STRUCTURE OF SINGULARITIES OF WIGHTMAN-DISTRIBUTIONS AND THE WILSON-ZIMMERMANN-EXPANSION RESPECTIVELY LIGHTCONE-EXPANSION

Siegfried Schlieder
Max-Planck-Institut für Physik und Astrophysik
München, Germany

Abstract: For a relativistic quantum field theory of WIGHTMAN type necessary and sufficient conditions are formulated for the existence of a WILSON-ZIMMERMANN-expansion (short distance expansion). One can deduce for those fields, which fulfil our sufficient conditions from the short-distance expansion a lightcone expansion.

1. Introduction

A field theory in which the principle of contact interaction shall be valid needs for the definition of the interaction terms products of field quantities taken at the same position.

If one intends to apply this principle also in a relativistic quantum field theory one has to define products of field operators at the same position. The difficulties which arise in this procedure are very well known. They have their origin in the distributive character of the field operators.

In the case of the free field $A_0(x)$ with $x = (x_0, x_1, x_2, x_3)$ however it is well known how one can get $A_0^n(x)$, $n = 2, 3, \ldots$ in the form of WICK products of the field. In the last years a heuristic ansatz for the products of field-operators at the same position in the case for interacting fields has been proposed especially by WILSON and ZIMMERMANN [1, 2, 3, 4] and by BRANDT [5]. For instance for a real scalar field WILSON and ZIMMERMANN assume the expansion

$$A(x+\chi_1)A(x+\chi_2)\cdots A(x+\chi_n)=\sum_{j=1}^{m} s_j(\chi_1,\chi_2,\cdots\chi_n)\,B_j(x)+R_m(x;\chi_1,\chi_2,\cdots,\chi_n)$$

The $B_j(x)$ are field operators relatively local to $A(x)$, the $s_j(\chi_1,\cdots,\chi_n)$ functions which become in general singular for $\chi_j \to 0$, $j = 1, 2, \ldots, n$, so that

$$\lim_{\chi_1,\cdots\chi_n\to 0} \frac{s_{j+1}(\chi_1,\cdots,\chi_n)}{s_j(\chi_1,\cdots,\chi_n)} = 0 .$$

With respect to the operator $R_m(x;\chi_1,\cdots,\chi_n)$ it is assumed that

$$\lim_{\chi_1,\cdots\chi_n\to 0} \frac{R_m(x;\chi_1,\cdots,\chi_n)}{s_m(\chi_1,\cdots,\chi_n)} = 0 .$$

Thereby the operators $B_j(x)$ are candidates for $A^n(x)$. ZIMMERMANN [2] and BRANDT [5] have proved the validity of such an expansion for certain examples in perturbation theory, WILSON and ZIMMERMANN [3] gave conditions under which the expansion is valid and they discussed several consequences of it; furthermore LOWENSTEIN [6] has proved the expansion rigorously for the THIRRING model.

The aim of the main part of this talk is to study the question of the existence of a short distance expansion (synonym for WILSON-ZIMMERMANN expansion) for the class of relativistic quantum field theories fulfilling WIGHTMAN's conditions. To keep the formalism as simple as possible we restrict our attention to the cases of a real scalar field A(x) and the products of only two operators.

We intend to use the characterisation of the field theory by WIGHTMAN distributions

$$\begin{aligned}&(\Omega, A(x_1)\cdots A(x_j)A(x_{j+1})\cdots A(x_n)\Omega)\\ &= \mathcal{W}_n(x_1,\cdots,x_j,x_{j+1},\cdots,x_n) = W_n(\xi_1,\cdots,\xi_j,\cdots,\xi_{n-1})\end{aligned}$$

with $\xi_j = x_{j+1} - x_j$ $\qquad j = 1, 2, \ldots n - 1.$

If a short distance expansion exists, then the structures of the singularities for the n-point functions should show a certain uniform behaviour, independent of n and independent of j for $\xi_j \to 0$. This is the content of section 2. In section 3 sufficient conditions are given for an uniform structure of the singularities. Section 4 connects a set of WIGHTMAN distributions showing an uniform singular behaviour with the WILSON-ZIMMERMANN expansion. Relations between a short distance expansion and a lightcone expansion are the content of section 5.

2. Necessary Conditions for the Existence of the Short-Distance Expansion

According to the remarks in the introduction one can show that the assumption of the existence of the WILSON-ZIMMERMANN expansion implies that singularities appearing for $\xi_j \to 0$ in

$$W_n(\xi_1,\cdots,\xi_j,\cdots,\xi_{n-1})$$

have their counterparts in singularities appearing in $W_4(\xi_1,\xi_2,\xi_3)$ for $\xi_3 \to 0$

For the proof of this statement one has to assume that the expansion is an expansion for operators. If it is an expansion which yields for $\xi_j \to 0$ only a sum of bilinear forms, then the singularities appearing for $\xi_j \to 0$ in W_n must be found again for $\xi_1, \xi_3 \to 0$ simultaneously in $W_4(\xi_1, \xi_2, \xi_3)$. The proofs can be found in [7] .

3. Conditions for an uniform structure of the singularities

The problem is to show, that the singularities which arise if in

$$W_n(\xi_1, \dots, \xi_j, \dots \xi_{n-1})$$

$\xi_j \to 0$ are essentially not dependent on n and on j. More precisely: It is intended in the following to give such conditions, that in the limit above no other singularities can appear, than those which arise in

$$W_4(\xi_1, \xi_2, \xi_3)$$

for $\xi_3 \to 0$. Regarding the cluster properties of the n-point-functions, one recognises easily that for n even, $n > 4$, at least the same singluarities as in the case W_4 and $\xi_3 \to 0$ must appear. What sufficient conditions, however, can be given so that no other and probably more serious singularities appear if n is growing?

Before formulating them it may be of advantage to give at first an idea of the procedure:

Clearly the singularity structure of WIGHTMAN distributions is characterized by the behaviour of the analytic WIGHTMAN functions near the boundary. In the forward tubes τ_n^+ these holomorphic functions can be interpreted as scalar-products of holomorphic states: For instance [see 8]

$$W_4(\xi_1, \xi_2, \xi_3) = \left(\Phi_2(\bar{z}_1, \bar{z}_0), \Phi_2(z_2, z_3)\right)$$

with $\qquad \xi_i = z_i - z_{i-1} \; , \; \xi_i \in \tau^+$

and $\qquad \Phi_2(z, z') = A(z)\, A(z')\,\Omega = \int e^{i(pz + p'z')}\, \tilde{A}(p)\, \tilde{A}(p')\, dp\, dp' \cdot \Omega$

with $\quad z' - z \in \tau^+ \; , \; z \in \tau^+ .$

Now one makes the assumption, that W_4 can be written as

$$W_4(\xi_1, \xi_2, \xi_3) = \sum_{k=1}^{l} G_{4,k}(\xi_1, \xi_2, \xi_3)\, u_k(\xi_3) \quad \text{with} \quad \xi_j \in \mathbb{C}^4$$

where the u_k carry the singularities at $\xi_3 = 0$, and the $G_{4,k}$ are not only holomorphic for $\{\xi_1, \xi_2, \xi_3\} \in \tau_3^+$ but also in $\{\xi_1, \xi_2\} \in \tau_2^+ , \xi_3 \in \tau^+ \cup \mathcal{U}_r(0)$ with $\mathcal{U}_r(0)$ as a real neighbourhood of 0.

Under certain conditions it is possible to regard also the $G_{4,k}$ k = 1, 2, ..., l as scalar-products of states which are holomorphic even at $\zeta_3 = 0$.

$W_n(\zeta_1, \zeta_2, \cdots, \zeta_{n-1})$ then can be regarded as a sum of l terms, where each again can be interpreted as a scalar-product of states. The states on the right in these scalar-products can be identified with those, which appear in $G_{4,k}$, k = 1, 2, ..., l, to the right. So a corresponding expansion of W_n into l terms

$$W_n(\zeta_1, \cdots, \zeta_{n-1}) = \sum_{k=1}^{l} G_{n,k}(\zeta_1, \cdots, \zeta_{n-1})\, u_k(\zeta_{n-1})$$

with $\zeta_j \in \mathbb{C}^4$ and u_k as before is possible.

$G_{n,k}$ are holomorphic for $\{\zeta_1, \cdots, \zeta_{n-1}\} \in \tau^+_{n-1}$ and also holomorphic for $\{\zeta_1, \cdots, \zeta_{n-2}\} \in \tau^+_{n-2}$ and $\zeta_{n-1} \in \tau^+ \cup \mathcal{U}_r(0)$. One can generalise this procedure by allowing that the $G_{4,k}$ respective the $G_{n,k}$ are not holomorphic, however, meromorphic with the condition that a pole at $\zeta_3 = 0$ respectively $\zeta_{n-1} = 0$ is excluded. This procedure shows that the structure of singularities appearing in W_n for $\zeta_{n-1} \to 0$ must appear also in W_4 for $\zeta_3 \to 0$.

After that one can try to use the invariance of the WIGHTMAN function $\mathcal{W}_n(z_1, \cdots, z_n)$ under permutation of its arguments to get a similar structure of the singularities for the case that not the last variable $\zeta_{n-1} \to 0$ however an arbitrary variable $\zeta_j \to 0$; j = 1, 2, ... n-1.

Now I wish to give conditions under which the described procedure works. At first the most simple case is treated:

(A) Let us assume that $W_4(\zeta_1, \zeta_2, \zeta_3)$ is such, that a function $r(\zeta) \neq 0$ exists with the following properties:

a) r is holomorphic in τ^+

b) $r(\zeta) = \overline{r(-\bar{\zeta})}$

c) r is invariant under the (homogeneous) Lorentz group

d) $W_4(\zeta_1, \zeta_2, \zeta_3)\, r(\zeta_3)$ has a (locally unique) analytic continuation to the points $\{\zeta_1, \zeta_2\} \in \tau^+_2$, $\zeta_3 \in \mathcal{U}_r(0)$ where $\mathcal{U}_r(0)$ is a real neighbourhood of 0 independent of $\{\zeta_1, \zeta_2\}$ and $W_4(\zeta_1, \zeta_2, 0)\, r(0) \neq 0$

e) $r(\zeta) \in \mathcal{D}'(\mathcal{U}_r(0))$. (D $(U_r(0))$ test-functions with support in $U_r(0)$).

<u>Theorem 1.</u>: Assuming the conditions (A) $W_n(\xi) \tau(\xi_j)$ has an (locally unique) analytic continuation to the points $\xi \in \tau^+_{n-2}\ , \xi_j = 0$.

For the proof of the theorem see 7 .
Examples for this simple case characterized by the conditions (A) are:
The free scalar field with mass 0: $\Box A_0(x) = 0$
Wickproducts of this field
Exponentials of the free field with arbitrary mass in 2-dimensions.
For the more general case, where expansion of the 4-point-functions contains a finite number of terms it is suitable to split the assumptions into three groups. (A') replaces (A):

(A') It is assumed that an expansion of W_4

$$W_4(\xi_1,\xi_2,\xi_3) = \sum_{k=1}^{l} G_{4,k}(\xi_1,\xi_2,\xi_3)\, u_k(\xi_3)$$

exists with the following properties:

a_1.) $u_k(\xi)\ ,\ \ k = 1,2,\cdots,l$ are holomorphic in τ^+, they are Lorentzinvariant and

$$u_k(\xi) = \overline{u_k(-\bar{\xi})}$$

a_2.) $G_{4,k}(\xi_1,\xi_2,\xi_3)$ are holomorphic in τ^+_3 and can be continued as holomorphic functions to the points $\{\xi_1,\xi_2\} \in \tau^+_2\ , \xi_3 = 0$ (in a locally unique manner).

a_3.) $\{u_1, u_2, \ldots, u_l\}$ are linearily independent over the field of the meromorphic functions at $\xi = 0$.

The second group of conditions (B') shall guarantee, that an expansion

$$\Phi_2(z, z+\xi) = \sum_{k=1}^{l} \Phi_{2,k}(z, z+\xi)\, u_k(\xi)$$

exists where the $\Phi_{2,k}$ are vectorstates holomorphic for $\{z,\xi\} \in \tau^+ \times [\tau^+ \cup U_r(0)]$.
Since (B') is more or less of technical nature, it will not be given here explicitly (see [9]).

From (A') and (B') one derives

<u>Theorem 2.</u>: There exists an expansion $W_n(\xi_1,\cdots,\xi_{n-1}) = \sum_1^l G^{(n-1)}_{n,k}(\xi_1,\cdots,\xi_{n-1})\, u_k(\xi_{n-1})$
with $G^{(n-1)}_{n,k}(\xi_1,\cdots,\xi_{n-1})$ holomorphic in $\{\xi_1,\cdots,\xi_{n-2}\} \in \tau^+_{n-2}\ , \xi_{n-1} \in [\tau^+ \cup U_r(0)]$

To get the corresponding expansion for an arbitrary argument ζ_j

$$W_n(\zeta_1,\cdots,\zeta_j\cdots,\zeta_{n-1}) = \sum_{k=1}^{L} G_{n,k}^{(j)}(\zeta_1,\cdots,\zeta_j,\cdots,\zeta_{n-1})\, u_k(\zeta_j)$$

with $G_{n,k}$ holomorphic in $\zeta_j \in [\tau^+ \cup \mathcal{U}_r(0)]$ one can formulate the following sufficient conditions:

(C') It exists an expansion

$$W_n(\zeta_1,\cdots,\zeta_j,\cdots,\zeta_{n-1}) = \sum_{k=1}^{\infty} H_{n,k}^{(j)}(\zeta_1,\cdots,\zeta_j,\cdots,\zeta_{n-1})\, w_k(\zeta_j)$$

with

c$_1$.) $w_k(\zeta_j) = \overline{w_k(-\overline{\zeta}_j)}$, $k = 1,2,3,\cdots$

c$_2$.) $H_{n,k}^{(j)}$ holomorphic for $\zeta_j \in [\tau^+ \cup \mathcal{U}_r(0)]$, $\zeta_{\nu} \in \tau_{n-2}^+$.

Remark: All known examples of WIGHTMAN distributions fulfill the conditions (A') (B') and (C'). For the details see [9].

4. Short Distance Expansion

Given a theory as described before one is able to use the holomorphy of $\{G_{n,k}^{(j)}\}$ at $\zeta_j = 0$ to formulate a short distance expansion for $A(x_j)A(x_{j+1})$ with $x_{j+1}-x_j \to 0$ and express the singularities by $u_k(\zeta_j)$.

Technically there are different versions of the short distance expansion possible ([7], [9]). I will give here the simplest one:

The WILSON-ZIMMERMANN expansion

$$A(x+\chi)\, A(x-\chi) = \sum_{j=1}^{m} s_j(\chi)\, B_j(x) + R_m(x,\chi)$$

exists as an expansion in bilinear forms over $\overline{\mathcal{J}} \times \mathcal{J}$, where $\mathcal{J}$ is the linear hull of the vectors

$$\{\Omega, \{\Phi_1(z_1')\}, \{\Phi_2(z_1'', z_2'')\}, \cdots, \{\Phi_n(z_1^{(n)},\cdots,z_n^{(n)})\}, \cdots\} \quad ; \quad \mathcal{J} \text{ is}$$

dense in H.

5. Lightcone Expansion

From the Lorentzinvariance of the $\{u_k(\zeta_j)\}$ and their independence it follows the Lorentzinvariance of the $\{G_{n,k}^{(j)}(\zeta_1,\cdots,\zeta_j,\cdots\zeta_{n-1})\}$. Using this property one is able to enlarge the assumed holomorphy domain of the $\{G_{n,k}^{(j)}\}$. The interesting part of this enlargement is, that the real neighbourhood $\mathcal{U}_r(0)$ of the origin goes over into a real neighbourhood of the lightcone. Using the holomorphy of

$G^{(j)}_{n,k}(\xi_1,\cdots,\xi_j,\cdots,\xi_{n-1})$ for ξ_j in this neighbourhood one is able to formulate an expansion of $A(x+\chi)A(x-\chi)$ with χ on the lightcone (the $\{u_k(\xi_j)\}$ are again responsible for the singularities) (see [9], [10], [11]).

References:

1. Wilson, K.: Cornell Report (1964); Phys. Rev. 179, 1499 (1969).
2. Zimmermann, W.: Commun. math. Phys. 6, 161 (1967); 10, 325 (1968).
3. Wilson, K., Zimmermann, W.: Commun. math. Phys. 24, 87 (1972).
4. Otterson, P., Zimmermann, W.: Commun. math. Phys. 24,107 (1972).
5. Brandt, R.A.: Ann. Phys. (N.Y.) 44, 221 (1967), 52, 122 (1969).
6. Lowenstein, J.H.: Commun. math. Phys. 16, 265 (1970).
7. Schlieder, S., Seiler, E.: Commun. math. Phys. 31, p 137-159 (1973).
8. Jost, R.: The General Theory of Quantized Fields. Providence (AMS) 1965.
9. Baumann, K.: Thesis 1974 and Prepr.: Januar 1975.
10. Kühn, J., Seiler, E.: Commun. math. Phys. 33, 253-257 (1973) and
11. Kühn, J.: Thesis 1974.