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Publications

This thesis is based on the following publications:

[1] M. Baggio, J. de Boer and K. Holsheimer
Hamilton-Jacobi Renormalization for Lifshitz Spacetime
JHEP **1201**, 058 (2012), arXiv:1107.5562 [hep-th].

[2] M. Baggio, J. de Boer and K. Holsheimer
Anomalous Breaking of Anisotropic Scaling Symmetry in the Quantum Lifshitz Model
JHEP **1207**, 099 (2012), arXiv:1112.6416 [hep-th].

[3] M. Baggio, J. de Boer and K. Papadodimas
A non-renormalization theorem for chiral primary 3-point functions
JHEP **1207**, 137 (2012), arXiv:1203.1036 [hep-th].

[4] M. Baggio, J. de Boer, J. I. Jottar and D. R. Mayerson
Conformal Symmetry for Black Holes in Four Dimensions and Irrelevant Deformations
JHEP **1304**, 084 (2013), arXiv:1210.7695 [hep-th].

Contents

Chapter 1

Introduction

The framework of quantum field theory has proven to be extremely useful in describing a large number of diverse phenomena, ranging from elementary interactions in high-energy physics to collective excitations of condensed matter systems. It is also the underlying theme of this thesis, whose aim is to study various aspects of quantum field theories and their applications to physical systems. In the first part of the thesis we will take some small steps towards the development of a holographic framework to describe strongly coupled non-relativistic field theories. As we will see, such theories are thought to be relevant for various setups appearing in condensed matter and statistical physics, such as cold atoms at unitarity and smectic liquid crystals. We will then move on to study non-renormalization theorems in quantum field theories that enjoy supersymmetry. Such theorems are concerned with the coupling constant dependence of certain field theory observables, and are potentially relevant for phenomenology. Moreover, they can be used to compute physical quantities in the strongly coupled regime from the knowledge of the weakly coupled results, so they provide ways to test various dualities between strongly coupled and weakly coupled theories. Lastly, we will study some recent proposals relating black hole entropy to two dimensional field theories, which may shed some light on the microscopic description of certain asymptotically flat black holes. The relevance of such a description comes from the fact that correctly reproducing the entropy formula from a sum over microstates provides a stringent consistency check on theories of quantum gravity such as string theory.

The ideas that we develop in this thesis can at first appear quite heterogeneous; it is the purpose of this introduction to show that in fact they all fit together into a coherent picture. For this, we need to describe the basic ingredients that will be used throughout the thesis, in particular the idea of *deformation* of a quantum field theory and the concept of *holography*.

Quantum field theories are typically very difficult to solve exactly, and several perturbative methods have been developed over the years. The underlying idea is to consider first a related theory that can be solved exactly, such as a theory with no interactions, or a theory with a lot of symmetry. If the deviation of such a theory from the original one we wish to study is sufficiently "small", the observables of the latter can be recovered as a power series in some small parameter, where the leading term is given by the answer of the exactly solvable model.

A large class of quantum field theories, including the Standard Model, can be analyzed (at least in some energy regime, as we will discuss in the next section) by treating the interactions as perturbations of a free theory. In this case the classical Lagrangian $\mathcal{L}$ is split in two parts

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \tag{1.1}$$

where $\mathcal{L}_{\text{free}}$ is the free (quadratic) part of the Lagrangian and $\mathcal{L}_{\text{int}}$ contains the interaction terms. Such terms typically involve the product of various fundamental fields at the same point. For example, the celebrated interaction term of quantum electrodynamics reads

$$\mathcal{L}_{\text{int}} = e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu}, \tag{1.2}$$

where e is the electric charge, which determines the strength of the electromagnetic interactions and therefore plays the role of a coupling constant,[1] while ψ and A_μ represent the electron and photon fields respectively.[2] When $e = 0$, the electrons do not interact with the electromagnetic field. Solving the theory in this case is quite easy: electrons and photons freely travel through space without seeing each other. In order to compute the corrections to the free result as a power-series in the coupling constant, we can use so-called Feynman diagrams: a physical observable $\mathcal{O}$ can be expanded as

$$\mathcal{O} = o_0 + o_1 e^2 + \sum_{n=2}^{\infty} o_n (e^2)^n, \tag{1.3}$$

and the o_n's can be determined by computing a finite number of Feynman diagrams, which amounts to performing a finite number of integrals. However, both the number and the complexity of these diagrams rapidly increases with the order. Luckily enough, the physical electric charge of the electrons in our universe is small

[1] To be more precise, the dimensionless coupling constant of QED is the fine-structure constant, which is related to the electric charge by $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$.

[2] In the following, we will sometimes refer to such terms as operators, since they do become operators acting on the Hilbert space of the theory upon quantization.

enough so that higher-order corrections give increasingly smaller contributions to the final result; keeping only few terms in the series above leads to extremely accurate predictions. For example, the measurement of the dimensionless magnetic moment (or g-factor) can be compared to the QED predictions obtained by truncating the series to fourth order, and an agreement to within ten parts in a billion has been found.[3]

More generally, $\mathcal{L}_{\text{int}}$ will contain a number of interaction terms whose strength is determined by "coupling constants", which we will collectively denote by g, and observables in the interacting theory can in principle be computed as power-series in g. From a physical point of view, we can think of these terms as deformations of the exactly solvable starting point, which in our case is a free theory. This will in fact be the main theme of this thesis: in a nutshell, we will tackle a variety of physical problems by mapping them into deformations of quantum field theories, and we will try to extract meaningful results by employing perturbative techniques.

In the rest of this brief introduction we broadly describe the main common ingredients that will be employed in the rest of the thesis. We first introduce the renormalization group (RG), which allows us to classify deformations in terms of their behavior under change of the energy scale of the process under consideration. Particular attention will be given to the RG fixed points, which describe scale invariant theories. Then we move on to the description of the concept of holography, which is a well-established framework to describe strongly coupled field theories on the one hand, and quantum theories of gravity on the other. Finally, we give an outline of the thesis and review the main original results that we derive in the subsequent chapters, and see how they all fit into the general framework described in this introduction. More specific details and background material will be given at the beginning of each chapter.

1.1 Relevant, marginal and irrelevant deformations

Even when the coupling constants are small, the calculation of Feynman diagrams involves loop integrals over all the possible momenta of intermediate virtual states. These integrals are typically divergent, so in order to extract physically meaningful results we need to find a way to regulate them. One very powerful idea that has emerged to cure these divergences is the *renormalization group*, which is based on

[3]However, the power-series of QED is believed to be divergent, and can be at best considered an asymptotic series. The truncated answer will give increasingly accurate results only up to the order $n \sim 137$, and then it will start to diverge. We will discuss this in more detail in chapter 2.

the observation that quantum field theories should not be regarded as fundamental theories valid at all energy scales, rather they come with an intrinsic energy cutoff. This cutoff can be thought of as the (inverse) lattice spacing in statistical physics or the energy scale at which new particle interactions become important in high-energy physics. In any case, it is possible to isolate the high-energy (ultraviolet) degrees of freedom and describe their effects on low-energy observables through *effective* interactions between the low-energy degrees of freedom.

The most important outcome of the renormalization group analysis is that the coupling constants appearing in the effective Lagrangian should be thought of as being dependent on the energy scale of the process under consideration. It is indeed possible to derive differential equations controlling the energy scale dependence of the coupling constants and correlation functions of the theory, known as Callan–Symanzik equations. In this sense, different Lagrangians do not just describe different physical systems, but sometimes the same system at different energy scales. As a consequence, the renormalization group produces a flow across different effective descriptions as we change the typical energy scale of the processes under consideration.

In general, the renormalization group tends to produce all the possible interaction terms compatible with the symmetries of the problem, even when these terms do not appear in the Lagrangian of the fundamental microscopic theory. Typically, there are infinitely many such terms, so the problem of computing observable quantities would seem to be intractable at first sight, since we would need to take into account arbitrarily complicated interactions between the fields of the effective theory, and we would need an infinite number of experiments to fix all the coupling constants. However, the contribution of most of these interaction terms can be shown to be under control whenever the energy scale of the process under consideration is much smaller than the cutoff scale.

In fact, the possible interaction terms can be divided in two main classes: those whose coupling grows as the energy is decreased are called *relevant*, while those whose coupling decreases are called *irrelevant*. A typical example of a relevant coupling is the mass term for a scalar field in four dimensions, while an important example of an irrelevant coupling is given by the four-fermion interaction of Fermi's theory of weak interactions. In the case where the coupling remains constant under the renormalization group flow, the corresponding operators are called *marginal*. Such operators are typically very difficult to come by, because the condition that the coupling remains constant as a function of the energy scale requires highly non-trivial cancellations of various quantum effects. It is in fact common for operators that look marginal at the classical level to become either *marginally relevant* (such as the interaction term in QCD) or *marginally irrele-*

vant when quantum corrections are included. As we will see, theories that enjoy supersymmetry do have *exactly marginal* operators.

It turns out that in most cases the number of relevant and marginal operators compatible with the symmetries of the problem is finite. From the point of view of the low-energy effective theory, this means that it is sufficient to measure a finite number of parameters (such as the electric charge at zero momentum) in order to extract meaningful predictions from the theory. We refer to such theories as *renormalizable*, and the renormalization group explains why they are so ubiquitous in the description of physical systems at low energy. Irrelevant couplings give rise to corrections that are suppressed by positive powers of E/M_{cutoff}, where E is the energy scale of the process considered, while M_{cutoff} is the mass scale appearing in the irrelevant coupling constant. One very important example is general relativity, where gravitational interactions around flat space are described by an irrelevant operator. In this case, the cutoff scale is the Planck mass $M_P \approx 1.22 \times 10^{19}\, GeV$, which is larger than any energy scale that can be directly probed by present day experiments.

1.1.1 Conformal field theories and their deformations

In this thesis, quantum field theories with conformal symmetry will play a prominent role. These theories, on top of the usual symmetries such as Lorentz symmetry (or some non-relativistic counterpart), are also invariant under rescalings of the coordinates.[4] One of the consequences is that these theories do not change under the renormalization group flow, in the sense that the various coupling constants present in the theory do not depend on the energy scale.

The world around us seems very far from being scale invariant, and in fact many of the things we measure have a characteristic size. Scaling symmetry might thus appear very peculiar and unphysical. However, it turns out that scale invariant phenomena are actually quite common. In fact, many macroscopic systems tend to exhibit "critical" behavior, in the sense that certain observable quantities are described by probability distributions that do not have an intrinsic scale. Quantities as diverse as citations in scientific papers and sizes of earthquakes are in fact described by power-law distributions, which look the same at any scale. Scale invariance is also extremely important in statistical physics, where it is at the heart

[4] It is believed that relativistic theories with scale invariance are also invariant under the bigger conformal group, which includes special conformal transformations. This has been proven to be the case in two dimensions, and some progress has been recently made in four dimensions. For this reason and for the sake of simplicity, in this introduction we decided to use the term "conformal" rather loosely, and it will refer both to theories with only scaling symmetry and to proper conformal field theories.

of the effective description of physical systems undergoing a phase transition: at the transition points, also known as critical points, the correlation functions of the system exhibit power-law behavior, signaling scale invariant physics.

Conformal field theories provide a very powerful framework to account for various properties of these critical points in the continuum limit (that is when we consider length scales that are much bigger than the typical lattice spacing). They are also typically found at the endpoints (UV or IR) of the renormalization group flow. Furthermore, two-dimensional conformal field theories play a fundamental role in perturbative string theory, where they describe excitations propagating on the string. More surprisingly, recent developments have shown that they also play a prominent role in quantum gravity, via various holographic dualities that will be described in the next section.

In this thesis, conformal field theories will provide the starting point for many of the perturbative analyses that we will perform. In fact, while these theories are not necessarily free, their large symmetry group make them relatively simple to study compared to non-conformal theories. The applications that we will consider are however very diverse, and range from RG flows in theories with non-relativistic scaling symmetry to black hole physics. We refer to the last section of this introduction for a more detailed explanation of the role of conformal field theories in the context of this work.

1.2 The AdS/CFT duality

The Bekensten–Hawking formula for the entropy of black holes has led to the proposal [5, 6] that the degrees of freedom of gravitational theories can be encoded "holographically" in a theory with one dimension less. The gauge/string dualities are a concrete realization of this idea, where certain quantum theories of gravity in $d+1$ dimensions are claimed to be equivalent (or dual) to ordinary quantum field theories (typically gauge theories) in d dimensions. The power of the duality lies in the fact that it exchanges the weakly coupled regime of one description with the strongly coupled regime of the other, providing therefore a powerful tool to study strong-coupling problems both in field theory and quantum gravity. They have in fact advanced our understanding of various classic problems in theoretical physics, such as confinement, transport properties of strongly coupled QFTs and black hole physics.

The best understood example is probably the duality between type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N}=4$ supersymmetric Yang–Mills theory in four dimensions [7]. The latter enjoys conformal symmetry, so the correspondence is often called AdS/CFT duality. The amount of evidence substantiating this duality that has been accumulated so far is overwhelming.

Various generalizations involving different backgrounds and CFTs have also been proposed. The most well-understood cases are those that arise from the decoupling limit of certain brane solutions in string theory. These branes can be described in terms of closed strings, where they correspond to gravitational objects similar to black holes, and the decoupling limit dynamically isolates the near-horizon region. They can also be described in terms of open strings, in which case the decoupling limit corresponds to a low-energy limit of the effective field theory describing open string interactions on the brane. The two descriptions are equivalent, and by carefully tracking the decoupling of various modes it is possible to derive explicit gauge/gravity dualities.

In order to build some intuition and set the terminology that we will use in the thesis, we briefly describe the correspondence between $\mathcal{N} = 4$ super Yang–Mills theory and type IIB string theory on $\mathrm{AdS}_5 \times S^5$ in more detail. We use the notation of [8] and we refer to it for further details. Let us start by describing the free parameters of the two theories. In the field theory side, we can obviously choose the gauge group, which we take to be $SU(N)$. The rank N then corresponds to one free parameter. The only other free parameter is the gauge coupling constant g_{YM}.[5] On the string theory side we also have only two independent dimensionless numbers, the (integer) flux N of the self-dual 5-form F_5 across the internal sphere, which essentially determines the ratio between the AdS radius and the Planck length, and the string coupling constant g_s.

The 5-form flux corresponds to the rank of the gauge group of the CFT, while the gauge and string couplings are related by

$$g_{YM}^2 = 4\pi\, g_s \tag{1.4}$$

The power of the duality becomes apparent when we discuss the regime of validity of various descriptions. First of all, loop diagrams in the field theory contribute with powers of

$$\lambda = g_{YM}^2 N, \tag{1.5}$$

where we introduced the 't Hooft coupling λ. The perturbative field theory regime, where the expansion in terms of Feynman diagrams becomes meaningful, then corresponds to $\lambda \ll 1$.[6] On the gravity side, the ratio between the AdS radius L

[5]In principle, we can also add a θ term to the theory, which corresponds to the axion in the bulk, but for the sake of simplicity and brevity we will not include it in our discussion.

[6]If N is large, there are some further simplifications that we will not discuss here. Once again, the interested reader is encouraged to look at [8] for further details and references.

and the string length ℓ_s is given by

$$\frac{L}{\ell_s} = \lambda^{1/4}. \tag{1.6}$$

It turns out that the contribution of string states is negligible when the radius of curvature of the background is much bigger than the string scale, that is when $\lambda \gg 1$. However, quantum corrections are not suppressed unless the ratio between L and the Planck length, which as we said is controlled by N, is large. This leads us to consider the so called large N limit. If both N and λ are much greater than 1, classical supergravity computations become reliable.

We reach the conclusion that classical type IIB supergravity on AdS$_5$ should provide a good perturbative description of the large N limit [9] of $\mathcal{N} = 4$ super Yang–Mills in the strongly coupled regime. On the other hand, the computation of observables in the CFT at weak coupling can in principle be used to study type IIB string theory at strong coupling. The duality therefore exchanges the weakly coupled and strongly coupled regimes, a fact that makes it extremely useful and very difficult to prove at the same time.

We also notice that the isometry group of AdS$_5$, which is $SO(2,4)$, precisely matches the conformal group in 3+1 dimensions. This fact will become particularly important in the first part of this thesis, where we consider generalizations of the correspondence to non-relativistic theories. Also in that case, the symmetry group in the field theory side will correspond to the isometry group of the dual gravitational background.

1.2.1 Relation between the observables

The basic observables in a quantum field theory are the correlation functions of (gauge invariant) local operators $O_I(x)$

$$\langle O_{I_1}(x_1) \dots O_{I_n}(x_n) \rangle. \tag{1.7}$$

All the information contained in these correlation functions can be conveniently collected in a generating functional called the *partition function*. The idea is to consider a classical source ϕ^I for each operator O_I. The object

$$Z_{\mathrm{QFT}}[\phi^I] = \left\langle \exp\left(- \int d^d x \sqrt{-g}\, \phi(x)^I O_I(x) \right) \right\rangle \tag{1.8}$$

allows us to recover the correlation functions by means of functional differentiation:

$$\langle O_{I_1}(x_1) \dots O_{I_n}(x_n) \rangle = \left(- \frac{\delta}{\delta \phi^{I_1}(x_1)} \right) \cdots \left(- \frac{\delta}{\delta \phi^{I_n}(x_n)} \right) \log Z_{\mathrm{QFT}}[\phi^I] \Big|_{\phi^I = 0}. \tag{1.9}$$

As we discussed before, the computation of these correlation functions (or equivalently the partition function) in interacting theories relies on perturbative methods such as Feynman diagrams. The holographic dualities provide us with an alternative means to construct the generating functional. The prescription is as follows: we start from a gravitational theory that has AdS_{d+1} spacetime as a classical solution.[7] Since AdS_{d+1} has a (conformal) boundary, the quantization of such a theory must be supplemented by "boundary conditions". For each classical field in the gravitational theory Φ^I we need to specify the behavior at the boundary $\Phi^I|_{boundary}(x) = \phi^I(x)$, where x is a coordinate on the boundary.[8] The quantization of the theory then leads to a partition function $Z_{\text{string}}[\phi^I]$ that is a functional of such boundary conditions. The main statement of the gauge/string duality is that this partition function is precisely the generating functional of a quantum field theory living in d dimensions [10, 11]:

$$Z_{\text{string}}[\phi^I] = Z_{\text{QFT}}[\phi^I]. \tag{1.10}$$

The formula above implies a one-to-one correspondence between classical gravity fields and operators of the dual CFT, also known as *field/operator correspondence*. The power of this formula comes from the fact that since the duality, as explained before, exchanges the strongly coupled regime with the weakly coupled regime of the dual description, we can use perturbative methods on one side of the duality to compute observables on the other side of the duality in the strongly coupled regime.

In this thesis we mostly use the duality in the regime where classical gravity computations are reliable. In this case, the string theory partition function is dominated by the contributions coming from the saddle points, which are nothing else than the classical solutions. As a consequence, we have

$$Z_{\text{string}}[\phi^I] \approx \exp(-S[\phi^I]), \tag{1.11}$$

where $S[\phi^I]$ is the gravitational on-shell action as a functional of the boundary conditions. We will see that the object $S[\phi^I]$ is divergent and, before it can be used to compute correlation functions, needs to be renormalized. This is the analog of the renormalization procedure in standard quantum field theory, and will be described in more detail in chapter 2.

When we have correctly renormalized the on-shell action and properly identified the boundary conditions at the conformal boundary, correlation functions of the

[7] In many cases involving supergravity, such spacetimes can be argued to be solutions of the full string theory.

[8] To be more precise, we would need to take into account the radial behavior of the various fields, since they typically either diverge or vanish as we go to the boundary. For the sake of simplicity, we ignore these issues in the introduction and we will be more careful when we actually perform computations in specific examples in the next chapter.

dual strongly coupled field theory can be computed by taking functional derivatives of $S[\phi^I]$ as follows:

$$\langle O_{I_1}(x_1)\ldots O_{I_n}(x_n)\rangle = -\left(-\frac{\delta}{\delta\phi^{I_1}(x_1)}\right)\cdots\left(-\frac{\delta}{\delta\phi^{I_n}(x_n)}\right)S[\phi^I]\Big|_{\phi^I=0}. \qquad (1.12)$$

1.2.2 The holographic renormalization group

As we said before, while conformal field theories are interesting in their own right, in the end we want to learn something about theories that are *not* conformal. The techniques illustrated above also allow us to discuss possible conformal field theory deformations in the holographic context. Suppose that we want to introduce a deformation by the operator $\mathcal{O}$, which is dual to the gravitational field Φ via the field/operator correspondence. Since coupling constants are nothing else than constant sources for their operators, we can simply set the source (or boundary condition) for Φ in (1.12) to some constant value at the end of the computation, instead of taking it to zero. In this way we obtain the correlators of a (strongly coupled) field theory deformed by the operator $\mathcal{O}$.

In this setting, irrelevant operators correspond to those deformations that grow as we approach the boundary. This typically means that the spacetime is no longer asymptotically AdS. In this situation, renormalizing the on-shell action is particularly difficult, since one typically needs an infinite number of counterterms. Relevant deformations on the other hand vanish at the boundary, and they only change the solution in the interior. Marginal deformations are of course also present, and the comments that we made in the field theory context apply here as well, including the distinction between marginally relevant and irrelevant operators. In any case, if the deformation is not exactly marginal, conformal invariance is broken, so we expect a non-trivial renormalization group flow. While the renormalization group parameter in field theory is the energy scale, in holography it is the extra (radial) coordinate that we get by going from d dimensions to $d+1$. By looking at the radial behavior of the fields,[9] we can learn something about the renormalization group flow of the dual field theory. In particular, in the presence of relevant deformations, the solution might flow to another AdS solution in the interior, which then corresponds to a non-trivial IR fixed point of the dual field theory. In the course of this thesis, we will also encounter irrelevant deformations. This is a much more problematic situation, because it calls for a new UV description of the theory. Unfortunately, the RG flow is irreversible in going from the UV to the IR, so the information provided by the knowledge of the IR theory and the irrelevant deformation is typically not enough to pinpoint a UV completion.

[9] Assuming of course that the classical gravity approximation remains reliable across the flow.

We will illustrate some of the implications of this observation in the last chapter, where we discuss irrelevant deformations in the context of black holes.

1.3 Outline and summary of the main results

In this last section we briefly give an outline of the thesis and describe the main results.

1.3.1 Lifshitz holography

In chapter 2 and 3, we describe some developments in extending the holographic dictionary outside the realm of asymptotically AdS spacetimes.

Chapter 2 describes the techniques developed in [1], in which we took some small steps towards the definition of holographic dualities where the the background gravitational geometry is Lifshitz instead of AdS. We decided to study these constructions because they are conjectured to describe strongly coupled non-relativistic field theories, which are potentially interesting for phenomenological applications in condensed matter and statistical physics.

In particular, we describe how the problem of holographic renormalization can be tackled in this case and show in one particular example how these techniques allow us to determine whether certain boundary conditions are allowed or not. While we do not provide a complete framework for Lifshitz holography, we present some evidence that such a framework does indeed exist. For example, simple field theory toy models with non-relativistic scaling symmetry can be deformed by relevant operators that induce an RG flow to a relativistic theory in the IR. We will show that such a possibility is also present for strongly coupled non-relativistic field theories with Lifshitz duals. We will in fact show that it is possible to holographically turn on a marginally relevant deformation of the Lifshitz UV fixed point that flows to AdS (which is believed to be dual to a relativistic field theory) in the IR. In doing so, we will come across some interesting physical phenomena familiar in standard field theory, such as certain non-analyticities in the free energy as a function of the coupling constant. While the general formalism was published in [1], most of the results that we describe in this chapter are yet unpublished.

From a condensed matter perspective, deformations like the one studied in chapter 2 are interesting because they allow us to move away from the (quantum) critical point. In particular, they can be used to probe the finite temperature phase diagram in the vicinity of the critical point and study phenomena such as finite temperature crossovers. In the holographic context, this is tantamount to studying black hole solutions that approach the marginally relevant deformation of Lifshitz spacetime that we have discussed.

In chapter 3 we give further evidence that gravity theories on Lifshitz backgrounds define non-relativistic field theories by computing the analog of the Weyl anomaly in this anisotropic setting. In particular, we provide a full characterization of the anisotropic scaling anomaly for a general (parity invariant) theory with Lifshitz scaling symmetry and $z = 2$, and show that it is given by two possible structures, one containing only time derivatives and the other containing only spatial derivatives. We then compute the anomaly of two non-relativistic models, one defined by standard field theory methods, the other defined holographically. We will show that a striking phenomenon occurs: while the two theories are in principle completely unrelated, one being free and the other being strongly coupled, they both produce only the term in the anomaly containing time derivatives. This hints at the possibility that gravity duals described by Einstein gravity cannot produce the second structure in the anomaly, a state of affairs similar to the $a = c$ result in standard AdS/CFT. On the other hand, it has recently been shown [12] that Horava–Lifshitz gravity duals do produce this structure. Furthermore, anomalies typically control universal field theory properties that can be in principle measured, such as the Casimir energy in two-dimensional conformal field theories. If these results could be carried over to Lifshitz field theories, the anomaly could be used to constrain (or rule out) gravity models for non-relativistic strongly coupled field theories.

1.3.2 Non-renormalization theorems

In chapter 4 we discuss certain marginal deformations of conformal field theories with a large amount of supersymmetry. The corresponding coupling constants can be thought of as living on a geometrical space, where each point represents a different conformal field theory. We will study how certain quantities vary as we move in this geometrical space. More specifically, we will show that the structure constants of chiral operators are covariantly constant on this space.

In more technical terms, we provide a non-renormalization theorem [3] for the structure constants of chiral operators for general $\mathcal{N} = (4, 4)$ supersymmetric conformal field theories in two dimensions and $\mathcal{N} = 4$ supersymmetric conformal field theories in four dimensions. We will also discuss some extensions to less supersymmetric multiplets.

Non-renormalization theorems have proven to be extremely useful in various phenomenological setups, both in field theory and string theory, where they constrain perturbative and non-perturbative contributions to numerous physical observables. For example, a powerful non-renormalization theorem in $\mathcal{N} = 1$ supersymmetric field theories protects the radiative corrections to the Higgs mass, making supersymmetry a possible resolution of the hierarchy problem. Non-renormalization theorems also allow the precise counting of microstates of various supersymmetric black holes, the most notable example being probably the Strominger–Vafa black hole. In our case, they provide a powerful tool to test the AdS/CFT correspondence. In fact, the computations on the gravity side and field theory side cannot be compared in general, since they are performed at different points in the moduli space. However, protected quantities do not depend on such moduli, and should be the same in both descriptions.

1.3.3 Black hole entropy

In chapter 5 we show how certain black hole constructions that recently appeared in the literature can be understood in terms of irrelevant deformations. The motivation comes in this case from the problem of explaining the entropy of black holes microscopically. For many supersymmetric black holes in string theory, the entropy has been successfully accounted for by studying the degeneracy of certain protected states in dual CFTs. In that case supersymmetry played an extremely important role, because the degeneracy of such states is a topological quantity that is constant in the moduli space of the theory, so that it can be computed exactly in the weakly coupled regime of the CFT.

Surprisingly enough, the entropy of generic black holes, even far away from extremality, seems to be of a CFT form, that is it strongly resembles Cardy's asymptotic growth of states in a two-dimensional CFT. This "numerological" observation has sparked a lot of interest in trying to find some CFT description for the microstates of general black holes. This has led to some interesting developments, such as the Kerr/CFT correspondence, which among other things uncovered a "hidden" $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry of the scalar wave equation in the near-horizon region, which is reminiscent of conformal symmetry in two dimensions.

Later, this "hidden conformal symmetry" has been given a geometric interpretation by replacing the original (asymptotically flat) black hole with a different geometry that, while preserving the near-horizon properties, has a different asymptotic behavior. It is the purpose of this last chapter to elucidate the relation between the original black hole and this "subtracted" geometry: we will show that there is a RG flow between the two geometries, which can be interpreted in CFT language as being driven by an irrelevant deformation [4]. As we discussed before, irrelevant deformations are generically problematic for predictability, so the results derived in this last chapter might indicate a problem with the Kerr/CFT proposal.

Chapter 2

Lifshitz Holography

In this chapter we begin our journey into deformations of quantum field theories by studying a marginally relevant operator of a certain strongly coupled non-relativistic field theory. The latter enjoys a peculiar scaling symmetry, named Lifshitz symmetry, where time is treated differently from space. We will employ the framework of holography to study this deformation, where this scaling symmetry is realized geometrically as the isometry group of a gravitational background. Geometries with this particular symmetry are called Lifshitz spacetimes; they were originally introduced as possible holographic dual descriptions of non-relativistic field theories in [13, 14] and have since appeared in many different setups, for example as IR geometries [15]. However, holography for these spacetimes is not yet standing on the same firm footing as ordinary AdS spacetimes, and it is therefore necessary to determine to what extend the usual AdS/CFT techniques can be applied to Lifshitz spacetimes.[1]

There are some indications, coming mainly from Schrödinger holography [18], that one needs non-local counterterms to remove divergences in the on-shell value of the action. Therefore, we decided to explore the nature of the divergences that appear in Lifshitz spacetimes when computing the on-shell value of the effective action using the Hamilton–Jacobi method, which turns out to be more efficient in this case than using the Fefferman-Graham expansion, which rapidly becomes quite intractable.

Normally, in order to perform holographic renormalization, one needs to first say something about the boundary conditions for the fields. Here however we will approach the problem from a different perspective. If we require that all divergences

[1] See also [16, 17].

should be canceled by local counterterms, this will automatically enforce particular boundary conditions for the fields. More precisely, we will find that particular local covariant quantities made out of the bulk fields have to scale in a specific way as we approach the "boundary" of Lifshitz spacetime.

In this chapter, we will briefly review the results of [1] and apply them to the special case $z = 2$. In particular, we will focus on a deformation that induces a flow to AdS in the IR, first identified in [13]. It is possible to argue that this deformation is marginally relevant from the point of view of the dual field theory. However, it induces large logarithmic corrections at leading order in the metric field, so it is not a priori clear that such a deformation has the right to be called "asymptotically Lifshitz". We will address this issue by showing that only local counterterms with no explicit dependence on the radial coordinate are needed in order to cancel the divergences, contrary to what has been previously claimed in the literature [19, 20]. However, a new interesting phenomenon arises: the renormalized on-shell action turns out to be a non-analytic function of the marginally relevant coupling constant. We will see that this behavior is perfectly consistent with our intuitions coming from asymptotically free couplings in quantum field theory, in particular QCD. This provides further evidence that Lifshitz geometries are dual to field theories with a well-defined UV completion.

The outline is as follows. In section 2.1 we will motivate our interest in field theories with Lifshitz scaling by briefly reviewing their use in condensed matter physics. In section 2.2 we review the Hamilton–Jacobi method and apply it to the non-derivative terms in the boundary effective action. We will show that it is possible to extract the local part of the on-shell action order by order in the fields. In section 2.3 we perform some non-trivial consistency checks by explicitly computing the on-shell action for scalar perturbations of the metric and gauge fields up to third order. We will provide evidence that the divergences are canceled order by order in the value of the boundary fields by our local counterterms. We will also compare our results with the ones presented in [19], and will discuss certain discrepancies. In section 2.4 we will perform an all-order analysis of a marginally relevant deformation, similar in spirit to "improved perturbation theory" in standard quantum field theory, and we will show that divergences are indeed removed by our construction, so that such deformation can be turned on using only local counterterms. Furthermore we will discuss the convergence of the counterterm perturbation series and relate it to a non-analyticity of the renormalized on-shell action as a function of the marginally relevant coupling constant. In the final section, we will describe some open issues and possible ideas for future research projects. Results for general z and derivative counterterms are discussed in the appendix.

2.1 Introduction to Lifshitz field theories

Condensed matter physics deals with systems that consist of a very large number of constituents, of the order of $\sim 10^{23}$. Solving the microscopic quantum many-body problem is therefore an extremely daunting task. However at low energies, or at distances much larger than the lattice spacing, it is often possible to describe the resulting effective degrees of freedom with the formalism of quantum field theory.

Quantum field theories are especially useful in the proximity of phase transitions, where they become scale invariant. In particular, quantum systems can exhibit phase transitions even at zero temperature, which are characterized by a non-analytic behavior of certain physical quantities as some external parameters are varied, such as an external magnetic field or doping. We refer to this situation as a *quantum phase transition* [21], and the points in parameter space where these transitions occur are named *quantum critical points.*

As we said, physics at a quantum critical point is scale invariant, but in many interesting physical systems it need not be Lorentz invariant. More precisely, the usual scale invariance given by

$$t \to \lambda t, \qquad x \to \lambda x, \tag{2.1}$$

is replaced by a "dynamical scaling"

$$t \to \lambda^z t, \qquad x \to \lambda x, \tag{2.2}$$

where z is called dynamical critical exponent, and it determines to what extent time scales differently from space.

The focus of this chapter is on particular models with $z = 2$. This kind of scaling arises in many interesting physical models, for example the Rokhsar–Kivelson quantum dimer model [22], which was conjectured in [23] to be in the same universality class as the so called *quantum Lifshitz model* [24]. The latter is a non-relativistic free field theory described by the Euclidean action

$$S = \int d\tau d^2x \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{\kappa^2}{2}(\nabla^2\phi)^2 \right), \tag{2.3}$$

which arises also as the field theory description of Lifshitz points at finite temperature, and where κ parametrizes a line of fixed points [25]. We will have much more to say about this particular theory in the next chapter, where we will compute the non-relativistic analog of the conformal anomaly.

This toy model is free, but we can deform it in interesting ways, for example by adding the operator $(\nabla\phi)^2$. This operator is relevant and it induces a flow to a relativistic theory in the IR. In other words, this model can be seen as a theory with Lifshitz scaling symmetry in the UV that flows to a theory with relativistic scaling (i.e. $z = 1$) in the IR.

Since holography has enjoyed some success in the description of strongly coupled field theories relevant to condensed matter systems (see [26] and references therein), it is natural to try to apply holographic techniques to explore whether the RG flow described in the previous paragraph has a strongly coupled counterpart. From the bulk point of view, we are looking for geometries that interpolate between AdS in the interior and Lifshitz at the boundary. It was shown in [13] that such geometries exist, and they were interpreted in [19] as being driven by a *marginally relevant* perturbation of the purported dual strongly coupled Lifshitz field theory.[2] However, the question of whether this marginally relevant deformation should be included in the standard holographic dictionary for Lifshitz field theories is still open, and this provides the motivation for this chapter.

2.2 The HJ approach to holographic renormalization

In this section we will describe the general problem of holographic renormalization [11, 28, 29, 30, 31, 32, 33] and focus in particular on the Hamilton–Jacobi approach [31, 34, 35].

As described in the general introduction, the AdS/CFT correspondence gives us a prescription to compute field theory observables, namely correlation functions of gauge invariant operators, in terms of quantities defined in the bulk. In particular, the object that allows us to compute correlation functions of the dual field theory is the classical gravitational on-shell action $S[\phi]$, written as a functional of the boundary conditions as explained in the general introduction.

However, $S[\phi]$ is typically divergent, and needs to be made finite with some renormalization prescription before it can be used to compute anything. This procedure is the analog of standard renormalization in quantum field theory, where the effective action receives quantum corrections that diverge and need to be regulated by adding suitable local counterterms. The gravitational counterpart is that the gravitational on-shell action needs to be made finite by the addition of counterterms preserving general covariance.

The divergences in quantum field theory that require renormalization come from ultraviolet effects, in particular they do not depend on the state of the theory.

[2]See also [27] for a discussion of similar flows when $z \neq 2$.

Analogously, divergences in the bulk are large volume effects and are expected to be independent of the particular state in the interior; they should therefore depend only on "near-boundary" data. This is in fact true in standard AdS/CFT in the presence of relevant deformations.

We will now describe how we can use the Hamiltonian formulation of general relativity to determine the functional form of the counterterm action in terms of the boundary fields. We write the on-shell action as

$$S = S_{loc} + \Gamma, \tag{2.4}$$

where S_{loc} is local and contains all the divergences[3] of S as we approach the boundary $r \to \infty$. The object Γ will be identified with the effective action of the dual field theory.

To illustrate the method, we briefly turn to classical mechanics. In this case, the on-shell action as a function of the boundary value of the fields obeys the so-called Hamilton–Jacobi equation. Typically, the boundary value problem is formulated in terms of initial and final conditions, that is we fix

$$\mathbf{q}(t_i) = \mathbf{q_i}, \tag{2.5}$$

$$\mathbf{q}(t_f) = \mathbf{q_f}, \tag{2.6}$$

where $\mathbf{q}$ represents the coordinates of the system. We then find the solution[4] $\mathbf{q}(t)$ that obeys these conditions. The on-shell action is defined as the classical action computed on this trajectory, seen as a function of the initial and final conditions:

$$S_{\mathbf{q_i},t_i}(\mathbf{q_f}, t_f) \equiv S[\mathbf{q}(\mathbf{t})]. \tag{2.7}$$

To simplify notation, we will omit the subscript from the final conditions as well as the dependence on the initial conditions, so that the on-shell action is simply as $S(\mathbf{q}, t)$. It can be proven that this object satisfies a non-linear partial differential equation called the Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \mathbf{p}, t\right) = 0, \qquad \mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \tag{2.8}$$

where H is the Hamiltonian of the system.

In a classical field theory, the "coordinates" $\mathbf{q}$ are replaced by fields ϕ. We have in mind a gravitational theory described by general relativity, and for applications to

[3] This is a little imprecise, given that Γ will typically exhibit logarithmic divergences, that can be related to anomalies. This will be discussed in detail in the next chapter.

[4] We are assuming that this problem is well-posed, that is there is a unique solution satisfying the boundary conditions. Furthermore, we require that the solutions originating from $\mathbf{q}_i$ at time t_i do not intersect each other in the time interval $[t_i, t_f]$. For more information, see [36].

holography the time parameter t will be replaced by a radial coordinate r representing a space-like foliation of our spacetime. The Hamiltonian for a gravitational system in the ADM formulation can be written as

$$H = \int_{\Sigma_r} d^d x \sqrt{-\gamma} \left(N\mathcal{H} + N^a \mathcal{H}_a \right), \tag{2.9}$$

where Σ_r is a hypersurface of constant r, while N and N^a are the usual lapse and shift functions. The Hamiltonian densities $\mathcal{H}$ and $\mathcal{H}_a$ will depend on the particular gravitational theory that is considered. General covariance implies the constraints

$$\mathcal{H}(\phi, \pi, r) = 0, \tag{2.10}$$

$$\mathcal{H}_a(\phi, \pi, r) = 0, \tag{2.11}$$

where π are the canonical momenta associated to ϕ. When we use the relation between the momenta and the on-shell action

$$\pi = \frac{\delta S}{\delta \phi}, \tag{2.12}$$

the constraints above become functional differential equations for $S[\phi]$.

We now focus on the first constraint (2.10). When we plug in the relation (2.12), the resulting equation becomes the gravitational analog of (2.8), so we will refer to it as the Hamilton–Jacobi equation. Since we want to keep the discussion general, we will not specify explicitly the form of $\mathcal{H}$, which depends on the particular gravitational theory considered as well as the matter fields. However, the dependence on the canonical momenta π is quadratic for a very large class of such theories, which includes the particular theories we will be concerned with in this chapter. As a consequence, the Hamiltonian constraint can generically be cast in the form

$$\mathcal{H} = \{S, S\} - \mathcal{L}, \tag{2.13}$$

where the bracket $\{F, G\}$ is a bilinear and symmetric operation that encodes the (quadratic) dependence on the canonical momenta, and $\mathcal{L}$ is the Lagrangian density computed on the slice Σ_r. Solving the Hamilton–Jacobi equation is as difficult as solving the full non-linear system of equations of motion, but the structure of the equation allows us to effectively exploit some simplifying assumptions that will make the problem tractable. In fact, using (2.4) and the bilinearity of the bracket operation, we can write

$$\{S_{loc}, S_{loc}\} - \mathcal{L} + 2\{S_{loc}, \Gamma\} + \{\Gamma, \Gamma\} = 0. \tag{2.14}$$

Since we are assuming that S_{loc} contains all the divergences as we approach the boundary, the "local" part of the equation above should decouple from the rest, so that

$$0 = \{S_{loc}, S_{\text{loc}}\} - \mathcal{L}, \tag{2.15}$$

$$0 = 2\{S_{loc}, \Gamma\} + \{\Gamma, \Gamma\}. \tag{2.16}$$

This splitting is not unique, in that we are free to move finite local counterterms from S_{loc} to Γ; this is simply related to a renormalization scheme ambiguity. However, it might happen that the requirement that S_{loc} removes the divergences is incompatible with such a splitting. We will analyze this possibility in detail in the next chapter, and show that it is related to a scaling anomaly of the dual field theory.

Notice that S generically depends also on the "initial conditions", that is the value of the fields on a radial slice Σ_r in the interior of our spacetime. However, since we expect divergences to be a large volume phenomenon as we explained before, S_{loc} should depend only on boundary data. It is not obvious a priori that this is true, so we can use this requirement as a consistency check on our formalism.

The problem becomes tractable thanks to two simplifying conditions on S_{loc}. Firstly, locality of the expected divergences implies that S_{loc} is a *local* functional of the induced fields on the slice Σ_r. Secondly, it is easy to argue that the second Hamiltonian constraint (2.11) can be solved by requiring that S (and in turn S_{loc}) is invariant under diffeomorphisms on the slice Σ_r. Therefore S_{loc} should be a local covariant functional of the induced fields on the slice. As we will see, this allows us to solve (2.15) explicitly and determine the counterterm action for the on-shell action.

In the next sections we will describe how this formalism can be applied to spacetimes with anisotropic scale invariance.

2.2.1 Lifshitz spacetime and the Einstein–Proca action

Lifshitz spacetime is a proposed gravitational dual to a field theory with a UV fixed point with anisotropic scaling symmetry (2.2). As in the case of standard AdS/CFT, this symmetry should be encoded in the gravitational theory as a subset of the isometry group of the classical geometry associated to the vacuum state. It is therefore natural to consider a spacetime of the form [13]

$$ds^2 \;=\; dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^{\,2}. \tag{2.17}$$

This metric is indeed invariant under the so-called Lifshitz algebra [37], which consists of time translations, spatial translations, spatial rotations, and anisotropic scale transformations (2.2) (with a simultaneous shift in the radial coordinate

$r \mapsto r - \log \lambda$). Notice however that, unlike so-called Schrödinger spacetimes, the Lifshitz spacetime is *not* invariant under Galilean boosts $x \mapsto x + vt$. We will eventually work in 3+1 bulk spacetime dimensions, but we keep the dimension d arbitrary for as long as possible.

By computing the Einstein tensor for the metric in (2.17), it is clear that such a geometry cannot be a solution of pure Einstein gravity (possibly with a cosmological constant). It is necessary to introduce some form of matter that can support a non-trivial stress-tensor. Of course the choice is not unique, but in this thesis we will focus on a particularly simple setup consisting of Einstein gravity with a cosmological constant plus a massive vector field [38]. This massive vector field will necessarily be non-zero in the background in order to support the Lifshitz spacetime, and takes the form

$$A \;=\; e^{zr} dt. \tag{2.18}$$

Since this vector field breaks Lorentz invariance on the constant r slices, it is natural to think of it as being responsible for the breaking of Lorentz invariance in the dual field theory.

The graviton and massive vector fields comprise a solution to the Einstein–Proca action $S = S_{\text{grav}} + S_A$, with

$$S_{\text{grav}} = \int d^{d+1}x\sqrt{-g}\,(R - 2\Lambda) + \int d^d\xi\sqrt{-\gamma}\;2K, \tag{2.19}$$

$$S_A = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right), \tag{2.20}$$

where we used the convention $16\pi G = 1$. In order to find Lifshitz spacetime as a solution, we must pick our parameters to be

$$\Lambda = -\frac{1}{2}(z^2 + z + 4), \qquad m^2 = 2z, \qquad \alpha_0 = -2\frac{z-1}{z}. \tag{2.21}$$

The equations of motion are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2}T_{\mu\nu}, \qquad \nabla_\mu F^{\mu\nu} = m^2 A^\nu, \tag{2.22}$$

where $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_A}{\delta g_{\mu\nu}}$ is the Proca stress tensor, given by

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + m^2 A_\mu A_\nu - \frac{1}{4}g_{\mu\nu}\left(F_{\rho\lambda}F^{\rho\lambda} + 2m^2 A_\rho A^\rho\right) \tag{2.23}$$

As we explained in the previous section, our aim is to construct a finite on-shell action for an appropriate class of asymptotically Lifshitz spacetimes, where this notion will be made more precise in the following. The momentum constraint is given by

$$\mathcal{H}_a = -2D^b\pi_{ab} - A_a D_b E^b + F_{ab}E^b = 0, \tag{2.24}$$

where the quantities π^{ab} and E^a are the canonical momenta dual to the induced metric γ_{ab} and induced vector A_a respectively. The Hamiltonian constraint is

$$\mathcal{H} = -\left(\pi_{ab}\pi^{ab} - \frac{1}{d-1}\pi^2\right) - \frac{1}{2}E^a E_a - \frac{1}{2m^2}(D_a E^a)^2 - \mathcal{L} = 0, \tag{2.25}$$

where $\mathcal{L} = R - 2\Lambda - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}m^2 A_a A^a$ is the Lagrangian restricted to Σ_r.

We can obtain a functional differential equation for S by using

$$\pi^{ab}(r) \;=\; \frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}}(r), \qquad\qquad E^a(r) \;=\; \frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta A_a}(r). \tag{2.26}$$

The ensuing equation can be cast in the form (2.14) by defining

$$\begin{aligned}(\sqrt{-\gamma})^2\,\{F,G\} \equiv{}& -\left(\gamma_{ac}\gamma_{bd} - \frac{1}{d-1}\gamma_{ab}\gamma_{cd}\right)\frac{\delta F}{\delta\gamma_{ab}}\frac{\delta G}{\delta\gamma_{cd}} \\ &- \frac{1}{2}\gamma_{ab}\frac{\delta F}{\delta A_a}\frac{\delta G}{\delta A_b} - \frac{1}{2m^2}D_a\frac{\delta F}{\delta A_a}\,D_b\frac{\delta G}{\delta A_b}.\end{aligned} \tag{2.27}$$

As we mentioned earlier, one typically chooses an ansatz that is covariant [31, 35, 34], such that the momentum constraint is automatically satisfied. Solving the HJ equation thus reduces to solving the Hamiltonian constraint. In the present case, the most general covariant ansatz one can take is

$$S_{loc} \;=\; \int_{\Sigma_r} d^dx\sqrt{-\gamma}\,U(\alpha) + (\text{derivative terms}). \tag{2.28}$$

The quantity $\alpha \equiv A_a A^a$ is the only scalar that one can construct from the metric and the vector field containing no derivatives. In the present chapter, we will discuss only counterterms with no spacetime derivatives; see appendix 2.A for a discussion of the general case, including spacetime derivatives. Furthermore, in the following we will set $z = 2$ and $d = 3$

2.2.2 Solving the Hamilton constraint

We shall now set out to find the counterterms at the level of no spacetime derivatives by solving the local part of the HJ equation. As we explained above, this boils down to solving the local part of the Hamiltonian constraint,

$$\begin{aligned} 0 \cong\ & \{S_{loc}, S_{loc}\} - \mathcal{L} && (2.29) \\ =\ & \frac{3}{8}U^2 - \left(\frac{1}{2}\alpha^2 + 2\alpha\right)\left(\frac{\partial U}{\partial \alpha}\right)^2 - \frac{1}{2}\alpha U \frac{\partial U}{\partial \alpha} - (z^2 + z + 4) + z\alpha, && (2.30) \end{aligned}$$

The idea is to expand the function $U(\alpha)$ as a power-series in $\alpha - \alpha_0$

$$U(\alpha) = \sum_n u_n (\alpha - \alpha_0)^n, \tag{2.31}$$

so that u_0 corresponds to the "boundary cosmological constant" of the pure Lifshitz background. The Hamiltonian constraint will then be a power-series as well

$$\mathcal{H}_{loc} \ = \ \sum_n \mathcal{H}_n \, (\alpha - \alpha_0)^n. \tag{2.32}$$

This allows us to solve the Hamiltonian constraint order by order by requiring that $\mathcal{H}_n = 0$ for every n.

At zeroth-order, we find the equation

$$\frac{3}{8}u_0^2 + \frac{1}{2}u_0 u_1 + \frac{3}{2}u_1^2 - 12 = 0, \tag{2.33}$$

and we notice that u_0 is not determined by this equation, only the relation between u_1 and u_0 is fixed. Similarly, the equations $\mathcal{H}_n = 0$ give the relations between u_{n+1} and the u_j with $j \leq n$, but do not help in determining u_0 itself. This fact is worrying at first sight, since it is unlike the classic case discussed in [31] where the analog of u_0 is in fact determined uniquely by the power-series expansion. We will come back to this issue in due time, for now we will determine this coefficient by looking at the pure Lifshitz background and requiring that the effective on-shell action be finite. It is easy to see that this gives

$$u_0 = 6. \tag{2.34}$$

We will now show that only for this value of u_0 can we expect Γ to be finite.

Let us consider the radial equation for the fields in the Hamiltonian formalism:

$$\begin{aligned} \partial_r \gamma_{ab} \ &= \ \frac{\delta H}{\delta(\sqrt{-\gamma}\,\pi^{ab})} \ = \ -2\pi_{ab} + \gamma_{ab}\pi, && (2.35) \\ \partial_r A_a \ &= \ \frac{\delta H}{\delta(\sqrt{-\gamma}\,E^a)} \ = \ -E_a + \frac{1}{m^2}D_a D_b E^b. && (2.36) \end{aligned}$$

The canonical momenta are given by

$$\begin{aligned} \pi_{ab} \ &= \ \frac{1}{2}\gamma_{ab}U - A_a A_b \frac{\partial U}{\partial \alpha} - \frac{1}{\sqrt{-\gamma}}\frac{\delta \Gamma}{\delta \gamma^{ab}}, && (2.37) \\ E_a \ &= \ 2A_a \frac{\partial U}{\partial \alpha} + g_{ab}\frac{1}{\sqrt{-\gamma}}\frac{\delta \Gamma}{\delta A_b}. && (2.38) \end{aligned}$$

If we assume that Γ is subleading with respect to S_{loc} as $r \to \infty$, the leading order behavior for the fields should be simply given by

$$\partial_r \gamma_{ab} = \left(\frac{1}{2}U - \alpha \frac{\partial U}{\partial \alpha}\right)\gamma_{ab} + 2A_a A_b \frac{\partial U}{\partial \alpha} + \ldots, \tag{2.39a}$$

$$\partial_r A_a = -2A_a \frac{\partial U}{\partial \alpha} + \ldots. \tag{2.39b}$$

where the ellipses denote terms coming from Γ that can be neglected for large r. In this region, we have

$$\partial_r \gamma_{tt} = 4\gamma_{tt} + \ldots, \tag{2.40}$$

$$\partial_r \gamma ij = 2\gamma_{ij} + \ldots, \tag{2.41}$$

$$\partial_r A_0 = 2A_0, \tag{2.42}$$

while using the fact that in this limit $\alpha \to \alpha_0$, we can rewrite (2.39b) as

$$\partial_r A_0 = -2u_1 A_0 + \ldots, \tag{2.43}$$

and comparing with (2.42) we get

$$u_1 = -1. \tag{2.44}$$

We can now use (2.33) to get

$$u_0 = 6. \tag{2.45}$$

Since the equation is quadratic, there is another solution $u_0 = -14/3$ to the equation, but this can be excluded as well by repeating the analysis above for the metric γ_{ab}.

We have therefore shown that only the choice $u_0 = 6$ is compatible with the assumption that Γ is subleading with respect to S_{loc}. In fact, any other choice would imply power-law divergences in the effective action.

2.2.3 Recursive relation

We can now set up the computation order by order in perturbation theory. We have already discussed $\mathcal{H}_0$, which gives u_1 in terms of u_0. It turns out that $\mathcal{H}_1$ vanishes identically, so from this condition we are not able to determine u_2. The $\mathcal{H}_2$ condition reads

$$6u_2^2 + u_2 - \frac{5}{8} = 0, \tag{2.46}$$

which has two solutions $u_2 = -\frac{5}{12}$ and $u_2 = \frac{1}{4}$. We will argue later that we need to choose the latter solution, which corresponds to the non-normalizable mode of the vector field.

Having determined u_0, u_1 and u_2, we can write a general recursion relation for the coefficient u_n in terms of the lower ones:

$$u_n = -\frac{1}{4}\Big(\sum_{i=1}^{n-1}\left(\frac{3}{8}-\frac{1}{2}i(n-i)-\frac{1}{2}i\right)u_i u_{n-i} \tag{2.47}$$

$$+\sum_{i=1}^{n-2}\left(\frac{1}{2}(i+1)-(i+1)(n-i)\right)u_{i+1}u_{n-i} \tag{2.48}$$

$$+\sum_{i=2}\left(\frac{3}{2}(i+1)(n+1-i)\right)u_{i+1}u_{n+1-i}\Big) \tag{2.49}$$

While finding a solution in closed form appears to be a difficult task, we can see that all the coefficients are unambiguously determined by this method, as was claimed without proof in [1]. The first few coefficients are

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\dots$
6	-1	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{3}{64}$	$-\frac{1}{128}$	$\frac{1}{256}$	$-\frac{11}{2048}$	$\dots$

(2.50)

However, the coefficients quickly become very large, as can be seen from Figure 2.1, which shows a log-plot of the first 1000 coefficients computed using Mathematica. We now have two main questions:

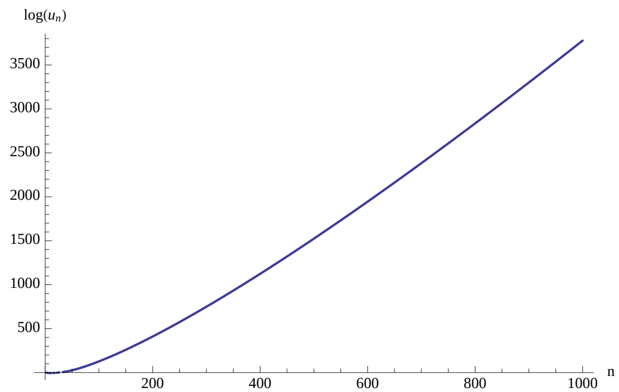


Figure 2.1: *Log plot of the first 1000 coefficients u_n, showing that they grow quite fast with n.*

- Does the counterterm action remove the divergences from the effective action?
- Does the counterterm action converge as a power-series in $(\alpha - \alpha_0)$?

In the next section we address the first question in perturbation theory. In the subsequent section we will answer the first question non-perturbatively and show that the counterterm series has a zero radius of convergence. This will be linked to a non-analytic behavior of the renormalized on-shell action, which in turn can interpreted physically in terms of a standard field theory argument provided by Dyson [39].

2.3 Renormalized On-Shell Action: Perturbative Analysis

Up to this point we did not impose any boundary conditions on our space of solutions. In this section we will analyze the Einstein–Proca equations in the constant perturbation sector up to third order. In doing so, we will set up a renormalization scheme that will allow us to show that divergences in the fourth-order on-shell action are indeed removed using our formalism.

2.3.1 Perturbative analysis of the equations of motion

First, we discuss the first-order solution obtained in [40] and then we set out to solve the second-order equations. The purpose of finding these solutions is to perform a non-trivial check of the counterterms that we found in 2.2.2. Again, we focus on the non-derivative sector throughout this chapter, so we shall restrict our analysis of the linearized field equations to constant modes which only depend on the radial coordinate r.

In view of the results of [16], it is particularly convenient to parametrize the perturbations using the frame field formalism. However, the results will be independent of the particular parametrization chosen, and in fact we will recover the results of [1]. Our parametrization for the frame fields e^I, $I = 0, \ldots, 3$ and vector field A is as follows:

$$e^0 = e^{2r}\Big(1 + \varepsilon\, f_1(r) + \varepsilon^2\, f_2(r) + \ldots\Big)\, dt, \tag{2.51}$$

$$e^1 = e^{r}\,\Big(1 + \varepsilon\, k_1(r) + \varepsilon^2\, k_2(r) + \ldots\Big)\, dx, \tag{2.52}$$

$$e^2 = e^{r}\,\Big(1 + \varepsilon\, k_1(r) + \varepsilon^2\, k_2(r) + \ldots\Big)\, dy, \tag{2.53}$$

$$e^3 = dr, \tag{2.54}$$

$$A = \Big(1 + \varepsilon\, j_1(r) + \varepsilon^2 j_2(r) + \ldots\Big)\, e^0. \tag{2.55}$$

The metric is just given by

$$ds^2 = -e^0 \otimes e^0 + \sum_{i=1}^{3} e^i \otimes e^i. \tag{2.56}$$

We use the small parameter ε to keep track of the order in the perturbative expansion. Notice that our choice for e^3 is equivalent to the radial gauge, namely the requirement that $g_{rr} = 1$ and $g_{r\mu} = 0$ for $\mu \neq r$.

First-order solution.

As was noted in [40], the first order field equations for constant perturbations reduce to the following four equations.

$$0 = 2j_1'' - 3f_1' + 11j_1' + 12j_1 + 6k_1', \tag{2.57}$$
$$0 = 2f_1'' + 7f_1' - 3j_1' - 12j_1 + 2k_1', \tag{2.58}$$
$$0 = 2k_1'' + 6k_1' + j_1' + f_1' + 4j_1, \tag{2.59}$$
$$0 = 6k_1' + 3f_1' + j_1'. \tag{2.60}$$

The first three equations are linear second order ordinary differential equations, so the general solution will depend on six parameter. The fourth equation, however, is a constraint that will reduce the number of free parameters to five, which we will denote by $c_1, ..., c_5$. In fact, it can be easily shown that the first two equations and the fourth imply the third, so that this system of equations is equivalent to two second order and one first order differential equations. The general solution is given by

$$\begin{aligned} j(r) &= -\left(c_1 + c_2\, r\right)\, e^{-4r} + c_3, \\ f(r) &= \frac{1}{24}\left(4c_1 - 5c_2 + 4c_2\, r\right)\, e^{-4r} + \left(2c_3\, r + c_4\right), \\ k(r) &= \frac{1}{48}\left(4c_1 + 5c_2 + 4c_2\, r\right)\, e^{-4r} + \left(-c_3\, r + c_5\right). \end{aligned} \tag{2.61}$$

In this case, the modes c_1 and c_2 are normalizable, while c_3, c_4, and c_5 are non-normalizable. Notice in particular that c_4 and c_5 correspond to linearized diffeomorphisms, generated by the vector field

$$\xi = c_4\, t\, \partial_t + c_5\, x^i\, \partial_i. \tag{2.62}$$

Since α is a scalar, it cannot depend on c_4 and c_5 at linear order. In fact it is easy to see that, at this order

$$\alpha - \alpha_0 = 2\,\varepsilon\, j(r) + O(\varepsilon^2). \tag{2.63}$$

The c_2 mode should be related to the mass of a black hole solution, as suggested by the asymptotically Lifshitz black holes considered for example in [38, 41].

We notice at this point that the mode c_3 shifts the value of $\alpha - \alpha_0$, while giving a logarithmic correction to the leading order behavior of g_{tt} and g_{ij}. This clearly suggests that this mode should correspond to a *marginal* deformation of the dual field theory, even though it remains unclear at this point whether the corresponding operator is marginally relevant or irrelevant. We will clarify this issue in the course of this section.

It is now easy to compute the renormalized on-shell action Γ to first order in this perturbation:

$$\Gamma = \varepsilon \frac{2}{3} c_2 + O(\epsilon^2), \tag{2.64}$$

so we see that the normalizable mode c_2 contributes to the free energy of the system, giving additional evidence that this mode should be related to an asymptotically Lifshitz black hole solution. This result matches the one in [40], where a restricted set of counterterms was employed, which turns out to be equivalent to ours at linear order. It is also easy to show that the second-order divergences are correctly canceled by our counterterms.

Second-order solution.

The second order field equations are given by

$$\begin{aligned} 0 \; = \; & 8j_1^2 + 16j_2 + f_1'^2 + 4f_2' + 2f_1'j_1' + j_1'^2 + 4j_1\left(2f_1' + j_1'\right) + 4j_2' - 24k_1k_1' \\ & + 4k_1'^2 + 24k_2' - 8k_1k_1'' + 4f_1\left(8j_1 + f_1' + 2j_1' + 12k_1' + 4k_1''\right) + 8k_2'' \end{aligned} \tag{2.65}$$

$$\begin{aligned} 0 \; = \; & 8j_1^2 + 16j_2 + 16f_1f_1' - 32k_1f_1' + f_1'^2 - 16f_2' + 8k_1j_1' + 2f_1'j_1' + j_1'^2 \\ & + 4j_1\left(8k_1 + 2f_1' + j_1'\right) + 4j_2' - 16k_1k_1' - 4f_1'k_1' - 16k_2' + 4f_1f_1'' \\ & - 8k_1f_1'' - 4f_2'' - 4k_1k_1'' - 4k_2'' \end{aligned} \tag{2.66}$$

$$\begin{aligned} 0 \; = \; & -f_1'^2 + 2f_2' + 4f_1j_1' + 4j_2' + 4f_1k_1' - 4k_1k_1' + 2j_1'k_1' + f_1'\left(j_1' + 2k_1'\right) \\ & + 4k_2' + j_1\left(2f_1' + 4k_1' + f_1''\right) + f_2'' + f_1j_1'' + j_2'' \end{aligned} \tag{2.67}$$

$$\begin{aligned} 0 \; = \; & -3f_1f_1' + \frac{1}{4}f_1'^2 + 3f_2' + \frac{1}{2}f_1'j_1' + \frac{1}{4}j_1'^2 + j_1\left(2f_1' + j_1'\right) + j_2' - 6k_1k_1' \\ & + 2f_1'k_1' + k_1'^2 + 6k_2'. \end{aligned} \tag{2.68}$$

Just like the first-order equations, these consist of one first-order differential equation and three second order ones, so again there are five integration constants. As we expect, the first order solution appears as a source for the second order equations, and the only modes that can appear in the second-order functions j_2, f_2 and k_2 are products of the modes we had already found at first order. The solution is thus given by

$$\begin{aligned} j_2(r) &= \left(j_1 + j_2\, r + j_3\, r^2\right)\, e^{-4r} + j_4 + j_5\, r + j_6\, r^2 + \left(j_7 + j_8\, r + j_9\, r^2\right)\, e^{-8r}, \\ f_2(r) &= \left(f_1 + f_2\, r + f_3\, r^2\right)\, e^{-4r} + f_4 + f_5\, r + f_6\, r^2 + \left(f_7 + f_8\, r + f_9\, r^2\right)\, e^{-8r}, \\ k_2(r) &= \left(k_1 + k_2\, r + k_3\, r^2\right)\, e^{-4r} + k_4 + k_5\, r + k_6\, r^2 + \left(k_7 + k_8\, r + k_9\, r^2\right)\, e^{-8r}. \end{aligned} \tag{2.69}$$

The general second-order solution is parametrized by 27 coefficients j_i, f_i, and k_i, but as we mentioned before only 5 of them are truly independent. In fact, most of them will be fixed in terms of the coefficients c_i of the first order equation.

Notice that the second order solution contains modes that appeared at first order and that were identified with sources. More precisely j_4 corresponds to the constant term in $j_1(r)$, namely c_3, while f_4 and k_4 correspond to the constant terms in $f_1(r)$ and $k_1(r)$ respectively. It is easy to see by inspection that these three coefficients are arbitrary in the second order solution. This is to be expected: at each order in perturbation theory, we are free to shift the sources by higher-order terms; this is part of our choice of perturbation scheme for marginal and irrelevant operators. In the following, we will impose that the constant part of j, f and k is not changed by higher order terms. This will decrease the number of undetermined parameters to two, which turn out to affect only the normalizable part of the solution. This in perfect agreement with the expectation that the relation between the non-normalizable modes and the normalizable modes is non-local in the bulk, and depends on the choice of state.

For completeness and future use, we display the leading part of the second-order solution where we turn off the constant piece of f and k:

$$j_2 = c_3^2\, r + \dots \tag{2.70}$$

$$f_2 = 3c_3^2\, r + 3c_3^2\, r^2 + \dots \tag{2.71}$$

$$k_2 = -2c_3^2\, r + \dots \tag{2.72}$$

where the ellipses stand for terms suppressed by e^{-4r} or more.

The on-shell action to second order for this perturbation is

$$\Gamma = \varepsilon \frac{2}{3}\, c_2 + \varepsilon^2 \left(c_2' + \frac{15}{8} c_1 c_3 + \frac{63}{32} c_2 c_3 \right), \tag{2.73}$$

where c_2' is a second-order correction to the vacuum expectation value, and the remaining part of the expression shows the typical source/vev coupling at second order.

It is also possible to check that the third order divergence is canceled to this order, as it should.

nth-order solution.

It is conceptually straightforward to extend this procedure to higher order, even though the computations become quite tedious. However, we explicitly solved the equations to third order for the deformation we are considering and checked that the fourth order divergences cancel. This suggests that our proposal for the counterterm action suffices to remove the divergences that arise in a perturbative computation where the sources are treated as infinitesimal quantities. Furthermore, inspection of the higher-order solutions suggests a pattern for the leading term at each order for j, namely

$$j_n = c_3 (c_3\, r)^{n-1} + \ldots , \tag{2.74}$$

where the ellipses denote terms that are subleading.[5] We can resum all these leading terms to obtain (we will set $\epsilon = 1$ in the following, since we do not take the sources to be infinitesimal anymore)

$$A = (1 + \sum_{i=1}^{n} j_n) e^0 = (1 + \frac{1}{\frac{1}{c_3} - r} + \ldots) e^0 . \tag{2.75}$$

We see that increasingly high powers of logarithms resum to an inverse logarithm, a behavior reminiscent of asymptotically free couplings in quantum field theory. Furthermore, this behavior precisely reproduces the inverse logarithmic behavior of the marginally relevant deformation studied in [19] (from now on we will refer to this paper as CHK). Additional evidence is provided by resumming the leading order part of f, which behaves like

$$f_n = (n+1)(c_3\, r)^n , \tag{2.76}$$

so that

$$e^0 = e^{2r}(1 + \sum_{i=1}^{n} f_n) = e^{2r} \left(\frac{1}{(c_3\, r - 1)^2} + \ldots \right) . \tag{2.77}$$

We see that the resummation leads to a *leading* inverse logarithmic behavior in the metric. Furthermore, this behavior precisely matches the one found in [19] for the marginally relevant perturbation considered there.

By changing coordinates (while reinstating the length scale ℓ of Lifshitz spacetime) as

$$\frac{\ell}{z} = e^r , \tag{2.78}$$

[5] Notice however that subleading terms for j_n will in general be dominant with respect to the leading term in j_k with $k < n$.

we can write see that

$$\frac{1}{c_3} - r = \log(z\Lambda), \tag{2.79}$$

where we have introduced the "dynamically generated scale" $\Lambda = \frac{e^{1/c_3}}{\ell}$.

In CHK, the authors proposed a peculiar set of counterterms to renormalize the on-shell action, which explicitly depend on the radial coordinate (albeit only with logarithmic terms). Therefore we turn to a comparison between the two approaches in the next section.

2.3.2 Comparison with CHK

We have established in the previous section that the counterterms determined with the Hamilton–Jacobi method are sufficient to cancel the divergences order-by-order in perturbation theory, at least up to fourth order. However, the authors of [19] have proposed different counterterms to renormalize the action, which have an explicit dependence on the radial coordinate r. More in detail, they propose the following form[6] for the counterterm action:

$$S_{c.t.} = \int_{\Sigma_r} d^d x \sqrt{-\gamma} \sum_{i=0}^{2} \tilde{u}_i (\alpha - \alpha_0)^i, \tag{2.80}$$

where the $\tilde{u}_i$ are explicit functions of r, namely

$$\tilde{u}_0 = 6 - \frac{8}{3} \frac{1}{(\frac{1}{c_3} - r)^2} + \dots \tag{2.81}$$

$$\tilde{u}_1 = -1 - \frac{8}{3} \frac{1}{(\frac{1}{c_3} - r)} + \dots \tag{2.82}$$

$$\tilde{u}_2 = -\frac{5}{12} + \frac{35}{36} c_3 + \dots \tag{2.83}$$

Here we have shown only the leading departure from the constant piece. The question is whether the explicit r dependence can be replaced with higher order terms in $(\alpha - \alpha_0)$ as we propose. In order to compare with our perturbative analysis, we need to expand these counterterms in powers of c_3. At leading order we have

$$\tilde{u}_0 = 6 - \frac{8}{3} c_3^2 + \dots \tag{2.84}$$

$$\tilde{u}_1 = -1 - \frac{8}{3} c_3 + \dots \tag{2.85}$$

$$\tilde{u}_2 = -\frac{5}{12} + \frac{35}{36} c_3 + \dots \tag{2.86}$$

[6] Here we have changed notation in order to better compare the results of [19] with our results.

so we see that the first discrepancies arise at second order in the sources. In fact, notice that even the constant piece in $\tilde{u}_2$ does not match our results, but rather corresponds to the choice of the normalizable branch for the vector-field mode. It is easy to see that, to second order in the sources,

$$\lim_{r\to 0}\int_{\Sigma_r} d^d x\sqrt{-\gamma}\sum_{i=0}^{2}(u_i-\tilde{u}_i)(\alpha-\alpha_0)^i = 0. \tag{2.87}$$

This means that, at least to this order, the necessity of the explicit r dependence in the CHK counter-term action comes entirely from the choice of the "normalizable" branch for the coefficient of the second order term. Notice that while both choices are compatible with the finiteness of the on-shell action, they lead to different canonical momenta when functionally differentiated, and we have already argued that the correct choice that reproduces the large r behavior of the canonical momenta is the one we adopted.

It is not difficult to see that the third order discrepancy can also be absorbed by adding the appropriate third order counterterm $u_3(\alpha-\alpha_0)^3$ that we determined in the previous section. This strongly suggests that r-dependent counterterms are not really necessary, and can be replaced by local counterterms that do not contain any explicit r dependence. However, the price that we have to pay is that we need an *infinite* number of terms in order to renormalize the action. This means that if we really want to turn on the marginally relevant perturbation, we need to perform a non-perturbative analysis in order to determine whether the on-shell action is finite. This will be the focus of the next section.

2.4 Renormalized On-Shell Action: All-Orders Results

In the previous sections we described a systematic procedure to compute the counterterm action for the Einstein–Proca system around a Lifshitz solution with $z=2$. Furthermore, we analyzed a marginally relevant perturbation of the system and gave evidence that the divergences are correctly removed order by order in the source by local counterterms. However, in order to remove divergences for a finite source, we need to keep an infinite number of terms in the counterterm action. As a consequence we need to study its convergence property and study whether divergences are indeed removed for a finite perturbation from the on-shell action.

2.4.1 Convergence of the counterterm action

At the end of section 2.2.3, we asked whether the series

$$\sum_i u_i(\alpha - \alpha_0)^i, \tag{2.88}$$

which appears in the counterterm action, has a finite radius of convergence. The behavior in Figure 2.1 looks naively linear, suggesting that the u_n's obey a power-law, which would in turn imply a *finite* radius of convergence. However, we will show analytically that the coefficients grow as $n!$ for large n, implying instead a *zero* radius of convergence.

In order to extract the large-n behavior from (2.47), we will assume that for large but fixed n, the sum on the right-hand side is dominated by the terms where u_{n-1} appear. This is certainly true if the coefficients grow sufficiently fast with n, so that terms like $u_{n-1}u_0$ dominate over terms like $u_k u_{n-k-1}$ for $0 < k < n-1$. This assumption is justified by the empirical behavior shown in Figure 2.1.

Keeping only the dominant terms on the right-hand side of (2.47), we get

$$u_n = \frac{1}{8}(n-1)u_{n-1} + \dots, \tag{2.89}$$

which can be immediately solved to give

$$u_n \approx c\left(\frac{1}{8}\right)^{n-1}\Gamma(n) \tag{2.90}$$

We therefore see that the leading behavior of u_n at large n is factorial. The prefactor c cannot be determined by the equation, but should be determined by the "initial condition" u_0. However, since we cannot extrapolate the result above to small n, the only viable way to determine c is by using numerics.

As a further consistency check on our assumptions, we can interpolate the coefficients computed numerically with an ansatz of the form

$$a\, n \log n - b\, n + c \log n + d. \tag{2.91}$$

The least-square fit (performed using the Mathematica function FindFit) on the region of coefficients between $n = 2000$ and $n = 3000$ gives

$$a \approx 1.000, \quad b \approx 3.079, \quad c \approx -10.75, \quad d \approx 23.07 \tag{2.92}$$

compatible with $a = 1$ and $b = 1 - \log\frac{1}{8} \approx 3.079$ implied by (2.90). We see that we can also determine the overall multiplicative constant e^d and the coefficient of the power-law term n^c.

The appearance of a divergent series might be worrisome at first. However, divergent series are ubiquitous in physics, and in particular in quantum field theory, where many observables are represented by power-series in the coupling constant that have zero radius of convergence. In general, the coefficients o_n associated to a power expansion of a certain observable O behave for very large n as [42]

$$o_n \sim K a^n n! n^b. \tag{2.93}$$

where K, a and b are given constants. It is apparent that (2.90) is of this form. Typically these series are believed to be asymptotic, which means that there exist constants K_N such that [42]

$$\left| O(\alpha) - \sum_{n=0}^{N} o_n \alpha^n \right| < K_{N+1} \alpha^{N+1}, \tag{2.94}$$

that is, given a truncation of the series, this approaches the exact result as $\alpha \to 0$ "sufficiently fast". In this case, it is often possible to argue that the truncation error as a function of N decreases until a critical N_*, given by

$$N_* \sim \frac{1}{|a|\alpha}. \tag{2.95}$$

Beyond this point, the truncation error starts to increase and the truncated series becomes less and less reliable.

These results suggest that if our series is asymptotic, the optimal truncation is obtained at

$$N_* \sim \frac{8}{\alpha - \alpha_0}. \tag{2.96}$$

In order to test this, it is interesting to consider what happens when $\alpha - \alpha_0 = 1$, or equivalently $\alpha = 0$. This corresponds to the AdS_4 solution of the system, and in that case we know what the counterterm action should be:

$$S_{c.t.}^{(\mathrm{AdS})} = \int d^3x \sqrt{-\gamma} \frac{4}{L_{\mathrm{AdS}}}. \tag{2.97}$$

It is very easy to see that, in our conventions, $L_{\mathrm{AdS}} = \sqrt{\frac{3}{5}}$, so that the "boundary cosmological constant" is

$$\frac{4}{L_{AdS}} = 4\sqrt{\frac{5}{3}} \approx 5.16... \tag{2.98}$$

On the other hand, using the fact that $N_* \sim 8$, we get

$$U(\alpha = 0) \approx \sum_{n=0}^{8} u_n \approx 5.16... \tag{2.99}$$

in perfect agreement with (2.98). Furthermore, we can see that the truncated series behaves as expected by plotting the function

$$\log(1 + \Delta(N)), \tag{2.100}$$

where $\Delta(N)$ measures the truncation error and is given by

$$\Delta(N) = \left| \sum_{n=0}^{N} u_n - 4\sqrt{\frac{5}{3}} \right|. \tag{2.101}$$

The result is shown in figure 2.2, and confirms our expectations that the series is asymptotic. The non analyticity of the counterterm series around $\alpha = \alpha_0$ suggests that something goes wrong in the vicinity of this point. We will see that something similar happens for the renormalized on-shell action in the next section, where we will also try to give this phenomenon a physical interpretation in the spirit of an argument first provided by Dyson in the context of QED.

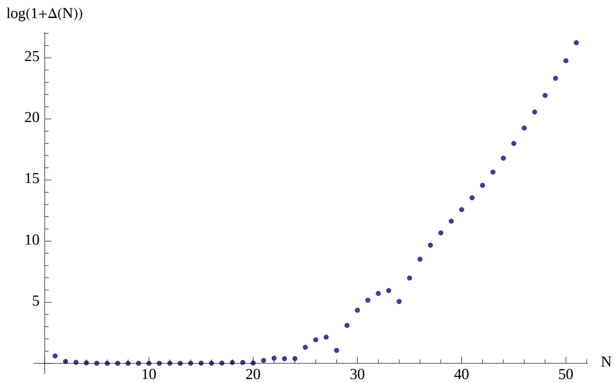


Figure 2.2: *This plot shows the logarithm of the truncation error when we include N terms in the sum. It is apparent that after $N \sim 10$, the error becomes exponentially big as expected on general grounds.*

2.4.2 The renormalized on-shell action

We turn our attention to the renormalized on-shell action $\Gamma = S - S_{loc}$. We argued in section 2.2 that this object obeys the functional differential equation

$$2\{S_{loc}, \Gamma\} + \{\Gamma, \Gamma\} = 0. \tag{2.102}$$

Since we expect Γ to be finite in the limit $r \to \infty$, it is not difficult to see from (2.27) that the second piece $\{\Gamma, \Gamma\}$ should be subdominant with respect to the first in this limit. As a consequence, we will extract the leading contribution to Γ by studying the equation

$$\{S_{loc}, \Gamma\} = 0, \tag{2.103}$$

and then argue that the corrections from the $\{\Gamma, \Gamma\}$ piece can be neglected.

Since we are turning on a perturbation that is constant on a slice, Γ will be necessarily of the form

$$\Gamma = \int d^3x \sqrt{-\gamma} W(\alpha). \tag{2.104}$$

After some easy manipulations, it is possible to cast (2.103) in the form

$$\beta(\alpha) W'(\alpha) - \phi(\alpha) W(\alpha) = 0. \tag{2.105}$$

The functions β and ϕ are given by

$$\beta(\alpha) = -\frac{1}{2}\alpha U(\alpha) - (\alpha^2 + 4\alpha) U'(\alpha), \tag{2.106}$$

$$\phi(\alpha) = -\frac{3}{4} U(\alpha) + \frac{1}{2}\alpha U'(\alpha). \tag{2.107}$$

The notation comes from the observation that

$$\partial_r \alpha = \beta(\alpha), \tag{2.108}$$

$$\partial_r \log(\sqrt{-\gamma}) = \phi(\alpha), \tag{2.109}$$

so that β and ϕ play the role of β-functions for the "effective coupling constants" α and $\log(\sqrt{-\gamma})$ respectively.

We have reduced equation (2.103) to a ordinary first order differential equation for W, which can be immediately solved by

$$W(\alpha) = C \exp\left(\int d\alpha \frac{\phi(\alpha)}{\beta(\alpha)}\right), \tag{2.110}$$

where C is an arbitrary constant coming from the constant additive ambiguity in defining the integral. Using the results of the previous sections for $U(\alpha)$, it is easy to see that

$$\frac{\phi(\alpha)}{\beta(\alpha)} = \frac{8}{(\alpha - \alpha_0)^2} + \frac{10}{\alpha - \alpha_0} + \ldots, \tag{2.111}$$

where the ellipses stand for positive (or zero) powers of $\alpha - \alpha_0$. Integrating the expression above immediately gives us

$$W(\alpha) = C \exp\left(-\frac{8}{\alpha - \alpha_0}\right) (\alpha - \alpha_0)^{10} \left(1 + O(\alpha - \alpha_0)\right) \tag{2.112}$$

This expression is non analytic around $\alpha = \alpha_0$, confirming our suspicion that the non-analyticity of the counterterm action would show up also in the renormalized on-shell action.

Furthermore, we are in a position to check the finiteness of Γ. Using the asymptotic expansion of [19]

$$\alpha - \alpha_0 \approx -\frac{2}{\log(\Lambda z)} - \frac{(\lambda - 3) - 5\log(-\log(\Lambda z))}{\log^2(\Lambda z)} + \dots \tag{2.113}$$

we immediately find

$$\Gamma = \int d^3x\sqrt{-\gamma}W(\alpha) \to \tilde{C}\,(\ell\Lambda)^4, \tag{2.114}$$

where $\tilde{C}$ is a constant proportional to C that cannot be determined by the near-boundary analysis. Since the result is finite, subleading terms in $W(\alpha)$ will not contribute to the result. In fact, it is not difficult to show that terms like $\{\Gamma, \Gamma\}$ contribute to W with higher powers of $\exp(-8/(\alpha - \alpha_0))$, so they do not affect the analysis above. This has to be contrasted with the counterterm action, where all the subleading terms in $U(\alpha)$ contribute to the divergences and only a perturbative analysis was possible.

We have therefore determined that the non-convergent power-series of counterterms determined by the Hamilton–Jacobi method is sufficient to remove the divergences from the on-shell action. A determination of $\tilde{C}$ as a function of the IR data and Λ would allow us to study the free energy, energy and entropy of asymptotically Lifshitz black hole solutions in the spirit of [19]. However, such an analysis would not be straightforward, since the $\{\Gamma, \Gamma\}$ contributions that are negligible for large r definitely become important in the interior, and we would need to keep track of them to determine the aforementioned relation between UV and IR data; as a consequence we leave this for future work.

2.4.3 Improved perturbation theory

We have shown that both the renormalized on-shell action and the counterterm action are non-analytic functions of $\alpha - \alpha_0$ around 0. In quantum field theory, the non-analyticity of observables as a function of the coupling constant is related to interesting non-perturbative effects. In order to get some intuition on the possible physical reasons that lead to the divergences that we found in our example, we briefly turn our attention to QED.

The first hint that the perturbation series of QED had to be divergent for any non-zero value of the coupling constant came from a paper of Dyson [39]. The argument goes as follows: a physical observable O can be written in perturbation theory as

$$O(\alpha) = o_0 + o_1\alpha + o_2\alpha^2 + \dots \tag{2.115}$$

where the coupling constant $\alpha \propto e^2$ is simply the electric charge squared. If the series was convergent for a (positive) non-zero value of α, then O would be analytic around $\alpha = 0$ and could be continued to negative values of α. In this case, charges of the same sign would attract each other and the classical limit would lead to a Coulomb potential with the "wrong" sign. That would imply that the ground state is unstable against the creation of particles, so observables are expected to be ill defined.

In our case, the role of the coupling constant is played by $g^2 \sim \alpha - \alpha_0$. It is clear that the renormalized on-shell action is well-defined only if we approach the UV limit from positive g^2. This is guaranteed by the β-function:

$$\partial_r(\alpha - \alpha_0) = \beta(\alpha) \approx -\frac{1}{2}(\alpha - \alpha_0)^2 + \dots \tag{2.116}$$

The expression above implies that the r-derivative of the coupling constant must be negative as we approach 0, which is only possible if the coupling constant itself is positive for sufficiently large r. This provides evidence that the non-analyticity of the on-shell action can be explained field-theoretically by Dyson's argument.

The β-function also explains why we were able to resum all the leading logarithms in the perturbative expansion. This is completely analogous to what happens in QCD, where the leading logarithmic terms that one finds at each order in perturbation theory can be resummed to give inverse logarithms [43].

2.5 Discussion and Conclusions

We have found a new and systematic method for simultaneously determining the boundary conditions on the one hand, and finding the counterterm action for asymptotically Lifshitz spacetimes on the other hand.

Using this method, we have have analyzed a marginally relevant perturbation of Lifshitz spacetime, both perturbatively and non perturbatively, and determined that it should be included in the class of allowed boundary conditions. Furthermore, the counterterm action is a local functional of the fields by construction, and in particular it is independent of the radial cut-off, unlike some previous approaches [19, 20].

Although we focused on constant perturbations and $z = 2$, the Hamilton–Jacobi analysis can be used to find higher-derivative counterterms for general z as well. See Appendix 2.A for a review of the more general results derived in [1].

There are many further directions to explore. The first would be to extend our analysis to the interior of Lifshitz spacetime, which would provide the relation between boundary data (sources) and vevs. This would allow us to study asymptotically Lifshitz black holes in the spirit of [19], which in turn would enable us to examine finite-temperature crossovers and explore the non-relativistic field theory phase diagram in the vicinity of the quantum critical point.

It would also be interesting to study how observables other than the free energy, such as correlation functions, are modified by this marginally relevant deformation. Furthermore, the UV Lifshitz fixed point provides a UV completion for a relativistic CFT deformed by an irrelevant deformation, and it would be interesting to study how the non-local divergences that are expected in this case [44] resum to local quantities from the point of view of the UV theory.

2.A Results for general z and higher derivative counterterms

In this appendix, we briefly review the more general results of [1] for arbitrary z, and we discuss higher-derivative counterterms.

2.A.1 General z and scalar field coupling

It is possible to repeat the analysis presented in this chapter for general z. At the non-derivative level, there is no obstruction to solving the Hamilton–Jacobi equation perturbatively in $\alpha - \alpha_0$. The only difference is that the coefficients u_n are functions of z:

$$S_{loc} = \int d^dx\sqrt{-\gamma}\left(2(z+1) - \frac{z}{2}(\alpha - \alpha_0) + \frac{z^2(2z-5+\beta_z)}{16(z^2-1)}(\alpha-\alpha_0)^2 + \dots\right), \tag{2.117}$$

where $\beta_z = \sqrt{9z^2 - 20z + 20}$ and the ellipses denote higher-order derivatives as well as terms that are of higher order in $\alpha - \alpha_0$.

The other important difference with respect to the $z = 2$ case is that the mode associated to the massive vector field is relevant for $z < 2$, so only a *finite* number of counterterms is necessary. In fact, the perturbative analysis can be repeated in this case and it shows that divergences are indeed removed to third order in the perturbations.

Counterterms for a scalar field coupled to the Einstein–Proca model were also constructed in [1]. This amounts to adding a term to the classical action of the form

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right), \tag{2.118}$$

where $V(\phi) = \frac{1}{2}\mu^2\phi^2 + O(\phi^3)$. In this case, an interesting phenomenon arises for specific values of the scalar field mass μ^2: we find that there is an obstruction to solving the local part of the Hamiltonian constraint, which leaves a finite reminder. This leads to logarithmic divergences in the renormalized on-shell action, which in turn can be related to anomalies [2]. This will be the main topic of the next chapter.

It should be noted that the energy flux is an irrelevant operator for any $z > 1$, and the vector field mode is irrelevant for $z > 2$. As a consequence, we expect multi-trace deformations, implemented by non-local counterterms, to play a role in their holographic renormalization [44].

2.A.2 Higher derivative counterterms

Derivative counterterms can also be analyzed with the Hamilton–Jacobi technique, even though computations become a little bit more tedious. We start by writing down an ansatz for the local part of the on-shell action:

$$\mathcal{L}_{loc} = U(\alpha) + \mathcal{C}(\alpha)D_aA^a + \mathcal{D}(\alpha)A^aA^bD_aA_b + \Phi(\alpha)R + \dots \tag{2.119}$$

Of course, there are other two-derivative terms as well as higher-derivative terms in the ansatz, but for the purpose of illustrating our method these terms will suffice. We assume that $\partial\Sigma_r = \emptyset$, so that we need to specify the possible counterterms only up to total derivatives. We perform a derivative expansion,

$$\mathcal{L}^{(0)}_{loc} = U(\alpha), \tag{2.120}$$
$$\mathcal{L}^{(1)}_{loc} = \mathcal{C}(\alpha)D_aA^a + \mathcal{D}(\alpha)A^aA^bD_aA_b, \tag{2.121}$$
$$\mathcal{L}^{(2)}_{loc} = \Phi(\alpha)R + \dots \tag{2.122}$$

and

$$\mathcal{L}^{(0)} = -2\Lambda - \frac{m^2}{2}\alpha, \tag{2.123}$$
$$\mathcal{L}^{(2)} = R - \frac{1}{4}F_{ab}F^{ab}. \tag{2.124}$$

The non-derivative level (level zero) has already been covered in Section 2.2, so let us go directly to the level of one spacetime derivative.

One derivative.

At level one we have only two possible structures, The canonical momenta are given by

$$\begin{aligned}\pi^{(1)ab} &= \frac{1}{\sqrt{-\gamma}}\frac{S^{(1)}_{loc}}{\gamma_{ab}} \\ &= \left(\frac{1}{2}\mathcal{D} - \mathcal{C}'\right)\left(A^aA^b(D\cdot A) - 2A^cA^{(a}D^{b)}A_c + \gamma^{ab}(A^cA^dD_cA_d)\right),\end{aligned} \tag{2.125}$$

and

$$E^{(1)a} = (2\mathcal{C}' - \mathcal{D})\left(A^a(D\cdot A) - A_bD^aA^b\right). \tag{2.126}$$

The Hamilton constraint can be solved if

$$\mathcal{D} = 2\mathcal{C}'. \tag{2.127}$$

The resulting term in $\mathcal{L}^{(1)}_{loc}$ is just a total derivative,

$$\mathcal{C}D_a A^a + 2\mathcal{C}' A^a A^b D_a A_b = D_a(\mathcal{C}A^a), \tag{2.128}$$

and can be discarded.

Two derivatives: the ΦR term.

Since the ΦR term does not mix with the other two-derivative terms, the equations that determine $\Phi(\alpha)$ do not get contributions from other two derivative terms. Therefore, we can consistently focus on this sector alone. The canonical momenta are

$$\pi^{(2)ab} = \frac{1}{2}\gamma^{ab}\mathcal{L}^{(2)}_{loc} - \frac{\delta(\Phi R)}{\delta\gamma_{ab}} + \dots \tag{2.129}$$

$$E^{(2)a} = 2\Phi' R A^a + \dots \tag{2.130}$$

We now want to compute the coefficient of the R term. Only terms with R or a not contracted $R_{\mu\nu}$ can produce a R term in the final expression:

$$\pi^{(2)}_{ab} = \frac{1}{2}g_{ab}\Phi R - \Phi' A_a A_b R - R_{ab}\Phi + \dots, \tag{2.131}$$

$$E^{(2)a} = 2\Phi' R A^a + \dots \tag{2.132}$$

Therefore we have

$$\begin{aligned} &2\{S^{(0)}_{loc}, S^{(2)}_{loc}\} - \mathcal{L}^{(2)} = \\ &\quad R\left(-\frac{1}{4}\Phi U + \frac{1}{2}A^2(\Phi' U - \Phi U') + (4A^2 + A^4)\Phi' U' + 1\right) + \dots = 0. \end{aligned} \tag{2.133}$$

Again, we expand Φ in power series in $(\alpha - \alpha_0)$ where $\alpha = A^2$,

$$\Phi = b_0 + b_1(\alpha - \alpha_0) + b_2(\alpha - \alpha_0)^2 + \dots, \tag{2.134}$$

and we plug this result into (2.133). We obtain

$$b_0 = \frac{1}{z}. \tag{2.135}$$

A similar computation for b_1 yields

$$b_1 = \frac{5z - 2 + \beta_z}{4(z+1)(z - 2 + \beta_z)}. \tag{2.136}$$

There does not seem to be a continuous ambiguity for the higher order coefficients.

Let us briefly discuss an important feature of (2.133). The function Φ satisfies a first-order differential equation, therefore it seems somewhat strange that we were able to determine the coefficient b_0 uniquely, which amounts to specifying the initial condition. The reason for this is that, since we want to compute the polynomial part of the on-shell action, we are using a power-series expansion. Nevertheless, the general solution of the differential equation might not be polynomial, so by requiring that our solution is a polynomial, we are effectively determining the initial condition. We illustrate this phenomenon with a toy example. Consider the differential equation:

$$xf'(x) + af(x) + 1 = 0. \tag{2.137}$$

If $a \neq 0$ this has the following general solution:

$$-\frac{1}{a} + Ax^{-a}, \tag{2.138}$$

where A is an arbitrary constant. If $a \neq 0, -1, -2, \ldots$, then the solution is not polynomial and using a Taylor expansion amounts to choosing $A = 0$. Nevertheless, if the coefficient a is a negative integer, the solution is indeed a polynomial but A is undetermined. This would show up as an ambiguity in defining the on-shell action, and it typically leads to anomalies as we will explain in the following chapter.

Equation (2.133) can be cast in a form similar to the toy model we just considered:

$$\left((\alpha^2 + 4\alpha)U' + \frac{1}{2}\alpha U\right)\Phi' + \left(\frac{1}{2}\alpha U' - \frac{1}{4}U\right)\Phi + 1 = 0. \tag{2.139}$$

The coefficient of Φ' is simply $-\partial_r \alpha$, and since Lifshitz spacetime is a solution to the equations of motion, we have that this coefficient vanishes as $\alpha \to \alpha_0$. This feature is very general and it explains why the HJ method is able to fix the derivative counterterms.

Chapter 3

Lifshitz Anomaly

In this chapter we continue our study of Lifshitz models with anisotropic scaling symmetry. We now consider a different type of deformation, that is we change the geometry on which the theory lives. Studying the theory on a non-trivial manifold is in principle a daunting task, but here we will focus our attention on a specific problem that turns out to be tractable, namely we ask ourselves to what extent anisotropic scale invariance is broken.

We have already hinted in the previous chapter that the non-relativistic analog of the Weyl anomaly, derived in the context of ordinary AdS/CFT in [28], might be present for 3+1 dimensional Lifshitz spacetimes with dynamical exponent $z = 2$. Conformal anomalies play an important role in relativistic field theories, especially in two dimensions, where various physical quantities display a universal behavior that is governed by the central charge only, such as the Casimir energy or the logarithmic contribution to the entanglement entropy. As we illustrated in the previous chapter, many interesting physical systems, such as smectic liquid crystals or cold atoms at unitarity, exhibit non-relativistic scaling symmetry. As a consequence, if similar results carried over to non-relativistic field theories, they could be used as a guiding principle to construct (or rule out) bottom-up holographic models for such systems. In this chapter we therefore take the first steps in this direction, showing that $z = 2$ Lifshitz field theories in 2+1 dimensions do indeed lead to anomalies.

The purpose of this chapter is manifold. In section 3.1, we give a complete characterization of the possible structures that can appear in the non-relativistic scaling anomaly. We describe how this quantity can be extracted in field theory, via the heat-kernel method, and in holography, using the Hamilton–Jacobi approach.

In section 3.2 we revisit the quantum Lifshitz model introduced in the previous chapter; we first couple it to a non-trivial background and we proceed to determine the anomaly by computing various coefficients in the heat-kernel expansion. In section 3.3 we describe the computation of the anomaly for strongly coupled non-relativistic theories that admit bulk duals described by gravity coupled to a massive vector field.

The main result of this chapter is that, while there are two structures that can in principle appear in the anomaly, the two models we consider here only contribute to structure containing time derivatives. In the final section we will offer some possible interpretations of this result.

3.1 The Anisotropic Scaling Anomaly

In this section we introduce the concept of anisotropic scaling anomaly, which we will sometimes refer to as the "Lifshitz anomaly". We first describe general concepts valid for any theory with non-relativistic scaling symmetry. We then illustrate how this anomaly can be computed in the standard path integral quantization of quantum field theory using the heat-kernel method. Finally, we describe how the anomaly shows up in the Hamilton–Jacobi approach to holographic renormalization described in the previous chapter.

3.1.1 Generalities

As we explained in the previous chapter, theories with Lifshitz scaling symmetry are characterized by the dynamical critical exponent z that appears in the transformation rule (2.2). The theories we are concerned with in this chapter will couple to a non-trivial background metric, so it is useful to describe these transformations in terms of an auxiliary three-dimensional metric

$$ds^2 = N^2 dt^2 + h_{ij} dx^i dx^j. \tag{3.1}$$

As we discussed previously, non-relativistic theories have a notion of time, so our metric above exhibits a preferred time foliation.[1] In fact, the metric (3.1) keeps its form under diffeomorphisms in time, $t \mapsto \tau(t)$, and in space, $x^i \mapsto \xi^i(\vec{x})$. The anisotropic scale transformation is implemented in this language by the following transformation rules for the metric

$$N \to e^{z\omega} N, \qquad h_{ij} \to e^{2\omega} h_{ij}. \tag{3.2}$$

[1] We do not include a shift N^i , because it can be removed locally by a foliation preserving diffeomorphism, e.g. $t \mapsto \tau(t)$ and $x^i \mapsto \xi^i(t, \vec{x})$.

If we allow ω to be an arbitrary function of t and x^i, the transformation above describes a *local* anisotropic scale transformation.

We now consider a general classical field theory, which is coupled to the metric (3.1). We can achieve this for example by replacing spatial derivatives with covariant derivatives with respect to the spatial metric h_{ij} and analogously for time derivatives. However, there might be different, non minimal ways to do so, and we will discuss one such possibility in the conclusions section. In any case, if we perform an *infinitesimal* rescaling (3.2), where $\omega = \delta\rho$ is an infinitesimal quantity (so that we keep only terms that are linear in $\delta\rho$), the classical action S transforms as

$$\delta S = \int dt d^2x \left(\delta N \frac{\delta S}{\delta N} + \delta h_{ij} \frac{\delta S}{\delta h_{ij}} \right) = \int dt d^2x \, \delta\rho \left(z N \frac{\delta S}{\delta N} + 2 h_{ij} \frac{\delta S}{\delta h_{ij}} \right) . \quad (3.3)$$

If $\delta S = 0$, then the theory is invariant under local anisotropic scale transformations at the classical level. This condition has the following physical interpretation: if we define the energy density $\mathcal{E}$ and momentum flux (spatial stress tensor) Π_{ij} as

$$\mathcal{E} = \frac{2}{N\sqrt{h}} N^2 \frac{\delta S}{\delta N^2}, \qquad \Pi^{ij} = \frac{2}{N\sqrt{h}} \frac{\delta S}{\delta h_{ij}}, \quad (3.4)$$

we see that local anisotropic scale invariance implies

$$z\mathcal{E} + \Pi^i_i = 0, \quad (3.5)$$

which is the non-relativistic analog of the tracelessness condition $T^a{}_a = 0$.

So far our discussion has been classical. At the quantum level, the classical action S is replaced by a quantum effective action, which we call W. Since the metric (3.1) is an external background field, that is we do not path-integrate over it, W will be a functional of this field $W[N, h_{ij}]$. Therefore it still makes sense to consider the variation δW under (3.2), which will still given by (3.3); the only difference is that we now interpret the variation with respect to N and h_{ij} as the quantum expectation values of the energy density $\langle \mathcal{E} \rangle$ and the spatial stress tensor $\langle \Pi^{ij} \rangle$ respectively, so that we have

$$\delta W = \int dt d^2x \, N\sqrt{h} \, \delta\rho \left(z \langle \mathcal{E} \rangle + \langle \Pi^i_i \rangle \right) . \quad (3.6)$$

Even when $\delta S = 0$, this does *not* imply that $\delta W = 0$; in fact, in the process of quantizing the theory, one often encounters divergences that need to be renormalized. As discussed in chapter 1, such a procedure typically involves the introduction of a scale in the problem, and it is not guaranteed that scale-independent quantities at the classical level will remain so after the renormalization procedure has been carried out. As we will see both in field theory and in holography, the right-hand side of (3.6) can be non-zero, and we have

$$2\langle\mathcal{E}\rangle + \langle\Pi^i_i\rangle = \mathcal{A} \tag{3.7}$$

where $\mathcal{A}$ is by definition the *anomaly*. In principle, the anomaly depends on the renormalization scheme, for example it is affected by local counterterms that can be added to the classical action. However, it cannot be completely removed by such counterterms, and as we will see a "part of it" is indeed renormalization-scheme independent.

Contrary to the relativistic case, where anomalies are present only for even dimension, in the non-relativistic setting anomalies can also be generated in odd dimension. That this is in principle possible can be seen quite easily by dimensional analysis. The analogue of the trace of the stress-tensor for non-relativistic field theories has dimension $z + d - 1$. A term with a time derivatives and b spatial derivatives, on the other hand, has dimension $az + b$. For generic z, there can only be contributions to the conformal anomaly with $a = 1$ and $b = d - 1$. However, such terms have an odd number of time derivatives, so they break time reversal invariance; such an anomaly may appear only in theories that break this symmetry. The theories we consider in this chapter are time-reversal invariant, so they will not be anomalous for generic z. However there are special values of z for which other values of a, b are allowed: for example, if $d = z + 1$, terms with either $(a, b) = (2, 0)$ or $(a, b) = (0, 2z)$ can appear.

In the remaining part of this chapter we will consider theories with $z = 2$, $d = 3$. In this case the argument above shows that in principle we can have terms with either two time or four spatial derivatives. While there are many such terms that one can write down, the more detailed analysis that we give below shows that the anomaly is generated by a total of two linearly independent and non-trivial structures.

3.1.2 Analysis of the possible terms

Just by dimensional analysis (i.e. by requiring invariance under constant rescalings) we can see that there are many terms of the right dimension that can appear in the anomaly. When $z = 2$, ∂_t has dimension two and ∂_i has dimension one, and we are interested in terms of dimension four that are covariant under reparametrizations of t and reparametrizations of x^i. These reparametrizations should not mix t and x^i since as we explained that would ruin the form of D.

Examples of terms of the right dimension are[2]

$$
\begin{aligned}
&N^{-2}\partial_t h_{ij} G^{ijkl} \partial_t h_{kl}, \\
&R^2, \qquad N^{-1}\Delta N\, R, \qquad (N^{-1}\Delta N)^2, \\
&h^{ij}h^{kl}(N^{-1}\partial_i N)(N^{-1}\partial_k N)(N^{-1}\nabla_j \partial_l N). \\
&\ldots
\end{aligned}
\tag{3.8}
$$

where $G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}$ is the DeWitt metric and λ an arbitrary real number (in General Relativity $\lambda = \frac{1}{D-2}$, where D is the spacetime dimension). Terms with one time derivative and two space derivatives cannot appear because of time reversal symmetry. All these terms are scale invariant, but the anomaly should also obey the Wess–Zumino consistency condition. This comes about from the observation that the anomaly is obtained by functionally differentiating the functional W. The requirement that functional derivatives commute then imposes some constraints on the anomaly. Concretely, if we denote the variation with parameter ω by δ_ω, we have

$$
\delta_{\omega_1}\delta_{\omega_2} W = \delta_{\omega_2}\delta_{\omega_1} W, \tag{3.9}
$$

which in terms of the anomaly reads

$$
\delta_{\omega_1} \int dt d^2x\, N\sqrt{h}\,\omega_2\,\mathcal{A} = \delta_{\omega_2} \int dt d^2x\, N\sqrt{h}\,\omega_1\,\mathcal{A}. \tag{3.10}
$$

This condition imposes strong restrictions on the possible terms that can appear in $\mathcal{A}$. The computation is straightforward: one writes down all the terms of the right dimension and determines which combinations solve the condition above by explicitly taking functional variations. This is done explicitly in appendix 3.A.1, and here we only report the final result:[3] the anomaly is given by the following

[2] A complete classification can be found in Appendix 3.A.1.

[3] As shown in [45], the problem of finding Wess–Zumino consistent structures can be phrased in terms of cohomology, very much like the relativistic case: one defines a nilpotent operator s acting on local functionals that implements local scale transformations. In this language, the Wess–Zumino condition turns out to be equivalent to the requirement that the anomaly is s-closed (that is annihilated by s). Trivial terms that can be removed by local counterterms turn out to be s-exact (that is they are of the form sQ for some local functional Q), so that the non-trivial structures are computed by the cohomology of the complex defined by s. It should be noted that the analysis of [45] appeared earlier than the one in [2], which is presented here, because the latter preprint contained a mistake that was corrected in the published version.

two independent non-trivial structures

$$\begin{aligned}\mathcal{A} \;=\; & C_1 \frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right) \\ & + C_2 \left(R + \frac{1}{N}\Delta N - h^{ij}\left(\frac{1}{N}\partial_i N\right)\left(\frac{1}{N}\partial_j N\right)\right)^2 \\ & + (\text{trivial total derivatives}).\end{aligned} \tag{3.11}$$

Here the trivial total derivatives can be canceled by appropriate local counterterms. The values of C_1 and C_2 are model dependent, and will be computed in two different models later in the chapter.

3.1.3 Heat-kernel expansion

In this section we will employ the notation and results of [46]. We consider the quantization of a general Lifshitz field theory. If the action is quadratic in the canonical momenta, a path-integral quantization can be carried out without additional complications, so that the effective action W is simply given by

$$e^{-W} = \int \mathcal{D}\phi\, e^{-S}, \tag{3.12}$$

where ϕ is a shorthand notation for all the dynamical fields that are present in S. Even though the action S is classically scale invariant, the functional measure $\mathcal{D}\phi$ might not be so. While determining this anomalous behavior might seem a very complicated task in principle, it turns out that a one-loop computation is sufficient (see [46] and references therein). As a consequence, we only consider quadratic terms in the classical action

$$S = \int d^3x N\sqrt{h}\,\phi\, D\,\phi, \tag{3.13}$$

where D is a self-adjoint operator with respect to some scalar product (implied by the expression above) and with some suitable domain of definition. Here we are not considering sources for our fields ϕ, but it is not difficult to show that their presence would not affect the analysis below. The integral is Gaussian and can be computed explicitly. It is in fact given by the formal expression

$$W = \frac{1}{2}\log\det(D), \tag{3.14}$$

where $\det(D)$ is the determinant of the operator D appearing in equation (3.13). The expression above is only formal because the operator acts on an-infinite dimensional space, so the product over the eigenvalues need not converge.[4] In order to give meaning to the determinant, we employ ζ-function regularization. We will illustrate the idea on a toy example and then extend it to more general operators.

A toy example

Consider an operator A whose spectrum is $\lambda_n = n$, $n = 1, 2, \ldots$. The "determinant" of this operator, following the intuition that comes from the finite-dimensional case, should be given by the product of the eigenvalues

$$\det(A) = \prod n, \tag{3.15}$$

which is however divergent. In order to give meaning to the previous expression, we first transform the infinite product into an infinite sum by taking the logarithm. This corresponds to taking the trace of $\log A$, that is

$$\log\det(A) = \mathrm{Tr}(\log A) = \sum \log n. \tag{3.16}$$

The trace is divergent and consequently the operator $\log A$ is not trace-class. The idea is to introduce a family of operators parametrized by a complex parameter s and whose trace is well-defined at least in a region of the complex $s-$plane. We then *define* the trace of the original operator by analytically continuing the answer and then taking appropriate limits. Concretely, we consider the operators A^{-s}, which are well-defined due to the self-adjointness of A [47]. If we take a sufficiently large positive s (in our case $\mathrm{Re}(s) > 1$), the trace is well-defined and is given by

$$\mathrm{Tr}(D^{-s}) = \sum n^{-s} = \zeta(s), \tag{3.17}$$

where $\zeta(s)$ is the usual Riemann zeta function. The series $\sum n^{-s}$ converges only when $\mathrm{Re}(s) > 1$, but it defines an analytic function of s that can be analytically continued to all complex values of $s \neq 1$. If we consider $\zeta'(s)$, and we use the series $\sum n^{-s}$ even when $\mathrm{Re}(s) < 1$, we immediately see that

$$\zeta'(0) \text{``}=\text{''} -\sum \log n, \tag{3.18}$$

where the equality is only formal because we are not allowed to use the sum representation when $\mathrm{Re}(s) \leq 1$. Since the left-hand side is well-defined and finite, and it formally reproduces the series we would like to give meaning to, it is natural to *define* our functional determinant as

$$\log\det(A) = -\zeta'(0) = \log\sqrt{2\pi}. \tag{3.19}$$

[4] We are using the term "eigenvalue" a bit loosely, since the operator does not necessarily have a discrete spectrum. However, the precise nature of the spectrum will not be important in what follows.

At this point, the reader might be a little worried about the uniqueness of the previous result. It is in fact *not* unique, as we could have used for example many different inequivalent analytic continuations. However, quantum field theories are more constrained than our toy example, in that the regularization procedure should respect the various symmetries of the system, such as reparametrization invariance and gauge symmetry. Furthermore, divergences should be removed by *local* counterterms. Respecting all these requirements at once puts strong restrictions on the possible regularization prescriptions that can be employed, and $\zeta-$function regularization certainly succeeds in this respect. Ultimately we are interested in computing the non-relativistic anomaly and, as shown before, different renormalization schemes related by finite local counterterms can change only its trivial part. As a consequence, $\zeta-$function regularization appears entirely adequate for the purpose of computing the anomaly.

The general construction

We extend the construction above to general operators by defining the generalized $\zeta-$function as

$$\zeta(s, f, D) = \mathrm{Tr}_{L^2}(f D^{-s}), \tag{3.20}$$

where s is an arbitrary positive number and L^2 an appropriate function space on which D^{-s} is trace-class, at least for sufficiently large $\mathrm{Re}(s)$. The regularized effective action is given by [46]

$$W = -\frac{1}{2}\zeta'(0, 1, D) - \frac{1}{2}\log(\mu^2)\zeta(0, 1, D), \tag{3.21}$$

where $\zeta'(0, f, D) = \partial_s \zeta(s, f, D)|_{s=0}$ and the term proportional to $\log(\mu^2)$ can be shown to be local and is related to a renormalization scheme ambiguity. The effective action as defined above can be shown to be finite, but it is still very difficult to compute. However, we are only interested in how this object transforms under anisotropic rescalings, and as we will see this piece of information is encoded in local quantities that can be computed exactly.

The first step is to define the so-called heat kernel

$$K(\epsilon, f, D) = \mathrm{Tr}_{L^2}(f\, e^{-\epsilon D}), \tag{3.22}$$

where f is an arbitrary function of t and x^i, and ϵ is an arbitrary positive parameter. It is not difficult to show that the zeta function $\zeta(s, f, D)$ is related to the object defined above by a Mellin transformation:

$$\zeta(s, f, D) \;=\; \Gamma(s)^{-1} \int_0^\infty d\epsilon\, \epsilon^{s-1}\, K(\epsilon, f, D), \tag{3.23}$$

$$K(\epsilon, f, D) \;=\; \frac{1}{2\pi i} \oint ds\, \epsilon^{-s}\, \Gamma(s)\, \zeta(s, f, D). \tag{3.24}$$

In principle K depends on the global behavior of the operator D (the trace can be written as a sum over the spectrum of the operator, which is determined by global properties); however there is an asymptotic series of the form

$$K(\epsilon, f, D) \sim \sum_{k=0}^{\infty} \epsilon^{\frac{k}{2}-1} \tilde{a}_k(f, D), \tag{3.25}$$

where $\tilde{a}_k(f, D)$ can be computed *locally* from N and h_{ij}. By repeating the analysis of [46] section 7.1, it is possible to show that the variation of the renormalized effective action under an infinitesimal anisotropic local scale transformation $h \to (1 + 2\delta\rho)h$, $N \to (1 + 2\delta\rho)N$, is given by [5]

$$\delta W = -2\, \tilde{a}_2(\delta\rho, D). \tag{3.26}$$

As explained above, this will be a local functional of N and h; we will therefore write

$$\tilde{a}_2(f, D) = \int dt d^2x N \sqrt{h} f\, a_2(N, h_{ij}), \tag{3.27}$$

where $a_2(N, h_{ij})$ is a local function that depends on N and h_{ij}.

Using the previous expression in (3.26) and comparing with (3.6), we immediately see that

$$\mathcal{A} = -2\, a_2(N, h_{ij}). \tag{3.28}$$

Therefore in order to compute the anomalous transformation of W under local anisotropic scale transformations, we need to extract the coefficient of order ϵ^0 in the heat-kernel expansion of the operator D, that is a_2.

3.1.4 Hamilton–Jacobi analysis

We explained in the previous section how one can compute the anomaly in the field theory side using the path-integral formulation. In this section we describe how this same anomaly can be computed holographically in the Hamilton–Jacobi framework.

Recall from section 2.2 that we want to extract the divergent part S_{loc} of the (bare) on-shell action by solving the functional differential equation

$$\{S_{loc}, S_{loc}\} - \mathcal{L} = 0, \tag{3.29}$$

where the notation is as in chapter 2. However, it might happen that there is no solution to the problem above compatible with the requirement that S_{loc} is indeed

[5] The factor 2 comes from the factor 4 in $D \to e^{-4\rho} D$ under scale transformations.

local. However, recall that our purpose is to remove divergences in the on-shell action; as a consequence, we will impose the weaker condition that at least the "divergent" terms in (3.29) be canceled. A term F is called divergent if

$$\lim_{r\to\infty} \sqrt{\gamma}\, F = +\infty. \tag{3.30}$$

It might still happen (and it does happen as we will see) that we get a finite remainder $\mathcal{H}_{\mathrm{rem}}$ that cannot be removed by local counterterms. As a consequence, the Hamilton–Jacobi equation for the effective action Γ in the large r limit, where as argued before we can ignore $\{\Gamma, \Gamma\}$, must be corrected by

$$2\{S_{loc}, \Gamma\} \approx -\mathcal{H}_{\mathrm{rem}}. \tag{3.31}$$

The symbol "$\approx$" means an equality in the large r limit.

We will show in the following that if the boundary conditions are chosen appropriately, the expression above for large r becomes

$$2\sqrt{\gamma}\{S_{loc}, \Gamma\} \approx zN\frac{\delta \Gamma}{\delta N} + 2h_{ij}\frac{\delta \Gamma}{\delta h_{ij}}. \tag{3.32}$$

By comparing with (3.7), we find that the Lifshitz anomaly is given by

$$\mathcal{A} = -\lim_{r\to\infty} \sqrt{\gamma}\, \mathcal{H}_{\mathrm{rem}}. \tag{3.33}$$

In other words, the holographic computation of the anisotropic scale anomaly involves the determination of the remainder term in the Hamiltonian constraint, just as in the relativistic case.

3.2 Field-theoretic Calculation

In this section we turn to the computation of the anomaly of the quantum Lifshitz model described in the previous chapter. First we need to couple this field theory to the non-trivial metric background (3.1). We therefore consider the following generalization of the action (2.3):

$$S = \int dt d^2x N\sqrt{h}\left(\frac{1}{2}N^{-2}(\partial_t\phi)^2 + \frac{1}{2}(\Delta\phi)^2\right). \tag{3.34}$$

In this expression, $\Delta = \nabla_i\nabla^i$ is constructed out of the covariant derivatives of the spatial metric h_{ij}. By integrating by parts and ignoring boundary terms[6], the action can be written as

$$S = \int dt d^2x\, N\sqrt{h}\, \phi\, D\, \phi, \tag{3.35}$$

[6]In the following, we will assume that the theory is defined on a manifold without boundary.

where D is given by

$$D = -\frac{1}{N\sqrt{h}}\partial_t N^{-1}\sqrt{h}\,\partial_t + \frac{1}{N}\Delta N\Delta \tag{3.36}$$

The model defined in (3.34) is classically invariant under local anisotropic scale transformations (3.2), with the additional requirement that the scalar field ϕ transform trivially

$$\phi \to \phi \tag{3.37}$$

It is very easy to show that the operator D transforms as

$$D \to e^{-4\omega} D \tag{3.38}$$

so that the action (3.34) is indeed invariant under *local* anisotropic rescalings.

We will prove that the classical anisotropic invariance of this model is broken at the quantum level. In particular, we will use the heat-kernel method described in the previous section to show that

$$\mathcal{A} = \frac{1}{1536\pi}\frac{1}{N^2}\left(16h^{ij}N\partial_t(N^{-1}\dot{h}_{ij}) + 5(h^{ij}\dot{h}_{ij})^2 - 10h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li}\right) \tag{3.39}$$
$$+\frac{1}{480\pi}\frac{1}{N}\nabla_i J^i, \tag{3.40}$$

where a dot indicates ∂_t and $\nabla_i J^i$ is a "trivial" total derivative, by which we mean that it can be removed by adding appropriate local counterterms. In fact the anomaly can be written in the simpler form

$$\mathcal{A} = \frac{1}{128\pi}\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right) \tag{3.41}$$

when appropriate counterterms are added to the action (3.34).

As an aside, one of the reasons for the particular interest in this model is that the ground-state wave functional is invariant under time-independent conformal transformations in space. All equal-time correlators can be computed using the machinery of a two-dimensional field theory [25, 24]. One may thus naively expect that the anomalous breaking of anisotropic scaling symmetry (2.2) is somehow related to the two-dimensional Weyl anomaly $\langle T^i{}_i\rangle \propto R$, see e.g. §5.A of [48]. We find, however, that this is *not* the case: the anomaly involves only derivatives with respect to the *time* coordinate, whereas the two-dimensional Ricci scalar R obviously contains only spatial derivatives.

As we explained above, the heat kernel can be expanded as

$$K(\epsilon, f, D) = \sum_{k\geq 0}\epsilon^{\frac{k}{2}-1}\int dtd^2x\,N\sqrt{h}\,f\,a_k(N, h_{ij}), \tag{3.42}$$

where $a_k(N, h_{ij})$ is a local function of N and h_{ij}. To evaluate this we need a suitable basis; it is customary to use the rescaled Fourier modes so that they are orthonormal with respect to the measure that includes the $N\sqrt{h}$ factor. Nevertheless, as pointed out in [49], the cyclicity of the trace allows us to use the usual flat Fourier modes. We thus find

$$K = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x \, e^{-i\omega t - ikx} f e^{-\epsilon D} e^{i\omega t + ikx}. \tag{3.43}$$

We can conjugate the Fourier mode to the left to get the expression

$$K = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x \, f e^{-\epsilon D_2}, \tag{3.44}$$

where D_2 is obtained from D by shifting the derivatives as follows:

$$\partial_t \to \partial_t + i\omega \qquad\qquad \partial_i \to \partial_i + ik_i. \tag{3.45}$$

The most singular term in the heat kernel is the one where we keep only the terms in D_2 without derivatives, leading to

$$\frac{1}{\epsilon}\tilde{a}_0(f, D) = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x f e^{-\epsilon(N^{-2}\omega^2 + (k^2)^2)}, \tag{3.46}$$

where $k^2 \equiv h^{ij}k_ik_j$. This expression is readily evaluated to yield the first term in the heat kernel expansion:

$$\tilde{a}_0(f, D) = \frac{1}{16\pi} \int dt d^2x N\sqrt{h} f(t, x). \tag{3.47}$$

Computing the subleading terms is now straightforward, though somewhat involved. We shall write

$$D_2 = D_2^0 + D_2^{\text{int}} \tag{3.48}$$

where D_2^0 is the piece we isolated above that contains ω^2 and k^4, and D_2^{int} the remainder. We then expand the exponential of D_2^{int}. This contains an explicit factor of ϵ, but ω counts as $\epsilon^{-1/2}$ and k as $\epsilon^{-1/4}$ in the Gaussian integral, as it is obvious from the fact that

$$\int d\omega dk \, k \, e^{-\epsilon(\omega^2 + (k^2)^2)} \, \omega^s \, k^r \propto \epsilon^{-1-\frac{s}{2}-\frac{r}{4}}. \tag{3.49}$$

This means that D_2^{int} has a term which scales as $\epsilon^{-1/4}$, and to get to the finite term one needs to expand D_2^{int} up to fourth order, so that we get terms up to k^{12}. This computation might thus appear extremely daunting; however, the problem becomes tractable if we consider the time-derivative and space-derivative sectors separately. This is consistent because the anomaly can only have structures involving either *two* time derivatives or *four* spatial derivatives.

The two-derivative anomaly

In order to compute the two-derivative contribution to the anomaly, and in turn C_1, it is sufficient to consider metrics that only depend on t, and not on x^a. Thus we can drop all the terms with spatial derivatives ∂_i in $D_2^{\rm int}$. Moreover, by changing the coordinate t if necessary, we can take $N = 1$. With these assumptions, we have

$$D_2^0 = \omega^2 + (k^2)^2, \tag{3.50}$$

$$D_2^{\rm int} = -i\omega(\partial_t + \frac{1}{\sqrt{h}}\partial_t\sqrt{h}) - \frac{1}{\sqrt{h}}\partial_t\sqrt{h}\partial_t. \tag{3.51}$$

The power-counting argument above shows that we need to expand to second order in $D_2^{\rm int}$. Since $D_2^{\rm int}$ and D_2^0 do not commute, we use the following formula:

$$\begin{aligned} e^{A+B} =& e^A + \int_{0\le\alpha\le1} d\alpha\, e^{\alpha A} B\, e^{(1-\alpha)A} \\ &+ \int_{0\le\alpha+\beta\le1} d\alpha\, d\beta\, e^{\alpha A} B\, e^{\beta A} B\, e^{(1-\alpha-\beta)A} + \mathcal{O}(B^3). \end{aligned} \tag{3.52}$$

We find the following contribution to a_2:[7]

$$\tilde{a}_2(f, D) \;=\; \frac{-1}{1536\pi}\int dt d^2x\sqrt{h}f\left\{16h^{ij}\ddot{h}_{ij} + 5(h^{ij}\dot{h}_{ij})^2 - 10h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li}\right\} + \dots \tag{3.53}$$

where the ellipses denote possible four-derivative contributions. To reinstate N, we simply need to change $dt \to dtN$ and $\partial_t \to N^{-1}\partial_t$. We can remove the first term in the right-hand side of the expression above by adding local counterterms, as explained in detail in appendix 3.A.1. We thus obtain the two-derivative contribution to the anomaly

$$\frac{1}{128\pi}\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right). \tag{3.54}$$

Using (3.11) we see that

$$C_1 = \frac{1}{128\pi}. \tag{3.55}$$

The four-derivative anomaly

We now determine the four derivative contribution, and in turn C_2. As explained in Appendix 3.A.1, there are 6 possible terms that can appear, 5 of which are total derivatives. These structures are distinguished by a metric of the form $h_{ij} = e^{2f(x)}\delta_{ij}$ and $N = e^{g(x)}$, which can be used to greatly simplify the computation.

[7] This computation is in principle quite lengthy. However, since there are only few terms that can appear, one can work this out for a diagonal h_{ij} and then reconstruct the full answer.

The expression for $D_2^{\rm int}$ is considerably more involved, but it is straightforward to derive it by conjugating the Fourier modes to the left as explained above for the second-derivative case. Furthermore we need to use the appropriate generalization of (3.52) to fourth order in B. Showing these expressions would not provide any conceptual clarification, so we decided to present directly the final result of the computation. The four-derivative contribution to the anomaly is thus

$$\mathcal{A} = \frac{1}{480\pi}\frac{1}{N}\nabla_i\left(-5(\partial^i N)R + 3(\partial^i N)(\frac{1}{N}\Delta N) + 2(\partial^j N)(\frac{1}{N}\nabla_j\partial^i N) - 5\partial^i\Delta N\right). \tag{3.56}$$

It is interesting to note that this result is a total derivative and, as predicted by the Wess–Zumino consistency condition, it is orthogonal[8] to the non-trivial total derivative $\mathcal{J}$ defined in equation (3.89). As a consequence, this term can be removed by a local counterterm and we conclude that

$$C_2 = 0. \tag{3.57}$$

In Appendix 3.A.3 we present an alternative derivation of $C_2 = 0$.

The **anomaly**

In summary, the Lifshitz model (3.34) exhibits an anomaly under anisotropic local scale transformations, which after the addition of appropriate counterterms is given by

$$\mathcal{A} = \frac{1}{128\pi}\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right). \tag{3.58}$$

It is striking that the anomaly involves only time derivatives. So far, it is unclear to us why this happens. It is also in contrast to the naive expectation that the anomaly is somehow related to the trace anomaly of a two-dimensional conformal field theory, as we mentioned at the beginning of this section.

3.3 Holographic Calculation

In the previous section we showed that a theory with anisotropic scaling symmetry has an anisotropic scaling symmetry anomaly parametrized by two central charges, denoted by C_1 and C_2. We computed these central charges for a particular model defined by the action (3.34). In this section we show that these central charges can be computed holographically for the Lifshitz spacetime considered in [16, 1].

[8] To see what we mean by 'orthogonal', we refer to the appendix 3.A.1.

3.3.1 Boundary conditions and anomaly

From the field theory side, we know that the volume form has a definite scaling weight $[\mathrm{Vol}_d] = [dt\, d^{d-1}x] = z+d-1$. In the dual gravitational picture this weight is translated to a radial scaling, such that

$$z+d-1 \;=\; [\mathrm{Vol}_d] \;=\; \frac{\partial_r\sqrt{\gamma}}{\sqrt{\gamma}} \;=\; \partial_r \log N + \frac{1}{2}\partial_r \log\det h \tag{3.59}$$

Here, N is the lapse function and h_{ij} is the induced metric on a spatial slice of Σ_r. We assume that the spatial metric is of the form $h_{ij} = e^{2r}\hat{h}_{ij}$, where $\hat{h}_{ij}$ has a finite limit as $r\to\infty$. It then follows that the lapse scales as $N \sim e^{zr}$. This puts a restriction on the degrees of freedom contained in the metric. It implies in particular that we must turn off the off-diagonal mode in γ_{ti} that scales as e^{2zr}; in terms of the linearized modes discussed in [40, 16, 1], one needs to kill the c_{1i} mode. This naturally leads us to consider only deformations with a preferred time foliation, as suggested also in [16].

Let us redefine N and h_{ij} to be the renormalized lapse and induced metric,[9]

$$N = \lim_{r\to\infty} e^{-zr}(-\gamma^{tt})^{-1/2}, \tag{3.60}$$

$$h_{ij} = \lim_{r\to\infty} e^{-2r}\gamma_{ij}, \tag{3.61}$$

such that the renormalized volume form is given by $N\sqrt{h} = \lim_{r\to\infty} e^{-(z+d-1)r}\sqrt{\gamma}$. With these conditions, it is straightforward to see that:

$$2\sqrt{\gamma}\,\{S_{\mathrm{loc}},\Gamma\} \approx 2z\,N^2\frac{\delta\Gamma}{\delta N^2} + 2h_{ij}\frac{\delta\Gamma}{\delta h_{ij}} + z\hat{A}_t\frac{\delta\Gamma}{\delta\hat{A}_t} \approx -\sqrt{\gamma}\,\mathcal{H}_{\mathrm{rem}}, \tag{3.62}$$

where $\hat{A}_t = \lim_{r\to\infty} e^{-zr}A_t$. As noted in the previous chapter, the "vector field mode" c_3 requires an infinite set of counterterms to be properly renormalized holographically. Furthermore, this mode introduces logarithmic divergences in the metric sector, spoiling our definition of anisotropic conformal infinity (3.60). For this reason, we will turn off this mode[10] by setting $c_3 = 0$. In particular notice the important relation:

$$A_t = e^{2r}N + \text{subleading}, \tag{3.63}$$

where the subleading terms scale as e^{-4r}. In the frame field language of [16], this corresponds to $\delta(A_A) = \delta(A_a e^a_A) = 0$. Notice that with these boundary conditions,

[9] Again, we do not consider a shift N^i, as it can locally be removed by a foliation-preserving diffeomorphism.

[10] As shown in [1], it is possible to consistently impose this condition when higher order non-linear corrections are considered, and we believe there is no obstruction at the full non-linear level.

Γ becomes a functional of N and h only. Therefore we have:

$$\left.\frac{\partial\Gamma}{\partial N}\right|_{h=\text{const}} = \frac{\delta\Gamma}{\delta N} + \frac{\delta\Gamma}{\delta\hat{A}_t}\frac{\partial\hat{A}_t}{\partial N}, \tag{3.64}$$

where the variations on the right are unconstrained, while the variation on the left represents the total variation of Γ with respect to N. Therefore (3.62) becomes

$$2N\frac{\partial\Gamma}{\partial N} + 2h_{ij}\frac{\delta\Gamma}{\delta h_{ij}} = -\sqrt{-\gamma}\,\mathcal{H}_{\text{rem}}. \tag{3.65}$$

By comparing with (3.7), we find that the anomaly is indeed given by

$$\mathcal{A} = -\lim_{r\to\infty} e^{4r}\,\mathcal{H}_{\text{rem}}. \tag{3.66}$$

3.3.2 Holographic anomaly

The computation of $\mathcal{H}_{\text{rem}}$ is as follows. We first write the most general ansatz for S_{loc}, such as

$$S_{loc} = \int d^dx\sqrt{\gamma}\left\{U(\alpha) + \mathcal{F}_1\,R + \mathcal{F}_2\,D_aA_b\,D^aA^b + \mathcal{F}_3\,D_aA_b\,D^bA^a + \dots\right\}, \tag{3.67}$$

where $U(\alpha)$ was discussed in the previous chapter, and each coefficient $\mathcal{F}_i$ is a function of α. The ellipses denote additional two-derivative terms that can be constructed out of covariant derivatives of A_a and higher derivative terms. More details can be found in [2].

Solving the Hamilton constraint recursively by expanding the various functions of α as a power-series in $\alpha-\alpha_0$ as explained in the previous section, one finds a remainder of the form

$$\mathcal{H}_{\text{rem}} = -D_aA_b\,D^bA^a + \frac{1}{2}(D_aA^a)^2 + \frac{1}{8}R^2 - \frac{1}{4}R_{ab}R^{ab} + \dots, \tag{3.68}$$

where the ellipses denote four-derivative terms involving the vector field A_a. Using the definitions in 3.3.1 to write the two-derivative piece in terms of h_{ij} and N, and plugging the result in (3.66), we get the following contribution to the anomaly

$$\mathcal{A} = \frac{\ell^2}{64\pi G}\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right) + \dots, \tag{3.69}$$

and using (3.11) we conclude that

$$C_1 = \frac{1}{128\pi}\frac{2\ell^2}{G} \tag{3.70}$$

We reinstated the four-dimensional Newton's constant G and the curvature length scale ℓ. The four-derivative piece is more complicated to analyze, but we can extract C_2 by extracting the coefficient of the square of the two-dimensional Ricci scalar $({}^2\!R)^2$. Writing the three dimensional Ricci tensor in terms of the two-dimensional one gives

$$\mathcal{H}_{\text{rem}} \;=\; \ldots + \frac{1}{4}\,{}^2\!R_{ij}{}^2\!R^{ij} - \frac{1}{8}\,({}^2\!R)^2 + \ldots, \tag{3.71}$$

where we have not shown terms that involve derivatives acting on N. Then we can use the off-shell identity that relates the Ricci tensor to the Ricci scalar, ${}^2\!R_{ij} = \frac{1}{2}{}^2\!R\, h_{ij}$, which is specific to two dimensions. When we plug this into (3.71), we find that the Ricci-squared terms cancel and we have (equation (3.11)):

$$C_2 = 0, \tag{3.72}$$

which agrees with the field theory computation.

Notice that for the purpose of computing C_2, which was the aim of this section, we did not have to compute the complete answer that includes trivial total derivative. In fact, a full analysis of the counterterm action and remainder at the four-derivative level, while conceptually straightforward, would be rather involved. Nevertheless, the complete answer has been computed using the results of [16] in [45], in perfect agreement with our result $C_2 = 0$. In conclusion, the holographic anomaly is given by

$$\mathcal{A} \;=\; \frac{\ell^2}{64\pi\, G}\,\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right). \tag{3.73}$$

3.4 Discussion and conclusions

In this chapter we computed the anisotropic scaling anomaly of two Lifshitz theories, one defined using a standard field theory quantization of an explicit classical action (3.34), the other defined using the holographic correspondence. A precise definition of Lifshitz holography is still lacking, and a microscopic characterization of the strongly coupled field theories dual to Lifshitz spacetime is not known. It is therefore a priori not very meaningful to compare the two anomalies. Nevertheless, we found that the anomalies are quite similar. In both cases one of the two possible central charges vanishes, and as a consequence the two anomalies are directly proportional to each other. The ratio of the two anomalies, in the conventions used in this chapter, is $2\ell^2/G$, with ℓ the curvature radius of the Lifshitz spacetime and G the 4d Newton constant. It would be interesting to evaluate this quantity in explicit string theory embeddings of Lifshitz spacetimes to see how it scales with the various integer fluxes, as this will provide some measure of the effective number of degrees of freedom of the dual field theory.

It is quite mysterious that the conformal anomaly only involves time derivatives, it is even mysterious that there exists a conformal anomaly at all. According to [37], the dynamical critical exponent is in general renormalized, and as soon as $z = 2+\epsilon$ a (time-reversal invariant) conformal anomaly can no longer be written down. So either there is some unknown mechanism that protects the value of $z = 2$, or the conformal anomaly can somehow be removed in the full quantum theory. Further work will be required to clarify this issue.

It is also clearly of interest to explore other systems with anisotropic scale invariance to examine whether the conformal anomaly is still of the same form. In particular, whenever one has a Lifshitz solution in a theory with Chern-Simons type terms, time reversal symmetry is broken and it is logically possible to have contributions with an odd number of time derivatives to the conformal anomaly. It is in principle straightforward to extend the analysis in appendix 3.A.1 to determine whether there are non-trivial terms of this type and we leave this as an exercise.

Some progress on these issues was recently reported in [12], building on the results of [45] that partly overlapped with our [2]. The authors showed that the second possible structure in the anomaly, the one involving spatial derivatives, can be generated in field theory by adding the following non-minimal coupling between the scalar field ϕ and the metric

$$S = \ldots + \int dtd^2x\, N\sqrt{h}\, \left(R + \frac{1}{N}\Delta N - h^{ij}\left(\frac{1}{N}\partial_i N\right)\left(\frac{1}{N}\partial_j N\right)\right)^2 \phi^2. \quad (3.74)$$

Employing the $\zeta-$function regularization discussed above, the authors argued that $C_2 \neq 0$ in this model. In view of this result, it is natural to wonder what kind of gravity models in the bulk can reproduce the non-trivial four derivative term in the anomaly. In [12] it was shown that various nonprojectable versions of Horava–Lifshitz gravity [50, 51] in the bulk do indeed generate non-zero C_2.

With these partial results, we may speculate that only theories with $C_2 = 0$ can be formulated in terms of bulk gravity duals with full local relativistic invariance, while models with $C_2 \neq 0$ require non-relativistic theories in the bulk such as Horava–Lifshitz gravity. It would be extremely interesting to explore this relationship further, and especially determine how the purported "hidden relativistic symmetry" of theories with $C_2 = 0$ can be understood from the field theory side.

As mentioned before, one of the main uses of the conformal anomaly is that it is a relatively simple quantity of a field theory which sometimes controls certain universal properties. For example, in the relativistic case, in $d = 2$ the conformal anomaly completely fixes the free energy at high temperatures, and it also controls the logarithmic contributions in the entanglement entropy in $d = 2, 4$. Whether similar universal properties also exist for non-relativistic field theories is an interesting open problem that we hope to come back to in the future.

3.A Appendices

3.A.1 Classification of possible terms in the anomaly

In this appendix we explore to what extent it is possible to remove total derivatives from the anomaly. This is achieved by adding appropriate scale invariant counterterms to the action that are not invariant under *local* scale transformations. Clearly, we can discuss the two-derivative and the four-derivative terms separately. Let us start with the former; there are only three possible scale-invariant terms that we can construct with two time derivatives:

$$h^{ij}\frac{1}{N}\partial_t(\frac{1}{N}\partial_t h_{ij}), \qquad \frac{1}{N^2}(h^{ij}\dot{h}_{ij})^2, \qquad h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li}. \tag{3.75}$$

It is straightforward to see that the two combinations

$$h^{ij}\frac{1}{N}\partial_t(\frac{1}{N}\partial_t h_{ij}), \qquad \frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right), \tag{3.76}$$

are invariant under *local* scale transformations (up to total derivatives). These two terms are related by partial integration, and we now show that it is indeed possible to "partially integrate" inside the anomaly by adding an appropriate counterterm to the action. The most general form of the anomaly at the two derivative level is:

$$\delta W = \int \delta\rho \left\{ a\frac{1}{N}h^{ij}\partial_t(\frac{1}{N}\dot{h}_{ij}) + b\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right)\right\}. \tag{3.77}$$

The presence of the factor $\delta\rho$ prevents us from doing partial integration directly. Let us add the following counterterm to the action:

$$W' = W + c\int N\sqrt{h}\frac{1}{N^2}(h^{ij}\dot{h}_{ij})^2. \tag{3.78}$$

It is then easy to check that

$$\delta W' = \int \delta\rho \left\{ (a-8c)\frac{1}{N}h^{ij}\partial_t\frac{1}{N}\dot{h}_{ij} + (b+8c)\frac{1}{N^2}\left(h^{ij}h^{kl}\,\dot{h}_{ik}\dot{h}_{jl} - \frac{1}{2}(h^{ij}\dot{h}_{ij})^2\right)\right\} \tag{3.79}$$

Therefore we can pick $c = a/8$ and get rid of the first term, which is tantamount to integrating by parts, or discarding total derivatives in the anomaly. For instance, in the field theory analysis, we went from (3.53) to (3.54) using this procedure. In particular, we had $a = 1/48\pi$ and $b = -5/384\pi$, such that $b + 8c = a + b = 1/128\pi$.[11]

[11] Bear in mind that there was an extra factor -2 coming from the relation between $\tilde{a}_2(\delta\rho, D)$ and the integrated anomaly, cf. (3.26).

Let us now consider the four derivative level. In this case we are interested in terms of the form $\nabla_i J^i$ in the anomaly. We ask ourselves to what extent it is possible to remove them by adding local counterterms G to the action. Both the total derivatives and the local counterterms must be scale invariant, therefore there is only a finite number of them. Let us choose a basis:

$$J_a^i \qquad a = 1, \dots, N \tag{3.80}$$

$$G_b \qquad b = 1, \dots, M. \tag{3.81}$$

The Weyl variation of a linear combination $\sum_b q_b\, G_b$ can be written, after partial integration, as:

$$\delta \sum_b q_b\, G_b = \frac{\omega}{N} \sum_{ab} M_{ab}\, q_b\, \nabla_i J_a^i, \tag{3.82}$$

If the variation of the effective action reads

$$\delta W = \int N \sqrt{h}\, \omega \left(\mathcal{A} + \sum_a c_a \nabla_i J_a^i \right), \tag{3.83}$$

we can get rid of the total derivatives if we can solve the system of linear equations:

$$M_{ab}\, q_b = c_a. \tag{3.84}$$

If we are to remove all the possible total derivatives that can appear, the number of rows N of the matrix M_{ab} must be less than or equal to the number of columns M, and the rank of the matrix should be maximal. It is easy to check that there are 6 possible functionally independent scale invariant currents J^i, and we choose the following basis:

$$\begin{array}{lll} J_1^i = N \partial^i R & J_2^i = (\partial^i N) R & J_3^i = (\partial^i N)(\frac{1}{N}\partial_j N)(\frac{1}{N}\partial^j N) \\ J_4^i = (\partial^i N)(\frac{1}{N}\Delta N) & J_5^i = (\partial^j N)(\frac{1}{N}\nabla_j \partial^i N) & J_6^i = \partial^i \Delta N. \end{array} \tag{3.85}$$

Analogously, there are 12 functionally independent scale invariant counterterms, and we choose the basis:

$$\begin{array}{ll} G_1 = R^2 & G_2 = \Delta R \\ G_3 = (\frac{1}{N}\Delta N) R & G_4 = (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N) R \\ G_5 = ((\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N))^2 & G_6 = (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N)(\frac{1}{N}\Delta N) \\ G_7 = (\frac{1}{N}\Delta N)^2 & G_8 = (\frac{1}{N}\partial^i N)(\frac{1}{N}\partial^j N)(\frac{1}{N}\nabla_i \partial_j N) \\ G_9 = (\frac{1}{N}\partial^i N)\frac{1}{N}\partial_i \Delta N & G_{10} = \frac{1}{N}\nabla_i \partial_j N \frac{1}{N}\nabla^i \partial^j N \\ G_{11} = \frac{1}{N}\Delta^2 N & G_{12} = \frac{1}{N}\partial^i N \partial_i R \end{array} \tag{3.86}$$

While we have many more possible counterterms than currents, it is important to notice that not all the counterterms are independent, since we can always partially integrate inside the action. This means that some linear combinations of counterterms will have the same Weyl transformation. Furthermore, there can be Weyl invariant combinations of counterterms that do not help in removing total derivatives from the anomaly.

By taking the Weyl variation of the 12 terms G_b, it is straightforward to compute the matrix M, which is given by:

$$M_{ab} = \begin{pmatrix} -4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ -4 & -2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & -8 & -6 & 0 & -5 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -8 & 2 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 & 0 & -2 & 4 & -4 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & 4 & 0 & -4 & 4 & 0 & 2 \end{pmatrix} \tag{3.87}$$

It is easily checked that M does *not* have maximal rank (which would be 6), but it has rank 5. In fact, M has a 7 dimensional space of null vectors, which is spanned by the 6 total derivatives $\nabla_i J^i$ and a Weyl invariant term:

$$\delta \int \sqrt{h}\, \nabla_i J^i_a = 0, \qquad \delta \int N\sqrt{h} \left(R + \frac{1}{N}\Delta N - \frac{1}{N^2}\partial_i N \partial^i N \right)^2 \; = \; 0. \tag{3.88}$$

Since the rank of M is 5, the Weyl variation of the most general counterterm spans a 5 dimensional subspace of the 6 dimensional space generated by $c_a \, \nabla_i J^i_a$. That means that we can find an orthonormal basis (with respect to the usual Euclidean scalar product δ_{ab}) for the currents where 5 are trivial (i.e. removable by counterterms) and 1 is non-trivial. In other words, we look for 5 vectors e_a such that $e_a = M_{ab} q_b$ admits a solution. If we now take u_a to be the null vector of the transpose of M_{ab}, it is obviously orthogonal to all the e_a since $u_a e_a = e_a M_{ab} q_b = 0$. We define the non-trivial current $\mathcal{J}^i$ to be:

$$\mathcal{J}^i = u_a J^i_a = J^i_1 - J^i_2 + J^i_4 + J^i_5 + 2J^i_6. \tag{3.89}$$

However, we will presently show that this current does *not* obey the Wess–Zumino consistency condition, therefore it cannot appear in the anomaly.

Wess–Zumino consistency condition and $\mathcal{J}^i$

The goal of this section is to figure out whether all possible terms that we found above satisfy the Wess–Zumino consistency conditions. To this end, we shall compute the quantities

$$\Omega_a \equiv \delta_1 \int \sqrt{h}\,\omega_2 \nabla_i J^i_a - \delta_2 \int \sqrt{h}\,\omega_1 \nabla_i J^i_a \tag{3.90}$$

$$= \int \delta_2\big(\sqrt{h}\,J^i_a\big)\,\partial_i\omega_1 - \int \delta_1\big(\sqrt{h}\,J^i_a\big)\,\partial_i\omega_2 \tag{3.91}$$

for each $a = 1,..,6$. The main idea of this analysis is to find all possible linear combinations of the Ω's such that

$$\sum_{a=1}^{6} c_a \Omega_a = 0 \tag{3.92}$$

If the vector space spanned by the vectors $\{\vec{c}\}$ is six dimensional, all J^i_a's are Wess–Zumino-consistent. If, on the other hand, this vector space is five-dimensional then we must conclude that one of the J^i_a's is inconsistent. Since we already know that five currents can be generated by varying appropriate local scale invariant terms, these are manifestly consistent. Therefore the inconsistent current, if present, must be the non-trivial current of equation (3.89).

The way we shall carry out this computation is by first computing the first term in (3.91). The second term in (3.91) is then obtained from the first one by replacing the derivatives that act on ω_1 for derivatives that act on ω_2 by means of partial integration.

We shall start with Ω_1. The first term in (3.91) is

$$\delta_2\big(\sqrt{h}\,J^i_1\big)\,\partial_i\omega_1 = \sqrt{h}\,\big(-\partial^i\omega_2\,NR - \partial^i\Delta\omega_2\,N\big)\,\partial_i\omega_1 \tag{3.93}$$

The second term is then

$$\delta_1\big(\sqrt{h}\,J^i_1\big)\,\partial_i\omega_2 = \sqrt{h}\,\big(-\partial^i\omega_1\,NR - \partial^i\Delta\omega_1\,N\big)\,\partial_i\omega_2 \tag{3.94}$$

$$= \sqrt{h}\,\big(-\partial^i\omega_2\,NR - \nabla^i\nabla^j\big(\partial_j\omega_2\,N\big)\big)\,\partial_i\omega_1 \tag{3.95}$$

$$= \sqrt{h}\big(-\partial^i\omega_2\,NR - \partial^i\Delta\omega_2\,N \tag{3.96}$$

$$- \Delta\omega_2\,\partial^i N - \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\,\partial_i\omega_1 \tag{3.97}$$

so that

$$\Omega_1 = \int \sqrt{h}\,\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\,\partial_i\omega_1 \tag{3.98}$$

Similarly, from J^i_2:

$$\delta_2\big(\sqrt{h}\,J^i_2\big)\,\partial_i\omega_1 = \sqrt{h}\,\big(\partial^i\omega_2\,NR - \Delta\omega_2\,\partial^i N\big)\,\partial_i\omega_1 \tag{3.99}$$

$$\delta_1\big(\sqrt{h}\,J^i_2\big)\,\partial_i\omega_2 = \sqrt{h}\,\big(\partial^i\omega_1\,NR - \Delta\omega_1\,\partial^i N\big)\,\partial_i\omega_2$$

$$= \sqrt{h}\,\big(\partial^i\omega_2\,NR + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\,\partial_i\omega_1 \tag{3.100}$$

$$\Omega_2 = -\int \sqrt{h}\,\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\,\partial_i\omega_1 \tag{3.101}$$

From J_3^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_3^i\big)\,\partial_i\omega_1 &= \sqrt{h}\,\big(\partial^i\omega_2\,\partial_j N\,\partial^j N + 2\partial_j\omega_2\,\partial^i N\,\partial^j N\big)\;\partial_i\omega_1 && (3.102)\\
\delta_1\big(\sqrt{h}\,J_3^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\big(\partial^i\omega_1\,\partial_j N\,\partial^j N + 2\partial_j\omega_1\,\partial^i N\,\partial^j N\big)\;\partial_i\omega_2\\
&= \sqrt{h}\,\big(\partial^i\omega_2\,\partial_j N\,\partial^j N + 2\partial_j\omega_2\,\partial^j N\,\partial^i N\big)\;\partial_i\omega_1 && (3.103)\\
\Omega_3 &= 0 && (3.104)
\end{aligned}$$

From J_4^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_4^i\big)\,\partial_i\omega_1 &= \sqrt{h}\,\big(\partial^i\omega_2\,\Delta N + 2\partial_j\omega_2\,\tfrac{1}{N}\partial^i N\partial^j N + \Delta\omega_2\,\partial^i N\big)\;\partial_i\omega_1 && (3.105)\\
\delta_1\big(\sqrt{h}\,J_4^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\big(\partial^i\omega_1\,\Delta N + 2\partial_j\omega_1\,\tfrac{1}{N}\partial^i N\partial^j N + \Delta\omega_1\,\partial^i N\big)\;\partial_i\omega_2\\
&= \sqrt{h}\,\big(\partial^i\omega_2\,\Delta N + 2\partial_j\omega_2\,\tfrac{1}{N}\partial^j N\partial^i N - \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 && (3.106)\\
\Omega_4 &= \int \sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 && (3.107)
\end{aligned}$$

From J_5^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_5^i\big)\,\partial_i\omega_1 &= \sqrt{h}\,\big(\partial_j\omega_2\,\nabla^i\partial^j N + \partial^i\omega_2\,\tfrac{1}{N}\partial^j N\partial_j N + \nabla_j\partial^i\omega_2\,\partial^j N\big)\;\partial_i\omega_1 && (3.108)\\
\delta_1\big(\sqrt{h}\,J_5^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\big(\partial_j\omega_1\,\nabla^i\partial^j N + \partial^i\omega_1\,\tfrac{1}{N}\partial^j N\partial_j N + \nabla_j\partial^i\omega_1\,\partial^j N\big)\;\partial_i\omega_2\\
&= \sqrt{h}\,\Big(\partial_j\omega_2\,\nabla^i\partial^j N + \partial^i\omega_2\,\tfrac{1}{N}\partial^j N\partial_j N - \nabla_j\big(\partial^{(i}\omega_1\,\partial^{j)} N\big)\Big)\;\partial_i\omega_1 && (3.109)\\
\Omega_5 &= \int \sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 && (3.110)
\end{aligned}$$

From J_6^i:

$$\begin{aligned}
\delta_2\big(\sqrt{h}\,J_6^i\big)\,\partial_i\omega_1 &= \sqrt{h}\,\partial^i\big(2\partial_j\omega_2\,\partial^j N + \Delta\omega_2\,N\big)\;\partial_i\omega_1\\
&= \sqrt{h}\;\big(\partial^i\Delta\omega_2\,N + \Delta\omega_2\,\partial^i N + 2\partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 && (3.111)\\
\delta_1\big(\sqrt{h}\,J_6^i\big)\,\partial_i\omega_2 &= \sqrt{h}\,\partial^i\big(2\partial_j\omega_1\,\partial^j N + \Delta\omega_1\,N\big)\;\partial_i\omega_2\\
&= \sqrt{h}\,\big(\partial^i\Delta\omega_2\,N - \Delta\omega_2\,\partial^i N\big)\;\partial_i\omega_1 && (3.112)\\
\Omega_6 &= 2\int \sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1 && (3.113)
\end{aligned}$$

We thus find that each Ω_a is a multiple of

$$\int \sqrt{h}\;\big(\Delta\omega_2\,\partial^i N + \partial^i\big(\partial_j\omega_2\,\partial^j N\big)\big)\;\partial_i\omega_1, \qquad (3.114)$$

which means that there is one linear combination that does *not* satisfy the Wess–Zumino consistency conditions. In other words, all but one of the six J_a^i's can be made consistent. Since we have already found that five of the six J_a^i's can be canceled by variations of local terms, the one that cannot be canceled (which we called $\mathcal{J}^i$) must be inconsistent. We can make this more precise by noticing that the consistency equation

$$c_1 - c_2 + c_4 + c_5 + 2c_6 \;=\; 0 \tag{3.115}$$

describes a five-dimensional hypersurface of consistent linear combinations $c_a J_a^i$. The set of all such c_a-vectors can be defined as those that are orthogonal to the *inconsistent* vector, v_a say, such that $c_a v_a = 0$. The inconsistent vector is

$$\vec{v} \;=\; \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 2 \end{pmatrix} \tag{3.116}$$

As a consistency check on our computations, notice that this is precisely the five-dimensional hypersurface that we mentioned above, which may be defined as all vectors that are orthogonal to u_a (as defined in (3.89)). Namely, the vector u_a is the same as the inconsistent vector, i.e. $u_a = v_a$. The fact that $\mathcal{J}^i$ does not satisfy the Wess–Zumino condition means that it cannot appear as the variation of either local or non-local terms. The fact that there are precisely five total-derivative terms in the anomaly, all of which can be canceled by variations of local terms, was also noted in [45].

3.A.2 Role of the massive vector on the field theory side

In this section we explore the conformal invariance of the field theory Lifshitz model from a different perspective. In particular, we will show that a preferred timelike vector n^μ plays a very similar role to the vector field A^μ appearing in the bulk.

Our set-up is the following three-dimensional scalar model with critical exponent $z = 2$ [24],

$$S \;=\; \int d^2x\, dt\, \mathcal{L} \;=\; \frac{1}{2}\int d^2x\, dt \Big(\dot{\phi}^2 - (\Delta\phi)^2\Big). \tag{3.117}$$

The operator Δ is the spatial Laplacian $\Delta = \delta^{ij}\partial_i\partial_j$ and the dot denotes differentiation with respect to (imaginary) time, $\dot{\phi} = \partial_t\phi$. The Noether current density $(J_a)^b$ corresponding to the infinitesimal diffeomorphism $x^a \mapsto x^a + \varepsilon^a$ is given via the usual definition[12]

$$\delta_\varepsilon S \;=\; \int d^2x\, dt\, (J_a)^b\, \partial_b \varepsilon^a, \tag{3.118}$$

[12] We use the notation $x^t = t$, i.e. the index a runs over $a = t, 1, 2$.

The current $(J_t)^a$ generates time reparametrizations and $(J_i)^a$ generates the spatial ones; their components are given by

$$
\begin{aligned}
(J_t)^t &= -\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\Delta\phi)^2 && (3.119)\\
(J_t)^i &= \partial^i\dot{\phi}\,\Delta\phi - \dot{\phi}\,\partial^i\Delta\phi && (3.120)\\
(J_i)^t &= -\dot{\phi}\,\partial_i\phi && (3.121)\\
(J_i)^j &= \delta_i^j\,\mathcal{L} + \partial_i\partial^j\phi\,\Delta\phi - \partial_i\phi\,\partial^j\Delta\phi && (3.122)
\end{aligned}
$$

where $\partial^i = \delta^{ij}\partial_j$. One thing we see here is that $(J_t)^t = -\mathcal{E}$, where $\mathcal{E}$ is the Hamiltonian/energy density. The conservation law reads

$$\partial_b(J_a)^b \;=\; -\partial_a\phi\,(\ddot{\phi} + \Delta^2\phi) \;\approx\; 0 \tag{3.123}$$

The symbol $\approx$ denotes weak equality, i.e. equality up to terms that vanish on shell. The 'gauge' parameter that generates the Lifshitz scaling is $\varepsilon^t = 2\varepsilon\,t$ and $\varepsilon^i = \varepsilon\,x^i$ (ε is just a small real number). The condition for scale invariance is

$$2\,(J_t)^t + (J_i)^i \;=\; \partial_i\left(-2\partial^i\phi\,\Delta\phi\right) \tag{3.124}$$

whose right-hand side is not zero but a total divergence. The conserved current $\mathcal{S}^a$ associated to scale invariance of the theory is

$$\mathcal{S}^t \;=\; 2t(J_t)^t + x^i(J_i)^t \qquad\qquad \mathcal{S}^i \;=\; 2t(J_t)^i + x^j(J_j)^i \tag{3.125}$$

Note that we cannot interpret the J's as comprising an energy momentum tensor, since it would be far from being symmetric.

If we couple the Lifshitz model to N and h_{ij}, we can easily write down the condition for conformal invariance, however the relation between the bulk (with its complete metric and the extra gauge field) and the field theory model is rather obscure. Clearly, the bulk metric does not couple to the energy momentum tensor of the field theory as defined through J_t and J_i, since that tensor is not even symmetric.

So we will now make a more precise proposal about the relation between the two. We introduce a three-dimensional metric $g_{\mu\nu}$ and a unit timelike vector n_a so that $n_a n^a = -1$. Define the projector $h_a{}^b = g_a{}^b + n_a n^b$, which is orthogonal to n^a, and

$$\Delta\phi \equiv \partial_a(h^{ab}\partial_b\phi) + \frac{1}{2}h^{ab}h^{cd}\,\partial_a\phi\,\partial_b h_{cd} \tag{3.126}$$

then we can couple the Lifshitz model to h_{ab} and n_a via the covariant action

$$S = \int d^2x\,dt\sqrt{-g}\left((n^a\partial_a\phi)^2 - (\Delta\phi)^2\right). \tag{3.127}$$

This action is conformally invariant under

$$\delta n_a = 2\omega\,n_a, \qquad \delta h_{ab} = 2\omega\,h_{ab} \tag{3.128}$$

This is why it is useful to introduce ${h_a}^b$, since the three-dimensional metric itself would transform as $\delta g_{ab} = -4\omega\, n_a n_b + 2\omega\, h_{ab}$ (using the completeness relation $g_{ab} = -n_a n_b + h_{ab}$). Of course, all of this is not very profound. We have merely replaced the spatial metric h_{ij} by the projection of the metric in the plane perpendicular to unit normal n_a.

The claim is that (3.127) describes the coupling of the Lifshitz model to a metric and a gauge field in exactly the same way as one would expect from the bulk description.

With this fully covariant action, we can define a symmetric "stress tensor" by varying it with respect to g_{ab}. Due to the presence of n_a, this stess-tensor is not conserved though. The precise equation that expresses general covariance of the theory reads

$$2D_b \frac{\delta S}{\delta g_{ab}} = (D^a n_b)\frac{\delta S}{\delta n_b} - D_b\left(n^a \frac{\delta S}{\delta n_b}\right), \tag{3.129}$$

where on the left hand side we recognize the covariant derivative of the stress-tensor. The background field n_a is the quantity that breaks the general covariance of the theory, which explains why this equation has a right-hand side.

In view of (3.128), to write the conformal anomaly in covariant variables, we also need a variation in terms of n_a

$$\sqrt{-g}\mathcal{A} = (-4n_a n_b + 2h_{ab})\frac{\delta S}{\delta g_{ab}} + 2n_a \frac{\delta S}{\delta n_a}. \tag{3.130}$$

This is exactly the same as the bulk equation with n_a playing the role of the asymptotic value of A_a. When we choose $n_t = N$, $g_{tt} = -N^2$, $g_{ti} = 0$ and $g_{ij} = h_{ij}$, the conformal anomaly becomes the expression we have been using all along.

3.A.3 Alternative computation of C_2

In this section we provide an alternative computation of C_2. Since the structure multiplying C_2 contains R, we can take $N = 1$ and assume that h_{ij} does not depend on t but does depend on x^i. With these assumptions, the ω integral separates out, yielding a factor of $\sqrt{\pi/\epsilon}$. What is left is to study the operator $\exp(-\epsilon\nabla^2)$. Now we can roughly think of the standard heat kernel expansion as the Laplace transform of the spectral density. So by taking the inverse Laplace transform we can reconstruct the spectral density. The inverse transform of ϵ^a is $s^{-1-a}/\Gamma(-a)$. Next, we can integrate this against $\exp(-\epsilon s^2)$ to obtain

$$\frac{\epsilon^{a/2}\Gamma(-a/2)}{2\Gamma(-a)}. \tag{3.131}$$

This suggests that if the operator ∇ has heat kernel expansion

$$\sum_{n\geq -1} \epsilon^n L_n \tag{3.132}$$

then ∇^2 has the expansion

$$\sum_{n\geq -1} \epsilon^{n/2} \frac{\Gamma(-n/2)}{2\Gamma(-n)} L_n. \tag{3.133}$$

The term with $n = 1$, which would contribute to the anomaly, vanishes due to the Gamma function. Therefore the coefficient of the R^2 term vanishes[13]. We conclude that the coefficient C_2 in (3.11) vanishes as well. Notice that while this method is simpler than a direct computation, it is not powerful enough to determine the total derivative terms.

3.A.4 Divergent terms in the heat-kernel expansion

The methods of section 3.2 allow us to compute the divergent part of the heat-kernel expansion for the model considered in equation (3.34). We present the results here for completeness:

$$K(\epsilon, f, D) \sim \frac{1}{\epsilon}\tilde{a}_0(f, D) + \frac{1}{\sqrt{\epsilon}}\tilde{a}_1(f, D) + O(\epsilon^0), \tag{3.134}$$

where

$$\tilde{a}_0(f, D) = \frac{1}{16\pi} \int dtd^2xN\sqrt{h}\, f(t, x), \tag{3.135}$$

$$\tilde{a}_1(f, D) = \frac{1}{48\pi^{3/2}} \int dtd^2xN\sqrt{h}\, f(t, x) \left(R - \frac{1}{N}\Delta N\right). \tag{3.136}$$

[13] See also [52] for a rigorous proof of this statement.

Chapter 4

Non-renormalization theorems

In this chapter we will study exactly marginal deformations. These are driven by operators that remain marginal even when all the quantum effects are accounted for. As we explained in the introduction, this requires very non-trivial cancellations between the quantum corrections, and without enough symmetry it seems very unlikely that such cancellations will accidentally take place in interacting theories. However, there is a particular class of field theories that do indeed have enough symmetry for this to happen, those that enjoy supersymmetry. As we will see, supersymmetry allows us to constrain the dimension of certain "protected" operators in terms of numbers that characterize their representations under the symmetry group. This comes about because there are special representations that are "shorter" than generic ones, where the highest-weight state is annihilated by some supercharges. In order for the conformal dimension to change, the number of states in the representation would need to jump discontinuously. [1]

Some of these protected operators are such that the conformal dimension satisfies $\Delta = d$, so that they are exactly marginal. As a consequence we can use these operators to deform the conformal field theories to "nearby" field theories that are still conformal. The marginal coupling constants can be thought of as coordinates in a geometrical space, which we call *moduli space*.[2] In this sense, such conformal

[1] This is not exactly true: short representations can combine into longer representations in which the conformal dimension is not protected anymore. We will comment on this possibility in the discussion session.

[2] This is also known as conformal manifold.

field theories do not live in isolation, and as argued in the general introduction it is interesting to study how various physical properties change as we move in the moduli space, since this might give rise to interesting phenomenological constraints or might serve as a test of gauge/gravity dualities. In particular, we will focus on certain protected operators called chiral operators, and show that their structure constants "do not depend" on the position on this manifold if we have enough supersymmetry.

More specifically, we will be able to prove that the most general 3-point function of chiral primaries in two-dimensional $\mathcal{N} = (4, 4)$ theories is not renormalized, completing the proof initiated in [53]. Second, we will obtain a short, and in our view simpler, proof of the non-renormalization theorem for 1/2 BPS chiral primary 3-point functions for $\mathcal{N} = 4$ SCFTs in four dimensions. Our presentation provides a unified treatment of both cases, based on superconformal Ward identities and the structure of the representations of the superconformal algebra.

We also prove a few more results:

i) 3-point functions of half-chiral primary states in 2d $\mathcal{N} = (4, 4)$ SCFTs are not renormalized

ii) 3-point functions of chiral primaries in 2d $\mathcal{N} = (0, 4)$ SCFTs are not renormalized.

iii) "Extremal" n-point functions of 1/2 BPS operators in 4d $\mathcal{N} = 4$ SCFTs are not renormalized

iv) 3-point functions involving one 1/4 BPS and two 1/2 BPS operators in 4d $\mathcal{N} = 4$ SCFTs are not renormalized.

Notice that our results are non-perturbative in the coupling constant of the theory and hold for any gauge group, and in particular they do not depend on a large N limit.

The chapter is organized as follows: in section 4.1 we present some necessary background material, mostly on marginal deformations of CFTs, Ward identities, the structure of short multiplets and their 3-point functions. In section 4.2 we outline the main proof of the non-renormalization theorem in general context. In section 4.3 we provide a detailed proof of the theorem for 2d $\mathcal{N} = (4, 4)$ SCFTs. In section 4.4 we present a detailed proof of the theorem for 4d $\mathcal{N} = 4$ theories. In the remaining sections and appendices we provide various additional details.

4.1 Introduction and preliminary material

One of the earliest checks of the AdS/CFT correspondence [7, 10, 11] was the matching of 3-point functions of chiral primaries. This was first done [54] for the duality between the $\mathcal{N} = 4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and later [55, 56, 57, 58] for the duality between the two dimensional $\mathcal{N} = (4,4)$ D1/D5 CFT and IIB string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$. The matching of 3-point functions is non-trivial because they are not fully determined by symmetry considerations.

Notice that a priori the matching did not have to work: even if it did not work, it would not indicate a problem with the AdS/CFT correspondence. The bulk and boundary computations of 3-point functions are performed at different points of the moduli space (i.e. different values of the coupling constants), and in general there is no reason to expect that such computations should give the same answer. The fact that the computations do indeed agree strongly suggests that these 3-point functions are actually independent of the coupling constant. In other words, there should exist a "non-renormalization theorem" for 3-point functions of chiral primaries in superconformal field theories with sufficient amount of supersymmetry.

For the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ and the $\mathcal{N} = 4$ SYM a proof of such a non-renormalization theorem was given in a series of works [59, 60, 61, 62, 63, 64]. The proof relies on the formalism of analytic superspace, and here we provide a simpler proof that does not make use of this machinery.[3] In the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ with $\mathcal{N} = (4,4)$ supersymmetry a (partial) non-renormalization theorem was proven in [53] using elementary techniques. This theorem is partial because it does not include the most general case of 3-point function of chiral primaries, but only the case of "extremal" correlation functions, and here we complete the theorem.

In the remaining part of this section we review the basic ingredients that go into the proof of the non-renormalization theorem.

4.1.1 Conformal perturbation theory

Our goal is to understand the coupling constant dependence of certain correlation functions. Changing a coupling constant g in a CFT corresponds to deforming the CFT by an exactly marginal operator $\mathcal{O}$. Correlators in the deformed theory can be computed from *integrated* correlators in the undeformed CFT. We have

[3] In [65] such a proof was proposed. However we believe that the arguments in that paper are actually not sufficient in order to prove the non-renormalization theorem. More explanations about this can be found at the end of section 4.4.1, in particular see footnote 12.

schematically

$$\frac{\partial}{\partial g}\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \sim \int d^dx\,\langle \mathcal{O}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle. \tag{4.1}$$

This is only schematic because the integral has to be regularized due to UV divergences when x approaches the other insertions. Because of these divergences and the need for regularization, marginal deformations at second order do not commute, we refer the reader to [66, 53, 67] for more details.[4] Physically this can be understood as a certain kind of operator mixing: under marginal deformations there is an ambiguity of coupling-constant dependent redefinitions of operators with the same quantum numbers.

The picture that we should keep in mind is that in general the moduli space $\mathcal{M}$ (i.e. the space of marginal couplings of the CFT) is a higher dimensional manifold and the local operators of the CFT are sections of vector bundles over $\mathcal{M}$. So more precisely instead of (4.1), what we have is that

$$\nabla_g\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \equiv \int d^dx\,\langle \mathcal{O}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle. \tag{4.2}$$

In general $[\nabla_{g_1},\nabla_{g_2}]\neq 0$, which expresses that there is non-trivial operator mixing over the moduli space. The bundles on which operators take values have have non-trivial connection which enters this covariant derivative.

In this chapter we will prove that certain correlation functions of chiral primaries do not depend on the couplings of the CFT. More precisely, what we need to show is that the covariant derivative of such correlators with respect to the couplings is zero. This is the "covariant" way to phrase the non-renormalization of correlation functions, which is unambiguous with respect to coupling constant dependent operator redefinitions.

Actually we will prove a stronger statement. We will not only show that, in certain supersymmetric CFTs, and for specific choices of the operators $\mathcal{O}_1\dots\mathcal{O}_n$, the RHS of (4.2) vanishes, but we will show that the *integrand* on the RHS of (4.2) is zero. This is a sufficient condition for the LHS to vanish. The integral is supposed to be carefully regularized, and the operators are never brought on top of each other, so there is no subtlety with possible "contact terms" (see also footnote 4).

[4] An alternative approach is to attribute this phenomenon to the presence of "contact terms", as explained in [68, 69]. Instead, the point of view we are adopting is that CFT correlators are only defined at distinct points and hence "contact terms" play absolutely no role. From this point of view operator mixing comes from the *definition* of the regularized integrated correlators, as was nicely discussed in [66]. The two approaches are equivalent, but we find it conceptually more clear to follow [66] and to avoid talking about contact terms. Hence, in the entirety of this chapter we will never bring two local operators on the same spacetime point.

Let us then emphasize once more that if

$$\langle \mathcal{O}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = 0 \tag{4.3}$$

for distinct points, then it is guaranteed that the correlator $\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle$ does not change under marginal deformations by $\mathcal{O}$.

4.1.2 Superconformal Ward identities

For a general strongly coupled CFT there is no reason to expect the vanishing of a correlator of the form (4.3). The simplest reason for a correlator to exactly vanish is because of a symmetry of the theory. For example, if the CFT has an (unbroken) global $U(1)$ symmetry, then a correlator is automatically zero if the charges of the inserted operators do not satisfy $\sum q_i = 0$. In a sense, our proof will be based on similar conservation conditions, coming from the supersymmetric (and superconformal) charges of the theory.

Symmetries in CFTs are expressed in terms of Ward identities. In the case of a global internal symmetry with a conserved current J_μ we define the charge as $R = \int d^{d-1}x\, J_0(x)$ and then we can show that for any set of local operators we have

$$\sum_{i=1}^{n}\langle \mathcal{O}_1(x_1)\dots[R,\mathcal{O}_i](x_i)\dots\mathcal{O}_n(x_n)\rangle = 0. \tag{4.4}$$

For global internal symmetries, this is the only type of Ward identity that we have.

The situation is richer for conserved "currents" with additional spacetime indices. For example, let us consider the stress energy tensor which satisfies $\partial^\mu T_{\mu\nu} = 0$. Consider an arbitrary vector field $V_\mu(x)$ and construct the operator $j^V_\mu(x) = V^\nu(x)T_{\mu\nu}(x)$. Using that $T_{\mu\nu}$ is conserved and symmetric we have that $\partial^\mu j^V_\mu(x) = \frac{1}{2}(\partial^\mu V^\nu + \partial^\nu V^\mu)T_{\mu\nu}$. Combining this with the tracelessness of $T_{\mu\nu}$ we conclude that any vector field which satisfies $\partial^\mu V^\nu + \partial^\nu V^\mu = \omega(x)g^{\mu\nu}$ leads to a conserved current j^V_μ. Of course this is the condition for a conformal Killing vector field. Provided that $V_\mu(x)$ does not grow too fast at infinity, this can be used to define corresponding charges $R^V = \int d^{d-1}x j^V_0(x)$ and corresponding Ward identities, characterized by the choice of V. These conformal Ward identities are slightly more complicated than the ones for global internal symmetries, but are of course very well understood.

In this chapter we will mostly use the superconformal Ward identities, i.e. the identities that follow from the existence of a supercurrent operator in the CFT. This is an operator of dimension $d - \frac{1}{2}$ and two Lorentz indices, a vector index μ and a spinor index a. Let us denote this operator as $G_{\mu a}$. We can construct (fermionic) conserved currents out of the supercurrent by contracting it with a spinor valued field $\psi^a(x)$ as

$$j_\mu^\psi(x) = \psi^a(x) G_{\mu a}(x). \tag{4.5}$$

The condition for j_μ^ψ to be conserved is that $\psi^a(x)$ must be a conformal Killing spinor. We also need to impose that it does not grow too fast as $|x| \to \infty$ in order for the corresponding charge $\int d^{d-1}x \, j_0^\psi(x)$ to be well-defined. Then we find the following possibilities. The first possibility is to take $\psi^a(x)$ to be a constant spinor independent of x. Then the charges $\int j_0^\psi$ are the usual supercharges that we denote by $\mathbf{Q}$. The second possibility is to take $\psi^a(x)$ to be linear in x and then the corresponding charges turn out to be the "superconformal partners" of $\mathbf{Q}$ that we denote by $\mathbf{S}$.[5] For a general $\psi^a(x)$ which grows at most linearly at infinity the Ward identities have the schematic form

$$\begin{aligned} &\sum_i \psi(x_i) \langle \mathcal{O}_1(x_1) \ldots [\mathbf{Q}, \mathcal{O}_i\}(x_i) \ldots \mathcal{O}_n(x_n) \rangle \\ +&\sum_i \psi'(x_i) \langle \mathcal{O}_1(x_1) \ldots [\mathbf{S}, \mathcal{O}_i\}(x_i) \ldots \mathcal{O}_n(x_n) \rangle = 0. \end{aligned} \tag{4.6}$$

Here we have not shown explicitly the spinor indices of ψ and how they are contracted with the supercharges in order not to clutter the notation. Also by $[\ldots, \ldots\}$ we mean commutator or anticommutator depending on whether the operator $\mathcal{O}$ is bosonic or fermionic.

It is important to notice that we can always choose $\psi^a(x)$ to vanish at some particular point x_i and then the corresponding term proportional to $[\mathbf{Q}, \mathcal{O}\}(x_i)$ does not contribute to the Ward identity. This observation is quite crucial and it is a basic fact on which our proof is based. Notice also that if the operators $\mathcal{O}$ are superconformal primaries, we have $[\mathbf{S}, \mathcal{O}\}(x_i) = 0$ and the Ward identity becomes particularly simple.

4.1.3 Chiral primary 3-point functions

Now we come to the correlators, whose non-renormalization we aim to prove. These are 3-point functions of chiral primary operators, that is operators belonging to "short" multiplets of the superconformal algebra. In theories with extended

[5] Our notation in this section is rather loose. By $\mathbf{S}$ we simply mean the "superconformal partner" of $\mathbf{Q}$ in the sense that they both come from the same supercurrent. In two dimensional notation we would have that $\mathbf{Q} \sim G_{-\frac{1}{2}}$ while $\mathbf{S} \sim G_{+\frac{1}{2}}$. In 4d SCFTs if by $\mathbf{Q}$ we denote one of the left-chiral supercharges Q_a then the corresponding $\mathbf{S}$ which comes from the same supercurrent is right-chiral $\mathbf{S} \sim \overline{S}_{\dot{a}}$. We hope the notation is not too confusing; more details on the superconformal Ward identities for 4d SCFTs can be found in [65, 67].

supersymmetry such operators must fall into representations of the non-abelian R-symmetry. For example, in the $\mathcal{N}=4$ SYM the R-symmetry is $SU(4)$ while in 2d CFTs with $\mathcal{N}=(4,4)$ it is $SU(2)_L \otimes SU(2)_R$. Hence the chiral primary operators are labeled by the representation $\mathcal{R}$ of the R-symmetry and also by a set of additional indices $\vec{m}$ that denote the specific element of the representation. As we will explain later, the general structure of the 3-point function is

$$\langle \phi_I^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\ \phi_J^{(\mathcal{R}_2,\vec{m}_2)}(x_2)\ \phi_K^{(\mathcal{R}_3,\vec{m}_3)}(x_3)\rangle = C_{IJK} \times (\text{group theoretic factors}) \tag{4.7}$$

where the indices I, J, K label various irreps of the R-symmetry group. The only dynamical information is in the coefficients C_{IJK}, which are precisely the coefficients whose independence of the coupling we need to prove. The "group theoretic factors" above contain both R-symmetry related factors, as well as the x-dependence of the correlator which is completely fixed by conformal invariance.

Given the general form (4.7) of these 3-point functions it becomes clear that we can isolate the desired coefficient C_{IJK} by evaluating the correlator for specific alignments of the $\vec{m}$'s, as long as the corresponding group theoretic factor is non-zero. In particular, as we will explain in more detail later, it is possible to choose the operator at x_2 to be a "highest weight" state in the representation $\mathcal{R}_2$ and the one at x_3 to be a "lowest weight" state in $\mathcal{R}_3$, while the one at x_1 will be "mixed" i.e. will have weight $\vec{m}$ which is neither highest nor lowest. So we have that

$$C_{IJK} \sim \langle \phi_I^{(\mathcal{R}_1,\vec{m})}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle, \tag{4.8}$$

where $+, -$ denote the highest and lowest weight state respectively.

The constant of proportionality depends on group theoretic factors and is not relevant for us as long as it is non-zero. Also notice that from the point of view of chiral primaries in $\mathcal{N}=1$ theories, the operator at x_2 would be "chiral primary", the one at x_3 would be "anti-chiral primary" while the one at x_1 would be neither chiral nor anti-chiral.

4.1.4 Null vectors in short multiplets

Before we proceed we need to make one more observation. The highest weight state of a short representation is annihilated by some of the supercharges. The lowest weight state is annihilated by the conjugate supercharges. However, "intermediate" weight states in short representations are generally not annihilated by *any* of the supercharges.

Even so, these intermediate states satisfy "nullness conditions", by which we mean that certain linear combinations of superconformal descendants of intermediate weight states in the multiplet are zero. These can be derived by starting with the nullness conditions of the highest weight state $[\mathbf{Q}, \phi^{(\mathcal{R},+)}\} = 0$ and acting on it with lowering operators of the R-symmetry algebra. Using the Jacobi identity these operators act both on the $\mathbf{Q}$ and on the chiral primary. Acting with such lowering operators repeatedly we get conditions which have the following general form

$$[\mathbf{Q}\,,\,\phi^{(\mathcal{R},\vec{m})}\} = \sum_i c_i\,[\mathbf{Q}'_i\,,\,\phi^{(\mathcal{R},\vec{m}'_i)}\}, \tag{4.9}$$

where $\mathbf{Q}'_i$ are supercharges with R-symmetry weights different from those of $\mathbf{Q}$ and of course some of the c_i's may be zero. The operators $\phi^{(\mathcal{R},\vec{m}'_i)}$ are in the same multiplet as $\phi^{(\mathcal{R},\vec{m})}$ but have different R-symmetry weight.

This condition will perhaps become more clear once we study it in specific theories.

4.1.5 Supersymmetric marginal deformations

The final element that we need is that the marginal deformations that we are interested in are of special kind, they are deformations that preserve not only conformal invariance but also supersymmetry. Imposing that superconformal invariance is preserved implies that the marginal operator must be a descendant of an (anti)-chiral primary. Let us illustrate this with a few examples.

In 2d $\mathcal{N}=(2,2)$ theories, the supersymmetric marginal deformations are of the form $\{\mathbf{G}^-_{-\frac{1}{2}}, [\overline{\mathbf{G}}^-_{-\frac{1}{2}}, \phi]\}$ and $\{\mathbf{G}^+_{-\frac{1}{2}}, [\overline{\mathbf{G}}^+_{-\frac{1}{2}}, \overline{\phi}]\}$ where $\phi, \overline{\phi}$ are chiral primaries in the (c,c) and (a,a) rings respectively, with conformal dimension $(\frac{1}{2},\frac{1}{2})$, and also of the form $\{\mathbf{G}^-_{-\frac{1}{2}}, [\overline{\mathbf{G}}^+_{-\frac{1}{2}}, \psi]\}$ and $\{\mathbf{G}^+_{-\frac{1}{2}}, [\overline{\mathbf{G}}^-_{-\frac{1}{2}}, \overline{\psi}]\}$ where $\psi, \overline{\psi}$ are chiral primaries in the (a,c) and (c,a) rings respectively, again with conformal dimension $(\frac{1}{2},\frac{1}{2})$.

Another example is the $\mathcal{N}=4$ SYM in 4d. There is only one (complex) marginal coupling $\mathcal{O}_\tau$ preserving the full $\mathcal{N}=4$ supersymmetry, corresponding to changes of the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. The moduli space of this CFT is the upper half τ plane, modded out by the appropriate S-duality group. The operator $\mathcal{O}_\tau$ is the (holomorphic part of the) Lagrangian density. The important thing for us is that it can be written as

$$\mathcal{O}_\tau = \{\mathbf{Q}, [\mathbf{Q}, \{\mathbf{Q}, [\mathbf{Q}, \mathrm{Tr}(Z^2)]\}]\}, \tag{4.10}$$

where Z is one of the complex adjoint scalars. Here we did not write explicitly the indices of the supercharges. More details can be found in appendix 4.A.2. Notice that these supercharges are all of the same chirality so they (anti)-commute and their order is not important.

Instead of giving more examples, let us emphasize the main point: supersymmetric marginal operators can be written as

$$\mathcal{O} = \{\mathbf{Q}, \Lambda\}, \tag{4.11}$$

where Λ is *some* operator and $\mathbf{Q}$ is a supercharge that annihilates either highest, or lowest weight states. The operator Λ is a descendant of chiral primaries of specific conformal dimension (the details depend on the theory).

Finally, let us recall that the marginal operator has to be a singlet of the R-symmetry of the theory. If not, it would break part of the supersymmetry.

4.2 Outline of the proof

Now we have collected all the ingredients and we can put them together to give an outline of the proof. The (theory-specific) details will be presented in the next sections.

Step 1: We isolate the dynamical part of the 3-point function by aligning the chiral primaries so that one of them is highest weight, the other lowest and the third intermediate. So we have

$$C_{IJK} \sim \langle \phi_I^{(\mathcal{R}_1,\vec{m})}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle. \tag{4.12}$$

Step 2: We write the marginal operator corresponding to the change of a marginal coupling g as $\mathcal{O} = \{\mathbf{Q}, \Lambda\}$. Hence we would like to prove the vanishing of

$$\nabla_g C_{IJK} \sim \int d^d x\, \langle \{\mathbf{Q}, \Lambda\}(x)\ \phi_I^{(\mathcal{R}_1,\vec{m})}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle. \tag{4.13}$$

Let us denote the *integrand* by $\mathcal{I}$, on which we now focus.

Step 3: Without loss of generality we can assume that $\mathbf{Q}$ annihilates the highest weight operator at x_2. Then we use the superconformal Ward identity (4.6) with a spinor $\psi^a(x)$ vanishing at x_3 to move $\mathbf{Q}$ away from the point x. The result is[6]

$$\mathcal{I} \sim \langle \Lambda(x)\ [\mathbf{Q}, \phi_I^{(\mathcal{R}_1,\vec{m})}\}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle. \tag{4.14}$$

The important point here is that there is no other contribution to the Ward identity.[7]

[6] Again, by $[\ldots,\ldots\}$ we mean the commutator (or anticommutator) if the operator is bosonic (or fermionic).

[7] Notice that ϕ_I, ϕ_J, ϕ_K are all superconformal primaries, so they are annihilated by the $\mathbf{S}$'s.

Step 4: We use the "nullness condition" (4.9) for the operator at x_1 to rewrite this as

$$\mathcal{I} \sim \sum_{\prime} \langle \Lambda(x)\ [\mathbf{Q}', \phi_I^{(\mathcal{R}_1,\vec{m}')}\}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle. \tag{4.15}$$

where $\vec{m}'$ is some other element of the same representation and $\mathbf{Q}'$ supercharges with R-symmetry weight different from those of $\mathbf{Q}$.

Step 5: The set of supercharges $\mathcal{A}$ can be partitioned into two disjoint sets $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$, where the charges in $\mathcal{A}_+$ annihilate the highest weight states and the charges in $\mathcal{A}_-$ annihilate the lowest weight states. If $\mathbf{Q}' \in \mathcal{A}_+$ then we use the Ward identity with a spinor vanishing at x_3 to move $\mathbf{Q}'$ away from x_1. If $\mathbf{Q}' \in \mathcal{A}_-$ then we choose a spinor which vanishes at x_2. In both cases we have

$$\mathcal{I} \sim \sum_{\prime} \langle \{\mathbf{Q}', \Lambda\}(x)\ \phi_I^{(\mathcal{R}_1,\vec{m}')}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3)\rangle. \tag{4.16}$$

Step 6: Remarkably the quantum numbers conspire in such a way that in the theories that we study $\{\mathbf{Q}', \Lambda\} = 0$. Hence

$$\mathcal{I} = 0 \qquad \Rightarrow \qquad \nabla_g\, C_{IJK} = 0. \tag{4.17}$$

This completes the proof.

Here we have skipped many theory-dependent details, which will be presented in the next sections.

4.3 Two-dimensional CFTs with $\mathcal{N} = (4,4)$ supersymmetry

In this section we present the non-renormalization theorem for 3-point functions of chiral primaries in two-dimensional $\mathcal{N} = (4,4)$ superconformal field theories, generalizing and completing the results of [53].

In the first subsection we describe the short multiplets in these theories and review the general form of the 3-point function of chiral primaries. In the second subsection we prove the non-renormalization theorem.

4.3.1 Short representations and their 3-point functions

The R-symmetry of the $\mathcal{N} = (4,4)$ superconformal algebra is $SU(2)_L \otimes SU(2)_R$. The left moving supercharges are denoted by $\mathbf{G}^{ar}_{-\frac{1}{2}}$. Here the index $a = \pm$ denotes the J^3 eigenvalue with respect to the left-moving $SU(2)_L$ R-symmetry, while the index $r = \pm$ denotes the eigenvalue of the supercharge under a left $SU(2)$ outer automorphism of the $\mathcal{N} = 4$ algebra. The right-moving supercharges have similar structure. We refer the reader to [53] for more details.

Representations of the algebra are labeled by the conformal dimension $\{h, \bar{h}\}$ and the R-symmetry representation $\{j, \bar{j}\}$ of the superconformal primaries[8] of the multiplet. Notice that a given multiplet contains several superconformal primaries which differ by their $SU(2)_L \otimes SU(2)_R$ quantum numbers. Unitarity requires

$$h \geq j, \qquad \bar{h} \geq \bar{j}. \tag{4.18}$$

Multiplets that saturate the bound are "short" and are usually called "chiral primary" multiplets.

To simplify notation, in the following we will sometimes write only the quantum numbers of the left-moving sector. For a multiplet characterized by conformal dimension h and R-symmetry quantum number j, we have the following set of superconformal primaries

$$\phi^{(j,m)}, \qquad m = -j, \ldots, +j, \tag{4.19}$$

which differ by their J^3 eigenvalue m. All these operators are superconformal primaries, they have conformal dimension h and can be recovered from the "highest weight" state of the multiplet by acting with $SU(2)_L$ lowering operators

$$\phi^{(j,m)} \sim \overbrace{[J^-, \ldots [J^-}^{j-m}, \phi^{(j,j)}] \ldots]. \tag{4.20}$$

The "highest weight" operator of a short multiplet $\phi^{(j,j)}$ is annihilated by some of the supercharges

$$[\mathbf{G}^{+r}_{-\frac{1}{2}}\ ,\ \phi^{(j,j)}\} = 0, \qquad r = +, - \tag{4.21}$$

and similarly for the "lowest weight" one $\phi^{(j,-j)}$

$$[\mathbf{G}^{-r}_{-\frac{1}{2}}\ ,\ \phi^{(j,-j)}\} = 0, \qquad r = +, - \tag{4.22}$$

The other members of the short multiplet $\phi^{(j,m)}$ with $m \neq \pm j$ *are not* annihilated by any of the left moving supercharges. They do however satisfy nullness conditions, which can be derived by starting with $[\mathbf{G}^{+r}_{-\frac{1}{2}}\ ,\ \phi^{(j,j)}_I\} = 0$ and acting with lowering operators J^-. This leads to the following relation[9]

$$[\mathbf{G}^{+r}_{-\frac{1}{2}}\ ,\ \phi^{(j,n)}_I\} \quad \sim \quad [\mathbf{G}^{-r}_{-\frac{1}{2}}\ ,\ \phi^{(j,n+1)}_I\}, \tag{4.23}$$

[8] i.e. operators annihilated by all $\mathbf{G}^{ab}_n$, $n > 0$.

[9] This equation is proven in appendix 4.A.3.

where the constant of proportionality is nonzero as long as $n < j$.

Notice that here there is some potentially confusing terminology: from the $\mathcal{N} = (4,4)$ point of view, all the operators $\phi^{(j,m)}$ are sometimes called "chiral primaries", since they all belong to the same short multiplet. If however we consider an $\mathcal{N} = (2,2)$ subalgebra then the operator $\phi^{(j,j)}$ would be called "chiral", the operator $\phi^{(j,-j)}$ "antichiral" and the other operators $\phi^{(j,m)}$ with $m \neq \pm j$ would be neither chiral nor antichiral.

Let us write the general form of the 3-point function of chiral primary operators. We have

$$\begin{gathered}\langle \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3)\rangle = C_{IJK} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 \\ \overline{m}_1 & \overline{m}_2 & \overline{m}_3 \end{pmatrix} \\ \times \frac{1}{x_{12}^{(j_1+j_2-j_3)} x_{23}^{(j_2+j_3-j_1)} x_{13}^{(j_1+j_3-j_2)}} \frac{1}{\bar{x}_{12}^{(\bar{j}_1+\bar{j}_2-\bar{j}_3)} \bar{x}_{23}^{(\bar{j}_2+\bar{j}_3-\bar{j}_1)} \bar{x}_{13}^{(\bar{j}_1+\bar{j}_3-\bar{j}_2)}}.\end{gathered} \tag{4.24}$$

Here we did not write explicitly the $SU(2)_L \otimes SU(2)_R$ quantum numbers on the LHS of the equation.

The x-dependence in (4.24) is fixed by conformal invariance in terms of the conformal dimension of the operators. The dependence on the quantum numbers $(j, m\,; \bar{j}, \overline{m})$ is fixed by the $SU(2)_L \otimes SU(2)_R$ R-symmetry and is expressed by the 3-j symbols presented above. All the dynamical information is encoded in the coefficient C_{IJK}, which as we can see only depends on the choice of chiral primary representations I, J, K and not on the specific representatives from each of them (i.e. does not depend on the R-symmetry quantum numbers $m, \overline{m}$).

Our goal is to show that the constants C_{IJK} do not depend on the coupling constants of the CFT.

Going back to the distinction between a "chiral primary" in $\mathcal{N} = (2,2)$ theories and a "chiral primary" in $\mathcal{N} = (4,4)$ theories let us notice the following: in $\mathcal{N} = (2,2)$ theories R-charge conservation requires that the three operators satisfy the condition $j_3 = j_1 + j_2$ (or permutations) - and similarly for the right-moving sector. These would be "extremal" 3-point functions of chiral primaries from the $\mathcal{N} = (4,4)$ point of view. However in $\mathcal{N} = (4,4)$ theories there are also 3-point functions of chiral primaries that are not extremal.

In [53] a non-renormalization theorem for 3-point functions was proven for this special "extremal" case. The 3-point function can be viewed as a 3-point function of chiral primaries of an $\mathcal{N} = (2,2)$ subalgebra. In the more general case where $j_3 \neq j_1 + j_2$ this is not possible. There is no way to align all three operators so that they are all in the chiral ring of a given $\mathcal{N} = (2,2)$ subalgebra. Nevertheless, the "non-extremal" 3-point functions also seem to be protected and thus should obey some non-renormalization theorem, which we will prove in the next subsection.

4.3.2 The non-renormalization theorem in 2d

Theories with $\mathcal{N} = (4,4)$ supersymmetry in two dimensions have a moduli space of marginal deformations which is locally of the form $\frac{SO(n,4)}{SO(n)\times SO(4)}$ [68]. Here n is the number of chiral primary multiplets that transform in the $(1/2, 1/2)$ representation of the $SU(2)_L \otimes SU(2)_R$ R-symmetry group.

Let us consider the 3-point function of operators which belong to chiral primary multiplets

$$\langle \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3) \rangle, \tag{4.25}$$

where for simplicity we do not write any R-symmetry indices. Let us also consider a marginal operator $\mathcal{O}$ corresponding to the change of a marginal coupling constant g. By definition we have

$$\nabla_g \langle \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3) \rangle \equiv \int d^2x \, \langle \mathcal{O}(x)\ \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3) \rangle. \tag{4.26}$$

As discussed above, in order to prove that the 3-point functions are independent of the coupling we have to show that the expression above vanishes. We will actually prove a stronger statement, namely that

$$\mathcal{I} \equiv \langle \mathcal{O}(x)\ \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3) \rangle = 0 \tag{4.27}$$

even without integrating over x. We will follow the steps outlined in section 4.2. In order to prove this we will use two properties of the $\mathcal{N} = (4,4)$ algebra

First, we exploit the $SU(2)_L \otimes SU(2)_R$ structure of the correlator (4.24). If we simply want to compute the 3-point function C_{IJK}, or rather to prove that it is independent of the coupling, we are free to evaluate the correlator for any alignment of the operators for which the 3j symbols are non-vanishing. Hence we will choose the representatives of the other chiral primaries in the following way

$$\mathcal{I} \sim \langle \mathcal{O}(x)\ \phi_I^{(j_1,n)}(x_1)\ \phi_J^{(j_2,j_2)}(x_2)\ \phi_K^{(j_3,-j_3)}(x_3) \rangle, \tag{4.28}$$

where $n = j_3 - j_2$. The constant of proportionality is some (non-vanishing) group-theoretic factor which is of no interest for our argument. Notice that from the point of view of an $\mathcal{N} = 2$ subalgebra the operator at x_2 is "chiral primary", the operator at x_3 is "anti-chiral primary" while the operator at x_1 is neither chiral on antichiral.

Second, without loss of generality[10] we can assume that the marginal operator can be written as $\mathcal{O} = \{\mathbf{G}^{+r}_{-\frac{1}{2}}\,,\,[\overline{\mathbf{G}}^{+s}_{-\frac{1}{2}}\,,\,\overline{\phi}]\}$ where $\overline{\phi}$ is an element of a chiral primary multiplet of conformal weight $\left(\frac{1}{2},\frac{1}{2}\right)$ and which is aligned to have $(J^3,\overline{J^3}) = \left(-\frac{1}{2},-\frac{1}{2}\right)$.

Then we have that

$$\mathcal{I} \sim \langle\left(\{\mathbf{G}^{+r}_{-\frac{1}{2}}\,,\,[\overline{\mathbf{G}}^{+s}_{-\frac{1}{2}}\,,\,\overline{\phi}]\}\right)(x)\;\phi_I^{(j_1,n)}(x_1)\;\phi_J^{(j_2,j_2)}(x_2)\;\phi_K^{(j_3,-j_3)}(x_3)\rangle. \tag{4.29}$$

Using a superconformal Ward identity (4.6) for $\mathbf{G}^{+r}_{-\frac{1}{2}}$ with a conformal Killing spinor vanishing at the point x_3 we find that this can be written as

$$\mathcal{I} \sim \langle[\overline{\mathbf{G}}^{+s}_{-\frac{1}{2}}\,,\,\overline{\phi}](x)\;[\mathbf{G}^{+r}_{-\frac{1}{2}}\,,\,\phi_I^{(j_1,n)}\}(x_1)\;\phi_J^{(j_2,j_2)}(x_2)\;\phi_K^{(j_3,-j_3)}(x_3)\rangle, \tag{4.30}$$

where the constant of proportionality in this expression is different from zero. Here we used that $\mathbf{G}^{+r}_{-\frac{1}{2}}$ annihilates the operator at x_2.

Now we use the nullness condition (4.23) to rewrite it as

$$\mathcal{I} \sim \langle[\overline{\mathbf{G}}^{+s}_{-\frac{1}{2}}\,,\,\overline{\phi}](x)\;[\mathbf{G}^{-r}_{-\frac{1}{2}}\,,\,\phi_I^{(j_1,n+1)}\}(x_1)\;\phi_J^{(j_2,j_2)}(x_2)\;\phi_K^{(j_3,-j_3)}(x_3)\rangle. \tag{4.31}$$

Finally we use a superconformal Ward identity for $\mathbf{G}^{-r}_{-\frac{1}{2}}$ with a conformal Killing spinor which vanishes at the point x_2. All other operators do not contribute because they are annihilated by $\mathbf{G}^{-r}_{-\frac{1}{2}}$, hence we find

$$\mathcal{I} = 0. \tag{4.32}$$

This proves that 3-point functions of chiral primaries are independent of the coupling constant.

Notice that it would not be possible to apply a similar argument to prove non-renormalization of 4- and higher point functions of chiral primaries (unless they are extremal [53]), which is of course consistent, since we know that such correlators *do* depend on the coupling constants.

4.4 Four-dimensional $\mathcal{N} = 4$ SCFTs

The same type of argument can be used to prove the non-renormalization of 3-point functions of 1/2 BPS chiral primaries in four-dimensional SCFTs with $\mathcal{N} = 4$ supersymmetry.

[10] This is a general property of $\mathcal{N} = (4,4)$ SCFTs which was discussed in detail in [53].

4.4.1 Short representations

Now the R-symmetry is $SU(4)$. We choose a basis for its Cartan subalgebra. The short representations that we are interested in are those with Dynkin labels $[0,k,0]$, Lorentz spin $(j,\bar{j})=(0,0)$ and conformal dimension $\Delta=k$. These are the "1/2 BPS" operators of the $\mathcal{N}=4$ algebra. In terms of Young tableaux for $SU(4)$ these representations correspond to tableaux with k columns of length 2 (we refer to appendix 4.A.2 for more details). As before we denote the superconformal primaries of such a multiplet by

$$\phi^{(k,\vec{m})}, \tag{4.33}$$

where now $\vec{m}$ labels the weight of the state inside the $SU(4)$ multiplet (i.e. $\vec{m}$ are the eigenvalues of the state under the Cartan generators). Of special interest will be the highest and lowest weight states of any given representation, which we call $\phi^{(k,\pm)}$. For example, in some conventions highest weight operators are $\mathrm{Tr}(Z^k)$ and their multi-trace products.

Let us recall some group theory (more details are given in appendix 4.A.1 and 4.A.2). We denote by E_i the generators of $SU(4)$ corresponding to positive simple roots, or raising operators. The highest weight state satisfies $[E_i\,,\,\phi^{(k,+)}]=0$. Other operators in the same $SU(4)$ multiplet can be recovered starting from $\phi^{(k,+)}$ and acting with the lowering operators $E_i^\dagger$

$$\phi^{(k,\vec{m})}\sim[E^\dagger_{i_n}\,,\dots[E^\dagger_{i_1}\,,\,\phi^{(k,+)}]\dots], \tag{4.34}$$

where the product is some specific combination of the "negative simple roots", perhaps with repeated appearances.

Of course equivalently we can start from the lowest weight state and get the same state by acting with "raising" operators.

$$\phi^{(k,\vec{m})}\sim[E_{i_n}\,,\dots[E_{i_1}\,,\,\phi^{(k,-)}]\dots]. \tag{4.35}$$

It is a group-theoretic fact that in a tensor product of the form $[0,k_1,0]\otimes[0,k_2,0]$ any representation of the form $[0,k_3,0]$ appears either one time or none.[11] Hence the general form of a 3-point function is

$$\begin{aligned}\langle\phi_I^{(k_1,\vec{m}_1)}(x_1)\,\phi_J^{(k_2,\vec{m}_2)}(x_2)\,\phi_K^{(k_3,\vec{m}_3)}(x_3)\rangle &= C_{IJK}\,\mathbf{G}(k_1,\vec{m}_1;k_2,\vec{m}_2;k_3,\vec{m}_3)\\ &\times\frac{1}{|x_{12}|^{k_1+k_2-k_3}|x_{23}|^{k_2+k_3-k_1}|x_{13}|^{k_1+k_3-k_2}}\end{aligned} \tag{4.36}$$

[11] If k_1,k_2,k_3 satisfy the triangle (in)-equality and $\frac{k_1+k_2+k_3}{2}$ is an integer, then the representation appears one time. Otherwise it does not appear.

where $\mathbf{G}(k_1, \vec{m}_1; k_2, \vec{m}_2; k_3, \vec{m}_3)$, is the (unique) $SU(4)$ Clebsh–Gordan coefficient for three representations of the type $[0, k, 0]$ i.e. a group-theoretic factor. The dynamical information is encoded in the coefficient C_{IJK}.

Notice that, as emphasized previously in the chapter, it is only the highest and lowest weight states of the short multiplets that are annihilated by supercharges. "Intermediate weight" states are generally not annihilated by any of the supercharges (though they lead to certain "nullness conditions" as explained earlier). For example, while the superconformal primary operators of the form

$$C_{i_1 \dots i_n} \mathrm{Tr}(\phi^{i_1} \cdots \phi^{i_k}), \tag{4.37}$$

with C symmetric and traceless, are members of 1/2 BPS multiplets. However, for a generic choice of such symmetric traceless C, they are *not* annihilated by any supercharges. Only if C is chosen so that the corresponding operator is highest or lowest weight state with respect to $SU(4)_R$ is the operator annihilated by 1/2 of the supercharges.[12]

4.4.2 The non-renormalization theorem in 4d

First let us choose a basis of the left chiral supercharges so that they have definite weight under the Cartan subalgebra.[13] We denote these left chiral supercharges as $\mathbf{Q}^i_a$ where the index $i = 1, \dots 4$ is the $SU(4)$ and a the Lorentz index.

The theory has an exactly marginal operator $\mathcal{O}_\tau$ corresponding to the change of the complexified coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. As mentioned before and explained in detail in appendix 4.A.2 this operator can be written as

$$\mathcal{O}_\tau = (\mathbf{Q})^4 \phi^{(2,+)}, \tag{4.38}$$

where only four of the left-chiral supercharges act on the highest weight state. The notation $(\mathbf{Q})^4$ means the nested (anti)-commutator, as in equation (4.10), we hope this is obvious. Notice that the left chiral supercharges anticommute among themselves so we do not need to worry about the order with which they act on an operator.

The set of left chiral supercharges $\mathcal{A}$ can be partitioned into two disjoint sets $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$, where the charges in $\mathcal{A}_+$ annihilate the highest weight states of the 1/2 BPS multiplets and the charges in $\mathcal{A}_-$ annihilate the lowest weight states.

[12]In [65] it was incorrectly assumed that all superconformal primaries of the 1/2 BPS multiplet are annihilated by half of the supercharges, hence the proposed proof of the non-renormalization theorem in [65] is incomplete.

[13]And also a definite weight under the J_3 of the $SU(2)_L$ part of the Lorentz group.

The set of supercharges which appear in (4.38) is simply $\mathcal{A}_-$, and any other left chiral supercharge in $\mathcal{A}_+$ annihilates the operator $\phi^{(2,+)}$. This will be important below.

Consider now the change of a 3-point function under a deformation by $\mathcal{O}_\tau$. We will show that

$$\mathcal{I} \equiv \langle \mathcal{O}_\tau(x)\ \phi_I(x_1)\ \phi_J(x_2)\ \phi_K(x_3)\rangle = 0. \tag{4.39}$$

As before we can choose the $SU(4)$ alignment of the operators in such a way that

$$\mathcal{I} \sim \langle \mathcal{O}_\tau(x)\ \phi_I^{(k_1,\vec{m}_1)}(x_1)\ \phi_J^{(k_2,+)}(x_2)\ \phi_K^{(k_3,-)}(x_3)\rangle, \tag{4.40}$$

where the operator at x_2 is a highest weight state, the one at x_3 is lowest weight and the one at x_1 is of some general weight in the representation k_1. Using the form of the marginal operator we have

$$\mathcal{I} \sim \langle \Big((\mathbf{Q})^4 \phi^{(2,+)} \Big)(x)\ \phi_I^{(k_1,\vec{m}_1)}(x_1)\ \phi_J^{(k_2,+)}(x_2)\ \phi_K^{(k_3,-)}(x_3)\rangle. \tag{4.41}$$

Notice that the four supercharges acting on the operator at x are all left chiral so they (anti)-commute and their order is not important. As we mentioned above we call this set of supercharges $\mathcal{A}_-$. Also notice that all of these four supercharges annihilate the operator at x_3.

We take one of them, let us call it $\mathbf{Q}^\star$ and move it away using the Ward identity. We choose the conformal Killing spinor to vanish at the point x_2. Hence the correlator becomes

$$\mathcal{I} \sim \langle \Big((\mathbf{Q})^3 \phi^{(2,+)} \Big)(x)\ \ \Big([\mathbf{Q}^\star, \phi_I^{(k_1,m_1)}] \Big)(x_1)\ \phi_J^{(k_2,+)}(x_2)\ \phi_K^{(k_3,-)}(x_3)\rangle. \tag{4.42}$$

Now we will use the analogue of (4.23) coming from the fact that $\mathbf{Q}^\star$ annihilates the highest weight state of the representation k_1, that is.

$$[\mathbf{Q}^\star, \phi_I^{(k_1,\vec{m}_1)}] = \sum_{j\neq\star} [\mathbf{Q}^j, \mathcal{X}_j], \tag{4.43}$$

where all supercharges in the sum on the RHS are left chiral and different from $\mathbf{Q}^\star$ and $\mathcal{X}_j$ is either one of the elements of the multiplet $\phi_I^{(k_1,\vec{m}_j)}$ or perhaps zero.[14]. This important relation is proven in appendix 4.A.3

Next, for each of these $\mathbf{Q}^j$'s we apply the Ward identity (4.6) again. There are two possibilities:

1) $\mathbf{Q}^j$ is in $\mathcal{A}_-$: in this case we use (4.6) with a spinor vanishing at x_2. We do not get any contribution from x_3 because the operator is annihilated by the

[14]In either case the operator $\mathcal{X}$ is annihilated by the $\overline{S}$'s.

supercharges in $\mathcal{A}$. We do not get any contribution from x because the supercharge is already there, so it squares to zero.

2) $\mathbf{Q}^j$ is in $\mathcal{A}_+$: then this supercharge annihilates operators of the form $\phi^{(k,+)}$. Then we use (4.6) superconformal Ward identity with a spinor vanishing at x_3 and we get zero.

So in all cases the contribution is zero. Hence

$$\mathcal{I} = 0 \qquad \Rightarrow \qquad \nabla_\tau C_{IJK} = 0. \tag{4.44}$$

Exactly the same argument can be applied for the marginal operator $\overline{\mathcal{O}}_\tau \equiv (\overline{\mathbf{Q}})^4 \phi^{(2,-)}$. So all in all the 3-point functions are not renormalized and this completes the proof.

Notice that this argument fails, as expected, if we try to prove the non-renormalization of n-point functions of chiral primaries with $n > 3$ (unless they are "extremal"). The last step of the proof relied on the fact that there was at most one operator which was not annihilated by the supercharge involved in the Ward identity. We chose the Killing spinor to vanish at the point where this operator was inserted. If there had been more operators not annihilated by the supercharge, it would not have been possible to simultaneously "hide" their contributions to the Ward identity by choosing the Killing spinor appropriately.

4.5 Extremal correlators

Similar arguments can be used to show that a certain class of higher n-point functions are not renormalized. These are the so-called "extremal correlators" i.e. correlators where all chiral primaries are aligned to be "highest weight" except for one that is aligned to be "lowest weight" and which ensures R-charge neutrality

$$\langle \phi_1^{(\mathcal{R}_1,+)}(x_1)\; \phi_2^{(\mathcal{R}_2,+)}(x_2) \ldots \phi_n^{(\mathcal{R}_n,-)}(x_n)\rangle. \tag{4.45}$$

Charge conservation shows that the operators must satisfy $\Delta_n = \sum_{i=1}^{n-1} \Delta_i$.

That such correlators are not renormalized in 2d $\mathcal{N} = (4,4)$ theories was proven in [53]. The proof was based on the observation that in these theories a marginal operator can always be written as $\mathcal{O} = [\mathbf{G}^{-r}_{-\frac{1}{2}}, \Lambda]$. Then we can consider

$$\langle \mathcal{O}(z)\; \phi_1^{(\mathcal{R}_1,+)}(x_1)\; \phi_2^{(\mathcal{R}_2,+)}(x_2) \ldots \phi_n^{(\mathcal{R}_n,-)}(x_n)\rangle, \tag{4.46}$$

and use a Ward identity with a spinor vanishing at x_n to move the supercharges away from z. The operators at $x_1, \ldots, x_{n-1}$ do not contribute since they are annihilated by $\overline{\mathbf{G}}^{-r}_{-\frac{1}{2}}$ and the operator at x_n does not contribute because of the choice of the spinor in the Ward identity. Hence this correlator vanishes and the desired result is proven.

Let us quickly repeat the similar statement in $\mathcal{N} = 4$ SYM. In that theory we have two marginal operators, corresponding to changes of the coupling constant g and the θ-angle, which as explained before can be combined into the holomorphic and anti-holomorphic operators $\mathcal{O}_\tau, \overline{\mathcal{O}}_\tau$. One of these operators can be written as

$$\overline{\mathcal{O}}_\tau = (\overline{\mathbf{Q}})^4 \mathrm{Tr}(\overline{Z}^2), \tag{4.47}$$

where the supercharges $\overline{\mathbf{Q}}$ annihilate highest weight states of $SU(4)$. Hence we can use the Ward identity with a spinor vanishing at x_n to show that the analogue of (4.46) in $\mathcal{N} = 4$ vanishes. To complete the proof of the non-renormalization we also need to show that the same correlator vanishes for the marginal operator $\mathcal{O}_\tau$. We can use the fact that in $\mathcal{N} = 4$ theories this marginal operator can also be written as

$$\mathcal{O}_\tau = (\mathbf{Q})^4 \mathrm{Tr}(\overline{Z}^2), \tag{4.48}$$

where the $\mathbf{Q}$'s are supercharges of left chirality. This may look confusing when compared to (4.47) and against our intuition from theories with less supersymmetry, but it is indeed a true statement (explained in appendix 4.A.2).[15] The four supercharges in (4.48) annihilate the highest weight states of $SU(4)$ of the form $\phi^{(k,+)}$. Hence the Ward identities can be used as above to show that the correlator vanishes.

All in all we have proved that extremal n-point functions of 1/2 BPS chiral primaries in four-dimensional $\mathcal{N} = 4$ SCFTs are not renormalized.

4.6 Other extensions

In this section we list some immediate generalizations of our results.

4.6.1 Half-chiral states in 2d $\mathcal{N} = (4, 4)$

Interestingly, the argument in section 4.3 relied only on one sector, say the left moving one, of the CFT. This implies that the same argument goes through without changes when applied to 3-point functions of operators that are in short multiplets of the left-moving $SU(2)_L$ and long multiplets on the right-moving one.

[15] Notice that the four $\mathbf{Q}$'s in (4.48) *are not* the complex conjugates of the supercharges in (4.47).

Such operators are of the form (chiral, anything). Our argument shows that their 3-point functions are not renormalized as a function of the coupling constants. Notice that these states are related by spectral flow to states of the form (Ramond ground state, anything) which are precisely the microstates of the Strominger-Vafa black hole [70]. It would be interesting to explore the possible applications of this result.

Notice however that our arguments show that the 3-point functions of such states do not renormalize as a function of the coupling *assuming* that they remain chiral primaries during the deformation (i.e. that short multiplets do not combine and lift from the BPS bound). We have not addressed the issue of whether BPS states lift or not under marginal deformations.

4.6.2 3-point functions in 2d $\mathcal{N} = (0, 4)$ SCFTs

Another interesting case is that of two-dimensional CFTs with $(0, 4)$ supersymmetry. In string theory they arise on the worldvolume of bound states of M2/M5 branes wrapped on Calabi-Yau compactifications of M-theory and are relevant for the computation of the entropy of certain supersymmetric black holes [71, 72].

Theories with $\mathcal{N} = (0, 4)$ supersymmetry are not very well understood, but it is clear that on their "supersymmetric side" they have operators in short representations, which are the analogue of the (anything, chiral) operators in $(4, 4)$ CFTs. Our claim is that 3-point functions of such operators are not renormalized as a function of the coupling constants. This follows immediately from our proof, if we also remember that marginal operators in these theories can be written as $\mathcal{O} = [\overline{\mathbf{G}}^{\pm r}_{-\frac{1}{2}}, \phi]$ and its conjugate, where ϕ is "chiral primary" with respect to the right moving supersymmetric side. Also, notice that the statement holds only for operators which do not lift from the BPS bound as we vary the coupling.

4.6.3 Less supersymmetric multiplets in 4d

It would be interesting to generalize our results to 1/4 and 1/8 BPS operators in four dimensional $\mathcal{N} = 4$ SCFT. Unfortunately, the group theory structure of the correlators is much more intricate in this case. For example, the product of three 1/4 BPS scalar operators, which sit in $[q, p, q]$ representations of the $SU(4)$ R-symmetry group, contains many trivial representations. As an example, the product of three $[1, 2, 1]$ representations contains 5 distinct trivial representations. This means that the corresponding 3-point functions are not determined by a single numerical coefficient, unlike what happened in the 1/2 BPS case.

As a consequence, the first step of choosing an alignment cannot be carried out in general. It is interesting to explore whether the rest of the proof extends at least for specific alignments. So let us consider a general 3-point function, aligned in a convenient way, and let us try to derive some necessary conditions for our proof to hold. It is clear that the highest-weight of such operators should be annihilated by at least one supercharge, so let us consider the product of three 1/8 BPS operators, so that the change of their 3-point function generated by $\mathcal{O}_\tau$ reads

$$\mathcal{I} \sim \langle \Big((\mathbf{Q})^4 \phi^{(2,+)} \Big)(x)\ \phi_I^{(\mathcal{R}_1,\vec{m}_1)}(x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3) \rangle. \tag{4.49}$$

The charges appearing in $(\mathbf{Q})^4$ are either $\mathbf{Q}^3$ or $\mathbf{Q}^4$. We can take one of the two[16] $\mathbf{Q}^4$'s (which annihilate the operator at x_3, since it is a lowest-weight) and move it using a Ward identity with a conformal Killing spinor that vanishes at x_2:

$$\mathcal{I} \sim \langle \Big((\mathbf{Q})^3 \phi^{(2,+)} \Big)(x)\ [\mathbf{Q}^4, \phi_I^{(\mathcal{R}_1,\vec{m}_1)}](x_1)\ \phi_J^{(\mathcal{R}_2,+)}(x_2)\ \phi_K^{(\mathcal{R}_3,-)}(x_3) \rangle. \tag{4.50}$$

The null condition applied to the operator at x_1 will generically give supercharges $\mathbf{Q}^i$ with $i = 1, 2, 3$, therefore if we want to use a Ward identity to argue that $\mathcal{I}$ vanishes, the operator at x_2 and x_3 should be 1/2 BPS operators.[17] As a consequence, the proof seems to work only for the case $1/8 \otimes 1/2 \otimes 1/2$.

A simple application of the Berenstein–Zelevinsky triangles shows that a product of the form $[p, q, r] \otimes [0, k_1, 0] \otimes [0, k_2, 0]$ contains the trivial representation only if $p = r$, which implies that the the operator at x_1 must be 1/4 BPS. In this case, if the trivial representation does appear, it appears only one time and the relative Clebsh–Gordan coefficient is unique. Furthermore, since the highest weight of a 1/4 BPS operator is also annihilated by a right chiral supercharge $\overline{\mathbf{Q}}_4$, the proof works for the marginal operator $\overline{\mathcal{O}}_\tau$ as well.

Summarizing, we were able to generalize the non-renormalization proof to the 3-point function of one 1/4 BPS operator and two 1/2 BPS operators, but the proof seems to fail in more general cases.

4.7 Discussion and Conclusions

We proved the non-renormalization of certain correlation functions of chiral primary operators in 4d $\mathcal{N} = 4$ and 2d $\mathcal{N} = (4, 4)$ superconformal field theories. Our

[16] Remember that the supercharges $\mathbf{Q}^i$ are spinors, so they also carry a Lorentz index.

[17] If the highest-weight is annihilated by $\mathbf{Q}^i$ with $i = 1, 2$, a simple argument based on unitarity bounds [73] shows that it must be annihilated by $\overline{\mathbf{Q}}_i$ with $i = 3, 4$ as well.

proof was based on the superconformal Ward identities and not on superspace arguments. While equivalent to the latter, we find that the direct proof offers some conceptual advantages.

It would be interesting to explore further more general correlators, for example three point functions of 1/4 BPS operators, and see whether an argument for their non-renormalization can be found, or alternatively to identify specific examples of such correlators whose weak and strong coupling values differ.

In this chapter we have not addressed an interesting phenomenon: under continuous deformations of conformal field theories it is possible for short multiplets to combine into long ones and to lift from the BPS bound. By requiring that the spectrum of operators varies continuously, one can derive certain "selection rules" for the types of states that can combine. These rules mostly rely on representation theory of the superconformal algebra and comparing certain (combinations of) characters of the representations and can be encoded into what is called the "index" of the superconformal theory [74]. However, we have some additional information: the deformation of the theory is generated by a marginal operator, which is itself a descendant of a chiral primary. It would be interesting to explore whether this imposes any additional constraints on the possible combinations of short multiplets into long ones, besides those imposed by the superconformal index. We hope to revisit this question in future work.

4.A Appendices

4.A.1 Roots and weights

In this appendix we review some basic facts about Lie algebras in order to set notation. In every finite dimensional Lie algebra $\mathfrak{g}$, characterized by a set of hermitian generators T_a, there is a maximal subset of commuting generators called *Cartan subalgebra*, spanned by H_i, $i = 1, \ldots, m$, where m is called the *rank* of the algebra.

In a finite-dimensional representation D of the Lie algebra, the generators are represented by matrices; the Cartan generators can be simultaneously diagonalized, i.e. we can find a basis of vectors $|\mu\rangle$ such that

$$H_i|\mu\rangle = \mu_i|\mu\rangle, \tag{4.51}$$

where the *weight vectors* μ's are m-component vectors with components μ_i. A weight is *positive* if its last non-zero component is positive and *negative* if its last non-zero component is negative.[18] In particular, a weight μ^h such that $\mu^h - \mu$ is positive for every weight μ is called *highest weight*. If the representation is irreducible, the highest weight is unique.

The Lie algebra is a vector space spanned by its generators $|T_a\rangle$, so we can consider the *adjoint representation*, defined by the action of the algebra on itself:

$$T_a|T_b\rangle = |[T_a, T_b]\rangle. \tag{4.52}$$

The basis in which the Cartan subalgebra is diagonal is spanned by $\{H_i, E_\alpha\}$, and we have

$$[H_i, H_j] = 0, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \alpha \cdot H. \tag{4.53}$$

The weights α of the adjoint representation are called *roots*. A root is called *simple* if it is positive and cannot be written as a sum of other positive roots. It is possible to prove that the simple roots are linearly independent and complete, so the number of simple roots is equal to the rank of the algebra m. We will label the simple roots by α^j, $j = 1, \ldots, m$.

Given an irreducible representation D and a weight μ, the state $E_\alpha|\mu\rangle$ has weight $\mu' = \mu + \alpha$ if $E_\alpha|\mu\rangle \neq 0$. We will refer to the E_{α^j} as raising operators and $E_{-\alpha^j} = E^\dagger_{\alpha^j}$ as lowering operators. In particular, the highest weight is annihilated by the raising operators:

$$E_{\alpha^j}|\mu^h\rangle = 0, \tag{4.54}$$

[18] It is customary to define positive weights as having the first non-zero component positive. Nevertheless, our definition is more convenient for $SU(N)$ groups.

since $\mu^h + \alpha$ is not a weight if α is positive. It is possible to show that

$$\frac{2\alpha^j \cdot \mu^h}{\alpha^j \cdot \alpha^j} = \ell^j \tag{4.55}$$

where the ℓ^j are non-negative integers called *Dynkin coefficients*.

It is convenient to introduce a basis of weight vectors μ^j such that

$$\frac{2\alpha^j \cdot \mu^k}{\alpha^j \cdot \alpha^j} = \delta^{jk} \tag{4.56}$$

so that the highest weight can be written as $\mu^h = \sum_j \ell^j \mu^j$. The μ^j's are called *fundamental weights*. Given the highest weight state, all the states in its irreducible representation can be obtained by acting with lowering operators:

$$E_{-\alpha^{j_1}} E_{-\alpha^{j_2}} \cdots E_{-\alpha^{j_n}} |\mu^h\rangle \tag{4.57}$$

where α^{j_k} are simple roots. The procedure stops when a state of zero norm is reached. Therefore an irreducible representation is completely characterized by its highest weight state and can be reconstructed by acting on this state with lowering operators associated to simple roots.

As a simple application, notice that if a state has weight $\mu = \sum_i k^i \mu^i$ with $k^j = 0$ for a given j, it is annihilated by the lowering operator $E_{-\alpha^j}$. In fact, we have

$$\langle \mu | E_{\alpha^j} E_{-\alpha^j} |\mu\rangle = \alpha^j \cdot \mu \langle \mu |\mu\rangle = 0, \tag{4.58}$$

so that $E_{-\alpha^1} |\mu\rangle$ is a zero-norm state.

4.A.2 1/2 BPS multiplets in $\mathcal{N} = 4$

A detailed analysis of the short multiplets in $\mathcal{N} = 4$ can be found in [73]. Let us start with some group-theoretic elements. The R-symmetry group of the $\mathcal{N} = 4$ algebra in 4 dimensions is $SU(4)$. Its Lie algebra has rank 3, and the Cartan generators are given by:

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.59}$$

$$H_2 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.60}$$

$$H_3 = \frac{1}{\sqrt{24}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{4.61}$$

The weights of the fundamental representation are given by

$$v^1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{24}} \end{pmatrix}, \quad v^2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{24}} \end{pmatrix}, \quad v^3 = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{12}} \\ \frac{1}{\sqrt{24}} \end{pmatrix}, \quad v^4 = \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{\sqrt{24}} \end{pmatrix}. \tag{4.62}$$

the roots by

$$\alpha^1 = v^1 - v^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha^2 = v^2 - v^3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad \alpha^3 = v^3 - v^4 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \tag{4.63}$$

and the fundamental weights by

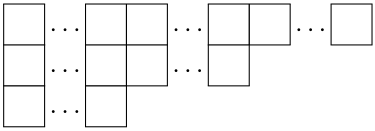
$$\mu^1 = v^1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{24}} \end{pmatrix}, \quad \mu^2 = v^1 + v^2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mu^3 = v^1 + v^2 + v^3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}, \tag{4.64}$$

so that $\frac{2\alpha^j \cdot \mu^k}{\alpha^j \cdot \alpha^j} = \delta^{jk}$. Every irreducible representation is uniquely characterized by the Dynkin label $[k_1, k_2, k_3]$, meaning that the highest weight is $\mu^h = k_1\mu^1 + k_2\mu^2 + k_3\mu^3$. The complex conjugate of the representation $[k_1, k_2, k_3]$ is $[k_3, k_2, k_1]$.

We will denote the raising operators E_{α^1}, E_{α^2} and E_{α^3} by E_1, E_2 and E_3 respectively, and the corresponding lowering operators by $E_1^\dagger$, $E_2^\dagger$ and $E_3^\dagger$.

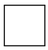
The highest weight for the fundamental representation is $v^1 = \mu^1$, therefore the Dynkin label is simply $[1, 0, 0]$. Sometimes it is convenient to denote representations by their dimension $\mathbf{d}$, so that the fundamental representation $[1, 0, 0]$ is denoted by **4** and its complex conjugate $[0,0,1]$ by $\bar{\mathbf{4}}$. Finally, the six-dimensional representation $[0,1,0]$, or **6**, corresponds to the fundamental representation of $SO(6)$ through the local isomorphism $SO(6) \approx SU(4)$.

Representations $[k_1, k_2, k_3]$ can be represented in terms of Young tableaux with k_3 columns with 3 boxes, k_2 columns with 2 boxes and k_1 columns with 1 box:



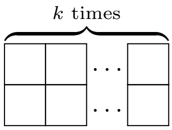
(4.65)

In particular, the fundamental representation $[1,0,0]$ is denoted by



(4.66)

and the representations $[0,k,0]$ by a Young tableau with $2k$ boxes:



(4.67)

We refer to [75] for more details.

The $[0,k,0]$ multiplet

The representations of the form $[0,k,0]$ are of particular importance, since the 1/2 BPS multiplets $\phi^{(k,\vec{m})}$ in the $\mathcal{N}=4$ theory sit in such representations. The vector $\vec{m}$ denotes the weight associated to a particular state in the representation. The highest and lowest weight states are denoted respectively by $\phi^{(k,+)}$ and $\phi^{(k,-)}$.

These representations can be constructed by taking tensor products of k $[0,1,0]$ representations. The $[0,1,0]$, or **6**, representation can be obtained as the antisymmetric product of two **4** representations. It is usually more convenient to work with a $SO(6)$ notation ϕ^i, $i = 1, \ldots, 6$. The six scalar fields of $\mathcal{N}=4$ super Yang–Mills sit in this representation. The irreducible representations $[0,k,0]$ for the chiral primaries correspond to traceless symmetric tensors $C_{i_1 \ldots i_k}$:

$$C_{i_1 \ldots i_n} \mathrm{Tr}(\phi^{i_1} \cdots \phi^{i_k}), \tag{4.68}$$

where the trace is over the $SU(N)$ gauge group. The highest weight state in this notation is

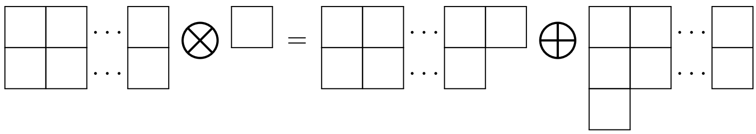
$$\mathrm{Tr}\left(Z^k\right) = \mathrm{Tr}\left((\phi^1 + i\phi^2)^k\right), \tag{4.69}$$

while the lowest weight is

$$\mathrm{Tr}\left(\bar{Z}^k\right) = \mathrm{Tr}\left((\phi^1 - i\phi^2)^k\right). \tag{4.70}$$

The left-chiral supercharges $\mathbf{Q}$ sit in the fundamental representation of $SU(4)$, and we will use a basis $\mathbf{Q}^i$, $i = 1, \ldots, 4$ corresponding to the weights v^i, $i = 1, \ldots, 4$ defined in equation (4.62) (in this section we ignore the Lorentz indices).

When we act with $\mathbf{Q}$ on ϕ we obtain a tensor product representation that can be decomposed as the sum of two irreducible representations as follows



(4.71)

or

$$[1, 0, 0] \otimes [0, k, 0] = [1, k, 0] \oplus [0, k-1, 1] \tag{4.72}$$

Using the $\mathcal{N} = 4$ algebra and the condition $\Delta = k$, it is easy to see that the highest weight in $[1, k, 0]$, namely $[\mathbf{Q}^1, \phi^{(k,+)}]$, has zero norm. Furthermore, from equation (4.58) we have $[E_1^\dagger, \phi^{(k,+)}] = 0$, which means that:

$$[E_1^\dagger, [\mathbf{Q}^1, \phi^{(k,+)}]] = [[E_1^\dagger, \mathbf{Q}^1], \phi^{(k,+)}] = [\mathbf{Q}^2, \phi^{(k,+)}]. \tag{4.73}$$

Therefore $[\mathbf{Q}^2, \phi^{(k,+)}]$ belongs to the null representation as well, being a descendant of the highest weight $[\mathbf{Q}^1, \phi^{(k,+)}]$. Therefore we will write

$$[\mathbf{Q}^1, \phi^{(k,+)}] = 0, \qquad [\mathbf{Q}^2, \phi^{(k,+)}] = 0. \tag{4.74}$$

Analogously, we have

$$[\mathbf{Q}^3, \phi^{(k,-)}] = 0, \qquad [\mathbf{Q}^4, \phi^{(k,-)}] = 0. \tag{4.75}$$

Finally, notice that the decomposition of $[k_1, k_2, k_3] \otimes [k_1', k_2', k_3']$ into a sum of irreducible representations contains the trivial representation if and only if $[k_1', k_2', k_3']$ is the complex conjugate representation of $[k_1, k_2, k_3]$, that is $[k_3, k_2, k_1]$. In particular, the tensor product $[0, k, 0] \otimes R$, where R is an arbitrary (not necessarily irreducible) representation, contains the trivial representation if and only if R contains the representation $[0, k, 0]$.

The $[0, 2, 0]$ multiplet

We summarize some (well known) facts about the $[0, 2, 0]$ 1/2 BPS multiplet of $\mathcal{N} = 4$ SYM. This multiplet is special because it contains the conserved currents and also the marginal operators.

The highest weight of the multiplet is the operator $\mathrm{Tr}(Z^2)$, where $Z = \phi^1 + i\phi^2$. This operator is annihilated by 1/2 of the left chiral and 1/2 of the right chiral supercharges. Here we use the notation $\mathbf{Q}^i_a$, $\overline{\mathbf{Q}}_{j,\dot{a}}$ where i, j are $SU(4)$ indices and $a, \dot{a}$ are $(1/2, 0)$ and $(0, 1/2)$ Lorentz spinor indices. The operator $\mathrm{Tr}(Z^2)$ is annihilated by the left chiral $\mathbf{Q}^1_a, \mathbf{Q}^2_a$ and the right chiral $\overline{\mathbf{Q}}_{3,\dot{a}}, \overline{\mathbf{Q}}_{4,\dot{a}}$, and is not annihilated by the rest of the supercharges.

Let us consider the four left chiral supercharges which do not annihilate the operator $\mathrm{Tr}(Z^2)$, namely $\mathbf{Q}^3_a, \mathbf{Q}^4_a$ where the spinor indices can be $a = 1, 2$. We notice that according to the $\mathcal{N} = 4$ superconformal algebra, these operators anticommute among themselves. Hence if we consider a nested (anti)-commutator of these supercharges, then the order in which the supercharges appear is not important and we can bring them to any desired order. The marginal operator $\mathcal{O}_\tau$ can then be written as

$$\mathcal{O}_\tau = \{\mathbf{Q}^4_1, [\mathbf{Q}^4_2, \{\mathbf{Q}^3_1, [\mathbf{Q}^3_2, \mathrm{Tr}(Z^2)]\}]\}. \tag{4.76}$$

It is straightforward to check using the superconformal algebra that this operator is Lorentz scalar, conformal primary and has $\Delta = 4$. Similarly, if we act on it with the four right chiral supercharges which do not annihilate it we get the conjugate marginal operator

$$\overline{\mathcal{O}}_\tau = \{\overline{\mathbf{Q}}_{2,\dot{1}}, [\overline{\mathbf{Q}}_{2,\dot{2}}, \{\overline{\mathbf{Q}}_{1,\dot{1}}, [\overline{\mathbf{Q}}_{1,\dot{2}}, \mathrm{Tr}(Z^2)]\}]\}. \tag{4.77}$$

Similar statements hold for the conjugate operator $\mathrm{Tr}(\overline{Z}^2)$, which is the $SU(4)$ lowest weight state of the $[0, 2, 0]$ multiplet. This operator is also annihilated by 1/2 of the left chiral and 1/2 of the right chiral supercharges, more specifically it is annihilated by $\mathbf{Q}^3_a, \mathbf{Q}^4_a$ and $\overline{\mathbf{Q}}_{1,\dot{a}}, \overline{\mathbf{Q}}_{2,\dot{a}}$. If we act on it with the four left chiral supercharges which do not annihilate it we have

$$\mathcal{O}_\tau = \{\mathbf{Q}^2_1, [\mathbf{Q}^2_2, \{\mathbf{Q}^1_1, [\mathbf{Q}^1_2, \mathrm{Tr}(\overline{Z}^2)]\}]\}, \tag{4.78}$$

while acting with the right chiral supercharges

$$\overline{\mathcal{O}}_\tau = \{\overline{\mathbf{Q}}_{4,\dot{1}}, [\overline{\mathbf{Q}}_{4,\dot{2}}, \{\overline{\mathbf{Q}}_{3,\dot{1}}, [\overline{\mathbf{Q}}_{3,\dot{2}}, \mathrm{Tr}(\overline{Z}^2)]\}]\}. \tag{4.79}$$

The expressions (4.76) and (4.79) are manifestly related by complex conjugation. On the other hand, the fact that $\mathcal{O}_\tau$ (and similarly $\overline{\mathcal{O}}_\tau$) can be written either as (4.76) or (4.78) is less obvious and special to $\mathcal{N} = 4$ theories.

The reason that we went into such a detailed presentation here is because the marginal operators in the $\mathcal{N} = 4$ have some special properties, which differ from those encountered in theories with less supersymmetry. If we think of the operator $\mathrm{Tr}(Z^2)$ as a "chiral primary" and that of $\mathrm{Tr}(\overline{Z}^2)$ as an "anti-chiral", we notice that both the holomorphic $\mathcal{O}_\tau$ and antiholomorphic $\overline{\mathcal{O}}_\tau$ marginal operators can be written as descendant of either the chiral or the anti-chiral primary. This is in contrast to what happens in less supersymmetric theories, where the holomorphic deformations are paired with descendants of chiral primaries and anti-holomorphic with descendants of anti-chiral.

Similar special properties of marginal operators are encountered in 2d $\mathcal{N} = (4,4)$ theories, as explained in detail in [53].

4.A.3 Null states and short multiplets

In this appendix we prove the null conditions (4.23) and (4.43). The proof is very similar in both cases, and we begin with the two-dimensional case which is technically simpler.

Structure of null conditions in $\mathcal{N} = (4,4)$

For simplicity we drop all extra indices/boldface notation and denote the supercharges by $G^\pm \equiv \mathbf{G}^{\pm r}_{-\frac{1}{2}}$, $J \equiv J^-$ and $\phi = \phi^{(j,j)}$, i.e. the highest weight state in the (short) representation. Also for simplicity we assume that the highest weight state is bosonic (if fermionic some commutators have to be replaced by anticommutators). By definition we have $[G^+, \phi] = 0$. What we want to prove is that

$$[G^+, \overbrace{[J, \dots [J,}^{n} \phi] \dots] \sim [G^-, \overbrace{[J, \dots [J,}^{n-1} \phi] \dots]. \tag{4.80}$$

We will prove it recursively. For $n = 1$ we have

$$[G^+, [J, \phi]] = [[G^+, J], \phi] + [J, [G^+, \phi]] = [G^-, \phi], \tag{4.81}$$

where we used that the second term is zero and the algebra relation $[G^+, J] = G^-$.

Next, let us assume that the condition is true for n and show that it also true for $n+1$. We have

$$\begin{aligned}
[G^+, \overbrace{[J, \dots [J,}^{n+1} \phi] \dots] &= [[G^+, J], \overbrace{[J, \dots [J,}^{n} \phi] \dots] + [J, [G^+, \overbrace{[J, \dots [J,}^{n} \phi] \dots] && (4.82)\\
&= [G^-, \overbrace{[J, \dots [J,}^{n} \phi] \dots] + [J, [G^-, \overbrace{[J, \dots [J,}^{n-1} \phi] \dots]. && (4.83)
\end{aligned}$$

To get this we used the algebra $[G^+, J] = G^-$ and the inductive hypothesis. Now we commute G^- to the left and we have

$$[G^+, \overbrace{[J, \dots [J,}^{n+1} \phi] \dots] = [G^-, \overbrace{[J, \dots [J,}^{n} \phi] \dots] + [[J, G^-], \overbrace{[J, \dots [J,}^{n-1} \phi] \dots]. \tag{4.84}$$

Now from the algebra we have $[J, G^-] = 0$, so we have proved the desired relation.

Structure of null conditions for $\mathcal{N}=4$

We now move to the four dimensional case where we want to prove (4.43), which reads

$$[\mathbf{Q}^\star, \phi_I^{(k_1,\vec{m}_1)}] = \sum_{j\neq\star}[\mathbf{Q}^j, \mathcal{X}_j]. \tag{4.85}$$

Here all supercharges are left chiral. We have chosen a basis of supercharges that have definite weight under the Cartan subalgebra. Let us consider one of the supercharges that annihilate a highest weight state $\phi^{(k,+)}$ (namely either $\mathbf{Q}^1$ or $\mathbf{Q}^2$) and call it $\mathbf{Q}^\star$. Hence we have

$$[\mathbf{Q}^\star, \phi^{(k,+)}] = 0. \tag{4.86}$$

In this case equation (4.85) is trivially satisfied.

Let us prove equation (4.85) in the case where the operator is the first $SU(4)$ "descendant" i.e. $[E_i^\dagger, \phi^{(k,+)}]$. We have

$$[\mathbf{Q}^\star, [E_i^\dagger, \phi^{(k,+)}]] = [E_i^\dagger, [\mathbf{Q}^\star, \phi^{(k,+)}]] + [[\mathbf{Q}^\star, E_i^\dagger], \phi^{(k,+)}] = [\mathbf{Q}', \phi^{(k,+)}]. \tag{4.87}$$

The first term is zero while the term $[\mathbf{Q}^\star, E_i^\dagger] = \mathbf{Q}'$ is another supercharge. However the important point is that the $SU(4)$ weight of the supercharge $\mathbf{Q}'$ is equal to the weight of $\mathbf{Q}^*$ minus the root α_i, so definitely $\mathbf{Q}' \neq \mathbf{Q}^\star$. Hence (4.85) is proven in this case.

In general, let us assume that the relation is true for an n descendant, that is

$$[\mathbf{Q}^\star, [E_{i_1}^\dagger, [\ldots, [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots] = \sum_{i\neq\star}[\mathbf{Q}^i, \phi^{(k,\vec{m}_i)}], \tag{4.88}$$

where the weight of each $\mathbf{Q}^i$ is strictly smaller than that of $\mathbf{Q}^\star$. We now show that the relation holds for an $n+1$ descendant as well. We have

$$\begin{aligned}[\mathbf{Q}^\star, [E_i^\dagger, [E_{i_1}^\dagger, \ldots [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots] &= [E_i^\dagger, [\mathbf{Q}^\star, [E_{i_1}^\dagger, \ldots [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots] \\ &+ [[\mathbf{Q}^\star, E_i^\dagger], [E_{i_1}^\dagger, \ldots [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots].\end{aligned} \tag{4.89}$$

By using the inductive hypothesis (4.88) on the right hand side, we have

$$\begin{aligned}&[\mathbf{Q}^\star, [E_i^\dagger, [E_{i_1}^\dagger \ldots, [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots] = \\ &\qquad [E_i^\dagger, \sum_{j\neq\star}[\mathbf{Q}^j, \phi^{(k,\vec{m}_j)}]] + [\mathbf{Q}', [E_{i_1}^\dagger \ldots [E_{i_n}^\dagger, \phi^{(k,+)}]\ldots]\end{aligned} \tag{4.90}$$

where the weight of $\mathbf{Q}' \equiv [\mathbf{Q}^\star, E_i^\dagger]$ is strictly smaller than the weight of $\mathbf{Q}^\star$. A further manipulation gives

$$\begin{aligned}[\mathbf{Q}^\star, [E_i^\dagger, [E_{i_1}^\dagger \ldots [E_{i_n}^\dagger, \phi^{(k,+)}] \ldots] &= \sum_{j\neq\star} [\mathbf{Q}^j, [E_i^\dagger, \phi^{(k,\vec{m}_j)}]] \\ &+ \sum_{j\neq\star} [[E_i^\dagger, \mathbf{Q}^j], \phi^{(k,\vec{m}_j)}] + [\mathbf{Q}', [E_{i_1}^\dagger \ldots [E_{i_n}^\dagger, \phi^{(k,+)}] \ldots], \end{aligned} \tag{4.91}$$

and since $\mathbf{Q}'' \equiv [E_i^\dagger, \mathbf{Q}^j]$ has a smaller weight than $\mathbf{Q}^j$, we have proved the desired relation. It is trivial to repeat the above steps for $\mathbf{Q}^3$ and $\mathbf{Q}^4$ by starting with the lowest weight $\phi^{(k,-)}$ and working "upwards".

Chapter 5

Conformal Symmetry for Black Holes

In this chapter we change gears and we turn our attention to a problem that seems at first completely unrelated to what we discussed so far, namely black hole entropy. As illustrated in the introduction, however, the holographic dictionary allows us to translate questions about gravity into questions about field theory.

In fact, we will be concerned with some recent constructions where the asymptotically flat region of certain four dimensional black holes is replaced with a different, and for certain aspects simpler, asymptotic region. This can be thought of as "putting the black hole in a box", and it makes manifest a certain hidden conformal symmetry that seems to underlie the dynamics of generic asymptotically flat black holes in the near horizon region. As we will discuss, such a hidden conformal symmetry is thought to be responsible for a conformal field theory interpretation of black hole entropy even far from extremality. More relevant to us, we will show that the subtraction procedure that changes the asymptotic region can naturally be cast in the language of field theory, and is implemented by irrelevant deformations of a conformal field theory.

This chapter is organized as follows. In section 5.1 we give a brief introduction to the the observations that suggest that the entropy of generic black holes can be explained in terms of CFTs. In section 5.2 we review the STU model and construct a four-parameter family of four-charged, non-rotating black holes with different asymptotics but the same thermodynamics, and explicitly identify the original (asymptotically flat) geometry and the subtracted geometry (which has been proposed to be the "same" black hole put in a box) as members of this family. Then, in section 5.3, we perform a linear analysis to determine the perturbations needed to flow from the subtracted to the original geometry. In section 5.4 we uplift the subtracted geometry and the linear perturbations thereof to $5d$; we then consistently reduce to an effective three-dimensional description to easily identify the irrelevant operators and sources through the standard AdS/CFT dictionary. The determination of the sources gives us a clear criterion for the window in which the effective IR CFT description is valid. Finally, in section 5.5, we summarize and discuss our findings. Various details of our calculations and useful formulae are collected in the appendix.

5.1 Introduction to black hole entropy and CFTs

Since the discovery that black holes carry entropy [76, 77, 78], the theoretical physics community has spent considerable efforts in trying to understand its microscopic origin. Supersymmetric black holes are relatively easy to analyze: their entropy can be computed by counting the degeneracy of BPS states of weakly-coupled dual CFTs [70]. This is possible because such BPS states, as we discussed in the previous chapter, are protected against renormalization, and they can be counted in a point in the moduli space where the CFT is weakly coupled, even though the black hole regime corresponds to the strongly coupled region of the moduli space. These computations were later extended to other setups, including near-extremal black holes [79, 80, 81, 82, 83, 84], and a precise agreement was found.

It was also noticed long ago [85, 86] that the entropy formula can be written in many cases as

$$S = 2\pi\sqrt{\frac{c}{6}\Delta_L} + 2\pi\sqrt{\frac{c}{6}\Delta_R}, \tag{5.1}$$

suggestive of Cardy's asymptotic growth of states in a CFT even *far away* from extremality. This suggests that a CFT interpretation might be given to the entropy of generic black holes. However, the identification cannot be as direct as in the case of (near-)extremal black holes; for example, generic black holes (including Schwarzschild and Kerr black holes) have negative specific heat and slowly evaporate via Hawking radiation. Since thermal states in CFTs have positive specific heat, it seems necessary to first "isolate" the black hole degrees of freedom from the environment by putting the system in a box that can reflect Hawking radiation back, so that a state of thermal equilibrium can be reached. While this procedure can be carried out quite naturally in the extremal and near-extremal cases, by taking appropriate decoupling limits, the procedure for general black holes stands on shakier grounds.

Hidden conformal symmetry

Various concrete CFT constructions have been recently proposed in the context of (near-)extremal Kerr black holes, and these and related developments go under the name of Kerr/CFT correspondence. In this section we will only review the salient conceptual points that motivate our work; for a comprehensive review and an exhaustive list of references we refer the reader to [87]. In a surprising development, it was noted in [88] that the massless wave equation for a probe scalar in a (generic) black hole background admits a $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry when certain offending terms are removed. These terms are once again negligible only in certain limits (near-extremal, near extreme rotating, low energy [89, 90, 91, 92, 93, 88, 87]). However reminiscent of conformal symmetry this approximate $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry may be, the terms "breaking" this symmetry are not small for general black holes and thus cannot justifiably be ignored.

Even so, this hidden conformal symmetry is very suggestive, and the idea that it may play a role in explaining black hole entropy is substantiated by the following observations: this conformal symmetry is broken by global effects, due to various periodicities in the coordinates, reminiscent to what happens when we put a CFT on a torus. In fact, a detailed analysis leads to the identification of a left temperature and a right temperature, given by

$$T_L = \frac{M^2}{2\pi J}, \qquad T_R = \frac{\sqrt{M^4 - J^2}}{2\pi J}, \tag{5.2}$$

where M and J are the black hole mass and angular momentum respectively. The low-energy scattering amplitudes then coincide with the correlators of a $2d$ CFT with these temperatures, lending credence to the idea that such theories might play a role for the microscopic description of these black holes. Furthermore, the black hole entropy turns out to be given by Cardy's formula if one assumes that the left and right central charges are given by the value derived at extremality, that is $c = 12J$. These observations have been extended to Reissner–Nordström black holes [94, 95] and Kerr–Newman black holes [96, 97, 98].

The "subtracted" geometry

A recent development by Cvetič and Larsen [99, 100] provides further evidence for an approximate CFT description of general black holes in four and five dimensions far from extremality. They construct a so-called "subtracted" geometry, where the warp factor of the geometry is modified. Thus, the asymptotics of the black hole are changed from asymptotically flat to asymptotically conical [101], but the thermodynamic properties of the black hole are left untouched. This subtracted geometry can then be intuitively thought of as implementing the idea of "putting the black hole in a box". On the one hand, this proposal generalizes the "hidden conformal symmetry" by providing an exact $SL(2,\mathbb{R})^2$ symmetry directly in the geometry. Furthermore, the resulting "subtracted" geometry can be uplifted one dimension higher to a geometry that is locally a product of AdS_3 and a two-sphere. Thus, a $2d$ CFT description of this black hole subtracted geometry is immediately obvious.

What is less obvious is the relation between the subtracted geometry and the original, asymptotically flat one, and in particular how this relation would be visible in the CFT description. Further developments [101] have made some progress in this direction by showing that the subtracted geometry can be obtained as a scaling limit of the original geometry. In this chapter, we wish to address this problem and provide further evidence of the $2d$ CFT description of general asymptotically flat, non-rotating, four-charge black holes in four dimensions first constructed in [102, 103] (and later extended to the rotating case in [104]). If the $2d$ CFT description of the subtracted geometry is related to some IR limit of the asymptotic flat original geometry, then it is natural to expect this original geometry to be described by the CFT plus some irrelevant deformation. In the following sections, we will argue that this is precisely the case.

5.2 The STU model

The STU model is a four-dimensional $\mathcal{N}=2$ supergravity theory coupled to three vector multiplets [105, 106, 107]. Its Lagrangian is given by[1]

$$\begin{aligned}\mathcal{L}_4 = R\star_4\mathbb{1} &- \frac{1}{2}H_{ij}\star_4 dh^i\wedge dh^j - \frac{3}{2f^2}\star_4 df\wedge df - \frac{f^3}{2}\star_4 F^0\wedge F^0\\ &- \frac{1}{2f^2}H_{ij}\star_4 d\chi^i\wedge d\chi^j - \frac{f}{2}H_{ij}\star_4\left(F^i+\chi^iF^0\right)\wedge\left(F^j+\chi^jF^0\right)\\ &+\frac{1}{2}C_{ijk}\,\chi^iF^j\wedge F^k+\frac{1}{2}C_{ijk}\,\chi^i\chi^jF^0\wedge F^k+\frac{1}{6}C_{ijk}\,\chi^i\chi^j\chi^kF^0\wedge F^0\,,\end{aligned}\tag{5.3}$$

where the fields f and h^i $(i=1,2,3)$ are scalars, χ^i are pseudoscalars, and F^0 and F^i are $U(1)$ gauge field strengths. The metric H_{ij} on the scalar moduli space is diagonal with entries $H_{ii}=(h^i)^{-2}$, and the symbol C_{ijk} is pairwise-symmetric in

[1]We mostly follow the notation and conventions of reference [108], which we found particularly useful.

its indices with $C_{123} = 1$ and zero otherwise. The h^i fields are constrained by the relation $h^1 h^2 h^3 = 1$, which must be solved before taking variations of the action. Our conventions for Hodge duality as well as some useful expressions can be found in appendix 5.A.1.

In the following we shall be concerned with solutions where the pseudoscalars χ^i are set to zero. This is not in general a consistent truncation, inasmuch as the pseudoscalar equations of motion then imply the constraints

$$-f\, H_{ij} \star_4 F^0 \wedge F^j + \frac{1}{2} C_{ijk}\, F^j \wedge F^k = 0\,. \tag{5.4}$$

In order to fulfill these conditions we will consider solutions where F^0 is purely electric and the F^i are purely magnetic. If we restrict to this case, we can write a simpler action from which we can derive the equations of motion, given by the Lagrangian

$$\mathcal{L} = -\frac{1}{2\kappa^2}\left[R - \frac{e^{-\eta_0}}{4} F^0_{\mu\nu} F^{0\,\mu\nu} - \frac{1}{2}\sum_{i=1}^{3}\left(\nabla_\mu \eta_i \nabla^\mu \eta_i + \frac{e^{2\eta_i - \eta_0}}{2} F^i_{\mu\nu} F^{i\,\mu\nu}\right)\right], \tag{5.5}$$

where $\kappa^2 = 8\pi G_4$ (κ has units of length), and we have introduced the shorthand notation

$$\eta_0 \equiv \eta_1 + \eta_2 + \eta_3\,. \tag{5.6}$$

The scalar fields η_i $(i = 1, 2, 3)$ are related to the scalars in (5.3) through

$$h^i = e^{\frac{1}{3}\eta_0 - \eta_i}\,, \tag{5.7}$$

$$f = e^{-\frac{1}{3}\eta_0}\,. \tag{5.8}$$

The corresponding equations of motion read[2]

$$0 = \nabla_\mu \nabla^\mu \eta_i + \frac{1}{4}\left[e^{-\eta_0} F^0_{\mu\nu} F^{0\mu\nu} + e^{-\eta_0}\sum_{j=1}^{3}\left(1 - 2\delta_{ij}\right) e^{2\eta_j} F^j_{\mu\nu} F^{j\mu\nu}\right], \tag{5.9}$$

$$0 = \nabla_\mu\left(e^{-\eta_0} F^{0\,\mu\nu}\right), \tag{5.10}$$

$$0 = \nabla_\mu\left(e^{-\eta_0 + 2\eta_i} F^{i\,\mu\nu}\right), \tag{5.11}$$

$$\begin{aligned} G_{\mu\nu} = \frac{1}{2}\Bigg[&\sum_{i=1}^{3}\left(\nabla_\mu \eta_i \nabla_\nu \eta_i - \frac{g_{\mu\nu}}{2}\nabla_\lambda \eta_i \nabla^\lambda \eta_i\right) + e^{-\eta_0}\left(F^{0\,\rho}_{\mu} F^0_{\nu\rho} - \frac{g_{\mu\nu}}{4} F^0_{\lambda\rho} F^{0\,\lambda\rho}\right) \\ &+ e^{-\eta_0}\sum_{i=1}^{3} e^{2\eta_i}\left(F^{i\,\rho}_{\mu} F^i_{\nu\rho} - \frac{g_{\mu\nu}}{4} F^i_{\lambda\rho} F^{i\,\lambda\rho}\right)\Bigg]. \end{aligned} \tag{5.12}$$

[2]We note that the purely electric configurations of [100, 101] solve the equations of motion following from the action obtained from (5.5) by dualizing the fields F^i as $F^i \to -e^{\eta_0 - 2\eta_i}\left(\star_4 F^i\right)$.

5.2.1 Static ansatz

In the present context we will be interested in static, spherically symmetric black hole backgrounds. As discussed above, in order to fulfill the constraint (5.4) we furthermore consider an electric ansatz for F^0 and a magnetic ansatz for the F^i. Explicitly, our ansatz for the metric and matter fields reads

$$
\begin{aligned}
ds_4^2 &= -\frac{G(r)}{\sqrt{\Delta(r)}}dt^2 + \sqrt{\Delta(r)}\left(\frac{dr^2}{X(r)} + d\theta^2 + \frac{X(r)}{G(r)}\sin^2\theta\, d\phi^2\right) && (5.13)\\
A^0 &= A_t^0(r)\, dt && (5.14)\\
A^i &= B_i \cos\theta\, d\phi && (5.15)\\
\eta_i &= \eta_i(r)\,, && (5.16)
\end{aligned}
$$

where the constants B^i $(i = 1, 2, 3)$ are the magnetic charges. Einstein's equations are easily seen to imply $G(r) = \gamma X(r)$, where $\gamma = \mathit{const}$, and also $X''(r) = 2$. Hence, without loss of generality we set

$$
X(r) = G(r) = r^2 - 2mr\,. \tag{5.17}
$$

Given this ansatz, we first notice that the equation for F^0 implies

$$
F^0_{rt} = q_0 \frac{e^{\eta_0}}{\sqrt{\Delta}}\,, \tag{5.18}
$$

where the constant q_0 is the electric charge (up to normalization). The scalar equations then reduce to

$$
0 = \Big(r(r-2m)\eta_i'\Big)' - \frac{e^{\eta_0}}{2\sqrt{\Delta}}\left[q_0^2 + \sum_{j=1}^{3}(2\delta_{ij}-1)\, B_j^2\, e^{2(\eta_j-\eta_0)}\right]. \tag{5.19}
$$

Finally, one notices that the independent information contained in Einstein's equations amounts to one second order and one first order equation. These can be taken to be

$$
0 = \frac{\Delta''}{\Delta} - \frac{3}{4}\left(\frac{\Delta'}{\Delta}\right)^2 + (\eta_1')^2 + (\eta_2')^2 + (\eta_3')^2 \tag{5.20}
$$

$$
\begin{aligned}
0 = {} & \left(\frac{\Delta'}{2\Delta}\right)^2 - \frac{2(r-m)}{r(r-2m)}\frac{\Delta'}{\Delta} + \frac{4}{r(r-2m)} + (\eta_1')^2 + (\eta_2')^2 + (\eta_3')^2\\
& - \frac{e^{\eta_0}}{r(r-2m)\sqrt{\Delta}}\left[q_0^2 + \sum_{i=1}^{3} e^{-2(\eta_0-\eta_i)}B_i^2\right].
\end{aligned} \tag{5.21}
$$

The first of these equations is a linear combination of the (t, t) and (ϕ, ϕ) components of Einstein's equations, while the first order constraint is the (r, r) component.

5.2.2 A family of static black hole solutions

Quite remarkably, it is possible to diagonalize the full non-linear system of equations. To this end we introduce new fields ϕ_0, ϕ_i defined as

$$\phi_0(r) = \frac{1}{2}\log\left(\frac{\Delta(r)}{m^4}\right) - \eta_1(r) - \eta_2(r) - \eta_3(r) \tag{5.22}$$

$$\phi_1(r) = \frac{1}{2}\log\left(\frac{\Delta(r)}{m^4}\right) - \eta_1(r) + \eta_2(r) + \eta_3(r) \tag{5.23}$$

$$\phi_2(r) = \frac{1}{2}\log\left(\frac{\Delta(r)}{m^4}\right) + \eta_1(r) - \eta_2(r) + \eta_3(r) \tag{5.24}$$

$$\phi_3(r) = \frac{1}{2}\log\left(\frac{\Delta(r)}{m^4}\right) + \eta_1(r) + \eta_2(r) - \eta_3(r)\,. \tag{5.25}$$

Taking suitable linear combinations of the scalar and Einstein's equations one finds

$$0 = \Big(r\,(r-2m)\,\phi_0'(r)\Big)' + 2\left(\frac{q_0^2}{m^2}\,e^{-\phi_0(r)} - 1\right) \tag{5.26}$$

$$0 = \Big(r\,(r-2m)\,\phi_i'(r)\Big)' + 2\left(\frac{B_i^2}{m^2}e^{-\phi_i(r)} - 1\right)\,. \tag{5.27}$$

Upon solving these decoupled equations one has the solution for the original fields $\Delta(r)$ and $\eta_i(r)$, and the solution for F^0 is then given by (5.18). Hence, we have effectively diagonalized the full non-linear system.

We have obtained general solutions to the decoupled equations (5.26)-(5.27),[3] each of which depends on two arbitrary integration constants. These generic solutions are not regular at the horizon $r = 2m$, but upon imposing regularity they reduce to

$$\phi_0^{\text{reg}}(r) = \log\left[\frac{q_0^2}{4m^4}\,\frac{\left(a_0^2\,r + 2m\right)^2}{1+a_0^2}\right] \tag{5.28}$$

$$\phi_j^{\text{reg}}(r) = \log\left[\frac{B_j^2}{4m^4}\,\frac{\left(a_j^2\,r + 2m\right)^2}{1+a_j^2}\right], \tag{5.29}$$

where the four independent constants a_0, a_i parametrize a family of static black hole solutions. Close to the horizon, one finds

$$\phi_0^{\text{reg}}(r\to 2m) = \log\Big(\frac{q_0^2}{m^2}\,\left(1+a_0^2\right)\Big) + \mathcal{O}\big(r-2m\big)\,, \tag{5.30}$$

$$\phi_j^{\text{reg}}(r\to 2m) = \log\Big(\frac{B_j^2}{m^2}\,\left(1+a_j^2\right)\Big) + \mathcal{O}\big(r-2m\big)\,. \tag{5.31}$$

[3] The general static solutions of the STU model have been constructed in [109] using different techniques. The hidden conformal symmetry of the Klein-Gordon equation on these backgrounds is studied in [110].

Similarly, in the asymptotic region $r \to \infty$

$$\phi_0^{\text{reg}}(r \to \infty) = \begin{cases} \log \frac{r^2}{m^2} + \mathcal{O}(1) \,, & a_0 \neq 0 \\ \log \frac{q_0^2}{m^2} \,, & a_0 = 0 \end{cases} \tag{5.32}$$

$$\phi_j^{\text{reg}}(r \to \infty) = \begin{cases} \log \frac{r^2}{m^2} + \mathcal{O}(1) \,, & a_j \neq 0 \\ \log \frac{B_j^2}{m^2} \,, & a_j = 0 \end{cases} \tag{5.33}$$

Going back to the original fields η_i and Δ, the solution reads

$$\Delta(r) = \frac{\sqrt{q_0^2 B_1^2 B_2^2 B_3^2}}{16m^4} \prod_{I=0}^{3} \frac{a_I^2\, r + 2m}{\sqrt{1 + a_I^2}} \tag{5.34}$$

$$e^{2\eta_1(r)} = \left|\frac{B_2 B_3}{q_0 B_1}\right| \sqrt{\frac{(1+a_0^2)\,(1+a_1^2)}{(1+a_2^2)\,(1+a_3^2)} \frac{\left(a_2^2\, r + 2m\right)\left(a_3^2\, r + 2m\right)}{\left(a_0^2\, r + 2m\right)\left(a_1^2\, r + 2m\right)}} \tag{5.35}$$

$$e^{2\eta_2(r)} = \left|\frac{B_1 B_3}{q_0 B_2}\right| \sqrt{\frac{(1+a_0^2)\,(1+a_2^2)}{(1+a_1^2)\,(1+a_3^2)} \frac{\left(a_1^2\, r + 2m\right)\left(a_3^2\, r + 2m\right)}{\left(a_0^2\, r + 2m\right)\left(a_2^2\, r + 2m\right)}} \tag{5.36}$$

$$e^{2\eta_3(r)} = \left|\frac{B_1 B_2}{q_0 B_3}\right| \sqrt{\frac{(1+a_0^2)\,(1+a_3^2)}{(1+a_1^2)\,(1+a_2^2)} \frac{\left(a_1^2\, r + 2m\right)\left(a_2^2\, r + 2m\right)}{\left(a_0^2\, r + 2m\right)\left(a_3^2\, r + 2m\right)}} \,. \tag{5.37}$$

It is worth emphasizing that, depending on how many of the constants a_0, a_i are non-zero, the asymptotic behavior of $\Delta(r)$ in our family of solutions can be of the form $\Delta(r \to \infty) \sim r^\gamma$, with $\gamma = 0, 1, \ldots, 4$. In particular, when $a_0 = a_i = 0$ (i.e. $\gamma = 0$) we obtain an asymptotically $\text{AdS}_2 \times S^2$ black hole solution. When $\gamma \neq 0$, the metric displays a "Lifshitz-covariance" of the form $t \to \lambda^z t$, $r \to \lambda^{2\theta/\gamma} r$, $ds^2 \to \lambda^\theta ds^2$ in the $r \gg 2m$ region, where the dynamical exponent z and the hyperscaling violation exponent[4] θ are related by $\theta = \left(\frac{\gamma}{\gamma - 2}\right) z$. We have checked explicitly that our family of solutions satisfies all the coupled equations of motion. In particular, the first order constraint (5.21) is satisfied identically, and places no restriction on the values of the constants a_0, a_i.

5.2.3 The "original" and "subtracted" geometries

The family of solutions found in section 5.2.2 contains as a particular case the solutions dubbed "original" and "subtracted" in [99, 100].[5] The original solution

[4]These metrics are in a sense a "global" version of the planar black brane solutions that have been used to model condensed matter systems displaying hyperscaling violation; for some representative works, see [111, 112, 113] and references therein.

[5]As we have mentioned our solutions are related to the purely electric solutions of [99, 100] by the duality transformation $F^i \to -e^{\eta_0 - 2\eta_i} \left(\star_4 F^i\right)$. In particular the conformal factor $\Delta(r)$ and the scalars $\eta(r)$ are unaffected by this transformation, and it is in this sense that we use the same terminology to refer to the full solutions.

is given in terms of functions

$$p_I(r) = r + 2m \sinh^2 \delta_I \,, \tag{5.38}$$

$(I = 0, 1, 2, 3)$ and it reads

$$\Delta(r) = \prod_{I=0}^{3} p_I(r) \tag{5.39}$$

$$e^{-\eta_i(r)} = p_i(r) \sqrt{\frac{p_0(r)}{p_1(r)p_2(r)p_3(r)}} \tag{5.40}$$

$$F^0_{rt} = \, -m \frac{\sinh(2\delta_0)}{p_0(r)^2} \,. \tag{5.41}$$

We then see that the original geometry is asymptotically flat in the $r \to \infty$ (i.e. $r \gg 2m$) region. Comparing with our general solution we can easily read off the electric and magnetic charges and the parameters a_I in terms of the δ_I:

$$q_0^{\rm orig} = -m \sinh(2\delta_0) \,, \quad B_i^{\rm orig} = m \sinh(2\delta_i) \,, \tag{5.42}$$

$$a_I^{\rm orig} = \frac{1}{\sinh(\delta_I)} \,. \tag{5.43}$$

Similarly, the so-called subtracted geometry is given by

$$\Delta(r) = (2m)^3 \left[\left(\Pi_c^2 - \Pi_s^2 \right) r + 2m \, \Pi_s^2 \right] \tag{5.44}$$

$$e^{\eta_i(r)} = \frac{1}{\sqrt{\Delta(r)}} \prod_{j \neq i} B_j \tag{5.45}$$

$$F^0_{tr} = \, -\frac{16m^4}{\Delta^2(r)} \Pi_c \Pi_s B_1 B_2 B_3 \,, \tag{5.46}$$

where the B_i are the magnetic charges as before, and we can read off the electric charge as $q_0 = -16m^4 \left(B_1 B_2 B_3\right)^{-1} \Pi_c \Pi_s$.[6] This solution is asymptotically conical for $r \to \infty$ [101]. Comparing with our general solution, we learn that the subtracted geometry has

$$a_0^{\rm subt} = \sqrt{\frac{\Pi_c^2 - \Pi_s^2}{\Pi_s^2}} \,, \qquad a_i^{\rm subt} = 0 \,. \tag{5.47}$$

As shown in [99], the thermodynamics of the original and subtracted solutions matches if the parameters Π_c and Π_s are given as follows:

$$\Pi_c = \prod_{I=0}^{3} \cosh \delta_I \,, \qquad \Pi_s = \prod_{I=0}^{3} \sinh \delta_I \,. \tag{5.48}$$

[6] Upon dualizing, we obtain a generalization of the solution presented in [99, 100] where we allow for a set of four independent $U(1)$ charges.

As we indicated above, depending on how many of the parameters a_I are zero the (large-r) asymptotic behavior of the conformal factor $\Delta(r)$ changes. While the asymptotically flat original geometry has all $a_I \neq 0$ and $\Delta_{\text{orig}} \sim r^4$ for large r (i.e. $r \gg 2m$), we have shown that the subtracted solution has $a_1 = a_2 = a_3 = 0$ and therefore the conformal factor scales linearly $\Delta_{\text{subt}} \sim r$. Figure 5.1 illustrates how we can smoothly interpolate between the subtracted and original geometries by dialing the parameters a_I. In particular notice that when all $\delta_i \gg 1$ a region emerges where the two solutions match to a very good approximation. It is in this sense that we refer to our family of solutions as an interpolating flow, with the different curves in figure 5.1 corresponding to different points in the space of couplings on a putative dual field theory. In fact, in this limit (also known as dilute-gas approximation) one can think of the subtracted solution as coming from a decoupling limit of the original solution, as we will discuss in the next sections.

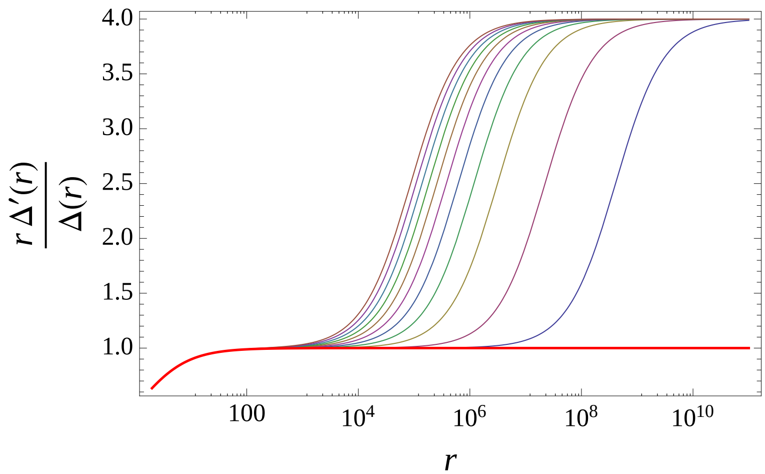


Figure 5.1: *Log plot of* $\gamma(r) \equiv \frac{d\log(\Delta)}{d\log r}$ *for the general solution* (5.34)*. The bottom red curve with* $\gamma(r \gg 2m) = 1$ *corresponds to the subtracted geometry* ($a_1 = a_2 = a_3 = 0$)*, while the various curves with* $\gamma(r \gg 2m) = 4$ *correspond to the original geometry with different values for* $a_1 = a_2 = a_3 \equiv 1/\sinh(\delta)$*. The different curves have increasingly larger values of* δ *towards the right, so we see that the original and subtracted geometries agree over a broader range in* r *as the magnetic charges* $B \sim \sinh(2\delta)$ *increase.*

5.3 Interpreting the flow between the original and subtracted geometries

In the previous section we described a four-parameter family of *exact* static solutions of the STU model that interpolates between the original and subtracted geometries; depending on the choice of parameters, this family also includes geometries with different asymptotic behavior from that of the original geometry. We can thus view the subtracted geometry loosely as an IR endpoint of an RG flow starting from the original geometry.
It is noteworthy that our solution implements explicitly the scaling limit discussed in [101] that extracts the subtracted solution from the original one. In the present section we will interpret this scaling limit as a flow between the original and subtracted geometries, while setting the stage for the AdS/CFT discussion to follow in section 5.4.

Even though (5.34)-(5.37) (with F^0 given by (5.18)) is an exact solution of the full nonlinear equations, we find it instructive to discuss its linearized version. On the one hand the linearized analysis makes the discussion of regularity at the horizon cleaner, since this is related to a choice of state in the holographic context. Secondly, the sources that one gets by linearizing our family of exact solutions do not necessarily correspond to the sources of irrelevant perturbation theory, as we will discuss in detail below. Lastly, while in generic situations exact solutions to the nonlinear equations are not available, the principles behind the linearized analysis still apply. In particular, we will exhibit the existence of linearized modes of the subtracted geometry that start the flow to the original geometry when their sources are chosen correctly. As we will explicitly show in section 5.4, upon uplifting the solutions to $5d$, these modes will turn out to be dual to irrelevant operators that deform the conformal field theory dual to the subtracted geometry.

5.3.1 Linearized analysis

We start our analysis by linearizing the field equations around the subtracted solution. Since the full non-linear equations of motion are diagonalized by the fields ϕ_I in (5.22)-(5.25), we can simply consider

$$\phi_I = \phi_I^{\text{subt}} + \delta\phi_I \, , \tag{5.49}$$

where the $\delta\phi_I$ will be our linearized perturbations. The linearized equations are then

$$r(r-2m)\delta\phi_0'' + 2(r-m)\delta\phi_0' - 2\frac{q_0^2 B_1^2 B_2^2 B_3^2}{\Delta^2}\delta\phi_0 = 0\,, \tag{5.50}$$

$$r(r-2m)\delta\phi_i'' + 2(r-m)\delta\phi_i' - 2\delta\phi_i = 0\,. \tag{5.51}$$

The equations for the $\delta\phi_i$'s are particularly simple, and we focus on those first. Changing to a new radial variable $x = \frac{r}{m} - 1$, these equations take the form

$$(1-x^2)\delta\phi_i''(x) - 2x\delta\phi_i'(x) + 2\delta\phi_i(x) = 0\,, \tag{5.52}$$

which is a Legendre equation whose general solution is

$$\delta\phi_i(x) = \alpha_i\, x + \beta_i\left(\frac{x}{2}\log\left(\frac{x+1}{x-1}\right) - 1\right). \tag{5.53}$$

Of these two solutions, only one is regular at the horizon (which is located at $x=1$), therefore we must set $\beta_i = 0$, or

$$\delta\phi_i = \alpha_i\left(\frac{r}{m} - 1\right). \tag{5.54}$$

Using the same variable x, and defining the parameters b and c as

$$b = \frac{\Pi_c^2 - \Pi_s^2}{2\sqrt{2}\Pi_c\Pi_s}\,, \qquad c = \frac{\Pi_c^2 + \Pi_s^2}{2\sqrt{2}\Pi_c\Pi_s}\,, \tag{5.55}$$

the equation for $\delta\phi_0$ becomes

$$(1-x^2)\delta\phi_0''(x) - 2x\delta\phi_0'(x) + \frac{1}{(bx+c)^2}\delta\phi_0(x) = 0\,. \tag{5.56}$$

The solution that is regular at the horizon in this case is given by

$$\delta\phi_0 = \alpha_0\frac{cx+b}{bx+c} = \alpha_0\frac{(\Pi_c^2+\Pi_s^2)r - 2m\Pi_s^2}{(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2}\,. \tag{5.57}$$

Notice that the condition of regularity will translate into a functional relation between the normalizable and non-normalizable modes in the standard holographic setting.

5.3.2 Perturbation theory and determination of the sources

In the previous section we showed that the solutions dubbed "original" and "subtracted" fit in a four-parameter family of solutions parametrized by a_I. In particular, recall that we have

$$a_i^{\text{orig}} = \frac{1}{\sinh\delta_i}\,, \qquad a_i^{\text{subt}} = 0\,, \tag{5.58}$$

while the two a_0's are both different from zero. In order to go from the subtracted to the original geometry, we need to "turn on" the parameters a_i and change the parameter a_0. We would like to understand this in terms of a flow that is started by linearized fluctuations around the subtracted background. This suggests that the sources α_I should be directly related to the parameters a_I. However, since we are turning on an irrelevant mode, at each order in perturbation theory higher powers of r will be generated, therefore we need to treat the sources as infinitesimal quantities. It is easy to see that linearizing the general solution ϕ_i around $a_i^2 = 0$, at first order in a_i^2 one gets:

$$\phi_i = \phi_i^{\text{subt}} + a_i^2 \left(\frac{r}{m} - 1 \right) + \dots \,, \tag{5.59}$$

and we recognize the second term on the right-hand side as being the linearized perturbation of the previous subsection. Notice that the higher order terms do not contain terms linear in r, so the sources obtained by linearizing in a_i^2 are equivalent to the sources that one would obtain by extracting the coefficient of order r in a power-series expansion. Therefore we should identify

$$\alpha_i = (a_i^{\text{orig}})^2 = \frac{1}{\sinh^2 \delta_i} \,. \tag{5.60}$$

Analogously, we have

$$\phi_0 = \phi_0^{\text{subt}} + (a_0^2 - (a_0^{\text{subt}})^2) \frac{\Pi_s^2}{\Pi_c^2} \frac{(\Pi_c^2 + \Pi_s^2) r - 2m\Pi_s^2}{(\Pi_c^2 - \Pi_s^2) r + 2m\Pi_s^2} \,, \tag{5.61}$$

and as a consequence

$$\alpha_0 = \left((a_0^{\text{orig}})^2 - (a_0^{\text{subt}})^2 \right) \frac{\Pi_s^2}{\Pi_c^2} \,. \tag{5.62}$$

Notice however that the sources are in general *not* infinitesimal. They do become infinitesimal in the limit where the three parameters δ_i become very large. In fact in this limit we obtain particularly simple expressions for the sources:

$$\alpha_i \approx 4\, e^{-2\delta_i} \,, \tag{5.63}$$

$$\alpha_0 \approx -4 \sum_i e^{-2\delta_i} \,. \tag{5.64}$$

Therefore, to leading order we have the relation:

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2} \sum_I \delta\phi_I = 0, \tag{5.65}$$

that is, the metric is not changed to leading order in the parameters $e^{-2\delta_i}$.

Looking at the behavior of the linearized modes for very large r, one is led to the suspicion that the $\delta\phi_i$'s correspond to irrelevant perturbations while $\delta\phi_0$ seems to be associated to a marginal perturbation. In section 5.4 we will show that this suspicion is correct (after a suitable change of basis), and we will compute the quantum numbers of the operators in the dual CFT$_2$ that we need to turn on to start the flow to the asymptotically flat original black hole. It is important to notice that these irrelevant perturbations do change the value of the matter fields in the interior, and in particular they are finite (i.e. non zero) at the horizon. This is in contrast to the extremal case, where irrelevant perturbations die off quickly in the interior and do not change the value of the fields at the horizon.

Notice also that since we are turning on irrelevant deformations, there is no intrinsic (i.e. coordinate invariant) way to extract the sources for the dual operators. Their precise definitions must be supplemented with a perturbation scheme to compute higher order corrections. For example, it is easy to see that our choice for the α_i's is compatible with a scheme where the linear term in r does not receive higher order corrections; however if we used a different radial coordinate, for example $r' = r + c$ where c is a constant, this would not be true anymore. This ambiguity has an analog in quantum field theory, where the question of whether a source of an operator receives quantum corrections or not depends on the renormalization scheme. This ambiguity is obviously not present to leading order in perturbation theory. We will revisit this issue at the end of section 5.4.3, where we will describe other possible choices for the sources.

5.3.3 Range of validity of the linear approximation

In many applications, one does not have the exact solutions, and often it is even impossible to solve the linearized equations exactly. In fact, in many interesting situations only the linearized modes in the asymptotic region are available, and a numerical treatment becomes necessary. It is therefore useful to investigate how one could approach this problem from a numerical perspective; we will then be able to compare the numerical results with the analytic results of the previous sections. The first step is to find a region that can be identified with the asymptotic region of the subtracted geometry (that is $\frac{r}{m} \gg 1$) but where the modes that start the flow to the original geometry are still small, so that they can be treated perturbatively. From the discussion in the previous sections, it is clear that this region should be

$$1 < \frac{r}{m} \ll \frac{1}{\alpha} \,. \tag{5.66}$$

where α is the smallest of the α_i's. Furthermore, we argued that the α_i's are related to the parameters δ_i, so that when the latter are large, $\alpha_i \approx 4\, e^{-2\delta_i}$. As anticipated in the previous section, this is when the three charges B_1, B_2, and B_3 are large compared to the fourth, q_0 .

Since we expect the difference between the original and subtracted solutions for the ϕ_i's to be linear in this intermediate region, it is possible to determine the sources for these three modes by means of a linear interpolation, as shown in figure 5.2. The slope of the linear function turns out to be $4e^{-2\delta_i}$, perfectly matching the results of the previous section.

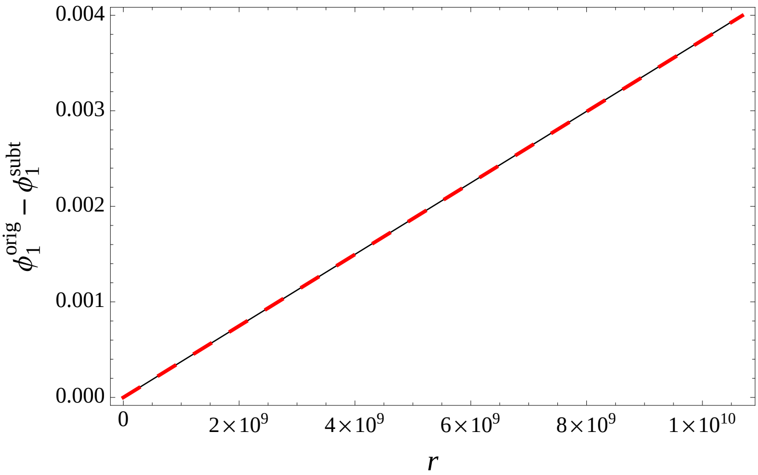


Figure 5.2: *The dashed red line represents the difference between the original and subtracted fields ϕ_1 . The solid black line is a linear function with slope $4e^{-2\delta_1}$. For this plot, we chose $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 15$, $m = 1$, and the domain is $r \in [100, 10^{-4}e^{2\delta_1}]$.*

We can also plot ϕ_0 and the function $\alpha_0 \frac{(\Pi_c^2+\Pi_s^2)r-2m\Pi_s^2}{(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2}$. We see in figure 5.3 that the correct source for this mode is $\alpha_0 = -\alpha_1 - \alpha_2 - \alpha_3 = -4(e^{-2\delta_1} + e^{-2\delta_2} + e^{-2\delta_3})$, confirming once again the analysis of the previous section. Before closing this section it is worth emphasizing that, by turning on different combinations of the sources, we can also flow to the various geometries with Lifshitz-like scaling discussed in section 5.2.2.

5.4 Uplifting and AdS/CFT interpretation

In this section we uplift our $4d$ solutions to five dimensions, where an AdS/CFT interpretation of the flow is possible. As shown in [99], the subtracted geometry uplifts to a BTZ black hole, which is asymptotically $\mathrm{AdS}_3 \times S^2$. The linearized perturbations of the previous section uplift to linearized perturbations of the BTZ

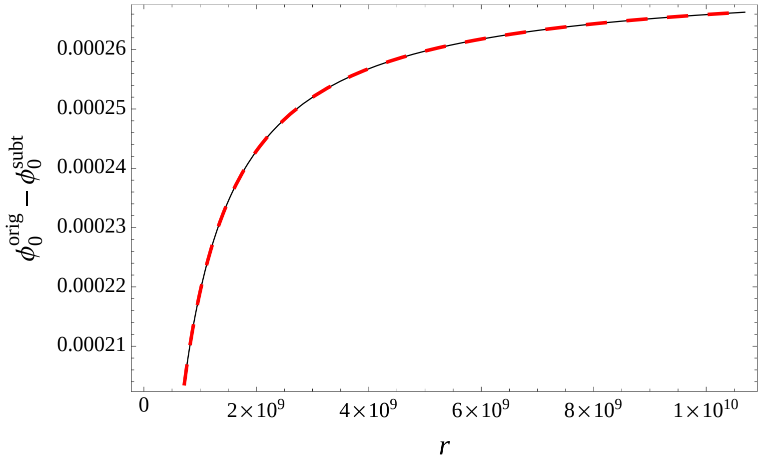


Figure 5.3: *The dashed red line represents the difference between the original and subtracted fields ϕ_0. The solid black line is the function $4(e^{-2\delta_1}+e^{-2\delta_2}+e^{-2\delta_3})\frac{(\Pi_c^2+\Pi_s^2)r-2m\Pi_s^2}{(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2}$. For this plot, we chose $\delta_0=\delta_1=\delta_2=\delta_3=15$, $m=1$, and the domain is $r\in[100,10^{-4}e^{2\delta_1}]$.*

black hole, and we will explicitly show them to be dual to three irrelevant scalar operators with conformal weights $(h,\bar{h})=(2,2)$. This allows us to give a more precise description of the dynamical realization of the conformal symmetry for the charged $4d$ black holes under study, while clarifying at the same time the limitations of this program.

5.4.1 The $5d$ Lagrangian and equations of motion

As shown in [108], the STU model (5.3) can be obtained by dimensional reduction of the following $5d$ Lagrangian:

$$\mathcal{L}_5 = R_5 \star_5 \mathbb{1} - \frac{1}{2}H_{ij}\star_5 dh^i\wedge dh^j - \frac{1}{2}H_{ij}\star_5 \tilde{F}^i\wedge\tilde{F}^j + \frac{1}{6}C_{ijk}\,\tilde{F}^i\wedge\tilde{F}^j\wedge\tilde{A}^k, \quad (5.67)$$

where H_{ij} and C_{ijk} are defined as in (5.3). The $4d$ and $5d$ line elements are related by

$$ds_5^2 = f^{-1}ds_4^2 + f^2\left(dz+A^0\right)^2\,, \quad (5.68)$$

and the vector fields by

$$\tilde{A}^i = \chi^i(dz+A^0)+A^i\,. \quad (5.69)$$

The form of the h^i scalars in our general four-parameter family of solutions is given in (5.177)-(5.179). In particular, uplifting the subtracted solution (5.44)-(5.46) we discover that the $5d$ scalar fields are constant in this case:[7]

$$h^1_{\text{subt}} = \left(\frac{B_1^2}{B_2 B_3}\right)^{1/3}, \qquad h^2_{\text{subt}} = \left(\frac{B_2^2}{B_1 B_3}\right)^{1/3}, \qquad h^3_{\text{subt}} = \left(\frac{B_3^2}{B_1 B_2}\right)^{1/3}. \tag{5.70}$$

As anticipated, the $5d$ subtracted geometry asymptotes to $\text{AdS}_3 \times S^2$, and this will allow us to interpret the flow we found in the four-dimensional STU theory in terms of deformations of the CFT living on the boundary of the AdS_3 factor. The strategy we will follow consists of performing a consistent Kaluza–Klein reduction of the $5d$ theory on the two-sphere to obtain an effective $(2+1)$-dimensional theory. We will discover that the solutions of this effective theory with constant scalars correspond locally to AdS_3, and one of them uplifts precisely to the subtracted geometry in five dimensions. Linearizing the theory around this solution will then allow us to identify the dual operators associated with the flow between the original and subtracted geometries.

Before proceeding further we note that the model (5.67) is slightly inconvenient in that the scalar fields satisfy the constraint $h^1 h^2 h^3 = 1$ which must be solved before taking variations of the action. Hence, we choose to work instead with unconstrained scalars Ψ and Φ defined through [108]

$$h^1 = e^{\sqrt{\frac{2}{3}}\Psi}, \quad h^2 = e^{-\frac{\Psi}{\sqrt{6}} - \frac{\Phi}{\sqrt{2}}}, \quad h^3 = e^{-\frac{\Psi}{\sqrt{6}} + \frac{\Phi}{\sqrt{2}}}, \tag{5.71}$$

in terms of which

$$\begin{aligned}\mathcal{L}_5 = R_5 \star_5 \mathbb{1} - \frac{1}{2}\star_5 d\Psi \wedge d\Psi - \frac{1}{2}\star_5 d\Phi \wedge d\Phi \\ - \frac{1}{2} H_{ij}(\Psi, \Phi) \star_5 \tilde{F}^i \wedge \tilde{F}^j + \frac{1}{6} C_{ijk}\, \tilde{F}^i \wedge \tilde{F}^j \wedge \tilde{A}^k .\end{aligned} \tag{5.72}$$

The equations of motion for the matter fields are then

$$0 = d\left(H_{ij} \star_5 \tilde{F}^j\right) - \frac{C_{ijk}}{2} \tilde{F}^j \wedge \tilde{F}^k \tag{5.73}$$

$$0 = d\big(\star_5 d\Psi\big) - \frac{1}{2}\frac{\delta H_{ij}}{\delta \Psi} \star_5 \tilde{F}^i \wedge \tilde{F}^j \tag{5.74}$$

$$0 = d\big(\star_5 d\Phi\big) - \frac{1}{2}\frac{\delta H_{ij}}{\delta \Phi} \star_5 \tilde{F}^i \wedge \tilde{F}^j . \tag{5.75}$$

Similarly, Einstein's equations read

$$\begin{aligned}G_{\mu\nu} = \frac{1}{2}\Big[\nabla_\mu \Psi \nabla_\nu \Psi - \frac{g_{\mu\nu}}{2}\nabla_\lambda \Psi \nabla^\lambda \Psi + \nabla_\mu \Phi \nabla_\nu \Phi - \frac{g_{\mu\nu}}{2}\nabla_\lambda \Phi \nabla^\lambda \Phi \\ + H_{ij}\left(\tilde{F}^i_{\mu}{}^{\rho} \tilde{F}^j_{\nu\rho} - \frac{g_{\mu\nu}}{4}\tilde{F}^i_{\lambda\rho}\tilde{F}^{j\,\lambda\rho}\right)\Big]\end{aligned} \tag{5.76}$$

[7] Without loss of generality, in order to simplify the notation we assume that the magnetic charges satisfy $B_i > 0$ from now on. In the general case, the absolute value of various expressions involving products of the B_i should be considered when appropriate.

and we recall that the only non-vanishing components of $H_{ij}(\Psi,\Phi)$ are given by $H_{ii}(\Psi,\Phi) = \left(h^i(\Psi,\Phi)\right)^{-2}$.

5.4.2 Consistent Kaluza–Klein reduction

The general structure of the uplifted line element (5.68) is

$$\begin{aligned} ds_5^2 &= e^{\frac{\eta_0}{3}} ds_4^2 + e^{-\frac{2\eta_0}{3}} \left(dz + A^0\right)^2 \\ &= e^{\frac{\eta_0}{3}} \sqrt{\Delta(r)} \left(\frac{dr^2}{X(r)} - \frac{G(r)}{\Delta(r)} dt^2 + \frac{e^{-\eta_0}}{\sqrt{\Delta(r)}} \left(dz + A^0\right)^2 \right) + e^{\frac{\eta_0}{3}} \sqrt{\Delta(r)}\, ds^2\left(S^2\right) \end{aligned} \tag{5.77}$$

where we assume that A^0 has no legs on the sphere directions (i.e. it is purely electric). It is easy to show that the subtracted geometry uplifts to a BTZ$\times S^2$ black hole. Nevertheless, it is more convenient to take a more general route that will allow us to characterize the general linear perturbations around the uplifted geometry. A Kaluza–Klein (KK) Ansatz that includes all our uplifted solutions is

$$ds_5^2 = ds^2_{\text{string}}(M) + e^{2U(x)} ds^2(Y) \tag{5.78}$$

$$\tilde{F}^i = -B_i \sin\theta\, d\theta \wedge d\phi \tag{5.79}$$

$$\Psi = \Psi(x) \tag{5.80}$$

$$\Phi = \Phi(x)\,. \tag{5.81}$$

Here, M is the $(2+1)$-dimensional "external" manifold with coordinates $x = \{r,t,z\}$, and some metric that we keep arbitrary, and Y is the "internal" (compact) manifold, namely the two-sphere with radius ℓ_S and coordinates $y = \{\theta,\phi\}$. We pick the orientation such that the volume form on Y is $\text{vol}_2 = \ell_S^2 \sin\theta\, d\theta\wedge d\phi$. The subscript "string" in $ds^2_{\text{string}}(M)$ is meant to remind us that the theory that will come out of the reduction will not be immediately in the $(2+1)$-dimensional Einstein frame, but rather in what could be called string frame. After performing the reduction we will translate the effective theory to Einstein frame before performing the AdS/CFT analysis. The radius of the sphere ℓ_S is set by the equations of motion to be:

$$\ell_S = \left(B_1 B_2 B_3\right)^{1/3}\,. \tag{5.82}$$

Notice however that we have the freedom to rescale U in the reduction. The choice above guarantees that the radius of the reduced $3d$ (locally) AdS_3 metric (in the final $3d$ Einstein frame) is equal to the radius of the (locally) AdS_3 factor in the $5d$ geometry.

The details of the (consistent) KK reduction can be found in appendix 5.A.2. Reducing the $5d$ equations of motion one finds that all reference to the two-sphere drops out, and the resulting three-dimensional equations of motion (5.147), (5.148), (5.171) and (5.172) follow from the effective (string frame) action

$$S_{\text{string}} = -\frac{1}{16\pi G_3}\int d^3x\sqrt{|g|}\,e^{2U}\Big[R + \frac{2}{\ell_S^2}e^{-2U} - \frac{e^{-4U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi) + 2\,(\nabla U)^2 - \frac{1}{2}\,(\nabla\Psi)^2 - \frac{1}{2}\,(\nabla\Phi)^2\Big]. \quad (5.83)$$

The three dimensional Newton's constant G_3 is fixed in terms of the normalization of the $5d$ action and the volume of the internal manifold, which are in turn related to the $4d$ Newton's constant G_4:

$$G_3 = \frac{1}{4\pi\ell_S^2}G_5 = \frac{R_z}{2\ell_S^2}G_4\,. \quad (5.84)$$

Here, R_z is the radius of the circle on which we reduce to go from the $5d$ theory (5.67) to the $4d$ STU model, and is in principle arbitrary. The next step consists in passing to three-dimensional Einstein frame by performing a Weyl rescaling of the metric on M. Denoting with a subscript (E) the quantities in Einstein frame, the transformation we need is

$$ds^2_{\text{string}}(M) = e^{-4U}ds^2_{(\mathrm{E})}(M)\,, \quad (5.85)$$

which in particular implies

$$R = e^{4U}\left[R_{(E)} + 8\Box_{(E)}U - 8\left(\nabla_{(E)}U\right)^2\right]. \quad (5.86)$$

It follows that the Einstein frame effective action is (after dropping a surface term)

$$S_{(\mathrm{E})} = -\frac{1}{16\pi G_3}\int d^3x\sqrt{|g_{(\mathrm{E})}|}\Big[R_{(E)} - 6\left(\nabla_{(E)}U\right)^2 - \frac{1}{2}\left(\nabla_{(E)}\Psi\right)^2 - \frac{1}{2}\left(\nabla_{(E)}\Phi\right)^2 + \frac{2}{\ell_S^2}e^{-6U} - \frac{e^{-8U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi)\Big]. \quad (5.87)$$

In a slight abuse of notation, we will drop the subscript (E) from now on because we will be working exclusively in Einstein frame. The equations of motion are then

$$G_{\mu\nu} = -\frac{g_{\mu\nu}}{\ell_S^2}\left[-e^{-6U} + \frac{e^{-8U}}{4}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^2}H_{ii}(\Psi,\Phi)\right] + \frac{1}{2}\tilde{T}_{\mu\nu} \tag{5.88}$$

$$0 = \nabla_\mu\nabla^\mu\Psi - \frac{e^{-8U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}(\Psi,\Phi)}{\delta\Psi} \tag{5.89}$$

$$0 = \nabla_\mu\nabla^\mu\Phi - \frac{e^{-8U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}(\Psi,\Phi)}{\delta\Phi} \tag{5.90}$$

$$0 = \nabla_\mu\nabla^\mu U - \frac{e^{-6U}}{\ell_S^2} + \frac{e^{-8U}}{3}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi)\,, \tag{5.91}$$

where we defined the "kinetic" part of the stress tensor as

$$\tilde{T}_{\mu\nu} = \nabla_\mu\Psi\nabla_\nu\Psi - \frac{g_{\mu\nu}}{2}(\nabla\Psi)^2 + \nabla_\mu\Phi\nabla_\nu\Phi - \frac{g_{\mu\nu}}{2}(\nabla\Phi)^2 + 12\nabla_\mu U\nabla_\nu U - 6g_{\mu\nu}(\nabla U)^2\,. \tag{5.92}$$

5.4.3 Asymptotically AdS$_3$ solutions and dual operators

We will consider solutions where the scalars take constant values $U = \bar{U}$, $\Psi = \bar{\Psi}$, $\Phi = \bar{\Phi}$, so that $\tilde{T}_{\mu\nu} = 0$. In such a background, the equations (5.88)-(5.91) reduce to

$$G_{\mu\nu} = -\Lambda_{\text{eff}}\, g_{\mu\nu} \tag{5.93}$$

$$0 = \sum_{i=1}^{3} B_i^2 \left.\frac{\delta H_{ii}(\Psi,\Phi)}{\delta\Psi}\right|_{\bar{\Psi},\bar{\Phi}} \tag{5.94}$$

$$0 = \sum_{i=1}^{3} B_i^2 \left.\frac{\delta H_{ii}(\Psi,\Phi)}{\delta\Phi}\right|_{\bar{\Psi},\bar{\Phi}} \tag{5.95}$$

$$e^{2\bar{U}} = \frac{1}{3}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^2}H_{ii}(\bar{\Psi},\bar{\Phi})\,, \tag{5.96}$$

where the effective cosmological constant Λ_{eff} is given by

$$\Lambda_{\text{eff}} = \frac{1}{\ell_S^2}\left[-e^{-6\bar{U}} + \frac{e^{-8\bar{U}}}{4}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^2}H_{ii}(\bar{\Psi},\bar{\Phi})\right] = -\frac{e^{-6\bar{U}}}{4\ell_S^2}\,, \tag{5.97}$$

and we used (5.96) in the last equality. In three dimensions, the only solutions to Einstein's equations with negative cosmological constant are locally AdS$_3$; the effective AdS$_3$ length L in our case is then given by

$$L^2 = -\frac{1}{\Lambda_{\text{eff}}} = 4\,e^{6\bar{U}}\,\ell_S^2\,. \tag{5.98}$$

There is in fact a unique solution to equations (5.94)-(5.96) for the scalars, given by

$$e^{\bar{U}} = \left(\frac{B_1 B_2 B_3}{\ell_S^3}\right)^{1/3} = 1\,, \qquad e^{\bar{\Psi}} = \left(\frac{B_1^2}{B_2 B_3}\right)^{\frac{1}{\sqrt{6}}}\,, \qquad e^{\bar{\Phi}} = \left(\frac{B_3}{B_2}\right)^{\frac{1}{\sqrt{2}}}\,. \tag{5.99}$$

Notice in particular that $\bar{U} = 0$. Comparing with (5.70)-(5.71), we see that these are precisely the values corresponding to the subtracted geometry.[8] Moreover, as it follows from (5.93), the metric of this three-dimensional solution is locally AdS_3 with radius

$$L = 2\, e^{3\bar{U}}\ell_S = 2\ell_S = 2\,(B_1 B_2 B_3)^{1/3}\,. \tag{5.100}$$

We will describe the global properties of the solution that corresponds to the subtracted geometry in the following subsection.

We can determine the operator content of the dual field theory from the action (5.87): we have the stress tensor coupling to the massless graviton, and three scalar operators that couple to the boundary values of U, Ψ, Φ. Following the standard AdS/CFT dictionary, in order to compute the conformal dimensions of these operators we need to obtain the masses of the linearized bulk fields around the solution corresponding to the subtracted geometry. Linearizing the equations we find that the fluctuations of the three bulk scalars decouple and in fact satisfy the same equation:

$$0 = \nabla_\mu \nabla^\mu \delta F - \frac{8}{L^2}\,\delta F\,, \tag{5.101}$$

where δF stands for any of δU, $\delta\Psi$, $\delta\Phi$. Therefore, the masses are given by

$$m^2_{\delta U} = m^2_{\delta\Psi} = m^2_{\delta\Phi} = \frac{8}{L^2}\,, \tag{5.102}$$

and according to the standard dictionary we conclude that the three scalar operators in the dual theory are irrelevant, with conformal dimension $\Delta = 4$.

[8] For completeness, the explicit form of the $3d$ scalars in our general family of solutions is given in (5.183)-(5.185).

5.4.4 Irrelevant deformation of the CFT

Finally, we relate the $4d$ modes of section 5.3, parametrized by the α_I's, to the linearized modes of the $3d$ theory. The scalars are

$$\delta U = \frac{1}{6m}\left(\alpha_1+\alpha_2+\alpha_3\right)(r-m) \tag{5.103}$$

$$\delta \Psi = \frac{1}{2\sqrt{6}m}\left(2\alpha_1-\alpha_2-\alpha_3\right)(r-m) \tag{5.104}$$

$$\delta \Phi = \frac{1}{2\sqrt{2}m}\left(\alpha_3-\alpha_2\right)(r-m)\,, \tag{5.105}$$

corresponding to non-normalizable modes. To identify the marginal mode, a little work is required. As explained in [100], the uplifted subtracted geometry can be cast in the BTZ form with the change of coordinates (we work in a gauge where $A_0 \to 0$ as $r \to \infty$):

$$\rho^2 = \frac{R_z^2}{\ell_S^4}\Delta(r) = \frac{R_z^2}{\ell_S^4}(2m)^3\Big(\left(\Pi_c^2-\Pi_s^2\right)r+2m\Pi_s^2\Big) \tag{5.106}$$

$$t = \frac{R_z}{2\ell_S^4}(2m)^3\left(\Pi_c^2-\Pi_s^2\right)t_3 \tag{5.107}$$

$$z = -R_z\phi_3\,, \tag{5.108}$$

so that the metric reads

$$ds^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{L^2\rho^2}dt_3^2+\frac{L^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2+\rho^2\left(d\phi_3+\frac{\rho_+\rho_-}{L\rho^2}dt_3\right)^2, \tag{5.109}$$

with the position of the inner (ρ_-) and outer (ρ_+) horizons given by

$$\rho_+ = \frac{16m^2R_z}{L^2}\Pi_c\,, \qquad \rho_- = \frac{16m^2R_z}{L^2}\Pi_s\,. \tag{5.110}$$

The left- and right-moving temperatures are then

$$T_L = \frac{\rho_++\rho_-}{2\pi L^2} = \frac{8m^2R_z}{\pi L^4}\left(\Pi_c+\Pi_s\right)\,, \qquad T_R = \frac{\rho_+-\rho_-}{2\pi L^2} = \frac{8m^2R_z}{\pi L^4}\left(\Pi_c-\Pi_s\right)\,, \tag{5.111}$$

and the black hole mass, angular momentum, entropy density and temperature are

$$M = \frac{1}{8G_3}\left(\frac{\rho_+^2+\rho_-^2}{L^2}\right) = \frac{32m^4R_z^2}{L^6G_3}\left(\Pi_c^2+\Pi_s^2\right) \tag{5.112}$$

$$J = \frac{1}{8G_3}\left(\frac{2\rho_+\rho_-}{L}\right) = \frac{64m^4R_z^2}{L^5G_3}\Pi_c\Pi_s \tag{5.113}$$

$$S = \frac{(4\pi\rho_+)}{8G_3} = \frac{8\pi m^2R_z}{L^2G_3}\Pi_c \tag{5.114}$$

$$T = \frac{2T_LT_R}{T_L+T_R} = \frac{8m^2R_z}{\pi L^4}\left(\frac{\Pi_c^2-\Pi_s^2}{\Pi_c}\right)\,. \tag{5.115}$$

When we uplift the perturbations (5.54) and (5.57), we find that they superficially destroy the BTZ asymptotics. This is due to the fact that all the independent perturbations in the $4d$ theory involve a change in the metric. However, since the $3d$ linearized Einstein's equations decoupled from the matter fields, the solution must still be locally AdS_3 and the uplifted perturbations must correspond to a change in the BTZ parameters up to diffeomorphisms. Indeed, we can perform a linearized diffeomorphism

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,, \tag{5.116}$$

that brings the metric to the original BTZ form, with the change of parameters:

$$\delta\rho_+ = \Big(\alpha_0+\sum_i\alpha_i\Big)\frac{4m^2R_z}{L^2}\Pi_c \tag{5.117}$$

$$\delta\rho_- = -\Big(\alpha_0+\sum_i\alpha_i\Big)\frac{4m^2R_z}{L^2}\Pi_s\,. \tag{5.118}$$

Incidentally, this shows that the marginal mode is non-normalizable from the AdS_3 perspective, and that there is a non-trivial change of basis between the independent $4d$ modes and the $3d$ modes. We can translate this into a change of mass and angular momentum of the BTZ black hole:

$$\delta M = \delta\left(\frac{\rho_+^2+\rho_-^2}{8G_3L^2}\right) = \Big(\alpha_0+\sum_i\alpha_i\Big)\frac{16m^4R_z^2}{G_3L^6}\left(\Pi_c^2-\Pi_s^2\right) \tag{5.119}$$

$$\delta J = 0\,. \tag{5.120}$$

Notice that the variations of the physical BTZ parameters vanish when

$$\alpha_0 = -\sum_i\alpha_i\,. \tag{5.121}$$

As we now explain, we can choose our sources so that they satisfy the relation above, and this corresponds to a scheme where the entropy of the black hole does not change order by order in perturbation theory.

Recall that the precise relation between the parameters α_I and the parameters a_I that describe the family of exact black hole solutions depends on the renormalization scheme, as explained at the end of section 5.3.2. The choice (5.60)-(5.61) is one possibility, but here we will present an alternative that is more natural from the point of view of AdS/CFT. In quantum field theory one has the freedom to redefine the sources at each order in perturbation theory, so that

$$J = J_0 + \lambda J_1 + \ldots + \lambda^n J_n + \ldots\,, \tag{5.122}$$

where λ is the coupling constant. It is customary to choose a scheme where

$$J = J_0\,, \tag{5.123}$$

i.e. where the source is not renormalized. From the point of view of AdS/CFT, this means that the coefficient of $\rho^{\Delta-d}$ that corresponds to the source of the dual operator does not change at higher order in perturbation theory. This corresponds to the choice

$$\alpha_i = \frac{\Pi_c^2 - \Pi_s^2}{\Pi_c^2 \sinh^2\delta_i - \Pi_s^2 \cosh^2\delta_i} \approx 4e^{-2\delta_i}\,, \tag{5.124}$$

showing once again that the leading contribution is independent of the scheme. It is possible to do the same for the metric mode associated to the dual stress tensor, but in this context it seems more natural to choose a scheme where the $4d$ metric does not change at the horizon order by order in perturbation theory. This yields

$$\alpha_0 = -\sum_{i=1}^{3} \alpha_i \approx -4\sum_{i=1}^{3} e^{-2\delta_i}\,, \tag{5.125}$$

which once again agrees with the previous results to leading order in $e^{-2\delta_i}$. Notice that this choice corresponds to keeping the physical parameters of the BTZ black hole fixed, so that at first order the marginal mode associated to the metric is turned off. We conclude that the flow to the original geometry is started by turning on three irrelevant operators in the dual CFT.

5.4.5 Irrelevant mass scale and range of validity of the CFT description

Finally, we can determine the mass scale set by the irrelevant deformations, which represents the UV cutoff of the dual field theory. Consider the asymptotic behavior of the field δU:

$$\delta U = \frac{\ell_S^4 \sum_{i=1}^{3} \alpha_i}{3R_z^2(2m)^4\,(\Pi_c^2 - \Pi_s^2)}\rho^2 + \ldots\,. \tag{5.126}$$

As a consequence, the source of the operator dual to U reads

$$J_U = \frac{L^8 \sum_{i=1}^{3} \alpha_i}{48 R_z^2 (2m)^4 \left(\Pi_c^2 - \Pi_s^2\right)} = \frac{1}{12\pi^2 T_L T_R} \sum_i \alpha_i \,. \tag{5.127}$$

Recall that the temperature in a CFT sets an infrared cutoff, while the mass scale of the irrelevant deformation sets an ultraviolet cutoff. Equation (5.127) shows that when the α_i's are of order 1, the infrared cutoff and the ultraviolet cutoff are of the same order, so that there is no regime where the conformal field theory description is meaningful. On the other hand, when the α_i's become small (or $\delta_i \gg 1$), an energy window appears where perturbation theory on the CFT should be a good description of the system:

$$1 < \frac{E^2}{T_L T_R} \ll \frac{1}{\alpha} \,. \tag{5.128}$$

This is the CFT analog of the condition (5.66) that we have identified in the $4d$ system.

We can phrase the result above in terms of standard effective field theory. The contributions of irrelevant couplings to a process characterized by an energy scale E are typically suppressed by powers of E/M, where M is the UV cutoff set by the irrelevant couplings. In our case we have

$$M^2 \approx \frac{1}{\alpha} T_L T_R \,, \tag{5.129}$$

and this should be compared to the IR cutoff of the system, that is the temperature. We can argue this as follows: the expectation value of an operator A in a thermal field theory is given by

$$\mathrm{tr}(\rho\, A) = \sum_i e^{-\beta E_i} \langle E_i | A | E_i \rangle, \tag{5.130}$$

where we used the fact that ρ is a thermal density matrix. The corrections coming from the irrelevant operators will become important for matrix elements $\langle E|A|E\rangle$ where $E \gtrsim M$. The previous expression tells us that these contributions are multiplied by a factor $e^{-\beta M}$; as a consequence, if we want the computations in effective field theory to be reliable, we need to ask that $\beta M \gg 1$. In this sense, M is very large when the α's are small, opening up a range of energies where effective field theory becomes meaningful. It is precisely in this region that CFT (plus perturbation theory) becomes a good description of the system.

5.5 Discussion and conclusions

It has been recently argued that certain questions involving the entropy and thermodynamics of four-dimensional asymptotically flat non-extremal black holes can be elucidated by replacing the original geometry by one with a different conformal factor, dubbed subtracted geometry. The replacement modifies the asymptotics while preserving the near-horizon behavior of the original black hole, in such a way that the role of an underlying conformal symmetry becomes manifest, shedding light on the form of the entropy for black holes away from extremality [99, 100, 101]. Building on these works, we have shown that four-dimensional, static, asymptotically flat non-extremal black holes with one electric and three magnetic charges can be connected to their corresponding subtracted geometry by a flow which we have constructed explicitly in the form of an interpolating family of solutions. Upon uplifting the construction to five dimensions the subtracted geometry asymptotes to $\mathrm{AdS}_3 \times S^2$, and an AdS/CFT interpretation of the flow is readily available as the effect of irrelevant perturbations in the conformal field theory dual to the AdS_3 factor. In particular, we have identified the quantum numbers of the deformations responsible for the flow and showed that they correspond to three scalar operators with conformal weights $(h, \bar{h}) = (2, 2)$.

As discussed in detail in section 5.3 and 5.4, the mass scale associated to such irrelevant perturbations becomes very large compared to the temperature when the magnetic charges are large. In this limit, it is reasonable to expect that some dynamical questions can be approximately answered by means of perturbation theory in the CFT_2. At least in the static limit, our construction then puts the procedure followed in [99, 100] on a somewhat more concrete footing. On the other hand, away from this limit the ultraviolet cutoff set by the irrelevant deformations becomes of the same order of the infrared cutoff set by the temperature, and the dual CFT captures an increasingly smaller subset of the dynamics, making the usefulness of such an approach doubtful.

It would clearly be of interest to extend our analysis to include rotating four-dimensional black holes. The first steps in this direction were taken in [114], where the general flow between original and subtracted geometries, including the rotating case, was constructed using various duality transformations. The reduction to three dimensions appears to be technically more challenging: in the $5d$ uplifted geometry the two-sphere S^2 is fibered non-trivially over AdS_3, and the modes that start the flow presumably involve non-trivial harmonics on the sphere. However, the radial dependence of such modes in the rotating case suggests that, in addition to the $(2, 2)$ operators that we found in the present chapter, we need to consider also a $(1, 2)$ operator [114].

It would also be very interesting to set up the perturbation scheme in the dual CFT_2 and determine what observables of the $4d$ black hole can be reliably computed in terms of perturbation theory in the irrelevant couplings. Effective field theory makes sense only up to the scale set by the irrelevant deformations. However, our perturbations can be resummed geometrically to all orders, allowing us to go beyond the region where perturbation theory is meaningful and reach the asymptotically flat region. From the field theoretic perspective, it is then natural to wonder whether this allows us to say something about the regime where effective field theory breaks down. Similar questions can be considered for the black holes with Lifshitz-like asymptotics that can be obtained by turning on only a subset of the irrelevant deformations.

5.A Conventions and useful formulae

5.A.1 Hodge duality

Let ω be a p-form in D-dimensions,

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}\,. \tag{5.131}$$

We define the action of the Hodge star on the basis of forms as

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{1}{(D-p)!}\epsilon_{\nu_1\dots\nu_{D-p}}{}^{\mu_1\cdots\mu_p}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-p}}\,, \tag{5.132}$$

where $\epsilon_{\mu_1\dots\mu_D}$ are the components of the Levi-Civita tensor. Equivalently, in components we find

$$(\star\,\omega)_{\mu_1\dots\mu_{D-p}} = \frac{1}{p!}\epsilon_{\mu_1\dots\mu_{D-p}\,\nu_1\dots\nu_p}\,\omega^{\nu_1\dots\nu_p}\,. \tag{5.133}$$

If $\varepsilon_{\mu_1\dots\mu_D}$ denotes the components of the Levi-Civita *symbol* (a tensor density), we have

$$\epsilon_{\mu_1\dots\mu_D} = \sqrt{|g|}\,\varepsilon_{\mu_1\dots\mu_D} \quad \Leftrightarrow \quad \epsilon^{\mu_1\dots\mu_D} = \frac{(-1)^t}{\sqrt{|g|}}\varepsilon^{\mu_1\dots\mu_D} \tag{5.134}$$

where t denotes the number of timelike directions, and we have adopted the convention that the Levi-Civita symbol ε with up or down indices is the same. The volume element is given by

$$\star\mathbb{1} = \sqrt{|g|}\,d^Dx \equiv \mathrm{vol}_D \qquad \Rightarrow \qquad \star\mathrm{vol}_D = (-1)^t\mathbb{1}\,. \tag{5.135}$$

A useful observation is that, for any two p-forms A and B,

$$\star A \wedge B = \star B \wedge A = \frac{1}{p!}\,A^{\mu_1\cdots\mu_p}B_{\mu_1\dots\mu_p}\,\mathrm{vol}_D\,. \tag{5.136}$$

Similarly, if ϕ is a scalar it follows

$$d{\star}d\phi = (-1)^{D-1}\,\nabla_\mu\nabla^\mu\phi\,\mathrm{vol}_D \qquad \Rightarrow \qquad \star d{\star}d\phi = (-1)^{t+D-1}\,\nabla^\mu\nabla_\mu\phi\,, \tag{5.137}$$

while a one-form A with field strength $F = dA$ satisfies

$$\star d{\star}dA = \star d{\star}F = (-1)^{t+1}\,\nabla_\nu F^\nu{}_\lambda\, dx^\lambda\,. \tag{5.138}$$

5.A.2 Details of the Kaluza–Klein reduction

Here we provide further details on the reduction of the $5d$ theory (5.72) on the two-sphere. As described in the main text, our KK ansatz is

$$
\begin{aligned}
ds_5^2 &= ds^2_{\text{string}}(M) + e^{2U(x)} ds^2\,(Y) && (5.139)\\
\tilde{F}^i &= \,-\,B_i\,\sin\theta\, d\theta \wedge d\phi && (5.140)\\
\Psi &= \Psi(x) && (5.141)\\
\Phi &= \Phi(x)\,, && (5.142)
\end{aligned}
$$

where M is the (2+1)-dimensional external manifold with coordinates $x = \{r, t, z\}$ and Y is the two-sphere with radius ℓ_S and coordinates $y = \{\theta, \phi\}$. We pick the orientation such that the volume form on Y is $\mathrm{vol}_2 = \ell_S^2 \sin\theta\, d\theta \wedge d\phi$.

Because the field strengths $\tilde{F}^i$ are purely magnetic, and proportional to the volume form of the two-sphere, the vector equations (5.73) are satisfied trivially in our ansatz and do not yield lower-dimensional equations of motion. Let us now consider the reduction of the scalar equations (5.74)-(5.75). In order to reduce the coupling of the scalars to the U(1) field strengths it is useful to notice that via (5.136) our ansatz implies

$$
\star_5 \tilde{F}^i \wedge \tilde{F}^i = \frac{1}{2!}\tilde{F}^{i\,\mu\nu}\tilde{F}^i_{\mu\nu}\,\mathrm{vol}_5 = \frac{B_i^2}{\ell_S^4} e^{-4U(x)}\,\mathrm{vol}_5 = \frac{B_i^2}{\ell_S^4} e^{-2U(x)}\,\mathrm{vol}_3 \wedge \mathrm{vol}_2\,. \quad (5.143)
$$

Next, we note that for any one-form A with support in M

$$
\star_5 A = e^{2U(x)} \star_3 A \wedge \mathrm{vol}_2\,. \quad (5.144)
$$

In particular, if Ψ is a scalar in M, applying this result to $d\Psi$ we find

$$
\star_5 d\Psi = e^{2U(x)} \star_3 d\Psi \wedge \mathrm{vol}_2\,. \quad (5.145)
$$

The decomposition of the scalar Laplacian then follows:

$$
d\star_5 d\Psi = e^{2U(x)}\Big[d\left(\star_3 d\Psi\right) + 2dU(x) \wedge \star_3 d\Psi\Big] \wedge \mathrm{vol}_2\,. \quad (5.146)
$$

Plugging this result together with (5.143) into (5.74)-(5.75) we find the effective $3d$ equations for the scalar fields on M:

$$
\begin{aligned}
0 &= d\left(\star_3 d\Psi\right) + 2dU \wedge \star_3 d\Psi - \frac{e^{-4U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Psi}\mathrm{vol}_3 && (5.147)\\
0 &= d\left(\star_3 d\Phi\right) + 2dU \wedge \star_3 d\Phi - \frac{e^{-4U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Phi}\mathrm{vol}_3\,. && (5.148)
\end{aligned}
$$

Equivalently, in component notation we have

$$0 = \nabla_\mu \nabla^\mu \Psi + 2\nabla_\mu U \nabla^\mu \Psi - \frac{e^{-4U}}{2} \sum_{i=1}^{3} \frac{B_i^2}{\ell_S^4} \frac{\delta H_{ii}(\Psi, \Phi)}{\delta \Psi} \tag{5.149}$$

$$0 = \nabla_\mu \nabla^\mu \Phi + 2\nabla_\mu U \nabla^\mu \Phi - \frac{e^{-4U}}{2} \sum_{i=1}^{3} \frac{B_i^2}{\ell_S^4} \frac{\delta H_{ii}(\Psi, \Phi)}{\delta \Phi} \,. \tag{5.150}$$

We now turn our attention to the reduction of the $5d$ Einstein's equations (5.76). In order to reduce the Ricci tensor, we first study the decomposition of the spin connection and the curvature two-form. Let $\hat{e}^M$ denote the $5d$ local Lorentz frame, and $M, N, \ldots$ denote the flat indices on the $5d$ manifold. Denoting by $a, b, \ldots$ the flat indices on M, and by $\alpha, \beta, \ldots$ the flat indices on the compact manifold Y, our choice of vielbein reads

$$\hat{e}^a = e^a \tag{5.151}$$

$$\hat{e}^\alpha = e^U e^\alpha \,, \tag{5.152}$$

where e^a and e^α are orthonormal frames for M and Y, respectively. Denoting by ${\omega^a}_b$ the spin connection associated with M and by ${\omega^\alpha}_\beta$ the spin connection appropriate to Y, solving the torsionless condition for the $5d$ spin connection $\hat{\omega}$ we find

$${\hat{\omega}^a}_b = {\omega^a}_b \tag{5.153}$$

$${\hat{\omega}^\alpha}_\beta = {\omega^\alpha}_\beta \tag{5.154}$$

$${\hat{\omega}^\alpha}_a = P_a \, e^\alpha \,, \tag{5.155}$$

where we introduced the shorthand

$$P_a \equiv e^U (\partial_a U) \,. \tag{5.156}$$

It is useful to notice that ${\hat{\omega}^a}_\alpha \wedge {\hat{\omega}^\alpha}_b = P^a P_b \, \eta_{\alpha\beta} \, e^\alpha \wedge e^\beta = 0$. Next, let Θ denote the curvature two-form. Then, on the $5d$ manifold we have ${\hat{\Theta}^M}_N = d{\hat{\omega}^M}_N + {\hat{\omega}^M}_P \wedge {\hat{\omega}^P}_N$. Computing the different components we find

$${\hat{\Theta}^a}_b = {\Theta^a}_b \tag{5.157}$$

$${\hat{\Theta}^\alpha}_\beta = {\Theta^\alpha}_\beta - P_a P^a \eta_{\beta[\gamma} {\delta^\alpha}_{\sigma]} \, e^\sigma \wedge e^\gamma \tag{5.158}$$

$${\hat{\Theta}^\alpha}_a = {\delta^\alpha}_\gamma \, (\nabla_c P_a) \, e^c \wedge e^\gamma \,. \tag{5.159}$$

The antisymmetrization symbol [...] used above includes a factor of $1/2!$, and ∇_a denotes the connection on M. From these expressions we can identify the non-vanishing components of the Riemann tensor, defined as ${\hat{\Theta}^M}_N = \frac{1}{2!} {\hat{R}^M}_{NPQ} \, \hat{e}^P \wedge$

$\hat{e}^Q$:

$$\hat{R}^a{}_{bcd} = R^a{}_{bcd} \tag{5.160}$$

$$\hat{R}^\alpha{}_{\beta\gamma\delta} = e^{-2U}\mathcal{R}^\alpha{}_{\beta\gamma\delta} - 2e^{-2U}P_aP^a\delta^\alpha{}_{[\gamma}\eta_{\delta]\beta} \tag{5.161}$$

$$\hat{R}^\alpha{}_{a\beta b} = -\,\delta^\alpha{}_\beta\, e^{-U}\nabla_b P_a \tag{5.162}$$

$$\hat{R}^a{}_{\alpha b\beta} = -\,\eta_{\alpha\beta}\, e^{-U}\nabla_b P^a\,. \tag{5.163}$$

In the above notation $R^a{}_{bcd}$ are the components of the Riemann tensor of the external manifold M, and $\mathcal{R}^\alpha{}_{\beta\gamma\delta}$ those of the Riemann tensor of the compact manifold Y. Finally, for the decomposition of the Ricci tensor $\hat{R}_{MN} = \hat{R}^P{}_{MPN}$ we find

$$\begin{aligned}\hat{R}_{ab} &= R_{ab} - d_Y e^{-U}\nabla_b P_a \\ &= R_{ab} - d_Y\left(\nabla_b\nabla_a U + \nabla_a U\nabla_b U\right)\end{aligned} \tag{5.164}$$

$$\begin{aligned}\hat{R}_{\alpha\beta} &= e^{-2U}\mathcal{R}_{\alpha\beta} - (d_Y-1)\,e^{-2U}P_aP^a\,\eta_{\alpha\beta} - e^{-U}\nabla_c P^c\,\eta_{\alpha\beta} \\ &= e^{-2U}\mathcal{R}_{\alpha\beta} - d_Y\left(\nabla_a U\nabla^a U\right)\eta_{\alpha\beta} - \left(\nabla^a\nabla_a U\right)\eta_{\alpha\beta}\end{aligned} \tag{5.165}$$

$$\hat{R}_{a\alpha} = 0\,, \tag{5.166}$$

where d_Y is the dimension of the compact manifold ($d_Y = 2$ in our case). In particular, for the Ricci scalar $\hat{R} = \eta^{MN}\hat{R}_{MN}$ it follows that

$$\hat{R} = R + e^{-2U}\mathcal{R} - 2d_Y\nabla^a\nabla_a U - d_Y\left(1+d_Y\right)\nabla^a U\nabla_a U\,, \tag{5.167}$$

where R is the scalar curvature on M, and $\mathcal{R}$ that of Y. Since the two-sphere has radius ℓ_S we have $\mathcal{R}_{\alpha\beta} = \eta_{\alpha\beta}/\ell_S^2$. Setting $d_Y = 2$ in the above expressions we find that the only non-vanishing components in our reduction are

$$\hat{R}_{ab} = R_{ab} - 2\big(\nabla_b\nabla_a U + \nabla_a U\nabla_b U\big) \tag{5.168}$$

$$\hat{R}_{\alpha\beta} = \left(\frac{e^{-2U}}{\ell_S^2} - \nabla^a\nabla_a U - 2\nabla_a U\nabla^a U\right)\eta_{\alpha\beta}\,. \tag{5.169}$$

The Ricci scalar is then given by

$$\hat{R} = R + \frac{2}{\ell_S^2}e^{-2U} - 4\nabla^a\nabla_a U - 6\nabla^a U\nabla_a U\,. \tag{5.170}$$

Using the decomposition of the Ricci tensor, from the components of the $5d$ Einstein's equations in the directions of the external manifold M we get (using flat indices on M)

$$\begin{aligned}R_{ab} &= 2\big(\nabla_b\nabla_a U + \nabla_a U\nabla_b U\big) \\ &\quad + \frac{1}{2}\left[\nabla_a\Psi\nabla_b\Psi + \nabla_a\Phi\nabla_b\Phi - \frac{\eta_{ab}}{3}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi)\right].\end{aligned} \tag{5.171}$$

Similarly, noting that with flat indices $\tilde{F}^{i\ P}_{\alpha}\tilde{F}^{i}_{\beta P} = \left(e^{-4U}B_i^2/\ell_S^4\right)\eta_{\alpha\beta}$, from the components of the $5d$ Einstein's equations in the directions of Y we find

$$\nabla^a\nabla_a U + 2\nabla_a U\nabla^a U - \frac{e^{-2U}}{\ell_S^2} + \frac{e^{-4U}}{3}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi) = 0\,. \tag{5.172}$$

Since all reference to the two-sphere dropped out from the equations of motion, the proposed truncation is consistent. Finally, we point out that the resulting three-dimensional equations of motion (5.147), (5.148), (5.171) and (5.172) can be obtained from the following effective action (in string frame):

$$S_{\text{string}} = -\frac{1}{16\pi G_3}\int d^3x\sqrt{|g|}\,e^{2U}\bigg[R + \frac{2}{\ell_S^2}e^{-2U} - \frac{e^{-4U}}{2}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi) + 2\left(\nabla U\right)^2 - \frac{1}{2}\left(\nabla\Psi\right)^2 - \frac{1}{2}\left(\nabla\Phi\right)^2\bigg]. \tag{5.173}$$

5.A.3 The general solution in terms of $5d$ and $3d$ fields

In terms of the $4d$ fields (5.13)-(5.16), our four-parameter family of solutions was given in (5.34)-(5.37) (with F^0 given by (5.18)). The h^i fields appearing in the $5d$ theory (and also in the $4d$ STU model) are related to the diagonal fields (5.22)-(5.25) through

$$h^1 = \exp\left[\frac{1}{6}\left(2\phi_1 - \phi_2 - \phi_3\right)\right] \tag{5.174}$$

$$h^2 = \exp\left[\frac{1}{6}\left(2\phi_2 - \phi_1 - \phi_3\right)\right] \tag{5.175}$$

$$h^3 = \exp\left[\frac{1}{6}\left(2\phi_3 - \phi_1 - \phi_2\right)\right]. \tag{5.176}$$

Hence, in our general family of solutions (5.28)-(5.29) they read

$$h^1(r) = \left(\frac{B_1^2}{|B_2B_3|}\frac{\sqrt{(1+a_2^2)\,(1+a_3^2)}}{1+a_1^2}\frac{\left(a_1^2\,r+2m\right)^2}{(a_2^2\,r+2m)\,(a_3^2\,r+2m)}\right)^{1/3} \tag{5.177}$$

$$h^2(r) = \left(\frac{B_2^2}{|B_1B_3|}\frac{\sqrt{(1+a_1^2)\,(1+a_3^2)}}{1+a_2^2}\frac{\left(a_2^2\,r+2m\right)^2}{(a_1^2\,r+2m)\,(a_3^2\,r+2m)}\right)^{1/3} \tag{5.178}$$

$$h^3(r) = \left(\frac{B_3^2}{|B_1B_2|}\frac{\sqrt{(1+a_1^2)\,(1+a_2^2)}}{1+a_3^2}\frac{\left(a_3^2\,r+2m\right)^2}{(a_1^2\,r+2m)\,(a_2^2\,r+2m)}\right)^{1/3}. \tag{5.179}$$

We recall that the $5d$ line element is given in terms of the $4d$ one by (5.77). Similarly, in terms of the decoupled fields the $3d$ scalars are given by

$$U = \frac{\phi_1 + \phi_2 + \phi_3}{6} + \log \frac{m}{\ell_S} \tag{5.180}$$

$$\Psi = \frac{1}{2\sqrt{6}} \left(2\phi_1 - \phi_2 - \phi_3\right) \tag{5.181}$$

$$\Phi = \frac{1}{2\sqrt{2}} \left(\phi_3 - \phi_2\right) , \tag{5.182}$$

so in our general solution we obtain

$$U(r) = \frac{1}{3} \log\left[\frac{1}{\sqrt{(1+a_1^2)\,(1+a_2^2)\,(1+a_3^2)}} \left(\frac{a_1^2}{2m} r + 1\right) \left(\frac{a_2^2}{2m} r + 1\right) \left(\frac{a_3^2}{2m} r + 1\right)\right] \tag{5.183}$$

$$\Psi(r) = \frac{1}{\sqrt{6}} \log\left[\frac{B_1^2}{|B_2 B_3|} \frac{\sqrt{(1+a_2^2)\,(1+a_3^2)}}{1+a_1^2} \frac{\left(a_1^2\, r + 2m\right)^2}{(a_2^2\, r + 2m)\,(a_3^2\, r + 2m)}\right] \tag{5.184}$$

$$\Phi(r) = \frac{1}{\sqrt{2}} \log\left[\left|\frac{B_3}{B_2}\right| \sqrt{\frac{1+a_2^2}{1+a_3^2}} \left(\frac{a_3^2\, r + 2m}{a_2^2\, r + 2m}\right)\right] . \tag{5.185}$$

Conclusions and Outlook

In this final chapter we want to review the main results that were derived in this thesis, and indicate some possible directions for future research. Chapter 2 focused on the application of holography to systems with non-relativistic scaling symmetry. These field theories are relevant in many physical setups, where they describe processes near a (quantum) critical point. We focused on systems with Lifshitz scaling symmetry, which appears for example in the study of smectic liquid crystals and quantum dimer models, and took some steps towards the extension of the holographic dictionary to theories with such symmetry. The long term goal of this program is at least twofold. On the one hand we would like to build a framework to define and study non-relativistic strongly coupled systems that may be relevant to condensed matter and statistical physics. On the other hand, such a framework provides examples of holographic setups with background spacetimes that are not asymptotically AdS. We have indeed provided evidence that many of the usual techniques developed in the AdS/CFT context, such as holographic renormalization, carry over to the non-relativistic case. Furthermore, we have shown that in our specific example we can turn on a (marginally) relevant deformation, which allows us to move away from the quantum critical point and opens up the possibility of studying interesting phenomena such as finite temperature crossovers.

In chapter 3 we continued our study of theories with non-relativistic scaling symmetry, and we showed that they exhibit the non-relativistic analog of the Weyl anomaly. In the case $z = 2$ in 2+1 dimensions we have identified two possible structures that can appear in the anomaly and cannot be removed by local counterterms, and computed them in a field theory model and in a holographic model. The next logical step would be to extend the analysis to theories in higher dimensions as well as theories that break time reversal symmetry. Anomalies have proven to be an extremely useful tool in relativistic theories, where they control various universal contributions to physical quantities, such as the Casimir energy or the logarithmic corrections to the entanglement entropy. As a consequence, these anomaly coefficients are experimentally accessible, for example by measuring how the vacuum energy varies as the size of the system is changed. It is interesting to notice that the holographic models we considered, which were based on Einstein gravity plus matter in the bulk, seem able to produce only one of the possible two structures in the anomaly, and there are indications that the other structure requires some kind of non-relativistic gravity theory in the bulk, such as Horava–Lifshitz gravity. More work will be required to clarify the implications of these observations.

In chapter 4 we studied non-renormalization theorems in theories with a large amount of supersymmetry. These theorems constrain the coupling constant dependence of various observable quantities, and are therefore extremely useful both for phenomenological applications, where they impose cancellations between radiative corrections to various parameters of the theory, and for theoretical reasons, providing quantities that are exactly computable at strong coupling. In particular, by proving a non-renormalization theorem we have shown that some AdS/CFT predictions concerning the behavior of three-point functions of chiral primary operators at strong coupling are indeed confirmed. Various issues remain to be studied, such as the extension of our techniques to less supersymmetric multiplets and the possibility of lifting from the BPS bound. Counting the BPS states that survive at strong coupling is important for example for the microscopic computation of the entropy of supersymmetric black holes. Group theory imposes strong restrictions on how multiplets can lift, which are captured by a mathematical object known as "index", but there are examples where this information is not enough. For instance, the most general index for the $\mathcal{N} = 4$ theory is not able to reproduce the entropy of supersymmetric black holes in AdS_5, indicating that the index is oversubtracting BPS states; clearly some dynamical input is needed to correctly account for the black hole microstates. The techniques that we presented in chapter 4 do indeed make use of some dynamical information, namely the structure of the possible marginal deformations, and it is conceivable that they might impose further constraints on which multiplets can lift. We definitely hope to come back to this question in the near future.

Finally, in chapter 5 we explored some aspects of a recent development relating to black hole entropy, which goes under the name of Kerr/CFT correspondence. The entropy of various supersymmetric and asymptotically AdS black holes can be accounted for microscopically thanks to the AdS/CFT correspondence. However, realistic asymptotically flat black holes appear to be much more difficult to study, so it is quite mysterious that the entropy of many of these generic black holes can be cast in a form reminiscent of Cardy's formula for the asymptotic growth of states in a CFT. The Kerr/CFT program, among other things, uncovered a "hidden" conformal symmetry of the scalar wave equation in the near-horizon region of these black hole backgrounds, which becomes manifest when certain offending terms are removed. This can also be understood in terms of a subtraction procedure, where the asymptotically flat region is replaced by a different, more controllable, region while preserving the near-horizon properties. Upon uplifting to one dimension higher, the resulting subtracted geometry turns out to be an asymptotically $\mathrm{AdS}_3 \times S^2$ black hole, and a CFT interpretation is therefore obvious. We have shown that the subtraction procedure corresponds in the field theory side to ignoring some irrelevant deformations of the CFT. While this is justified when some of the charges are sufficiently large, in the generic case the UV cutoff set by these irrelevant deformations is of the same order of magnitude as the temperature of the CFT, so that the irrelevant modes are excited by the thermal background and cannot justifiably be ignored. Recently, it was shown that similar considerations apply to the rotating case as well, so it is definitely important to further explore the possible implications of our result for the Kerr/CFT program.

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Summary

The 20th century has witnessed a tremendous amount of progress in our understanding of the fundamental laws of nature. The development of quantum mechanics on the one hand and general relativity on the other hand opened up the possibility to describe microscopic interactions between elementary particles as well as gravitational interactions between macroscopic bodies with an unprecedented precision.

Our most fundamental understanding of the microscopic world is represented by the *standard model*, which describes electromagnetic, weak and strong interactions in the unified framework of quantum field theory. Despite its numerous experimental successes, there are strong indications, both theoretical and experimental, that the standard model is incomplete. For example, some of the free parameters of the theory must be tuned very finely in order to match experiments, an issue known as hierarchy problem. Furthermore the indirect observation of dark matter points towards the existence of yet to be discovered particles.

From a theoretical point of view, an issue with the standard model is that it does not contain gravitational interactions. At the macroscopic level, gravity is well described by general relativity, which has been very successful in astrophysics and cosmology. However, reconciling such a theory with the principles of quantum mechanics has proven to be extremely challenging. The standard attempts at quantizing the theory are indeed plagued with mathematical inconsistencies that cannot be fixed without giving up predictability. As a consequence, one of the main challenges of present-day research in high-energy physics is to find a consistent theory of *quantum gravity*. Despite the successes of both the standard model and general relativity in their own regimes of applicability, such a venture is rather problematic. Current technology does not allow us to directly probe the energies at which both quantum and gravitational effects are relevant. In the absence of hints coming from experiments, the development of a quantum theory of gravity must therefore rely on mathematical consistency and thought experiments.

The leading candidate for a quantum theory of gravity is *string theory*, whose main underlying idea is to give up the notion that particles are point-like objects, replacing them instead with one-dimensional strings. This theory has passed many non-trivial consistency checks, and has provided important theoretical insights into the microscopic structure of gravitational objects such as black holes. It must be noted, however, that our understanding of string theory is still incomplete, especially at the non-perturbative level.

One of the most spectacular developments coming from string theory is represented by the so-called gauge/gravity dualities. These dualities state that certain quantum theories of gravity, such as those defined on backgrounds that are asymptotically anti-de Sitter (AdS), can be entirely described by ordinary quantum field theories living in one dimension less. This provides a concrete realization of the holographic principle, which had already emerged in the study of black hole physics. Furthermore, the dictionary between the two theories exchanges the strongly coupled regime of one theory with the weakly coupled regime of the dual theory. As a consequence, standard perturbative field theory methods can be used to probe the strongly coupled quantum structure of the dual gravity theory, while classical gravitational computations can give us insights into strongly-coupled phenomena in quantum field theory, such as confinement and transport properties of the quark-gluon plasma.

The research directions presented in this thesis all fit together in the framework of holography. In chapter 2 and 3 we take some small steps towards the development of a holographic dictionary for spacetimes that are *not* asymptotically AdS. More specifically, we study the situation where the background spacetime is asymptotically Lifshitz, a particular gravitational setup believed to be dual to strongly coupled theories that are non-relativistic. On the one hand, we hope that attempts at extending the holographic dictionary to more general spacetimes will eventually lead to the development of a satisfactory holographic description of realistic gravitational setups. On the other hand, these constructions are potentially interesting for applications in condensed matter and statistical physics, where many non-trivial phenomena enjoying non-relativistic symmetry seem to elude a weakly-coupled field theory descriptions.

In chapter 4 we turn to relativistic field theories that enjoy a large amount of supersymmetry, a particular symmetry relating bosons and fermions and which might very well be a symmetry of nature. We show that various observable quantities of such theories do not receive quantum corrections, or in more technical language we prove non-renormalization theorems. Similar theorems have proven to be extremely useful for phenomenological applications, where they impose cancellations between radiative corrections on the one hand and allow the exact computation of

certain observables at strong coupling on the other. In our case, the theorems we prove allow us to test some predictions coming from the holographic correspondence, providing further consistency checks of such duality.

In the last chapter we study black holes and their entropy, once again in the holographic context. We are concerned in particular with recent developments towards the microscopic understanding of the entropy of realistic (asymptotically flat) black holes. In the (somewhat unrealistic) asymptotically AdS scenario, the holographic dualities described above provide a complete understanding of the structure of the entropy. For example, the AdS/CFT duality explains why the formula for the entropy of three-dimensional asymptotically AdS black holes is identical to the one describing the entropy of two-dimensional conformal field theories (CFT), which are particular quantum field theories that "look the same" at all energy scales. Interestingly enough, the entropy formula for many realistic black holes can also be cast in such a form, but whether there is an underlying CFT explanation is currently unknown. Some recent attempts to elucidate possible relations between such black holes and two-dimensional CFTs involve the removal of the asymptotically flat region from the black hole geometry, a procedure that is often referred to as "putting the black hole in a box". In performing this "subtraction", a conformal symmetry emerges and a CFT description becomes immediately obvious. However, the relation between said CFT and the original asymptotically flat black hole is not entirely clear from the construction. In chapter 5 we address this problem by showing that removing the box we put the black hole in corresponds to turning on an irrelevant (i.e. unimportant at low energies) deformation in the CFT. While the corrections coming from this deformation can be justifiably ignored in a certain region of the parameter space, generically this is not the case and large corrections are expected. Since irrelevant deformations are problematic for predictability, our results call for a refinement of the ideas underlying the subtraction procedure.

Samenvatting

De twintigste eeuw heeft een enorme vooruitgang meegemaakt in het begrip van de fundamentele natuurwetten. De ontwikkeling van de kwantummechanica enerzijds en de algemene relativiteitstheorie aan de anderzijds opende de mogelijkheid om zowel microscopische interacties tussen elementaire deeltjes als gravitationele interactie tussen macroscopische hemellichamen te beschrijven met ongekende precisie.

Onze meest fundamentele kennis van de microscopische wereld wordt beschreven door het standaard model van de deeltjesfysica. Dit model beschrijft de elektromagnetisch kracht en de zwakke en sterke kernkrachten in het kader van de kwantumveldentheorie. Ondanks zijn talrijke experimentele successen zijn er sterke aanwijzingen, zowel theoretisch als experimenteel, dat het standaard model onvolledig is. Zo moet een van de vrije parameters van de theorie zeer fijn worden afgestemd om met experimenten overeen te komen. Deze kwestie staat bekend als het hiërarchie probleem. Ook wijst de indirecte waarneming van donkere materie op het bestaan van nog te ontdekken elementaire deeltjes.

Een probleem met het standaard model vanuit een theoretisch oogpunt is dat het geen gravitationele interacties beschrijft. Op een macroscopisch niveau wordt de zwaartekracht goed beschreven door de algemene relativiteitstheorie, hetgeen zeer succesvol is geweest in de astrofysica en kosmologie. Echter, de combinatie van algemene relativiteitstheorie met de principes van de kwantummechanica heeft bewezen uitermate lastig te zijn. De standaard manieren om van een klassieke theorie een kwantumtheorie te maken worden inderdaad geplaagd met wiskundige tegenstrijdigheden. Deze tegenstrijdigheden kunnen niet worden opgelost zonder dat men de voorspellende eigenschap van de theorie moet opgeven. Als gevolg hiervan is een van de belangrijkste uitdagingen van het hedendaagse onderzoek in de hoge energie fysica om een consistente theorie van de *kwantumzwaartekracht* te vinden. Ondanks de successen van zowel het standaard model als de algemene relativiteitstheorie in hun eigen domein van toepasbaarheid is een dergelijke onderneming vrij problematisch. De huidige technologie staat ons niet toe om rechtstreeks metingen te doen op energieschalen waarop zowel kwantum als gravitationele effecten relevant zijn. Bij het ontbreken van aanwijzingen uit experimenten moet de ontwikkeling van een kwantumtheorie van de zwaartekracht dus vertrouwen op wiskundige consistentie en gedachte-experimenten.

De belangrijkste kandidaat voor een kwantumtheorie van de zwaartekracht is de *snaartheorie*, waarvan de belangrijkste achterliggende gedachte is het opgeven van het idee dat elementaire deeltjes puntdeeltjes zijn; deze worden vervangen door ééndimensionale deeltjes: snaren. Snaartheorie heeft aan vele niet-triviale consistentie-checks voldaan en het heeft belangrijke theoretische inzichten gegeven in de microscopische structuur van gravitationele objecten zoals zwarte gaten. Het moet echter worden opgemerkt, dat ons begrip van de snaartheorie nog onvolledig is, met name op het niet-perturbatieve niveau.

Een van de meest spectaculaire ontwikkelingen uit de snaartheorie is die van zogenaamde ijk/zwaartekracht dualiteiten. Deze dualiteiten verklaren dat bepaalde kwantumtheorieën van de zwaartekracht, zoals die gedefiniëerd op achtergronden die asymptotisch anti-de Sitter (AdS) zijn, volledig kan worden beschreven door gewone kwantumveldentheorieën in één ruimtedimensie lager. Dit geeft een concrete realisatie van het holografisch principe dat voortkwam uit de fysische studie van zwarte gaten. Bovendien vormt de vertaling tussen de twee theorieën een uitwisseling van het sterk gekoppelde regime van de ene theorie met het zwak gekoppelde regime van de andere (duale) theorie. Als gevolg hiervan, kunnen standaard methoden van de perturbatieve veldentheorie worden gebruikt om de sterk gekoppelde kwantumstructuur van de duale zwaartekrachttheorie te bestuderen. Aan de andere kant kunnen klassieke zwaartekacht berekeningen ons inzicht geven in sterk-gekoppelde fenomenen in kwantumveldentheorie, zoals "confinement" en transporteigenschappen van het quark-gluon plasma.

Het gehele onderzoek dat in dit proefschrift wordt gepresenteerd past in het kader van holografie. In hoofdstuk 2 en 3 nemen we enkele kleine stappen in de richting van de ontwikkeling van een holografisch woordenboek voor ruimtes die *niet* asymptotisch AdS zijn. In het bijzonder bestuderen we de situatie waarin de ruimtes asymptotisch Lifshitz zijn. Dit zijn bepaalde gravitationele configuraties die verondersteld worden duaal te zijn aan sterk-gekoppelde niet-relativistische theorieën. Aan de ene kant hopen we dat pogingen tot uitbreiding van de holografische vertaling naar meer algemene ruimtes uiteindelijk zal leiden tot de ontwikkeling van een bevredigende holografische beschrijving van realistische gravitationele configuraties. Aan de andere kant zijn deze constructies mogelijk interessant voor toepassingen binnen de gecondenseerde materie en statistische fysica, waar veel niet-triviale verschijnselen een niet-relativistische symmetrie lijken te vertonen terwijl deze niet kunnen worden beschreven door een zwak gekoppelde veldentheorie.

In hoofdstuk 4 kijken we naar relativistische veldentheorieën die veel supersymmetrie bevatten. Supersymmetrie is een bepaalde symmetrie die bosonen en fermionen aan elkaar relateert en het is mogelijk een symmetrie van de natuur. We laten zien dat verschillende waarneembare grootheden van dergelijke theorieën geen kwantumcorrecties ontvangen. Soortgelijke stellingen hebben bewezen bijzonder nuttige fenomenologische toepassingen te leveren waarbij enerzijds stralingscorrecties tegen elkaar wegvallen en waarbij anderzijds exacte berekeningen van bepaalde grootheden bij sterke koppeling uitvoerbaar zijn. In ons geval, geven de stellingen die wij bewijzen de mogelijkheid om sommige voorspellingen uit de holografische correspondentie te toetsen.

In het laatste hoofdstuk bestuderen we zwarte gaten en hun entropie, wederom in holografische context. We zijn in het bijzonder geïnteresseerd in de recente ontwikkelingen in de richting van het microscopische begrip van de entropie van realistische (asymptotisch vlakke) zwarte gaten. In het (weinig realistische) asymptotisch AdS scenario geeft de zojuist beschreven holografische dualiteit een volledig begrip van de structuur van de entropie. Zo verklaart de AdS/CFT dualiteit waarom de formule voor de entropie van driedimensionale asymptotisch-AdS zwarte gaten identiek zijn aan de beschrijving van de entropie van tweedimensionale conforme veldentheorieën (zgn. CFT's). CFT's zijn bijzondere kwantumveldentheorieën zijn die ër hetzelfde uitzienöp alle energie-schalen. Interessant genoeg kan de entropie formule voor vele realistische zwarte gaten ook worden beschreven in een soortgelijke vorm, maar of er een onderliggende CFT verklaring is, is op dit moment niet duidelijk. Sommige recente pogingen om mogelijke relaties tussen deze zwarte gaten en twee-dimensionale CFT's te verhelderen maken gebruik een methode waarbij het asymptotisch vlakke gebied van de geometrie van het zwarte gat wordt verwijderd. Dit is een procedure die vaak wordt aangeduid als “het zwarte gat in een doos te stoppen”. Bij het uitvoeren van deze “verwijdering” ontstaat een conforme symmetrie en zodoende wordt een CFT beschrijving direct duidelijk. De relatie tussen de CFT en het originele asymptotisch vlakke zwarte gat zijn echter niet geheel duidelijk vanuit een deze opzet. In hoofdstuk 5 van dit proefschift pakken we dit probleem aan door aan te tonen dat het verwijderen van de “doos” correspondeert met het aanzetten van een irrelevante (d.w.z. onbelangrijk bij lage energieën) deformatie in de CFT. Terwijl de correcties uit deze deformatie gerechtvaardigd genegeerd zou kunnen worden in een bepaald gebied in de parameterruimte, is dit in het algemeen duidelijk niet het geval. Aangezien irrelevante deformaties problematisch zijn voor het voorspelbare karakter van een theorie, pleiten onze resultaten voor een verfijning van de ideeën die ten grondslag liggen aan de bovengenoemde procedure.

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