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RISING CROSS-SECTIONS

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A B S T R A C T

Amplitudes with total cross-sections rising asymptotically like $(\log s)^{\beta_+}$ require shrinkage of the diffraction peak. For $0 < \beta_+ \leq 1$, it is sufficient to have a leading positive signature trajectory which is regular at $t = 0$, but not a constant. For $1 < \beta_+ \leq 2$, the trajectory must have a branch point at $t = 0$. Explicit amplitudes with complex Pomeranchuk trajectories are given which are compatible with t channel unitarity. The limiting case of maximal absorption is considered briefly.

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I. INTRODUCTION

Recent and future experiments with the new accelerators and storage rings are expected to shed some light on the asymptotic properties of scattering amplitudes. Whatever may be the outcome of these experiments, within the framework of dispersion theory, it is of interest to explore the constraints imposed upon the amplitudes by specific assumptions about the asymptotic form of total cross-sections.

In this lecture, we consider the possibility that total cross-sections rise with increasing energy. We know from the Froissart bound [1)] that

$$\sigma_{tot} \leq O\left((\log s)^2\right) . \tag{1.1}$$

For an asymptotic increase of the cross-section corresponding to

$$\sigma_{tot} \sim (\log s)^{\beta_+} , \tag{1.2}$$

we find that some shrinkage of the diffraction peak is required. If $\beta_+ \leq 1$, this minimal shrinkage is at most logarithmic and can be obtained with a leading positive signature trajectory [*)] $\alpha(t)$ which is a regular analytic function near $t = 0$. However, for $\beta_+ > 1$, we have a quite different situation. Within the framework of complex angular momentum methods, we can show that the trajectory function must have a branch point at $t = 0$. In particular, it should be an appropriate branch of a function like

$$\alpha(t) = 1 + \text{const.}\, t^{1/\beta} , \text{ with } \beta_+ \leq \beta \leq 2 \tag{1.3}$$

near $t = 0$. Unless $\beta = 2$, we need special new branch points ("hiding cuts") in order to remove those branches of $\alpha(t)$, which violate s channel unitarity, from the physical sheet of the complex angular momentum plane [2)].

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*) We use the word "trajectory" for a singular surface of the continued partial wave amplitude $F(t,\lambda)$. The character of this surface is not specified, a priori. It may be a branch point surface or a pole surface.

The complex Pomeranchuk trajectories of the form (1.3) are rather natural in connection with impact parameter representations of the amplitude. We explore this connection, and we give special examples of amplitudes with rising cross-sections which are also compatible with t channel unitarity. Finally, we discuss the limiting case of maximal absorption.

II. RISING CROSS-SECTIONS AND SHRINKAGE

We consider in the following only the positive signature amplitude, assuming that the highest negative signature trajectory has an intercept well below $\lambda = 1$ [3]. For the forward amplitude, we make then the ansatz

$$F_+(s,0) \propto is(\log s)^{\beta_+} + \frac{\pi}{2}\beta_+ s(\log s)^{\beta_+ - 1} + \dots \tag{2.1}$$

for $s \to \infty$, which is consistent with the dispersion relations for $F_+ = F + \bar{F}$, where F and $\bar{F}$ are the amplitudes for particle and anti-particle scattering respectively.

The minimal shrinkage required by the condition $\sigma_{el} \leq \sigma_{tot}$ can be seen from the inequality

$$\int_{-s}^{0} dt \, |f(s,t)|^2 \leq \frac{s \, \mathrm{Im}\, F(s,0)}{|F(s,0)|^2} \, , \tag{2.2}$$

where

$$f(s,t) \equiv \frac{F(s,t)}{F(s,0)} \, . \tag{2.3}$$

With the ansatz (2.1), and the assumption that there is no negative signature trajectory with $\alpha_-(0) = 1$, [3] we have for $s \to \infty$

$$s\left(\mathrm{Im}\, F(s,0)\right)^{-1} \sim (\log s)^{-\beta_+} \, . \tag{2.4}$$

If $\beta_+ > 0$, and hence the total cross-section increases with a power of $\log s$, we see that the bound (2.2) requires some shrinkage of the diffraction peak. Hence, in this case, the leading singularity at $\lambda = 1$ in the complex angular momentum plane cannot be a fixed (i.e., t independent) one *).

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*) As required here, a fixed singularity by itself would, of course, violate t channel unitarity, but this could be avoided by introducing the appropriate shielding cuts [4].

For $0 < \beta_+ \leq 1$, it is sufficient to have a trajectory $\alpha(t)$ which is regular near $t = 0$. In particular, $\beta_+ = 1$ requires a non-vanishing linear term :

$$\alpha(t) = 1 + \alpha'(0)\,t + \cdots , \qquad \alpha'(0) > 0 . \tag{2.5}$$

On the other hand, if $\beta_+ > 1$, a regular trajectory function $\alpha(t)$ is no more sufficient in order to produce the necessary shrinkage as indicated by the inequality (2.2). A trajectory $\alpha(t)$ with a branch point at $t = 0$ is now required. At least, this is the case if the shrinkage is to be generated by the properties of the function $\alpha(t)$, which is the case for pole and branch point surfaces of the continued partial wave amplitude $F(t,\lambda)$. There could be other possibilities only if we allow essential singularities to be present at $(t,\lambda) = (0,1)$. Here we do not consider this case. Essential singularities are considerably restricted by t channel unitarity, and, already for this purpose, they must have a very special t dependent character [5].

It is well known that trajectory functions $\alpha(t)$ do not inherit the left-hand cuts of the partial wave amplitude $F(t,\lambda)$. They can have branch points at a point like $t = 0$ only if there are two or more singular surfaces which cross there, and only if these surfaces have the <u>same character</u> [6]. Then, we can have

$$\alpha(t) = \alpha(0) + \sum_{j=1}^{\infty} c_j\, t^{j/n} , \qquad n \geq 2 \tag{2.6}$$

near $t = 0$ without the partial wave amplitude having a singularity at this point which is not allowed. The n branches of the singular surface (2.6) must appear in a completely symmetric fashion in $F(t,\lambda)$. For $t \to 0$, we have then

$$\alpha(t) = \alpha(0) + \text{const.}\, t^{\frac{m}{n}} . \tag{2.7}$$

However, if additional, very special branch points $\lambda = \alpha_c(t)$ are present in $F(t,\lambda)$, we can arrange for some of the branches of the function (2.6) to be in unphysical sheets of the λ plane. With such additional "hiding cuts", we can even have any number β in place of the ratio n/m in Eq. (2.7). For example, we may have

$$\alpha(t) = \alpha(0) + \text{const.}\, t^{1/\beta} \tag{2.8}$$

near $t = 0$ if the amplitude $F(t,\lambda)$ has a branch point of the type $(\lambda - \alpha_c(t))^{\beta}$ with $\alpha_c(t) = \alpha(0) + O(t)$. This hiding cut can then remove almost all of the branches of (2.8) from the physical sheet.

In order to have sufficient shrinkage for amplitudes with rising cross-sections as in Eq. (1.2), we need a branch point in $\alpha(t)$ of the form (2.7) or (2.8) with $\alpha(0) = 1$,

$$\beta_+ \leq \beta \leq 2 , \tag{2.9}$$

and a branch with $\mathrm{Re}\,\alpha(t) < 1$ for $t < 0$. In addition, if $\beta < 2$, there are always also branches with $\mathrm{Re}\,\alpha(t) > 1$ for $t < 0$. These branches must be removed from the physical sheet with the help of the hiding cuts mentioned above. They would violate the bound

$$\mathrm{Im}\, F(s,t) \leq \mathrm{Im}\, F(s,0) \quad \text{for} \quad t \leq 0 . \tag{2.10}$$

There is also the familiar bound

$$|F(s,t)| \leq O(s^{1+\sqrt{at}}) \quad \text{for} \quad 0 \leq t \leq t_0 , \tag{2.11}$$

where $\sqrt{a}$ is the radius appearing in $L_{max} = \frac{1}{2}\sqrt{as}\log s$, the maximal orbital angular momentum for which partial waves are relevant for $s \to \infty$ [7]. Hence branches of $\alpha(t) = 1 + ct^{1/\beta}$ with $\mathrm{Re}\, c > 0$ for $t \geq 0$ are only allowed in the physical sheet if $\beta \leq 2$ and $\mathrm{Re}\, c < \sqrt{a}$ for $\beta = 2$.

If the amplitude $F(t,\lambda)$ has no hiding cut, then all branches of $\alpha(t)$ are in the physical sheet. Since for $1 < \beta < 2$ there is always a branch with $\mathrm{Re}\,\alpha(t) > 1$ for $t < 0$, the bound (2.10) implies then that $\beta = 2$, and we are left with trajectories of the form

$$\alpha(t) = 1 + c\sqrt{t} + O(t) \tag{2.12}$$

near $t = 0$. In this case, $\alpha(t)$ is real for $t \geq 0$ ($t \leq t_0$ = threshold) and complies with the bound (2.11) for $c < \sqrt{a}$ as well as with (2.10) because $\mathrm{Re}\,\alpha(t) \leq 1$ for $t \leq 0$.

III. EXPLICIT AMPLITUDES

In this and the following Section, we consider explicit examples of amplitudes with complex trajectories of the form (2.12) which give rise to increasing total cross-sections. In particular, we are interested in showing that one can write down amplitudes which also comply with the usual analyticity and unitarity requirements of s and t channels.

In earlier publications [8),9)], we have shown that a rather general class of amplitudes with complex trajectories can be expressed in the form

$$F_+(t,\lambda) \propto \int_0^1 d\xi \, \frac{\rho_+(\xi,t)}{\sqrt{(\lambda-1)^2-\xi^2 a t}} \tag{3.1}$$

for (t,λ) near $(0,1)$. The high energy limit of $F_+(s,t)$ is then obtained from the transform

$$F_+(s,t) \sim \frac{1}{2\pi i}\int d\lambda \, s^\lambda S_+(\lambda) K(t,\lambda) F_+(t,\lambda). \tag{3.2}$$

We write here

$$S_+(\lambda) = i \exp\left(-\frac{i\pi}{2}(\lambda-1)\right), \tag{3.3}$$

and absorb the factor $K(t,\lambda)$ in the definition of $F_+(t,\lambda)$ *). Then, we find for $s \to \infty$

$$F_+(s,t) \sim is\int_0^1 d\xi \, \rho_+(\xi,t) J_0\left(\xi\sqrt{-at}\left(\log s-\frac{i\pi}{2}\right)\right). \tag{3.4}$$

For fixed values of [10)]

$$\tau = -at(\log s)^2, \tag{3.5}$$

the leading asymptotic terms are given by [2)]

$$\begin{aligned} F_+(s,t) \sim \; & is\int_0^1 d\xi \, \rho_+(\xi,t) J_0(\xi\sqrt{\tau}) \\ & -\frac{\pi}{2} s\int_0^1 d\xi \, \rho_+(\xi,t) \frac{\xi\sqrt{\tau}}{\log s} J_1(\xi\sqrt{\tau}) + \cdots . \end{aligned} \tag{3.6}$$

Note that this is an expansion in powers of $\sqrt{-at}$, and not of $\sqrt{-at}\,\log s = \sqrt{\tau}$. For fixed values of t, other terms in Eq. (3.4) cannot be neglected.

We are interested in amplitudes which reduce to Eq. (2.1) in the limit of forward scattering. In particular, let us consider the case $\beta_+ = 2$. Appropriate forms can be obtained from the representation (3.1) by choosing

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*) $K(t,\lambda) = q^{-2\lambda}(t)\,\Gamma(\lambda+\frac{3}{2})\left[\Gamma(\lambda+1)\sqrt{\pi}\,\sin\frac{\pi}{2}\lambda\right]^{-1}$

$$\rho_+(\xi,t) = \frac{1}{at}\varphi(\xi) \tag{3.7}$$

with $\varphi(\xi)$ satisfying the conditions

$$\int_0^1 d\xi\, \varphi(\xi) = 0 \tag{3.8}$$

and

$$\int_0^1 d\xi\, \xi^2 \varphi(\xi) > 0 . \tag{3.9}$$

Then we find for $t \to 0$

$$F_+(t,\lambda) \sim \frac{1}{(\lambda-1)^3}\, \frac{1}{2} \int_0^1 d\xi\, \xi^2 \varphi(\xi) , \tag{3.10}$$

and $F_+(t,\lambda \neq 1)$ is regular at $t = 0$. The asymptotic forward amplitude becomes

$$F_+(s,0) \sim \frac{1}{4}\int_0^1 d\xi\, \xi^2\varphi(\xi) \left\{ is(\log s)^2 + \pi s \log s \right\} + \cdots . \tag{3.11}$$

In order to give a simple and explicit example of an amplitude which complies also with t channel unitarity, we write

$$\varphi(\xi) \propto \xi\, N(\beta,\gamma) \left\{ 1 - \frac{1}{\sqrt{\beta}} \frac{1-\sqrt{\gamma}}{\sqrt{\beta}-\sqrt{\gamma}}\, \Theta(\sqrt{\beta}-\xi) + \frac{1}{\sqrt{\gamma}} \frac{1-\sqrt{\beta}}{\sqrt{\beta}-\sqrt{\gamma}}\, \Theta(\sqrt{\gamma}-\xi) \right\} , \tag{3.12}$$

where β and γ are real parameters satisfying the conditions

$$0 < \beta < 1 , \quad 0 < \gamma < 1 , \quad \beta \neq \gamma . \tag{3.13}$$

The normalization factor N can be chosen to be

$$N = \frac{1}{16} \left\{ 1 - \beta^{3/2} \frac{1-\sqrt{\gamma}}{\sqrt{\beta}-\sqrt{\gamma}} + \gamma^{3/2} \frac{1-\sqrt{\beta}}{\sqrt{\beta}-\sqrt{\gamma}} \right\} \tag{3.14}$$

in order to have

$$\frac{1}{4}\int_0^1 d\xi\, \xi^2 \varphi(\xi) = 1 . \tag{3.15}$$

The ansatz (3.12) satisfies Eqs. (3.8) and (3.9), and in addition

$$\int_0^1 d\xi\, \xi^{-1} \varphi(\xi) = 0$$

so that $F_+(t, \lambda = 1)$ is also regular at $t = 0$. The continued partial wave amplitude is of the form

$$F_+(t,\lambda) \propto \frac{N}{(at)^2} \Big\{ \sqrt{(\lambda-1)^2 - at} - \frac{1}{\sqrt{\beta}} \frac{1-\sqrt{\gamma}}{\sqrt{\beta}-\sqrt{\gamma}} \sqrt{(\lambda-1)^2 - \beta a t} \tag{3.16}$$
$$+ \frac{1}{\sqrt{\gamma}} \frac{1-\sqrt{\beta}}{\sqrt{\beta}-\sqrt{\gamma}} \sqrt{(\lambda-1)^2 - \gamma a t} \Big\} .$$

It has the square-root branch points at positions corresponding to Eq. (2.12). These branch points are compatible with t channel unitarity without requiring a threshold in the trajectory function or shielding devices [5].

Since the model (3.16) is meant only as a mathematical example, we do not discuss here the details of its high energy limit. From Eq. (3.6) we see that the leading term for $s \to \infty$ and fixed values of τ is given by

$$F_+(s,t) \propto i s (\log s)^2 \frac{N}{-\tau} \Big\{ \frac{J_1(\sqrt{\tau})}{\sqrt{\tau}}$$
$$- \frac{1-\sqrt{\gamma}}{\sqrt{\beta}-\sqrt{\gamma}} \frac{J_1(\sqrt{\beta\tau})}{\sqrt{\tau}} + \frac{1-\sqrt{\beta}}{\sqrt{\beta}-\sqrt{\gamma}} \frac{J_1(\sqrt{\gamma\tau})}{\sqrt{\tau}} \Big\} , \tag{3.17}$$

which reduces to $is(\log s)^2$ for $t = 0$.

IV. LIMITING CASES

More familiar physical models are obtained if we choose the weight function $\varphi(\xi)$ in Eq. (3.7) to be of the form [2)]

$$\varphi(\xi) \propto \delta'(\xi-\xi_0) \tag{4.1}$$

with $0 < \xi_0 < 1$, or superpositions thereof. We obtain then for the continued partial wave amplitude in the neighbourhood of $(t, \lambda) = (0,1)$

$$F_+(t,\lambda) \propto \left[(\lambda-1)^2 - \xi_0^2 a t\right]^{-3/2}. \tag{4.2}$$

As it stands, this expression does not comply with the requirements of t channel unitarity if continued to $t \geq t_0$. But it should be viewed as a degenerate limit for small $|t|$ of several complex pole and related branch-point trajectories which are different from each other for larger values of $|t|$ and which are separately compatible with the t channel constraints. We have discussed similar pole-cut relations in previous publications [2),6),9)].

The partial wave amplitude (4.2) gives rise to the high energy limit

$$\begin{aligned} F_+(s,t) &\propto i s\left(\log s - \frac{i\pi}{2}\right) \frac{J_1(\xi_0\sqrt{-at}\,\log s)}{\xi_0\sqrt{-at}} \\ &= i s(\log s)^2 \frac{J_1(\xi_0\sqrt{\tau})}{\xi_0\sqrt{\tau}} + \frac{\pi}{2} s \log s\, J_0(\xi_0\sqrt{\tau}) \\ &\quad + O(at), \end{aligned} \tag{4.3}$$

where $\tau = -at(\log s)^2$.

In Eq. (4.3) we have also expanded the asymptotic expression in powers of $\underline{at}$, exhibiting the leading terms for $s \to \infty$ and fixed values of τ.

The expression (4.3) may be compared with the impact parameter representation [11)]

$$F(s,t) \sim \frac{s}{2} \int_0^{\sqrt{a}\log s} db\, b\, \frac{\eta(s,b)e^{2i\delta(s,b)} - 1}{2i} J_0(b\sqrt{-t}), \tag{4.4}$$

which we write in the form [2)]

$$F(s,t) \sim s \int_0^1 d\xi \, \psi(\xi,s) \, J_0(\xi\sqrt{\tau}) \tag{4.5}$$

with

$$b = \xi\sqrt{a}\log s \quad \text{and} \quad \sqrt{\tau} = \sqrt{-at}\log s \, . \tag{4.6}$$

The radius $\sqrt{a}$ has been defined before.

We have the identification

$$\begin{aligned} \mathrm{Re}\,\psi(\xi,s) &= (\log s)^2 \xi \frac{a}{4} \eta \sin 2\delta \, , \\ \mathrm{Im}\,\psi(\xi,s) &= (\log s)^2 \xi \frac{a}{4} (1-\eta\cos 2\delta) \, . \end{aligned} \tag{4.7}$$

The model (4.3) corresponds to a weight function ψ with the high energy limit

$$\lim_{s\to\infty} \psi(\xi,s) \propto i(\log s)^2 \xi \, \Theta(-\xi+\xi_0) + O(\log s), \tag{4.8}$$

where we consider ξ as a fixed parameter [*)].

In the following, we write simply $\eta(s,\xi)$ and $\delta(s,\xi)$ in the place of $\eta(s,b(\xi,s))$ and $\delta(s,b(s,\xi))$, where $b(s,\xi)$ is given in Eq. (4.6). Using Eq. (4.7), we see then that

$$\lim_{s\to\infty} \eta(s,\xi)\sin 2\delta(\xi,s) = 0 \tag{4.9}$$

and

$$\lim_{s\to\infty} \{1-\eta(s,\xi)\cos 2\delta(s,\xi)\} = \text{const.} \tag{4.10}$$

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*) In the representation (4.5) the variables ξ and $\sqrt{\tau}$ are conjugate to each other like b and $\sqrt{-t}$ in the usual impact parameter representation (4.4).

for fixed $0 \leq \xi \leq \xi_0$, and zero for $\xi > \xi_0$. We note that $b(s,\xi) \to \infty$ for $s \to \infty$.

In terms of $\eta(s,\xi)$, the inelastic cross-section is given by

$$\sigma_{inel} \sim 2\pi a (\log s)^2 \int_0^1 d\xi\, \xi\, (1-\eta^2(s,\xi)) , \tag{4.11}$$

which is maximal for $\eta \to 0$ in the high energy limit.

Suppose we require that the inelastic cross-section approaches the largest value *)

$$\sigma_{inel} \sim \pi a (\log s)^2 \tag{4.12}$$

which is compatible with the bound $L \leq L_{max} = 1/2\sqrt{as}\log s$ for the relevant orbital angular momenta in the s channel partial wave expansion. Then Eqs. (4.9) and (4.10) imply

$$\lim_{s\to\infty} \psi(\xi,s) = i\frac{a}{4}(\log s)^2\,\xi \tag{4.13}$$

for the leading asymptotic term, and for fixed values of τ, the high energy limit of the amplitude becomes **)

$$F(s,t) \underset{s\to\infty}{\sim} i s (\log s)^2 \frac{a}{8}\,\frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}} + O(s\log s) , \tag{4.14}$$

which corresponds to the expression (4.3) with $\xi_0 = 1$ and the factor $a/4$ [2]. The leading term of the dispersive (real) part is given by the dispersion relation. We see that the requirement (4.12) is sufficient to determine the asymptotic form of the

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*) High energy cosmic ray experiments may also give important bounds on the asymptotic increase of σ_{inel}.

**) Iteration schemes in the t channel often lead to amplitudes which overshoot the Froissart bound. If these are then unitarized by further iteration in the s channel, we are likely to get an amplitude corresponding to Eq. (4.14) or (4.3) because complete absorption is required in order to restore unitarity [12].

amplitude for fixed values of τ. The total cross-section becomes in this case *)

$$\sigma_{tot} \sim 2\sigma_{inel} \sim 2\pi a (\log s)^2 . \tag{4.15}$$

In the high energy limits considered in this and in the previous section, we have not specified the normalization of the variable s in the logarithmic factors. Although this is a question which affects secondary terms only, and which is, therefore, more model dependent, it is of course important from a phenomenological point of view. In particular, at presently available energies, we expect that possible logarithmically increasing terms in the amplitudes are superimposed upon constant ones or others which are due to secondary singularities in the angular plane.

V. CONCLUSIONS

Let us sum up the conclusions of our discussion of rising cross-sections within the framework of dispersion theory. We find the following .

1. Any increase of the total cross-section with a power of $(\log s)$ requires some shrinkage of the diffraction peak.

2. If $\sigma_{tot} \sim (\log s)^{\beta_+}$ with $0 < \beta_+ \leq 1$, it is sufficient to have a positive signature trajectory which is regular at $t = 0$. For $\beta_+ = 1$, a trajectory of the form $\alpha(t) = 1 + \alpha'(0)t + \ldots$ with $\alpha'(0) > 0$ is required near $t = 0$. Whenever $\beta_+ > 0$, it is not sufficient to have a fixed (and shielded) singularity at $\lambda = 1$.

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*) We use the normalization

$$\sigma_{tot} \sim \frac{16\pi}{s} \operatorname{Im} F(s,0) ,$$

$$F(s,t) = \frac{1}{2} \{ F_+(s,t) + F_-(s,t) \}.$$

Complete saturation of the Froissart bound would imply $\sigma_{tot} \sim 4\pi a(\log s)^2$ and $\sigma_{inel} \sim 0$. This case corresponds to $\delta(s,\xi) \to \pi/2$ and $\eta(s,\xi) \to 1$ for $s \to \infty$, ξ = fixed ; it also gives rise to a unique limit of the amplitude for fixed values of τ, which is given by Eq. (4.14) except for a factor two [10),13)].

3. If $\sigma_{tot} \sim (\log s)^{\beta_+}$ with $\beta_+ > 1$ ($\beta_+ \leq 2$ in general), then the positive signature trajectory near $t = 0$ must have a square-root branch point : $\alpha(t) = 1 + \text{const.}\sqrt{t} + O(t)$ *), if no hiding cuts are present. With hiding cuts, we can have trajectories like $\alpha(t) = 1 + \text{const.}t^{1/\beta}$ with $\beta_+ \leq \beta \leq 2$, provided only branches with $\text{Re}\,\alpha(t) < 1$ for $t < 0$ are in the physical sheet of the complex angular momentum plane.

A trajectory $\alpha(t)$ may, of course, have a branch point at $t = 0$ also in cases where this is not required by s channel unitarity [6].

We have shown that amplitudes with complex trajectories of the form (1.3) are rather naturally related to Bessel function representations. Examples can be written down which are compatible with the usual requirements of analyticity and unitarity in both channels. We have also considered limiting cases which correspond to the maximal allowed asymptotic value of the inelastic cross-section.

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*) We have assumed here that any shrinkage of the diffraction peak, which may be required, is due to the properties of the trajectory function $\alpha(t)$. This is the case for pole and branch point trajectories. In some situations, where β_+ is not an integer, there may be other possibilities if complicated essential singularities with t dependent character are allowed. We have not explored these possibilities. For $\beta_+ = 2$, it is sufficient to use the bound (2.2) in order to show that a trajectory of the form (1.3) is required [10].

We also assume that $\alpha_-(0) < 1$ for the leading negative signature trajectory [3].

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