D **76** 104014
[67] Ruser M 2005 Vibrating cavities: a numerical approach *J. Opt. B: Quantum Semiclassical Opt.* **7** S100–15
[68] Ruser M 2006 Numerical approach to the dynamical Casimir effect *J. Phys. A: Math. Gen.* **39** 6711–23
[69] Ruser M 2006 Numerical investigation of photon creation in a three-dimensional resonantly vibrating cavity: Transverse electric modes *Phys. Rev.* A **73** 043811
[70] Rosu H C 2005 Quantum vacuum radiation and detection proposals *Int. J. Theor. Phys.* **44** 493–528
[71] Sorge F 2006 Dynamical Casimir Effect in a kicked box *Int. J. Mod. Phys.* A **21** 6173–82
[72] Mendonça J T, Brodin G and Marklund M 2008 Vacuum effects in a vibrating cavity: time refraction, dynamical Casimir effect, and effective Unruh acceleration *Phys. Lett.* A **372** 5621–4
[73] Srivastava Y N, Widom A, Sivasubramanian S and Pradeep Ganesh M 2006 Dynamical Casimir effect instabilities *Phys. Rev.* A **74** 032101
[74] Shvartsburg A B 2005 Optics of nonstationary media *Physics–Uspekhi* **48** 797–823
[75] Mendonça J T and Guerreiro A 2005 Time refraction and the quantum properties of vacuum *Phys. Rev.* A **72** 063805
[76] Tozzo C, Krämer M and Dalfovo F 2005 Stability diagram and growth rate of parametric resonances in Bose-Einstein condensates in one-dimensional optical lattices *Phys. Rev.* A **72** 023613
[77] Ciuti C, Bastard G and Carusotto I 2005 Quantum vacuum properties of the intersubband cavity polariton field *Phys. Rev.* B **72** 115303
[78] Lawandy N M 2006 Scattering of vacuum states by dynamic plasmon singularities: generating photons from vacuum *Optics Lett.* **31** 3650–2
[79] De Liberato S, Ciuti C and Carusotto I 2007 Quantum vacuum radiation spectra from a semiconductor microcavity with a time-modulated vacuum Rabi frequency *Phys. Rev. Lett.* **98** 103602
[80] Takashima K, Hatakenaka N, Kurihara S and Zeilinger A 2008 Nonstationary boundary effect for a quantum flux in superconducting nanocircuits *J. Phys. A: Math. Theor.* **41** 164036
[81] Bialynicki-Birula I and Bialynicka-Birula Z 2008 Electromagnetic radiation by gravitating bodies *Phys. Rev.* A **77** 052103

[82] Carusotto I, Antezza M, Bariani F, De Liberato S and Ciuti C 2008 Optical properties of atomic Mott insulators: From slow light to dynamical Casimir effects *Phys. Rev.* A **77** 063621
[83] Saito H and Hyuga H 2008 Dynamical Casimir effect for magnons in a spinor Bose-Einstein condensate *Phys. Rev.* A **78** 033605
[84] Bialynicki-Birula I and Bialynicka-Birula Z 2008 Dynamical Casimir effect in oscillating media *Phys. Rev.* A **78** 042109
[85] Zagoskin A M, Il'ichev E, McCutcheon M W, Young J F and Nori F 2008 Controlled generation of squeezed states of microwave radiation in a superconducting resonant circuit *Phys. Rev. Lett.* **101** 253602
[86] Sarkisyan E M, Petrosyan K G, Oganesyan K B, Hakobyan A A, Saakyan V A, Gevorkian S G, Izmailyan N S and Hu C K 2008 Detection of Casimir photons with electrons *Laser Phys.* **18** 621–4
[87] Westlund P O and Wennerström H 2005 Photon emission from translational energy in atomic collisions: A dynamic Casimir-Polder effect *Phys. Rev.* A **71** 062106
[88] Passante R, Persico F and Rizzuto L 2007 Nonlocal field correlations and dynamical Casimir–Polder forces between one excited- and two ground-state atoms *J. Phys. B: At. Mol. Opt. Phys.* **40** 1863–74
[89] Rizzuto L, Passante R and Persico F 2007 Nonlocal properties of dynamical three-body Casimir–Polder forces *Phys. Rev. Lett.* **98** 240404
[90] Vasile R and Passante R 2008 Dynamical Casimir–Polder force between an atom and a conducting wall *Phys. Rev.* A **78** 032108
[91] Dodonov A V, Dodonov V V and Mizrahi S S 2005 Separability dynamics of two-mode Gaussian states in parametric conversion and amplification *J. Phys. A: Math. Gen.* **38** 683–96
[92] Andreata M A and Dodonov V V 2005 Dynamics of entanglement between field modes in a one-dimensional cavity with a vibrating boundary *J. Opt. B: Quantum Semiclassical Opt.* **7** S11–20
[93] Schützhold R and Tiersch M 2005 Decoherence versus dynamical Casimir effect *J. Opt. B: Quantum Semiclassical Opt.* **7** S120–5
[94] Dodonov V V, Andreata M A and Mizrahi S S 2005 Decoherence and transfer of quantum states of field modes in a one-dimensional cavity with an oscillating boundary *J. Opt. B: Quantum Semiclassical Opt.* **7** S468–79
[95] Celeri L C, Pascoal F, de Ponte M A and Moussa M H Y 2008 Number of particle creation and decoherence in the nonideal dynamical Casimir effect at finite temperature *Preprint* arXiv: 0805.0786
[96] Man'ko V I 1991 The Casimir effect and quantum vacuum generator *J. Sov. Laser Res.* **12** 383–5
[97] Sassaroli E, Srivastava Y N and Widom A 1994 Photon production by the dynamical Casimir effect *Phys. Rev.* A **50** 1027–34
[98] Schützhold R, Plunien G and Soff G 1998 Trembling cavities in the canonical approach *Phys. Rev.* A **57** 2311–8
[99] Crocce M, Dalvit D A R and Mazzitelli F D 2001 Resonant photon creation in a three-dimensional oscillating cavity *Phys. Rev.* A **64** 013808
[100] Senitzky I R 1960 Dissipation in quantum mechanics. The harmonic oscillator *Phys. Rev.* **119** 670–9
[101] Schwinger J 1961 Brownian motion of a quantum oscillator *J. Math. Phys.* **2** 407–32
[102] Haus H A and Mullen J A 1962 Quantum noise in linear amplifiers *Phys. Rev.* **128** 2407–13
[103] Lax M 1966 Quantum noise. IV. Quantum theory of noise sources *Phys. Rev.* **145** 110–45
[104] Dodonov V V and Man'ko V I 1987 Density matrices and Wigner functions of quasiclassical quantum systems *Group Theory, Gravitation and Elementary Particle Physics (Proceedings of Lebedev Physics Institute, vol 167)* ed A A Komar (Commack: Nova Science) p 7–101
[105] Dodonov V V 2003 Parametric excitation and generation of nonclassical states in linear media *Theory of Nonclassical States of Light* ed V V Dodonov and V I Man'ko (London: Taylor & Francis) pp 153–218
[106] Dodonov V V, Man'ko O V and Man'ko V I 1995 Quantum nonstationary oscillator: models and applications *J. Russ. Laser Res.* **16** 1–56
[107] Dodonov V V, Man'ko O V and Man'ko V I 2002 Reduced density matrices of oscillator systems *Preprint* arXiv: quant-ph/0210030
[108] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
[109] Faddeev L D 1958 On the relation between the S-matrix and potential for the one-dimensional Schroedinger equation *Sov. Phys. – Doklady* **3** 747–751
[110] Born M and Wolf E 1975 *Principles of Optics* (Cambridge: Cambridge University Press) section 1.6.5
[111] Ginzburg V L 1967 *Propagation of Electromagnetic Waves in Plasmas* (Moscow: Nauka) pp 292-4
[112] Dodonov A V and Dodonov V V 2005 Resonance generation of photons from vacuum in cavities due to strong periodical changes of conductivity in a thin semiconductor boundary layer *J. Opt. B: Quantum Semiclassical Opt.* **7** S47–58
[113] McNeil K J and Walls D F 1975 Possibility of observing enhanced photon bunching from two photon emission *Phys. Lett.* A **51** 233–4

[114] Sotskii B A and Glazachev B I 1981 Entropy properties of superrandom fields *Opt. Spectrosc.* **50** 582–4
[115] Dodonov V V, Man'ko O V and Man'ko V I 1994 Photon distribution for one-mode mixed light with a generic Gaussian Wigner function. *Phys. Rev.* A **49** 2993–3001
[116] Dodonov V V and Man'ko V I 1994 New relations for two-dimensional Hermite polynomials *J. Math. Phys.* **35** 4277–94
[117] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic Press) pp 463–74
[118] Müller J 1939 Untersuchungen über Electromagnetische Hohlraume *Hochfrequenztechnik und Elektroakustik* **54** 157–61
[119] Bethe H A and Schwinger J 1943 Perturbation theory for cavities *National Defense Research Committee report D1-117* (Cornell University)
[120] Casimir H B G 1951 On the theory of electromagnetic waves in resonant cavities *Philips Res. Rep.* **6** 162–82
[121] Hsieh H-T, Goldey J M and Brown S C 1954 A resonant cavity study of semiconductors *J. Appl. Phys.* **25** 302–7
[122] Spencer E G, LeCraw R C and Ault L A 1957 Note on cavity perturbation theory *J. Appl. Phys.* **28** 130–2
[123] Karpova O V 1959 On an absolute method of measurement of dielectric properties of a solid using a Π-shaped resonator *Sov. Phys. – Solid State* **1** 220–8
[124] Eldumiati I I and Haddad G I 1972 Cavity perturbation techniques for measurement of the microwave conductivity and dielectric constant of a bulk semiconductor material *IEEE Trans. Microwave Theory Tech.* **MTT-20** 126–32
[125] Chao S-H 1985 Measurements of microwave conductivity and dielectric constant by the cavity perturbation method and their errors *IEEE Trans. Microwave Theory Tech.* **MTT-33** 519–26
[126] Peligrad D-N, Nebendahl B, Kessler C, Mehring M, Dulčić A, Požek M and Paar D 1998 Cavity perturbation by superconducting films in microwave magnetic and electric fields *Phys. Rev.* B **58** 11652
[127] Peligrad D-N, Nebendahl B, Mehring M, Dulčić A, Požek M and Paar D 2001 General solution for the complex frequency shift in microwave measurements of thin films *Phys. Rev.* B **64** 224504
[128] Arndt G D, Hartwig W H and Stone J L 1968 Photodielectric detector using a superconducting cavity *J. Appl. Phys.* **39** 2653–6
[129] Grabtchak S and Cocivera M 1998 Microwave response due to light-induced changes in the complex dielectric constant of semiconductors *Phys. Rev.* B **58** 4701–7
[130] Van Bladel J 1964 *Electromagnetic Fields* (New York: McGraw-Hill) chap 10
[131] Landau L D and Lifshitz E M 1982 *Electrodynamics of Continuous Media* (Moscow: Nauka) section 90
[132] Braggio C, Bressi G, Carugno G, Dodonov A V, Dodonov V V, Ruoso G and Zanello D 2007 Resonance frequency shift in a cavity with a thin conducting film near a conducting wall *Phys. Lett.* A **363** 33–7
[133] Dodonov A V and Dodonov V V 2001 Nonstationary Casimir effect in cavities with two resonantly coupled modes *Phys. Lett.* A **289** 291–300
[134] Braggio C and Carugno G 2008 private communication
[135] Aspnes D E 1983 Recombination at semiconductor surfaces and interfaces *Surface Science* **132** 406–21
[136] Kawakubo T and Yamamoto K 2008 A standard Hamiltonian formulation for the dynamical Casimir effect *Preprint* arXiv: 0808.3618