MICRO-LOCAL CALCULUS

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Introduction

The methods of micro-local calculus are to characterize a function by the system of differential equations which it satisfies and to analyze the micro-local structure of the system. We are mostly concerned with holonomic systems, because they are the systems whose solutions form a vector space of finite dimension. Almost all functions which we encounter in Physics or Mathematics are solutions of holonomic systems.

In this paper an outline of the theory of holonomic systems is given and the micro-local properties of Feynman integrals are discussed as its application.

In section 1 we explain some basic notions concerning holonomic systems such as holonomy diagram, geometric and analytic interaction, order and principal symbol.*) In section 2 we give explicit forms of functions determined by holonomic systems. Then in the last section we apply our theory to the Feynman integrals.

1. *Holonomic systems and holonomy diagram.*

First we review a micro-local calculus of holonomic systems.

Consider a system of differential (or micro-differential) equations on a manifold X with an unknown function $u(x)$;

$$\mathcal{M} : P_1(x, D)u(x) = \cdots = P_N(x, D)u(x) = 0$$

$P_j(x, D)$ $(j = 1,\cdots,N)$ are differential (resp. micro-differential) operators. Denoting by $\mathcal{D}$ (resp. $\mathcal{E}$) the sheaf of differential (resp. micro-differential) operators, we write by $\mathcal{J}$ the ideal of $\mathcal{D}$ (resp. $\mathcal{E}$) generated by $P_1(x, D),\cdots, P_N(x, D)$, so that $\mathcal{D}u = \mathcal{D}/\mathcal{J}$ (resp. $\mathcal{E}u = \mathcal{E}/\mathcal{J}$).

*) As for details of holonomic systems, see S-K-K [1], [4], [5], where we have used "maximally overdetermined system" and "Lagrangean manifold" in place of "holonomic system" and "holonomic manifold", respectively.

We analyze its structure micro-locally, that is, locally on the cotangent bundle T^*X of X.

The *symbol ideal* J is the ideal of the sheaf $\mathcal{O}_{T^*X}$ of holomorphic functions on the cotangent vector bundle T^*X of X, which is generated by principal symbols of operators in $\mathcal{J}$. The common zeros Λ of all functions in the symbol ideal J is said to be the *characteristic variety* of the system $\mathcal{M}$. It is known that the codimension of the characteristic variety does not exceed the dimension of X, say n. A *holonomic system* is, by definition, a system whose characteristic variety has codimension n. In this case, its characteristic variety Λ is holonomic, that is, the fundamental 1-form $\omega = \Sigma\xi_j dx_j$ of T^*X vanishes on it. Let $\Lambda = \cup\Lambda_j$ be a decomposition of Λ into irreducible components. Λ_j is called a holonomic component. Λ_j is the conormal bundle of its image Y_j by the projection to X.

If the symbol ideal J is reduced, in other words, if J coincides with the ideal of functions which vanish on Λ, then we say $\mathcal{M}$ is *very good*. If J is reduced at generic points of Λ_j's, then $\mathcal{M}$ is said to be *simple*.

When there exists an irreducible analytic set S of dimension $(n-1)$ contained in Λ_j and Λ_k $(j \neq k)$, we say Λ_j and Λ_k have a *geometric interaction* at S. If, moreover, $\mathcal{M}$ is not a direct sum of two $\mathcal{E}$-Modules whose characteristic varieties are Λ_j and Λ_k respectively in a neighborhood of a generic point of S, then we say that Λ_j and Λ_k have an *analytic interaction*.

A geometric interaction is important because no other interaction between irreducible components occurs by virtue of the following fact; if two holonomic systems $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic outside of $(n+2)$ co-dimensional analytic set, then they are globally isomorphic. By these data, we write a *holonomy diagram*, which consists of dots and segments joining them. Dots represent the holonomic components Λ_j of characteristic variety and segments represent the analytic interactions between the holonomic components.

We explain the order of a generator of a simple holonomic system on each irreducible component Λ_j. There is a micro-differential operator $P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \cdots$ of order m belonging to $\mathcal{J}$ such that $dP_m(x, \xi) \equiv \varphi\,\omega \mod J$ for some function φ on Λ_j. Here

$\omega = \Sigma \xi_j dx_j$ is the fundamental 1-form. It is shown that $\varphi^{-1}[P_{m-1} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}] - \frac{m-1}{2}$ is a constant function on Λ_j independent of the choice of P and called the *order* of u on Λ_j. The holonomy diagram with orders on each component gives us a fundamental information on the holonomic system.

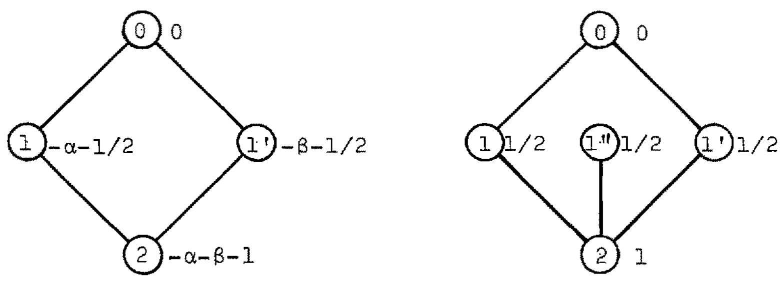
A microfunction solution $u(x)$ of a given holonomic system on each Λ_j can be written $u(x) = P(x, D)\delta(x_1, \cdots, x_r)$ by a suitable elliptic micro-differential operator P, if we take a local coordinate system $(x_1, \cdots, x_n)$ so that Λ_j is a conormal bundle of a submanifold defined by $x_1 = \cdots = x_r = 0$. $(2\pi)^{-r/2}\sigma(P)|_{\Lambda_j}(d\xi_1 \cdots d\xi_r dx_{r+1} \cdots dx_n / dx_1 \cdots dx_n)^{1/2}$ is said to be the *principal symbol* of $u(x)$. Here, $dx_1 \cdots dx_n$ and $d\xi_1 \cdots d\xi_r dx_{r+1} \cdots dx_n$ are regarded as volume elements on X and Λ_j, respectively. It was shown that this is invariant by a coordinate transformation. The order of a generator u coincides with the homogeneous degree of the principal symbol with respect to the fiber coordinate ξ. The principal symbol $\sigma(u)$ satisfies the following system of differential equations of order 1 on Λ_j so that it can be characterized as its solution up to constant multiple: $(H_{P_m} + (P_{m-1} - \frac{1}{2} \Sigma \frac{\partial^2 P_m}{\partial x_i \partial \xi_i}))[\sigma(u)(dx_1 \cdots dx_n)^{1/2}] = 0$ for any $P = P_m(x, D) + P_{m-1}(x, D) + \cdots$ in $\mathcal{J}$. Here, $H_{P_m} = \Sigma(\frac{\partial P_m}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial P_m}{\partial x_i} \frac{\partial}{\partial \xi_i})$.

Microfunction solutions of a simple holonomic system at a generic point of holonomic components form a vector space of one dimension, and the principal symbol characterizes them.

At a generic point of Λ_j, a simple holonomic system is determined by its order. That is, if two systems of equations $\mathcal{M}_\nu = \mathcal{E} u_\nu$ $(\nu=1,2)$ with the same order are given, then there exists locally an elliptic operator $Q(x, D)$ of *order* 0 such that $u_1 = Qu_2$ gives an isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$.

If Λ_j and Λ_k intersect regularly at an (n-1)-dimensional analytic set S (that is, in a neighborhood of generic point of S, Λ_j and Λ_k are non singular and the tangent space of S is the intersection of those of Λ_j and Λ_k) and $\mathcal{M}$ is very good, we call it the regular interaction. Then, $\mathfrak{M}$ is determined by the orders e_j, e_k on Λ_j and Λ_k. If $e_j - e_k - \frac{1}{2} \neq 0, 1, 2, \cdots$, then the support of any non zero microfunction solution is not contained in Λ_j. That is, the solution propagates from Λ_k to Λ_j. On the contrary, if two systems $\mathfrak{M}_\nu = \mathcal{E}/\mathcal{J}_\nu$ ($\nu = 1,2$) are given with the same orders e_j and e_k, $\mathcal{J}_1 = \mathcal{J}_2$ on Λ_{j_1} implies $\mathcal{J}_1 = \mathcal{J}_2$ on $\Lambda_j \cup \Lambda_k$, under the same condition $e_j - e_k - \frac{1}{2} \neq 0, 1, \cdots$. That is, the system itself propagates from Λ_j to Λ_k. Therefore, if the difference is not a half integer, they propagate in both directions. In the real domain, S divides Λ_j (resp. Λ_k) into two components. Microfunction solutions at a generic point of S form a vector space of two dimentions, and two relations among the four principal symbols of a microfunction solution can be written down. This leads us to the global determination of principal symbols of microfunction solutions by means of analytic interactions. (See [5].)

Example



(a) (b)

The number enclosed with a circle signifies the codimension of the image of holonomic components to the base manifold. The number beside a circle is the order on the corresponding holonomic manifold. Example (a) : $X = \mathbb{C}^2$, $(xD_x - \alpha)u = (yD_y - \beta)u = 0$, ⓪, ①, ①', ② are the conormal bundles of X, $x = 0$, $y = 0$, $x = y = 0$, resp. Example (b) : $X = \mathbb{C}^2$, $(D_x - D_y)xy(x+y)u = (D_x - D_y)D_x xu = (D_x - D_y)D_y yu = (D_x x + D_y y)u = 0$, ⓪, ①, ①', ①", ② are the conormal bundles of X, $x = 0$, $y = 0$, $x+y = 0$, $x = y = 0$, resp. ; In this case, ⓪ and ①" have a geometric interaction but no analytic one.

2. *Explicit forms of functions with simple singular spectrum.*

When a function $u(x)$ satisfies a simple holonomic system and the characteristic variety is not complicated, we can give an explicit form of functions. (In this section, the conormal of the total manifold is omitted.)

a) In the case the characteristic variety is the conormal bundle of a hypersurface $f(x) = 0$, $u(x)$ can be written as $u(x) = \varphi(x)f(x)^{\alpha} + \psi(x)$ where $\varphi(x)$ and $\psi(x)$ are functions analytic near the hypersurface. The order is $-\alpha-\frac{1}{2}$. When α is a non negative integer, $f(x)^{\alpha}$ must be replaced by $f(x)^{\alpha}\log f(x)$. If $u(x)$ is considered as hyperfunction, $f(x)^{\alpha}$ is interpreted as $(f(x)\pm i0)^{\alpha}$.

b) In the case the characteristic variety is the union of the conormal bundle of hypersurface $f(x) = 0$ and the conormal bundle of $f(x) = g(x) = 0$, $u(x)$ can be written as $\varphi(x)g(x)^{\alpha}\delta^{(m)}(f(x)) + \psi(x)$.

c) In the case the characteristic variety is the union of conormal bundles of two hypersurfaces $f(x) = 0$ and $g(x) = 0$ which have a contact of order 2, by modifying $f(x)$ and $g(x)$, we can set $g(x) = f(x) - h(x)^2$ with some function $h(x)$. Thus, we can take a local coordinate system $(x_1,\cdots,x_n)$ such that $f(x) = x_1$ and $h(x) = x_2$. Then, $u(x)$ satisfies micro-locally the following system of equations; $(x_1D_1 + \frac{1}{2}x_2D_2 - \alpha)v = 0$, $((D_2 + 2x_2D_1)D_2 - 4\beta D_1)v = 0$, $D_3v = \cdots = D_nv = 0$ by setting $u(x) = R(x, D)v(x)$ for a micro-differential operator R. If we denote by e_0, e_1, the orders of $u(x)$ at the conormal bundles of $f(x)=0$ and $g(x)=0$, we can say $u(x)$ is of the following form $u(x) = \varphi(x)v(x) + \psi(x)D_2D_1^{-1}v(x) + \phi(x)$ for some analytic functions $\varphi(x)$, $\psi(x)$ and $\phi(x)$. Solving the above equation, $v(x)$ and $D_2D_1^{-1}v(x)$ is given by

$$v(x) = f(x)^{-\frac{e_0+1}{2}}g(x)^{-\frac{e_1}{2}-\frac{1}{4}}Q_{e_0}^{e_1+1/2}(h(x)/\sqrt{f(x)})$$

and

$$D_2D_1^{-1}v(x) = f(x)^{-\frac{e_0}{2}}g(x)^{-\frac{e_1}{2}-\frac{1}{4}}Q_{e_0-1}^{e_1+1/2}(h(x)/\sqrt{f(x)}).$$

Here, Q is the associated Legendre function. Especially, if $e_0 = 0$ and $e_1 = -\frac{1}{2}$,

$$u(x) = \varphi(x)f(x)^{-1/2}\log\frac{h(x) - f(x)^{1/2}}{h(x) + f(x)^{1/2}} + \psi(x)\log g(x) + \phi(x).$$

In the case $e_0 = -1$ and $e_1 = -\frac{1}{2}$,

$$u(x) = \varphi(x)f(x)^{1/2}\log\frac{h(x) - f(x)^{1/2}}{h(x) + f(x)^{1/2}} + \psi(x)f(x)^{1/2} + \phi(x).$$

The preceding discussion on holonomic system is applied to a microlocal study of Feynman integrals.

3. *Feynman Integrals.*

Consider a Feynman diagram D which consists of n external lines, n' internal lines joining n'' vertices (parametrized by r, ℓ and j, resp.). Internal lines have directions. $j^{\pm}(\ell)$ signifies the end and the source of an internal line r. The incidence number $[j, \ell] = 1, -1, 0$ according to whether $j = j^{+}(\ell)$, $j^{-}(\ell)$ or else. The set of $(p_1, \cdots, p_n; u_1, \cdots, u_n)$ in the cotangent bundle $T^*(R^{\nu})^n$ of momentum space satisfying the following *Landau equation* is called *positive Landau holonomic manifold* and denoted by Λ_D^+; The relations $\sum_{r\to j} p_r + \sum_{\ell}[j : \ell]k_\ell = 0$ and $u_r = v_j$, where $j = 1, \cdots n''$ and r runs external lines ending at j, hold for some ν-vectors k_ℓ, v_j and scalars α_ℓ ($\ell = 1, \cdots, n'$; $j = 1, \cdots, n''$) which satisfy

$$k_\ell^2 = m_\ell^2, \quad v_{j^+(\ell)} - v_{j^-(\ell)} = \alpha_\ell k_\ell, \qquad \alpha_\ell > 0.$$

A set of $(p_1, \cdots, p_n ; u_1, \cdots, u_n)$ which satisfies the same equation replacing the last positivity condition $\alpha_\ell > 0$ with $\alpha_\ell \neq 0$ is denoted by Λ_D and called *Landau holonomic manifold* for D. The projection of Landau holonomic manifold on the momentum space $(R^{\nu})^n$ is called *Landau manifold* and denoted by L_D.

The Feynman integral $F_D(p_1, \cdots, p_n)$ is given by

$$\int \left(\prod_\ell (m^2 - k_\ell^2 - i0)\right)^{-1} \prod_j \delta^{\nu}\left(\sum_{r\to j} p_r + \sum_\ell [j : \ell]k_\ell\right) \prod_\ell d^{\nu}k_\ell .$$

This integral satisfies a holonomic system. It is shown that the singular spectrum of a Feynman integral $F_D(p_1, \cdots, p_n)$ is the union of the positive Landau holonomic manifolds of graphs obtained by contractions of internal lines of D.

For a diagram D, set $cd(D) = \#$ internal lines $+\nu(\#$ vertices $- \#$ internal lines).

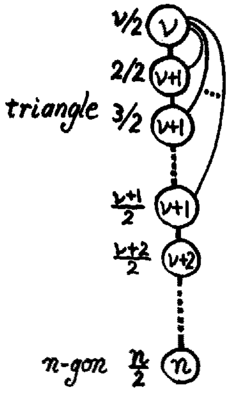
Theorem. Let D' be a diagram obtained by contracting internal lines of D. If, for (p, u) in $\Lambda^{+}_{D'}$, we can solve uniquely (possibly finite) k_ℓ and α_ℓ from the Landau equation for D'. Then, the order of F_D on $\Lambda_{D'}$ is a half of cd(D'). Especially, if all vertices are external, the above condition holds and this formula is true.

This theorem is obtained by the formula concerning the order of an integral : the order of an integral is the order of the integrand minus a half of the number of the integration variables.

Theorem. If D' is a diagram obtained by contractions of internal lines of D and D" is obtained by contraction of *one* internal line, say ℓ_0, from D', and if the Landau equation for D' in which $k^2_{\ell_0} = m^2_{\ell_0}$ is replaced by $\alpha_{\ell_0}(k^2_{\ell_0} - m^2_{\ell_0}) = 0$ gives α_ℓ and k_ℓ from $(p_1, \cdots, p_n, u_1, \cdots, u_n)$, then $\Lambda_{D'}$ and $\Lambda_{D''}$ have a regular interaction. The above condition is again satisfied when all vertices are external.

In this case, the difference of the orders on $\Lambda_{D'}$ and $\Lambda_{D''}$ is one half. Therefore, there exists a non trivial microfunction solution whose support is contained in $\overline{\Lambda}_{D'}$, but no solutions with support in $\overline{\Lambda}_{D''}$ in a neighborhood of $\overline{\Lambda}_{D'} \cap \overline{\Lambda}_{D''}$. That is, any solution on $\overline{\Lambda}_{D'}$ can be uniquely continued to some solution on $\overline{\Lambda}_{D''}$.

In the sequel, we assume all vertices are external. We say D is a *large diagram* if $cd(D') \geq (\nu+1)b_0$ for any subgraph D' of D. Here b_0 is the number of connected components of D'. The codimension (in $R^{\nu n}$) of the Landau manifold L_D of a large diagram D coincides with cd(D). For example, the holonomy diagram for the Feynman integral of one loop with n-vertices is as follows.



Each circle denotes the Landau holonomic manifold, and the number beside a circle indicates the order. Just one route from the full diagram is shown here. The full diagram contains many routes corresponding to the way of contracting internal lines successively. In the neighborhood of the intersections between Landau surfaces, the situation in 2 c) occurs.

On the Landau holonomic manifold of a large diagram D' obtained by contraction of D, the Feynman integral F_D is micro-locally equal to a δ-function on the Landau manifold $L_{D'}$, multiplied by an analytic function. This amplitude can be obtained by solving only an algebraic equation. In fact, the amplitude of the Feynman integral is determined by its principal symbol, and the principal symbol of F_D on $\Lambda_{D'}$ is given by :

$$\frac{\text{const.}}{\prod_{\ell''}(m_{\ell''}^2-k_{\ell''}^2)}\sqrt{\frac{\prod_{\ell'} d\alpha_{\ell'} \prod_j d^\nu p_j \prod_{j'} d^\nu v_{j'} \prod_{\ell'} d^\nu k_{\ell'} \prod_{\ell''} d^\nu k_{\ell''}}{\prod_{\ell'} d(m_{\ell'}^2-k_{\ell'}^2)\prod_j d^\nu(p_j+\Sigma[j:\ell]k_\ell)\prod_{\ell'} d^\nu(\alpha_{\ell'} k_{\ell'}-\Sigma[j:\ell]v_j)}}$$

where ℓ', j' are internal lines and vertices on D' and ℓ'' are contracted internal lines.

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