RESEARCH ARTICLE

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Gibbs' paradox and black-hole entropy

Abstract In statistical mechanics Gibbs' paradox is avoided if the particles of a gas are assumed to be indistinguishable. The resulting entropy then agrees with the empirically tested thermodynamic entropy up to a term proportional to the logarithm of the particle number. We discuss here how analogous situations arise in the statistical foundation of black-hole entropy. Depending on the underlying approach to quantum gravity, the fundamental objects to be counted have to be assumed indistinguishable or not in order to arrive at the Bekenstein–Hawking entropy. We also show that the logarithmic corrections to this entropy, including their signs, can be understood along the lines of standard statistical mechanics. We illustrate the general concepts within the area quantization model of Bekenstein and Mukhanov.

Keywords Black-hole entropy, Logarithmic corrections, Quantum gravity

1 Introduction

Black holes are fascinating objects that have not yet revealed all their secrets. If described by Einstein's classical theory of relativity, they are characterized by an event horizon which encloses a region from which nothing, not even light, can escape. If quantum theory on a black-hole background is considered in addition, it is found that black holes emit thermal radiation [1]. Black holes thus play a key role in the search for a quantum theory of gravity [2].

Our contribution deals with the black-hole entropy and its interpretation. We are especially interested in logarithmic corrections to the Bekenstein–Hawking

Dedicated to the 60th birthday of Bahram Mashhoon

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formula of black-hole entropy and their relation to similar terms in ordinary statistical mechanics. By highlighting their role in the discussion of Gibbs' paradox, we give an interpretation of these terms that should encompass all cases discussed in the literature (see [3] and the references cited therein).

Let us, however, first give a brief introduction to this subject; more details can be found in [4] and many other references. The concept of black-hole entropy first arose from formal analogies of mechanical black-hole laws with the laws of thermodynamics. The First Law of black-hole mechanics reads[1]

$$dM = \frac{\kappa}{8\pi G}dA + \Omega_H dJ + \Phi dq, \tag{1}$$

where M is the black-hole mass, A the area of the event horizon, Ω_H its angular velocity, J the angular momentum, Φ the electric potential, and q the electric charge of the black hole (if it has a charge). The quantity κ denotes the surface gravity of the black hole. For a Kerr black hole, κ is explicitly given by the expression

$$\kappa = \frac{\sqrt{(GM)^2 - a^2}}{2GMr_+} \quad \xrightarrow{a \to 0} \quad \frac{1}{4GM} = \frac{GM}{R_S^2}, \tag{2}$$

where

$$r_+ = GM + \sqrt{(GM)^2 - a^2}$$

denotes the location of the event horizon. In the Schwarzschild limit $a \to 0$, one recognizes the well-known expression for the Newtonian gravitational acceleration. ($R_S \equiv 2GM$ there denotes the Schwarzschild radius.)

Since within the classical theory, the area A of the event horizon never decreases, this suggests a formal analogy to the Second Law of thermodynamics, where the entropy, S, of a closed system never decreases. This is re-enforced by the analogy of (1) with the First Law of thermodynamics:

$$dE = TdS - pdV + \mu dN ; \tag{3}$$

M, in particular, corresponds to E. If we tentatively identify S with a constant times A, the temperature should be proportional to the surface gravity.

In the classical theory, this correspondence would remain purely formal. Its physical significance is revealed by taking quantum theory into account: black holes radiate with a temperature proportional to $\hbar$, the Hawking temperature [1],

$$T_{BH} = \frac{\hbar c^3}{8\pi k_B GM} \approx 6.17 \times 10^{-8} \left(\frac{M_\odot}{M}\right) \text{ K}. \tag{4}$$

The black-hole entropy is then found from (1) to read

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar} = k_B \frac{A}{4l_P^2} ; \tag{5}$$

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[1] Here and in most of the following expressions we set $c = 1$.

here, l_P denotes the Planck length,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\mathrm{m}. \tag{6}$$

For a Schwarzschild black hole with mass M (to which we shall mostly restrict ourselves), one has

$$S_{BH} \approx 1.07 \times 10^{77} k_B \left(\frac{M}{M_\odot}\right)^2. \tag{7}$$

In conventional units, this reads

$$S_{BH} \approx 1.5 \times 10^{54}\,\frac{\mathrm{J}}{\mathrm{K}}\,\left(\frac{M}{M_\odot}\right)^2. \tag{8}$$

Since the entropy of the Sun is of the order of $10^{57} k_B$, it would experience an increase in 20 orders of magnitude in entropy after collapsing to a black hole.[2] Gravitational collapse thus ensues an enormous increase of entropy.

It is a big challenge for any approach to quantum gravity to provide a microscopic derivation of black-hole entropy. The aim is to identify fundamental quantum gravitational entities which can be counted in Boltzmann's sense to yield the entropy. Both string theory and quantum general relativity have provided partial answers; the fundamental entities can there be D-branes or spin networks [2]. The picture is, however, far from being complete. In fact, one suffers from an embarrassment of riches, as Steven Carlip has called it [5]: there are many, not obviously related, approaches which yield the same result (5). There thus seems to be a universal principle behind all of them, a principle that is still veiled.

A general mechanism which could provide such a universal principle is connected with the notion of entanglement entropy. If one divides Minkowski spacetime into two different regions and considers quantum correlations across these regions, there is a non-vanishing entanglement entropy that is proportional to the *area* that divides these regions [6; 7; 8]. This result has been discussed as a support for the area law (5) in black-hole physics. But what could give the entangled quantum degrees of freedom? Previous work uses quantum fields on a background [6; 7; 8]. A universal result could perhaps be obtained from the quasi-normal modes which are typical for the black hole itself [9; 10]. These quasi-normal modes describe the characteristic perturbations of a black hole before it reaches its final unique stationary state. No entanglement entropy, however, has yet been calculated in this case.

An area law for the entanglement entropy is also found in analogous situations in statistical mechanics, for example in the case of general bosonic harmonic lattice systems [11]. This enforces its universal nature.

As mentioned above, there exist various microscopic derivations of black-hole entropy. In many of these derivations, logarithmic corrections to (5) are found if one goes beyond the leading order of the combinations. These corrections are proportional to $\ln S_{BH}$, but both the sign and the exact coefficient vary. We shall address below these terms in more detail, but turn before to a discussion of analogous terms in statistical mechanics.

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[2] In reality, only stars with masses bigger than about $3M_\odot$ collapse to a black hole.

Fig. 1 A *box* with an ideal gas of particles is divided into two parts with equal volume and particle number

2 Gibbs' paradox and logarithmic corrections

Statistical mechanics provides a microscopic explanation of thermodynamical relations. As has already been emphasized by Josiah Willard Gibbs, one arrives at the Boltzmann entropy only when dividing the number of permutations by $N!$, where N is the number of particles. The Boltzmann entropy coincides with the expression for the entropy found in thermodynamics, which is known to be empirically correct. The particles are thus counted as being indistinguishable, a procedure that receives its justification only from quantum theory. With this "Indistinguishability Postulate" one then gets an entropy which is additive, at least approximately (see below). This problem in counting states is discussed in many places, see, for example, [12] and [13].

To illustrate this situation, we consider a system of N free particles in classical statistical mechanics. The partition sum of this model is given by

$$Z = \int d^{3N}q \, d^{3N}p \, \exp\left(-\frac{H}{k_{\mathrm{B}}T}\right) = V^N (CmT)^{3N/2},\tag{9}$$

where

$$H = \sum_{i=1}^{3N} \frac{p_i^2}{2m}$$

(all masses being equal), and C is a constant which is independent of the nature of the atoms. For the entropy one gets, using standard formulae of statistical mechanics,

$$\begin{aligned}
S &= k_{\mathrm{B}} \ln Z + k_{\mathrm{B}} T \frac{\partial \ln Z}{\partial T} \\
&= k_{\mathrm{B}} N \left(\ln V + \frac{3}{2} \ln(CmT) + \frac{3}{2} \right).
\end{aligned}\tag{10}$$

This expression for the entropy is *not* additive, that is, the entropy does not double if volume and particle number are doubled. It would thus be in conflict with thermodynamics.

Consider now the following consequence of this formula. We have a box filled with an ideal gas of free particles, see Fig. 1.

A partition divides the box into two equal parts, each of which is characterized by volume V and particle number N. If one removes the partition at constant temperature, one gets from (10) the following increase in entropy:

$$\Delta S = 2k_{\mathrm{B}} N \ln 2.\tag{11}$$

On the other hand, in phenomenological thermodynamics one would expect that $\Delta S = 0$ because the situation is reversible: removing and re-inserting the partition is a reversible process, since the state of the gas does not change. This discrepancy

is known as Gibbs' paradox [14]. It is usually remedied by assuming that the particles are indistinguishable and therefore dividing the partition sum Z by $N!$, the number of particle permutations. Instead of (9) one then gets the expression

$$Z = \frac{V^N (CmT)^{3N/2}}{N!}. \tag{12}$$

In order to calculate the new entropy expression, we use Stirling's formula, which is valid for large particle numbers $N \gg 1$,

$$\ln N! = N \ln N - N + \frac{1}{2} \ln N + \frac{1}{2} \ln(2\pi) + \mathcal{O}\left(\frac{1}{N}\right). \tag{13}$$

Instead of (10) we then find the expression for the "Gibbs entropy",

$$\begin{aligned}
S_{\text{Gibbs}} &= S - k_{\text{B}} \ln N! \\
&\approx k_{\text{B}} N \left(\ln \frac{V}{N} + \frac{3}{2} \ln(CmT) + \frac{5}{2} - \frac{\ln N}{2N} - \frac{\ln(2\pi)}{2N} \right).
\end{aligned} \tag{14}$$

(This is sometimes called the "Sackur–Tetrode equation" [13].) Apart from the last two terms (which are very small), this expression for the entropy is now additive. Removing the partition in the box described above, we now get for the change in entropy the result

$$\Delta S \approx \frac{1}{2} k_{\text{B}} \ln N \ll 2 k_{\text{B}} N \ln 2, \tag{15}$$

which, in contrast to (11), is almost zero. Up to a term proportional to $\ln N$, the result of statistical mechanics now coincides with the thermodynamical result $\Delta S = 0$.

The fact that there is not an exact coincidence can easily be understood: the term proportional to $\ln N$ describes fluctuations. If the partition is removed, fluctuations with larger magnitude than in the presence of the partition become possible; thus, a little more states become available. In this sense, the removal of the partition is not quite reversible. As discussed in detail in [14], this situation corresponds to a "microscopic preparation", where N identical particles are initially placed on each side of the partition at the same temperature. If, instead, one makes a "macroscopic preparation" (with knowledge only about the pressure and the temperature, but with no information about the exact value of N), one finds the exact result $\Delta S = 0$ upon removing the partition.

It is important to emphasize in this connection the important difference between identity and indistinguishability [12]. In classical mechanics, different particles are not identical even if they are indistinguishable; in principle, they can be identified and have therefore to be counted separately.[3] In quantum theory, on the other hand, one does not have "particles", but only field modes. If one has, for example, a wave packet with two bumps, the exchange of the bumps describes the same

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[3] To quote from Otto Stern's paper [15]: "The conception of atoms as particles losing their identity cannot be introduced into the classical theory without contradiction. This is possible only on the ground of the non-classical ideas of quantum theory."

state—in this sense both states (before and after the exchange) are identical. It is only this identity that justifies the division of the partition sum by $N!$.

Quite generally, one can write the ensemble entropy related to a reduced density operator as a sum of the averaged physical entropy (which is a definite function of volume, temperature, etc.) plus the entropy of missing information, the latter being usually much smaller than the former [12]. Consider, for example, the case of a grand-canonical ensemble, for which the density operator reads

$$\rho = \frac{1}{Z} \exp\left(-\frac{H - \mu N}{k_\mathrm{B} T}\right), \qquad (16)$$

where μ is the chemical potential. Only the mean particle number is specified here, because the system is assumed to be in contact with a particle reservoir. If this contact is closed, the particle number assumes a definite value, but this value is unknown. The corresponding relative entropy of missing information about this value is of order $\ln N / N$, which just corresponds to the entropy increase in (15).

To conclude this section, we briefly discuss a simple model which can also serve as an analogy for the black-hole case. Consider a set of N spin-1/2 particles out of which n point up and $N - n$ point down:

Since N is assumed to be given, we have a microcanonical ensemble. We define the entropy as the logarithm of the number of configurations with n spins up and $N - n$ spins down,

$$S = \ln \binom{N}{N - n} = \ln \binom{N}{n}. \qquad (17)$$

In a realistic setting, n could correspond to the magnetization as the given macroscopic quantity.

Consider first the "equilibrium case" $n = N/2$. Using (13), one gets from (17), neglecting terms of order $1/N$,

$$S = N \ln 2 - \frac{1}{2} \ln N + \frac{1}{2} \ln \frac{2}{\pi}. \qquad (18)$$

Defining $S_0 \equiv N \ln 2$, we see that the relative contribution of the second term is just of order $\ln N / N$. In contrast to the above, it comes with a minus sign; the reason is that it does not describe missing information because N is fixed from the very beginning (since we have here a microcanonical ensemble). Note that we can approximately write

$$S \approx S_0 - \frac{1}{2} \ln S_0.$$

In the general case (17) we get (assuming both n and N to be large numbers)

$$S = -N(w \ln w + (1 - w) \ln(1 - w)) - \frac{1}{2} \ln(Nw(1 - w)) - \frac{1}{2} \ln(2\pi), \qquad (19)$$

Fig. 2 Comparison of the exact entropy (17) with the approximations with, Eq. (21), and without, Eq. (20), the logarithmic correction term for $N = 20$. The difference between the exact expression and (21) is hardly noticeable

Fig. 3 Symbolic attachment of bits to the surface of a black hole

where $w = n/N$. Defining now

$$S_0 \equiv -N\left(w\ln w + (1-w)\ln(1-w)\right),\qquad(20)$$

we find

$$S = S_0 - \frac{1}{2}\ln S_0 - \frac{1}{2}\ln\left(\frac{2\pi w(1-w)}{\alpha}\right),\qquad(21)$$

where $\alpha = (w-1)\ln(1-w) - w\ln w$.

Figure 2 compares the exact expression for the entropy with S_0 and S according to (19). One easily sees that (19) is an excellent approximation unless n or N are small numbers.

3 Logarithmic corrections to black-hole entropy

The big challenge in understanding black-hole entropy is to provide a microscopic interpretation for it. This is possible only in quantum gravity, a theory which presently does not exist in a complete form. Two major approaches within which the interpretation of the entropy can be tackled are quantum general relativity and string theory [2]. For example, loop quantum gravity, which is a particular case of quantum general relativity, gives a discrete spectrum for an appropriately defined area operator; the area of the event horizon can then only assume discrete values [2; 16]. Before we turn to this case, we demonstrate the essential features in the context of a much simpler model: we assume the presence of an equidistant area spectrum as put forward by Bekenstein and Mukhanov [17]. Such a spectrum can also be found from quantum geomtrodynamics [18]. It is given by

$$A_N = (4\ln k)l_{\mathrm{P}}^2 N,\qquad(22)$$

where k is an integer number > 1. The intuitive picture is that the horizon is divided into small cells with area $(4\ln k)l_{\mathrm{P}}^2$, see Fig. 3.

In each cell there are "spins" which can assume k different values. The simplest case is $k = 2$, so one bit of information can be put on each cell, cf. Wheeler's notion of "it from bit" [19]. Inspecting the spin model discussed at the end of the last section, one recognizes that one can identify the Bekenstein–Hawking entropy (5) with the leading term $S_0 = N\ln 2$ of the equilibrium entropy. This is not possible for the non-equilibrium case (19), which sounds reasonable, since one would expect that the spins are equally distributed on the surface of a big black hole. Taking into account the corrections to S_0 from (18), one arrives at

the following expression for the black-hole entropy in the Bekenstein–Mukhanov model when using (22):

$$S = \frac{A_N}{4l_{\mathrm{P}}^2} - \frac{1}{2}\ln\frac{A_N}{4l_{\mathrm{P}}^2} + \frac{1}{2}\ln\frac{2}{\pi} + \frac{1}{2}\ln(\ln 2). \tag{23}$$

The logarithmic correction term has a negative sign because one has here a micro-canonical ensemble—the value of the area is *fixed*, and one therefore has a slight increase of information from the knowledge of the microstate compared to (5). The situation would be different in a grand-canonical setting in which the area can fluctuate and only the mean value of A is known; then the sign of the logarithmic term would be positive, corresponding to missing information. This difference has been clearly emphasized by Gour [20].

The analogue of the black-hole mass M is in statistical mechanics the energy E; the analogue of the area A is the particle number N. Situations in which A is fixed lead to an increase of information by going beyond the highest order in N, whereas situations in which A fluctuates lead to a decrease of information (of information about the exact value of A).

In this simple model we can also evaluate the exact value for the entropy by making use of (17). Inserting there $N = A_N/4l_{\mathrm{P}}^2\ln 2$ and $n = N/2$, one gets

$$S = \ln\frac{\left(\frac{A_N}{4l_{\mathrm{P}}^2\ln 2}\right)!}{\left[\left(\frac{A_N}{8l_{\mathrm{P}}^2\ln 2}\right)!\right]^2}. \tag{24}$$

It may be of interest to compare this exact expression with the approximate expression (23) in order to see how good the approximation is. Consider, for example, a small black hole with $N = 20$. We then have $A_{20} \approx 55.45\ l_{\mathrm{P}}^2$, and the number of states is

$$\binom{20}{10} = 184756,$$

which corresponds to the exact entropy $S = \ln 184756 \approx 12.127$. The approximate entropy, as found from (23), is $S \approx 12.139$, which is only slightly larger than the exact value. (The dominant term $A/4l_{\mathrm{P}}^2$ is about 13.86.) Thus, although this black hole is rather small, the approximation found by using Stirling's formula is still quite good. This black hole has a radius $R_{\mathrm{S}} \approx 2.1\ l_{\mathrm{P}}$, a mass $M \approx 1.05\ m_{\mathrm{P}}$, where $m_{\mathrm{P}} = \hbar/l_{\mathrm{P}}$ is the Planck mass and, from (4), a temperature $T_{\mathrm{BH}} \approx 4.7 \times 10^{17}$ GeV. Once the black hole approaches the Planck scale, the whole approximation breaks down and one would have to use the full quantum theory of gravity [2].

As we have seen, the relative contribution of the logarithmic correction term is negligible even for relatively small black holes. Even a primordial black hole with $M \approx 10^{-18}M_{\odot}$ (which could have been formed in the early universe) gives a logarithmic correction with relative contribution only of 4.4×10^{-40}.

In the Bekenstein–Mukhanov model, logarithmic contributions to the main contribution to the entropy appear naturally, see (23). Such terms also arise from various approaches to quantum gravity [3]. Let us concentrate on two of them: loop quantum gravity and string theory.

In loop quantum gravity, black-hole entropy follows from counting all possible punctures of a spin network with the horizon. A spin network is characterized by a collection of quantum numbers j and m, where $j \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and $m = -j, \ldots, j$. The combinatorial problem is difficult [21; 22; 23]. The entropy turns out to be proportional to the area only if the exchange of nodes in the spin network produces a *different* state, that is, if the counting is performed without dividing by the corresponding number of permutations. Otherwise the entropy would come out to be proportional to the *square root* of the area instead of the area itself. This would be in conflict with the laws of black-hole mechanics, which correspond to the level of thermodynamics. If the nodes are treated as distinguishable, the proportionality to the area is found. The exact expression (5) can only be recovered if an unknown parameter of loop quantum gravity (the Barbero–Immirzi parameter β) is chosen appropriately. A logarithmic correction term arises if one imposes in addition a "spin projection constraint" of the form $\sum_i m_i = 0$. It turns out to be of the same form as in (23), that is, it comes with a factor $-1/2$. As we have discussed above, the reason for the minus sign is the fact that the area of the horizon is assumed to be fixed in this approach and that the additional constraint therefore can only lead to an increase of information (decrease of entropy), different from the case where the area fluctuates.

In string theory, the situation is different. There, the Bekenstein–Hawking entropy (5) is recovered by counting states of D-branes in a weak-field situation without black holes but duality-related to a situation with black holes, see, for example, [24] and the references therein. Corrections are also found, and they start in many cases with a term proportional to $\ln A$. The signs of these terms vary, which again seems to depend on whether area is fixed or not. In contrast to loop quantum gravity, invariance under permutations is assumed, that is, the fundamental "particles" are assumed to be indistinguishable.

As the analogy with the above examples shows, the terms proportional to $\ln N$ come mainly into play through the application of Stirling's formula beyond the highest order. They are thus not necessarily of "quantum origin", but can arise already from classical statistical mechanics (as can be seen from the fact that they are not proportional to $\hbar$).

In most of the above foundations of black-hole entropy, the black hole is considered to be in a state corresponding to a microcanonical ensemble. The canonical ensemble (black hole in a heat bath) is undefined in an asymptotically flat spacetime because it would yield a negative specific heat and formal energy fluctuations with a negative variance. Therefore, logarithmic corrections cannot be computed in this case. The situation improves if the black hole is put in a box [25] or in anti-de Sitter spacetime [3]. As discussed in detail by Don Page, logarithmic terms can easily show up by going from a microcanonical to a canonical ensemble and vice versa. This shows that "entropies need to be defined carefully before there is any unambiguous meaning to logarithmic corrections" [3].

We conclude this section by presenting some numerical examples for the size of the logarithmic corrections. We assume that we have the relation (in units of k_B)

$$S = S_{BH} - \frac{1}{2} \ln S_{BH} + \cdots. \tag{25}$$

Let us consider, for example, the galactic black hole, which lurks in the centre of our Milky Way and which has a mass $M \approx 3.6 \times 10^6 M_\odot$ [26]. From (7) one gets

$$S_{\mathrm{BH}} \approx 2 \times 10^{67} \, \frac{\mathrm{J}}{\mathrm{K}}, \qquad (26)$$

which is, of course, enormous compared to any laboratory-scale entropy. It is even bigger than the entropy of the cosmic background radiation, which is known to dominate the non-gravitational entropy of the observable part of our Universe [12]. The galactic black hole also possesses angular momentum, which slightly reduces its entropy [in the extremal case, one would have half of the value in (25)]. The logarithmic correction leads to the following tiny decrease in entropy:

$$-\frac{1}{2} \ln S_{\mathrm{BH}} \approx -1.4 \times 10^{-21} \, \frac{\mathrm{J}}{\mathrm{K}}, \qquad (27)$$

which is about 7×10^{-89} of S_{BH}. This would be a negligible number even for laboratory scales!

4 Interpretation and conclusion

We have seen that various conceptual issues that arise in the counting of micro-scopic states for the black hole are fully analogous to ordinary statistical mechan-ics. We have shown, in particular, that logarithmic corrections to the Bekenstein–Hawking area law occur in a natural way. These corrections are not a priori of quantum nature, but have their origin in combinatorial relations such as Stirling's formula. The sign of a logarithmic term can be understood as either related to missing information (if it is positive) or increase of information (if it is negative), depending on whether the horizon area is fixed or not.

The traditional Gibbs paradox has been resolved by assuming that the micro-scopic particles are identical. While in the classical theory this is an ad hoc assump-tion without justification, it can be understood from quantum theory, which does not contain fundamental particles. This shows that one should get rid of classical pictures as much as possible [12].

In the case of black holes, the Bekenstein–Hawking area law (5) plays in a certain sense the role of the entropy expression in thermodynamics; microscopic derivations from quantum gravity are expected to recover it at leading order. It is therefore of interest to see whether a new type of Gibbs paradox may arise. Sur-prisingly, the situation seems to be opposite in loop quantum gravity and in string theory: whereas the former needs fundamental entities that are distinguishable, the latter works with indistinguishable structures in order to recover (5). In analogy with quantum theory one would have expected that the fundamental "particles" are identical, so the situation in loop quantum gravity needs perhaps some further understanding to become intuitive.

Black holes are open systems. They can thus only be understood if their inter-action with other degrees of freedom are consistently taken into account [27]. In quantum theory, this is known to lead to the emergence of classical properties through decoherence [28]. In a similar way, the black hole, which is fundamentally described by quantum theory, should assume classical properties by interacting

with other fields: these could be additional quantum fields or the quantum perturbations (the quasi-normal modes) of the black hole itself. The thermal nature of Hawking radiation can, for example, be understood as arising from decoherence [29]. The quantum entanglement between these other fields and the quantum gravitational states of the black hole could be at the heart of the black-hole entropy. The corresponding calculation should automatically avoid Gibbs' paradox and lead to further insight into the interpretation of the underlying quantum theory of gravity.

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