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*Research Article*

# Quasi-Analytical Solutions of DKP Equation under the Deng-Fan Interaction

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Quasianalytical solutions of Duffin-Kemmer-Petiau equation under scalar and vector Deng-Fan potentials are reported via a novel ansatz.

## 1. Introduction

A yet open challenge in the annals of wave mechanics is the equivalence or nonequivalence of Duffin-Kemmer-Petiau (DKP) and Klein-Gordon (KG) equations. While the latter solely describes relativistic spin-zero bosons, the former can investigate spin-one particles as well. This equation was introduced 1930s in search for a linear relativistic wave equation similar to Dirac equation for relativistic bosons [1–4]. In its present format, the DKP equation is normally represented in two five and ten-dimensional versions that respectively work for spin-zero and spin-one bosons. At the present, to our best knowledge, the equivalence of KG and spin-zero DKP equations is doubted [5–14]. This is also true for the other counterpart of DKP equation, that is the Proca equation which is a relativistic framework to study spin-one bosons. There are papers that investigate related problems via these equations and report motivating data [15–20]. Here, our focus is on the spin-zero version of the equation, which has many applications [21–25].

In solving wave equations of mechanics, various analytical techniques including supersymmetry quantum mechanics (SUSYQM), Nikiforov-Uvarov (NU), quantization rule, Lie algebras, WKB approximation, point canonical transformation (PCT), and series expansion have been applied [26–40]. Nevertheless, we face situations in which these methodologies do not work very well. A very successful approach in such cases is proposing a physical ansatz solution that is on the one hand consistent with the requirements of quantum mechanics and on the other hand satisfies the equation. This is in many cases a definitely cumbersome, and in some cases an impossible task. Here we follow this rich quasianalytical technique to solve the DKP equation under the Deng-Fan potential [41]. There are, however, a few papers which discuss this attractive potential. This is partially due to the complex structure of this interaction. As far as we know, the published works include references [42–46] that consider the potential under either relativistic or nonrelativistic equations. The structure of the present paper is as follows. In Section 2, we briefly introduce the DKP equation and a Pekeris-type approximation. In the next part, we introduce some transformations as well as an attractive ansatz by which the problem is solved in a quasianalytical manner.

## 2. The DKP Equation

The DKP Hamiltonian for scalar $U_s$ and vector $U_v^o$ interactions is

$$\left(\beta.\vec{p}c + mc^2 + U_s + \beta^0 U_v^o\right)\psi(\vec{r}) = \beta^0 E\psi(\vec{r}), \tag{2.1}$$

where

$$\psi(\vec{r}) = \begin{pmatrix} \psi_{\text{upper}} \\ i\psi_{\text{lower}} \end{pmatrix}, \tag{2.2}$$

and the upper and lower components, respectively, are

$$\psi_{\text{upper}} \equiv \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \qquad \psi_{\text{lower}} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{2.3}$$

The engaged matrices are

$$\beta^0 = \begin{pmatrix} \theta & \widetilde{0} \\ \overline{0}_T & 0 \end{pmatrix}, \qquad \beta^i = \begin{pmatrix} \widetilde{0} & \rho^i \\ -\rho_T^i & 0 \end{pmatrix}, \tag{2.4a}$$

with $\widetilde{0}$, $\overline{0}$ and $0$, respectively, being $2 \times 2$, $2 \times 3$, $3 \times 3$ zero matrices

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4b}$$

The equation, in (3+0)-dimensions, is written as [1–4]

$$\begin{gathered}\left(mc^2+U_s\right)\phi=(E-U_v^o)\varphi+\hbar c\vec{\nabla}.\vec{A},\\ \vec{\nabla}\phi=\left(mc^2+U_s\right)\vec{A},\\ \left(mc^2+U_s\right)\varphi=(E-U_v^o)\phi,\end{gathered}\tag{2.5}$$

where $\vec{A}=(A_1,A_2,A_3)$. In (2.4a) and (2.4b), $\psi$ is a simultaneous eigenfunction of $J^2$ and $J_3$, that is,

$$\begin{gathered}J^2\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}=\begin{pmatrix}L^2\psi_{\text{upper}}\\ (L+S)^2\psi_{\text{lower}}\end{pmatrix}=J(J+1)\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},\\ J_3\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}=\begin{pmatrix}L_3\psi_{\text{upper}}\\ (L_3+s_3)\psi_{\text{lower}}\end{pmatrix}=M\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},\end{gathered}\tag{2.6}$$

and the general solution is considered as

$$\psi_{JM}(r)=\begin{pmatrix}f_{nJ}(r)Y_{JM}(\Omega)\\ g_{nJ}(r)Y_{JM}(\Omega)\\ i\sum\limits_{L}h_{nJL}(r)Y_{JL1}^{M}(\Omega)\end{pmatrix},\tag{2.7}$$

where spherical harmonics $Y_{JM}(\Omega)$ are of order $J$, $Y_{JL1}^{M}(\Omega)$ are the normalized vector spherical harmonics, and $f_{nJ}, g_{nJ}$, and $h_{nJL}$ represent the radial wavefunctions. The above equations yield the following coupled differential equations [30–38]:

$$\begin{gathered}\left(E_{n,J}-U_v^0\right)F_{n,J}(r)=\left(mc^2+U_s\right)G_{n,J}(r),\\ \left(\frac{dF_{n,J}(r)}{dr}-\frac{J+1}{r}F_{n,J}(r)\right)=-\frac{1}{\alpha_J}\left(mc^2+U_s\right)H_{1,n,J}(r),\\ \left(\frac{dF_{n,J}(r)}{dr}+\frac{J}{r}F_{n,J}(r)\right)=\frac{1}{\zeta_J}\left(mc^2+U_s\right)H_{-1,n,J}(r),\\ -\alpha_J\left(\frac{dH_{1,n,J}(r)}{dr}+\frac{J+1}{r}H_{1,n,J}(r)\right)+\zeta\left(\frac{dH_{-1,n,J}(r)}{dr}-\frac{J}{r}H_{-1,n,J}(r)\right)\\ =\frac{1}{\hbar c}\left(\left(mc^2+U_s\right)F_{n,J}(r)-\left(E_{n,J}-U_v^0\right)G_{n,J}(r)\right),\end{gathered}\tag{2.8}$$

which give [39–41]

$$\begin{aligned}&\frac{d^2F_{n,J}(r)}{dr^2}\left[1+\frac{\zeta_J^2}{\alpha_J^2}\right]-\frac{dF_{n,J}(r)}{dr}\left[\frac{U_s'}{(m+U_s)}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)\right]\\&+F_{n,J}(r)\left[-\frac{J(J+1)}{r^2}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)+\frac{U_s'}{(m+U_s)}\left(\frac{J+1}{r}-\frac{\zeta_J^2}{\alpha_J^2}\frac{J}{r}\right)\right.\\&\left.-\frac{1}{\alpha_J^2}\left((m+U_s)^2-\left(E_{n,J}-U_v^0\right)^2\right)\right]=0,\end{aligned}\tag{2.9}$$

where $\alpha_J = \sqrt{(J+1)/(2J+1)}$, $f_{nJ}(r) = F(r)/r$, $g_{nJ}(r) = G(r)/r$, $h_{nJJ\pm1} = H_{\pm1}/r$ and $\zeta_J = \sqrt{J/(2J+1)}$. When $U_s = 0$, we recover the well-known formula [30–32]

$$\left(\frac{d^2}{dr^2}-\frac{J(J+1)}{r^2}+\left(E_{n,J}-U_v^0\right)^2-m^2\right)F_{n,J}(r)=0.\tag{2.10}$$

We consider the Deng-Fan vector and scalar potentials [42–46]

$$\begin{aligned}U_s&=V_0+\frac{V_1}{e^{\alpha r}-1}+\frac{V_2}{(e^{\alpha r}-1)^2},\\U_v^0&=u_0+\frac{u_1}{e^{\alpha r}-1}+\frac{u_2}{(e^{\alpha r}-1)^2}.\end{aligned}\tag{2.11}$$

By a change of variable of the form

$$F_{n,J}(r)=\sqrt{m+U_s}\,u_{n,J}(r),\tag{2.12}$$

(2.9) is written as

$$\begin{aligned}&\left[\frac{d^2}{dr^2}+\frac{U_s''}{2}\frac{1}{m+U_s}-\frac{3}{4}\frac{U_s'^2}{(m+U_s)^2}-\frac{J(J+1)}{r^2}+\frac{CU_s'}{A(m+U_s)}\frac{1}{r}\right.\\&\left.-\frac{1}{A\alpha_J^2}\left[(m+U_s)^2\left(E_{n,J}-U_v^0\right)^2\right]\right]u_{n,J}(r)=0,\end{aligned}\tag{2.13}$$

where

$$A=1+\frac{\zeta_J^2}{\alpha_J^2},\qquad C=J+1-\frac{\zeta_J^2}{\alpha_J^2}J.\tag{2.14}$$

From (2.11), (2.13) can be written as

$$\begin{aligned}
&\Bigg[\frac{d^2}{dr^2}+\frac{1}{2}\left(-\alpha^2V_1\frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}+2\alpha^2V_1\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^3}-2\alpha^2V_2\frac{e^{\alpha r}}{(e^{\alpha r}-1)^3}+6\alpha^2V_2\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^4}\right)\\
&\quad\times\frac{1}{\left((1/m)+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)\right)}\\
&\quad-\frac{3}{4}\frac{1}{\left(m+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)\right)^2}\left(-\frac{\alpha V_1e^{\alpha r}}{(e^{\alpha r}-1)^2}-\frac{2\alpha V_2e^{\alpha r}}{(e^{\alpha r}-1)^3}\right)^2-\frac{J(J+1)}{r^2}\\
&\quad+\frac{C}{A\left(m+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)\right)}\left(-\frac{\alpha V_1e^{\alpha r}}{(e^{\alpha r}-1)^2}-\frac{2\alpha V_2e^{\alpha r}}{(e^{\alpha r}-1)^3}\right)\frac{1}{r}\\
&\quad-\frac{1}{A\alpha_J^2}\left[\left(m+V_0+\frac{V_1}{e^{\alpha r}-1}+\frac{V_2}{(e^{\alpha r}-1)^2}\right)^2-\left(E_{n,J}-u_0-\frac{u_1}{e^{\alpha r}-1}-\frac{u_2}{(e^{\alpha r}-1)^2}\right)^2\right]\Bigg]u_{n,J}(r)=0.
\end{aligned}\tag{2.15}$$

Here, we use the following approximations for the centrifugal term [47]:

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(e^{\alpha r}-1)^2},\tag{2.16a}$$

$$\frac{1}{r}\approx\frac{\alpha}{e^{\alpha r}-1}.\tag{2.16b}$$

Equation (2.16a) is a quite logical alternative for $\alpha<0.1$ (see Figure 1), and (2.16a) and (2.16b) brings (2.15) into the form

$$\begin{aligned}
&\Bigg[\frac{d^2}{dr^2}+\left(-\frac{\alpha^2V_1}{2}\frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}+\alpha^2V_1\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^3}-\alpha^2V_2\frac{e^{\alpha r}}{(e^{\alpha r}-1)^3}+3\alpha^2V_2\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^4}\right)\\
&\quad\times\frac{1}{m+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)}\\
&\quad+\left(-\frac{3}{4}\alpha^2V_1^2\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^4}-3\alpha^2V_2^2\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^6}-3\alpha^2V_1V_2\frac{e^{2\alpha r}}{(e^{\alpha r}-1)^5}\right)\\
&\quad\times\frac{1}{\left(m+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)\right)^2}
\end{aligned}$$

$$
\begin{aligned}
&-J(J+1)\alpha^2\left(\frac{1}{e^{\alpha r}-1}\right)^2+\frac{C\alpha}{A}\left(\frac{1}{e^{\alpha r}-1}\right)\left(-\alpha V_1\frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}-2\alpha V_2\frac{e^{\alpha r}}{(e^{\alpha r}-1)^3}\right)\\
&\times\frac{1}{m+V_0+(V_1/(e^{\alpha r}-1))+\left(V_2/(e^{\alpha r}-1)^2\right)}\\
&-\frac{1}{A\alpha_J^2}\left\{\left(m+V_0+\frac{V_1}{e^{\alpha r}-1}+\frac{V_2}{(e^{\alpha r}-1)^2}\right)^2-\left(E_{n,J}-u_0-\frac{u_1}{e^{\alpha r}-1}-\frac{u_2}{(e^{\alpha r}-1)^2}\right)^2\right\}\Bigg]u_{n,J}(r)=0.
\end{aligned}
\tag{2.17}
$$

By introducing $z=e^{\alpha r}$ and making the transformation $u_{n,J}(z)=\phi_{n,J}(z)/(\sqrt{z})$, we obtain

$$
\begin{aligned}
&\Bigg[\frac{d^2}{dz^2}+\frac{1}{4}\frac{1}{z^2}+\left(-\frac{V_1}{2(m+V_0)}\frac{1}{z}+\frac{V_1}{(m+V_0)}\frac{1}{z-1}-\frac{V_2}{(m+V_0)}\frac{1}{z(z-1)}+\frac{3V_2}{(m+V_0)}\frac{1}{(z-1)^2}\right)\\
&\times\left(\frac{1}{(z+z_1)(z+z_2)}\right)\\
&+\left(-\frac{3V_1^2}{4(m+V_0)^2}-\frac{3V_2^2}{(m+V_0)^2}\frac{1}{(z-1)^2}-\frac{3V_2V_1}{(m+V_0)^2}\frac{1}{z-1}\right)\left(\frac{1}{(z+z_1)(z+z_2)}\right)^2\\
&-J(J+1)\frac{1}{z^2(z-1)^2}+\frac{1}{(z+z_1)(z+z_2)}\left\{-\frac{CV_1}{A(m+V_0)}\frac{1}{z(z-1)}-\frac{2CV_2}{A(m+V_0)}\frac{1}{z(z-1)^2}\right\}\\
&-\frac{1}{A\alpha^2\alpha_J^2}\left((m+V_0)^2(E_{n,J}-u_0)^2\right)\frac{1}{z^2}-\frac{1}{A\alpha^2\alpha_J^2}\left(V_1^2-u_1^2+2V_2(m+V_0)+2u_2(E_{n,J}-u_0)\right)\\
&\times\frac{1}{z^2(z-1)^2}\\
&-\frac{1}{A\alpha^2\alpha_J^2}(2V_1(m+V_0)+2u_1(E_{n,J}-u_0))\frac{1}{z^2(z-1)}-\frac{1}{A\alpha^2\alpha_J^2}(2V_2V_1-2u_1u_2)\frac{1}{z^2(z-1)^3}\\
&-\frac{1}{A\alpha^2\alpha_J^2}\left(V_2^2-u_2^2\right)\frac{1}{z^2(z-1)^4}\Bigg]\phi_{n,J}(z)=0,
\end{aligned}
\tag{2.18}
$$

which after decomposition of fractions gives

$$
\begin{aligned}
&\Bigg[\frac{d^2}{dz^2}+\frac{\lambda}{z}+\frac{\omega}{z^2}+\frac{s}{z-1}+\frac{q}{(z-1)^2}+\frac{p}{(z-1)^3}+\frac{N}{(z-1)^4}+\frac{g}{z+z_1}+\frac{h}{(z+z_1)^2}\\
&+\frac{\chi}{z+z_2}+\frac{\kappa}{(z-1)^2}\Bigg]\phi_{n,J}(z)=0
\end{aligned}
\tag{2.19}
$$

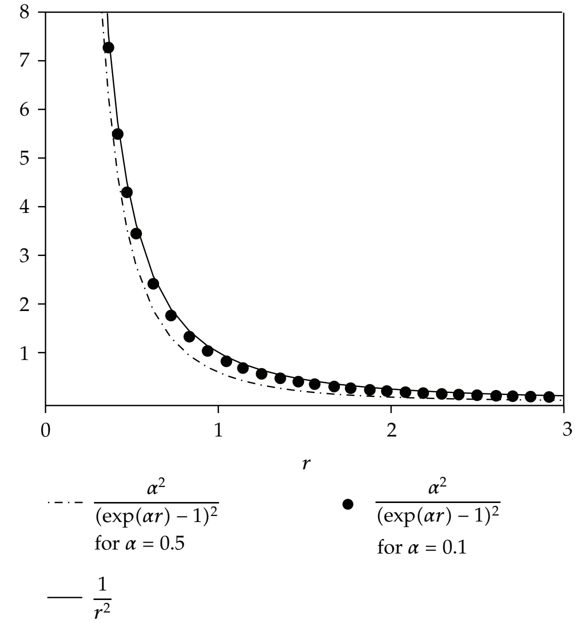


**Figure 1:** $1/r^2$ and its approximation for different value of $\alpha$.

with

$$z_1 = \frac{1}{2}\left(\frac{V_1 - 2(m+V_0)}{m+V_0} + \sqrt{\left(\frac{V_1 - 2(m+V_0)}{m+V_0}\right)^2 - 4\frac{m+V_0-V_1+V_2}{m+V_0}}\right),$$

$$z_2 = \frac{1}{2}\left(\frac{V_1 - 2(m+V_0)}{m+V_0} - \sqrt{\left(\frac{V_1 - 2(m+V_0)}{m+V_0}\right)^2 - 4\frac{m+V_0-V_1+V_2}{m+V_0}}\right),$$

$$\lambda = -\frac{V_1}{2z_1z_2(m+V_0)} + \frac{1}{z_1z_2}\left(\frac{V_2}{m+V_0} + \frac{CV_1}{A(m+V_0)}\right) + \frac{2V_1(m+V_0)}{A\alpha_J^2\alpha^2} + \frac{2u_1(E_{n,J}-u_0)}{A\alpha_J^2\alpha^2}$$
$$- 2J(J+1) - \frac{2}{A\alpha_J^2\alpha^2}\left(V_1^2 - u_1^2 + 2V_2(m+V_0) + 2u_2(E_{n,J}-u_0)\right) + \frac{3}{A\alpha_J^2\alpha^2}(2V_1V_2 - 2u_1u_2)$$
$$- \frac{4}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right) - \frac{2CV_2}{Az_1z_2(m+V_0)},$$

$$\omega = \frac{1}{4} - \frac{1}{A\alpha_J^2\alpha^2}\left((m+V_0)^2 - \left(E_{n,J}-u_0\right)^2\right) + \frac{2V_1(m+V_0)}{A\alpha_J^2\alpha^2} + \frac{2u_1(E_{n,J}-u_0)}{A\alpha_J^2\alpha^2} - J(J+1)$$

$$
\begin{aligned}
&- \frac{1}{A\alpha_J^2\alpha^2}\left(V_1^2 - u_1^2 + 2V_2(m+V_0) + 2u_2\left(E_{n,J} - u_0\right)\right)\\
&+ \frac{1}{A\alpha_J^2\alpha^2}(2V_1V_2 - 2u_1u_2) - \frac{1}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right),\\
S &= \frac{V_1}{(m+V_0)(z_1+1)(z_2+1)} + \frac{1}{(z_1+1)(z_2+1)}\left(-\frac{V_2}{m+V_0} - \frac{CV_1}{A(m+V_0)}\right)\\
&- \frac{3V_2(z_1+z_2+2)}{(z_1+1)^2(z_2+1)^2(m+V_0)} + \frac{6V_2^2(z_1+z_2+2)}{(z_1+1)^3(z_2+1)^3(m+V_0)^2}\\
&- \frac{3V_1V_2}{(m+V_0)^2(z_1+1)^2(z_2+1)^2} - \frac{2V_1(m+V_0)}{A\alpha_J^2\alpha^2} - \frac{2u_1\left(E_{n,J}-u_0\right)}{A\alpha_J^2\alpha^2}\\
&+ 2J(J+1) + \frac{2}{A\alpha_J^2\alpha^2}\left(V_1^2 - u_1^2 + 2V_2(m+V_0) + 2u_2\left(E_{n,J}-u_0\right)\right) - \frac{3}{A\alpha_J^2\alpha^2}(2V_1V_2 - 2u_1u_2)\\
&+ \frac{4}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right) + \frac{2CV_2(z_1z_2 + 2z_1 + 2z_2 + 3)}{A(m+V_0)(z_1+1)^2(z_2+1)^2},\\
q &= \frac{3V_2}{(m+V_0)(z_1+1)(z_2+1)} - \frac{3V_2^2}{(m+V_0)^2(z_1+1)^2(z_2+1)^2} - J(J+1)\\
&- \frac{1}{A\alpha_J^2\alpha^2}\left(V_1^2 - u_1^2 + 2V_2(m+V_0) + 2u_2\left(E_{n,J}-u_0\right)\right)\\
&+ \frac{2}{A\alpha_J^2\alpha^2}(2V_1V_2 - 2u_1u_2) - \frac{3}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right) - \frac{2CV_2}{A(m+V_0)(z_1+1)(z_2+1)},\\
g &= -\frac{V_1}{2z_1(z_1-z_2)(m+V_0)} + \frac{V_1}{(z_1+1)(z_1-z_2)(m+V_0)}\\
&+ \frac{1}{z_1(z_1-z_2)(z_1+1)}\left(\frac{V_2}{m+V_0} + \frac{CV_1}{A(m+V_0)}\right)\\
&- \frac{3V_2}{(m+V_0)(z_1+1)^2(z_1-z_2)} - \frac{6V_1^2}{4(m+V_0)^2(z_1-z_2)^3} - \frac{6V_2^2(-z_2+2z_1+1)}{(m+V_0)^2(z_1-z_2)^3(z_1+1)^3}\\
&+ \frac{3V_2V_1(-z_2+3z_1+2)}{(m+V_0)^2(z_1-z_2)^3(z_1+1)^2} - \frac{2CV_2}{Az_1(m+V_0)(z_1-z_2)(z_1+1)^2},\\
p &= -\frac{1}{A\alpha_J^2\alpha^2}(2V_1V_2 - 2u_1u_2) + \frac{2}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right),\\
N &= -\frac{1}{A\alpha_J^2\alpha^2}\left(V_2^2 - u_2^2\right),\\
h &= -\frac{3V_1^2}{4(m+V_0)^2(z_2-z_1)^2} - \frac{3V_2^2}{(m+V_0)^2(z_1-z_2)^2(z_1+1)^2} + \frac{3V_1V_2}{(m+V_0)^2(z_1-z_2)^2(z_1+1)},
\end{aligned}
$$

$$\begin{aligned}
\chi &= \frac{V_1}{2z_2(z_1-z_2)(m+V_0)} + \frac{V_1}{(z_2+1)(z_2-z_1)(m+V_0)} \\
&\quad + \frac{1}{z_2(z_2-z_1)(z_2+1)}\left(\frac{V_2}{m+V_0} + \frac{CV_1}{A(m+V_0)}\right) \\
&\quad - \frac{3V_2}{(m+V_0)(z_2+1)^2(z_2-z_1)} + \frac{6V_1^2}{4(m+V_0)^2(z_1-z_2)^3} - \frac{6V_2^2(-z_1+2z_2+1)}{(m+V_0)^2(z_2-z_1)^3(z_2+1)^3} \\
&\quad + \frac{3V_2V_1(-z_1+3z_2+2)}{(m+V_0)^2(z_2-z_1)^3(z_2+1)^2} - \frac{2CV_2}{Az_2(m+V_0)(z_2-z_1)(z_2+1)^2}, \\
\kappa &= -\frac{3V_1^2}{4(m+V_0)^2(z_2-z_1)^2} - \frac{3V_2^2}{(m+V_0)^2(z_1-z_2)^2(z_2+1)^2} + \frac{3V_1V_2}{(m+V_0)^2(z_1-z_2)^2(z_2+1)}.
\end{aligned} \tag{2.20}$$

Let us now consider an ansatz of the form

$$\phi_{n,J}(z) = f_n(z)\exp\left(g_J(z)\right), \tag{2.21}$$

with

$$\begin{aligned}
f_n(z) &= \begin{cases} 1, & n = 0, \\ \prod_{i=1}^{n}\left(z-\alpha_i^n\right), & n > 0, \end{cases} \\
g_J(z) &= \gamma\ln(z) + \beta\ln(z-1) + \frac{\eta}{z-1} + \xi\ln(z+z_1) + \delta\ln(z+z_2).
\end{aligned} \tag{2.22}$$

Substitution of the latter in (2.20) yields

$$\begin{aligned}
\phi''_{0,J}(z) = \Bigg[&\left(-2\beta\gamma - 2\gamma\eta + \frac{2\gamma\xi}{z_1} + \frac{2\gamma\delta}{z_2}\right)\frac{1}{z} + \left(\gamma^2-\gamma\right)\frac{1}{z^2} \\
&+ \left(2\beta\gamma + 2\gamma\eta + \frac{2\beta\xi}{1+z_1} + \frac{2\beta\delta}{1+z_2} + \frac{2\eta\xi}{(1+z_1)^2} + \frac{2\eta\delta}{(1+z_2)^2}\right)\frac{1}{z-1} \\
&+ \left(\beta^2 - \beta - 2\gamma\eta - \frac{2\eta\xi}{1+z_1} - \frac{2\eta\delta}{1+z_2}\right)\frac{1}{(z-1)^2} + (2\eta - 2\beta\eta)\frac{1}{(z-1)^3} + \eta^2\frac{1}{(z-1)^4}
\end{aligned}$$

$$+\left(-\frac{2\gamma\xi}{z_1}-\frac{2\beta\xi}{1+z_1}-\frac{2\eta\xi}{(1+z_1)^2}+\frac{2\xi\delta}{z_2-z_1}\right)\frac{1}{z+z_1}$$
$$+\left(\xi^2-\xi\right)\frac{1}{(z+z_1)^2}+\left(-\frac{2\delta\gamma}{z_2}-\frac{2\beta\delta}{1+z_2}-\frac{2\eta\delta}{(1+z_2)^2}-\frac{2\xi\delta}{z_2-z_1}\right)\frac{1}{z+z_2}$$
$$\left.+\left(\delta^2-\delta\right)\frac{1}{(z+z_2)^2}\right]\phi_{0,J}(z). \tag{2.23}$$

For the sake of simplicity, here we consider only the nodless solutions. Substitution of the proposed ansatz solution and equating the corresponding powers on both sides give

$$\begin{gathered}
-2\beta\gamma-2\gamma\eta+\frac{2\gamma\xi}{z_1}+\frac{2\gamma\delta}{z_2}=-\lambda,\\
\gamma^2-\gamma=-\omega,\\
2\beta\gamma+2\gamma\eta+\frac{2\beta\xi}{1+z_1}+\frac{2\beta\delta}{1+z_2}+\frac{2\eta\xi}{(1+z_1)^2}+\frac{2\eta\delta}{(1+z_2)^2}=-S,\\
\beta^2-\beta-2\gamma\eta-\frac{2\eta\xi}{1+z_1}-\frac{2\eta\delta}{1+z_2}=-q,\\
2\eta-2\beta\eta=-p,\\
\eta^2=-N\\
-\frac{2\gamma\xi}{z_1}-\frac{2\beta\xi}{1+z_1}-\frac{2\eta\xi}{(1+z_1)^2}+\frac{2\xi\delta}{z_2-z_1}=-g,\\
\xi^2-\xi=-h,\\
-\frac{2\delta\gamma}{z_2}-\frac{2\beta\delta}{1+z_2}-\frac{2\eta\delta}{(1+z_2)^2}-\frac{2\xi\delta}{z_2-z_1}=-\chi,\\
\delta^2-\delta=-\kappa.
\end{gathered}\tag{2.24}$$

For the fixed values of $V_1$, $u_1$, $\alpha$, and $m$, in particular, the system of ten equations (2.24) determines the sets of variables $E_{0,J}$, $V_0$, $V_2$, $u_0$, $u_2$, $\gamma$, $\beta$, $\xi$, $\eta$, and $\delta$. Therefore, the spectrum and eigenfunctions of the system are easily obtained for a particular system. For the higher states, the mathematical process is more cumbersome and complicated but can be followed by the same token we did here, that is, by choosing $f_1(z)=(z-\alpha_1^1)$ for the first node, $f_2(z)=(z-\alpha_1^2)(z-\alpha_2^2)$ for the second node, and so forth.

## 3. Conclusion

Motivation behind our study was the high number of spin-zero relativistic systems that we frequently face as well as the attractive structure of the Deng-Fan potential. The corresponding ordinary differential equation was too complicated to be solved by common analytical techniques. We, therefore, performed some novel transformations and applied an acceptable approximation to the centrifugal term. We next proposed an interesting physical ansatz solution by which we were able to find a quasi-analytical solution. Our results are particularly useful in particle and nuclear physics and can be directly used after prerequisite fits performed.

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