SSCL-N-783

Notes About the Beta Function

G. López

Abstract

These notes do not pretend to say something new about the betatron function, but rather they discuss the Courant-Snyder's parameters from a different point of view. In this approach, a constant of motion is associated to the beta differential equation when the gradient of the magnetic field is constant, and the explicit solutions are found for a FODO period of a lattice and for the triplet structure in the Interaction Region (IR).

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1.0 INTRODUCTION

The Courant-Snyder's parameters[1] have been very useful for lattice design of collider ring accelerators. These parameters, α, β, ψ(phase advance), ν(Q tune), ϵ(emittance), and γ, appear from a particular transformation of the Hill's equation. They are normally calculated through computer mapping programs of the lattice, using the periodicity property of the solution. Of all of these parameters, the most important is the β function since the others depend on this one. Thus, knowing the β function all over the accelerator, it means knowing all its linear properties. The nonlinearities that appear in the accelerator (magnetic field multipoles, multipoles elements, misalignment, etc.) produce a perturbation on these parameters. The main goal in accelerator design is to keep these nonlinearities small enough so that the beta-theory remains essentially valid. One of the most important derived parameters from the beta function is the emittance, ϵ. This is a constant of motion of the system and defines the area of the beam particles in the phase space. In addition, because of this constant, it is possible to know the evolution of the whole beam of charged particles in the accelerator. Hence, the constants of motion are particularly valuable in accelerator physics.

The following approach is based in the constants of motion associated to the beta differential equation. Firstly, the transformation is made in the Hill's equation to bring about the differential equation for the beta (or ω^2) function. The constant of motion is deduced when the magnetic field gradient is constant. Using this constant of motion, the examples of the FODO lattice and IR section are given.

2.0 HILL'S EQUATION AND ITS TRANSFORMATION

The Hill's equation governs the linear behavior of a particle in a collider accelerator, and it is given by

$$x'' + K(s)x = 0 , \tag{1}$$

where $x' = dx/ds$, and $K(s)$ is a periodic function. Proposing a solution of the form[2]

$$x(s) = A\omega(s)\cos(\psi(s)) , \tag{2}$$

the derivatives of this expression are

$$x' = A\omega' \cos(\psi) - A\omega\psi' \sin(\psi) , \tag{3}$$

and

$$x'' = A\omega'' \cos(\psi) - 2A\omega'\psi' \sin(\psi) - A\omega\psi'' \sin(\psi) - A\omega(\psi')^2 \cos(\psi) . \tag{4}$$

Substituting these in Eq. (1), it follows

$$\left[\omega'' - \omega(\psi')^2 + K(s)\omega\right] A\cos(\psi) - \left[2\omega'\psi' + \omega\psi''\right] A\sin(\psi) = 0 \ . \qquad (5)$$

Since "cos" and "sin" are independent functions, two differential equations are obtained,

$$\omega'' - \omega(\psi')^2 + K(s)\omega = 0 \qquad (6a)$$

and

$$2\omega'\psi' + \omega\psi'' = 0 \ . \qquad (6b)$$

Now, multiplying Eq. (6b) by ω, the following relation appears

$$\omega(2\omega'\psi' + \omega\psi'') = \frac{d}{ds}(\omega^2\psi') = 0 \qquad (6c)$$

which brings about the constant of motion

$$k = \omega^2(s)\psi'(s) \ . \qquad (7)$$

Using this constant of motion in Eq. (6a), an uncouple differential equation is obtained for the ω function,*

$$\omega'' + K(s)\omega - \frac{k^2}{\omega^3} = 0 \ . \qquad (8)$$

The constant A, in Eq. (2), is another constant of motion of the system. Substituting "$A\cos(\psi)$" from Eq. (2) in the first term of the right hand side of Eq. (3) and substituting "ψ'" from Eq. (7) in the second term of the right hand side of Eq. (3), the following two relations are obtained

$$A\cos(\psi) = \frac{x}{\omega} \qquad (9)$$

* Derivating Eq. (8) again and using the definition of the beta function, $\beta = \omega^2$, a third order linear differential equation can be obtained, $d^3\beta/ds^3 + 4K(s)\ (d\beta/ds) + 2(dK(s)/ds)\ \beta = 0$, but doing this, some information may be lost, and solutions must be substituted in the non linear differential equation, Eq. (8), to check that they are also solutions of this one. This process, even for K equal constant, is quite complicated. Therefore, instead of using these explicit solutions, it is common to find the solution from a mapping transformation which results from the Eqs. (2) and (3), the initial conditions, and the periodicity statement.

and

$$A\sin(\psi) = \frac{(\omega' x - x'\omega)}{k} \ . \tag{10}$$

Taking the square and adding both relations, the next expression is obtained,

$$A^2 = \frac{(\omega' x - x'\omega)^2}{k^2} + \frac{x^2}{\omega^2} \ . \tag{11}$$

From this expression, it is clear that the constant k can never be zero here since A is a finite constant of motion. The case $k = 0$ can be analyzed starting from Eq. (7) and Eq. (8) bringing about the identity transformation for Eq. (1).

3.0 CONSTANT OF MOTION ASSOCIATED TO THE ω FUNCTION

Using the definition

$$v = \frac{d\omega}{ds} \ , \tag{12a}$$

Eq. (8) is transformed into a nonautonomous dynamical system given by equations

$$\frac{dv}{ds} = \frac{k^2}{\omega^3} - K(s)\omega \tag{12b}$$

and

$$\frac{d\omega}{ds} = v \ . \tag{12c}$$

A constant of motion of this system, $E = E(s, \omega, v)$, satisfies the equation

$$\frac{dE}{ds} = 0 \ , \tag{13}$$

i.e., E satisfies the following partial differential equation

$$\frac{\partial E}{\partial s} + v\ \frac{\partial E}{\partial \omega} + \left[\frac{k^2}{\omega^3} - K(s)\omega\right] \frac{\partial E}{\partial v} = 0 \ . \tag{14}$$

However, only the case "$K =$ *constant*" will be considered. In this case, the system Eqs. (12a, b, c) is autonomous, and the term $\partial E/\partial s$ in Eq. (14) can be ignored bringing about the following equation

$$v \frac{\partial E}{\partial \omega} + \left[\frac{k^2}{\omega^3} - K\omega\right] \frac{\partial E}{\partial v} = 0 . \tag{15}$$

The solution of the preceding equation can be obtained through the characteristics method where the equations for the characteristics are

$$\frac{d\omega}{v} = \frac{dv}{k^2/\omega^3 - K\omega} = \frac{dE}{0} . \tag{16}$$

Using the first two terms, the following equation is obtained

$$v \frac{dv}{d\omega} = \frac{k^2}{\omega^3} - K\omega , \tag{17}$$

and its solution brings about the constant of motion,

$$E = v^2 + K\omega^2 + \frac{k^2}{\omega^2} . \tag{18}$$

In terms of the Courant-Snyder parameters,

$$\beta(s) = \omega^2(s) \tag{19}$$

and

$$\alpha(s) = -\frac{1}{2}\frac{d\beta}{ds} . \tag{20}$$

This constant is expressed as

$$E = \frac{\alpha^2 + K\beta^2 + k^2}{\beta} , \tag{21a}$$

where the relation

$$\frac{d\omega}{ds} = \pm\frac{1}{\sqrt{\beta}} \frac{d\beta}{ds} \tag{21b}$$

has been used. Notice that the function $\omega^2(s)$ can never be equal to zero here since E is a finite constant of motion.

Given the initial conditions, $x(0) = x_o$ and $x'(0) = x'_o$, relations Eqs. (2), (3), (7), (11), and (18) form a set of five algebraic relations,

$$
\begin{aligned}
x_o &= A\omega_o \cos\psi_o \ , \\
x'_o &= A\omega'_o \cos\psi_o - \frac{Ak}{\omega_o} \sin\psi_o \ ,
\end{aligned}
$$

$$
\begin{aligned}
A^2 &= \frac{x_o^2}{\omega_o^2} + \frac{1}{k}(\omega'_o x_o - x'_o \omega_o)^2 \ , \\
k &= \omega_o^2 \psi'_o \ ,
\end{aligned}
$$

and

$$
E = \omega_o'^2 + K\omega_o^2 + \frac{k^2}{\omega_o^2} \ ,
$$

where $\omega(0) = \omega_o$, $\omega'(0) = \omega'_o$, $\psi(0) = \psi_o$, and $\psi'(0) = \psi'_o$. However, the number of unknown parameters is six, $x_o, x'_o, \omega_o, \omega'_o, \psi_o$, and ψ'_o. Consequently, it is possible to choose one of these parameters freely, say ψ'_o. The common selection is

$$
\psi'_o = \frac{1}{\omega_o^2} \tag{22}
$$

which makes $k = 1$. Hence, the function ψ is given by

$$
\psi(s) = \psi_o + \int_0^s \frac{d\tau}{\beta(\tau)} \ . \tag{23}
$$

4.0 IMPLICATIONS OF THE CONSTANT OF MOTION E

The constant of motion Eq. (18) can be written in terms of the beta function in the following way

$$
E = \frac{1}{4\beta} \left(\frac{d\beta}{ds}\right)^2 + V(\beta) \ , \tag{24}
$$

where the function V is defined as

$$
V(\beta) = K\beta + \frac{1}{\beta} \ . \tag{25}
$$

The first term on the right hand side of Eq. (24) represent the "Kinetic" contribution to the constant of motion and the second one represents the "Potential" contribution. The potential is plotted in Figure 1. The trajectories in the space (β, β') are plotted in Figure 2. This constant of motion will be used to obtain some explicit solutions for the beta function.

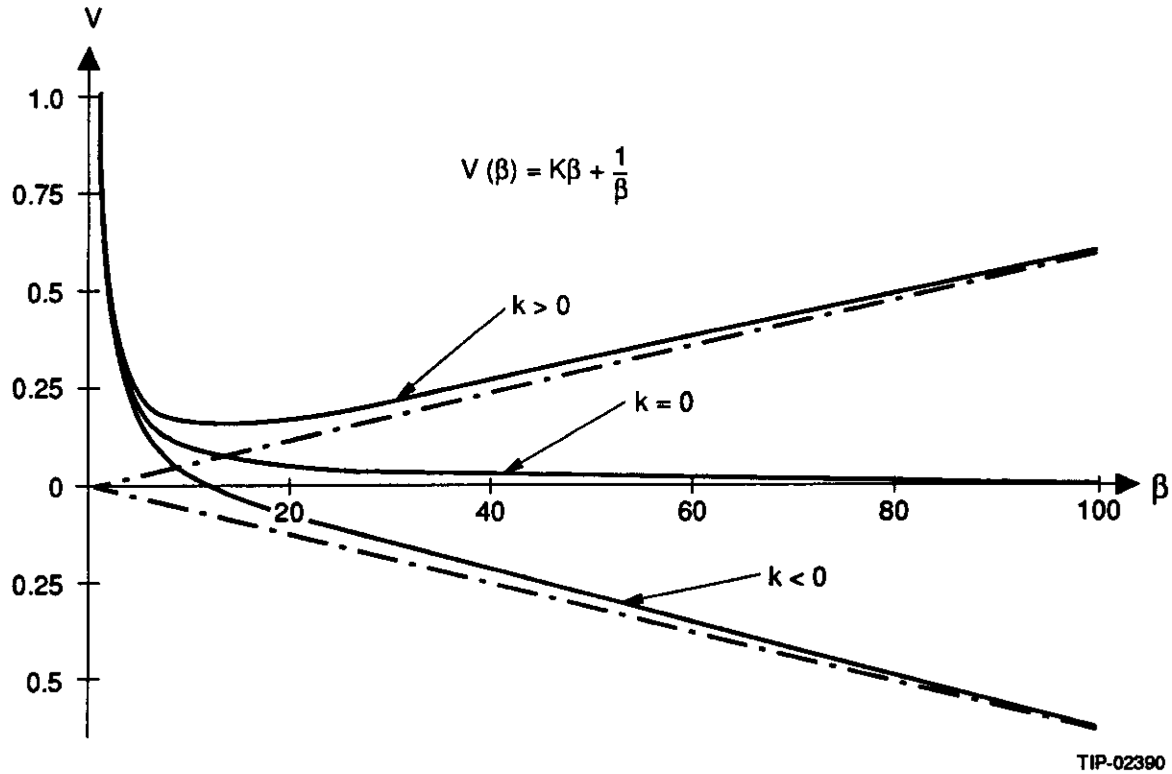

Figure 1. Potential Term in the Constant of Motion.

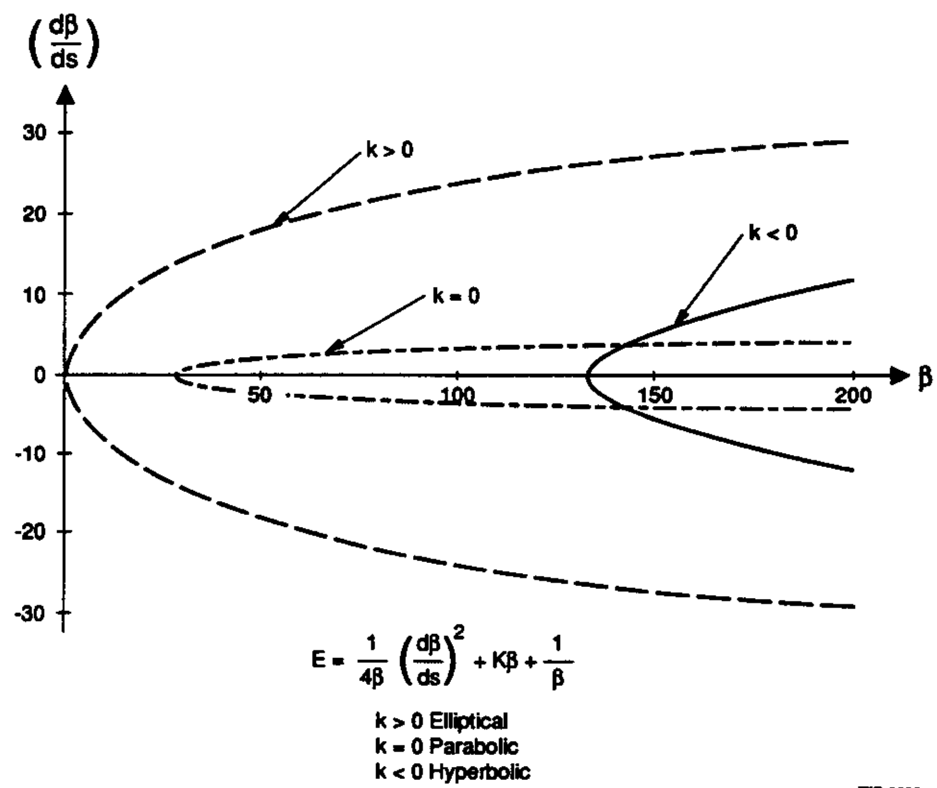

Figure 2. Curves in the Beta Phase Space for Constant "Energies."

4.1 Solution for the Case $K = 0$

From Eqs. (24) and (25), the constant of motion has the expression

$$E_o = \frac{1}{4\beta}\left(\frac{d\beta}{ds}\right)^2 + \frac{1}{\beta} \; . \tag{26}$$

The rate of change of the beta function is expressed in the following form

$$\frac{d\beta}{ds} = \pm 2\sqrt{-1 + E_o\beta} \tag{27}$$

which can be transformed in the integral

$$\frac{1}{2}\int \frac{d\beta}{\sqrt{-1+E_o\beta}} = \pm s + \lambda_o \; , \tag{28}$$

where λ_o is an arbitrary constant. Solving Eq. (28), the beta function is given by

$$\beta(s) = \frac{1}{E_o} + E_o(\pm s + \lambda_o)^2 \; , \tag{29a}$$

and its derivative is

$$\beta'(s) = 2E_o s \pm 2E_o\lambda_o \; . \tag{29b}$$

The constants E_o and λ_o are determinate by the initial conditions,

$$\beta(0) = \beta_o \tag{30a}$$

and

$$\beta'(0) = \beta'_o \; , \tag{30b}$$

and they have the following expressions

$$E_o = \frac{1}{4\beta_o}(\beta'_o)^2 + \frac{1}{\beta_o} \tag{31a}$$

and

$$\lambda_o = \pm\frac{\beta'_o}{2E_o} \; . \tag{31b}$$

Then the relations Eqs. (29a, b) can be written as

$$\beta(s) = \frac{1}{E_o} + E_o \left(s + \frac{\beta_o'}{2E_o} \right)^2 , \tag{32a}$$

and

$$\beta'(s) = 2E_o s + \beta_o' . \tag{32b}$$

4.2 Solution for the Case $K > 0$

The constant of motion is given by

$$E_1 = \frac{1}{4\beta} \left(\frac{d\beta}{ds} \right)^2 + K\beta + \frac{1}{\beta} , \tag{33}$$

and the integral obtained from this expression is

$$\frac{1}{2} \int \frac{d\beta}{\sqrt{-1 + E_1\beta - K\beta^2}} = \pm s + \lambda_1 , \tag{34}$$

where λ_1 is an arbitrary constant. Solving Eq. (34), the solution can be written as

$$\beta(s) = \frac{E_1}{2K} + \frac{1}{2K}\sqrt{E_1^2 - 4K} \ \sin\left(2\sqrt{K}(\pm s + \lambda_1)\right) \tag{35a}$$

and

$$\beta'(s) = \sqrt{\frac{E_1^2}{K} - 4} \ \cos\left(2\sqrt{K}(\pm s + \lambda_1)\right) . \tag{35b}$$

Assuming Eqs. (30a, b), the constants E_1 and λ_1 are given by

$$E_1 = \frac{1}{4\beta_o} \ \beta_o'^2 + K\beta_o + \frac{1}{\beta_o} \tag{36a}$$

and

$$\lambda_1 = \frac{1}{2\sqrt{K}} \arccos \left(\frac{\beta_o'}{\sqrt{\frac{E_1^2}{K} - 4}} \right) . \tag{36b}$$

4.3 Solution for the Case $K < 0$

Defining $\widetilde{K} = -K$, the constant of motion is given by

$$E_2 = \frac{1}{4\beta}\left(\frac{d\beta}{ds}\right)^2 - \widetilde{K}\beta + \frac{1}{\beta} \,. \tag{37}$$

The integral obtained from this expression is clearly

$$\frac{1}{2}\int \frac{d\beta}{\sqrt{-1 + E_2\beta + \widetilde{K}\beta^2}} = \pm s + \lambda_2 \,, \tag{38}$$

where λ_2 is an arbitrary constant. Again, solving the above integration and making some rearrangements, the following solution is obtained

$$\beta(s) = \frac{4\widetilde{K} + \left(E_2 - \Omega_o \exp\left(2\sqrt{\widetilde{K}}\, s\right)\right)^2}{4\widetilde{K}\Omega_o \exp\left(2\sqrt{\widetilde{K}}\, s\right)} \tag{39a}$$

and

$$\beta'(s) = -\frac{4\widetilde{K} + E_2^2}{2\sqrt{\widetilde{K}}\Omega_o} \exp\left(-2\sqrt{\widetilde{K}}\, s\right) + \frac{\Omega_o}{2\sqrt{\widetilde{K}}} \exp\left(2\sqrt{\widetilde{K}}\, s\right) \,, \tag{39b}$$

where the constants E_2 and Ω_o are given by

$$E_2 = \frac{\beta_o'}{4\beta_o} - \widetilde{K}\beta_o + \frac{1}{\beta_o} \tag{40a}$$

and

$$\Omega_o = \beta_o'^2\sqrt{\widetilde{K}} + 2\widetilde{K}\beta_o + E_2 \,. \tag{40b}$$

5.0 APPLICATION TO A FODO CELL

Assume $K(s)$ has a typical FODO lattice periodicity where one cell of the lattice is made up of alternated focusing defocusing quadrupoles with dipoles (or drift space) in between. K is constant per section, and the only places where there is a change is at the boundary of each element. Figure 3 shows an example of this particular case. At the boundaries, s_i, $i = 1,..5$, the functions $x(s), x'(s), \omega(s)$, and $\omega'(s)$ are all continuous. However, the second derivatives, $x''(s)$ and $\omega''(s)$ have a jump which can be calculated straightforward from Eqs. (1) and (8). Taking their limit values from the left and the right hand sides of the boundaries, these jumps are given by

$$|\delta x''(s_i)| = K_q \ x_i \ \ , \ i = 1, ..., 5 \tag{41a}$$

and

$$|\delta \omega''(s_i)| = K_q \ \omega_i \ \ , \ i = 1, ..., 5 \ , \tag{41b}$$

where $x_i = x(s_i)$ and $\omega_i = \omega(s_i)$ $i = 1, .., 5$.

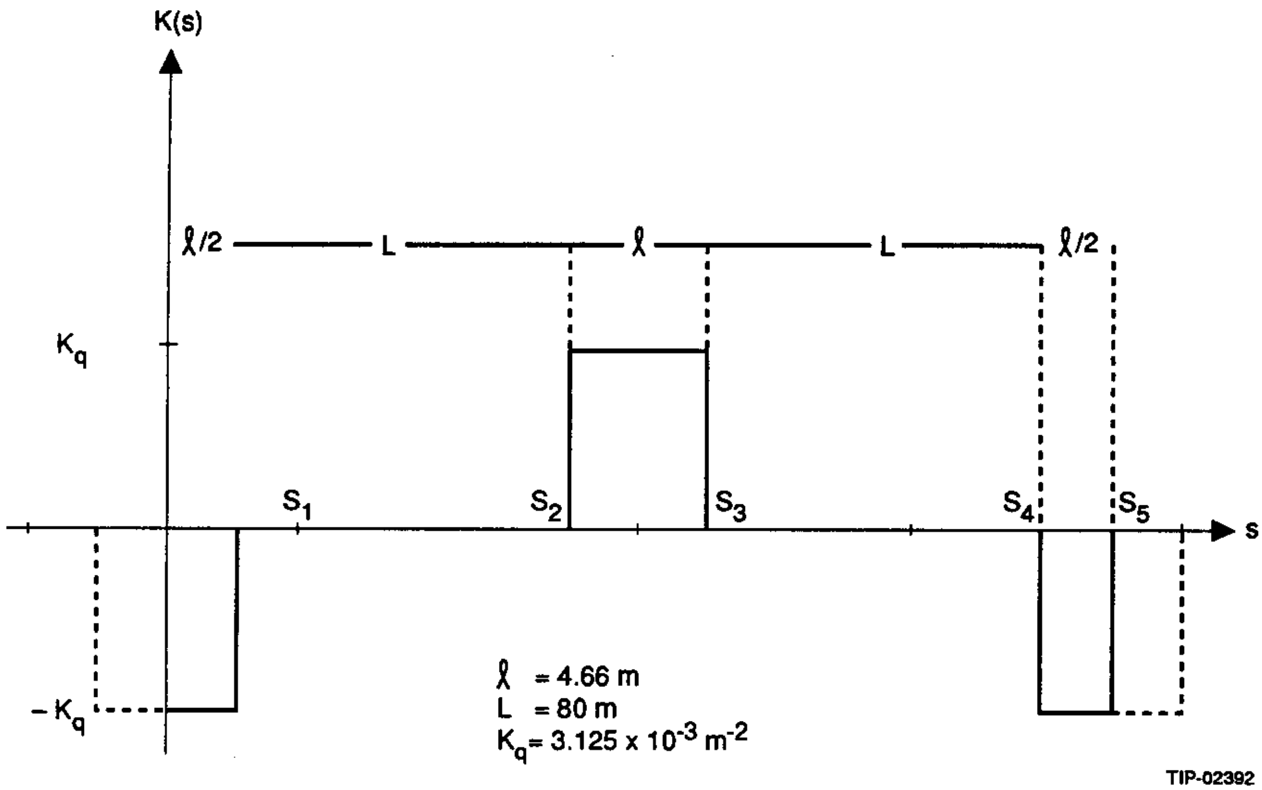

Figure 3. FODO Cell Lattice.

The above solution can be used in each section independently, and then they can be matched at the boundaries. But there is something else that is necessary to consider. If a periodic beta solution is required, the length of the quadrupoles, l, the length of the drift space, L, and the strength of the quadrupoles, Kq, must be correlated. Suppose that at $s = 0$, the conditions are $\beta(0) = \beta_o$ and $\beta'(0) = \beta'_o$. So at the point $s = s_5$ (see Figure 3), the final values must be, $\beta(s_5) = \beta_o$ and $\beta'(s_5) = \beta'_o$. Applying these conditions to Eq. (39a) and Eq. (39b), it follows

$$\beta_o = \frac{4\widetilde{K} + \left(E_2 - \Omega_o \exp\left(\sqrt{\widetilde{K}}\; l_D\right)\right)^2}{4\widetilde{K}\Omega_o \exp\left(\sqrt{\widetilde{K}}\; l_D\right)} \tag{42}$$

and

$$\beta_o' = -\frac{4\widetilde{K} + E_2^2}{2\sqrt{\widetilde{K}}\Omega_o} \exp\left(-\sqrt{\widetilde{K}}\; l_D\right) + \frac{\Omega_o}{2\sqrt{\widetilde{K}}} \exp\left(\sqrt{\widetilde{K}}\; l_D\right) , \tag{43}$$

where l_D is the length of the defocusing quadrupole magnet. The constants E_2 and Ω_o are calculated with the value of the functions, β and β', at the point $s = s_4$. Any of these relations can be used to calculate the length l_D since their equivalency is given through the two constant of motion E_2 and Ω_o. Using Eq. (43), the length is given by

$$l_D = \frac{1}{\sqrt{\widetilde{K}}} \log\left\{ \sqrt{\widetilde{K}}\Omega_o\beta_o' + \sqrt{\widetilde{K}\Omega_o^2\beta_o'^2 + \frac{4\widetilde{K} + E_2^2}{\Omega_o^2}} \right\} . \tag{44}$$

Applying the same criterion to the focusing magnet, $\beta(s_3) = \beta(s_2)$ and $\beta'(s_3) = -\beta'(s_2)$, and using the final conditions in Eqs. (35a) and (35b), the following relations are obtained

$$\beta_2 = \frac{E_1}{2K} + \frac{1}{2K}\sqrt{E_1^2 - 4K}\; \sin 2\sqrt{K}(\pm l_F + \lambda_1) \tag{45}$$

and

$$-\beta_2' = \sqrt{\frac{E_1}{K} - 4}\; \cos 2\sqrt{K}(\pm l_F + \lambda_1) , \tag{46}$$

where l_F is the length of the focusing quadrupole magnet. The constants E_1 and λ_1 are calculated with the value of the functions, β and β', at the point $s = s_2$. Again, the length of the focusing quadrupole, l_F, can be calculated with either of the above expression. Using Eq. (46), this length is given by

$$l_F = -\lambda_1 + \frac{1}{2\sqrt{K}} \arccos\left(-\frac{\beta_2'}{\sqrt{E_2^2/K - 4}} \right) . \tag{47}$$

Now, the length and the strength of the quadrupoles are the same. Thus, from Eqs. (44) and (47), the following relationship between the the dimension and variables appears

$$l = \log\left(\sqrt{\widetilde{K}}\Omega_o\beta_o' + \sqrt{\widetilde{K}\Omega_o^2\beta_o'^2 + \frac{4\widetilde{K} + E_2^2}{\Omega_o^2}} \right)$$

$$= -\lambda_1 + \frac{1}{2\sqrt{K}} \arccos\left(-\frac{\beta_2'}{\sqrt{E_2^2/K - 4}}\right) . \tag{48}$$

Notice that E_1, E_2, β_2', and Ω_o depend on the length of the drift space, L, and length of the quadrupole magnets, l. Thus, given the strength of the quadrupoles and the separation between them, Eq. (48) can be solved numerically to find its fixed point. This relation gives the periodicity condition in the particle motion. This fixed point as a function of $K_q = K = \widetilde{K}$ is shown in Figure 4, where the distance from the center of one magnet to another is always 90 m. This figure shows that for $K_q = 0.00308\ \mathrm{m}^{-2}$ (nominal parameter for the SSC quadrupoles), the length of the quadrupole magnet must be 5.2 m in order to have a periodic solution (also in agreement with its nominal value).[3]

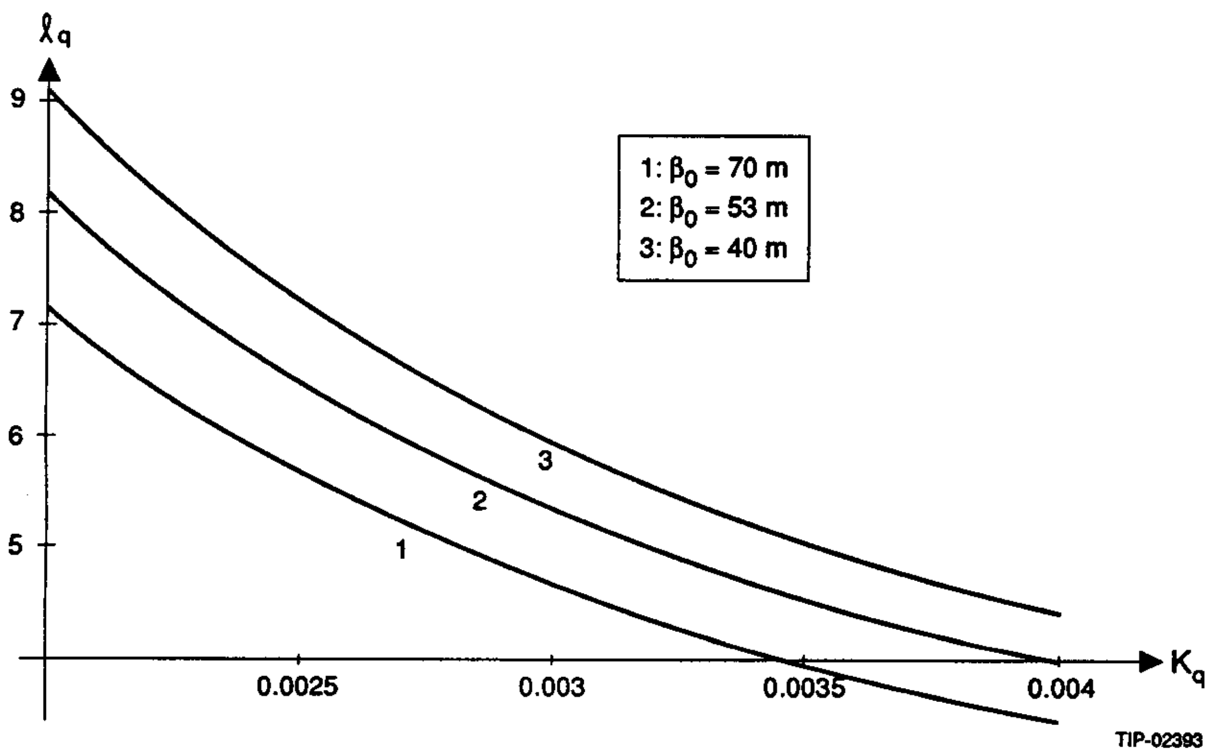

Figure 4. Relation between the Length of the Quadrupole and its Strength for a Periodic FODO Structure.

Using the above $l = 5.2$ m, $K_q = 0.00308\ \mathrm{m}^{-2}$, separation between centers of magnets of 90 m, and a small computer program, E-BETA-2, it is possible to carry out the beta function calculations for this lattice. The results can be seen from Figure 5 to Figure 9.

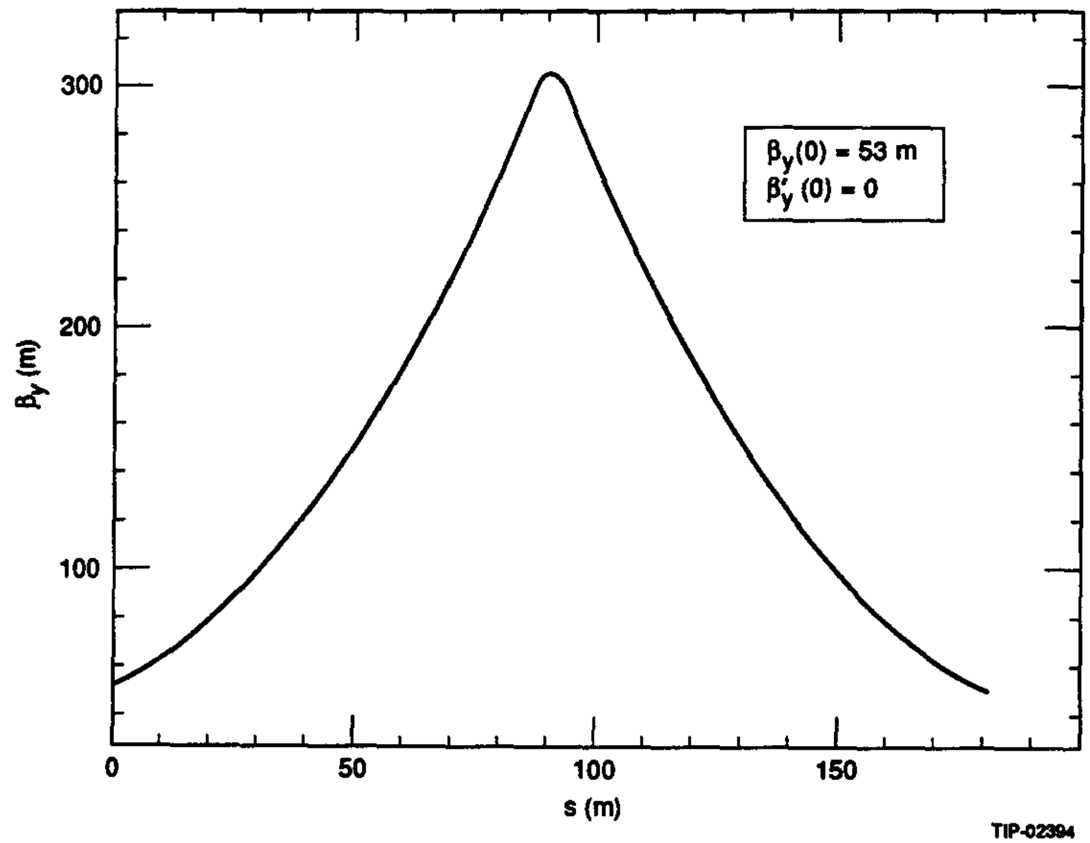

Figure 5. Vertical Beta Function in the FODO Structure.

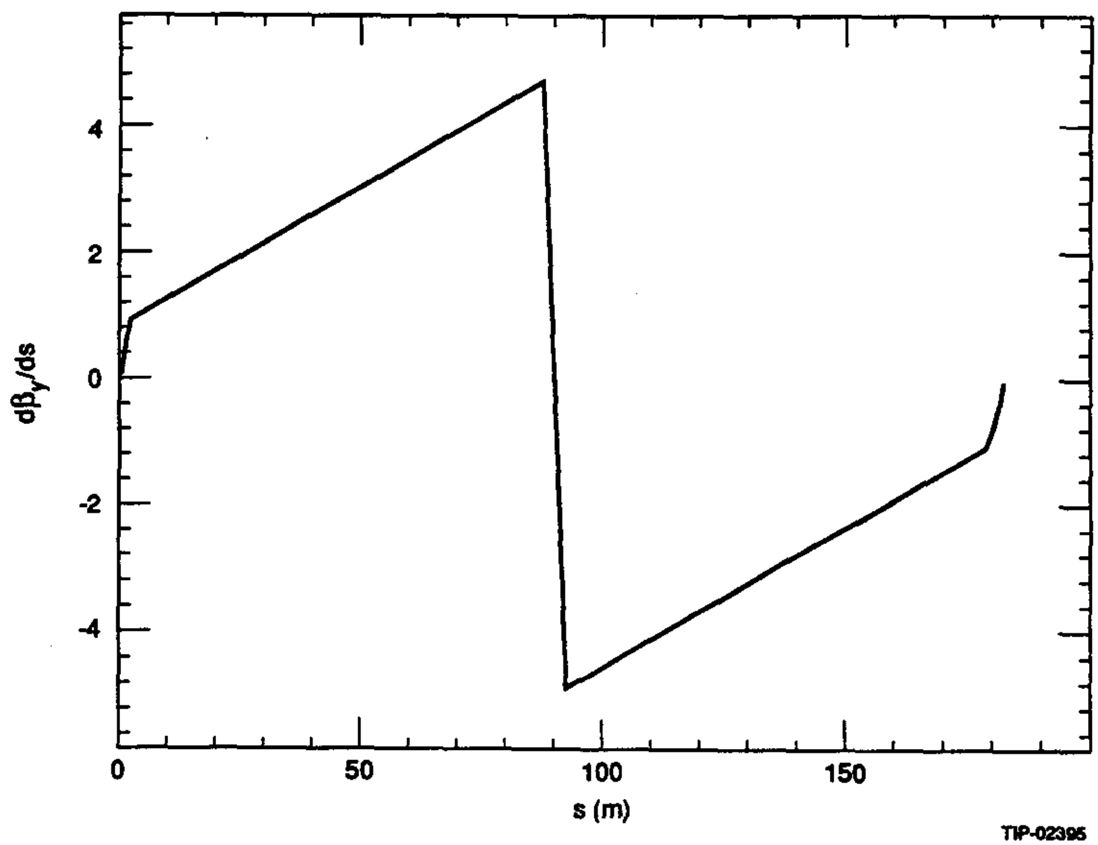

Figure 6. Vertical -2α Function in the FODO Structure.

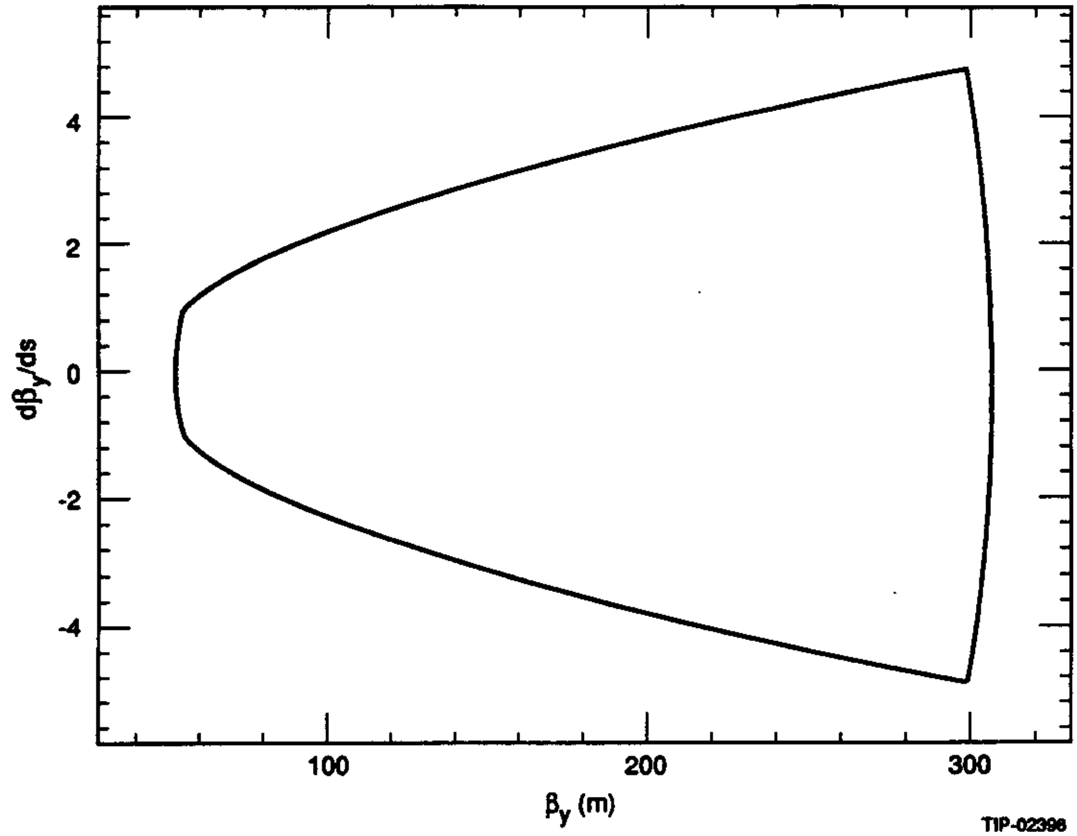

Figure 7. Motion in the Beta Phase Space for the FODO Structure.

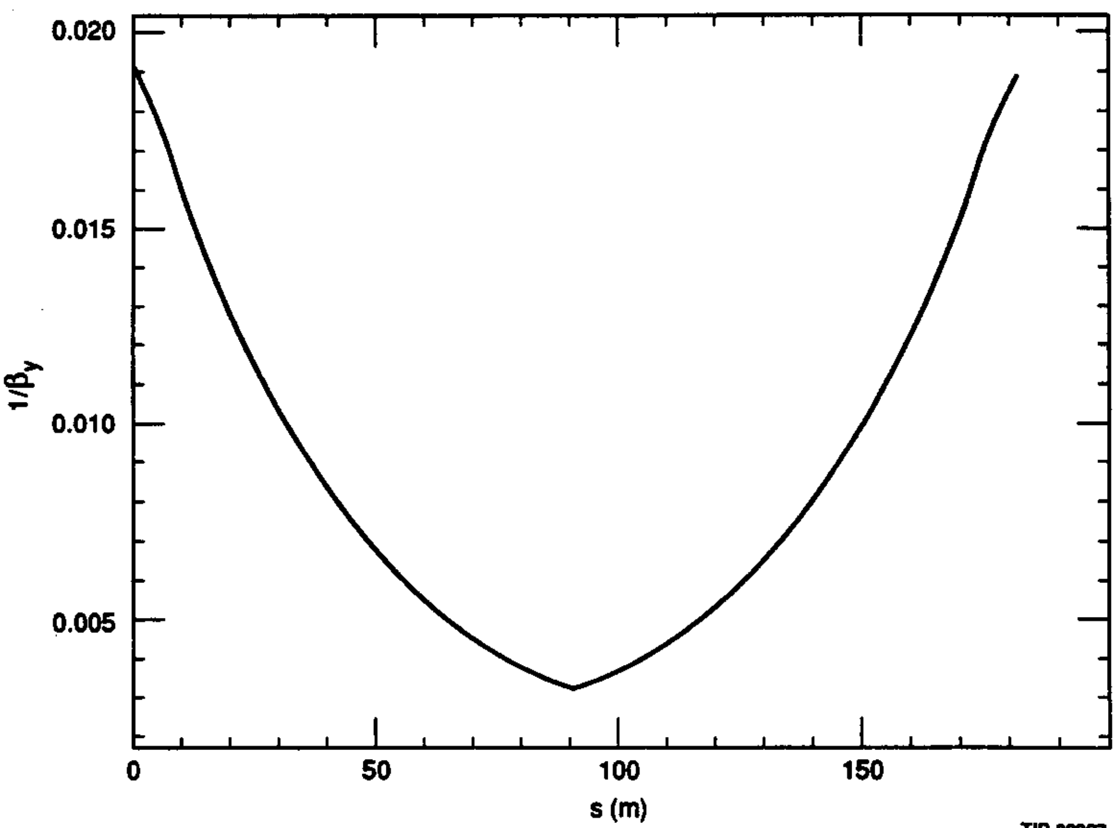

Figure 8. Related Beta Function in the FODO Cell.

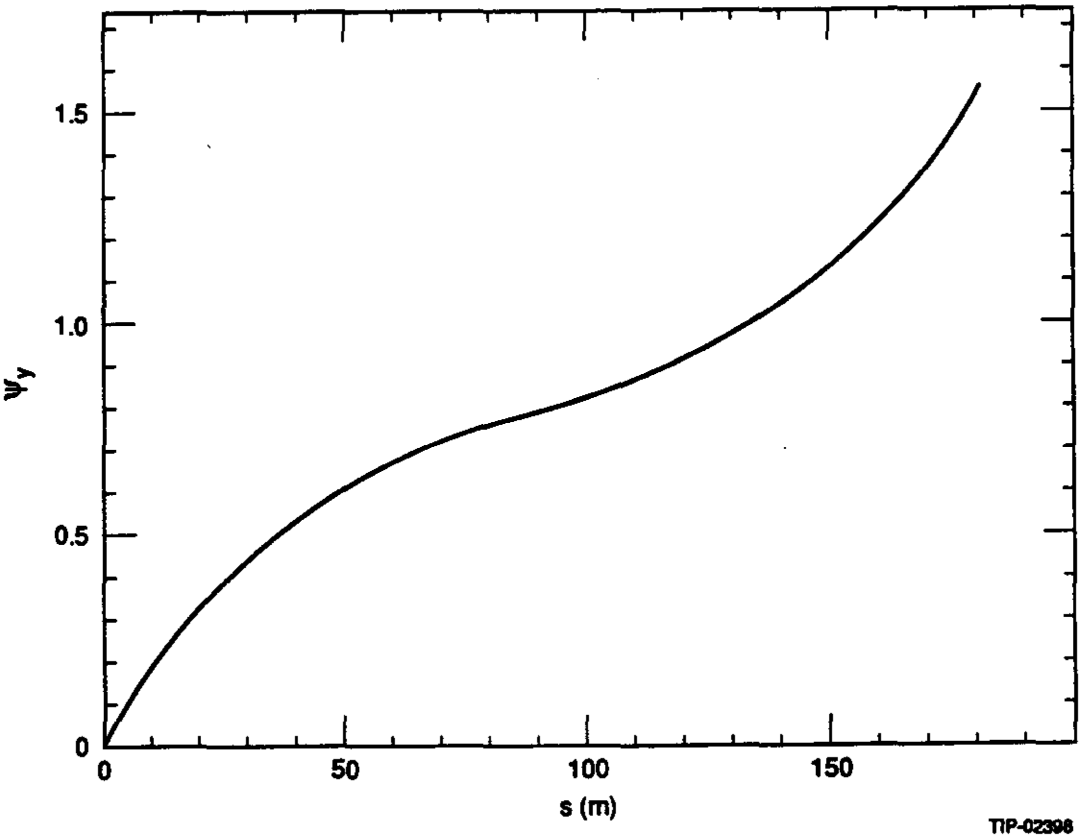

Figure 9. Phase Advance Behavior in the FODO Cell.

6.0 APPLICATION TO THE TRIPLET STRUCTURE

The triplet structure is the basic element for a collider accelerator in the Interaction Region (IR), and it is thought that it could become an important element for the design of synchrotron radiation sources.[4] This structure is made essentially of three strong quadrupoles with the configuration Defocusing-Focusing-Defocusing, DFD, for the horizontal plane (FDF for the vertical plane). However, each quadrupole could be subdivided according to some mechanical constraints. Figure 10 shows three IR triplet designs. The first one corresponds to the nominal configuration for the SSC.[3] The second corresponds to a possible configuration for a reduced maximum beta function in the triplet.[5] The last one corresponds to another possible configuration for a reduced maximum beta function and for a drastic increasing in the quadrupole lifetime due to radiation damage,[6] if collimators are used in the big gaps. In fact, this last example was made fixing the quadrupole gradients as well as the gaps between them (2.3 m, 1 m, and 0.8 m), and looking for the quadrupole lengths which have the optimum solution.

Given the free space between the Interaction Point (IP) and the first magnet, the gaps between magnets, the quadrupole lengths, and their strengths (as it is shown in Figure 10), the beta function can be found by using the solutions of the beta function in the corresponding region (Section 3). A unique solution is obtained with the initial conditions

$$\beta(0) = 0.5 \text{ m} \tag{49a}$$

and

$$\beta'(0) = 0 \; , \tag{49b}$$

and matching the solutions of different regions at the boundaries where a change in $K(s)$ occurs (the continuity of β and β' is used here). Using a short computer algorithm, E-BETA, the solution for the three cases in Figure 10 is shown in Figures 11 and 12.

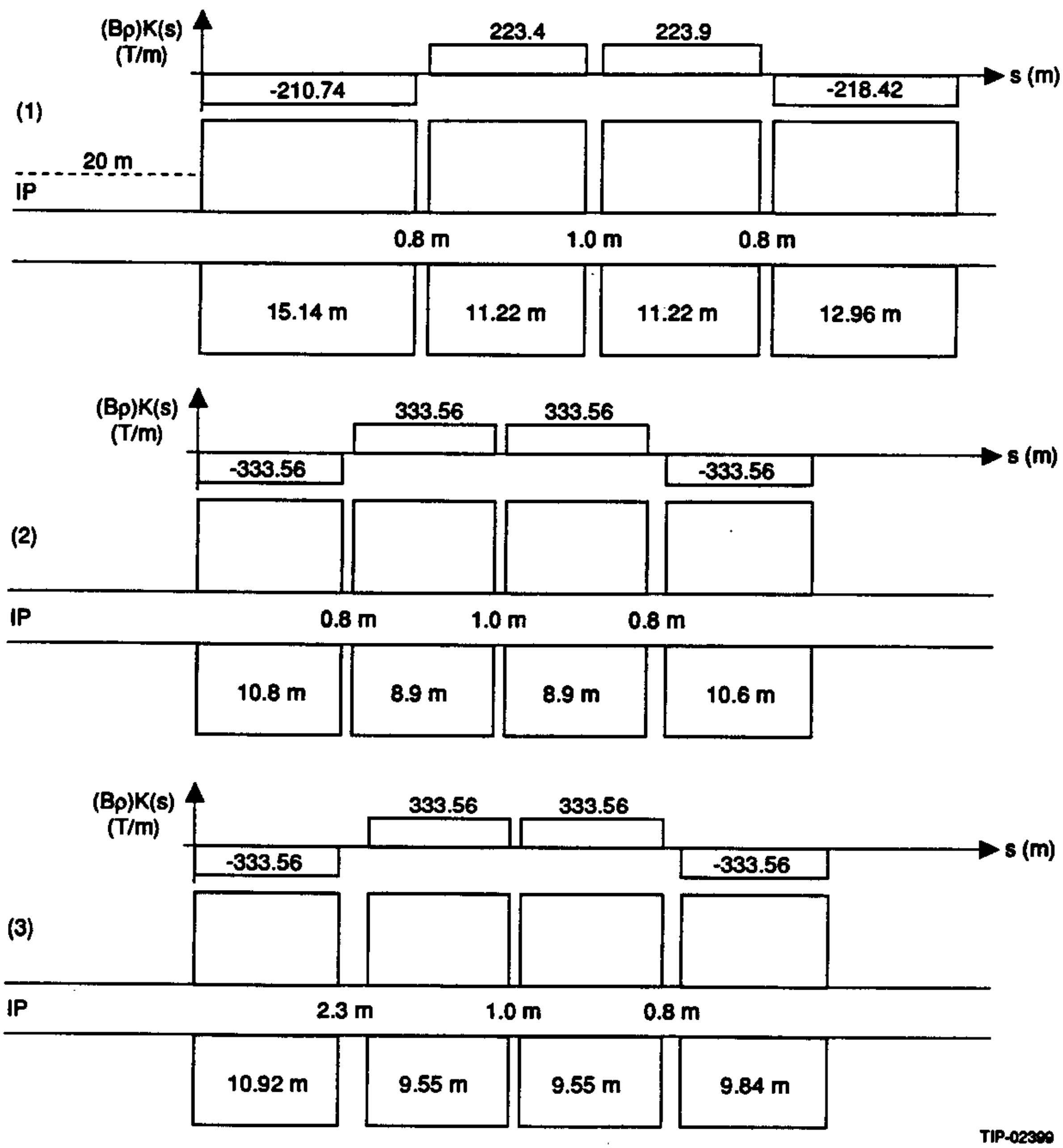

Figure 10. Three IR Triplet Designs. (1) Nominal Low-β SSC Design, (2) and (3) are Possible Different Examples.

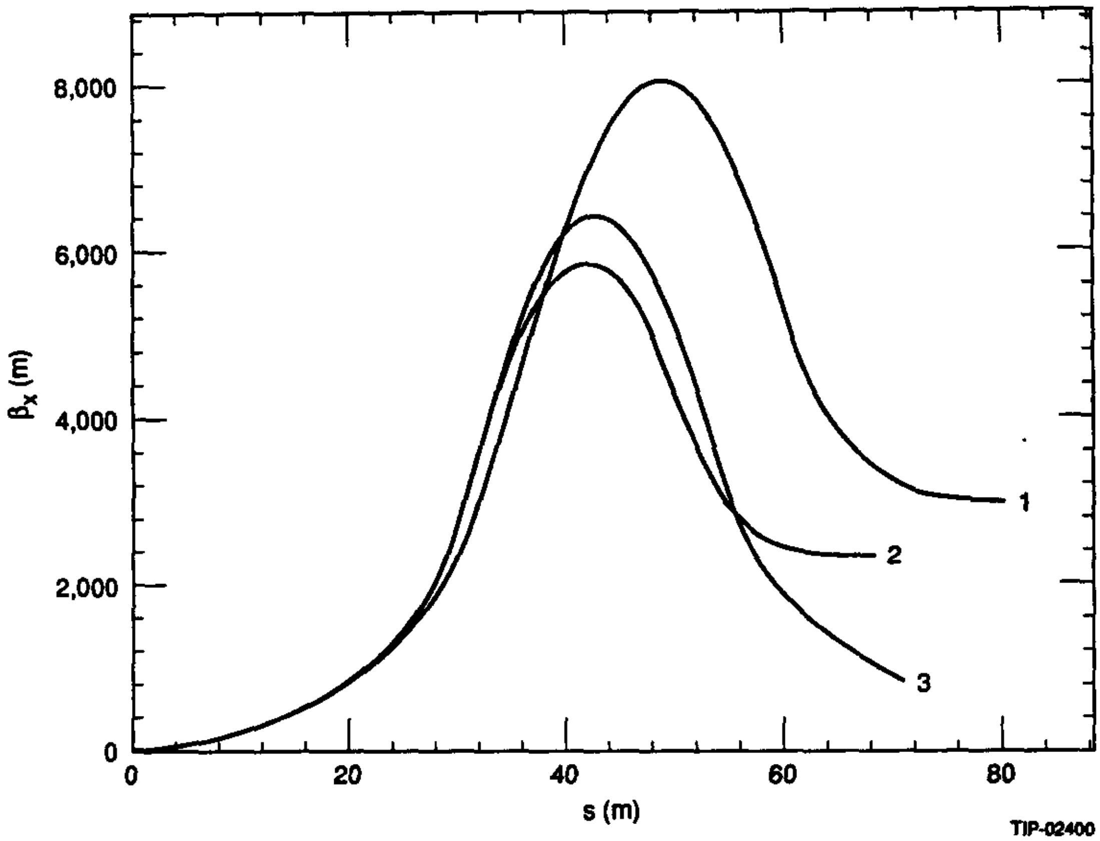

Figure 11. Horizontal Beta Function for the Examples of Figure 10.

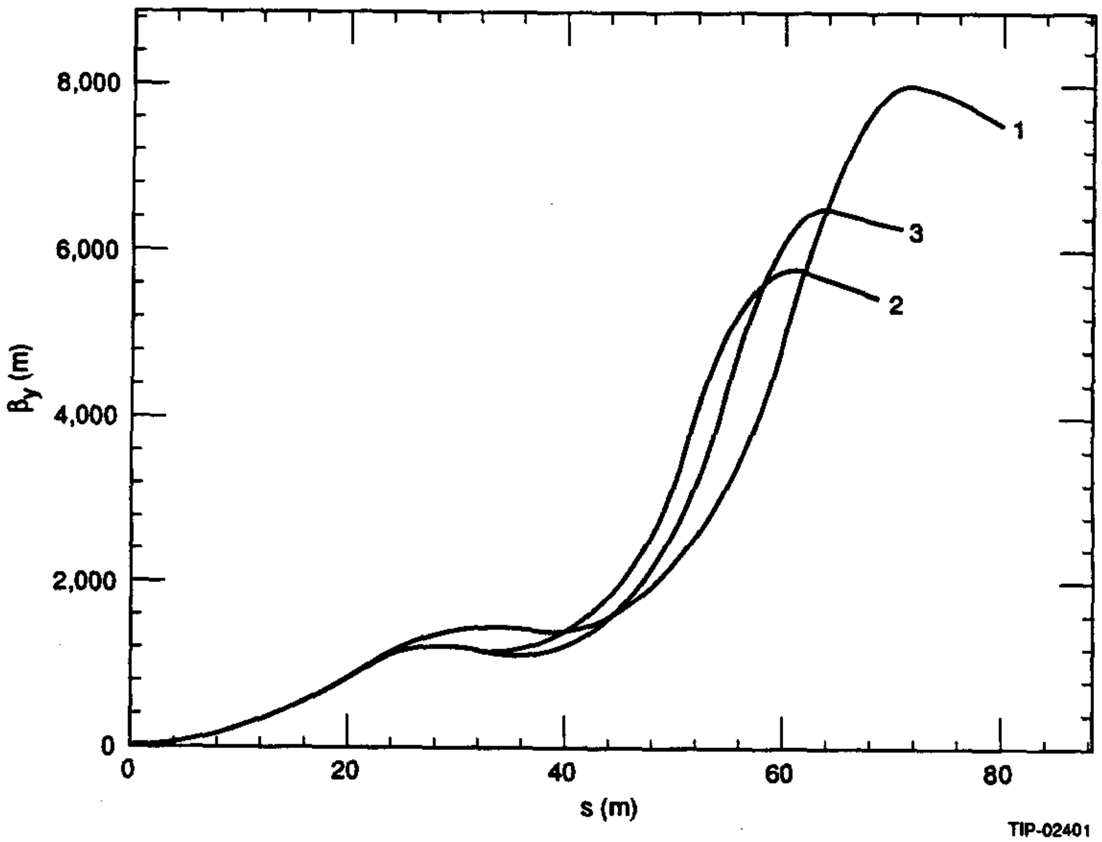

Figure 12. Vertical Beta Function for the Examples of Figure 10.

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