COMPLEX QUATERNIONS AND SPINOR REPRESENTATIONS OF DE SITTER GROUPS $SO(4,1)$ AND $SO(3,2)$

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(Received June 18, 1977)

The quaternion algebra over the field of complex numbers is used for the realization of transformations from the de Sitter groups $SO(4,1)$ and $SO(3,2)$ by the two-row matrices. Within this approach a technique of the so-called horospherical shifts is developed, which are of great importance in quantum field theory with the fundamental length. The spinor representations of both the groups are also constructed.

But the virtue of the quaternion lies not so much as yet in solving hard questions, as in enabling us to see the meaning of the question and of its solution.

J. C. Maxwell

1.

Quantum field theory with momentum space of constant curvature has been developed in the papers [1–7]. It has been shown that in the Bogolubov axiomatic approach [8–10] the ordinary flat momentum space off the mass shell can be replaced by a space of constant curvature. Quantum field theory based on this hypothesis is uncontradictory and represents an alternative to the traditional theory.

Both possible variants of the curved momentum space, which possess metrics

$$g_{KL}p^K p^L = p_0^2-p_1^2-p_2^2-p_3^2-p_4^2 = -1, \tag{1.1}$$

$$\hat{g}_{KL}p^K p^L = p_0^2-p_1^2-p_2^2-p_3^2+p_4^2 = 1, \tag{1.2}$$

and motion groups $SO(4, 1)$ and $SO(3,2)$ respectively, were considered.

Fourier-analysis employing the main series of the unitary irreducible representations of de Sitter groups allowed one to introduce an adequate configuration representation,

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the causality condition in the new configuration space being a straightforward generalization of the Bogolubov causality condition. There are two operations of translation, which are of importance in the theory with curved p-space. Namely, these are curved and orispherical translations. In the present paper we shall develop a convenient technique for describing these transformations with the help of hypercomplex realizations of the covering groups of de Sitter groups $SO(4,1)$ and $SO(3,2)$[1]. This realization will also serve as a basis for constructing spinor (finite-dimensional, non-unitary) representations of de Sitter groups [11]. The spinor representations are necessary for constructing equations of motion and configuration representation for particles with spin.

We call the hypercomplex realization a group of 2×2 matrices with elements that belong to the algebra $\{K\}$ of hypercomplex numbers (quaternions) over the field of complex numbers $\{Z\}$. The use of the complex quaternions is of importance: exactly in this very case a uniform treatment of all the real branches of the complex orthogonal group is possible. These branches turn into one another under analytic continuations in the components of the 5-momentum. In other words, we acquire an apparatus that enables us to treat uniformly both $SO(4,1)$ and $SO(3,2)$-variants of quantum field theory and analogues of their euclidean formulations.

The algebra $\{K\}$ comprises singular elements — devisors of zero. This fact is immediately connected with the existence of isotropic vectors in subspaces corresponding to the reduction onto pseudoorthogonal groups of lesser dimension. Classical real quanternions arise in constructing the universal covering of the $SO(4,1)$ group [12] (positively-defined quadratic form $p_4^2+p_1^2+p_2^2+p_3^2$ corresponding to the reduction $SO(4,1) \supset SO(3,1)$ coincides with the norm of the hamiltonian quaternion). Universal covering $SL(2, C)$ [13, 14] of the Lorentz group arises in reducing to the subalgebra of complex number $\{Z\}$, (see formula 2.11d).

It is worth noting that the quaternions were repeatedly considered as number systems for quantum mechanics [15]. In papers [16] the quaternions were applied to the consideration of the Lorentz-group representations and to the construction of wave equation in spaces of constant curvature.

2. *Basic definitions*

Let us consider an algebra $\{K\}$ of hypercomplex numbers of the form

$$a = a_0 + a_i\sigma_i \quad (i = 1, 2, 3), \tag{2.1}$$

where a_0, $a_i \in \{K\}$ and $\{\sigma_i\}$ is a system of three imaginary units with the table of multiplication

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{2.2}$$

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[1] A hypercomplex realization of the $O(3,2)$-group transformations in application to the field theory with curved p-space has been considered in the paper [7].

This system can be realized with the help of the Pauli matrices. Later we shall use this fact in construction of the spinor representations. The multiplication of the quaternions by a complex number $z \in \{Z\}$, addition and subtraction are defined as follows:

$$za = za_0 + za_i\sigma_i, \tag{2.3a}$$

$$a+b = (a_0+b_0)+(a_i+b_i)\sigma_i, \tag{2.3b}$$

$$a-b = (a_0-b_0)+(a_i-b_i)\sigma_i = a+(-b). \tag{2.3c}$$

The formula (2.2) insures the product of two quaternions to be a quaternion again.

The operation of quaternion conjugation is given by the formula

$$a \to \bar{a} = a_0 - a_i\sigma_i, \tag{2.4}$$

and for any $a, b \in \{K\}$

$$(\overline{ab}) = \bar{b}\,\bar{a}. \tag{2.5}$$

The quaternion norm is defined by the relation

$$||a|| = \bar{a}a = a\bar{a}. \tag{2.6}$$

It is evident that the norm is a complex number; it can be, in particular, negative or zero. In the case $||a|| \neq 0$ we can define the inverse to a element of the algebra $\{K\}$

$$a^{-1} = \frac{\bar{a}}{||a||}, \tag{2.7a}$$

$$aa^{-1} = a^{-1}a = 1. \tag{2.7b}$$

We shall call zero norm elements of the algebra $\{K\}$ singular elements or devisors of zero: the inverse a^{-1} is not defined for such elements.

Let us introduce the operation of hermitian conjugation of the quaternions[2]

$$a \to a^+ = a_0^* + a_i^*\sigma_i, \tag{2.8a}$$

$$(ab)^+ = b^+a^+, \tag{2.8b}$$

and also the operation of complex conjugation of the quaternions

$$a \to \hat{a} = a_0^* - a_i^*\sigma_i, \tag{2.9a}$$

$$(\widehat{ab}) = \hat{a}\hat{b}. \tag{2.9b}$$

A successive application of any of the two operations $-$, $+$ or $\wedge$ is equivalent to the application of the third one.

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[2] Asterisk denotes the ordinary complex conjugation.

For example

$$(\bar{a})^+ = \hat{a}. \tag{2.10}$$

Besides these we shall also need the operation of partial conjugation

$$a \to \tilde{a} = \sigma_3 \bar{a} \sigma_3 = a_0 + \sigma_1 a_1 + \sigma_2 a_2 - \sigma_3 a_3, \tag{2.11a}$$

$$\widetilde{(ab)} = \tilde{b}\tilde{a}. \tag{2.11b}$$

The algebra of the complex quaternions (2.1) contains three closed subalgebras which can be sorted out in the following way

$$a = a_0 + a_i \sigma_i, \quad a_\mu \in \{Z\}, \tag{2.12a}$$

$$b = b_0 + b_k i\sigma_k, \quad b_\mu \in \{R\}, \tag{2.12b}$$

$$c = c_0 + c_1 \sigma_1 + c_2 \sigma_2 + c_3 i\sigma_3, \quad c_\mu \in \{R\}, \tag{2.12c}$$

$$d = d_0 + d_1 i\sigma_3, \quad d_{0,1} \in \{R\}, \tag{2.12d}$$

$\{R\}$ being the field of the real numbers. The system (2.12d) is isomorphic to the ordinary complex numbers. The system (2.12b) coincides with the classical (real) quaternions [16]. Three imaginary units that build up the basis of the classical quaternions are related to σ_i by the formulae

$$i = -i\sigma_1, \quad j = i\sigma_2, \quad k = -i\sigma_3. \tag{2.13}$$

The system (2.12c), unlike the systems (2.12b) and (2.12d), contains divisors of zero. Further we shall see that the branches (2.12b) and (2.12c) arise in the natural way in analysing covering groups of the $SO(4,1)$ and $SO(3,2)$ groups.

Let us now consider 2×2 matrices

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.14}$$

a, b, c, d being quaternions of the form (2.1). We denote by u^+ the "hermitian conjugate" matrix

$$u^+ = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \tag{2.15}$$

one can easily verify that the determinant is uniquely defined for hypercomplex hermitian matrices ($u^+ = u$):

$$\det u = ad - bc = da - cb, \tag{2.16}$$

which is not valid for hypercomplex matrices in general. (The hermitian conjugation of hypercomplex matrices should be distinguished from the hermitian conjugation of quaternions that has been introduced earlier.)

3. *Hypercomplex realization of the universal covering group of the SO(4,1)-group*

Let us introduce hermitian matrices

$$\tau^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \tau^4 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (k = 1, 2, 3). \tag{3.1}$$

Now we can relate any 5-vector P_L to the corresponding matrix

$$\mathscr{P} = p_L \tau^L = \begin{pmatrix} -ip_4 & \bar{p} \\ p & ip_4 \end{pmatrix}, \quad p = p_0 - \vec{p}\vec{\sigma}. \tag{3.2}$$

It is obvious that

$$\mathscr{P}^+ = \mathscr{P}$$

$$\det \mathscr{P} = -g_{KL} p^K p^L. \tag{3.3}$$

Let u be a unitary quaternion matrix ($u^+ = u^{-1}$) of the form (2.13).

The conditions

$$u^+ u = 1, \tag{3.4a}$$

$$u u^+ = 1, \tag{3.4b}$$

produce two (equivalent) systems of restrictions on the elements of the matrix u:

$$a\bar{a} + b\bar{b} = c\bar{c} + d\bar{d} = 1, \quad a\bar{c} + b\bar{d} = 0, \tag{3.5a}$$

$$\bar{a}a + \bar{c}c = \bar{b}b + \bar{d}d = 1, \quad \bar{a}b + \bar{c}d = 0. \tag{3.5b}$$

Let us impose on the elements of the matrix u also the restrictions

$$a^+ = \bar{d}, \quad b^+ = \bar{c}. \tag{3.6}$$

One can easily prove that the set of matrices u that satisfy the relations (3.4), (3.6) makes up a 10 (real)-parameter group U. Transformations

$$\mathscr{P}' = u \mathscr{P} u^+, \; u \in U \tag{3.7}$$

conserve the quadratic form (3.3b). It is evident that matrices u and $-u$ induce the same transformation of the 5-vector p_L. Thus, the matrix group U is the universal covering in respect to the $SO(4,1)$-group.

Let us also define the matrix group V that is connected with U through a similarity transformation

$$V = SUS^{-1}, \tag{3.8}$$

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{3.9}$$

If a matrix $v \in V$

$$v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.10}$$

is related to a matrix $u \in U$ through the transformation (3.8) then we get the following correspondence between the elements of these matrices:

$$\alpha = \tfrac{1}{2}(a-ib+ic+d), \quad \beta = \tfrac{1}{2}(-ia+b+c+id), \tag{3.11}$$
$$\gamma = \tfrac{1}{2}(ia+b+c-d), \quad \delta = \tfrac{1}{2}(a+ib-ic+d).$$

The matrix v^{-1} is

$$v^{-1} = S^2 v^+ S^{-2} = \begin{pmatrix} \bar{\delta} & \bar{\beta} \\ \bar{\gamma} & \bar{\alpha} \end{pmatrix}. \tag{3.12}$$

The elements α, β, γ, δ are subject to the conditions

$$\alpha\bar{\delta}+\beta\bar{\gamma} = 1, \quad \alpha\bar{\beta}+\beta\bar{\alpha} = \gamma\bar{\delta}+\delta\bar{\gamma} = 0 \tag{3.13a}$$

or equivalently

$$\bar{\delta}\alpha+\bar{\beta}\gamma = 1, \quad \bar{\gamma}\alpha+\bar{\alpha}\gamma = \bar{\delta}\beta+\bar{\beta}\delta = 0. \tag{3.13b}$$

Moreover, the formulae (3.6) and (3.11) give

$$\alpha^+ = \bar{\alpha}, \quad \beta^+ = \bar{\beta}, \quad \gamma^+ = \bar{\gamma}, \quad \delta^+ = \bar{\delta}. \tag{3.14}$$

Let us introduce a matrix

$$\not{p} = p_L \pi^L = \begin{pmatrix} p_0 - p_4 & ip_K\sigma_K \\ -ip_K\sigma_K & p_0+p_4 \end{pmatrix}, \tag{3.15}$$

$$\det \not{p} = -g_{KL}p^K p^L, \tag{3.16}$$

where the hermitian matrices π^L are defined as follows:

$$\pi^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi^k = \begin{pmatrix} 0 & -i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \pi^4 = \begin{pmatrix} -1 & 0 \\ & 0 & 1 \end{pmatrix}. \tag{3.17}$$

The transformation of the matrix $\not{p}$ with a matrix $v \in V$

$$\not{p}' = v\not{p}v^+ \tag{3.18}$$

also conserves the quadratic form (3.16b).

Due to the relations (3.13), (3.14), the following lemma can be easily proved: any element $v \in V$ satisfying the condition $||\alpha|| \neq 0$ can be uniquely represented in the form

$$v = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \tag{3.19}$$

where

$$z = iz_K\sigma_K, \quad \mu = i\mu_K\sigma_K,$$
$$\xi = \xi_0 + i\xi_K\sigma_K, \quad ||\xi|| = 1,$$
$$t, z_K, \mu_K, \xi_0, \xi_K \in \{R\}. \tag{3.20}$$

The relations (3.20) stipulate that the elements of the matrix v belong to the subalgebra (2.12b), i.e. to the subalgebra of the classical quaternions.

The relation (3.19) immediately gives a natural parametrization for the elements of the matrix v

$$\alpha = e^{t/2}\xi, \quad \beta = e^{t/2}\xi\mu, \quad \gamma = e^{t/2}z\xi, \quad \delta = e^{t/2}z\xi\mu + e^{-t/2}\xi. \tag{3.21}$$

It is also important that due to the factorization (3.19) we get various subgroups of the group V.

Let us consider a subgroup $\Omega \subset V$ which includes the matrices of the following type:

$$\omega = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} e^{t/2} & 0 \\ e^{t/2}z & e^{-t/2} \end{pmatrix}. \tag{3.22}$$

This four-parametric subgroup has important physical applications [2,5] (see also [18]).

We call a set of points that arises from one fixed point of the hyperboloid (1.1) under the action of transformations

$$v_\omega = v^{-1}\omega v, \tag{3.23}$$

where ω runs over the whole subgroup Ω (cf. [13]), an orisphere of de Sitter space. Thus, the orisphere is defined by a point of de Sitter space and an element v of de Sitter group.

Supposing the 5-vector p_L is of the form $p = (0, 0, 0, 0, 1)$ and $v = \omega$, we derive from (3.18) a relation that maps the set of parameters of the group Ω onto the hyperboloid (1.1)

$$\not{p} = \omega \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \omega^+. \tag{3.24}$$

Having equated the corresponding elements of the matrices in (3.24), we get an orispherical coordinate system on this hyperboloid (cf. [2])

$$p_4 - p_0 = e^t, \quad p_4 + p_0 = e^{-t} - e^t z_k^2, \quad p_k = e^t z_k. \tag{3.25}$$

The group of transformations Ω induces on the hyperboloid (1.1) a certain group operation — a group of orispherical translations. When applied to a 5-vector, this operation will be denoted with the symbol $\oplus$:

$$p' = p \oplus q. \tag{3.26}$$

The formula (3.26) is equivalent to the relations

$$t' = t + s, \quad z'_K = e^{-s} z_K + w_K, \tag{3.27}$$

(t, z_K), (t', z'_K) and (s, w_K) being the orispherical coordinates of the 5-vectors p, p' and q, respectively.

4. Hypercomplex realization of the covering group of the SO(3,2)-group

Let us first note that the quadratic form (1.1) can be transformed into the form (1.2) by the substitution $p_4 \to ip_4$. This procedure can be also applied for deriving other formulae concerning the covering of the $SO(3,2)$-group. Thus, the analysis of the $SO(3,2)$-group is very much alike to that of the $SO(4,1)$-group.

Substituting, for example, $p_4 \to ip_4$ in (3,2) we get

$$\mathscr{P} = \begin{pmatrix} p_4 & \bar{p} \\ p & -p_4 \end{pmatrix} = p_L \tau^L, \tag{4.1}$$

$$\det \mathscr{P} = -\hat{g}_{KL} p^K p^L, \tag{4.2}$$

the hermitian matrices τ^L now being of the form

$$\tau^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \tau^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

The covering group U of the $SO(3,2)$-group is formed by the unitary matrices u that satisfy the conditions (3.5) and also the relations[3]

$$a^+ = \bar{d}, \quad b^+ = -\bar{c}. \tag{4.4}$$

The transition to the group V is now carried out with the help of the transformation

$$v = TuT^{-1}, \tag{4.5}$$

where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sigma_3 \\ \sigma_3 & 1 \end{pmatrix}. \tag{4.6}$$

The relation between the elements of the matrices v and u reads:

$$\alpha = \tfrac{1}{2}(a - b\sigma_3 - \sigma_3 c + \sigma_3 d\sigma_3), \quad \beta = \tfrac{1}{2}(a\sigma_3 + b - \sigma_3 c\sigma_3 - \sigma_3 d), \tag{4.7}$$
$$\gamma = \tfrac{1}{2}(\sigma_3 a - \sigma_3 b\sigma_3 + c - d\sigma_3), \quad \delta = \tfrac{1}{2}(\sigma_3 a\sigma_3 + \sigma_3 b + c\sigma_3 + d).$$

Th inverse matrix v^{-1} is

$$v^{-1} = T^2 v^+ T^{-2} = \begin{pmatrix} \tilde{\delta} & -\tilde{\beta} \\ -\tilde{\gamma} & \tilde{\alpha} \end{pmatrix}, \tag{4.8}$$

$\sim$ denoting the operation of partial conjugation (2.11).

The elements α, β, γ, δ are subject to the conditions

$$\alpha\tilde{\delta} - \beta\tilde{\gamma} = 1, \quad \gamma\tilde{\delta} - \delta\tilde{\gamma} = \beta\tilde{\alpha} - \alpha\tilde{\beta} = 0, \tag{4.9a}$$

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[3] The universal covering of the group $SO(3,2)$ has an infinite discrete centre. Nevertheless the "minimal" covering group is sufficient for the construction of all the spinor representations of the group $SO(3,2)$.

or equivalently

$$\tilde{\delta}\alpha-\tilde{\beta}\gamma = 1, \quad \tilde{\alpha}\gamma-\tilde{\gamma}\alpha = \tilde{\delta}\beta-\tilde{\beta}\delta = 0. \tag{4.9b}$$

Moreover, owing to the formulae (4.4) and (4.7), we have the restrictions

$$\alpha^+ = \tilde{\alpha}, \quad \beta^+ = \tilde{\beta}, \quad \gamma^+ = \tilde{\gamma}, \quad \delta^+ = \tilde{\delta}. \tag{4.10}$$

Let us now define a matrix $\not{p}$ according to the formula

$$\not{p} = p_L\pi^L = \begin{pmatrix} p_3+p_4 & p_0-p_1\sigma_1-p_2\sigma_2 \\ p_0+p_1\sigma_1+p_2\sigma_2 & p_3-p_4 \end{pmatrix}, \tag{4.11}$$

$$\det \not{p} = -\hat{g}_{KL}p^Kp^L. \tag{4.12}$$

The hermitian matrices π^L are defined as follows:

$$\pi^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi^{1,2} = \begin{pmatrix} 0 & -\sigma_{1,2} \\ \sigma_{1,2} & 0 \end{pmatrix}, \quad \pi^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pi^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.13}$$

A lemma analogous to that of the previous paragraph is valid for the transformations v

$$\not{p}' = v\not{p}v^+, \tag{4.14}$$

which belong to the group V. Namely, any element $v \in V$ satisfying the condition $||\alpha|| \neq 0$ can be uniquely represented in the form

$$v = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\begin{pmatrix} \xi & 0 \\ 0 & \tilde{\xi} \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \tag{4.15}$$

where

$$z = z_0+z_1\sigma_1+z_2\sigma_2, \quad \mu = \mu_0+\mu_1\sigma_1+\mu_2\sigma_2$$

$$\xi = \xi_0+\xi_1\sigma_1+\xi_2\sigma_2+i\xi_3\sigma_3, \quad ||\xi|| = \pm 1.$$

$$z_0,\ z_K,\ \mu_0,\ \mu_K,\ \xi_0,\ \xi_K,\ t \in \{R\}. \tag{4.16}$$

The relations (4.16) show that the parameters of the group V in the present case appertain to the branch (2.12c).

Everything told in the previous paragraph about the orispheres can be transferred to the case of the $SO(3,2)$-group.

Consideration of the subgroup $\Omega \subset V$ of the triangular matrices

$$\omega = \begin{pmatrix} e^{t/2} & 0 \\ ze^{t/2} & e^{-t/2} \end{pmatrix} \tag{4.17}$$

leads, in complete analogy to the formulae (3.24), (3.25), to the pseudooripherical coordinates on the hyperboloid (1.2)

$$p_3+p_4 = e^t, \quad p_3-p_4 = (z_0^2-z_1^2-z_2^2)e^t-e^{-t}, \quad p_{0,1,2} = z_{0,1,2}e^t \tag{4.18}$$

and to the pseudoorispherical translations

$$p' = p \oplus q, \tag{4.19a}$$

$$t' = t+s, \quad z'_K = e^{-s} z_K + w_K, \tag{4.19b}$$

(t, z_K), (t', z'_K), (s, w_K) being the pseudoorispherical coordinates of the vectors p, p' and q.

5. *Spinor representations of the groups SO(4,1) and SO(3,2)*

We get the lowest spinor representation of the $SO(4,1)$-group if we substitute in the hypercomplex matrices (3.4)–(3.6) Pauli matrices for the quaternions σ_i.

The matrices τ^L (3.1) are transformed under this substitution into matrices Γ^L that appertain to the algebra of Dirac matrices[4]

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^4 = \gamma^5 \quad (\mu = 0, 1, 2, 3). \tag{5.1}$$

These matrices are subject to the relation

$$\Gamma_K = -\frac{1}{4!} \varepsilon_{KLMNP} \Gamma^L \Gamma^M \Gamma^N \Gamma^P, \tag{5.2}$$

ε_{KLMNP} being the antisymmetric tensor ($\varepsilon_{01234} = 1$).

The set of 15 Dirac matrices divides in accordance with their transformation properties into $SO(4,1)$-vector and $SO(4,1)$-antisymmetric tensor

$$M^{KL} = \frac{i}{4} [\Gamma^K, \Gamma^L]. \tag{5.3}$$

The matrices (5.3) are infinitesimal operators of the lowest spinor representation of the $SO(4,1)$-group. Finite transformations of this group corresponding to the hypercomplex matrices u will be denoted by A. The relations (3.5), (3.6) lead to the following restrictions on the matrices A

$$\mathscr{C} A^T \mathscr{C}^{-1} A = 1, \quad \Gamma^0 A^+ \Gamma^0 A = 1, \tag{5.4}$$

where

$$\mathscr{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} = \Gamma_0 \Gamma_2 \Gamma_4, \quad C = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.5}$$

The restrictions (5.4) provide that the matrix of a finite transformation

$$\Lambda^L{}_M = \tfrac{1}{4} \operatorname{Sp} (\Gamma^L A \Gamma_M A^{-1})$$

satisfies the conditions

$$\Lambda^0{}_0 > 0, \quad \Lambda^L{}_M \Lambda^M{}_K = \delta^L{}_K.$$

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[4] We use the representation of Dirac matrices accepted in [10] (see formula 2.4.15).

In order to single out a connected group we shall also impose the condition $\det A = 1$, which is consistent with (5.4).

Thus, the group of the fourth-order matrices satisfying the conditions

$$\mathscr{C} A^T \mathscr{C}^{-1} A = 1, \quad \Gamma^0 A^+ \Gamma^0 A = 1, \quad \det A = 1 \tag{5.6}$$

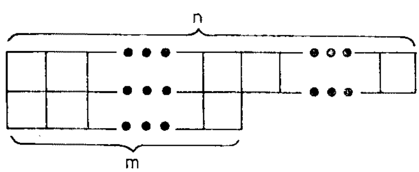
forms the universal covering of the $SO(4,1)$-group. Let us introduce spinors ξ_α, ξ^α, η_α, η^α which transform with the help of the matrices A, $(A^T)^{-1}$, A^*, $(A^+)^{-1}$ respectively. It is evident that owing to the relations (5.6) all these representations are equivalent. Therefore we restrict ourselves to the spinors with lower undotted indices. These indices should be raised with the help of the matrix $\mathscr{C}$ (5.5)

$$\xi_\alpha = \mathscr{C}_{\alpha\beta}\xi^\beta, \quad \xi^\alpha = (\mathscr{C}^{-1})^{\alpha\beta}\xi_\beta; \quad \mathscr{C}^T = -\mathscr{C}.$$

We should also mention that there are two invariant spin-tensors: the spin-tensor of the second rank $\mathscr{C}_{\alpha\beta}$ and the antisymmetric spin-tensor of the fourth rank $\varepsilon_{\alpha\beta\gamma\delta}$ ($\varepsilon_{1234} = 1$). The irreducible representations of the group $SO(4,1)$ are realized by the spinors, which satisfy the condition

$$(\mathscr{C}^{-1})^{\alpha\beta}\chi_{\alpha_1 \cdots \alpha \cdots \beta \cdots \alpha_n} = 0$$

in any pair of indices and, moreover, possess a certain symmetry under the permutation of indices. Owing to the existence of the antisymmetric spin-tensor $\varepsilon_{\alpha\beta\gamma\delta}$, we can construct all the spinor representations of the group $SO(4,1)$ using only the spinors of the symmetry that corresponds to the Young tableaux of the following type:



In the present paper we shall restrict ourselves to a more detailed consideration of the symmetric spinors which are of interest for physical applications.

A representation realized by a symmetric spinor of the rank n

$$\Phi_{\alpha_1 \cdots \alpha_n}$$

is of dimension $(n+1)(n+2)(n+3)/3!$, the Casimir operators C_1 and C_2 [19] acquire in such representations the following values

$$C_1 = \tfrac{1}{2}\hat{M}_{KL}\hat{M}^{KL} = \frac{n(n+4)}{2}, \quad C_2 = W_K W^K = \frac{n(n+4)}{2}\,\frac{(n+2)^2}{8}. \tag{5.7}$$

The expression for the generators $\hat{M}_{KL}$ reads

$$\hat{M}_{KL} = \sum_{i=1}^{n} (M_{KL})_i \tag{5.8}$$

and the operator W_K is defined by the relation

$$W_K = \frac{1}{8}\,\varepsilon_{KLMNS}\hat{M}^{LM}\hat{M}^{NS} = \frac{n+2}{4}\sum_{i=1}^{n}(\Gamma_K)_i. \tag{5.9}$$

Let us pass to a new numeration of spinor components

$$\Phi_{\alpha_1\cdots\alpha_n} \to F_{\varrho\sigma}\left(\frac{l}{2}, \frac{n-l}{2}\right), \tag{5.10}$$

where $\frac{l}{2}+\varrho$ — the number of indices 1, $\frac{l}{2}-\varrho$ — the number of indices 2, $\frac{n-l}{2}+\sigma$ — the number of indices 3, $\frac{n-l}{2}-\sigma$ — the number of indices 4.

The following bounds on l, ϱ, σ are evident

$$n \geqslant l \geqslant 0, \quad -\frac{l}{2} \leqslant \varrho \leqslant \frac{l}{2}, \quad -\frac{n-l}{2} \leqslant \sigma \leqslant \frac{n-l}{2}.$$

In the explicit form

$$F_{\varrho\sigma}\left(\frac{l}{2}, \frac{n-l}{2}\right) = \frac{\Phi_{\overbrace{1\cdots 1}^{\frac{l}{2}+\varrho}\;\overbrace{2\cdots 2}^{\frac{l}{2}-\varrho}\;\overbrace{3\cdots 3}^{\frac{n-l}{2}+\sigma}\;\overbrace{4\cdots 4}^{\frac{n-l}{2}-\sigma}}}{\sqrt{\left(\frac{l}{2}+\varrho\right)!\left(\frac{l}{2}-\varrho\right)!\left(\frac{n-l}{2}+\sigma\right)!\left(\frac{n-l}{2}-\sigma\right)!}}. \tag{5.11}$$

The coefficient in (5.11) arises owing to the demand that the scalar product of two spinors be invariant

$$\sum_{l=0}^{n} F^{\varrho\sigma}\left(\frac{l}{2}, \frac{n-l}{2}\right)F_{\varrho\sigma}\left(\frac{l}{2}, \frac{n-l}{2}\right) = \text{inv} = \frac{\Phi^{\alpha_1\cdots\alpha_n}\Phi_{\alpha_1\cdots\alpha_n}}{n!}. \tag{5.12}$$

The law of the transformation of these components reads

$$F'_{\varrho\sigma}\left(\frac{l}{2}, \frac{n-l}{2}\right) = \sum_{m=0}^{n} \mathscr{D}(A)^{m;\tau\varphi}_{l;\varrho\sigma}F_{\tau\varphi}\left(\frac{m}{2}, \frac{n-m}{2}\right), \tag{5.13}$$

where the "matrix" $\mathscr{D}(A)^{m;\tau\varphi}_{l;\varrho\sigma}$ is uniquely defined by the matrix A. One can easily see that in the case of the Lorentz transformations

$$\mathscr{D}(A)^{m;\tau\varphi}_{l;\varrho\sigma} = \mathscr{D}(A)^{l;\tau\varphi}_{l;\varrho\sigma}\delta^m_l, \tag{5.14}$$

that is the components with different l transform independently.

This numeration of the spinor components corresponds to the reduction $SO(4,1) \supset \supset SO(3,1)$. Hence, we easily derive that the irreducible representation of the $SO(4,1)$-group realized by a symmetric spinor of the rank n is decomposed into direct sum of irreducible representations of the Lorentz group

$$(n)_{SO(4,1)} = \sum_{l=0} \oplus \left(\frac{l}{2}, \frac{n-l}{2}\right). \tag{5.15}$$

In considering spinor representations of the $SO(3,2)$ group we get, due to the formula (4.2), the following 5-vector of Γ-matrices

$$\Gamma^{\mu} = \gamma^{\mu}, \quad \Gamma^{4} = i\gamma^{5}. \tag{5.16}$$

The generators M^{KL} are still defined by the relation (5.2), and instead of (5.3) we now have

$$\Gamma_{K} = -\frac{i}{4!}\varepsilon_{KLMNS}\Gamma^{L}\Gamma^{M}\Gamma^{N}\Gamma^{S}. \tag{5.17}$$

The matrices of finite transformations A, similarly to (5.6), satisfy the conditions

$$\mathscr{C}A^{T}\mathscr{C}^{-1}A = 1, \quad \Gamma^{0}\Gamma^{4}A^{+}\Gamma^{4}\Gamma^{0}A = 1, \quad \det A = 1. \tag{5.18}$$

It is obvious that, in exact analogy to the case of the $SO(4,1)$-group, all the lowest spinor representations are equivalent, the irreducible representations being realized by the spinors possessing zero convolution in any pair of indices and a certain symmetry which corresponds to the same type of Young tableaux, as in the case of the group $SO(4,1)$. Here we shall also consider only symmetric representations.

A symmetric spinor $\Phi_{\alpha_1 \ldots \alpha_n}$ of the rank n realizes the representation of dimension $(n+1)(n+2)(n+3)$ [3]. The generators $\hat{M}^{KL}$ are defined by the same formula (5.8) and the operator W_K appears to be

$$W_{K} = -i\,\frac{n+2}{4}\sum_{i=1}^{n}(\Gamma_{K})_{i}. \tag{5.19}$$

In the symmetric representations the Casimir operators C_1 and C_2 assume the values

$$C_{1} = \frac{n(n+4)}{2}, \quad C_{2} = -\frac{n(n+4)}{2}\,\frac{(n+2)^{2}}{8}. \tag{5.20}$$

The relations (5.10) — (5.15) are transferred to the case of the $SO(3,2)$-group without alteration. In particular, the decomposition of the irreducible, representation of the $SO(3,2)$-group reads

$$(n)_{SO(3,2)} = \sum_{l=0}^{n} \oplus \left(\frac{l}{2}, \frac{n-l}{2}\right). \tag{5.21}$$

The authors express their gratitude to V. G. Kadyshevsky and A. A. Logunov for interest in the work and useful remarks. The authors are also obliged to D. P. Zhelobenko, A. N. Leznov, S. Shoh. Mavrodiev, M. D. Mateev, and N. B. Skachkov for discussions.

APPENDIX

We shall list here the explicit form of the matrices which correspond to finite rotations in various planes.

I. $O(4,1)$-group

a) Three-dimensional rotation $u_{\vec{n}}(\omega)$ around the axis $\vec{n}$

$$u_{\vec{n}}(\omega) = \begin{pmatrix} \cos\frac{\omega}{2} + i(\vec{\sigma}\vec{n})\sin\frac{\omega}{2} & 0 \\ 0 & \cos\frac{\omega}{2} + i(\vec{\sigma}\vec{n})\sin\frac{\omega}{2} \end{pmatrix}. \tag{A.1}$$

b) Pure Lorentz transformation (boost) corresponding to the velocity $\vec{v} = \vec{n}\,\mathrm{th}\,\chi$

$$u_{0\vec{n}}(\chi) = \begin{pmatrix} \mathrm{ch}\frac{\chi}{2} - (\vec{\sigma}\vec{n})\,\mathrm{sh}\frac{\chi}{2} & 0 \\ 0 & \mathrm{ch}\frac{\chi}{2} + (\vec{\sigma}\vec{n})\,\mathrm{sh}\frac{\chi}{2} \end{pmatrix}. \tag{A.2}$$

c) Rotations $u_{4\vec{n}}(\delta)$ in the plane $(4, \vec{n})$ i.e. curved translations [3] by the vector $K_L = (0, \vec{n}\sin\delta, \cos\delta)$

$$u_{4\vec{n}}(\delta) = \begin{pmatrix} \cos\frac{\delta}{2} & -i(\vec{\sigma}\vec{n})\sin\frac{\delta}{2} \\ -i(\vec{\sigma}\vec{n})\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}. \tag{A.3}$$

d) Hyperbolic rotations $u_{04}(\psi)$ in the plane (0,4), i.e. curved translations by the vector $K_L = (\mathrm{sh}\,\psi, \vec{0}, \mathrm{ch}\,\psi)$

$$u_{04}(\psi) = \begin{pmatrix} \mathrm{ch}\frac{\psi}{2} & -i\,\mathrm{sh}\frac{\psi}{2} \\ i\,\mathrm{sh}\frac{\psi}{2} & \mathrm{ch}\frac{\psi}{2} \end{pmatrix}. \tag{A.4}$$

II. group $O(3,2)$

The matrices related to the items a) and b) in the case of the $O(3,2)$-group coincide with (A.1) and (A.2).

c) Hyperbolic rotations $u_{4\vec{n}}(\psi')$ in the plane $(4, \vec{n})$, i.e. curved translations by the vector $K_L = (0, \vec{n}\,\mathrm{sh}\,\psi', \mathrm{ch}\,\psi')$

$$u_{4\vec{n}}(\psi') = \begin{pmatrix} \mathrm{sh}\,\frac{\psi'}{2} & (\vec{\sigma}\vec{n})\,\mathrm{ch}\,\frac{\psi'}{2} \\ (\vec{\sigma}\vec{n})\,\mathrm{ch}\,\frac{\psi'}{2} & \mathrm{sh}\,\frac{\psi'}{2} \end{pmatrix}. \tag{A.5}$$

d) Rotations $u_{04}(\delta')$ in the plane (0,4), that is curved translations by the vector $K_L = (\sin\delta', \vec{0}, \cos\delta')$

$$u_{04}(\delta') = \begin{pmatrix} \cos\frac{\delta'}{2} & -\sin\frac{\delta'}{2} \\ \sin\frac{\delta'}{2} & \cos\frac{\delta'}{2} \end{pmatrix}. \tag{A.6}$$

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