Geometric Quantization of the Kepler Problem with a Magnetic Charge

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ABSTRACT

The geometric quantization scheme is applied to the compact Kaehler orbit manifolds of a one-parameter family of deformations of the Kepler problem. Thus we obtain the quantization of the magnetic charge and energy spectrum of the corresponding quantum problem. A new regularization of the standard Kepler problem is presented.

In this paper we use reduction and geometric quantization to compute the discrete quantum energy spectrum of the Hamiltonian system $(T^*\dot{R}^3,\ \Omega_\mu,\ H_\mu)$, where

$$T^*\dot{R}^3 = \{(p,q)\in R^3\times R^3 : q \neq 0\}$$

$$\Omega_\mu = \sum_{j=1}^{3} dp_j \wedge dq_j + \frac{\mu}{|q|^3}\sum_{j,k,l=1}^{3} \varepsilon_{jkl} q_j dq_k\, dq_l \tag{1}$$

$$H_\mu = \frac{1}{2}|p|^2 + \frac{\mu}{2|q|^2} - \frac{\alpha}{|q|}; \qquad \alpha,\mu\ R,\ \alpha > 0$$

Following Iwai and Uwano [1], we call (1) the MiC-Kepler problem (after McIntosh and Cisneros [2] who seem to have been the first to study it sistematically - see also [3]). The MIC-Kepler problem describes the motion of a charged particle in the presence of a Coulomb field, a centrifugal potential and a magnetic monopole with charge μ. We may treat (1) as a one-parameter family (μ) of deformations of the classical Kepler problem (μ=0) [4]. It is known that for any real value of μ, the symmetry group of the system is SO(4). If $\mu \neq 0$, then for any $E < 0$, all orbits lying on the energy hypersurface $H_\mu^{-1}(E)$, are periodic (when $\mu = 0$ one must regularize [5] - see also sect. 2).

In the present approach the quantization of the energy of the system (1) is possible only if one quantizes also the magnetic charge μ, and the μ-spectrum obtained coincides with the classical Dirac quantization of the magnetic monopole. This phenomenon occurs only in dimension three, and the geometric fact underlying it (responsible also for several recent developments in mathematical physics - e.g. self-duality) is the fact that the quadric surface in P^3 is decomposable (biholomorphic to $P^1 \times P^1$) or what amounts to the same thing, that $SO(4) = SO(3) \oplus SO(3)$. When the energy $E < 0$, and the magnetic charge μ take all allowed (quantum) values, the corresponding eigenspaces of wave functions carry all (finite dimensional) irreducible representations of SO(4) (see Theorem 1 below).

Reduction of the symplectic manifold $(T^*\dot{R}^3, \Omega_\mu)$, with respect to the flow of H_μ, followed by geometric quantization of the symplectic manifolds obtained, yields as quantum bundles all positive line bundles on $P^1 \times P^1$ (the flag manifold of SO(4)), whence the last statement is a special case of the Borel-Weil-Bott theorem.

Theorem 1. The Hamiltonian system (1) has a discrete quantitum energy spectrum if the magnetic charge μ is (half) integer.

$$\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots \tag{2}$$

If we fix a value of μ from (2) then the allowed negative energy levels are

$$E_N = -\alpha/_{2N^2} ; \qquad N = |\mu| + 1, |\mu| + 2, \ldots \tag{3}$$

with multiplicities

$$m(E_N) = N^2 - \mu^2. \tag{4}$$

The proof of the theorem (1) is based on the explicit description of the orbit manifolds of the MIC-Kepler problem. For a fixed $\mu \in R$ and $E < 0$, we denote by $\mathcal{O}_\mu(E)$ the orbit manifold of the flow of the Hamiltonian H_μ on the level set $H^{-1}(E)$. All orbits are periodic (after regularization at $\mu=0$), whence the flow of H_μ defines an S^1 action on $H^{-1}(-\infty,0)$, with momentum map H_μ. By the Marsden-Weinstein reduction theorem $\mathcal{O}_\mu(E) = H_\mu^{-1}(E)/_{S^1}$ is a symplectic manifold with reduced symplectic form $\Omega_\mu(E)$. We have

Theorem 2. Let $E < 0$, $\lambda = \sqrt{-8E}$. Then

(i) if $\lambda|\mu| < 2\alpha$ then $\mathcal{O}_\mu(E) = H_{\mu_1}^{-1}(E)/_{S^1} \cong P^1 \times P^1$.

(ii) if $\lambda|\mu| = 2\alpha$ then $H_\mu^{-1}(E) \cong P^1$ consists of fixed points of the flow.

(iii) if $\lambda|\mu| > 2\alpha$ then $H_\mu^{-1}(E) = \emptyset$.

Moreover, the reduced symplectic form on $\mathcal{O}_\mu(E)$ is

$$\Omega_\mu(E) = \frac{2\pi(2\alpha + \lambda\mu)}{\lambda}\omega_1 + \frac{2\pi(2\alpha - \lambda\mu)}{\lambda}\omega_2 , \tag{5}$$

where

$$\omega_j = \frac{i}{2\pi}\frac{d\zeta_j \wedge d\bar{\zeta}_j}{(1 + |\zeta_j|^2)^2} ; \quad j = 1,2; \; i^2 = -1 \tag{6}$$

for any pair (ζ_1, ζ_2) of nonhomogeneous coordinates on $P^1 \times P^1$.

We outline the proofs of Theorem 1 and Theorem 2 in the next section. Complete proofs appear in [6] .

Remark. The present approach to geometric quantization via orbit manifolds originates from Simms [7]. Related problems concerning interchangeability of quantization and reduction were studied recently in [8]. The opposite approach of lifting classical systems and then quantizing has recently been applied to a veriety of problems by Kibler et all (cf. [9] and references therein).

Outline of the proof of the theorems.

Let $(T^*\dot{R}^4, \Omega)$ denote the symplectic manifold

$$T^*\dot{R}^4 = \{(x,y) \in R^4 \times R^4 : x \neq 0\}, \; \Omega = \sum_{j=1}^{4} dy_j \wedge dx_j \tag{7}$$

For an arbitrary constant $\lambda > 0$, we introduce complex coordinates on $T^*\dot{R}^4$ (compare [1]).

$$z_1 = \lambda x_1 + y_2 + i(y_1 - \lambda x_2); \quad z_2 = \lambda x_3 - y_4 + i(y_3 + \lambda x_4) \qquad (8)$$
$$z_3 = \lambda x_1 - y_2 + i(y_1 + \lambda x_2); \quad z_4 = \lambda x_3 + y_4 + i(y_3 - \lambda x_4)$$

Obviously

$$\Omega = \frac{i}{4\lambda} dz \wedge d\bar{z} = \frac{i}{4\lambda} \sum_{j=1}^{4} dz_j \wedge d\bar{z}_j \qquad (9)$$

Thus $T^*\dot{R}^4 = C^4 \setminus D$ where,

$$D = \{z \in C^4 : z_1 = -\bar{z}_3;\ z_2 = -\bar{z}_4\}. \qquad (10)$$

We introduce three hamiltonians on $T^*\dot{R}^4$

$$H = \frac{1}{8|x|^2}(|y|^2 - 8\alpha), \quad \alpha \in R, \quad \lambda > 0 \qquad (11)$$

$$K = \frac{1}{2}(\lambda^2 |x|^2 + |y|^2) = \frac{1}{4}|z|^2 = \frac{1}{4}\sum_1^4 z_j \bar{z}_j \qquad (12)$$

$$M = \frac{1}{2}(x_1 y_2 - x_2 y_1 + x_3 y_4 - y_3 x_4) = \frac{1}{8\lambda}(|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2) \qquad (13)$$

The Hamiltonian system $(T^*\dot{R}^4, \Omega, H)$ is called the conformal Kepler problem. We note that the harmonic oscillator K, the moment M and the symplectic form Ω are well defined on C^4. Let $\dot{C}^4 = C^4 \setminus \{0\}$. We denote by K_t, M_s the flows of the Hamiltonian systems $(K, \Omega, \dot{C}^4)$, $(M, \Omega, \dot{C}^4)$ respectively.

Lemma 1. ([6]). For any $z \in \dot{C}^4$, s, $t \in R$ we have

$$K_t z = e^{i\lambda t} z = (e^{i\lambda t} z_1,\ e^{i\lambda t} z_2,\ e^{i\lambda t} z_s,\ e^{i\lambda t} z_4) \qquad (14)$$

$$M_s z = (e^{is/2} z_1,\ e^{is/2} z_2,\ e^{-is/2} z_3,\ e^{-is/2} z_4) \qquad (15)$$

Thus K_t, M_s define two commuting free actions of $U(1)$ on $\dot{C}^4$, where moment maps are $K : \dot{C}^4 \to R$, $M : \dot{C}^4 \to R$ given by formulae (12), (13). We denote by $J : \dot{C}^4 \to R^2 = u^*(1) \times u^*(1)$, the moment map of the action of $U(1) \times U(1)$ on $\dot{C}^4$ defined by $z \to K_t o M_s z$, thus

$$J(z) = (K(z), M(z)). \qquad (16)$$

We also remark that the $U(1) \times U(1)$ action defined by (14), (15) is free on the set $\dot{C}^4 \setminus (D' \cup D'')$ where,

$$D' = \{z : |z_1|^2 + |z_2|^2 = 0\} = \{z \in \dot{C}^4;\ K + 2\lambda M = 0\}$$
$$D'' = \{z : |z_3|^2 + |z_4|^2 = 0\} = \{z \in \dot{C}^4;\ K - 2\lambda M = 0\} \tag{17}$$

Lemma 2. ([1],[6]). Let $E < 0$, $\lambda = \sqrt{-8E}$. Then

$$H^{-1}(E) = K^{-1}(4\alpha) \cap T^*\dot{R}^4 \tag{18}$$

Moreover, on the level set (18) we have

$$4|x|^2 X_H = X_K \tag{19}$$

where X_H, X_K are the hamiltonian vector fields of H, K respectively. By Lemma 2, the flows of H, K on the level set (17) coincide up to a monotone change of parameter, whence the orbits of the harmonic oscillator K on $\dot{C}^4$ are extensions of the conformal Kepler problem. Thus we may treat $(\dot{C}^4,\Omega,K)$ as a regularization of $(T^*\dot{R}^4,\Omega,H)$. Another consequence of Lemma 1 and Lemma 2, is that all three flows of the hamiltonians H, K, M commute on $T^*\dot{R}^4$.

The following crucial result was established by Iwai and Uwano [1].

Proposition. Let $\mu \in R$. We have the U(1) action (15) on $M^{-1}(\mu)$ and

$$M^{-1}(\mu)/_{U(1)} \cong T^*\dot{R}^3.$$

Moreover, the result of reducing the conformal Kepler problem $(T^*\dot{R}^4,\Omega,H)$ with respect to the U(1) action (15) on the level set $M = \mu$, is the MIC-Kepler problem $(T^*\dot{R}^3,\Omega_\mu, H_\mu)$.

The set $\dot{C}^4 \setminus T^*\dot{R}^4$ is M_s-invariant, thus an orbit of $(T^*\dot{R}^3,\Omega_\mu,H_\mu)$ is closed if it is the image of closed orbits of $(T^*\dot{R}^4,\Omega,H)$ under reduction. The nonclosed orbits of $(T^*\dot{R}^3,\Omega_\mu,H_\mu)$ occur only at $\mu=0$. At the corresponding level set $M^{-1}(0)$ we have $(D' \cup D'') \cap M^{-1}(0) = \emptyset$, and the periodic K-orbits on $M^{-1}(0)$ are the closures of the H-orbits which are preimages of H_0 orbits. Thus we get a new regularization of the classical Kepler problem.

Now we observe that Lemma 2 and the Proposition imply

$$\mathcal{O}_\mu(E) = J^{-1}(4\alpha,\mu)/_{U(1)\times U(1)} \tag{20}$$

By (12), (13), (16) we get

$$J^{-1}(4\alpha,\mu)=\{z \in \dot{C}^4 : |z_1|^2+|z_2|^2=4(2\alpha+\lambda\mu);\ |z_3|^2+|z_4|^2=4(2\alpha-\lambda\mu)\} \tag{21}$$

whence

(i) $J^{-1}(4\alpha,\mu) \cong S^3\times S^3$ iff $\lambda|\mu| < 2\alpha$; and by (17) $U(1)\times U(1)$ acts freely.

(ii) $J^{-1}(4\alpha,\mu) \cong S^3$ iff $\lambda|\mu| = 2\alpha$, and by (17) $J^{-1}(4\alpha,\mu)\subset D'$ or D''.

(iii) $J^{-1}(4\alpha,\mu) = \emptyset$ if $\lambda|\mu| > 2\alpha$.

From formuli (14), (15) we see that in case (i) the map $p : J^{-1}(4\alpha,\mu) \to J^{-1}(4\alpha,\mu)/_{U(1)\times U(1)}$ is the direct product of two Hopf maps which can be written explicitely in coordinates as

$$p(z_1,z_2,z_3,z_4) = (\zeta_1,\zeta_2) = (z_2/_{z_1}, z_4/_{z_3}) \in P^1\times P^1 \tag{22}$$

where (ζ_1,ζ_2) are a pair of nonhomogeneous coordinates on $P^1\times P^1$.

A straight computation using (21), (22) and the defining relation

$$p^*\Omega_\mu(E) = \Omega_{/J^{-1}(4\alpha,\mu)}$$

yields theorem 2.

Now we outline the proof of theorem 1. We quantize the compact Kähler manifolds $(\mathcal{O}_\mu(E), \Omega_\mu(E))$.

By the standard procedure of geometric quantization [10], the allowed values of the parameters μ, E are exactly those for which there exists a quantum line bundle, i.e. a line bundle $L \to \mathcal{O}_\mu(E)$ such that

$$c_1(L) = \frac{1}{2\pi}\,\Omega_\mu(E) - \tfrac{1}{2}c_1(\mathcal{O}_\mu(E)) \in H^2(\mathcal{O}_\mu(E),Z) \tag{23}$$

is a nonnegative cohomology class. It is a well known that the cohomology class of the forms ω_1, ω_2 defined in formula (6) generate $H^2(P^1\times P^1,Z)$ and that a nonnegative class must have nonnegative coeficients in them, whence the condition (23) reduces to

$$c_1(L) = (N_1-1)\omega_1 + (N_2-1)\omega_2 \tag{24}$$

where N_1, N_2 are arbitrary positive integers.

Taking $\Omega_\mu(E)$ from (5) and using (24) we get

$$(N_1-1)\omega_1 + (N_2-1)\omega_1 = \frac{2\alpha+\lambda\mu}{\lambda}\,\omega_1 + \frac{2\alpha-\lambda\mu}{\lambda}\,\omega_2 - (\omega_1+\omega_2)$$

whence

$$\lambda = 4\alpha/(N_1+N_2)\,;\quad \mu = \tfrac{1}{2}(N_1-N_2) \tag{25}$$

Obviously μ can take exactly all (half)-integer values. Fixing a (half)-integer μ, we introduce a new (half)-integer variable

$$N = \tfrac{1}{2}(N_1+N_2)$$

Because of $N_1 = N+\mu \geq 1$, $N_2 = N-\mu \geq 1$, and $\lambda = \sqrt{-8E}$ formula (25) gives exactly formula (3). In order to obtain the multiplicities we apply the Riemann-Roch theorem for fixed $E_{N,\mu}$ to the corresponding bundle L_{N_1,N_2} and obtain

$$m(E_N) = \dim H^0(L_{N_1,N_2}, P^1\times P^1) = N_1N_2 = N^2-\mu^2,$$

which proves theorem 1.

We remark that using the degenerate situation of item (ii) in Theorem 2, one can also quantize μ, arriving at the same result.

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