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Perturbative stability of smooth strings

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Abstract

Polyakov's theory of surfaces embedded in Euclidean space-time with an extrinsic curvature term is stable under small fluctuations only if the number of dimensions is positive and less than twenty-six.

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Polyakov has proposed a theory of surfaces in which the action depends not only upon the area of the surface, but also on the way in which the surface is embedded in space-time, through its extrinsic curvature.[1-5] This theory serves as a model for flux sheets in QCD, particularly if it is possible to reach a phase in which surfaces are not "creased" over large distances, as they are in the Nambu model, but "smooth". This theory had arisen previously in the study of interfaces by Helfrich[6] and others.[7,8]

The Nambu string can be written in a way which exhibits a local conformal symmetry; in the quantum theory, this symmetry is only manifest in twenty-six dimensions.[9] Smooth surfaces do not appear to posssess any such conformal symmetry, so presumably they are consistent theories in a wide range of dimensions. In this work I compute in which dimensions smooth surfaces are stable under small fluctuations.

A better understanding of the role played by the Liouville theory in this model is gained along the way. Förster,[8] Polyakov,[1] and David[2] suggested that the Liouville action arises in the quantization of smooth surfaces. The Liouville term is proportional to $26 - d$, which led David[2] to conjecture that smooth surfaces are unstable in more than twenty-six dimensions. I show this is true, but find that the Liouville term does not show up where expected. Instead of being most important over large distances,[1,2,8] it dominates only at short distances.

In conformal gauge, where the metric $g_{ab} = \rho\,\delta_{ab}$, the action for smooth surfaces is

$$S = \frac{1}{2\alpha}\int d^2z\,\rho\left(\left(\frac{-1}{\rho}\partial^2 x\right)^2 + i\,\frac{\lambda^{ab}}{\rho}\left(\partial_a x\cdot\partial_b x - \rho\,\delta_{ab}\right)\right) + \int d^2z\,\mu\,\rho\; + \; S_{ghost}(\rho). \tag{1}$$

The surface is described by $x = x(z^a)$, where x is a vector in d Euclidean dimensions; the z^a, $a = 1, 2$, are the coordinates of the world sheet.

The first term in eq. (1) is the square of the extrinsic curvature for the surface, with α a dimensionless coupling constant. In the second term, λ^{ab} is a constraint field which fixes the metric to be that intrinsic to the surface. The third is the Nambu term, with a string tension equal to μ. Lastly, the contribution of the Fadeev-Popov ghosts for general coordinate invariance is represented by $S_{ghost}(\rho)$.

Suppose that one sets $\rho = 1$ in eq. (1) and drops the integration over ρ in the functional integral. This truncated theory is a sum not over all surfaces, but only flat ones. (By flat, I mean that the intrinsic curvature vanishes. As shown by the example of a cylinder, there are many surfaces that are curved extrinsically but not intrinsically.) I have studied this model of flat surfaces before,[10] and these calculations are of help here.

The x's appear quadratically in the action, so they can be integrated out to give

$$S_{eff}(\rho, \lambda^{ab}) = \frac{d}{2}\, tr\, log \left(\frac{-1}{\rho}\partial^2 \left(\frac{-1}{\rho}\partial^2 \right) - \frac{i}{\rho}\, \partial_a \lambda^{ab} \partial_b \right)$$

$$+ \int d^2 z \left(\mu\, \rho - \frac{i}{2\alpha}\, \rho\, \lambda^{ab} \delta_{ab} \right) + S_{ghost}(\rho). \tag{2}$$

The integration is with respect to the invariant measure on the world-sheet, $\int d^2 z\, \rho$.

I expand about the simplest possible background — an infinite, flat surface. Thus I assume that $\rho = \rho_0$ and $\lambda^{ab} = -i\, \lambda_0\, \delta^{ab}$, where ρ_0 and λ_0 are constants. To determine these constants it is necessary to calculate S_{eff} under this ansatz. The result is

$$S_{eff} = \rho_0\, A \left(\frac{-1}{\alpha_r}\lambda_0 + \mu + \frac{d}{8\pi}\lambda_0 \left(1 - log\left(\lambda_0\right)\right) \right)\,, \tag{3}$$

where A is the area of the surface. The renormalized coupling is α_r, $\alpha_r^{-1} = \alpha^{-1} -$

$(d/4\pi)\, log(\Lambda) + \ldots$, where Λ is a momentum cutoff. This relationship between α_r and α implies that the theory is asymptotically free.[1-4,7,8]

For the action to be stationary under variations of ρ_0 and λ_0, $\lambda_0 = -8\pi\mu/d = exp(-8\pi/(\alpha_r d))$; the bare string tension, μ, must be negative so that $\lambda_0 > 0$. The value of ρ_0 is not fixed, and can be taken as an arbitrary positive number. At the stationary point, $S_{eff} = 0$, so the renormalized string tension vanishes.

These results are similar to those found in the study of smooth surfaces in a large number of dimensions.[2,3] At large d, the corrections to S_{eff} are of order $1/d$ and so small. I work in an arbitrary number of dimensions, where this represents merely one of many possible solutions to the equations of motion.

What, then, is the value of analyzing stability about this special solution? Because the theory is asymptotically free, perturbation theory is valid at large momenta. But large momenta corresponds to small distances, and at short distances any (regular) surface is essentially flat. Thus I establish a necessary condition for stability.

Define

$$\rho = \rho_0 \left(1 + \sqrt{\frac{8\pi}{|d|}}\, \frac{\tilde{\rho}}{m}\right), \quad \lambda^{ab} = \frac{m}{\rho_0}\left(-im\,\delta^{ab} + \sqrt{\frac{8\pi}{|d|}}\, \tilde{\lambda}^{ab}\right), \tag{4}$$

where $m^2 = \rho_0\, \lambda_0 > 0$. Note that while the stationary point of λ^{ab} is imaginary, the fluctuations in $\tilde{\lambda}^{ab}$ are real.[11] This is required so that integration over the $\tilde{\lambda}^{ab}$ generates the proper delta-function constraint in the functional integral.

Expanding to quadratic order in $\tilde{\rho}$ and $\tilde{\lambda}$,

$$S_{eff}(\rho, \lambda^{ab}) = \frac{1}{2}\int\!\!\int \left(\tilde{\rho}\, \Delta^{-1}(\rho,\rho)\, \tilde{\rho} + 2\,\tilde{\rho}\, \Delta^{-1}(\rho,\lambda)\, \tilde{\lambda} + \tilde{\lambda}\, \Delta^{-1}(\lambda,\lambda)\, \tilde{\lambda}\right) + \ldots\,, \tag{5}$$

$\Delta^{-1}(\rho,\lambda) = \Delta^{-1}(\lambda,\rho)$. The indices on λ^{ab} are often dropped.

I start with the diagonal term for the metric field, $\Delta^{-1}(\rho,\rho)$. For this it is possible to ignore $\tilde{\lambda}$ and write

$$S_{ghost}(\rho) + \frac{d}{2}\, tr\ log\left(\frac{-1}{\rho}\partial^2\right) + \frac{d}{2}\, tr\ log\left(\frac{-1}{\rho}\partial^2 + \frac{m^2}{\rho_0}\right)$$
$$= \frac{1}{2}\iint \tilde{\rho}\,\Delta^{-1}(\rho,\rho)\,\tilde{\rho}\ +\ \ldots\,. \qquad (6)$$

The first two terms on the left hand side are identical to those in Polyakov's formulation of the Nambu model. Their only dependence on ρ is through the conformal anomaly, so they give the usual factor of the Liouville action.[9] The last is an integration over a massive mode.

Let the momentum be p^a: it is convienient to introduce a dimensionless variable P, which is proportional to the magnitude of the momentum, $P = p^2/m^2$, and the unit vector $\hat{p}^a$ along p^a. Then

$$\Delta^{-1}(\rho,\rho) \sim sign(d)\left(\frac{26-d}{6\,d}\,P - 2\,\frac{log(P)}{P} + \ldots\right)\,. \qquad (7)$$

The first term on the right hand side is the Liouville action to $\sim O(\tilde{\rho}^2)$, written in an unfamiliar way. The second is the leading contribution of the massive mode at large momentum. In eq. (7) and henceforth, I write only the dominant terms at large P; corrections are at most $\sim 1/log(P)$ times the terms written.

Fõrster,[8] Polyakov,[1] and David[2] proposed that the Liouville term appears in the quantum theory of smooth surfaces. They argued that since the Nambu term has fewer derivatives than the extrinsic curvature term, and as the Liouville theory arises from the quantization of the Nambu term, that the Liouville term should be most significant about zero momentum.

In contrast, eq. (7) shows that the Liouville term dominates $\Delta^{-1}(\rho,\rho)$ at *large* momentum. Because the contribution of the massive mode is so much smaller than that of the Liouville term at large P, presumably when $d \neq 26$ the Liouville term dominates $\Delta^{-1}(\rho,\rho)$ at large P to *all* orders in perturbation theory. Away from large P, however, the massive mode also contributes, and there is no reason why it cannot be as important as the Liouville term.

For example, consider the theory about $d = \infty$. (The model is unstable in this limit, but it serves to make a point.) For large d, eq. (6) can be used to compute $\Delta^{-1}(\rho,\rho)$ around zero momentum.[2] Though the the Liouville term contributes $-P/6$, as $P \to 0$ the massive mode gives $-1 + P/6 + \ldots$, so in all $\Delta^{-1}(\rho,\rho) \sim -1 + O(P^2)$: the Liouville term does not dominate at small momentum. This result, which supersedes that of ref. (2), is discussed at the end of the paper.

The other terms in Δ^{-1} can be found in a straightforward manner by expanding perturbatively in $\tilde{\rho}$ and $\tilde{\lambda}^{ab}$. At large P the off-diagonal elements are

$$\Delta^{-1}(\rho,\lambda^{ab}) \sim -i\, sign(d) \left(\frac{\delta^{ab}}{2} log(P) + \hat{p}^a \hat{p}^b \right) \,. \tag{8}$$

$\Delta^{-1}(\lambda,\lambda)$ can be read off directly from the results for flat surfaces:

$$\Delta^{-1}(\lambda,\lambda) \sim sign(d) \left(\frac{1}{4P} \left(K^1 + K^2\right) + \frac{1}{4P} log(P)\, K^3 - \frac{1}{2P} K^4 + \frac{2}{P} K^5 \right) \,. \tag{9}$$

The K's are matrices which span the space of two symmetric tensors, and are defined in eq. (4.8) of ref. (10). The momentum dependence of $\Delta^{-1}(\lambda,\lambda)$ follows from eq. (4.15) of ref. (10).

The theory is perturbatively stable if every eigenvalue of Δ^{-1} has a real part that is positive. These four eigenvalues can be determined with a little effort.

One eigenvalue of Δ^{-1} is $\sim sign(d)\, log(P)/(2P)$. For this eigenvalue to be positive so must the number of dimensions. Hence I assume that $d > 0$.

The characteristic equation for the remaining three eigenvalues is

$$y^3 - \left(cP - \frac{log(P)}{P}\right) y^2 + \frac{1}{2} log^2(P)\, y - \frac{1}{4P} log^3(P) = 0, \tag{10}$$

where y represents the eigenvalue, and the constant $c = (26 - d)/(6d)$. In eq. (10) I have dropped any terms which are small at large P. The only exception is that for y^2, since when $d = 26$, $c = 0$, and the term $\sim log(P)/P$ is dominant.

When the number of dimensions is not equal to twenty-six, the solutions to eq. (10) are given by

$$y \sim cP \ , \ \frac{1}{2cP} log^2(P) \ , \ \frac{1}{2P} log(P). \tag{11}$$

When d is less than twenty-six ($c > 0$), all eigenvalues are real and positive and the theory is stable. The theory is unstable when d is greater than twenty-six ($c < 0$), with two positive and two negative eigenvalues.

Twenty-six dimensions is a special case. There is one positive eigenvalue, and a pair of complex congugate eigenvalues:

$$y \sim \pm \frac{i}{\sqrt{2}} log(P) - \frac{3}{4P} log(P) \ \ , \ \frac{1}{2P} log(P) \ . \tag{12}$$

There are corrections to the complex conjugate eigenvalues that are of order one, etc., but these are all purely imaginary. The leading real part of these eigenvalues is given in eq. (12), and as it is negative the theory is not stable in twenty-six dimensions.

There is an easy way to understand these results. At least over short distances, smooth surfaces can be viewed as an amalgam of the Liouville theory (which results from integrating over the metric field) and the model of flat surfaces (as that contains the dynamics of the constraint field). The Liouville theory which arises is proportional to $26 - d$, so smooth surfaces are unstable in more than twenty-six dimensions.[2] On the other hand, flat surfaces are stable only if the number of dimensions is positive,[12] so this should hold for smooth surfaces as well. In twenty-six dimensions the Liouville theory vanishes, and detailed calculation is needed to show instability.

That smooth surfaces are only stable for $0 < d < 26$ is in one sense rather remarkable. For smooth surfaces, x is a vector, and so it is natural to expect that it should be possible to develop some sort of consistent solution at large d.[2,3] This would allow the theory to be studied not just at large but over all momenta. For the Liouville theory of the Nambu string, the correct large d limit is about *minus* infinity.[13] For flat surfaces, the correct large d limit is about *plus* infinity.[12] If Nature were kind, she would allow smooth surfaces to have a stable solution about either plus or minus infinity; instead, neither is.

The nature of the unstable solution about infinite d can be studied by rotating the contours of integration for $\tilde{\rho}$ and $\tilde{\lambda}$. As mentioned following eq. (7), in this limit $\Delta^{-1}(\rho,\rho)$ does not vanish about zero momentum. Further calculation shows that this is (usually) true for all components of the inverse propagator Δ^{-1}, as well as for those of the propagator Δ computed from Δ^{-1}.

This is unlike the Nambu model. There, the interactions of the metric field ρ are given by the Liouville theory.[9] In perturbation theory, which can be used for $d = -\infty$, the two-point function of ρ has logarithmic correlations over large distances.[13]

Let me make the dangerous assumption that qualitative properties of smooth surfaces for $0 < d < 26$ can be gleaned from the solution at infinite d. It is certainly natural to expect that correlations between the λ^{ab}'s are exponentially damped over large distances.[1-3,7,8] The solution at $d = \infty$ indicates that for smooth surfaces, interactions due to the extrinsic curvature term mix up the ρ and λ^{ab} fields together, so that in the end, neither have long-ranged correlations.

Not too much should be made of these differences. What is most important is the two-point function of the x's, since this measures the mean square size of the surface. For smooth surfaces about $d = +\infty$, generally this size increases logarithmically with the area of the surface, as it does for Nambu strings around $d = -\infty$.[13]

Nevertheless, that the model is more complicated in the infrared than first thought might indicate that Polyakov's goal — of reaching a phase in which surfaces are smooth instead of creased over large distances[1] — could be easier than first thought.

The solution about $d = +\infty$ will be presented in a seperate publication, along with details of the present study.

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[11] Since the absolute value of d enters in eq. (4), this also holds for negative values of d. Of course, the number of transverse degrees of freedom of a surface is $d-2$, so the model is only physical for $d > 2$. The model is defined formally when $d < 2$ by the Lagrangian of eq. (2). While $d < 2$ is unphysical, it is comforting to find that the the lower bound for stability in d, $d = 0^+$, is a safe distance from the values of interest, $d = 3$ and 4.

[12] Like smooth surfaces, the model of flat surfaces is asymptotically free. Following the methods of this work, flat surfaces can be shown to be stable at large momenta for any $d > 0$. Using the solution of ref. (10), they can also be shown to be stable over all momenta about $d = +\infty$.

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