The Geometry and Representation Theory of Superconformal Quantum Mechanics

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This dissertation is submitted for the degree of

Doctor of Philosophy

May 2016

Declaration

This dissertation is based on research carried out while a graduate student at the Department of Applied Mathematics and Theoretical Physics from October 2012 to April 2016. The work presented is my own and contains nothing which is the outcome of work done in collaboration, except as declared below and specified in the text. The material in section 5 is based on [1], and section 7 contains material which is unpublished in any form at the time of submission. This is wholly my own work. Section 6 is based on [2, 3] and was carried out in collaboration with my supervisor Prof. Nick Dorey.

No part of this work has been submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

Acknowledgements

I owe a tremendous debt of gratitude to the many people who have helped me over the past few years. First of all, I thank my parents for their unconditional love and for always being there when I needed them, and I thank my supervisor Nick Dorey, from whom I learned so much and who has helped me to become the academic I am today.

I have enjoyed stimulating academic conversations with any number of talented individuals. I learned important and/or fun things about maths and physics from discussions with Alex Arvanitakis, Chris Blair, Mike Blake, Daniele Dorigoni, Dejan Gajic, Chris Halcrow, Joe Keir, Chris King, Emanuel Malek, Jessie Muir, Sophie Renner, Kai Röhrig, Alasdair Routh, Arnab Rudra, Jeremy Sakstein, Adam Solomon, Saran Tunyasuvunakool, Carl Turner and Jack Williams. I particularly thank Alec Barns-Graham, David Tong and Kenny Wong for many insightful discussions from which I learned a lot.

I am grateful to my many friends in DAMTP and DPMMS for making the maths faculty such a fun and comfortable place to spend four years. I must especially thank Manda Stagg for being the best support a research group could hope to have. The community of Jesus College has been a wonderful escape from the world of research, and I am grateful to the staff, fellows and the MCR of 2012-2016 for ensuring this. I am grateful to the STFC for funding my studies. Last but certainly not least, I would like to thank Angela, Faye, Gemma, Helen, Jacob, James, Jenny, Katie, Katriel, Max, Sophie, Tatiana and Will, for being awesome.

Abstract

We study aspects of the quantum mechanics of nonlinear σ-models with superconformal invariance. The connection between the differential geometry of the target manifold and symmetries of the quantum mechanics is explored, resulting in a classification of spaces admitting $\mathcal{N} = (n, n)$ superconformal invariance with $n = 1, 2, 4$. We construct the corresponding superalgberas $\mathfrak{su}(1, 1|1)$, $\mathfrak{u}(1, 1|2)$ and $\mathfrak{osp}(4^*|4)$ explicitly. The low-energy dynamics of Yang-Mills instantons is an example of the latter and arises naturally in the discrete light-cone quantisation (DLCQ) of certain superconformal field theories. In particular, we study in some detail the quantum mechanics arising in the DLCQ of the six-dimensional (2,0) theory and four-dimensional $\mathcal{N} = 4$ SUSY Yang-Mills.

In the (2,0) case we carry out a detailed study of the representation theory of the light-cone superalgebra $\mathfrak{osp}(4^*|4)$. We give a complete classification of the unitary irreducible representations and their branching at the unitarity bound, and use this information to construct the superconformal index for $\mathfrak{osp}(4^*|4)$. States contribute to the index if and only if they are in the cohomology of a particular supercharge, which we identify as the L^2 Dolbeault cohomology of instanton moduli space with values in a real line bundle.

In the SUSY Yang-Mills case the target space is the Coulomb branch of an elliptic quiver gauge theory, and as such is a scale-invariant special Kähler manifold. We describe a new type of σ-model with $\mathcal{N} = (4, 4)$ superconformal symmetry and $U(1) \times SO(6)$ R-symmetry which exists on any such manifold. These models exhibit $\mathfrak{su}(1, 1|4)$ invariance and we give an explicit construction of the superalgebra in terms of known functions. Consideration of the spectral problem for the dilatation operator in these models leads to a deformation which we interpret, via an extension of the moduli space approximation, as an anti-self-dual spacetime magnetic field coupling to the topological instanton current.

Contents

1 Introduction

The unification of quantum mechanics with special relativity in the 1920's and 30's brought into existence the formalism of quantum field theory[1]. Over the next 50 years, development of the theoretical underpinnings of quantum field theory, along with new experimental data from particle accelerators, led physicists to formulate the Standard Model. This theory of the interactions of elementary particles has achieved immense experimental success and survived largely unchanged to this day.

However, this picture is not as satisfying as it first appears. The aforementioned experimental successes all concern phenomena which are *weakly coupled*. That is, the relevant physical laws contain a small expansion parameter with respect to which one can derive an asymptotic series for the observable of interest. Unfortunately, there are many *strongly coupled* physical phenomena for which no such expansion exists. The classic example is the strong interaction of QCD, leading to the confinement of colour charge in hadrons and mesons. In these circumstances, the methods of perturbation theory are useless and there are no direct analytic tools available for calculations[2]. A general study of the IR behaviour of non-abelian gauge theories suggests that such situations are ubiquitous in quantum field theory, so any more complete understanding will need new insight into non-perturbative phenomena.

One of the most powerful tools in understanding field theory beyond perturbation theory is *symmetry*. If a physical system is known to exhibit a symmetry, this has the dual simplifying effects of constraining the possible laws describing it, and reducing the complexity of the solutions to these laws. The former played a crucial role in the construction of the Standard Model. Sadly, the symmetries present in the Standard Model, or indeed any generic field theory, are not strong enough to offer any usable constraints on non-perturbative dynamics. This much is guaranteed by the Coleman-Mandula theorem [5], which restricts the symmetries of a generic field theory to a combination of the Poincaré group with global and gauge symmetries generated by a finite-dimensional compact semisimple Lie algebra.

There are a number of important caveats to Coleman-Mandula, of which two play a central role in what follows. The first is that in a theory with only massless particles, the Poincaré group

[1]Standard results and terminology from field theory not otherwise referenced can be found in [4]. In particular, chapter 1 of volume 1 has a nice review of the early development of quantum field theory.

[2]Of course, lattice QCD provides a powerful numerical approach.

can be enhanced to the *conformal* group. The new generators are spacetime transformations which rescale distances by a constant factor while preserving angles. In particular, there is a homogeneous rescaling of spacetime $x \to \lambda x$ known as *dilatation*. Such theories are far from uncommon, since any fixed point of the renormalisation group must be scale-invariant and hence, it is believed, conformally invariant [6]. In many cases, such as the UV limit of an asymptotically free gauge theory, the resulting fixed point is a free theory, but there are many known examples, especially at critical points in condensed matter [7] or derived from string theory[3] [8–11], where the fixed point is a strongly coupled field theory. The study of conformal field theories is therefore an important stepping stone towards more general field theory. Conformal symmetry offers some additional constraints on correlation functions, whose exploitation is the subject of a major ongoing research programme known as the bootstrap (see [12] for a review).

The second key caveat to Coleman-Mandula is *supersymmetry (SUSY)* [13,14]. This avoids the assumptions of the theorem by supposing the existence of symmetry generators forming a *Lie superalgebra*, rather than an ordinary Lie algebra (see appendix D). This innocent-looking addition turns out to have profound consequences. The new generators, which have spin 1/2 and form part of the *super-Poincaré algebra*, relate bosonic degrees of freedom to fermionic. This imposes stringent constraints on the Lagrangians of supersymmetric theories, resulting in cancellations between bosonic and fermionic contributions to various processes which greatly simplify the final results. When supersymmetry and conformal symmetry coincide, they form an even larger symmetry known as a *superconformal algebra*. Such field theories are quite special and unusually accessible to analytic techniques in the non-perturbative regime.

The extra analytic control offered by supersymmetry has made possible any number of advances in non-perturbative field theory. Though it's unclear to what extent many of the techniques used are directly applicable to non-supersymmetric theories, the hope is to learn enough about field theory in general to transfer the lessons back to the Standard Model. For example, supersymmetry is responsible for much of our current understanding of strongly coupled vacua and field theory dualities [15–18], the AdS/CFT correspondence [19–21], and large-N integrability in gauge theory [22].

One theme to emerge from supersymmetry is the importance of geometry in quantum field theory. The requirement that a field theory is supersymmetric can often be expressed in part

[3]This list is far from exhaustive.

by demanding that any scalar fields take values in a manifold with additional structure, such as a Kähler manifold [23–25]. This imposes constraints which extend to the structure of the space of vacua of a theory, known as the *moduli space*, and which are valid even for quantum corrected effective Lagrangians. Indeed, these constraints are often strong enough to determine a theory uniquely [15, 16]. This link to geometry is most direct in the context of *nonlinear σ-models* [23, 26–30]. These are theories of scalar and spinor fields which are supersymmetric generalisations of harmonic maps from spacetime to some *target* manifold. In particular, the quantum mechanical version, in which spacetime is just the time coordinate or *worldline* of a particle, generalises geodesic motion. The symmetries of these models are determined entirely by target space geometry, and properties such as extended supersymmetry can be understood in terms of additional constraints on the target [27, 31]. Furthermore, observables such as the energy levels of these models compute geometric and topological data about the target [28–30].

One approach to non-perturbative field theory which has perhaps been underappreciated is *discrete light-cone quantisation (DLCQ)* [32, 33]. Here one considers compactifying a field theory on a null circle. The results of this are rather surprising: provided that the compactification even makes sense [34], we obtain a non-relativistic quantum mechanical model with a finite number of degrees of freedom determined by the number k of discrete units of momentum flowing around the null circle. It should then be possible to recover information about the field theory by taking $k \to \infty$ [32, 35]. In the case of certain theories with maximal superconformal invariance, such as $\mathcal{N} = 4$ SUSY Yang-Mills in four dimensions and the six-dimensional (2,0) theory describing the worldvolume of M5-branes, it is possible to use string theory to give quite precise descriptions of this quantum mechanics [36–39]. In particular, a non-trivial superconformal algebra descends from the spacetime theory [40]. A natural goal is to describe this algebra in detail, and if possible to understand its representation theory and the spectrum of operators such as the dilatation. This spectrum should then feed back into the original field theory via the $k \to \infty$ limit.

The broad aim of this thesis is to understand superconformal quantum mechanics in as much detail as possible. The geometric viewpoint which has been applied so successfully to supersymmetric theories is especially helpful. We will extend these ideas to superconformal mechanics, with the aim of understanding the link between target space geometry and superconformal symmetry. This perspective also provides a powerful approach to deriving new types of superconformal σ-model. Once these models have been defined and their symmetry algebra understood, the next step is to understand their representation theory. We will be particularly interested in the spectrum of the dilatation operator, which contains special *BPS states* whose scaling dimensions are determined by their integer charges under other compact symmetries. The precise questions we choose to tackle are motivated by DLCQ, though we emphasise that the theory of superconformal quantum mechanics is an interesting area of mathematical physics in its own right, and that our results are equally relevant to this more general context.

The structure of this thesis is as follows. In sections 2, 3, and 4 we review the necessary physical background for our work. Section 2 is concerned with the link between symmetries of quantum mechanical σ-models and target space geometry. We review the old results explaining how quantisation leads to the algebra of differential forms on the target manifold M [29], and how additional structure on M naturally leads to extra supersymmetries [27]. In preparation for later discussion of the superconformal index, we also explain the Witten index. This is a quantity protected from a large class of supersymmetry-preserving deformations, which in the σ-model context computes a simple topological invariant of M [29].

In section 3 we review instantons in Yang-Mills theory. These enter into our story as the target spaces for the DLCQ models of $\mathcal{N} = 4$ SUSY Yang-Mills and of the (2,0) theory. We review the basics of the moduli space of instantons, its geometric structure as a hyper-Kähler manifold, and its construction as a quotient via ADHM [41,42]. We also review how instantons appear as solitons in higher dimensions, and their significance in supersymmetry as BPS objects [43] and in string theory as a low-energy description of certain configurations of D-branes [44,45].

In section 4 we describe the DLCQ procedure, first in the simple context of free scalar field theory then in the specific cases of $\mathcal{N} = 4$ SUSY Yang-Mills and the (2,0) theory. We explain how to determine the symmetry algebra preserved by DLCQ, with a particular emphasis on the superconformal case and how a non-trivial quantum mechanical superconformal algebra is left over. The description of the DLCQ models for $\mathcal{N} = 4$ and the (2,0) theory proceeds via string theory, and in particular the matrix description of M theory [35, 46]. We'll explain how the quantum mechanical models are seen to have instanton moduli spaces as their targets [36, 38], and in the Yang-Mills case how one can use string dualities to arrive at a completely explicit description of the geometry [37, 39]. We'll also comment on what problems we can hope to address in DLCQ for each model, with a view to motivating the work in the remainder of the thesis.

Section 5 begins our own work. We extend the exterior algebra formalism for quantum mechanical σ-models to the superconformal case, deriving constraints on the target geometry required to extend $\mathcal{N} = (1, 1)$, $(2, 2)$, and $(4, 4)$ supersymmetry to superconformal algebras. We give explicit geometric realisations of these algebras, and classify them in terms of the simple Lie superalgebras of [47–49]. We find that $\mathcal{N} = (1,1)$ supersymmetry extends to $\mathfrak{su}(1,1|1)$ superconformal invariance, $\mathcal{N} = (2,2)$ extends to $\mathfrak{u}(1,1|2)$, and $\mathcal{N} = (4,4)$ to $\mathfrak{osp}(4^*|4)$. In particular, consistency with DLCQ requires that the moduli space of instantons on $\mathbb{R}^4$ admits $\mathfrak{osp}(4^*|4)$ invariance, and we use the ADHM construction to show that this is indeed the case.

In section 6 we address the class of models to which the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills belongs. The target space can be described as the Coulomb branch of an auxiliary superconformal $\mathcal{N} = 2$ gauge theory in four dimensions, compactified on $\mathbb{R}^3 \times S^1$ [39]. As such, we study *special Kähler manifolds* [15, 50] and torus bundles over them [17, 51]. We review how special Kähler geometry arises from $\mathcal{N} = 2$ gauge theory, as well as an alternative more geometrically minded definition of [50]. This allows us to construct the target space quantum mechanics explicitly, and prove the highly non-obvious fact that it does indeed have the correct symmetry algebra, $\mathfrak{su}(1,1|4)$, to describe the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills. This construction leads us to define a novel quantum mechanical σ-model with $\mathcal{N} = (4,4)$ supersymmetry and a scale-invariant special Kähler manifold as its target. In a large class of examples including the Yang-Mills case, the special Kähler structure has a known exact form [52], so we can give exact expressions for all generators of $\mathfrak{su}(1,1|4)$. We can therefore hope to determine its representations explicitly, and in particular to find the spectrum of the dilatation operator. This problem appears to be singular, so we suggest a deformation whose spectral theory is better defined. This introduces a worldline magnetic field proportional to the target space holomorphic symplectic form, and in the Yang-Mills case we prove that the same deformation can be induced in DLCQ by a spacetime magnetic field coupling to instanton number density.

In section 7 we return to the DLCQ of the (2,0) theory, and more generally to systems with $\mathfrak{osp}(4^*|4)$ invariance. We classify the irreducible unitary representations, and in particular those which obey BPS shortening conditions. As a useful guiding example, we describe quantum mechanics on $\mathbb{C}^{2k}$ in some detail and show how the abstract representation theory is realised in practice. For systems whose target space geometry is not known explicitly, such as the moduli space of instantons on $\mathbb{R}^4$, the quantum mechanics will not be completely soluble and a characterisation of the BPS states may be the best bet. It is therefore crucial to understand the superconformal index [53, 54]. This is a generalisation of the Witten index to superconformal theories, and contains in some sense the most information that can be deduced about a theory purely from symmetries. Only BPS representations can contribute to the index, and we characterise exactly how they do so.

We also make contact with the geometric story by identifying these states with the L^2-Dolbeault cohomology of the target with values in a line bundle.

In section 8 we conclude with a summary of our main results and a discussion of the future directions they might lead in. Appendices A, B, C, and D review important mathematical background on constrained quantisation, Hodge theory, various manifold quotients, and Lie superalgebras, with a view to keeping this thesis as close to mathematically self-contained as possible. Appendix E is a complete listing of the generators and relations of all the superalgebras we construct.

2 Supersymmetric Quantum Mechanics

We begin with a thorough review of the class of supersymmetric quantum mechanical models on which our results are based. We will start by describing Witten's observation [28, 29] that the supersymmetry algebra can be represented by the exterior derivative, then show how this can be obtained by quantising a supersymmetrised version of geodesic motion. We then review the conditions under which our basic model admits extended supersymmetry. The key is to demand extra geometric structure on the target space [27], and the entirety of the extended supersymmetry algebra, including R-symmetries, emerges naturally in this context. Finally, we study the Witten index and the connection between supersymmetric vacua and target space topology.

2.1 The Basic Model

Our fundamental object of interest is the $\mathcal{N} = (1, 1)$ supersymmetry algebra. This contains a pair of supercharges Q and $Q^\dagger$ along with a Hamiltonian H, and obeys the algebraic relations[4]

$$\left\{Q, Q^\dagger\right\} = 2H, \qquad Q^2 = 0, \qquad [Q, H] = 0. \tag{2.1}$$

The name $\mathcal{N} = (1, 1)$ is really $(1+1)$-dimensional language, in that such an algebra can be obtained by dimensional reduction of a supersymmetry algebra with a single Majorana spinor supercharge. The meaning is subtle in quantum mechanics, but in particular implies that the supercharges are represented as Hermitian conjugates. There is also a $U(1)$ R-symmetry J_3 obeying

$$[J_3, Q] = \frac{1}{2}Q, \qquad [J_3, H] = 0.$$

The magnitude of this charge is an arbitrary convention: we choose $1/2$ as it is the natural normalisation for extended supersymmetry.

Such an algebra can be represented geometrically [28, 29]. Consider a Riemannian manifold[5] (M, g) of dimension n with exterior algebra $\Omega^*(M; \mathbb{C})$. The exterior algebra carries an L^2 inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\bar{\beta} = \frac{1}{p!} \int d^n X \sqrt{g}\, \alpha_{\mu_1 \ldots \mu_p} \bar{\beta}^{\mu_1 \ldots \mu_p}, \tag{2.2}$$

[4]It is our convention throughout this document not to display commutation relations which can be obtained as Hermitian conjugates of those which we have displayed.

[5]Standard results and terminology from differential geometry can be found in [55].

where α and β are forms of degree p and the product is defined to vanish when the degrees are unequal. $X^\mu : \mu = 1, \ldots, n$ are local coordinates on M and $*$ is the Hodge dual

$$(*\alpha)_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \ldots \nu_p \mu_1 \ldots \mu_{n-p}} \alpha^{\nu_1 \ldots \nu_p},$$

which satisfies $*^2 = (-1)^{p(n-p)}$. The product (2.2) makes the space of differential forms with finite norm into a Hilbert space $\mathcal{H}(M)$, and enables us to define adjoint operators. In particular, the adjoint of the exterior derivative[6] is the *coderivative*

$$\delta = d^\dagger = (-1)^{np+n+1} * d*, \tag{2.3}$$

satisfying $\delta^2 = 0$. Given this, the Laplacian is

$$\Delta = \{d, \delta\} = d\delta + \delta d = \Delta^\dagger. \tag{2.4}$$

Using $d^2 = \delta^2 = 0$, it's straightforward to verify that the $\mathcal{N} = (1, 1)$ supersymmetry algebra (2.1) can be represented by

$$Q = d, \qquad Q^\dagger = \delta, \qquad H = \frac{1}{2}\Delta. \tag{2.5}$$

The $U(1)$ R-symmetry can be taken to count half the degree of a form, which clearly assigns the correct charges to Q and $Q^\dagger$.

The above construction is rather neat but of limited value unless we can obtain the exterior algebra from the quantisation of a physical system. To see how to do this, first consider the case of a free non-relativistic particle moving on a curved manifold M (called the *target space*), the action for which is

$$S = \frac{1}{2} \int dt \, g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu. \tag{2.6}$$

As in the flat case, the Hilbert space of this model is $L^2(M)$ and the canonical momentum P_μ is represented as $-i\partial_\mu$. The classical Hamiltonian is $\frac{1}{2} g^{\mu\nu} P_\mu P_\nu$, which can (with a choice of operator ordering) be quantised as the Laplacian

$$\Delta\psi = -\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \right) \psi.$$

[6]Checking that this really is the adjoint requires the use of Stokes' theorem, so when M is not compact we need suitable decay conditions for forms 'at infinity'. More generally, we will often need to restrict to a subspace of $\mathcal{H}(M)$ to ensure that all operations on forms are well-defined. We will leave such restrictions implicit except where they have significant consequences, but note that a space of 'test' forms, which are smooth and decay exponentially at infinity, is more than sufficient.

We have thus reproduced the degree zero part of our geometric construction. Of course, the model (2.6) is not supersymmetric, so we will need to add some fermionic degrees of freedom. There are many ways to make this model supersymmetric, with different numbers of supercharges, potentials, extra fields and so forth, but there is one answer for $\mathcal{N} = (1,1)$ supersymmetry which is 'canonical' in the sense that it requires a minimal set of new fields and no additional structure on M. It is [26]:

$$S = \int dt \, \frac{1}{2} g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu + i g_{\mu\nu}(X) \psi^{\dagger\mu} \frac{D}{Dt} \psi^\nu - \frac{1}{4} R_{\mu\nu\rho\sigma}(X) \psi^{\dagger\mu} \psi^{\dagger\nu} \psi^\rho \psi^\sigma. \tag{2.7}$$

Here ψ^μ and $\psi^{\dagger\mu} = (\psi^\mu)^\dagger$ are complex conjugate fermionic variables which can be interpreted as Grassmann-odd sections of the cotangent bundle, D/Dt is the appropriate covariant derivative

$$\frac{D}{Dt} \psi^\mu = \nabla_{\dot{X}} \psi^\mu = \dot{\psi}^\mu + \dot{X}^\nu \Gamma^\mu_{\nu\rho} \psi^\rho, \tag{2.8}$$

and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. The model is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon X^\mu &= -\epsilon^\dagger \psi^\mu + \epsilon \psi^{\dagger\mu} \\ \delta_\epsilon \psi^\mu &= i \dot{X}^\mu \epsilon - \Gamma^\mu_{\nu\rho} \left(\delta_\epsilon X^\nu \right) \psi^\rho \end{aligned} \tag{2.9}$$

for complex Grassmann parameter ϵ: checking this is an extremely tedious but routine exercise in manipulating Christoffel symbols. Doing the same again with time-dependent ϵ leads us to the Noether supercharges[7]

$$Q^\dagger = g_{\mu\nu} \dot{X}^\mu \psi^\nu, \qquad Q = g_{\mu\nu} \dot{X}^\mu \psi^{\dagger\nu}. \tag{2.10}$$

The classical Hamiltonian is

$$H = \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + \frac{1}{4} R_{\mu\nu\rho\sigma} \psi^{\dagger\mu} \psi^{\dagger\nu} \psi^\rho \psi^\sigma. \tag{2.11}$$

Quantising this model requires some care. There are two major subtleties:

1. Since the fermions are bundle-valued, we have a choice of frame in which to evaluate their components. The naïve canonical prescription says that fermions commute with the bosonic canonical momentum, but this gives a different answer for different choices of frame.

2. According to Dirac's standard analysis of quantisation [56], the first-order fermion Lagrangian leads to a constraint $P_\psi \propto \psi^\dagger$ on phase space which must be handled appropriately.

In fact, the Faddeev-Jackiw prescription [57] (see appendix A.2) which we use to handle the second subtlety is simplest when the first-order fermion kinetic term has no X dependence. Up to local

[7] The reason for confusingly defining $Q \propto \psi^\dagger$ will be clear after quantisation.

$SO(n)$ and global scale transformations, the unique choice of frame achieving this is the *vielbein*. Recall that a vielbein is a set of n vectors e_a which are orthonormal in the sense that

$$g_{\mu\nu} e_a^\mu e_b^\nu = \delta_{ab}.$$

There is a dual basis of 1-forms e^a which satisfy

$$e_\mu^a e_a^\nu = \delta_\mu^\nu, \qquad e_a^\mu e_\mu^b = \delta_b^a,$$

and we use these to define 'flattened' fermions

$$\psi^a = e_\mu^a \psi^\mu.$$

Rewriting the Lagrangian (2.7) in a vielbein basis gives

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu + i\delta_{ab} \psi^{\dagger a} \dot{\psi}^b + i\psi^{\dagger a} \psi^b \dot{X}^\mu \omega_{ab\mu} - \frac{1}{4} \Omega_{abcd} \psi^{\dagger a} \psi^{\dagger b} \psi^c \psi^d, \tag{2.12}$$

where ω and Ω are the connection 1-forms and curvature 2-forms respectively:

$$(\omega_{ab})_\mu = \delta_{ac} e_\nu^c \nabla_\mu e_b^\nu, \qquad \Omega_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b = \frac{1}{2} \Omega_{abcd} e^c \wedge e^d := \frac{1}{2} R_{abcd} e^c \wedge e^d.$$

The Faddeev-Jackiw method instructs us to put the Lagrangian into first-order form, which requires the bosonic canonical momentum

$$P_\mu = \frac{\partial \mathscr{L}}{\partial \dot{X}^\mu} = g_{\mu\nu} \dot{X}^\nu + i\psi^{\dagger a} \omega_{ab\mu} \psi^b. \tag{2.13}$$

This gives

$$\tilde{\mathscr{L}} = P_\mu \dot{X}^\mu + i\delta_{ab} \psi^{\dagger a} \dot{\psi}^b - H$$

with Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} \left(P_\mu - i\psi^{\dagger a} \omega_{ab\mu} \psi^b \right) \left(P_\nu - i\psi^{\dagger c} \omega_{cd\nu} \psi^d \right) + \frac{1}{4} \Omega_{abcd} \psi^{\dagger a} \psi^{\dagger b} \psi^c \psi^d.$$

If we treat P_μ as an independent variable then it's straightforward to verify that the equations of motion following from $\tilde{\mathscr{L}}$ are equivalent to those of (2.12) and the dynamics is unconstrained on the configuration space $\left\{X^\mu, P_\nu, \psi^a, \psi^{\dagger b}\right\}$. We are now in position to apply the Faddeev-Jackiw prescription, which defines brackets such that the classical equations of motion are reproduced by the Heisenberg equation. This results in the nonzero commutation relations

$$[X^\mu, P_\nu] = i\delta_\nu^\mu, \qquad \left\{\psi^a, \psi^{\dagger b}\right\} = \delta^{ab}. \tag{2.14}$$

In particular, the intuition that ψ^a and $\psi^{\dagger a}$ should be canonically conjugate is reproduced. However, if we try to return to 'curved' fermions we find a mess:

$$[P_\mu, \psi^\nu] = -ie^a_\rho \left(\partial_\mu e^\nu_a\right) \psi^\rho.$$

This is hardly surprising since we expect P_μ to act like a partial derivative, which will clearly do non-covariant things with respect to basis changes. To remedy this, we make the field redefinition

$$\Pi_\mu = g_{\mu\nu}\dot{X}^\nu = P_\mu - i\psi^{\dagger a}\omega_{ab\mu}\psi^b. \tag{2.15}$$

In a 'flat' basis, it's immediate from (2.14) that Π_μ obeys

$$[\Pi_\mu, \psi^a] = i\omega^a{}_{b\mu}\psi^b, \qquad \left[\Pi_\mu, \psi^{\dagger a}\right] = i\omega^a{}_{b\mu}\psi^{\dagger b},$$

and the combination of the unwieldy commutation relation for P_μ in a curved basis and the equally unwieldy change of basis formula for connections guarantees

$$[\Pi_\mu, \psi^\nu] = i\Gamma^\nu_{\mu\rho}\psi^\rho, \qquad \left[\Pi_\mu, \psi^{\dagger\nu}\right] = i\Gamma^\nu_{\mu\rho}\psi^{\dagger\rho}, \qquad [\Pi_\mu, \Pi_\nu] = -R_{\rho\sigma\mu\nu}\psi^{\dagger\rho}\psi^\sigma. \tag{2.16}$$

The fact that Π_μ acts on fermions in a manner reminiscent of a covariant derivative and transforms nicely under basis changes motivates the name *covariant momentum*. Indeed, we will shortly see that Π_μ acts on Hilbert space as the covariant derivative. Finally, the curved fermions ψ^μ obey the usual conjugate relation

$$\left\{\psi^\mu, \psi^{\dagger\nu}\right\} = g^{\mu\nu}. \tag{2.17}$$

We can now construct the Hilbert space. As earlier, the bosonic coordinates lead to functions on M, and the key observation is that fermions extend this to differential forms [29]. To see this, identify $\psi^{\dagger\mu}$ as creation operators for fermionic degrees of freedom and ψ^μ as annihilation operators. We build a Fock space by demanding that a pure bosonic state $|f\rangle$ satisfies

$$\psi^\mu |f\rangle = 0.$$

A generic state is of the form

$$|\alpha\rangle = \frac{1}{p!}\psi^{\dagger\mu_1}\dots\psi^{\dagger\mu_p}\left|\alpha_{\mu_1\dots\mu_p}\right\rangle, \tag{2.18}$$

where $p \in \{1, \dots, n\}$ and $\left|\alpha_{\mu_1\dots\mu_p}\right\rangle$ are a collection of bosonic states, i.e functions on M. By virtue of fermionic statistics, this can be identified with the differential form

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\dots\mu_p}(X)dX^{\mu_1}\wedge\dots\wedge dX^{\mu_p}.$$

The commutation relation (2.17) can be used to extend the bosonic wavefunction inner product to the full Hilbert space, and it's easy to check that this coincides with the usual L^2 inner product (2.2) on forms. Furthermore, we find as a simple consequence of (2.16) that while P_μ is a partial derivative, Π_μ is a covariant derivative[8] in the sense that

$$\Pi_\mu \left|\alpha\right\rangle = -i \left|\nabla_\mu \alpha\right\rangle .$$

In summary, we obtain a dictionary between objects in quantum mechanics and differential geometry:

$$\begin{array}{c|c} \text{Quantum Mechanics} & \text{Differential Geometry} \\ \hline \text{Hilbert space } \mathcal{H} & \mathcal{H}(M) \subseteq \Omega^*(M;\mathbb{C}) \\ \left|\alpha\right\rangle \in \mathcal{H} & \alpha \in \Omega^* \\ \psi^{\dagger\mu} & dX^\mu \wedge \\ \psi^\mu & g^{\mu\nu} i_{\partial_\nu} \\ P_\mu & -i\partial_\mu \\ \Pi_\mu & -i\nabla_\mu \end{array} \tag{2.19}$$

where i_V is the contraction of a vector field V with a differential form

$$(i_V\alpha)(X_1, \ldots, X_{p-1}) = \alpha(V, X_1, \ldots, X_{p-1}).$$

In particular, we do indeed have $\psi^{\dagger\mu} = (\psi^\mu)^\dagger$. From this we can read off the action of the supercharges

$$\begin{aligned} Q &= i\psi^{\dagger\mu}\Pi_\mu = dX^\mu \wedge \nabla_\mu = d \\ Q^\dagger &= -i\Pi_\mu^\dagger \psi^\mu = -\nabla^\mu \circ i_{\partial_\mu} = \delta. \end{aligned} \tag{2.20}$$

We also have

$$\begin{aligned} \Delta = \left\{Q, Q^\dagger\right\} &= -R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\rho}\psi^\sigma\psi^\nu - i\Gamma^\mu_{\nu\rho}\psi^{\dagger\rho}\Pi_\mu\psi^\nu + i\Pi_\nu^\dagger\psi^{\dagger\mu}\Gamma^\nu_{\mu\rho}\psi^\rho + \Pi_\mu^\dagger g^{\mu\nu}\Pi_\nu - \partial_\mu\Gamma^\rho_{\nu\rho}\psi^{\dagger\mu}\psi^\nu \\ (\text{classically}) &= g^{\mu\nu}\Pi_\mu\Pi_\nu + \frac{1}{2}R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\nu}\psi^\rho\psi^\sigma = 2\times(2.11), \end{aligned}$$

where we used the Bianchi identity and the lack of classical distinction between Π_μ and $\Pi_\mu^\dagger$. This shows that it is consistent to quantise the classical Hamiltonian as the Laplace operator. Finally, there is a natural $U(1)$ R-symmetry

$$J_3 = \frac{1}{2}\left(g_{\mu\nu}\psi^{\dagger\mu}\psi^\nu - \frac{n}{2}\right) = \frac{1}{2}\left(p - \frac{n}{2}\right), \tag{2.21}$$

[8] This statement is not quite precise since Π_μ is not self-adjoint, rather $\Pi_\mu^\dagger = \frac{1}{\sqrt{g}}\Pi_\mu\sqrt{g} = \Pi_\mu - i\Gamma^\nu_{\mu\nu}$. Π_μ does act as a covariant derivative on forms, but not when integrating by parts in the L^2 norm (2.2).

where in the latter expression we've converted to geometric language with p the degree of a differential form. Taking everything together, we've shown how to reproduce the geometric construction of $\mathcal{N} = (1,1)$ supersymmetry by quantising the σ-model (2.7). In the following sections we'll see how to extend these ideas to understand how symmetries of the model interact with target space geometry, and begin to explore some basic observables of the quantum mechanics which contain topological information about the target space.

2.2 Extended Supersymmetry and R-symmetry

In this section we analyse the additional structure required of a target manifold M for the $\mathcal{N} = (1,1)$ σ-model (2.7) to admit extra supersymmetry without changing the field content. Our central result is the following:

Theorem

1. The σ-model admits the following extended supersymmetries:
   - $\mathcal{N} = (2,2)$ if and only if M is Kähler .
   - $\mathcal{N} = (4,4)$ if and only if M is hyper-Kähler .
2. - In the Kähler case there is an $SU(2) \times U(1)$ R-symmetry acting purely on fermions.
   - In the hyper-Kähler case this is extended to $SO(5)$.

Part 1 of this result was originally proved by Alvarez-Gaumé and Freedman [27] for the (1+1)-dimensional σ-model [26] of which (2.7) is the dimensional reduction, and the argument uses Lorentz and parity invariance to restrict the general ansatz for supersymmetry transformations. Such methods are unavailable in quantum mechanics, and indeed the possibilities for extended supersymmetry in quantum mechanical σ-models are much more varied than the above result suggests [58,59]. However, we can reach the same conclusion if we assume that the supersymmetry variation of X^μ is linear in the fermions $\psi^\nu, \psi^{\dagger\rho}$ and independent of time derivatives.

We begin by assuming that we have a new supercharge $\tilde{Q}$ with a generic operator form $\tilde{Q} = F(X, \Pi, \psi, \psi^\dagger)$. This can be expanded in fermions as

$$\tilde{Q} = \sum_{\mathbf{m},\mathbf{n}} F_{\mathbf{mn}}(X,\Pi)\psi^{\mathbf{m}}\psi^{\dagger\mathbf{n}},$$

where $\mathbf{m}$ and $\mathbf{n}$ are multi-indices so e.g $\psi^M = \psi^{\mu_1}\dots\psi^{\mu_p}$. Thus

$$\left[\tilde{Q}, X^\mu\right] = -i\sum_{\mathbf{m},\mathbf{n}} \frac{\partial F_{\mathbf{mn}}}{\partial \Pi_\mu}\psi^{\mathbf{m}}\psi^{\dagger\mathbf{n}}.$$

We have $\Pi_\mu = g_{\mu\nu}\dot{X}^\nu$ classically, so the supersymmetry variation of X^μ is only independent of time derivatives if $\partial F_{\mathbf{mn}}/\partial\Pi_\mu$ is independent of Π, i.e $F_{\mathbf{mn}}$ is at most linear in Π. It follows that the variation is linear in fermions if and only if all multi-indices of size $\neq 1$ are discarded. Finally, since the variation of X^μ must be real, we can arrange the remaining supersymmetry operators into the conjugate pair

$$\begin{aligned}\tilde{Q} &= \psi^{\dagger\mu} f_\mu{}^\nu(X)\Pi_\nu \\ \tilde{Q}^\dagger &= \Pi^\dagger_\nu \bar{f}_\mu{}^\nu(X)\psi^\mu,\end{aligned} \tag{2.22}$$

where diffeomorphism invariance demands that f is a tensor. If we calculate the variation of ψ and $\psi^\dagger$ we find that our transformations fall within the general ansatz of [27] (equation 5) and hence the (1+1)-dimensional result may be carried over.

The result of [27], and hence part 1 of our theorem, is proved in a rather brute-force manner by directly calculating the supersymmetry variation of (2.7) and deriving constraints on the tensor f such that the variation vanishes and the supersymmetry algebra is obeyed. Unfortunately, some amount of such brute force seems to be necessary in order to obtain a necessary and sufficient condition. If we are only interested in proving, say, that Kähler geometry is sufficient for $\mathcal{N} = (2,2)$ supersymmetry, then we can give a much simpler argument. First we recall some standard facts about Kähler geometry, the details of which can be found in [55, 60].

Definition

- A real manifold M of dimension $2n$ is *complex* if it can be covered by complex coordinate charts $Z^\imath : U \subseteq M \to \mathbb{C}^n$ whose transition functions are holomorphic. There is a tensor $I^\mu{}_\nu$, called the *complex structure*, which acts as multiplication by i on all basis vectors $\partial/\partial Z^\imath$ in any complex chart, and as $-i$ on conjugates. We say that M has *complex dimension* n.
- A complex manifold is *Kähler* if it has a Riemannian metric g satisfying

  $$g(X,Y) = g(IX, IY) \quad \forall X, Y \in \mathrm{Vect}(M)$$

  and the associated real (1,1) form $\omega_{\mu\nu} = g_{\mu\rho}I^\rho{}_\nu$, called the *Kähler form*, is closed. In particular, a Kähler manifold is always symplectic.

It follows that for a Kähler manifold the only nonzero components of the metric in complex coordinates are $g_{\imath\bar{\jmath}} = \bar{g}_{\bar{\imath}\jmath}$, the only non-vanishing Christoffel symbols are $\Gamma^\imath_{\jmath k}$ and $\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}$ and the only nonzero curvature components are those related by symmetry to $R_{\imath\bar{\jmath}k\bar{l}}$. Using complex coordinates and $g(X,Y) = g(IX,IY)$, it's immediate that the action (2.7) is invariant under the replacement $\psi \to I\psi$, so the combination of this replacement with the supersymmetry transformation (2.9) is another supersymmetry. Notice in particular that the corresponding Noether charge is indeed of the form (2.22) with $f = I$.

An alternative perspective on this derivation, and the one which we will follow throughout, is given by the Hilbert space picture. In terms of differential forms, the special feature of a complex manifold is the splitting of forms of a given degree r into forms of *bidegree* $(p,q) : p+q = r$ according to how many holomorphic or antiholomorphic differentials they contain. We denote the space of (p,q) forms by $\Omega^{p,q}(M;\mathbb{C})$. The exterior derivative also respects this splitting as it can be decomposed into the *Dolbeault operators*

$$d = \partial + \bar{\partial} := dZ^{\imath} \wedge \frac{\partial}{\partial Z^{\imath}} + d\bar{Z}^{\bar{\imath}} \wedge \frac{\partial}{\partial \bar{Z}^{\bar{\imath}}}, \tag{2.23}$$

which satisfy $\partial^2 = \bar{\partial}^2 = \{\partial, \bar{\partial}\} = 0$. Kähler geometry supplies a surprisingly rich algebraic structure known as the *Hodge identities* for these operators, which we exploit extensively to prove our theorem. To begin with, define operators $J_+^I : \Omega^{p,q} \to \Omega^{p+1,q+1}$, $J_-^I : \Omega^{p,q} \to \Omega^{p-1,q-1}$ and $J_3 : \Omega^{p,q} \to \Omega^{p,q}$ by

$$\begin{aligned} J_+^I(\alpha) &= \omega \wedge \alpha \\ J_-^I &= \left(J_+^I\right)^\dagger \\ J_3 &= \frac{1}{2}(p+q-n), \end{aligned} \tag{2.24}$$

where n is the complex dimension of our manifold. Notice that J_3 is exactly as defined in (2.21). It is no accident that we gave these operators names reminiscent of an $\mathfrak{su}(2)$ algebra. Indeed, using the quantum mechanical dictionary (2.19) we can write

$$\begin{aligned} J_+^I &= \frac{1}{2}\omega_{\mu\nu}\psi^{\dagger\mu}\psi^{\dagger\nu} \\ J_-^I &= \frac{1}{2}\omega_{\nu\mu}\psi^{\mu}\psi^{\nu} \\ J_3 &= \frac{1}{2}\left(\psi^{\dagger\mu}\psi_{\mu} - n\right), \end{aligned} \tag{2.25}$$

from which it is straightforward to derive the standard $\mathfrak{su}(2)$ algebra

$$\left[J_+^I, J_-^I\right] = 2J_3, \qquad \left[J_3, J_\pm^I\right] = \pm J_\pm^I.$$

The Hodge identities amount to a prescription for how the Dolbeault operators fit into multiplets of this $\mathfrak{su}(2)$. They read

$$\begin{aligned} \left[J^I_+, \partial\right] &= 0 & \left[J^I_+, \bar\partial\right] &= 0 \\ \left[J^I_+, \partial^\dagger\right] &= -i\bar\partial & \left[J^I_+, \bar\partial^\dagger\right] &= i\partial. \end{aligned} \tag{2.26}$$

The first line follows immediately from the fact that the Kähler form ω is closed, while the second is an easy application of the quantum mechanical dictionary (2.19). More precisely, we refine the dictionary to a complex coordinate system as follows. We set

$$\psi^{\dagger\imath} = dZ^\imath \wedge, \qquad \psi^{\dagger\bar\imath} = d\bar Z^{\bar\imath}\wedge,$$

so that if we write $Z^\imath = X^\imath + iY^\imath$ then e.g $\psi^{\dagger\imath} = \psi^{\dagger X^\imath} + i\psi^{\dagger Y^\imath}$. It follows that the annihilation operators are

$$\begin{aligned} \psi^\imath &= \left(\psi^{\dagger\bar\imath}\right)^\dagger = g^{\imath\bar\jmath} i_{\partial_{\bar\jmath}} \\ \psi^{\bar\imath} &= \left(\psi^{\dagger\imath}\right)^\dagger = g^{\bar\imath\jmath} i_{\partial_\jmath}, \end{aligned}$$

and the Dolbeault operators can be written as

$$\partial = i\psi^{\dagger\imath}\Pi_\imath, \qquad \bar\partial = i\psi^{\dagger\bar\jmath}\Pi_{\bar\jmath},$$

where $\Pi_\imath$ acts as the covariant derivative along $\partial_\imath$ and $\Pi_{\bar\imath}$ likewise along $\partial_{\bar\imath}$.

Observe that the Hodge identities imply that $(\partial, \bar\partial^\dagger)$ and $(\bar\partial, \partial^\dagger)$ form conjugate doublets of $SU(2)$. In fact they go much further than that, by virtue of the following easy corollaries:

$$\begin{aligned} 0 &= \left\{\partial, \bar\partial^\dagger\right\} = \left\{\bar\partial, \partial^\dagger\right\} \\ \frac{1}{2}\Delta &= \Delta_\partial := \left\{\partial, \partial^\dagger\right\} \\ \frac{1}{2}\Delta &= \Delta_{\bar\partial} := \left\{\bar\partial, \bar\partial^\dagger\right\}. \end{aligned} \tag{2.27}$$

These say that, on a Kähler manifold, the three different ways one could try to define a Laplacian are all equivalent. In particular, the Laplacian on a Kähler manifold must preserve the bidegree of a form. From a physical perspective they say more than this: these are the conditions we need to obtain $\mathcal{N} = (2, 2)$ supersymmetry. To see this, begin by defining an Hermitian $U(1)$ symmetry generator

$$R(\alpha) = \frac{1}{2}(p - q)\alpha, \tag{2.28}$$

where α is a (p,q) form. R generates the action of the complex structure on $\Omega^*(M;\mathbb{C})$, extended from that on vectors by the tensor product, via $U(\theta) = \exp(i\theta R)$ with $U(\pi) = I$. By analogy with [61] we define

$$Q^I = d^I = I^{-1} \circ d \circ I = i(\bar{\partial} - \partial) = -2i\,[R, d]\,, \tag{2.29}$$

thus obtaining the $\mathcal{N} = (2,2)$ supersymmetry algebra

$$\left\{Q, Q^\dagger\right\} = \left\{Q^I, Q^{\dagger I}\right\} = 2H, \qquad [Q, H] = \left[Q^I, H\right] = 0, \tag{2.30}$$

with all other anticommutators zero by virtue of (2.27). Furthermore, the Hodge identities (2.26) imply that $(Q, Q^{\dagger I})$ and $(Q^\dagger, Q^I)$ form doublets of $SU(2)$ while the Hamiltonian commutes with it, hence we have an $SU(2)$ R-symmetry. Finally, the $U(1)$ generator R commutes with H and $SU(2)$, since H preserves bidegree and the Kähler form is of type (1,1), while rotating Q and Q^I into each other.

In summary, we have shown that the σ-model (2.7) admits $\mathcal{N} = (2,2)$ supersymmetry, with $SU(2)\times U(1)$ R-symmetry acting purely on fermions, if and only if the target manifold M is Kähler. The generators are

$$\begin{aligned}
2H &= \Delta = -R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\rho}\psi^{\sigma}\psi^{\nu} - i\Gamma^{\mu}_{\nu\rho}\psi^{\dagger\rho}\Pi_\mu\psi^\nu + i\Pi_\nu\psi^{\dagger\mu}\Gamma^{\nu}_{\mu\rho}\psi^\rho + \Pi^\dagger_\mu g^{\mu\nu}\Pi_\nu - \partial_\mu\Gamma^{\rho}_{\nu\rho}\psi^{\dagger\mu}\psi^\nu \\
Q &= d = i\psi^{\dagger\mu}\Pi_\mu \\
Q^I &= i(\bar{\partial} - \partial) = -i\psi^{\dagger\mu}I^{\nu}{}_{\mu}\Pi_\nu \\
J^I_+ &= \omega\wedge = \frac{1}{2}\omega_{\mu\nu}\psi^{\dagger\mu}\psi^{\dagger\nu} \\
J_3 &= \frac{1}{2}\left(p + q - n\right) = \frac{1}{2}\left(\psi^{\dagger\mu}\psi_\mu - n\right) \\
R &= \frac{1}{2}\left(p - q\right) = \frac{i}{2}\omega_{\mu\nu}\psi^{\dagger\mu}\psi^\nu
\end{aligned} \tag{2.31}$$

and adjoints, where the first expression for each generator is geometric and the second is quantum mechanical. They have nonzero commutation relations

$$\begin{aligned}
\left[J^I_+, J^I_-\right] &= 2J_3 & \left[J_3, J^I_\pm\right] &= \pm J^I_\pm \\
\left[J^I_+, Q^\dagger\right] &= -Q^I & \left[J^I_+, Q^{\dagger I}\right] &= Q \\
[J_3, Q] &= \frac{1}{2}Q & \left[J_3, Q^I\right] &= \frac{1}{2}Q^I \\
[R, Q] &= \frac{i}{2}Q^I & \left[R, Q^I\right] &= -\frac{i}{2}Q \\
\left\{Q, Q^\dagger\right\} &= 2H & \left\{Q^I, Q^{\dagger I}\right\} &= 2H
\end{aligned} \tag{2.32}$$

Before moving on to the $\mathcal{N} = (4,4)$ case, note that the derivation above serves to highlight the strength of the link between target space geometry and σ-model symmetries in quantum mechanics. We've seen that $\mathcal{N} = (2,2)$ supersymmetry with $SU(2) \times U(1)$ R-symmetry essentially *is* Kähler geometry. Indeed, it would be possible to take the fact that our σ-model admits this structure as the definition of Kähler geometry. [31] adopts roughly this perspective in showing that the Hodge identities follow from reducing four-dimensional $\mathcal{N} = 1$ supersymmetry to quantum mechanics.

One might also wonder why $SU(2) \times U(1)$ is the correct R-symmetry, and whether there is more there to be found. Indeed, a generic four supercharge quantum mechanics with supersymmetry algebra

$$\{Q^m, Q^n\} = \delta^{mn} H$$

might be expected to admit $SO(4)$ R-symmetry, and it is easy to take complex combinations of the Q^m which obey the $\mathcal{N} = (2,2)$ algebra. This question is even more pertinent in view of a large number of papers exhibiting non-trivial four supercharge systems with $SO(4)$ R-symmetry, albeit the known geometric realisations require more than one complex structure [58, 59, 62]. The solution seems to rest with the precise form of σ-model we have chosen. Not only does it have at least $\mathcal{N} = (1,1)$ supersymmetry, it can also be obtained by dimensional reduction from 1+1 dimensions. In fact, in the Kähler case we can obtain it from dimensional reduction of a four-dimensional $\mathcal{N} = 1$ model of a chiral superfield with the same basic structure as (2.7) [31]. The four-dimensional model has a $U(1)$ R-symmetry, and under dimensional reduction the conjugate pair of Weyl spinor supercharges splits into two conjugate pairs of two-dimensional Majorana-Weyl spinors, giving $\mathcal{N} = (2,2)$ supersymmetry. Furthermore, the $SO(3)$ coming from rotations in the reduced spatial dimensions descends to an R-symmetry in the quantum mechanics, giving rise to the expected $SU(2) \times U(1)$. While it is by no means impossible that some accident occurs in dimensional reduction which allows this to extend to $SO(4)$, there is no natural reason to expect it and such an accident would probably not be generic, so we will not pursue this possibility.

We now proceed to the $\mathcal{N} = (4,4)$ case, which as we will see requires a hyper-Kähler target. We begin with the basics of hyper-Kähler geometry, an excellent reference for which is the PhD thesis [63].

Definition

A manifold M of real dimension $4n$ is *hyper-Kähler* if it has a triplet of complex structures

$I^a : a = 1, 2, 3$ obeying the quaternion algebra

$$I^a I^b = -\delta^{ab} \mathbb{1} + \epsilon^{abc} I^c \tag{2.33}$$

and M is Kähler with respect to each of these.

There are three Kähler forms $\omega^a_{\mu\nu} = g_{\mu\rho} I^{a\rho}{}_\nu$, leading to three copies of the $SU(2) \times U(1)$ R-symmetry described above (with a shared J_3), as well as three splits of the exterior derivative into Dolbeault operators

$$d = \partial^a + \bar{\partial}^a, \qquad d^a = i \left(\bar{\partial}^a - \partial^a \right). \tag{2.34}$$

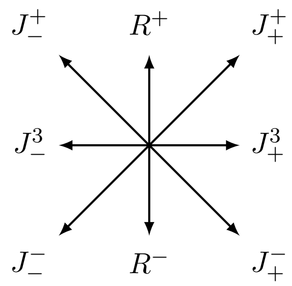
Verbitsky showed that the various R-symmetries combine together to form an $\mathfrak{so}(5)$ algebra [64]. The commutation relations are easily calculated using the quantum mechanical dictionary (2.19), explicit expressions (2.31) for the generators and the quaternion algebra (2.33). We find

$$\begin{aligned} \left[J^a_+, J^b_- \right] &= 2 \left(\delta^{ab} J_3 + i\epsilon^{abc} R^c \right) & \qquad \left[R^a, J^b_+ \right] &= i\epsilon^{abc} J^c_+ \\ \left[R^a, R^b \right] &= i\epsilon^{abc} R^c & \qquad \left[J_3, J^b_\pm \right] &= \pm J^b_\pm . \end{aligned} \tag{2.35}$$

While it is not immediately obvious that this algebra is $\mathfrak{so}(5)$, it can be made clear by defining

$\mathfrak{h}_0 = \langle J_3, R^3 \rangle$	Cartan subalgebra
$\Phi^+ = \langle R^+ = R^1 + iR^2, J^\pm_+ = J^1_+ \pm i J^2_+, J^3_+ \rangle$	Positive roots
$\Phi^- = \left(\Phi^+ \right)^\dagger$	Negative roots.

The corresponding root diagram is



which we recognise as that of $\mathfrak{so}(5)$. This gives our $SO(5)$ R-symmetry, which by virtue of the Hodge identities for each Kähler structure must commute with the Hamiltonian. Using the definition $Q^a = (I^a)^{-1} \circ d \circ I^a$ along with the Kähler result (2.30) and the quaternion algebra (2.33), it's easy to check that

$$\{Q, Q^a\} = \left\{ Q^a, Q^b \right\} = 0.$$

Using the dictionary (2.19) gives

$$\left[R^a, Q^b\right] = \frac{i}{2}\left(-\delta^{ab}Q + \epsilon^{abc}Q^c\right)$$

from which we obtain, via the Jacobi identity and results from the Kähler case,

$$\left\{Q^a, Q^{\dagger b}\right\} = 2\delta^{ab}H.$$

The remaining results follow from the known relations (2.32, 2.35) and the Jacobi identity:

$$\left[J_+^a, Q^b\right] = 0 \qquad \left[J_-^a, Q^b\right] = -\delta^{ab}Q^\dagger + \epsilon^{abc}Q^{\dagger c}.$$

To sum up, we've shown that hyper-Kähler manifolds admit $\mathcal{N} = (4,4)$ supersymmetry with $SO(5)$ R-symmetry acting on fermions only, and that such geometries are the only ones for which this occurs. The generators are

$$\begin{aligned}
2H &= \Delta = -R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\rho}\psi^\sigma\psi^\nu - i\Gamma^\mu_{\nu\rho}\psi^{\dagger\rho}\Pi_\mu\psi^\nu + i\Pi_\nu\psi^{\dagger\mu}\Gamma^\nu_{\mu\rho}\psi^\rho + \Pi^\dagger_\mu g^{\mu\nu}\Pi_\nu - \partial_\mu\Gamma^\rho_{\nu\rho}\psi^{\dagger\mu}\psi^\nu \\
Q &= d = i\psi^{\dagger\mu}\Pi_\mu \\
Q^a &= i(\bar\partial^a - \partial^a) = -i\psi^{\dagger\mu}I^{a\nu}{}_\mu\Pi_\nu \\
J_+^a &= \omega^a\wedge = \frac{1}{2}\omega^a_{\mu\nu}\psi^{\dagger\mu}\psi^{\dagger\nu} \\
J_3 &= \frac{1}{2}(r-2n) = \frac{1}{2}\left(\psi^{\dagger\mu}\psi_\mu - n\right) \\
R^a &= \frac{1}{2}(p-q) = \frac{i}{2}\omega^a_{\mu\nu}\psi^{\dagger\mu}\psi^\nu.
\end{aligned} \tag{2.36}$$

Here $4n$ is the real dimension of the target space, r is the degree of a form and (p,q) is its bidegree in complex coordinates adapted to I^a. The nonzero commutation relations are

$$\begin{aligned}
\left[J_+^a, J_-^b\right] &= 2\left(\delta^{ab}J_3 + i\epsilon^{abc}R^c\right) & [J_3, J_\pm^a] &= \pm J_\pm^a \\
\left[R^a, R^b\right] &= i\epsilon^{abc}R^c & \left[R^a, J_+^b\right] &= i\epsilon^{abc}J_+^c \\
\left[J_+^a, Q^\dagger\right] &= -Q^a & \left[J_+^a, Q^{\dagger b}\right] &= \delta^{ab}Q - \epsilon^{abc}Q^c \\
[J_3, Q] &= \frac{1}{2}Q & [J_3, Q^a] &= \frac{1}{2}Q^a \\
[R^a, Q] &= \frac{i}{2}Q^a & \left[R^a, Q^b\right] &= \frac{i}{2}\left(-\delta^{ab}Q + \epsilon^{abc}Q^c\right) \\
\left\{Q, Q^\dagger\right\} &= 2H & \left\{Q^a, Q^{\dagger b}\right\} &= 2\delta^{ab}H
\end{aligned} \tag{2.37}$$

We close this section by noting that the idea that Kähler geometry can be characterised by the fact that (2.7) admits $\mathcal{N} = (2,2)$ supersymmetry extends to the hyper-Kähler case. Indeed, [31] shows that the hyper-Kähler model, along with the full supersymmetry algebra and R-symmetry, can be obtained by dimensional reduction of a six-dimensional $\mathcal{N} = (1, 0)$ nonlinear σ-model. The $SO(5)$ R-symmetry arises naturally from rotations in the five spatial dimensions removed by the reduction.

2.3 The Witten Index and Topology

In this section we explore in more detail what our supersymmetric σ-model can tell us about its target manifold. Our object of interest is the *Witten index*, a sort of 'partial partition function' with cancellations between the contributions of bosonic and fermionic states, leading to some useful invariance properties. In the context of our σ-model (2.7) this index turns out to calculate the Euler characteristic of the target manifold, through a connection between supersymmetric ground states and Hodge theory.

Recall that the *thermal partition function* of a quantum theory with Hilbert space $\mathcal{H}$ and Hamiltonian H is

$$Z(\beta) = \mathrm{tr}_{\mathcal{H}} \left[e^{-\beta H} \right],$$

where $\beta = 1/T$ is the inverse temperature in units where Boltzmann's constant is 1. This object contains a great deal of detailed information about the theory, as is reflected in its complicated dependence on β and the parameters of H. However, the price of all this information is that Z is exceptionally difficult to calculate. Even in quantum mechanics we can't usually do better than a perturbative approximation.

An alternative approach is to redefine the partition function in such a way that it becomes calculable, at the cost of coarse-graining much of the information. The Witten index [29] is a simple example of such a coarse graining. To define it, we first introduce an operator F which counts the fermion number of a state.[9] The Witten index is [29]

$$I_W(\beta) = \mathrm{tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H} \right]. \tag{2.38}$$

For a generic theory this doesn't help very much, but for supersymmetric theories the factor of $(-1)^F$ makes all the difference. For example, consider a quantum mechanical model with the $\mathcal{N} = (1, 1)$ supersymmetry algebra (2.1). States annihilated by both Q and $Q^\dagger$ are also annihilated

[9] Strictly speaking, the count is only defined modulo 2 and one needs a convention to make F well-defined. In our σ-model the convention is that functions have fermion number zero.

by H. These are *supersymmetric vacua*, which contribute ± 1 to (2.38) according to whether they are bosonic or fermionic. On the other hand, if $|\phi\rangle = Q\,|\psi\rangle$ (or similar for $Q^\dagger$), then both $|\phi\rangle$ and $|\psi\rangle$ have the same energy since $[Q, H] = 0$, and their fermion numbers differ by 1 (mod 2). It follows that their contributions to (2.38) cancel out, so that

$$I_W = \#\,(\text{bosonic vacua}) - \#\,(\text{fermionic vacua})\,.$$

We see that I_W is actually independent of β. However, for a general supersymmetric theory this count need not be well-defined as both numbers may be infinite. We'll see some examples of this for our σ-model (2.7) shortly.

Notwithstanding the above issue, the Witten index is useful because it is invariant under a large range of supersymmetry-preserving parameter deformations of a theory [29]. This means it is often possible to calculate the index exactly, for example using a weak coupling expansion or large bare mass limit. Heuristically, this invariance follows from the fact that energies of states are continuous functions of the parameters of a theory. As parameters are varied, a bosonic vacuum state can acquire nonzero energy if and only if a fermionic vacuum does so in a pair, such that supersymmetry is preserved. Likewise, if a boson with positive energy loses this energy to become a vacuum, its partner fermion will do the same. In this way, the difference between the number of bosonic and fermionic vacua is always preserved. The one dangerous exception to this rule occurs when a change in parameters changes the asymptotics of a theory. In this instance, the Hilbert space itself changes and we can no longer rely on continuity arguments. We'll give examples of this shortly.

We now turn to an analysis of the Witten index for the supersymmetric σ-model (2.7). States of zero energy correspond to differential forms on the target space annihilated by the Laplace operator, known as *harmonic forms*. If the target space M is compact there are a number of theorems collectively known as Hodge theory (see appendix B) relating these harmonic forms to topology. The central result is:

Theorem

Harmonic forms are in 1-1 correspondence with de Rham cohomology classes

$$\mathscr{H}^r(M) \cong H^r_{dR}(M).$$

Denoting the dimension of H^r by h_r, we find that the Witten index is

$$I_W = \sum_{r=0}^{n} (-1)^r h_r = \chi(M), \tag{2.39}$$

where $\chi(M)$ is the Euler characteristic of M. Note that for a compact manifold the h_r are all finite, so the Witten index is well-defined. This illustrates the idea that the Witten index is a calculable but coarse-grained version of a partition function. In contrast, the thermal partition function contains complete information about the spectrum of the Laplace operator on M. Such information is known to be almost sufficient to fix M completely [65], but is exceptionally tough to compute in general. The fact that the index is a topological invariant in this case is a rather extreme example of parameter invariance, in the sense that the metric on M may be viewed as an infinite family of 'coupling constants' for the σ-model and all valid metrics on M lead to the same index. The connection between indices and topology of compact manifolds as described above can be greatly extended to give simple derivations of a number of well-known results, including Morse theory [28] and the Atiyah-Singer index theorem [30].

If M is not compact things become more complicated. As explained in appendix B, the connection between harmonic forms and cohomology is much more subtle in the non-compact case, so the identification with the Euler characteristic cannot be made. Furthermore, it is possible for the index to be ill-defined. To illustrate this, consider the two-dimensional target space $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Around each lattice point $(a, b) \in \mathbb{Z}^2$ we can define global polar coordinates (r_{ab}, θ_{ab}) so that $d\theta_{ab}$ is a well-defined, smooth, L^2-normalisable 1-form. It is a simple matter to check that each such $d\theta_{ab}$ is a harmonic form and that they are all linearly independent. Indeed, they correspond to different cohomology classes, so the Witten index receives a contribution of $-\infty$ from fermionic states. Using similar ideas in three target space dimensions, one can come up with an example where both bosonic and fermionic vacua contribute ∞, so we cannot even consistently assign the value ∞ to the index. We can also tweak this example to see a parameter deformation that does not preserve the index. If we instead suppose that the lattice is located at points $(\lambda a, \lambda b) : (a, b) \in \mathbb{Z}^2,\ \lambda \in \mathbb{R}$, then for $\lambda \in (0, \infty)$ the situation is identical to the above. On the other hand, as $\lambda \to 0$ all lattice points coalesce at the origin, while for $\lambda \to \infty$ all but one 'fly off to infinity'. In either limit, we have $I_W = -1$. These considerations suggest that some kind of regularisation is needed to make a sensible index. Our discussion of the superconformal index for $\mathfrak{osp}(4^*|4)$ in section 7.4 could be viewed as an example of this, where the addition of a special potential to the Laplacian transforms the ill-defined Witten index into the well-defined superconformal index.

3 Instantons

In this section we review some properties of Yang-Mills instantons. These are topologically non-trivial configurations of a non-abelian gauge field with self-dual field strength. We describe the moduli space of such solutions and its hyper-Kähler structure. The latter is most easily described using the ADHM construction [41], which gives an algebraic approach to obtaining instanton solutions and presents the moduli space as a hyper-Kähler quotient [42]. We show how quantum mechanics on instanton moduli space can emerge as a low-energy approximation to dynamics in certain five-dimensional gauge theories (an adaptation of [66] to instantons), and how this observation ties in with supersymmetry and the string-theoretic derivation of ADHM [44,45]. Our standard reference for instantons is the review [67]. For information on the coordinate-free formulation of gauge theory we refer to [55].

3.1 Basics and the Moduli Space

Consider an $SU(N)$ gauge theory[10] in four Euclidean dimensions $x^m : m = 1, \ldots, 4$. The simplest such theory is pure Yang-Mills, for which the action is

$$S_{YM} = \frac{1}{g^2} \|F\|^2 = -\frac{1}{g^2} \int_{\mathbb{R}^4} \mathrm{tr}\left\{F \wedge *F\right\} = -\frac{1}{2g^2} \int d^4x \, \mathrm{tr}\left\{F_{mn} F_{mn}\right\}. \tag{3.1}$$

Here g^2 is a coupling constant and F is the field strength, an $\mathfrak{su}(N)$ valued 2-form given by

$$F = DA = dA + A \wedge A = \frac{1}{2}\left(\partial_m A_n - \partial_n A_m + [A_m, A_n]\right) dx^m \wedge dx^n,$$

where $A_m = A_m^a T^a$ is the gauge potential and $T^a : a = 1, \ldots, N^2 - 1$ is a basis for $\mathfrak{su}(N)$ with $\mathrm{tr}\left\{T^a T^b\right\} = -\frac{1}{2}\delta^{ab}$ and $(T^a)^\dagger = -T^a$. $D = d + A$ is the gauge covariant derivative. The equations of motion following from (3.1) are

$$D * F = d * F + A \wedge *F - *F \wedge A = 0 \tag{3.2}$$

and the field strength obeys the Bianchi identity off-shell

$$DF = dF + A \wedge F - F \wedge A = 0. \tag{3.3}$$

The Yang-Mills equations are a system of coupled second-order nonlinear PDEs, and hence impossibly difficult to solve in general, but we can make progress by looking at a class of special solutions

[10] Other classical gauge groups can be treated analogously.

known as *instantons*. These were first introduced in [68, 69] in describing the infrared physics of gauge theories. Observe that if the field strength is *(anti-)self-dual*, $F = \pm * F$, then the Bianchi identity implies the Yang-Mills equations. Such solutions are called Yang-Mills (anti-)instantons.

These solutions saturate a *Bogomol'nyi bound* [70] for their action. We have

$$\begin{aligned} 0 \leq \|F \pm *F\|^2 &= -\frac{1}{g^2}\int \mathrm{tr}\left\{(F \pm *F) \wedge *(F \pm *F)\right\} \\ &= -\frac{2}{g^2}\int \mathrm{tr}\left\{F \wedge *F\right\} \mp \frac{2}{g^2}\int \mathrm{tr}\left\{F \wedge F\right\}, \end{aligned}$$

where, since $\|\cdot\|$ is positive, equality holds if and only if the configuration is (anti-)self-dual. We obtain the bound

$$S \geq \frac{8\pi}{g^2}|k|, \tag{3.4}$$

where k is the *instanton number*

$$k = -\frac{1}{8\pi^2}\int \mathrm{tr}\left\{F \wedge F\right\}. \tag{3.5}$$

The instanton number is a topological charge, and in particular $k \in \mathbb{Z}$ (see [71]). To see this, specialise to the case of $SU(2)$ and observe that any finite action field configuration must be asymptotically pure gauge at infinity

$$A \to g^{-1}dg \Rightarrow F \to 0 \text{ as } |x| \to \infty.$$

We can evaluate (3.5) as a surface integral over the 'sphere at infinity' using the Chern-Simons 3-form

$$C = A \wedge dA + \frac{2}{3}A \wedge A \wedge A, \qquad d\,\mathrm{tr}\,C = \mathrm{tr}\left\{F \wedge F\right\}. \tag{3.6}$$

Then

$$k = -\frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{tr}\,C = -\frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{tr}\left\{g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right\}. \tag{3.7}$$

In the latter integral we view the pure gauge configuration at infinity as a map $g : S^3_\infty \to SU(2) \cong S^3$, and the integrand is the pullback by g of the invariant Haar volume form on $SU(2)$. It follows that k computes the *topological degree* of the map g. This is an integer invariant which counts the 'number of times g wraps S^3_∞ around $SU(2)$'. Configurations with nonzero instanton number correspond to non-trivial principal $SU(N)$ bundles over $\mathbb{R}^4$, so cannot be expressed in terms of a single everywhere-smooth gauge potential.

Instanton solutions for large values of k and N are typically difficult to write down explicitly, but for $k = 1$ $SU(2)$ instanton there is a simple general formula [69]

$$A_m = 2g^{-1}\frac{(x - X)^n}{(x - X)^2 + \rho^2}\sigma_{mn}, \tag{3.8}$$

where $g \in SU(2)$, $\rho \in (0, \infty)$, $X^m \in \mathbb{R}^4$ and σ_{mn} are a self-dual set of elements of $\mathfrak{su}(2)$ derived from the Pauli matrices (see [67] for conventions on these). This solution is written in *regular gauge*, characterised by smoothness on $\mathbb{R}^4$ so that in some sense the singularity is 'at infinity', as shown by the rather slow $1/r$ falloff. There is an alternative, called *singular gauge*, in which the falloff at infinity is $1/r^3$ but there is a manifest singularity at $x = X$. In either case, the field strength is smooth everywhere with characteristic $1/r^4$ decay at infinity. We interpret X as the centre of the instanton configuration and ρ as its size. The arbitrary size is allowed because the Yang-Mills action is scale invariant: the transformation $A_m(x) \mapsto \lambda A_m(\lambda x)$ leaves the potential A, hence the field strength and action invariant. g^{-1} is a global gauge transformation. It is a genuine physical symmetry in the same sense that the global part of the electric $U(1)$ of QED is a genuine symmetry implying charge conservation, so must be included in the parameters of the solution. Overall, the solution has the following parameters

- Four coordinates of the centre, parameterising $\mathbb{R}^4$.
- Three dimensions of gauge transformations, parameterising $SU(2) \cong S^3$.
- A scale, giving $\mathbb{R}^+$.

Thus there is an overall eight-dimensional space of solutions, called the *moduli space* of one $SU(2)$ instanton. More generally, the space of k $SU(N)$ instantons also forms a moduli space, denoted $\mathcal{M}_{k,N}$. We will shortly see that this space is actually a hyper-Kähler manifold (with singularities) of dimension $4kN$ [42, 72, 73], and that we can give expressions for the metric and Kähler forms in terms of gauge theory data. Note that, in the special case of one $SU(2)$ instanton, all the moduli are consequences of symmetries of the underlying Yang-Mills theory, but this is certainly not true in general.

To study the geometry of the moduli space, we start by describing its tangent vectors. Intuitively, a tangent vector is an infinitesimal approximation to 'moving around' in a manifold, which suggests that we look at perturbations $A_m \mapsto A_m + \delta A_m$ of our instanton solution. The new configuration thus obtained should also be an instanton, so we solve the self-duality equation $F = *F$ to linear order in δA. The field strength transforms as

$$\delta F_{mn} = D_m \delta A_n - D_n \delta A_m, \tag{3.9}$$

where $D_m = \partial_m + A_m^{inst}$ is the background covariant derivative for the instanton configuration A^{inst}, so the linearised self-duality equations are

$$D_m \delta A_n - D_n \delta A_m = \epsilon_{mnpq} D_p \delta A_q. \tag{3.10}$$

We would like to call solutions of this equation tangent vectors (or *zero modes*), but this is not yet correct since if δA_m is a gauge transformation then it trivially solves (3.10), but a gauge transformation does not move us around moduli space. We need a way to 'remove the gauge part' from any zero mode, for which we introduce an inner product on perturbations

$$\left\langle \delta A, \delta' A \right\rangle = -2 \int d^4x \, \mathrm{tr} \left\{ \delta A_m \delta' A_m \right\}. \tag{3.11}$$

An infinitesimal gauge transformation is given by $A \mapsto A + D\Omega$ for some $\mathfrak{su}(N)$-valued scalar Ω, so orthogonality of a perturbation to gauge transformations requires

$$\int d^4x \, \mathrm{tr} \left\{ D_m \Omega \delta A_m \right\} = 0 \ \forall \Omega \quad \Leftrightarrow \quad D_m \delta A_m = 0. \tag{3.12}$$

Any perturbation satisfying both (3.10) and (3.12) is a genuine instanton zero mode. To make further progress we assume that the moduli space is a manifold, so has local coordinates X^μ : $\mu = 1, \ldots, 4kN$ with respect to which we can take derivatives. The X^μ are often called *collective coordinates*. Their existence can be proved directly, but we justify it a posteriori via the ADHM construction. Given this, the partial derivatives $\partial_\mu A_m$ are by definition solutions of (3.10) but needn't be gauge orthogonal. Instead, consider the combination

$$\delta_\mu A_m = \partial_\mu A_m - D_m \Omega_\mu, \tag{3.13}$$

where Ω_μ is a linearised gauge transformation parameter satisfying

$$\Box^{inst} \Omega_\mu = D_m \partial_\mu A_m.$$

Here $\Box^{inst} = D_m D_m$ is the gauge-covariant Laplacian, and solutions to this condition exist by Green's function methods. This combination is gauge orthogonal as it satisfies (3.12), and solves (3.10) since the equation is linear and all gauge transformations do so. $\delta_\mu A_m$ is therefore a genuine zero mode, and such quantities form a basis for the tangent space of $\mathcal{M}_{k,N}$.

From these tangent vectors along with the inner product (3.11) we obtain a natural metric on moduli space

$$g_{\mu\nu} = -2 \int d^4x \, \mathrm{tr} \left\{ \delta_\mu A_m \delta_\nu A_m \right\}. \tag{3.14}$$

Explicit expressions for this metric are not known in general, but for one $SU(2)$ instanton the solution (3.8) is sufficiently simple that the calculations can be done, resulting in

$$\mathcal{M}_{1,2} \cong \mathbb{R}^4 \times \frac{\mathbb{R}^4}{\mathbb{Z}_2}$$

with the standard Euclidean metric induced from $\mathbb{R}^8$. Here the $\mathbb{R}^4$ factor corresponds to the instanton centre, while the quotiented $\mathbb{R}^4 \sim S^3 \times \mathbb{R}^+$ is a combination of scale and global gauge parameters. The $\mathbb{Z}_2$ quotient occurs since the Weyl group $\mathbb{Z}_2 = \{\pm\mathbb{1}\}$ of $SU(2)$ does not act on an instanton solution. Observe that $\mathcal{M}_{1,2}$ is actually a hyper-Kähler manifold with an orbifold singularity. In general the singularities are more severe, and correspond to limiting submanifolds where one or more instantons shrink to zero size. The hyper-Kähler structure extends to all k and N, as we now describe.

The fact that $\mathbb{R}^4$ is a hyper-Kähler manifold allows us to induce almost complex structures on $\mathcal{M}_{k,N}$ obeying the quaternion algebra (2.33). Denote the complex structures and Kähler forms of $\mathbb{R}^4$ by $J^{am}{}_n$ and $\delta_{mn}J^{an}{}_p = J^a_{mp}$ respectively. A short calculation shows that $J^a_{mn}\delta_\mu A_n$ satisfies the equations (3.10) and (3.12), hence is a valid zero mode. It follows that we can write

$$J^a_{mn}\delta_\mu A_n = I^{a\nu}{}_\mu \delta_\nu A_m, \tag{3.15}$$

which defines the objects $I^{a\mu}{}_\nu$ as functions only of the collective coordinates. Furthermore, the structure of (3.15) implies that they are $(1,1)$ tensors and the algebra obeyed by the J^a_{mn} implies that they are actually almost complex structures on $\mathcal{M}_{k,N}$ obeying the quaternion algebra (2.33). This does not yet prove that $\mathcal{M}_{k,N}$ is hyper-Kähler, as we need to verify integrability of the almost complex structures and that the associated Kähler forms $\omega^a_{\mu\nu} = g_{\mu\rho}I^{a\rho}{}_\nu$ are closed. Rather than do this directly, in the next section we derive these results using the ADHM construction. In fact, we'll see that $\mathcal{M}_{k,N}$ is a special kind of hyper-Kähler manifold, in that it admits a *hyper-Kähler potential* [73]. Recall that, on a Kähler manifold, the Poincaré lemma implies that the Kähler form can locally be written as

$$\omega = -i\partial\bar{\partial}K, \tag{3.16}$$

where K is a real function which is unique up to addition of the real part of a holomorphic function [60]. In particular, on a hyper-Kähler manifold each Kähler form has an associated potential, and the manifold is said to admit a hyper-Kähler potential if all three such potentials can be chosen to coincide. This turns out to be rather a restrictive condition [63] but is true for $\mathcal{M}_{k,N}$.

3.2 The ADHM Construction

While the self-duality condition is a significant reduction in complexity from the full Yang-Mills equations, it is by no means obvious how to construct a general solution. However, it turns out that each instanton solution can be uniquely (up to gauge transformations) obtained from constrained algebraic data. That is, a configuration corresponding to k $SU(N)$ instantons, which in general has a rather complicated spatial dependence, can be obtained by taking any point in $\mathbb{R}^{4k(N+k)}$ satisfying the so-called ADHM equations [41]. We give a bare-bones review of the construction, omitting most of the details. In particular, we omit any proof of completeness: the original proof that all instantons are obtained in this way uses more advanced algebraic geometry, reviewed in [74]. For us, the most important result is that the ADHM construction is an instance of a hyper-Kähler quotient [42]. In particular, this provides a simple proof of many claimed properties of the moduli space [73].

We begin by describing the data involved in the ADHM construction. There is a pair of $k \times k$ complex matrices X and $\tilde{X}$, along with a $k \times N$ complex matrix q and an $N \times k$ matrix $\tilde{q}$, often written as q_i and $\tilde{q}^i : i = 1, \ldots, N$. This rewriting allows us to think of q^i as a vector in the $\bar{\mathbf{N}}$ of $SU(N)$, while $\tilde{q}^i$ is in the $\mathbf{N}$ and $X, \tilde{X}$ are neutral. In total these carry $4k^2 + 4kN$ real degrees of freedom. To show how to obtain instantons from these, it is useful to package the data in terms of quaternions. We refer the reader to [67] for the details of this, but the result is an $(N + 2k) \times 2k$ complex matrix $\Delta(x)$ depending linearly on the spatial coordinates x_m and the data $X, \tilde{X}, q, \tilde{q}$. For generic values of the data, the Hermitian conjugate $\Delta^\dagger : \mathbb{C}^{N+2k} \to \mathbb{C}^{2k}$ is a surjective map $\forall x$, with a null space of dimension N. By picking an orthonormal basis of this null space, we can form an $(N + 2k) \times N$ matrix U whose columns are this basis, such that

$$\Delta^\dagger U = 0 = U^\dagger \Delta, \qquad U^\dagger U = \mathbb{1}_N. \tag{3.17}$$

Given this data, the ADHM ansatz is

$$A_m = U^\dagger \partial_m U. \tag{3.18}$$

Note the similarity to a gauge transformation. Indeed, for $k = 0$ U is an $N \times N$ matrix, so the ansatz is pure gauge and we get the trivial solution $A = 0$.

For $k \neq 0$ it is by no means apparent that the ansatz (3.18) gives rise to an instanton solution. Indeed, without additional constraints on the data it does not. Furthermore, there is some redundancy in the ansatz since many possible matrices U lead to the same gauge field, possibly up to a gauge transformation. We omit the details, again referring the reader to [67], but it turns out that the required constraints can be written in terms of the original complex data $X, \tilde{X}, q_i, \tilde{q}^i$ as

$$\begin{aligned} &\left[X, X^\dagger\right] - \left[\tilde{X}, \tilde{X}^\dagger\right] + q_i q^{\dagger i} - \tilde{q}_i^\dagger \tilde{q}^i = 0 \\ &\left[X, \tilde{X}\right] + q_i \tilde{q}^i = 0. \end{aligned} \tag{3.19}$$

These are the *ADHM equations.* When they are satisfied, the data obey additional relations which allow a manifestly self-dual form for the field strength. Firstly, there is a factorisation

$$\Delta^\dagger \Delta = f^{-1} \otimes \mathbb{1}_2,$$

where f is an invertible $k \times k$ Hermitian matrix whose explicit form is unimportant for us. Secondly, the projection operator $P = UU^\dagger$ can be written as

$$P = \mathbb{1}_{N+2k} - \Delta f \Delta^\dagger.$$

With these identities in hand, a straightforward manipulation gives

$$F_{mn} = 4U^\dagger b \sigma_{mn} f b^\dagger U \tag{3.20}$$

where $b^\dagger = (0_{2k\times N},\ \mathbb{1}_{2k})$ and σ_{mn} is a self-dual $SU(2)$ generator as in (3.8), from which the self-duality of F follows. The most general transformation preserving F and the special form of b is

$$\Delta \mapsto \Lambda \Delta \Upsilon^\dagger, \qquad U \mapsto \Lambda U, \qquad f \mapsto \Xi f \Xi^\dagger,$$

with

$$\Lambda = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & \Upsilon \end{pmatrix}, \qquad \Upsilon = \Xi \otimes \mathbb{1}_2, \qquad \Xi \in U(k).$$

We also have ordinary gauge transformations $U \mapsto Ug : g \in SU(N)$. The transformation Ξ is allowed to depend on position and represents a redundancy in the description of an instanton by Δ, so is a kind of gauge symmetry for the ADHM data. In fact, if we unpack the various identifications then we find that X and $\tilde{X}$ transform in the adjoint of $U(k)$, q_i in the $\mathbf{k}$ and $\tilde{q}^i$ in the $\bar{\mathbf{k}}$.

To summarise, the space of physical ADHM data consists of complex matrices $X, \tilde{X}, q, \tilde{q}$ transforming as follows:

	X	$\tilde{X}$	q	$\tilde{q}$
$SU(N)$	$\mathbf{0}$	$\mathbf{0}$	$\bar{\mathbf{N}}$	$\mathbf{N}$
$U(K)$	**adj**	**adj**	$\mathbf{k}$	$\bar{\mathbf{k}}$

(3.21)

subject to the constraints (3.19) and modulo $U(k)$ gauge symmetry, leaving the expected $4kN$ degrees of freedom. Though it is far beyond the scope of this review, one can prove that this construction is sufficient in the sense that each instanton solution corresponds uniquely to a set of ADHM data as described above. In other words, the space of physical ADHM data is isomorphic to the instanton moduli space $\mathcal{M}_{k,N}$. The original proof [41, 74] uses twistor theory and sheaf cohomology, though more down-to-earth techniques were subsequently developed in [75].

Some intuition for this construction can be gained from the following discussion, adapted from [74]. An $SU(N)$ gauge potential can be thought of as a connection on a principal $SU(N)$ bundle over $\mathbb{R}^4$, or equivalently on an associated fundamental vector bundle $E \to \mathbb{R}^4$ with fibres $E_x \cong \mathbb{C}^N$. By embedding E into a larger trivial bundle $F = \mathbb{R}^4 \times \mathbb{C}^{N+2k}$ via the bundle map $\alpha : E \to F$, we can construct a connection as follows. α may be used to view a section s of E as a map $f_s : \mathbb{R}^4 \to \mathbb{C}^{N+2k}$, where it makes sense to take partial derivatives $\partial_m f_s$. This produces a collection of new sections of F, which will not in general lie in the image of α so cannot be thought of as sections of E. However, if we choose a projection map $P : F \to \text{image}(\alpha)$ then we have a connection

$$\nabla_m f_s = P \partial_m f_s.$$

Furthermore, if the projection P is orthogonal (with respect to the standard fibre metric on F and the metric induced by α on E) then the connection will be unitary. We have thereby constructed an $SU(N)$ gauge field. To make contact with the ADHM construction, choose a gauge for E so that the embedding α induces maps $U_x : \mathbb{C}^N \to \mathbb{C}^{N+2k}$. If we ensure that U preserves inner products, again with respect to standard choices on $\mathbb{C}^N$ and $\mathbb{C}^{N+2k}$, then we find

$$UU^\dagger = \left(UU^\dagger\right)^2, \qquad U^\dagger U = \mathbb{1}_N,$$

which are the familiar properties of (3.17). A quick calculation shows that such a choice of projection leads to the ADHM ansatz (3.18) for the connection, and once again the ADHM equations (3.19) are required so that the corresponding field strength is self-dual.

We now show how to view the ADHM construction as a hyper-Kähler quotient. The necessary differential geometry is reviewed in appendix C. The ADHM construction consists of data in the vector space $\mathbb{C}^{2k(N+k)}$, which is most certainly a hyper-Kähler manifold, modulo algebraic relations and a group quotient. This structure is reminiscent of the hyper-Kähler quotient, and we need only prove that the ADHM equations (3.19) correspond to moment map conditions for the $U(k)$ action on $\mathbb{C}^{2k(N+k)}$. To calculate moment maps we use matrix and vector components as coordinates, which we label

$$X^{\alpha}_{\ \beta}, \qquad \tilde{X}_{\alpha}^{\ \beta}, \qquad q_i^{\alpha}, \qquad \tilde{q}^i_{\alpha} \qquad \alpha, \beta = 1, \ldots, k,$$

and which transform under $U(k)$ as

$$\delta X^{\alpha}_{\ \beta} = [T^r, X]^{\alpha}_{\ \beta}, \qquad \delta \tilde{X}_{\alpha}^{\ \beta} = -\left[T^r, \tilde{X}\right]_{\alpha}^{\ \beta}, \qquad \delta q_i^{\alpha} = (T^r)^{\alpha}_{\ \beta}\, q_i^{\beta}, \qquad \delta \tilde{q}^i_{\alpha} = -\tilde{q}^i_{\beta}\, (T^r)^{\beta}_{\ \alpha}, \quad (3.22)$$

where T^r is a basis of generators for $U(k)$ in the fundamental. From these transformations we can read off the components of the induced vector fields X^r. We have a metric

$$ds^2 = \sum \left| dX^{\alpha}_{\ \beta} \right|^2 + \left| d\tilde{X}_{\alpha}^{\ \beta} \right|^2 + |dq_i^{\alpha}|^2 + \left| d\tilde{q}^i_{\alpha} \right|^2 \quad (3.23)$$

and a hyper-Kähler potential

$$K = \sum \left| X^{\alpha}_{\ \beta} \right|^2 + \left| \tilde{X}_{\alpha}^{\ \beta} \right|^2 + |q_i^{\alpha}|^2 + \left| \tilde{q}^i_{\alpha} \right|^2. \quad (3.24)$$

We also need the symplectic forms

$$\begin{aligned} \omega^1 &= i\left(dX^{\alpha}_{\ \beta} \wedge dX^{\dagger \beta}_{\alpha} + d\tilde{X}_{\alpha}^{\ \beta} \wedge d\tilde{X}^{\dagger \alpha}_{\ \ \beta} + dq_i^{\alpha} \wedge dq^{\dagger i}_{\alpha} + d\tilde{q}^i_{\alpha} \wedge d\tilde{q}^{\dagger \alpha}_i \right) \\ \eta &= dX^{\alpha}_{\ \beta} \wedge d\tilde{X}_{\alpha}^{\ \beta} + dq_i^{\alpha} \wedge d\tilde{q}^i_{\alpha}, \end{aligned} \quad (3.25)$$

where $\eta = \omega^2 + i\omega^3$ is the *holomorphic symplectic form* with respect to the preferred complex structure I^1. Picking a standard basis of elementary matrices for T^r, using the definition (C.1) of the moment map and plugging in vector components from (3.22), we find

$$\begin{aligned} \left(\mu^1\right)^{\alpha}_{\ \gamma} &= X^{\alpha}_{\ \beta} X^{\dagger \beta}_{\gamma} - X^{\dagger \alpha}_{\beta} X^{\beta}_{\ \gamma} - \tilde{X}_{\gamma}^{\ \beta} \tilde{X}^{\dagger \alpha}_{\ \ \beta} + \tilde{X}^{\dagger \beta}_{\ \ \gamma} \tilde{X}_{\beta}^{\ \alpha} + q_i^{\alpha} q^{\dagger i}_{\gamma} - \tilde{q}^{\dagger \alpha}_i \tilde{q}^i_{\gamma} \\ \left(\mu^{\eta}\right)^{\alpha}_{\ \gamma} &= X^{\alpha}_{\ \beta} \tilde{X}_{\gamma}^{\ \beta} - \tilde{X}_{\beta}^{\ \alpha} X^{\beta}_{\ \gamma} + q_i^{\alpha} \tilde{q}^i_{\gamma}, \end{aligned} \quad (3.26)$$

which are the component forms of the ADHM equations (3.19). It follows that the ADHM construction presents the instanton moduli space $\mathcal{M}_{k,N}$ as a hyper-Kähler quotient of $\mathbb{C}^{2k(N+k)}$ by $U(k)$. With some more detailed calculation, one finds that the hyper-Kähler structure of $\mathcal{M}_{k,N}$ defined by the quotient agrees with that derived directly in section 3.1 [73]. Furthermore, $\mathcal{M}_{k,N}$ admits a hyper-Kähler potential which is trivial to calculate from the ADHM perspective. Since the hyper-Kähler potential (3.24) on $\mathbb{R}^4$ is $U(k)$-invariant, we define the hyper-Kähler potential $\tilde{K}$ on $\mathcal{M}_{k,N}$ to be the restriction of K to a solution of the ADHM equations. That the resulting function really is a hyper-Kähler potential follows easily from the quotient definition of the metric and Kähler forms.

For completeness, we should note that we made the special choice of the $\xi = 0$ level set for the moment map in the quotient construction. Other choices would also produce a hyper-Kähler manifold, and would in fact resolve the singularities occuring in instanton moduli space. It turns out that taking $\xi \neq 0$ for the central $U(1) \subset U(k)$ corresponds to considering instantons in a non-commutative spacetime [76] where $[x_m, x_n] \propto \xi$. Thus ξ sets a fundamental 'minimal scale' below which instantons cannot shrink, hence resolving the moduli space singularities. Though one might think that the resulting smooth hyper-Kähler manifold is easier to work with, the minimal scale breaks conformal invariance so we will not use it.

3.3 Instantons as Solitons in $4+1$ Dimensions

In the remainder of this thesis we will often study quantum mechanics on the moduli space of instantons. The classical version of this problem arises naturally when thinking about Yang-Mills theory in $4+1$ dimensions. The equations of motion are

$$D^M F_{MN} = 0 \qquad M = 0, 1, 2, 3, 4,$$

which split into space x^m and time t components

$$\begin{aligned} 0 &= D_m F_{mn} - D_t F_{tn} \\ 0 &= D_m F_{tm}. \end{aligned} \tag{3.27}$$

Thus if we set

$$A_t = 0, \qquad A_m = A_m^{inst}(x)$$

then we see that an instanton solution is a static solution of the equations of motion. There is an off-shell conserved topological current

$$j = *\,\mathrm{tr}\,\{F \wedge F\} \tag{3.28}$$

whose corresponding charge $Q = \int d^4x\, j^0$ measures the instanton number of a configuration, and the mass of a static instanton is

$$M_{inst} = \frac{8\pi^2}{g^2} k. \tag{3.29}$$

We see from this that an instanton is very heavy at weak coupling $g^2 << 1$. This is typical behaviour for *solitons*, the generic name for such particle-like extended lumps of field configuration carrying topological charge (see [71] for a comprehensive review).

Since weakly coupled solitons are so heavy, a natural low-energy approximation is that they should be very slowly moving. A sensible ansatz for such slow motion supposes that the spatial gauge field remains an instanton configuration, but with moduli which evolve slowly with time.

This is an adaptation to instantons of the idea of the moduli space approximation, first applied to monopoles in [66]. More precisely, we set

$$A_m = A_m^{inst}(x, X(t))$$

with $\dot{X}$ small enough that the kinetic energy of the configuration is much less than the instanton mass. It is not consistent with such an ansatz to take $A_t = 0$ as this does not solve the Gauss equation $D_m F_{tm} = 0$ even to lowest order in $\dot{X}$. Instead, we set

$$A_t = \dot{X}^\mu \Omega_\mu,$$

where Ω_μ is the gauge-fixing parameter defined in (3.13). Then the Yang-Mills equations (3.27) are solved to linear order in $\dot{X}$ so we have an approximate solution for small velocities. To sum up, we've set

$$\begin{aligned} A_t &= \dot{X}^\mu \Omega_\mu & A_m &= A_m^{inst} \\ F_{tm} &= \dot{X}^\mu \delta_\mu A_m^{inst} & F_{mn} &= F_{mn}^{inst}, \end{aligned} \tag{3.30}$$

where $\delta_\mu A_m$ is defined in (3.13).

Substituting this ansatz into the Yang-Mills action produces an effective action for the dynamics of the collective coordinates X^μ. We find

$$S_X = \frac{1}{g^2} \int dt\, d^4x \,\mathrm{tr} \left\{ \dot{X}^\mu \dot{X}^\nu \delta_\mu A_m \delta_\nu A_m - \frac{1}{2} F_{mn} F_{mn} \right\}.$$

The second term gives the time integral of the instanton number, which is a constant shift to the effective Lagrangian and safely discarded. The first is more interesting as we recognise the spatial integral of zero modes as defining the moduli space metric (3.14), so that

$$S_X = -\frac{1}{2g^2} \int dt\, g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu. \tag{3.31}$$

We've shown that the moduli space approximation to slow-moving soliton dynamics is just geodesic motion on the soliton moduli space [66]. In particular, this reproduces the bosonic part of our σ-model (2.7). In the following section we'll see how supersymmetry can be incorporated from a similar perspective.

3.4 Instantons in Supersymmetry and String Theory

In this section we give a brief review of the properties of supersymmetric instantons, and their appearance and construction in string theory. Our intention is not to give a thorough account, but simply to introduce the key ideas we use in applications to discrete light-cone quantisation in section 4. Instanton solutions also exist in supersymmetric Yang-Mills theories, and are rather special in that they preserve a large amount of supersymmetry. These theories appear on the worldvolume of branes in string theory, and this perspective gives a simple alternative derivation of the ADHM construction.

To begin with, consider $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in four dimensions [77, 78] (see [67, 79] for conventions). As well as the Yang-Mills field this contains a pair of Weyl fermions $\lambda, \bar{\lambda}$ (which are not conjugate in the Euclidean theory) in the adjoint representation of $SU(N)$. The action is

$$S_{SYM} = \frac{1}{g^2} \int d^4x \,\mathrm{tr} \left\{ \frac{1}{2} F_{mn} F_{mn} - i \bar{\lambda} \bar{\not{D}} \lambda \right\}.$$

The equations of motion are

$$\begin{aligned} D_m F_{mn} &= \bar{\lambda} \bar{\sigma} \lambda \\ \bar{\not{D}} \lambda &= 0 \\ \not{D} \bar{\lambda} &= 0, \end{aligned}$$

which are solved by a bosonic instanton with $\lambda = \bar{\lambda} = 0$. The supersymmetry transformations of F are proportional to $\lambda, \bar{\lambda}$ so vanish automatically, but those of the fermions are more interesting

$$\delta\lambda = 2F_{mn}\sigma_{mn}\epsilon, \qquad \delta\bar{\lambda} = 2F_{mn}\bar{\sigma}_{mn}\bar{\epsilon}, \tag{3.32}$$

where $\epsilon, \bar{\epsilon}$ are independent Weyl spinor parameters, σ_{mn} is a self-dual $SU(2)$ generator and $\bar{\sigma}_{mn}$ is anti-self-dual. Consequently, if F is self-dual then the $\bar{\epsilon}$ transformations vanish and the instanton solution preserves half the supersymmetries. This is known as a *1/2-BPS* configuration.[11] The ϵ transformations result in nonzero values for the fermionic fields satisfying the covariant Dirac equation in the instanton background, so can be switched on while maintaining an exact classical solution. Indeed, a full and careful treatment of these zero modes (for which we refer the reader to [67]) leads to a fully supersymmetric formulation of the ADHM construction.

To get an $\mathcal{N} = (4, 4)$ supersymmetric version of the moduli space approximation in section 3.3 we need to work with a five-dimensional theory with an $SU(N)$ Yang-Mills part, so it has instanton solutions, and sixteen supercharges so that these instantons preserve eight. The unique choice is five-dimensional maximally supersymmetric Yang-Mills theory (*MSYM*). This has $\mathcal{N} = 2$ supersymmetry and $SO(5)$ R-symmetry, with sixteen fermions in the $\mathbf{4} \otimes \mathbf{4}$ of $SO(4, 1) \times SO(5)$ and

[11] More generally, a configuration is called $1/n$-BPS if it preserves 1 nth of the supersymmetries.

five scalars in the **5** of $SO(5)$, all of which transform in the adjoint of $SU(N)$. The minimal spinor in 4+1 dimensions is a pseudoreal four-complex-component Dirac spinor (see [80] for a comprehensive review of spinors in diverse dimensions). As such, the supercharges can be written as Q_α^A, where $A = 1, \ldots, 4$ is a fundamental index for $USP(4) \sim SO(5)$ and $\alpha = 1, \ldots, 4$ is a spinor index for $SO(4,1)$. Pseudoreality means that

$$\left(Q_\alpha^A\right)^\dagger := \bar{Q}_A^\alpha = C^{\alpha\beta}\Omega_{AB}Q_\beta^A, \tag{3.33}$$

where $C^{\alpha\beta}$ and Ω_{AB} are antisymmetric invariant tensors for $SO(4,1)$ and $SO(5)$ respectively. The tensor product of two pseudoreal representations is real, so in total there are sixteen real supercharges. Though it is beyond the scope of this thesis, a supersymmetric formulation of the ADHM construction makes it possible to apply the techniques of the moduli space approximation of section 3.3 to this maximally supersymmetric case. This is dealt with comprehesively in [67], and the outcome is that the effective action (3.31) is modified to the $\mathcal{N} = (4,4)$ supersymmetric version (2.7). That is, the slow motion of instanton-solitons in maximally supersymmetric Yang-Mills theory in five dimensions is governed by (2.7) with target space $\mathcal{M}_{k,N}$.

Poincaré supersymmetry algebras with more than one spinor supercharge often admit central extensions, meaning that conserved charges commuting with all other generators can appear on the right hand side of the supersymmetry algebra. For five-dimensional MSYM with a semisimple gauge group in a sector with zero VEVs for the adjoint scalar fields, the only such central charge is the topological $U(1)$ charge (3.28) corresponding to instanton number. Explicitly, we have

$$\left\{Q_\alpha^A, Q_\beta^B\right\} = -iP_M\left(\Gamma^M C\right)_{\alpha\beta}\Omega^{AB} + \frac{8\pi^2}{g^2}kC_{\alpha\beta}\Omega^{AB},$$

where Γ^M represent the $SO(4,1)$ Clifford algebra $\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN}$ and $k \in \mathbb{Z}$. Standard manipulations using $C^2 = \Omega^2 = -\mathbb{1}_4$ and the pseudoreality condition (3.33) give

$$\left\{Q_\alpha^A, (Q_\beta^B)^\dagger\right\} = \delta_B^A\left(m\Gamma^0 + i\frac{8\pi^2}{g^2}k\mathbb{1}_4\right)_\alpha^{\ \beta},$$

where we work in the rest frame of an on-shell state with mass m given by $P_M P^M = -m^2$. We also choose conventions so that

$$\Gamma^0 = -i\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.$$

For $\alpha = \beta,\ A = B$, the above anticommutator is a sum of terms of the form $MM^\dagger$ so must be positive definite. Thus

$$m \geq \frac{8\pi^2}{g^2}|k|,$$

with equality if and only if half the anticommutators vanish. But we have

$$\left\{Q^A_\alpha, (Q^A_\alpha)^\dagger\right\} = 0 \Leftrightarrow \left(Q^A_\alpha \pm (Q^A_\alpha)^\dagger\right)^2 = 0,$$

which occurs if and only if Q^A_α annihilates the state. This is an alternative derivation of the fact that an instanton is 1/2-BPS, and is a special case of a more general phenomenon where BPS states saturate a lower bound for the mass of a state with given charges[12]. We will see much more of this when we come to discuss superconformal algebras. The particular instance of a relation between BPS bounds and Bogomol'nyi bounds for soliton masses was first described in [43].

To close this section, we briefly describe how the preceding discussion fits inside string theory. There are extended $(p+1)$-dimensional objects, called *Dp-branes*, which are surfaces on which open fundamental strings can end. These should be thought of as dynamical objects, carrying field content dictated by the spectrum of open string theory with ends on the brane [81]. Furthermore, in type II superstring theories these objects preserve 16 of 32 supersymmetries and their dynamical content must respect this supersymmetry. At energies much less than the Planck mass m_{pl} and the string mass $m_s = 1/\sqrt{\alpha'}$, the field theory describing a coincident stack of N Dp-branes is maximally supersymmetric Yang-Mills theory in $(p+1)$-dimensions with Chern-Simons-like couplings to various Ramond-Ramond forms [82, 83]. In particular, there is a 1-form C_1 which is naturally sourced by D0-branes and which couples to D4-branes in type IIA via the topological current (3.28), so we consider the action

$$S_{D4} = S_{MSYM} + c\int_{D4} \mathrm{tr}\left\{F\wedge F\wedge C_1\right\},$$

where c is a coupling constant and F is the Yang-Mills field strength. The current (3.28) is the instanton number density, so the above coupling suggests a relation between the presence of D0-branes and instantons in five-dimensional MSYM. In fact, the relation is very precise [44, 45]: a stack of N coincident D4-branes with k D0-branes 'dissolved' in their worldvolume is described at energies $E << m_{pl},\ m_s$ by the dynamics of k instantons of $SU(N)$.

It is particularly illuminating to view this from the perspective of the D0-branes. In a suitable gauge their worldvolume is the time coordinate, so the field theory on their worldvolume is a supersymmetric quantum mechanics, with eight supercharges as dictated by the combined D0-D4 system. Open strings with both ends on a D0-brane contribute the field content of (0+1)-dimensional MSYM, namely a $U(k)$ gauge field along with nine real scalars and sixteen real fermions in the adjoint of $U(k)$. Under the breaking of $16 \to 8$ supersymmetries, these split into a vector

[12]See [79] for a complete analysis in the case of 4d Poincaré supersymmetry

multiplet containing the gauge field, eight fermions and five scalars, and a hypermultiplet with eight fermions and four scalars. There are also open strings with one end on a D0-brane and the other on a D4. These contribute hypermultiplets in the $(\mathbf{k}, \bar{\mathbf{N}}) \oplus (\bar{\mathbf{k}}, \mathbf{N})$ of $U(K) \times SU(N)$. From the (0+1)-dimensional perspective, the effective Yang-Mills coupling of a D4-brane is

$$\frac{1}{g_0^2} = \frac{V}{g_4^2},$$

where V is the volume of the D4-branes transverse to the D0-branes. Thus, in the infinite volume case of interest, excitations of the $SU(N)$ gauge field have infinite energy. This freezes their degrees of freedom and leaves an $SU(N)$ global symmetry in the D0-brane worldvolume theory.

If the D0-branes are supposed to correspond to instantons in the D4-brane field theory, then their vacuum configurations should describe static instantons. That is, we expect to find a correspondence between vacua of the D0-brane quantum mechanics and the moduli space $\mathcal{M}_{k,N}$. More precisely, we need to consider vacua for which the D0-branes remain 'dissolved' in the D4-branes, since any other situation cannot be interpreted as a state in the worldvolume gauge theory [44]. The scalars of the D0-brane quantum mechanics correspond to transverse fluctuations of the D0-brane worldvolume, and hence to motion of the D0-branes in space [84]. The five scalars of the vector multiplet are motions in the five dimensions transverse to the D4-branes, so we must take zero VEV for these scalars, while the four scalars of the hypermultiplet are motions in the four spatial dimensions of the D4-branes, so these can take nonzero VEVs. Generically, the situation where hypermultiplet scalars take nonzero VEVS but vector multiplet scalars do not is called the *Higgs branch* of an eight supercharge gauge theory, while the opposite situation is called the *Coulomb branch*. We are therefore interested in the Higgs branch vacua of the D0-brane gauge theory.

It is here that one of the many miracles of string theory occurs. Recall that the hypermultiplet matter content consists of a total of $4k^2$ scalars from the adjoint hypermultiplet and $4kN$ scalars from the (anti)fundamental hypermultiplets. We might choose to group the adjoint scalars into complex $k \times k$ matrices $X, \tilde{X}$ and the fundamental scalars into $k \times N$ and $N \times k$ complex matrices $q_i, \tilde{q}^i : i = 1, \ldots, N$. The condition that the potential for these scalars vanishes translates into the equations [44, 45]

$$\begin{aligned} \left[X, X^\dagger\right] - \left[\tilde{X}, \tilde{X}^\dagger\right] + q_i q^{\dagger i} - \tilde{q}_i^\dagger \tilde{q}^i &= 0 \\ \left[X, \tilde{X}\right] + q_i \tilde{q}^i &= 0. \end{aligned}$$

But these are nothing but the ADHM equations (3.19)! In fact, fermions can be included in this picture to arrive at the supersymmetric ADHM construction as described in [67]. We've seen that the Higgs branch vacuum equations, together with a quotient by the gauge group $U(k)$ to restrict to gauge-invariant physical vacuum states, precisely reproduces the ADHM construction of instantons. In fact, it's not unreasonable to view this stringy picture as a physical derivation of ADHM. Furthermore, it follows from similar considerations that the low-energy dynamics of D0-branes inside D4-branes are described by the supersymmetric instanton moduli space dynamics. In particular, this gives an alternative proof that $\mathcal{M}_{k,N}$ is hyper-Kähler, since we know from section 2 that $\mathcal{N} = (4, 4)$ supersymmetric quantum mechanics requires a hyper-Kähler target. This string theoretic picture of instanton dynamics will be crucial for the applications to discrete light-cone quantisation which we discuss in the following section.

4 Application to Discrete Light-Cone Quantisation

In this section we review *discrete light-cone quantisation* (*DLCQ*) [32], a major application area of our superconformal quantum mechanical models. The central idea of DLCQ is the compactification of a relativistic field theory on a null circle. The system splits as usual into a tower of Kaluza-Klein modes, which unusually have strictly positive null momentum. We'll briefly describe how, when evolved according to light-cone 'time', this results in a non-relativistic theory with a finite number of degrees of freedom for each KK mode. In the superconformal case the null compactification preserves a non-trivial superconformal subalgebra provided that the dilatation is augmented by a boost to fix the size of the null circle, and we'll review a general recipe to identify this subalgebra.

DLCQ of a general field theory has a number of issues which limit its utility [33, 34], but in maximally supersymmetric theories many of these can be circumvented. In particular, one can use the matrix formulation of M-theory [35,46] to give precise prescriptions for the quantum mechanical models arising from DLCQ of the six-dimensional (2,0) theory [36, 40] and from four-dimensional $\mathcal{N} = 4$ SUSY Yang-Mills [37–39]. We briefly review these constructions, with an emphasis on the final results as a starting point for our quantum mechanics.

4.1 The Basics of DLCQ

Consider a relativistic field theory in d spacetime dimensions $x^M : M = 0, \ldots, d-1$. One can choose light-cone coordinates

$$x_\pm = \frac{1}{\sqrt{2}}(x_0 \pm x_1)$$

with respect to which the Minkowski metric is

$$ds^2 = -2dx_+dx_- + dx^a dx^a,$$

where $x^a : a = 2, \ldots, d-1$ are spatial coordinates transverse to the light-cone. In the spirit of [85], we treat x_+ as light-cone 'time' and view the corresponding momentum $P_- = P_0 - P_1$ as the light-cone Hamiltonian. The basic idea of DLCQ is to compactify the null direction x_- on a circle of radius R_- [32]

$$(x_+, x_-, x^a) \sim (x_+, x_- + 2\pi R_-, x^a).$$

As one might imagine, such a manoeuvre has highly non-trivial effects on a field theory. To see how this works, consider the simple example[13] of a massless free scalar field ϕ with Lagrangian

$$\mathscr{L} = -\frac{1}{2}\partial_M\phi\partial^M\phi = \partial_+\phi\partial_-\phi - \frac{1}{2}\partial^a\phi\partial^a\phi.$$

We can expand the field in Fourier modes

$$\phi(x_+, x_-, x^a) = \sum_{k=-\infty}^{\infty} \phi_k(x_+, x^a)e^{ikx_-/R_-} \qquad\qquad \phi_k = \phi^*_{-k}$$

to obtain the equivalent Lagrangian

$$\mathscr{L} = \sum_{k,l} e^{i(k+l)x_-/R_-}\left(\frac{ik}{R_-}\phi_k\partial_+\phi_l - \frac{1}{2}\partial^a\phi_k\partial^a\phi_l\right).$$

The above Lagrangian is integrated d^dx to obtain an action. The integral over x_- is trivial using Fourier orthogonality, resulting in

$$\mathscr{L}' = \sum_k \left(\frac{ik}{R_-}\phi_k\partial_+\phi^*_k - \frac{1}{2}|\partial^a\phi_k|^2\right).$$

The above sum runs over all k, but using integration by parts with respect to x_+ it can easily be rewritten as

$$\mathscr{L}' = -\frac{1}{2}|\partial^a\phi_0|^2 + \sum_{k=1}^{\infty}\left(\frac{ik}{R_-}\phi^*_k\partial_+\phi_k - \frac{1}{2}|\partial^a\phi_k|^2\right).$$

This makes clear that the dynamical degrees of freedom are the modes with strictly positive light-like momentum. Furthermore, the kinetic terms have the characteristic form of a non-relativistic field theory[14] with mass

$$m_k = \frac{k}{R_-}$$

for the kth Kaluza-Klein mode. Since the null momentum P_+ is positive, it plays the role of a conserved 'particle number', and the system with fixed k may be treated using ordinary many-body quantum mechanics and the Schrödinger equation. Observe that the equations of motion for ϕ_k take the form of a Schrödinger equation

$$i\partial_+\phi_k = -\frac{1}{2m_k}\partial_a\partial^a\phi_k,$$

so the model is symmetric under the *Schrödinger group* [87]. We will discuss this in more detail in the context of superconformal field theory.

[13] Interacting examples are treated explicitly in e.g [32–34].

[14] The use of non-relativistic field theory in many-body physics is covered in [86].

Many of the conclusions of the above example can be generalised by considering the 1-particle on-shell condition $P_M P^M = -m^2$, which can be rewritten (for $P_+ \neq 0$) as

$$P_- = \frac{m^2 + (P^a)^2}{2P_+}.$$

In particular, the light-cone energy P_- is positive definite if and only if the null momentum P_+ is. Furthermore, single-valuedness of the wavefunction requires

$$P_+ = \frac{k}{R_-} = m_k \qquad k = 0, 1, \ldots.$$

We then have

$$P_- = \frac{m^2 + (P^a)^2}{2m_k}$$

which, in the case of a particle which is massless in the d-dimensional sense, is the usual non-relativistic free particle dispersion relation. It follows that the DLCQ of a massless theory is described by non-relativistic field theory, and hence by a many-body Schrödinger equation in the sector with conserved 'particle number' k [32].

We've seen that DLCQ appears to lead to a number of impressive simplifications of field theory. The reduction to a non-relativistic system with a finite number of degrees of freedom is in principle a vast improvement. Furthermore, one can show that the Fock vacuum coincides with the exact interacting vacuum for the light-cone Hamiltonian P_-. Finally, there is reason to believe that the original field theory in the so-called *infinite momentum frame* [88] can be recovered by taking $k \to \infty$ [32, 35]. Taken at face value, these statements offer the possibility of solving field theories non-perturbatively by considering finite-dimensional quantum mechanics! Of course, there are caveats to this bold statement. In the above analysis we largely neglected the modes of zero null momentum, but in general this is completely disallowed [34]. In an interacting theory, the zero modes are described by a field theory in $d-1$ dimensions and Feynman graphs in which they enter are generically all divergent, as a symptom of an infinitely strong effective coupling in the lower-dimensional theory. Even if this difficulty can be avoided, it is in general a very hard problem to determine the effective interactions in the quantum mechanics with k units of null momentum. Fortunately, for the maximally supersymmetric theories we study in sections 4.2 and 4.3, we'll see that supersymmetry allows us to circumvent these issues and give a precise, well-defined description in terms of a free superparticle moving on an appropriate instanton moduli space.

For now, we examine in more detail what DLCQ does to the symmetries of a theory, focusing on the superconformal case. Our analysis is based on that of [40], and we refer to [12] for a review of (super)conformal field theory. Since we impose periodic boundary conditions for all fields on the null circle, symmetries which do not act on spacetime are automatically preserved, so R-symmetries and global symmetries go through untouched. As for conformal and supersymmetries, we require that they preserve the constraint $x_- \sim x_- + 2\pi R_-$. Since none of the symmetries in question act periodically, we instead say that a symmetry generated by A is preserved if and only if $[A, P_+] = 0$. To analyse this condition we need an explicit presentation of the conformal algebra $\mathfrak{so}(d,2)$. There are $\frac{1}{2}(d+1)(d+2)$ generators given by the antisymmetric objects $M_{ij} : i, j = 0, \ldots, d+1$, satisfying

$$[M_{ij}, M_{kl}] = i\left(\eta_{ik}M_{jl} - \eta_{il}M_{jk} + \eta_{jl}M_{ik} - \eta_{jk}M_{il}\right)$$

with $\eta = \mathrm{diag}(-1, 1, \ldots, 1, -1)$. One obtains the usual presentation in terms of Lorentz generators M_{MN}, momenta P_M, special conformal generators K_M, and dilatation[15] T by defining

$$P_M = M_{d,M} + M_{d+1,M}, \qquad K_M = M_{d,M} - M_{d+1,M}, \qquad T = M_{d,d+1}.$$

It is immediate that all momenta P_M preserve the DLCQ condition, and the only special conformal generator which does so is

$$K_+ = K_0 + K_1.$$

Transverse rotations M_{ab} also clearly work. Less obvious are the *DLCQ dimension*

$$D = T + M_{01}, \tag{4.1}$$

which combines a longitudinal boost with a scale transformation to preserve the size of the null circle, and the transverse *Galilean boosts*

$$V_a = M_{0a} + M_{1a}.$$

The collection of generators $\mathfrak{g} = \{P_\pm,\ K_+,\ D,\ M_{ab},\ P_a,\ V_a\}$ forms the Schrödinger group. This is the symmetry group of the Schrödinger equation [87] and coincides with what we saw in the case of a free massless scalar. One can check that the transverse momenta P_a, boosts V_a and null momentum P_+ together form an ideal I, and we have

$$\frac{\mathfrak{g}}{I} \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(d-2),$$

where $\mathfrak{so}(d-2)$ consists of the transverse rotations and $\mathfrak{so}(2,1) = \{P_-,\ K_+,\ D\}$ with

$$[D, K_+] = -2iK_+, \qquad [D, P_-] = 2iP_-, \qquad [P_-, K_+] = 4iD.$$

[15] We reserve the standard notation D for the DLCQ dimension (4.1).

By setting $K = -\frac{1}{2}K_+$, $H = \frac{1}{2}P_-$ we get $\mathfrak{so}(2,1)$ in the form

$$[D, H] = 2iH, \qquad [D, K] = -2iK, \qquad [H, K] = -iD$$

used in the remainder of the text.

One might wonder why we have chosen to keep the transverse rotations M_{ab} rather than view them as part of the ideal I. The motivation for this comes from the applications of sections 4.2 and 4.3. In the context of instanton moduli space, it turns out that the generators of I only act non-trivially on the decoupled part of moduli space corresponding to the instanton centre of mass (see section 3), while those of $\mathfrak{so}(d-2)$ are more interesting [40]. Put another way, a subgroup of $\mathfrak{so}(d-2)$ will turn out to be an R-symmetry while I always corresponds to a global symmetry. It is therefore more important to treat $\mathfrak{so}(d-2)$ explicitly, though it should always be remembered that the additional boosts and translations are present.

We now turn to supersymmetry. This is fiddly to handle in generality since the details of spinor algebra are so dimension-dependent (see [80] for a review), so we consider the four-dimensional case. Others are analogous, and the six-dimensional case is done explicitly in [40]. We work with spinors of the conformal group $SO(4,2)$, so a spinor q_α contains both super-Poincaré and superconformal charges. α is a spinor index for $SO(4,2)$ and we suppress R-symmetry as it does not enter the analysis. We have

$$[M_{ij}, q_\alpha] = (\Gamma_{ij})_\alpha^{\ \beta} q_\beta,$$

where $\Gamma_{ij} \sim [\Gamma_i, \Gamma_j]$ are the spin generators of $SO(4,2)$ and Γ_i obey a Clifford algebra. It follows that

$$[P_+, q_\alpha] \sim (\Gamma_{40} + \Gamma_{50} + \Gamma_{41} + \Gamma_{51})_\alpha^{\ \beta} q_\beta := \Gamma_\alpha^{\ \beta} q_\beta,$$

so the preserved supercharges are those annihilated by Γ. One can pick a basis for the Clifford algebra with

$$\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}_2, \qquad \Gamma_4 = \gamma_5 \otimes \sigma_1, \qquad \Gamma_5 = \gamma_5 \otimes i\sigma_2,$$

where γ_μ are the usual gamma matrices for $SO(3,1)$ in the Weyl basis and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Then

$$\Gamma \propto \begin{pmatrix} 0 & 1+\sigma_1 \\ 1-\sigma_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

Each factor has half rank, so in total 3/4 of fermions are preserved by DLCQ. To identify them, we calculate the dilatation generator

$$\Gamma_{45} = \mathbb{1}_4 \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the light-cone boost

$$\Gamma_{01} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \otimes \mathbb{1}_2.$$

Those generators annihilated by both factors in (4.2) have DLCQ dimension zero, and have positive field theory dimension so are Poincaré supercharges. These cannot fit into a conventional superconformal algebra, and the Jacobi identity implies that their anticommutators with each other and with other fermionic charges generate the ideal I. Indeed, the full superalgebra preserved by DLCQ has an ideal I' consisting of I and these 'extra' supersymmetries. The supercharges annihilated only by the first factor of (4.2) have negative light-cone dimension and those annihilated only by the second have positive, so factoring out I' leaves a non-trivial superconformal algebra with half as many Poincaré and superconformal symmetries as the original spacetime algebra. In the following sections we'll describe this algebra in more detail for specific field theories.

4.2 DLCQ of the Six-Dimensional (2,0) Theory

The (2,0) theory of type A_{N-1} arises as the worldvolume description of a stack of N M5-branes in M theory at energy scales much less than the Planck mass [9]. Much like M theory itself, the (2,0) theory is intrinsically strongly coupled, has no known local Lagrangian description, and is extremely hard to study directly. Apart from some recent progress based on very general principles of superconformal field theory [89, 90], most results have been obtained via string dualities or the AdS/CFT correspondence [19, 52], which map (2,0) to better understood supergravity or field theories in certain limits. Fortunately, string dualities also allow for a precise description of the quantum mechanics arising from DLCQ [36].

We begin with the *BFSS matrix model* description of M theory. This asserts that the large k limit of the quantum mechanics describing k coincident D0-branes in type IIA superstring theory provides a light-cone Hamiltonian for M theory in the infinite momentum frame [35]. Furthermore, the same model at finite k is supposed to describe the DLCQ of M theory with k units of null momentum [46]. One can motivate this claim using the arguments of [91]. The basic idea is that compactification on an almost light-like circle of radius R_- can be related by a large Lorentz boost to compactification on a spatial circle of small radius R_s. In the limit where the almost light-like circle becomes null, the boost rapidity becomes infinite, and $R_s \to 0$, this should be equivalent to DLCQ. A full discussion of this equivalence can be found in [34]. More precisely, one needs to use this limit to identify scales and coupling constants between M theory on a null circle of radius R_-

and on a spatial circle of radius $R_s \to 0$. Since M theory on a spatial circle is equivalent to type IIA superstring theory [92, 93], this boils down to an identification of the string coupling g_s and mass scale m_s in terms of parameters of the original null-compactified M theory [91]

$$\begin{aligned} g_s &= (R_s R_- m_{pl}^2)^{3/4} \\ m_s &= R_s^{-1/4}(R_- m_{pl}^2)^{3/4} \\ R_s \tilde{m}_{pl}^2 &= R_- m_{pl}^2, \end{aligned}$$

where $\tilde{m}_{pl}$ is the Planck mass in the spatially compactified M theory. As $R_s \to 0$ we find that the string coupling becomes small while the Planck mass and the string mass become large. To complete the identification with D0-branes, we observe that the Ramond-Ramond 1-form of type IIA corresponds to the Kaluza-Klein photon coming from the reduction $11 \to 10$ dimensions [94,95]. It follows that a sector carrying k units of momentum in the compact direction corresponds to k units of Ramond-Ramond charge. But D0-branes are the only objects sourcing this Ramond-Ramond charge in type IIA string theory [82], so this must be a sector with k D0-branes. At energies well below the Planck and string masses, all gravitational and 'stringy' degrees of freedom decouple, leaving maximally supersymmetric (0+1)-dimensional Yang-Mills theory as described in section 3.4.

Now add N M5-branes to the picture. In the above construction, the M5-branes are taken to wrap the spatial circle and are described from the type IIA perspective by D4-branes. The Yang-Mills coupling is $g_4^2 = 8\pi^2 R_s$ [96], so in the sector with k units of compact momentum P_s we have

$$P_s = \frac{k}{R_s} = \frac{8\pi^2 k}{g_4^2} = m_{inst}.$$

That is, we expect the Kaluza-Klein modes to correspond to the instanton particles of section 3. In fact, we are exactly in the situation of D0-D4 quantum mechanics described in section 3.4 provided that we go to the D0-brane Higgs branch. Intuitively, this should occur because the new degrees of freedom from the D4-branes correspond to the Higgs branch, while the Coulomb branch is already present with only D0s. More explicitly, observe that the $U(k)$ gauge coupling is [82, 97]

$$g_{QM}^2 = \frac{1}{2\pi^2} l_s^{-3} g_s = \frac{1}{2\pi^2}(R_- m_{pl}^2)^3.$$

To decouple gravitational degrees of freedom and restrict the full M theory picture to the M5-brane worldvolume theory, we must take $m_{pl} \to \infty$, from which we have $g_{QM}^2 \to \infty$. If one writes out the D0-brane worldvolume quantum mechanics explicitly, one finds that this gives a large effective mass to Coulomb branch degrees of freedom while imposing the ADHM equations as a strict constraint. We conclude that the DLCQ of the (2,0) theory of type A_{N-1} in a sector with k units of null momentum is described by quantum mechanics on $\mathcal{M}_{k,N}$.

The reader might wonder how the issue of zero modes in DLCQ is resolved in this case. States carrying zero momentum around the circle are precisely the massless states of the lower-dimensional field theory, in this case five-dimensional MSYM. These states certainly interact with instantons and we might expect them to spoil our conclusions. Fortunately, supersymmetry dictates that the two-derivative effective action provided by the $\mathcal{N} = (4,4)$ supersymmetric σ-model on instanton moduli space cannot be renormalised [27, 98]. That we can ignore higher derivative corrections is guaranteed by the $g_{QM}^2 \to \infty$ limit, which is the quantum mechanical analogue of a low-energy limit and suppresses any higher derivative contributions. The conclusion is that the zero mode contributions are so tightly constrained that they pose no risk to our conclusions.

We now analyse the symmetries of this model. The (2,0) theory is superconformal, with $SO(6,2)$ conformal group, $SO(5)$ R-symmetry coming from rotations in the five dimensions transverse to the M5-branes, sixteen Poincaré supercharges and sixteen superconformal charges [9, 99]. These fit together to form the simple superalgebra $\mathfrak{osp}(6,2|4)$ (see appendix D for a review of superalgebras). The analysis of section 4.1 is done explicitly for this case in [40]. The outcome is a model with $SO(2,1)$ conformal group, $SO(5)\times SU(2)$ R-symmetry, eight supercharges and eight superconformal charges, combining to form the simple superalgebra $\mathfrak{osp}(4^*|4)$. An explicit presentation of $\mathfrak{osp}(4^*|4)$ is given in appendix D. The $SU(2)$ R-symmetry is a subgroup of the $SO(4)$ rotational symmetry of the transverse dimensions inside the D4-branes. In addition to $\mathfrak{osp}(4^*|4)$, the ideal I', in which the other $SU(2) \subset SO(4)$ acts as an R-symmetry for the 'extra' supersymmetries, and the global $SU(N)$ are preserved. In section 5 we prove that a large class of quantum mechanical σ-models are $\mathfrak{osp}(4^*|4)$-invariant, and use the ADHM construction to show directly that instanton quantum mechanics does indeed fit within this class.

Finally, we address the question of what this DLCQ description might tell us about the (2,0) theory. Our advantage here is that, while almost nothing is known directly about (2,0), the DLCQ model is entirely concrete. However, metrics on instanton moduli spaces are not known beyond some simple cases, so it appears too much to hope to solve the model. A more accessible task is to compute the spectrum of BPS states in DLCQ. As mentioned in section 3.4 and examined in detail in section 7, these are states annihilated by some fraction of the supercharges of a model, which in turn implies that their scaling dimension (in the superconformal case) is determined by their R-charges. This property offers some protection against quantum corrections and improves computability: since R-charges are integers, the dimension of a BPS state cannot vary continuously so may be computed in a convenient limit. Despite this, making progress with the (2,0) theory is sufficiently difficult that a full solution is not yet known, though partial results have been obtained using indices [100–103] and AdS/CFT [54, 104–106]. We expect the BPS state problem to be accessible to DLCQ. Since supercharges are carried over directly from $\mathfrak{osp}(6,2|4)$ to $\mathfrak{osp}(4^*|4)$, there should be a simple relation between states annihilated by a supercharge in each instance. In section 7 we give some initial results in this direction, by classifying all irreducible unitary representations of $\mathfrak{osp}(4^*|4)$ (those of $\mathfrak{osp}(6,2|4)$ having been classified in [107, 108]), describing their BPS content and identifying this with a form of cohomology of instanton moduli space. We then characterise the superconformal index of $\mathfrak{osp}(4^*|4)$, which contains in some sense the most information about a theory that can be deduced purely from knowing it has $\mathfrak{osp}(4^*|4)$ invariance.

4.3 DLCQ of Four-Dimensional $\mathcal{N} = 4$ SUSY Yang-Mills

Our second field theory of interest is maximally supersymmetric $SU(N)$ Yang-Mills theory in four dimensions. This arises as the worldvolume theory of N coincident D3-branes in type IIB superstring theory at energies much less than the Planck and string masses. In contrast to the (2,0) theory its action is very well known, and has a single dimensionless free parameter, the complexified Yang-Mills coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}.$$

Here g is the ordinary Yang-Mills coupling and θ is the topological θ-angle. $\mathcal{N} = 4$ SUSY Yang-Mills is known to be UV finite [25, 109], admits exact $SL(2;\mathbb{Z})$ electromagnetic duality [8, 110], has a well-understood AdS dual [19], and is believed to be integrable in the planar limit $N \to \infty$ [22]. We will therefore need to say something quite precise in DLCQ to provide any new information.

The DLCQ setup for $\mathcal{N} = 4$ SUSY Yang-Mills is rather similar to that of the (2,0) theory. This follows from the fact that the (2,0) theory compactified on a torus T^2_τ of complex structure τ, in the limit where the area $A(T^2_\tau) \to 0$, is described by $\mathcal{N} = 4$ SUSY Yang-Mills with coupling constant τ [8]. We therefore consider the (2,0) theory compactified on T^2_τ times a null circle and run the same argument as in the previous section, resulting in a system of k D0-branes inside N D4-branes whose worldvolume wraps T^2_τ. The Higgs branch of this system, and hence the DLCQ model, is described by quantum mechanics on the moduli space of k $SU(N)$ Yang-Mills instantons on $\mathbb{R}^2 \times T^2_\tau$ as $A(T^2_\tau) \to 0$ [37, 38].

As it stands, this prescription has the same weakness as the (2,0) DLCQ, in that explicit instanton moduli space metrics are hard to come by. Fortunately, in this case string dualities come to the rescue and allow a completely explicit description. First one takes a T-duality along T^2_τ, resulting in a system of k D2-branes wrapping the dual torus $\tilde{T}^2_\tau$ and N D2'-branes localised on $\tilde{T}^2_\tau$. The dual torus has $A(\tilde{T}^2_\tau) \propto 1/A(T^2_\tau)$ and a complex structure which is equivalent to τ under $SL(2,\mathbb{Z})$. The Higgs branch of the D2-D2' system, which should be isomorphic to the original instanton moduli space, is given by the moduli space of solutions to Hitchin's equations on $\tilde{T}^2_\tau$ with localised impurities [37]. Next we apply three-dimensional mirror symmetry [111], which equates the Higgs branch of one eight supercharge theory in three dimensions to the Coulomb branch of another dual theory. The low-energy dynamics on the D2-branes wrapping $\tilde{T}_\tau$ is effectively one-dimensional, so to apply mirror symmetry we take advantage of the fact that Higgs branches of eight-supercharge theories are dimension-independent to promote the D2-D2' system to a D4-D4' system with a (2+1)-dimensional non-compact intersection. Dimension-independence follows from a combination of two facts: the field content of hypermultiplets is dimension-independent, as are the non-renormalisation results of [98]. The procedure to obtain the dual can be understood via further string dualities [112] but for us the important thing is the result of [39].

To state this result, we start with an auxiliary four-dimensional $\mathcal{N} = 2$ gauge theory. More precisely, we consider the $\hat{A}_{N-1}$ elliptic quiver theory [113], whose quiver diagram is the Dynkin diagram for the affine Lie algebra $\hat{A}_{N-1}$ (see [114]). In practice, this means that we have an $\mathcal{N} = 2$ vector multiplet with gauge group[16]

$$G = U(1) \times \prod_{j=1}^{N} SU(k)_j, \tag{4.3}$$

along with hypermultiplets in the $(\bar{\mathbf{k}}, \mathbf{k})$ for each pair of adjacent factors $SU(k)_j \times SU(k)_{j+1} : j = 1, \ldots, N$ with $N+1 \equiv 1$. This theory is superconformal, with complexified $U(1)$ gauge coupling τ and $SU(k)$ couplings determined by Wilson lines wrapping T^2_τ in the DLCQ model (see [2] for a review of the details). The rank of the gauge group (4.3) is $g = kN - N + 1$ so, as we review in section 6.1, the theory has a Coulomb branch of complex dimension g whose metric is determined by a single holomorphic function which is known explicitly [15, 16, 52].

The DLCQ quantum mechanics has target space given by the Coulomb branch of the above theory defined on $\mathbb{R}^3 \times S^1_R$, where the radius R of S^1_R is proportional to $A(\tilde{T}^2_\tau)$ [39]. This Coulomb

[16] The system naïvely has gauge group $U(k)^N$ but the off-diagonal $U(1)$ factors freeze out [52].

branch is a torus bundle over the Coulomb branch of the uncompactified theory, and is a hyper-Kähler manifold of real dimension $4g$. In the limit $R \to \infty$ relevant to the DLCQ description of $\mathcal{N} = 4$ SUSY Yang-Mills, the torus fibres become small and the metric approaches a simple *semi-flat* form [17, 51] which is determined by the same known holomorphic function as is the base space metric. That is, we have a description of the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills which is completely explicit in that all geometric information about the target space is known.

The above discussion sounds extremely promising from the point of view of saying something precise about $\mathcal{N} = 4$ SUSY Yang-Mills. After all, a finite-dimensional quantum mechanical model with explicitly known Hamiltonian is about as tractable problem as one can hope to have in field theory. Ambitiously, we hope that this model can be used to directly attack the spectral problem for the field-theoretic dilatation operator, and to say something about the origin of the mysterious planar integrability. A less ambitious goal, which we fulfil in this thesis, is to understand the symmetries of the DLCQ model. The analysis of section 4.1 implies that the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills should have an $SO(2,1)$ conformal group along with $SO(6) \times U(1)$ R-symmetry, eight Poincaré supercharges and eight supeconformal charges, which combine together to form the simple superalgebra $\mathfrak{su}(1,1|4)$. In contrast to the case of the (2,0) theory, it is far from straightforward to show that the quantum mechanics described above actually has these symmetries. In particular, the (2,0) theory compactified on a torus of finite size $\propto R^{-1}$ is not conformal, and the R-symmetry is limited to the manifest $SO(5)$. Both of these issues must somehow resolve in the $R \to \infty$ limit, and we show in section 6 that this is indeed the case, giving an explicit construction of the full symmetry algebra at finite[17] R and showing how it enhances as $R \to \infty$. Even with these expressions to hand, the spectral problem for the dilatation operator is not obviously well-defined, and in section 6.5 we give some initial results establishing an alternative spectral problem which is better defined and hopefully contains the same information.

[17] At least in so far as the semi-flat metric is a valid description of the moduli space geometry at finite R.

5 The Geometry of Superconformal Quantum Mechanics

This section, based largely on [1], is the first signficant part of the author's work. We extend the geometric formalism for supersymmetric quantum mechanics, reviewed in section 2 and based around the exterior algebra structure of Hilbert space, to the superconformal case. This entails constructing differential operators which extend the Poincaré superalgebra built on the Laplacian, exterior derivative, and (hyper-)Kähler structure to a superconformal algebra. We will find that additional natural geometric constraints are imposed both to construct new generators and to ensure that their algebra closes, as summarised in the following:

Theorem

1. Let (M, g) be a Riemannian manifold admitting a *closed homothety* D, that is a vector field D satisfying
$$\mathcal{L}_D g = 2g, \qquad \mathcal{L}_D K = 2K, \qquad D_\mu = \partial_\mu K \tag{5.1}$$
for some function K on M. Then the σ-model (2.7) with target space M admits an $\mathfrak{su}(1,1|1)$ superconformal symmetry.
2. Let (M, g) as above be Kähler such that D is holomorphic and K is a Kähler potential. Then (2.7) has $\mathfrak{u}(1,1|2)$ symmetry.
3. Let (M, g) be hyper-Kähler such that D is triholomorphic and K is a hyper-Kähler potential. Then (2.7) has $\mathfrak{osp}(4^*|4)$ symmetry.

The mathematics of superalgebras is reviewed in appendix D and a complete list of generators and relations for the above algebras is found in appendix E. Observe that $\mathfrak{osp}(4^*|4)$ is exactly the expected superalgebra for instanton quantum mechanics based on the DLCQ arguments reviewed in section 4, and we will give an argument to show that instanton moduli space does indeed satisfy our theorem in section 5.4.

A similar analysis to ours was carried out for chiral supersymmetry in [62]. Many of their results are analogous to ours, and some of our initial ansätze for the forms of generators are inspired by their results. Our approach and calculational methods are quite distinct, however, and it is the author's view that the exterior algebra formalism is the best way to emphasise the naturalness of the underlying geometric structures. It is also worth noting that similar structures to those we uncover have been observed in the worldline formulation of p-form gauge fields [115], and it would be interesting to understand why this should be.

Throughout this section, our conventions are that the conformal algebra $\mathfrak{so}(2,1) \cong \mathfrak{su}(1,1) \cong \mathfrak{sl}(2;\mathbb{R})$ consists of Hermitian generators D, the *dilatation* operator, H, the *Hamiltonian*, and K, the *special conformal* generator. These satisfy

$$[D, H] = 2iH, \qquad [D, K] = -2iK, \qquad [H, K] = -iD. \tag{5.2}$$

In contrast to the usual field-theoretic treatment (see [12]), we will have nothing to say about the action of these generators on the 'spacetime' $\mathbb{R}$, but skip straight to the Hilbert space description. Before proceeding we give the simplest possible example of such a conformal algebra for free 1-particle quantum mechanics on $\mathbb{R}$. The Hamiltonian is $2H = P^2$ and can be completed to $\mathfrak{so}(2,1)$ by taking

$$D = \frac{1}{2}\left(XP + PX\right), \qquad K = \frac{1}{2}X^2. \tag{5.3}$$

In particular, note that H is the Laplace operator and D can be viewed as a vector field generating the scale transformation $X \mapsto \lambda X$. Furthermore, D is a closed homothety with respect to K. Our construction will be a generalisation of this simple idea.

5.1 The Riemannian Case: $\mathcal{N} = (1,1)$ and $\mathfrak{su}(1,1|1)$

We begin this section by constructing operators D and K which join the Hamiltonian $2H = \Delta$ to produce the conformal algebra (5.2). First we make an ansatz for the form of the dilatation operator. Since the σ-model is an (admittedly large) generalisation of the free model (5.3), we assume that the dilatation will also generalise that construction. More precisely, we suppose that D is represented by the flow of a vector field D on the target space M, namely[18]

$$\hat{D} = -\frac{i}{2}\left(\mathcal{L}_D - \mathcal{L}_D^\dagger\right), \tag{5.4}$$

where $\mathcal{L}_D^\dagger$ is defined formally for now and is included so that D is Hermitian. This formula is a direct extrapolation of (5.3), right up to the factor of $-i$ in common with the action of P and Π. Note that it is not clear that such an extrapolation is the only possibility, and we justify it by its consistency. Nor is it at all obvious from the quantum mechanical expression for D, obtained using Cartan's formula $\mathcal{L}_X = \{i_X, d\}$ and the dictionary (2.19), that this ansatz has any chance of

[18] We will usually confuse notation for an algebra element and its corresponding geometric object, so for example D is both the dilatation operator and a vector field. When this may cause confusion, we add a hat to the algebra element.

working. It is a vindication of the exterior algebra approach that the proof turns out to be rather straightforward.

To get the scaling dimension of the Hamiltonian, it is easiest to work instead with the supercharges. Recall that these satisfy

$$\left\{Q, Q^\dagger\right\} = 2H, \qquad Q = d, \qquad Q^\dagger = \delta,$$

so that consistency with the Jacobi identity requires

$$[\mathcal{L}_D, \delta] = -2\delta.$$

Using the fact that $[\mathcal{L}_D, d] = 0$ and $\delta = \pm * d *$, intuition suggests that the volume form must expand at a constant rate along the flow of D. It is possible to give a brute-force proof by computing $[\mathcal{L}_D, *d*]$ in components, but it's easier to first compute $[\mathcal{L}_D, *]$. Working in coordinates such that $D = \partial/\partial t$ with $t = X^1$, say, a short calculation gives

$$\left([\mathcal{L}_D, *]\,\alpha\right)_{\nu_1 \ldots \nu_{n-p}} = \frac{1}{p!}\left(\partial_t \epsilon^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_{n-p}}\right) \alpha_{\mu_1 \ldots \mu_p},$$

where α is a p-form. Now suppose that D is a homothetic vector field

$$\mathcal{L}_D g = 2g,$$

so that

$$\partial_t g^{\mu\nu} = -2g^{\mu\nu}, \qquad \partial_t \sqrt{\det g} = n\sqrt{\det g},$$

where $n = \dim_{\mathbb{R}} M$. From this we obtain

$$[\mathcal{L}_D, *]\,\alpha = (n - 2p) * \alpha, \tag{5.5}$$

which is enough to give us

$$[D, H] = 2iH, \qquad [D, Q] = iQ, \qquad \left[D, Q^\dagger\right] = iQ^\dagger.$$

The special conformal generator must generically be a degree zero differential operator since D and H have degrees one and two respectively and satisfy (5.2). Furthermore, since neither D nor H change fermion number, K must generically multiply each degree p of fermion by some function $K_p(X)$, each of which must be real so that K is Hermitian. The relation $[D, K] = -2iK$ is easily solved by requiring

$$\mathcal{L}_D K_p = 2K_p,$$

but $[H, K] = -iD$ is harder work. For simplicity we assume that all the K_p are equal, though we don't completely rule out other solutions. A few standard manipulations yield

$$[\Delta, K]\,\alpha = -\left\{d, i_{\tilde{dK}}\right\}\alpha + \{\delta, dK\wedge\},$$

where $\tilde{dK}^{\mu} = g^{\mu\nu}\partial_\nu K$ and we used the dictionary (2.19) to establish $(dK\wedge)^\dagger = i_{\tilde{dK}}$. But we are looking to obtain

$$[\Delta, K]\,\alpha = \left(\mathcal{L}_D^\dagger - \mathcal{L}_D\right),$$

so our problem is reminiscent of Cartan's formula. The solution is to take

$$D_\mu = \partial_\mu K,$$

so that our homothety is *closed*. This gives us everything we need to obtain the conformal algebra (5.2), and as an aside we find

$$K = \frac{1}{2} g^{\mu\nu}\partial_\mu K \partial_\nu K = \frac{1}{2}\|D\|^2. \tag{5.6}$$

Note that our conditions for D and K are exactly those of [62]. This is perhaps unsurprising as we have yet to make essential use of supersymmetry.

We are well on the way to establishing the first part of our theorem. To finish, define superconformal generators

$$\begin{aligned} S &= -i\,[K, Q] = idK\wedge \\ S^\dagger &= -i\left[K, Q^\dagger\right] = -ii_D. \end{aligned} \tag{5.7}$$

We also have the $U(1)$ R-symmetry generator J_3 (2.21) which counts fermion number, so Q and S have charge $+1/2$ while $Q^\dagger$ and $S^\dagger$ have $-1/2$. To complete the algebra we need a more explicit expression for the term $\mathcal{L}_D^\dagger$ appearing in the dilatation (5.4). Using (5.5) and the inner product formula (2.2), we obtain

$$\mathcal{L}_D^\dagger\alpha = (-1)^{p(n-p)+1} * \mathcal{L}_D * \alpha = (2p - n - \mathcal{L}_D)\,\alpha, \qquad (\alpha \in \Omega^p)$$

so that

$$\hat{D} = -i\mathcal{L}_D + i\left(p - \frac{n}{2}\right). \tag{5.8}$$

Computing the remaining commutators and verifying their consistency is a simple task.

This completes our proof of the first part of the theorem. A complete listing of all generators and relations, in both quantum mechanical and differential geometric form, can be found in appendix

E.1. The resulting superconformal algebra is a simple superalgebra which may be identified with $\mathfrak{su}(1,1|1)$. This means that the bosonic subalgebra is

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{u}(1)$$

and the fermion representation is $\mathbf{2}_+ \oplus \mathbf{2}_-$, where $\pm$ are $U(1)$ charges under J_3. $\mathfrak{su}(1,1|1)$ is the unique simple superalgebra with these properties [47–49].

5.2 The Kähler Case: $\mathcal{N} = (2,2)$ and $\mathfrak{u}(1,1|2)$

We now extend the above analysis to the case of Kähler geometry. In section 2.2 we constructed two additional supercharges (2.29) and $SU(2) \times U(1)$ R-symmetry generators (2.24, 2.28) obeying the Poincaré $\mathcal{N} = (2,2)$ supersymmetry algebra. We combine this with the new ingredients D, K, S, and $S^\dagger$ of the $\mathcal{N} = (1,1)$ superconformal algebra then examine any extra constraints for the algebra to close. First we define additional superconformal generators

$$\begin{aligned} S^I &= -i\left[K, Q^I\right] = \left(\partial - \bar{\partial}\right) K\wedge \\ S^{\dagger I} &= -i\left[K, Q^{\dagger I}\right] = -i i_{D^I}, \end{aligned} \tag{5.9}$$

where $D^{I\mu} = I^\mu{}_\nu D^\nu$. To verify that the supercharges Q^I and $Q^{\dagger I}$ have the correct scaling dimension, it's intuitively clear that we need the dilatation to respect the complex structure. More precisely, since D must commute with the R-symmetry generator $2R^I\big|_{\Omega^{p,q}} = p - q$ we find that $\mathcal{L}_D$ preserves the bidegree of a (p,q)-form. From this, it's simple to check that the vector field D is holomorphic

$$\mathcal{L}_D I = 0 \;\Rightarrow\; D = D^\imath(Z)\partial_\imath + D^{\bar\imath}(\bar{Z})\partial_{\bar\imath},$$

where $(Z^\imath, \bar{Z}^{\bar\imath})$ are complex coordinates. With two exceptions, the remaining commutation relations are either straightforward to calculate directly or are consequences of the Jacobi identity. The first exception is $\left\{Q, S^I\right\}$, which imposes a new constraint. We have

$$\left\{Q, S^I\right\}\alpha = -2\partial\bar{\partial}K \wedge \alpha,$$

which requires that $\partial\bar{\partial}K$ is a 2-form already present in the algebra. There are two options: either $\partial\bar{\partial}K = 0$, or K is a Kähler potential so that the above produces the Kähler form $\omega\wedge = J_+^I$. The first option is not feasible since then $K = f(Z) + \bar{f}(\bar{Z})$, so that $D_\imath = g_{\imath\bar\jmath}D^{\bar\jmath} = \partial_\imath K$ is holomorphic. This is nonsense, so we conclude that K must be a Kähler potential. However, the Kähler potential is unique only up to the addition of the real part of a holomorphic function, and in general may only be defined locally, so there is a question of which Kähler potential we mean. By the same argument as rules out $\partial\bar{\partial}K = 0$, the non-uniqueness of K plays havoc with the holomorphy of D, so this is an important question to settle. We give one possible solution below.

The second exception is $\left\{Q, S^{\dagger I}\right\}$, which closes onto the vector field D^I. It follows from the holomorphy, homothety and closure properties of D that D^I is a holomorphic isometry

$$\mathcal{L}_{D^I} I = \mathcal{L}_{D^I} g = \mathcal{L}_{D^I} \omega = 0.$$

Thus we obtain a new element

$$\hat{D}^I = -i\mathcal{L}_{D^I} \tag{5.10}$$

in our superconformal algebra, which turns out to be central. This allows the aforementioned relation $\left\{Q, S^{\dagger I}\right\}$ to close as

$$\left\{Q, S^{\dagger I}\right\} = D^I.$$

We can now return to the issue of uniqueness of the Kähler potential. A solution which often works is to observe that, since D^I is a holomorphic isometry, there is an associated moment map

$$i_{D^I}\omega = d\mu. \tag{5.11}$$

Explicit calculation shows that D is closed with respect to μ and that μ is a Kähler potential , so we can take $\mu = K$ to fix the Kähler potential up to a global constant (at least when $H^1(M) = 0$). This works provided that $\mu = \frac{1}{2}\left\|D\right\|^2$, or equivalently that $\mathcal{L}_D\mu = 2\mu$, which will be the case in all the examples we consider. In fact, we'll see in section 5.3 that this solution is forced on us in the hyper-Kähler case.

This completes our proof of the second part of the theorem. A complete listing of all generators and relations can be found in appendix E.2. The resulting algebra has bosonic part

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_C,$$

where $\mathfrak{u}(1)_R$ is an R-symmetry and $\mathfrak{u}(1)_C$ is central. The fermion representations are

$$(\mathbf{2} \otimes \mathbf{2})_+ \oplus (\mathbf{2} \otimes \mathbf{2})_-\,,$$

where $\pm$ are $\mathfrak{u}(1)_R$ charges. This algebra is not simple, but contains a simple subalgebra whose bosonic part is $\mathfrak{so}(2,1) \oplus \mathfrak{su}(2)$. Comparing with the lists of [47–49], we see that this simple algebra must be $\mathfrak{psu}(1,1|2)$. The addition of two $\mathfrak{u}(1)$ factors, one of which is not central and not generated by (anti)commutators, the other of which is central and is generated by anticommutators, results in the algebra $\mathfrak{u}(1,1|2)$. The removal of the two $\mathfrak{u}(1)$s corresponds to the '$\mathfrak{s}$' and '$\mathfrak{p}$' quotients respectively. More details on this can be found in appendix D.1.

5.3 The Hyper-Kähler Case: $\mathcal{N} = (4,4)$ and $\mathfrak{osp}(4^*|4)$

Finally we move to the case of hyper-Kähler geometry. We already constructed all the necessary operators in the Kähler case. Each of the operators Q^I, S^I, D^I, R^I defined with reference to a complex structure is triplicated using the three complex structures I^a. For example, Q^I is replaced by $Q^a : a = 1,2,3$. Doing this requires that the homothety D is triholomorphic and that K is a hyper-Kähler potential. Note however that the isometries D^a are not triholomorphic, rather we have the geometric identities

$$\begin{aligned} i_{D^a}\omega^b &= \delta^{ab} dK + \epsilon^{abc} d^c K \\ \mathcal{L}_{D^a}\omega^b &= -2\epsilon^{abc}\omega^c \\ \mathcal{L}_{D^a} I^b &= -2\epsilon^{abc} I^c \\ \left[D^a, D^b\right] &= -2\epsilon^{abc} D^c. \end{aligned} \tag{5.12}$$

The first line follows from the quaternion algebra (2.33), and the others follow from the first and the fact that D^a is an isometry and holomorphic with respect to I^a. As in (2.29), $d^c = i(\bar{\partial} - \partial)$ in complex coordinates adapted to I^c. In contrast to the Kähler case, the requirement that M admits a hyper-Kähler potential is not automatic and will be addressed later. We know from section 2.2 that the $SU(2) \times U(1)$ R-symmetries of each Kähler subalgebra combine together to form $SO(5)$, and from the geometry (5.12) we obtain

$$\begin{aligned} \left[D^a, D^b\right] &= 2i\epsilon^{abc} D^c \\ \left[D^a, J_+^b\right] &= 2i\epsilon^{abc} J_+^c \\ \left[D^a, S^b\right] &= 2i\epsilon^{abc} S^c. \end{aligned}$$

Every remaining commutation relation can be obtained from these geometric identities, results from the Kähler case and section 2.2 by use of Hermitian conjugation and the Jacobi identity. The fact that all of this fits together without any further modification or constraints as compared to the Kähler case, not to mention requiring so little energy to compute, is testament to the naturalness of the construction.

We still need to settle the issue of the existence of a hyper-Kähler potential. Fortunately, our class of models is particularly well suited to admitting such an object, as shown by the following result of [63]:

Theorem

Let M be a hyper-Kähler manifold. Then M has a hyper-Kähler potential if and only if there is an action of $SU(2)$ by vector fields $X^a : a = 1, 2, 3$, which permutes the complex structures and is such that $I^a X^a$ (no sum) is independent of the choice of a. Moreover, one choice of hyper-Kähler potential is the moment map $\mu : d\mu = i_{X^a}\omega^a$, which is also independent of a.

In our model this is certainly satisfied by the $SU(2)$ action of D^a. Observe that this is exactly the construction we used in the Kähler case (5.11) to solve uniqueness of the Kähler potential, and the same caveat applies. Provided that $\mathcal{L}_D\mu = 2\mu$, we can be sure that our structure is consistent.

At first glance it seems that we're done, but we can still do something to improve the presentation of our algebra. At present, the relations $\left[D^a, J^b_+\right] \neq 0$ suggest that the $SO(5)$ and $SU(2)$ R-symmetries do not decouple, but in fact if we choose the linear combination

$$T^a = D^a - 2R^a \tag{5.13}$$

then the T^a generate an $SU(2)$ which decouples from $SO(5)$ and assigns the same representations to the supercharges as do the D^a.

This completes the proof of the third and final part of our theorem. A full listing of the constructed algebra can be found in appendix E.3. It is a simple superalgebra with bosonic part

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(5)$$

and fermion representation $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{4}$. The unique such superalgebra is $\mathfrak{osp}(4^*|4)$ [47–49].

5.4 $\mathfrak{osp}(4^*|4)$ and Instanton Moduli Space

We now give an argument to show that the instanton moduli space $\mathcal{M}_{k,N}$ as described in section 3 has $\mathfrak{osp}(4^*|4)$ superconformal symmetry. This result was anticipated from the DLCQ perspective in section 4 and represents an explicit construction of the superalgebra described in [40]. Furthermore, it is perhaps intuitively unsurprising as we know that $\mathcal{M}_{k,N}$ has a scale symmetry inherited from the scale invariance of the Yang-Mills equations, and have already seen that it is a hyper-Kähler manifold so must admit $\mathcal{N} = (4,4)$ supersymmetry. The content of this section is therefore a verification that these symmetries interact 'nicely'.

Our argument is based on the hyper-Kähler quotient construction of section 3.2. To begin with, observe that $\mathbb{R}^{4k(N+k)}$ trivially admits $\mathfrak{osp}(4^*|4)$, with hyper-Kähler structure determined by (3.23, 3.24, 3.25), special conformal generator given by (3.24), and triholomorphic closed homothetic vector field

$$D = X^\alpha{}_\beta \frac{\partial}{\partial X^\alpha{}_\beta} + \tilde{X}_\alpha{}^\beta \frac{\partial}{\partial \tilde{X}_\alpha{}^\beta} + q_i^\alpha \frac{\partial}{\partial q_i^\alpha} + \tilde{q}^i_\alpha \frac{\partial}{\partial \tilde{q}^i_\alpha} + \text{complex conjugate.} \tag{5.14}$$

We have already described how to induce a hyper-Kähler structure on $\mathcal{M}_{k,N}$ via the quotient, and observed that it arises from a hyper-Kähler potential $\tilde{K} = K|_{\vec{\mu}^{-1}(0)}$. It remains to check that we can induce a homothety $\tilde{D}$ and its associated $SU(2)$ action $\tilde{D}^a$, and that these continue to interact correctly with the rest of the structure.

Observe first that the ADHM moment maps (3.26) are homogeneous of degree two under D. Thus we have $D(\vec{\mu}) = 0$ on the surface $\vec{\mu}^{-1}(0)$, so that D is a tangent vector to the constraint surface. Furthermore, D is manifestly $U(k)$-invariant, so $\tilde{D} = \pi_* D$ is a well-defined vector field on $\mathcal{M}_{k,N}$ which also satisfies $\mathcal{L}_{\tilde{D}}\tilde{K} = 2\tilde{K}$ and whose horizontal lift (C.2) is D. Notice that, had we chosen a different level set $\vec{\mu}^{-1}(\vec{\xi})$, D would not have restricted to the quotient space. Such a choice corresponds to a resolution of the moduli space using non-commutative spacetime [76], and the failure of this space to admit a homothety is a symptom of the minimum scale introduced by non-commutativity. Though it is not manifest from our presentation (3.26) of the ADHM equations, it can be shown that the moment maps are invariant under the $SU(2)$ action by D^a. Indeed, this invariance is manifest in the quaternionic notation of [67]. D^a are also $U(k)$ invariant, so the same argument as for D provides us with vector fields $\tilde{D}^a$ on $\mathcal{M}_{k,N}$ which still generate an $SU(2)$ action.

We now turn to the various interactions of the induced structures, all of which follow straightforwardly from the definitions. We calculate a few examples, starting with the fact that $\tilde{D}$ is a homothety. Consider

$$\mathcal{L}_{\tilde{D}}\left(\tilde{g}(\tilde{X},\tilde{Y})\right) = \left(\mathcal{L}_{\tilde{D}}\tilde{g}\right)(\tilde{X},\tilde{Y}) + \tilde{g}(\mathcal{L}_{\tilde{D}}\tilde{X},\tilde{Y}) + \tilde{g}(\tilde{X},\mathcal{L}_{\tilde{D}}\tilde{Y})$$

where $\tilde{X},\tilde{Y} \in T\mathcal{M}_{k,N}$. Now by the $U(k)$-invariance of horizontal lifts and the metric pullback by π, the left hand side is the same as $D(g(X,Y))$ where X,Y are the horizontal lifts of $\tilde{X},\tilde{Y}$. Using the Leibniz rule and $U(k)$-invariance of lifts again, as well as the homothety property of D, we find

$$\left(\mathcal{L}_{\tilde{D}}\tilde{g}\right)(\tilde{X},\tilde{Y}) = 2g(X,Y) = 2\tilde{g}(\tilde{X},\tilde{Y}),$$

so that $\mathcal{L}_{\tilde{D}}\tilde{g} = 2\tilde{g}$. An identical argument works for the symplectic forms, which in turn implies triholomorphy of $\tilde{D}$. Next we prove closure:

$$\tilde{g}(\tilde{D},\tilde{X}) = g(D,X) = dK(X) = X(K) = \tilde{X}(\tilde{K}),$$

where we used $U(k)$-invariance in the first and last equalities, and closure of D in the second. This completes the set of properties of $\tilde{D}$, and is therefore enough to imply all required properties of $\tilde{D}^a$. In particular, an identical argument to the closure of $\tilde{D}$ shows that the hyper-Kähler potential

$\tilde{K}$ is the shared moment map for the $SU(2)$ action of $\tilde{D}^a$. This completes the proof that instanton moduli space admits $\mathfrak{osp}(4^*|4)$ invariance.

It's worth remarking that this construction applies more generally. Given any manifold with $\mathfrak{osp}(4^*|4)$ invariance admitting a triholomorphic and isometric group action under which the homothety D and associated $SU(2)$ isometries are invariant and preserve a level set of the moment maps, one can use the hyper-Kähler quotient to construct another manifold with $\mathfrak{osp}(4^*|4)$ invariance.

6 Special Kähler Geometry and $\mathfrak{su}(1,1|4)$

In this section, based largely on [2,3], we consider quantum mechanics on another type of geometry known as *special Kähler*. This presents an interesting conundrum, as the geometry itself is a special case of Kähler geometry but the cotangent bundle is hyper-Kähler [50,116]. Large classes of special Kähler manifolds, which we dub *scale-invariant special Kähler (SISK)*, admit a closed holomorphic homothety, so by results of the previous section have $\mathfrak{u}(1,1|2)$ superconformal symmetry. However, this only has $\mathcal{N}=(2,2)$ supersymmetry, while the σ-model on the cotangent bundle has $\mathcal{N}=(4,4)$ but is not conformal in general. One might therefore wonder whether there is some model available which has the 'best of both worlds', in that it is both conformal and $\mathcal{N}=(4,4)$. It turns out that such a thing can be found and has $\mathfrak{su}(1,1|4)$ invariance. The construction involves taking a truncation of the cotangent bundle σ-model phase space. It can be understood as a discrete quotient of the cotangent bundle, leading to a torus bundle over a special Kähler base, followed by a limit in the corresponding σ-model where the size of the fibre becomes small, effectively freezing out the bosonic degrees of freedom associated to fibre coordinates. This leaves a novel type of σ-model on the base space with twice as many fermionic degrees of freedom as the standard type (2.7).

Such a construction is natural from the perspective of four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory. As reviewed in section 3.4, the vacua of such theories fall into two branches, the *Higgs* and *Coulomb* branches. Each branch is a manifold with singularities, and one can show that the Coulomb branch is special Kähler [25]. Furthermore, the Coulomb branch of the same theory defined on $\mathbb{R}^3\times S^1_R$ is a hyper-Kähler torus bundle over the four-dimensional Coulomb branch[19] [17], whose metric approaches that of the cotangent bundle in the limit $R\to\infty$ [51]. In the same limit, the torus fibres become small and the associated bosonic modes acquire infinite energy, effectively implementing the truncation of phase space. If the original four-dimensional theory is conformal then its Coulomb branch is SISK, giving us a large class of examples of $\mathfrak{su}(1,1|4)$ invariant σ-models. In particular, the DLCQ of $\mathcal{N}=4$ SUSY Yang-Mills theory fits within this framework and $\mathfrak{su}(1,1|4)$ is the expected subalgebra of $\mathfrak{psu}(2,2|4)$ preserved by DLCQ.

[19] Modulo a subtlety called the 'quadratic refinement' which we do not address.

6.1 Special Kähler Geometry: Local Description and Supersymmetric Gauge Theory

We begin with a review of the local[20] structure of special Kähler geometry and its appearance in four-dimensional $\mathcal{N}=2$ gauge theory. First, a definition[21]:

Definition

A manifold M is called *special Kähler* if it is a complex manifold and there exist local *special* complex coordinates a^I and a holomorphic function $\mathcal{F}(a)$, called the *prepotential*, such that the metric can be written as

$$ds^2 = \mathrm{Im}\left(\frac{\partial^2\mathcal{F}}{\partial a^I \partial a^J}\right) da^I d\bar{a}^J. \tag{6.1}$$

We will refer to this as the local definition of special Kähler geometry. A couple of words on notation are in order. As we will see later, special coordinates do not have tensorial transformation properties, so the indices I, J are not indices for holomorphic transformations. As such, when it does not cause confusion, we will not distinguish holomorphic and antiholomorphic indices. For later convenience we define

$$\mathcal{F}_{I_1 \ldots I_n} = \frac{\partial^n \mathcal{F}}{\partial a^{I_1} \ldots \partial a^{I_n}}.$$

The first and second derivatives have special notations

$$\tau_{IJ} = \mathcal{F}_{IJ}, \qquad a_I^D = \mathcal{F}_I \tag{6.2}$$

where a^D stands for the dual of a. The name refers to a form of electromagnetic duality transformation introduced by [15] as a means to analyse the strong coupling region of a Coulomb branch. We see that special Kähler manifolds are in fact Kähler, with potential and symplectic form given by

$$\begin{aligned} K &= \mathrm{Im}\left(a_I^D \bar{a}^I\right) \\ \omega &= \mathrm{Re}\left(da_I^D \wedge d\bar{a}^I\right). \end{aligned} \tag{6.3}$$

[20] In the context of supergravity, 'local' is often taken to refer to the slightly different structure of the moduli space of four-dimensional $\mathcal{N}=2$ supergravity [117]. In this context, our geometry is often called *rigid special kähler*. For us, local just means 'defined in a local coordinate patch'.

[21] This definition developed in a rather piecemeal fashion in the physics literature, but the earliest mention seems to be [24].

Now consider the Coulomb branch vacua of a four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. For brevity we assume this to contain only a vector multiplet, but the addition of matter hypermultiplets does not affect our conclusions [16]. The vector multiplet consists of a gauge field, two adjoint-valued Weyl fermions and a complex adjoint scalar ϕ [118]. The classical action has a scalar potential of the form

$$V(\phi) \propto \mathrm{tr}\left[\phi, \phi^\dagger\right]^2,$$

which must vanish for a supersymmetric vacuum configuration. For this to occur, we need ϕ and $\phi^\dagger$ to commute, which is achieved by taking ϕ to be gauge equivalent to a constant diagonal matrix. For a rank g gauge theory there are g independent eigenvalues which we label $a^I : I = 1, \ldots, g$. The space of all such diagonal matrices, modulo a residual action by the Weyl group W of G, forms the classical Coulomb branch moduli space. In particular, this moduli space carries a Euclidean metric, as can be read off from the scalar kinetic terms in the low-energy effective theory of fluctuations about the vacuum a^I. This means that $M_{cl} \cong \mathbb{C}^g/W$ is a special Kähler manifold, with the a^I forming local special coordinates and prepotential $\mathcal{F}_{cl} = 1/2(\tau_{cl})_{IJ}a^I a^J$ with $(\tau_{cl})_{IJ} = i\delta_{IJ}$. For generic values of a^I, the low-energy theory has an unbroken abelian gauge symmetry $U(1)^g$ with field strengths v^I_{mn}.

The quantum theory is more interesting. Supersymmetry constrains the bosonic low-energy effective action to the form [25, 119]

$$\mathscr{L} = \frac{1}{4\pi}\,\mathrm{Im}\,\tau_{IJ}\partial_m a^I \partial^m \bar{a}^J + \frac{1}{8\pi}\,\mathrm{Im}\,\tau_{IJ}v^I_{mn}v^{Jmn} + \frac{1}{8\pi}\,\mathrm{Re}\,\tau_{IJ}v^I_{mn}(*v^J)^{mn},$$

where τ_{IJ} is constant in the classical theory but picks up quantum corrections as a holomorphic function of the a^I. We see that the Coulomb branch is special Kähler if, as usual, we interpret the scalar kinetic terms as defining a metric. However, the quantum Coulomb branch no longer admits a global prepotential due to the presence of complex codimension-1 singularities around which the special coordinates and their duals (6.2) have non-trivial monodromies. The underlying theory has states carrying both electric and magnetic charges $(n_{e,I}, n^I_m) \in \mathbb{Z}^{2g}$, and in particular there are monopoles and dyons saturating the BPS bound $M \geq |Z|$, where

$$Z = a^I n_{e,I} + a^D_I n^I_m. \tag{6.4}$$

This formula is invariant under the $Sp(2g;\mathbb{Z})$ electromagnetic duality group, under which $(n_{e,I}, n^I_m)$ and (a^D_I, a^I) transform as vectors, and a subgroup of this duality is generated by the aforementioned monodromies of the Coulomb branch. Singular submanifolds occur when certain BPS monopoles and dyons become massless, and one can use duality transformations to give a weakly coupled description of this phenomenon.

Pushing this approach through leads to the *Seiberg-Witten solution.* This relies on the identification of the quantum Coulomb branch with a submanifold of the moduli space of hyperelliptic curves[22] of genus g. These curves are known as the *Seiberg-Witten curves.* This identification results in exact expressions for the quantum-corrected special Kähler structure in a large class of examples [15, 16, 121–124]. In this context, the local coordinates a^I and duals a_I^D are given by period integrals of a particular meromorphic 1-form on the Seiberg-Witten curve, and the $Sp(2g;\mathbb{Z})$ duality translates into the natural symmetry group of the intersection pairing of A and B cycles. Alternative perspectives on this solution have been developed, employing integrable systems [125], localisation [126, 127] and M-theory [52], further augmenting the list of known special Kähler structures. In particular, the case of the $\hat{A}_{N-1}$ quiver theory relevant to $\mathcal{N}=4$ supersymmetric Yang-Mills is known explicitly.

Now consider the same low-energy problem for the theory compactified on $\mathbb{R}^3 \times S^1_R$ as in [17], where R is the radius of the circle. Thinking classically for the moment, we still have the same collection of massless scalar VEVs a^I as in the four-dimensional case, but now there are extra moduli. The unbroken abelian gauge fields have gauge-invariant zero mode integrals around S^1_R, which we label θ^I_e. $Sp(2g;\mathbb{Z})$ duality also leads to abelian 'magnetic' gauge fields, with zero mode integrals $\theta_{m,I}$. These provide $2g$ real moduli in total. Both θ^I_e and $\theta_{m,I}$ are 2π-periodic, a fact which follows from charge quantisation. Thus, the new moduli parameterise the $2g$-dimensional fibres of a torus bundle $\mathcal{J}$ over the four-dimensional classical Coulomb branch. One can show that the classical metric on the fibres is flat, that the volume is proportional to $1/R$, and that the complex structure is given by τ_{cl} [17]. In particular, $\mathcal{J}$ is a hyper-Kähler manifold. There are natural complex coordinates

$$z_I = \theta_{m,I} - \tau_{IJ}\theta^J_e \tag{6.5}$$

for the fibres, and in terms of the 1-form

$$\delta z_I = d\theta_{m,I} - \tau_{IJ} d\theta^J_e \tag{6.6}$$

the bundle metric is

$$ds^2 = R \operatorname{Im}\tau_{IJ} da^I d\bar{a}^J + \frac{1}{4\pi^2 R}\left(\operatorname{Im}\tau^{-1}\right)^{IJ} \delta z_I \delta\bar{z}_J. \tag{6.7}$$

[22] We refer to [120] for the theory of Riemann surfaces.

The reader might wonder why we haven't called δz_I the derivative of z_I. Of course, in this classical case it is, but we'll now argue that these formulae generalise to the quantum setting, at least at large R, where τ is not constant and the distinction is important.

The nature of the quantum corrections depends on the compactification radius R. As well as the four-dimensional corrections leading to a non-constant τ, there are new corrections associated to the compactification. In particular, BPS states in four dimensions can be interpreted as three-dimensional instantons with Euclidean worldlines wrapping the circle. These provide corrections to the quantum picture of characteristic magnitude $\exp(-M_{BPS}R)$, where M_{BPS} is the four-dimensional BPS mass determined by (6.4). At finite R these corrections are important, and have the effect of smoothing out the singularities present in (6.7) for non-constant τ, leaving a smooth hyper-Kähler manifold. They are also rather complicated, requiring the twistor theoretic machinery of [51] to compute. On the other hand, as $R \to \infty$ all corrections which are specific to three dimensions drop out. It follows that the quantum corrections to the three-dimensional picture at large R reduce to those of the four-dimensional theory, with the simple upshot that we replace the constant classical τ with the a-dependent quantum τ in formulae (6.5, 6.6, 6.7). In particular, the quantum Coulomb branch is the torus bundle $\mathcal{J}$ over the four-dimensional quantum Coulomb branch, with *semi-flat* metric given by (6.7). The fact that the complex structure of the torus fibres is given by τ means that $\mathcal{J}$ corresponds to a fibration of the four-dimensional moduli space by the Jacobian variety of the Seiberg-Witten curve.

To sum up, we've seen that local special Kähler geometry in the sense of a Kähler structure derived from a prepotential occurs naturally in the low-energy theory of four dimensional $\mathcal{N} = 2$ gauge theories. Compactifying these theories on $\mathbb{R}^3 \times S^1$ leads to a low-energy theory which is a nonlinear σ-model with hyper-Kähler target given by a torus bundle over the four-dimensional Coulomb branch. At large compactification radius there is a simple explicit metric on this bundle, determined by the same prepotential $\mathcal{F}$ as the four-dimensional case. In fact it's possible to go further and show that the three-dimensional Coulomb branch is a complex integrable system [125]. The details of this, as well as the proof that the three-dimensional moduli space is hyper-Kähler (at least at large R), are easiest to understand in the global picture developed in the following section.

6.2 Special Kähler Geometry: Global Intrinsic Definition and Hyper-Kählerity of T^*M

In this section we review an alternative description of special Kähler geometry which is both global, in the sense that it makes no reference to a local coordinate system, and intrinsic, in the sense that it is defined in terms of structures on the tangent bundle only [50]. We'll see that this description is equivalent to the local construction of the previous section. In the global formalism it is easy to prove that the cotangent bundle of a special Kähler manifold is hyper-Kähler. We'll show that the semi-flat geometry (6.7) of the three-dimensional Coulomb branch $\mathcal{J}$ at large R can be understood as a quotient of $T^*\mathcal{M}$, where $\mathcal{M}$ is the four-dimensional Coulomb branch, by the BPS charge lattice. Furthermore, in this picture it's easy to check that the three-dimensional Coulomb branch is a complex integrable system. These ingredients, particularly the simple description of the hyper-Kähler structure of $\mathcal{J}$, are what we need to construct our $\mathfrak{su}(1,1|4)$-invariant quantum mechanics in sections 6.3 and 6.4.

The definition we work with is [50]

Definition

A Kähler manifold M is called *special Kähler* if it has an additional connection ∇ on TM which is

- Torsion free.
- Flat, $d_\nabla^2 = 0$.
- Symplectic, $\nabla\omega = 0$.
- *Special*, $d_\nabla I = 0$.

Here ω is the Kähler form, I is the complex structure and d_∇ is the exterior covariant derivative acting on TM-valued forms. The special Kähler condition can be written in local coordinates as

$$\partial_{[\rho} I^\mu{}_{\nu} + \Theta^\mu_{\sigma[\rho} I^\sigma{}_{\nu]} = 0,$$

where we denoted the components of ∇ by Θ to avoid confusion with the Levi-Civita connection. Note that this is not the same as $\nabla I = 0$.

The special connection ∇ can be used to prove the existence of local special coordinates a^I and prepotential $\mathcal{F}$ as in section 6.1. In particular, the Kähler structure is given by (6.1) and (6.3), so the intrinsic definition implies the usual local one [50]. Conversely, given a local prepotential $\mathcal{F}$

and special coordinates a^I, we can define a connection

$$\nabla \frac{\partial}{\partial a^I} = \mathcal{F}_{IJK} da^J \otimes \left(\operatorname{Im}\tau^{-1}\right)^{KL} \operatorname{Im} \frac{\partial}{\partial a^L}, \qquad \nabla \frac{\partial}{\partial \bar{a}^I} = \left(\nabla \frac{\partial}{\partial a^I}\right)^*,$$

which agrees with the formula for a special Kähler connection in special coordinates. Thus the familiar local description is equivalent to the global description. Rather unexpectedly, it follows from the above formula that $\mathcal{F}_{IJK}$ is a tensor, since it is the difference of the special Kähler and Levi-Civita connections.

We now turn to a sketch proof of the following

Theorem

The cotangent bundle T^*M of a special Kähler manifold M carries a canonical hyper-Kähler structure. Furthermore, if there is a ∇-flat lattice $\Lambda^* \subset TM$ whose dual is Lagrangian with respect to the holomorphic symplectic form η, then the quotient T^*M/Λ is a complex integrable system.

The first part of this theorem is due in local form to [116] and in global form to [50]. We will make extensive use of this hyper-Kähler structure so we prove it carefully. The second part is from [125] and we only sketch the idea.

First recall that, given any connection ∇ on a manifold M, we can define *horizontal lifts* of vectors and forms on M to vectors and forms on T^*M. To do this, let X^μ be coordinates on M and P_μ be the corresponding fibre coordinates on $T^*_X M$ obtained by writing 1-forms as $\alpha = P_\mu dX^\mu$. Then the horizontal lift of $\partial_\mu \in \Gamma(TM)$ to $T(T^*M)$ is given by

$$D_\mu = \frac{\partial}{\partial X^\mu} + P_\rho \Theta^\rho_{\mu\nu} \frac{\partial}{\partial P_\nu}.$$

Combined with the vertical vectors $\partial/\partial P_\mu$, these form a local frame for $T(T^*M)$. There is a dual coframe for $T^*(T^*M)$ given by the 1-forms

$$dX^\mu, \qquad \delta P_\mu = dP_\mu - P_\rho \Theta^\rho_{\mu\nu} dX^\nu. \tag{6.8}$$

In particular, we can apply this construction to the special Kähler connection ∇ and the local special coordinates a^I, z_I to obtain the frame

$$D_I = \frac{\partial}{\partial a^I} + \mathcal{F}_{IKL} \left(\operatorname{Im}\tau^{-1}\right)^{JL} \operatorname{Im} z_J \frac{\partial}{\partial z_K}, \qquad \frac{\partial}{\partial z_I} \tag{6.9}$$

and corresponding coframe

$$da^I, \qquad \delta z_I = dz_I - \mathcal{F}_{IKL} \left(\operatorname{Im}\tau^{-1}\right)^{JL} \operatorname{Im} z_J da^K. \tag{6.10}$$

The key point here is that the form δz_I obtained by horizontal lift coincides exactly with the form (6.6) appearing in the semi-flat metric.

We can now write down the canonical hyper-Kähler structure on T^*M. The metric is

$$ds^2 = \operatorname{Im}\tau_{IJ} da^I d\bar{a}^J + \left(\operatorname{Im}\tau^{-1}\right)^{IJ} \delta z_I \delta \bar{z}_J. \tag{6.11}$$

There is a complex structure I, which we call *preferred* as it makes a^I and z_I into holomorphic coordinates, with corresponding symplectic form

$$\omega^I = \frac{i}{2}\left(\operatorname{Im}\tau_{IJ} da^I \wedge d\bar{a}^J + \left(\operatorname{Im}\tau^{-1}\right)^{IJ} \delta z_I \wedge \delta \bar{z}_J\right). \tag{6.12}$$

The holomorphic symplectic form with respect to I is

$$\eta = da^I \wedge \delta z_I = da^I \wedge dz_I, \tag{6.13}$$

from which one can read off J, K, ω^J and ω^K. To finish the proof of the first part of the theorem, we must check that the complex structures so defined are integrable and obey the quaternion algebra (2.33). For integrability it is sufficient to check that the Kähler forms are closed [128], and the remaining checks are a simple algebraic exercise. Observe that the canonical hyper-Kähler metric (6.11) on T^*M coincides, up to a rescaling of variables, with the semi-flat metric (6.7) on $\mathcal{J}$. It is for this reason that we are interested in a quotient of T^*M.

We now sketch the second part of the theorem. The quotient of T^*M by the lattice Λ defines a complex symplectic manifold, since T^*M is complex symplectic and $\Lambda \subset T^*M$ is both complex and Lagrangian. In particular, the holomorphic symplectic form is still given in special coordinates by (6.13). Moreover, the flatness of Λ guarantees that the quotient does not interfere with the full hyper-Kähler structure obtained by lifting. The fibres of the quotient are complex tori parameterised by z_I and hence manifestly Lagrangian, and this is enough to make the quotient into a complex integrable system. We leave the details of this to [50, 125].

To close this section we make a few comments linking the above discussion back to the physical picture of section 6.1. The four-dimensional theory has a lattice $\Lambda \cong \mathbb{Z}^g \oplus \tau.\mathbb{Z}^g$ of BPS charges given by (6.4). The $Sp(2g;\mathbb{Z})$ monodromy transformations of this lattice are characteristic of the presence of a flat connection, which can be argued to be the special Kähler connection [50]. Pushing this idea through, one arrives at the conclusion that, in the large R limit for the theory on $\mathbb{R}^3 \times S^1_R$, we have

$$\mathcal{J} \cong \frac{T^*M}{\Lambda}.$$

This identification gives us everything we need to go ahead and construct quantum mechanics on $\mathcal{J}$, which we do in the following section.

6.3 Construction of Quantum Mechanics on $\mathcal{J}$

In this section we use the canonical hyper-Kähler structure described in the previous section to construct the action (2.7) for the quantum mechanical σ-model on $\mathcal{J}$. We'll put the resulting Lagrangian into manifestly $SO(5)$-invariant form then perform a Hamiltonian analysis of the system, including giving explicit expressions for the generators of $\mathcal{N} = (4,4)$ supersymmetry and $SO(5)$ R-symmetry.

First, we make a notational comment. The action (2.7) is written in tensorial language, so our convention up to now of using the same index $I = 1, \ldots, g$ for all types of coordinates is no longer sufficient. The difficulty arises because the special index I does not represent a tensorial transformation property, rather a transformation under electromagnetic $Sp(2g, \mathbb{Z})$ duality. There is a mathematically respectable way to deal with this by using a variation on the vielbein formalism to link special coordinates to generic holomorphic coordinates [117]. However, such a procedure is more complicated than we really need. Instead, we temporarily adopt the following convention. We work in special coordinates throughout, and use the index I for components in the D_I directions, $\bar{I}$ for $\bar{D}_I$ directions, I' for $\partial/\partial z_I$ and $\bar{I}'$ for $\partial/\partial \bar{z}_I$. If this is done carefully then no inconsistency can arise. Note for example that if we write $\mathcal{F}_{\bar{I}JK}$ then we do not mean that the first derivative is taken with respect to $\bar{a}^I$, this is just bookkeeping. Observe also that primed indices are naturally 'upstairs' when unprimed indices are 'downstairs'. We will drop primes and bars at the first opportunity.

The first step in constructing the action (2.7) is to calculate expressions for the connection and curvature of $\mathcal{J}$ in the non-coordinate basis given by (6.9) and (6.10). This is a laborious but completely routine procedure, so to spare the reader we just state the key ideas and the results. Recall that, given a generic frame e_A with dual coframe f^B, the connection components are defined by

$$\theta^C_{\ BA} = f^C(\nabla_{e_A} e_B) = \theta^C_{\ B}(e_A),$$

where $\theta^C_{\ B}$ are the connection 1-forms. We calculate these using the coordinate-free definition of the Levi-Civita connection

$$\begin{aligned} G(\nabla_X Y, Z) = \frac{1}{2}[&X(G(Y,Z)) + Y(G(Z,X)) - Z(G(X,Y)) \\ &+ G([X,Y], Z) + G([Z,X], Y) - G([Y,Z], X)] \qquad \forall X, Y, Z \in \Gamma(T\mathcal{J}). \end{aligned}$$

Note that the horizontal vectors D_I have nonzero Lie bracket with the vertical vectors given by

$$\begin{aligned}\left[D_I, \frac{\partial}{\partial z_{J'}}\right] &= \frac{i}{2}\mathcal{F}_{IK'L}\left(\operatorname{Im}\tau^{-1}\right)^{LJ'}\frac{\partial}{\partial z_{K'}}\\ \left[D_I, \frac{\partial}{\partial \bar{z}_{\bar{J}'}}\right] &= -\frac{i}{2}\left(\operatorname{Im}\tau^{-1}\right)^{L\bar{J}'}\frac{\partial}{\partial z_{K'}},\end{aligned}$$

and all other Lie brackets vanish. We obtain the connection components

$$\begin{aligned}\theta^K{}_{JI} &= -\frac{i}{2}\left(\operatorname{Im}\tau^{-1}\right)^{KL}\mathcal{F}_{IJL}\\ \theta^{KJ'I'} &= \frac{i}{2}\mathcal{F}_{LMN}\left(\operatorname{Im}\tau^{-1}\right)^{KL}\left(\operatorname{Im}\tau^{-1}\right)^{J'M}\left(\operatorname{Im}\tau^{-1}\right)^{I'N}\\ \theta_{K'}{}^{\bar{I}'}{}_J &= \frac{i}{2}\mathcal{F}_{JK'L}\left(\operatorname{Im}\tau^{-1}\right)^{L\bar{I}'}\\ \theta_{K'}{}^{J'}{}_I &= \frac{i}{2}\mathcal{F}_{IK'L}\left(\operatorname{Im}\tau^{-1}\right)^{J'L}.\end{aligned} \tag{6.14}$$

Observe that the connection components mixing holomorphic and antiholomorphic indices do not vanish since the D_I are not holomorphic. For the curvature, we use the form definition

$$\Omega^A{}_B = d\theta^A{}_B + \theta^A{}_C \wedge \theta^C{}_B = \frac{1}{2}\Omega^A{}_{BCD} f^C \wedge f^D.$$

We reserve the usual notation R for the base space Riemann tensor

$$R_{I\bar{J}K\bar{L}} = -\frac{1}{4}\left(\operatorname{Im}\tau^{-1}\right)^{M\bar{N}}\mathcal{F}_{IKM}\bar{\mathcal{F}}_{\bar{J}\bar{L}\bar{N}}. \tag{6.15}$$

Once we drop bars, we will further abuse notation by writing

$$R_{I\bar{J}K\bar{L}} = r_{IJKL}.$$

It is also convenient to use the special connection ∇ and base space tensor $\mathcal{F}_{IJK}$ to define a new totally symmetric base space tensor

$$\begin{aligned}G_{IJKL} &= -\frac{i}{2}\nabla_I\mathcal{F}_{JKL}\\ &= -\frac{i}{2}\mathcal{F}_{IJKL} + \frac{1}{4}\left(\operatorname{Im}\tau^{-1}\right)^{MN}\left(\mathcal{F}_{ILM}\mathcal{F}_{JKN} + \mathcal{F}_{JLM}\mathcal{F}_{IKN} + \mathcal{F}_{KLM}\mathcal{F}_{IJN}\right).\end{aligned} \tag{6.16}$$

Up to conjugation and the trivial symmetry $\Omega_{ABCD} = -\Omega_{ABDC}$, the nonzero curvature components are then

$$\begin{aligned}\Omega_{I\bar{J}K\bar{L}} &= R_{I\bar{J}K\bar{L}} & \Omega^{\bar{I}'}{}_{JK}{}^{\bar{L}'} &= -\left(\operatorname{Im}\tau^{-1}\right)^{\bar{I}'N}\left(\operatorname{Im}\tau^{-1}\right)^{\bar{L}'M} G_{JKMN}\\ \Omega_{\bar{I}J}{}^{K'\bar{L}'} &= R_{\bar{I}J}{}^{K'\bar{L}'} & \Omega^{\bar{I}'}{}_{J\bar{K}}{}^{L'} &= R^{\bar{I}'}{}_{\bar{K}}{}^{L'}{}_J\\ \Omega^{\bar{I}'J'K'\bar{L}'} &= R^{\bar{I}'J'K'\bar{L}'} & \Omega^{\bar{I}'J'}{}_{K\bar{L}} &= R^{\bar{I}'J'}{}_{K\bar{L}}\\ \Omega_{\bar{I}}{}^{J'}{}_K{}^{\bar{L}'} &= R^{\bar{L}'J'}{}_{K\bar{I}} & \Omega_{\bar{I}}{}^{J'}{}_{\bar{K}}{}^{L'} &= \left(\operatorname{Im}\tau^{-1}\right)^{J'\bar{M}}\left(\operatorname{Im}\tau^{-1}\right)^{L'\bar{N}}\bar{G}_{\bar{I}\bar{K}\bar{M}\bar{N}}.\end{aligned} \tag{6.17}$$

We can now read off the action (2.7). From the semi-flat metric (6.11) we obtain the bosonic kinetic terms

$$\mathscr{L}_{bose} = \operatorname{Im}\tau_{IJ}\dot{a}^I\dot{\bar{a}}^J + \left(\operatorname{Im}\tau^{-1}\right)^{IJ}\frac{\delta z_I}{dt}\frac{\delta\bar{z}_J}{dt}, \tag{6.18}$$

where

$$\frac{\delta z_I}{dt} = \dot{z}_I - \mathcal{F}_{IJK}\left(\operatorname{Im}\tau^{-1}\right)^{KL}\operatorname{Im} z_L\dot{a}^J$$

reflects the fact that we're working in the non-coordinate basis (6.9). We denote horizontal fermion components by χ^I and vertical components ζ_I, with e.g $\left(\chi^I\right)^\dagger = \bar{\chi}^{\dagger I}$. As promised, we have now dropped bars and primes in indices. We now indicate antiholomorphic objects with a bar over the object, so for example

$$\bar{\mathcal{F}}_{IJK} = \frac{\partial^3\bar{\mathcal{F}}}{\partial\bar{a}^I\partial\bar{a}^J\partial\bar{a}^K}.$$

The covariant time derivatives (2.8) can be read off from (6.14), giving

$$\begin{aligned}\frac{D\chi^I}{dt} &= \dot{\chi}^I - \frac{i}{2}\left(\operatorname{Im}\tau^{-1}\right)^{IL}\mathcal{F}_{JKL}\dot{a}^K\chi^J \\ &\quad + \frac{i}{2}\bar{\mathcal{F}}_{LMN}\left(\operatorname{Im}\tau^{-1}\right)^{IL}\left(\operatorname{Im}\tau^{-1}\right)^{JM}\left(\operatorname{Im}\tau^{-1}\right)^{KN}\frac{\delta z_K}{dt}\zeta_J \\ \frac{D\zeta_I}{dt} &= \dot{\zeta}_I + \frac{i}{2}\left(\operatorname{Im}\tau^{-1}\right)^{JL}\mathcal{F}_{IKL}\dot{a}^K\zeta_J \\ &\quad + \frac{i}{2}\mathcal{F}_{IJL}\left(\operatorname{Im}\tau^{-1}\right)^{LK}\frac{\delta\bar{z}_K}{dt}\chi^J.\end{aligned} \tag{6.19}$$

The resulting kinetic terms are quite messy, but can be cleared up somewhat by making the redefinition

$$\zeta^I = \left(\operatorname{Im}\tau^{-1}\right)^{IJ}\bar{\zeta}_J \tag{6.20}$$

and using the base space Christoffel symbols

$$\Gamma^I_{JK} = -\frac{i}{2}\mathcal{F}_{JKL}\left(\operatorname{Im}\tau^{-1}\right)^{IL}. \tag{6.21}$$

After making these substitutions we obtain

$$\begin{aligned}\mathscr{L}_{2fermi} &= i\operatorname{Im}\tau_{IJ}\left(\bar{\chi}^{\dagger J}D_t\chi^I + \bar{\zeta}^{\dagger J}D_t\zeta^I\right) \\ &\quad + i\left(\bar{\chi}^{\dagger J}\bar{\zeta}^M + \bar{\zeta}^{\dagger J}\bar{\chi}^M\right)\operatorname{Im}\tau_{IN}\left(\operatorname{Im}\tau^{-1}\right)^{KI}\bar{\Gamma}^N_{JM}\frac{\delta z_K}{dt} + \text{conjugate},\end{aligned} \tag{6.22}$$

where

$$D_t\chi^I = \dot{\chi}^I + \Gamma^I_{JK}\dot{a}^J\dot{\chi}^K$$

is the base space covariant derivative. It's worth noting that χ and ζ appear symmetrically in this expression, suggesting the possibility of combining them into a single object. We will shortly do so to make $SO(5)$ invariance manifest, but for now we move on to the curvature terms. From (6.17) we see that these split into two types according to whether they contain the tensor R or G. The latter gives

$$\mathscr{L}_{4fermi(a)} = 2\,\mathrm{Re}\left(G_{IJKL}\chi^{\dagger I}\chi^{J}\zeta^{\dagger K}\zeta^{L}\right) \tag{6.23}$$

and the former are

$$\begin{aligned}\mathscr{L}_{4fermi(b)} = r_{IJKL}\Big(&-\chi^{\dagger I}\chi^{K}\bar{\chi}^{\dagger J}\bar{\chi}^{L} + \chi^{\dagger I}\zeta^{K}\bar{\chi}^{\dagger J}\bar{\zeta}^{L} + \zeta^{\dagger I}\chi^{K}\bar{\zeta}^{\dagger J}\bar{\chi}^{L} \\ &- \zeta^{\dagger I}\zeta^{K}\bar{\zeta}^{\dagger J}\bar{\zeta}^{L} - \zeta^{\dagger I}\chi^{\dagger K}\bar{\chi}^{J}\bar{\zeta}^{L} + \chi^{I}\zeta^{K}\bar{\chi}^{\dagger J}\bar{\zeta}^{\dagger L}\Big).\end{aligned} \tag{6.24}$$

These terms are not especially enlightening as written, but we'll soon see how to put them in a manifestly $SO(5)$-invariant form. The full Lagrangian for the σ-model on $\mathscr{J}$ is the sum of (6.18), (6.22), (6.23) and (6.24).

To discuss R-symmetry we need to identify the transformation properties of the fundamental fermions χ^I and ζ^I under $SO(5)$. As coordinates, the bosons a^I and z_I are automatically invariant. The generators of $SO(5)$ are as in (2.36), and can be read off from the hyper-Kähler structure (6.11, 6.12, 6.13) using the rules

$$da^I \leftrightarrow \chi^{\dagger I}, \qquad \left(\mathrm{Im}\,\tau^{-1}\right)^{JI}\delta\bar{z}_I \leftrightarrow \zeta^{\dagger J}.$$

The precise formulae are not important to us, but observe that each generator follows the pattern

$$T \sim \mathrm{Im}\,\tau \times \text{holomorphic fermion} \times \text{antiholomorphic fermion}. \tag{6.25}$$

The fermions obey anticommutation relations

$$\left\{\chi^I, \bar{\chi}^{\dagger J}\right\} = \left(\mathrm{Im}\,\tau^{-1}\right)^{IJ} = \left\{\zeta^I, \bar{\zeta}^{\dagger J}\right\},$$

which follow from the same arguments as section 2.1 by temporarily switching to a vielbein basis. We see that the holomorphic and antiholomorphic fermions carry separate actions of $SO(5)$. Indeed, if we define

$$\psi^{IA} = \left(\chi^I, \chi^{\dagger I}, \zeta^I, \zeta^{\dagger I}\right) \tag{6.26}$$

then we see that[23] ψ^{IA} transforms in the $\mathbf{4}$ and $\bar{\psi}^{I\bar{A}} = \left(\psi^{IA}\right)^{\dagger}$ in the $\bar{\mathbf{4}}$. This can be made more explicit by observing that

$$\left\{\psi^{IA}, \bar{\psi}^{J\bar{B}}\right\} = \left(\mathrm{Im}\,\tau^{-1}\right)^{IJ}\delta^{A\bar{B}} \tag{6.27}$$

[23] By convention $A = 1, \ldots, 4$ are fundamental indices for $SU(4) \sim SO(6)$ and $\bar{A} = 1, \ldots, 4$ are antifundamental. $SO(5) \sim USP(4)$ has an antisymmetric invariant tensor Ω_{AB} such that the $\mathbf{4}$ and $\bar{\mathbf{4}}$ are equivalent via $\Omega_{AB}\delta^{B\bar{A}}$, where $\delta_{A\bar{B}} = \mathrm{diag}(1,1,1,1)$ is the invariant tensor of $SU(4)$, but we will keep the two separate notationally in view of the forthcoming extension to $SO(6)$. When we need an explicit formula, we take $\Omega_{23} = \Omega_{41} = 1$.

and defining R-symmetry generators

$$R^{A\bar{B}} = i\,\mathrm{Im}\,\tau_{IJ}\left(\psi^{IA}\bar{\psi}^{J\bar{B}} - \frac{1}{4}\delta^{A\bar{B}}\psi^{IC}\psi^{J}_{C}\right). \tag{6.28}$$

Observe that these actually generate an $SU(4)$ action, in the sense that

$$\begin{aligned}\left[R^{A\bar{B}}, R^{C\bar{D}}\right] &= i\left(\delta^{C\bar{B}}R^{A\bar{D}} - \delta^{A\bar{D}}R^{C\bar{B}}\right)\\ \left[R^{A\bar{B}}, \psi^{IC}\right] &= i\left(\delta^{C\bar{B}}\psi^{IA} - \frac{1}{4}\delta^{A\bar{B}}\psi^{IC}\right).\end{aligned} \tag{6.29}$$

However, this is not a symmetry of the action, as we demonstrate below. We can restrict to an action of $SO(5)$, which is a symmetry, by defining

$$\mathbb{R}^{AB} = R^{AB} + R^{BA}, \tag{6.30}$$

where we used $\Omega^{BC}\delta_{C\bar{B}}$ to convert an antifundamental index to fundamental. These ten generators obey

$$\begin{aligned}\left[\mathbb{R}^{AB}, \mathbb{R}^{CD}\right] &= i\left(\Omega^{BC}\mathbb{R}^{AD} + \Omega^{AC}\mathbb{R}^{BD} + \Omega^{AD}\mathbb{R}^{BC} + \Omega^{BD}\mathbb{R}^{AC}\right)\\ \left[\mathbb{R}^{AB}, \psi^{IC}\right] &= i\left(\Omega^{BC}\psi^{IA} + \Omega^{AC}\psi^{IB}\right)\end{aligned} \tag{6.31}$$

as appropriate to $USP(4)$, but the real proof that they generate a symmetry is to observe that any expression of the form $\Omega_{AB}v^A w^B$, where v^A and w^B are any objects transforming like ψ^{IA} under $SU(4)$, commutes with $\mathbb{R}^{AB}$ but not with $R^{A\bar{B}}$.

We can now put our Lagrangian into a manifestly $SO(5)$-invariant form. The bosonic kinetic terms (6.18) are already invariant. For the fermion kinetic terms (6.22) we need the $SO(5)$-invariant tensor Ω_{AB}. We find

$$\mathscr{L}_{2fermi} = i\,\mathrm{Im}\,\tau_{IJ}\bar{\psi}^{J}_{A}D_t\psi^{IA} - \frac{1}{2}\,\mathrm{Re}\left(\Omega_{\bar{A}\bar{B}}\bar{\mathcal{F}}_{JKL}\left(\mathrm{Im}\,\tau^{-1}\right)^{IJ}\bar{\psi}^{K\bar{A}}\bar{\psi}^{L\bar{B}}\frac{\delta z_I}{dt}\right). \tag{6.32}$$

The 4-fermion terms (6.23) involving G are also straightforward. They are

$$\mathscr{L}_{4fermi(a)} = -\frac{1}{12}\,\mathrm{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right), \tag{6.33}$$

where ϵ_{ABCD} is the Levi-Civita symbol with $\epsilon_{1234} = 1$, an invariant tensor for $SU(4)$. We will also refer to this term as the *chiral* part. The remaining terms (6.24) involving the base space Riemann tensor are more of a puzzle. The solution is to make a change of variables

$$\frac{\delta z_I}{\delta t} = \frac{\delta z_I}{dt} - \frac{1}{4}\mathcal{F}_{IJK}\Omega_{AB}\psi^{JA}\psi^{KB}, \tag{6.34}$$

from which we obtain

$$\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\frac{\delta z_I}{\delta t}\frac{\delta\bar z_J}{\delta t}=\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\frac{\delta z_I}{dt}\frac{\delta\bar z_J}{dt}-\frac{1}{2}\operatorname{Re}\left(\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\mathcal{F}_{ILK}\Omega_{AB}\psi^{LA}\psi^{KB}\frac{\delta\bar z_J}{dt}\right)$$
$$-\frac{1}{4}r_{IJKL}\Omega_{AB}\Omega_{\bar C\bar D}\psi^{IA}\psi^{KB}\bar\psi^{J\bar C}\bar\psi^{L\bar D}.$$

Using this to rewrite the bosonic kinetic terms (6.18), we see that the second term on the right is absorbed in cancelling the second term of (6.32), while the third term combines with the original Riemann tensor terms (6.24). Remarkably, these terms combine together into a manifestly $SO(5)$-invariant form. The end result of all this is the much neater equivalent Lagrangian

$$\begin{aligned}\mathscr{L}&=\operatorname{Im}\tau_{IJ}\dot a^I\dot{\bar a}^J+\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\frac{\delta z_I}{\delta t}\frac{\delta\bar z_J}{\delta t}\\&+i\operatorname{Im}\tau_{IJ}\bar\psi^J_A D_t\psi^{IA}-\frac{1}{12}\operatorname{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right)\\&-\frac{1}{2}r_{IJKL}\psi^{IA}\bar\psi^J_A\psi^{KB}\bar\psi^L_B.\end{aligned}\tag{6.35}$$

This Lagrangian has manifest $SO(5)$-invariance. Of course, since the change of variables (6.34) is itself $SO(5)$-covariant, one could in principle obtain an $SO(5)$-invariant form for the Lagrangian in terms of $\delta z/dt$. However, this would be less compact, and we'll see in the following that the expression we've chosen is the correct one with which to make the jump to $\mathfrak{su}(1,1|4)$.

To close this section, we complete a Hamiltonian analysis of (6.35) and provide explicit formulae for all symmetry generators. From (6.35) we compute the canonical momenta

$$\begin{aligned}P_I&=\frac{\partial\mathscr{L}}{\partial\dot a^I}\\&=\operatorname{Im}\tau_{IJ}\dot{\bar a}^J-\left(\operatorname{Im}\tau^{-1}\right)^{KJ}\left(\operatorname{Im}\tau^{-1}\right)^{LM}\mathcal{F}_{IKL}\operatorname{Im}z_M\frac{\delta\bar z_J}{\delta t}+i\operatorname{Im}\tau_{KJ}\bar\psi^J_A\Gamma^K_{IL}\psi^{LA}\\P^I&=\frac{\partial\mathscr{L}}{\partial\dot z_I}=\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\frac{\delta\bar z_J}{\delta t}.\end{aligned}\tag{6.36}$$

We also define a covariant version Π_I of P_I

$$\Pi_I=\operatorname{Im}\tau_{IJ}\dot{\bar a}^J=P_I-\left(\operatorname{Im}\tau^{-1}\right)^{LM}\mathcal{F}_{IJL}\operatorname{Im}z_M P^J-i\operatorname{Im}\tau_{KJ}\bar\psi^J_A\Gamma^K_{IL}\psi^{LA},\tag{6.37}$$

which should be compared to (2.15). By temporarily switching to a vielbein basis for the fermions, we can apply the Faddeev-Jackiw prescription (see appendix A.2) as in section 2.1 to obtain the commutation relations

$$\begin{aligned}\left[a^I,\Pi_J\right]&=i\delta^I_J&&\\ \left[z_I,\Pi_J\right]&=-2\operatorname{Im}z_K\Gamma^K_{IJ}&\left[z_I,P^J\right]&=i\delta^J_I\\ \left[\Pi_I,P^J\right]&=i\Gamma^J_{IK}P^K&\left[\Pi_I,\bar P^J\right]&=-i\Gamma^J_{IK}P^K\\ \left\{\psi^{IA},\bar\psi^{J\bar B}\right\}&=\delta^{A\bar B}\left(\operatorname{Im}\tau^{-1}\right)^{IJ}&\left[\Pi_I,\psi^{JA}\right]&=i\Gamma^J_{IK}\psi^{KA}.\end{aligned}\tag{6.38}$$

The Hamiltonian following from (6.35) is

$$
\begin{aligned}
H = &\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\Pi_I\bar{\Pi}_J \\
&+\frac{1}{12}\operatorname{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right)+\frac{1}{2}r_{IJKL}\psi^{IA}\bar{\psi}^J_A\psi^{KB}\bar{\psi}^L_B \\
&+\operatorname{Im}\tau_{IJ}P^I\bar{P}^J+\frac{1}{2}\operatorname{Re}\left(\mathcal{F}_{IJK}\Omega_{AB}\psi^{JA}\psi^{KB}P^I\right).
\end{aligned}
\tag{6.39}
$$

It follows from (6.38) that the momenta Π_I and P^I are $SO(5)$-invariant. In fact, they commute with the $SO(6)$ generators (6.28). In particular, the Hamiltonian is $SO(5)$-invariant as it must be, but is not $SO(6)$-invariant since the final term contains the symplectic form Ω_{AB}.

Finally, we consider the generators of $\mathcal{N}=(4,4)$ supersymmetry, given by (2.20) and (2.29). Using the complex structures determined by (6.12) and (6.13) we can read off the charges. For instance, returning to our original non-$SO(5)$-covariant notation, we have

$$
Q = i\chi^{\dagger I}\Pi_I + i\operatorname{Im}\tau_{IJ}\bar{\zeta}^{\dagger J}P^I+\frac{i}{2}\zeta^{\dagger L}\mathcal{F}_{JLM}\left(\chi^{\dagger M}\zeta^J+\zeta^{\dagger M}\chi^J\right)-\text{complex conjugate},
$$

along with similar expressions for Q^a, $Q^\dagger$, and $Q^{\dagger a}$. Taking suitable linear combinations[24] of these, we obtain supercharges which manifestly transform in the $\mathbf{4}$ of $SO(5)$

$$
Q^A = \psi^{IA}\Pi_I+\frac{1}{12}\epsilon^A{}_{\bar{B}\bar{C}\bar{D}}\bar{\mathcal{F}}_{IJK}\bar{\psi}^{I\bar{B}}\bar{\psi}^{J\bar{C}}\bar{\psi}^{K\bar{D}}+\operatorname{Im}\tau_{IJ}P^I\Omega^A{}_{\bar{B}}\bar{\psi}^{J\bar{B}},
\tag{6.40}
$$

along with the conjugate $\bar{Q}^{\bar{A}}=\left(Q^A\right)^\dagger$ which tranforms in the $\bar{\mathbf{4}}$. These obey the standard supersymmetry algebra

$$
\left\{Q^A,Q^B\right\}=0,\qquad \left\{Q^A,\bar{Q}^{\bar{B}}\right\}=\delta^{A\bar{B}}H.
$$

As with the Hamiltonian, the supercharges do not transform nicely under $SO(6)$ owing to the presence of the symplectic form Ω_{AB}. This completes our construction of the σ-model on $\mathcal{J}$.

6.4 Symmetry Enhancement at Zero Fibre Momentum and $\mathfrak{su}(1,1|4)$

In this section we consider a reduction of our σ-model to the sector of zero fibre momentum. This is motivated by the fact that the fibre becomes small in the limit $R\to\infty$ for compactification of a four-dimensional $\mathcal{N}=2$ gauge theory on $\mathbb{R}^3\times S^1_R$, as is relevant to the DLCQ of $\mathcal{N}=4$ SUSY Yang-Mills. We'll see that such a truncation can be made and automatically enhances the

[24] At least if we are slightly carefree with operator ordering. Strictly speaking, everything in this section is valid at the level of Poisson brackets, but we do not anticipate this being a real obstacle. For example, the Hamiltonian (6.39) is not chosen to be consistent with $H=\frac{1}{2}\Delta$, but we saw in section 2.1 that such a choice is possible.

$SO(5)$ R-symmetry to $SO(6)$. Assuming that the base special Kähler manifold is scale-invariant in an appropriate sense, we'll also show that this truncation allows a superconformal symmetry to develop. This is identified as $\mathfrak{su}(1,1|4)$, and we give an explicit construction of its generators and relations.

We start by checking that the truncation $P^I = \bar{P}^J = 0$ is consistent. This follows by observing that the commutation relations (6.38) are consistent with zero. Indeed, the Heisenberg equation $\dot{P}^I = i\left[H, P^I\right]$ reads

$$\left[H, P^I\right] = \left(\operatorname{Im}\tau^{-1}\right)^{JK}\left(\operatorname{Im}\tau^{-1}\right)^{IM}\operatorname{Re}\left(\Pi_J\bar{\mathcal{F}}_{KLM}P^L\right),$$

from which follow the $2g$ mutually commuting conserved charges

$$Q_{eI} = \operatorname{Re}\left(\tau_{IJ}P^J\right), \qquad Q_m^I = \operatorname{Re}P^I. \tag{6.41}$$

Setting these to zero gives the desired truncation. The existence of these $U(1)$ charges follows from the semi-flat structure of the metric (6.7), which has $2g$ commuting isometries corresponding to shifts of the coordinates θ_{mI} and θ_e^I. It is a general rule, following from the geometric picture of section 2.1, that any such isometry generates a symmetry.

Having made the truncation to zero fibre momentum, we find that all terms in the Hamiltonian (6.39) and supercharges (6.40) which are invariant only under $SO(5) \subset SO(6)$ drop out. The truncated model therefore exhibits an R-symmetry enhancement to an $SO(6)$ generated by (6.28), with new Hamiltonian and SUSY generators given by

$$\begin{aligned}
H &= \left(\operatorname{Im}\tau^{-1}\right)^{IJ}\Pi_I\bar{\Pi}_J \\
&\quad + \frac{1}{12}\operatorname{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right) + \frac{1}{2}r_{IJKL}\psi^{IA}\bar{\psi}^J_A\psi^{KB}\bar{\psi}^L_B \\
Q^A &= \psi^{IA}\Pi_I + \frac{1}{12}\epsilon^A{}_{\bar{B}\bar{C}\bar{D}}\bar{\mathcal{F}}_{IJK}\bar{\psi}^{I\bar{B}}\bar{\psi}^{J\bar{C}}\bar{\psi}^{K\bar{D}},
\end{aligned} \tag{6.42}$$

where the covariant momentum reduces to

$$\Pi_I = P_I - i\operatorname{Im}\tau_{KJ}\bar{\psi}^J_A\Gamma^K_{IL}\psi^{LA}.$$

Using (6.35) and (6.36) we also obtain a truncated Lagrangian

$$\begin{aligned}
\mathscr{L} &= \operatorname{Im}\tau_{IJ}\dot{a}^I\dot{\bar{a}}^J + i\operatorname{Im}\tau_{IJ}\bar{\psi}^J_A D_t\psi^{IA} \\
&\quad - \frac{1}{12}\operatorname{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right) - \frac{1}{2}r_{IJKL}\psi^{IA}\bar{\psi}^J_A\psi^{KB}\bar{\psi}^L_B
\end{aligned} \tag{6.43}$$

with manifest $SO(6)$ invariance.

We now turn to superconformal symmetry. We saw in section 5 that the presence of a conformal symmetry can be tied in to the existence of a homothetic vector field $D : \mathcal{L}_D g = 2g$ on the target manifold. Of course, by truncating to zero fibre momentum we have moved away from the manifestly geometric setting in which this was derived, but we'll see that this is still the right way to think about things. However, there is no reason to expect our target special Kähler manifold to have such a homothety. To make progress we need the following:

Definition

A manifold is called *scale invariant special kähler* (SISK) if it is special Kähler and has a prepotential satisfying

$$a^I a_I^D = 2\mathcal{F}. \tag{6.44}$$

It's important to ask whether this condition has any interesting solutions. An obvious one is flat space $\mathbb{C}^n$, where the prepotential is a quadratic polynomial in the a^I. In terms of Coulomb branches of four-dimensional gauge theories, this corresponds to the finite $\mathcal{N} = 4$ theory and as such is potentially of some physical interest, but we can do much better. The SISK condition follows if and only if the prepotential is homogeneous of degree two in a, so any function of the form

$$\mathcal{F} = \left(a^1\right)^2 f\left(\frac{a^I}{a^J}\right)$$

will do. A large class of such prepotentials are provided by physical examples. If the underlying four-dimensional $\mathcal{N} = 2$ gauge theory giving rise to a special Kähler Coulomb branch is superconformal then the Coulomb branch carries an action of the field theoretic dilatation operator, since the scale transformation of a zero energy state is a zero energy state. One can check that this action induces the SISK condition. For a quick justification of this, observe that the BPS mass formula $M \geq |Z|$, with Z as in (6.4), implies that both a^I and a_I^D have mass dimension one, since the charges n_{eI} and n_m^I are integer-valued and cannot scale. Since $a_I^D = \partial\mathcal{F}/\partial a^I$, it follows that $\mathcal{F}$ has mass dimension two. Consistency with the usual scale transformation rules for operators in conformal field theory (see [12]) demands that the scaling action is by the homothety

$$D = a^I \frac{\partial}{\partial a^I} + \bar{a}^I \frac{\partial}{\partial \bar{a}^I} \tag{6.45}$$

and that the prepotential satisfies the SISK condition (6.44). It follows that any four-dimensional $\mathcal{N} = 2$ superconformal field theory gives rise to a scale-invariant special Kähler manifold. In particular, the quiver theory corresponding to the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills is of this type [52].

More generally, any SISK manifold has a homothety of the form (6.45). This can be used to construct a dilatation operator

$$D = a^I \Pi_I + \bar{a}^I \bar{\Pi}_I, \tag{6.46}$$

and we need to find a special conformal generator which is consistent with this choice and with the Hamiltonian (6.42), in the sense that the $SO(2,1)$ algebra (5.2) is obeyed. In section 5 we showed that in the (hyper-)Kähler case we get a special conformal generator from a (hyper-)Kähler potential. One can easily check that the base space Kähler potential (6.3) works, so with

$$K = \operatorname{Im}\left(a_I^D \bar{a}^I\right) \tag{6.47}$$

we get an $SO(2,1)$ conformal algebra. In particular, the fundamental bosons a^I and $\bar{a}^I$ have dimension one

$$\left[D, a^I\right] = -ia^I,$$

as do the dual bosons a_I^D, in agreement with the physical argument above, while the fermions have dimension zero as a consequence of the SISK condition and (6.38). Observe also that the $SO(2,1)$ generators manifestly commute with the generators (6.28) of $SO(6)$.

Turning to supersymmetry, we can easily check that the supercharges (6.42) also have the correct scaling dimension. We define superconformal generators via $S^A = -i\left[K, Q^A\right]$, giving

$$S^A = \operatorname{Im} \tau_{IJ} \bar{a}^J \psi^{IA} \tag{6.48}$$

along with conjugates $\bar{S}^{\bar{A}} = -i\left[K, \bar{Q}^{\bar{A}}\right] = \left(S^A\right)^\dagger$. The scaling dimensions and $SO(6)$ transformation properties of these are as expected, as are the relations

$$\begin{aligned} &\left\{S^A, S^B\right\} = 0 && \left[K, S^A\right] = 0 \\ &\left\{S^A, \bar{S}^{\bar{B}}\right\} = \delta^{A\bar{B}} K && \left[H, S^A\right] = -iQ^A. \end{aligned}$$

It remains to check the relations of the form $\{Q, S\}$. Doing so reveals a $U(1)$ R-symmetry

$$\mathcal{R} = i\left(a^I \Pi_I - \bar{a}^I \bar{\Pi}_I\right) + \frac{1}{2} \operatorname{Im} \tau_{IJ} \psi^{IA} \bar{\psi}_A^J \tag{6.49}$$

with charges

a^I	$\bar{a}^{\bar{I}}$	ψ^{IA}	$\bar{\psi}^{\bar{I}\bar{A}}$	Q^A	S^A	$\bar{Q}^{\bar{A}}$	$\bar{S}^{\bar{A}}$
1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

With this definition, we have

$$\begin{aligned} &\left\{Q^A, S^B\right\} = 0 \\ &\left\{Q^A, \bar{S}^{\bar{B}}\right\} = \frac{1}{2}\delta^{A\bar{B}}\left(D - i\mathcal{R}\right) - R^{A\bar{B}}. \end{aligned}$$

This completes our construction. The resulting generators and relations are summarised in appendix E.4. The algebra we have obtained is a simple superalgebra with bosonic subalgebra

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(4)$$

and fermion representation $(\mathbf{2},\mathbf{4})_- \oplus (\mathbf{2},\bar{\mathbf{4}})_+$, where $\pm$ are $U(1)$ charges. Consulting the lists of [47–49] we identify this algebra with $\mathfrak{su}(1,1|4)$. These constructions apply in particular to the $\hat{A}_{N-1}$ quiver theory on $\mathbb{R}^3 \times S^1_R$, whose Coulomb branch in the large-R limit describes the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills. The brane construction of this model from the $(2,0)$ theory, described in section 4.3, suggests that at finite R the model should have eight supercharges, $SO(5)$ R-symmetry coming from rotations transverse to the branes, and should not be superconformal, while in the large-R limit the symmetry should enhance to $\mathfrak{su}(1,1|4)$. Our construction has exactly reproduced this expectation.

It's interesting to note that, despite our truncation taking us away from an explicitly geometric formulation, it is still evident that the geometry is there. The homothety (6.45) is closed with respect to K, and in fact satisfies all the conditions of the theorem of section 5, and the $U(1)$ R-symmetry corresponds to the holomorphic isometry ID. This suggests that, given a suitable formalism for the fermions, it may be possible to make this model manifestly geometric. Some of the structures in [117] involving vector bundles for $Sp(2g;\mathbb{Z})$ look promising in this regard.

6.5 Beyond Zero Fibre Momentum and Magnetic Fields

Finally we examine what can be said about our model without truncating to zero fibre momentum. The motivation for this comes from the spectral problem for the dilatation operator (6.46), particularly in the context of the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills. It is often convenient [3, 100] to make a basis change for the conformal algebra $\mathfrak{sl}(2;\mathbb{R})$ under which

$$iD \mapsto L_0 = \mu^{-1}(H + \mu^2 K),$$

with $\mu \in (0,\infty)$. The algebraic details of this transformation are considered in section 7.1. For now we note that L_0 should have a well-defined discrete spectrum provided that K is smooth and grows at infinity, as is the case for (6.47) in our models of interest. Indeed, for a flat Coulomb branch L_0 is nothing but a bosonic harmonic oscillator Hamiltonian. Unfortunately, for the curved Coulomb branch corresponding to the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills, the Kähler potential vanishes at $a = 0$ but its derivatives there have logarithmic singularities [52]. In fact, conformal invariance dictates that all singular submanifolds of $\mathcal{M}$ intersect at the origin. The effect of this is to concentrate the wavefunctions for eigenstates of L_0 at a highly singular point, rendering the spectral problem ill-defined without some form of regulator. One possibility is to go beyond the semi-flat metric, where singularities are smoothed out by new BPS corrections [17, 51], but the explicit loss of conformal invariance in this case is difficult to work around. Instead, in the remainder of this section we give an alternative regularisation which is intrinsic to the semi-flat geometry. We'll see that working at nonzero fibre momentum can shift the centre of the effective potential in L_0 to a generically non-singular point of $\mathcal{M}$, and that such a shift preserves a non-trivial symmetry algebra $\mathfrak{osp}(1,1|4)$ which is in some sense superconformal. We interpret this shift as coupling the σ-model (2.7) to a worldline magnetic field, and in terms of a spacetime magnetic field coupled to instanton number density in DLCQ. An adaptation of the moduli space approximation discussed in section (3.3) gives a concrete link between the two viewpoints. Furthermore, this regularisation will reveal an integrable limit whose signficance, particularly in the context of planar integrability in $\mathcal{N}=4$ SUSY Yang-Mills, is yet to be understood.

Consider now the sector of the σ-model (6.35) characterised by arbitrary fixed values of the conserved quantities (6.41). Although this sector is not conformally invariant, there is still a natural generalisation of L_0 given by

$$\begin{aligned}\tilde{L}_0 &= L_0 - \operatorname{Im}\tau_{IJ} P^I \bar{a}^J - \operatorname{Im}\tau_{IJ}\bar{P}^I a^J \\ &= \mu^{-1}\left(\operatorname{Im}\tau^{-1}\right)^{IJ}\Pi_I\bar{\Pi}_J + \mu^{-1}\operatorname{Im}\tau_{IJ}\left(P^I-\mu a^I\right)\left(\bar{P}^J-\mu\bar{a}^J\right) + \text{ fermions},\end{aligned} \tag{6.50}$$

which reduces to L_0 at zero fibre momentum. We see that the potential has been translated to have its zero at

$$a^I = \mu^{-1}P^I.$$

P^I can be determined as a function of the base space coordinates and the conserved charges (6.41), so this condition generically picks out a non-singular point of $\mathcal{M}$. It follows that eigenstates of $\tilde{L}_0$ will be localised away from singular points. This gives us a much better chance of making sense of the spectral theory of $\tilde{L}_0$, and we hope that a suitable limiting procedure would produce information about L_0.

If we are to use this regularisation then it's important to identify the symmetries it preserves. We naturally keep $SO(5)$ R-symmetry as generated by (6.30) as it is a symmetry of the full σ-model with dynamical fibre momentum, but $SO(6)$ is lost since $\tilde{L}_0$, like the full Hamiltonian (6.39), contains the invariant tensor Ω_{AB}. Similarly, $\tilde{L}_0$ is not part of an $\mathfrak{sl}(2;\mathbb{R})$ conformal algebra. Nevertheless, there is a preserved superalgebra with eight real supercharges

$$q^A = \mu^{-1/2}\left(Q^A + i\Omega^A{}_{\bar{B}}\bar{Q}^{\bar{B}}\right) + i\mu^{1/2}\left(S^A + i\Omega^A{}_{\bar{B}}\bar{S}^{\bar{B}}\right) \tag{6.51}$$

and $\bar{q}^{\bar{A}} = (q^A)^\dagger$. These transform in the $\mathbf{4}$ of $SO(5)$ and satisfy

$$\left[\tilde{L}_0, q^A\right] = q^A, \qquad \left\{q^A, \bar{q}^{\bar{A}}\right\} = 2\delta^{A\bar{A}}\tilde{L}_0 - 2i\Omega^{\bar{A}}{}_B\mathbb{R}^{AB},$$

with $\mathbb{R}^{AB}$ as in (6.30). Referring again to the lists of [47–49] we identify this superalgebra with $\mathfrak{osp}(1,1|4)$. We collect the generators and relations in appendix E.5. In particular, since the supercharges do not commute with $\tilde{L}_0$ it shares many features of superconformal algebras, even though it does not have a conformal subalgebra.

$\tilde{L}_0$ is not a symmetry generator in the original quantum mechanics on $\mathscr{J}$, so one might wonder whether it has any physical significance or is just a convenient mathematical trick. To answer this, we observe that $\tilde{L}_0$ can be obtained from the σ-model Hamiltonian (6.39) by the replacement $P^I \mapsto P^I - \mu a^I$, that is

$$\tilde{L}_0 = \mu^{-1}\, H|_{P\mapsto P-\mu a}\,.$$

The original model with Hamiltonian H describes a free particle moving on $\mathscr{J}$, so we interpret this deformation as coupling to a background anti-self-dual (ASD) magnetic field with potential and field strength

$$\begin{aligned}\mathscr{A} &= \mu\left(a^I dz_I + \bar{a}^I d\bar{z}_I\right)\\ \mathscr{F} &= \mu\left(da^I \wedge dz_I + d\bar{a}^I \wedge d\bar{z}_I\right) = \mu(\eta + \bar{\eta}),\end{aligned} \tag{6.52}$$

where η is the holomorphic symplectic form of $\mathscr{J}$ with respect to the preferred complex structure.

A natural question to ask is whether we can understand this magnetic field from the perspective of DLCQ and the low-energy dynamics of instantons in 4+1 dimensions. Recall from section 4.3 that quantum mechanics on $\mathscr{J}$ is obtained by compactifying the (2,0) theory on a torus of complex structure τ and on a null circle, leading to quantum mechanics on the moduli space of instantons on $\mathbb{R}^2 \times T^2_\tau$. This is a low-energy approximation to the dynamics of instantons in five-dimensional $\mathcal{N} = 2$ SUSY Yang-Mills on $\mathbb{R}^{2,1} \times T^2_\tau$. We can parameterise $\mathbb{R}^2 \times T^2_\tau$ with a pair of complex coordinates $u = x_1 + ix_2$ and $v = x_3 - ix_4$, where v is doubly periodic with periods determined by τ and the area of T^2_τ. $\mathbb{R}^2 \times T^2_\tau$ is naturally hyper-Kähler with holomorphic symplectic form

$$\theta = du \wedge dv.$$

Since instanton moduli space always contains a centre of mass factor identical to space, a reasonable guess is that the target space magnetic field (6.52) could be reproduced by coupling instantons to the ASD magnetic field

$$\begin{aligned} a &= \mu\left(udv + \bar{u}d\bar{v}\right) \\ f &= \mu\left(\theta + \bar{\theta}\right). \end{aligned} \tag{6.53}$$

This intuition can be strengthened by a more careful spacetime analysis in DLCQ [3,129,130], but all we need from this is that the appropriate coupling constant (the 'electric' charge of an instanton) is set by the light-cone momentum $p_+ = K/R_-$. Recall that K is also the instanton number, so the field (6.53) should couple to the instanton number density (3.28).

We will support the above discussion by adapting the moduli space approximation of section 3.3 to the case of a background ASD magnetic field coupling to (3.28). This derivation is valid for more general fields than (6.53) so we specialise only when necessary. We neglect supersymmetry, so our starting point is the action

$$\begin{aligned} S &= -\frac{1}{g^2}\int_{\mathbb{R}^{2,1}\times T^2} \mathrm{tr}\left\{F\wedge *F + \mu F\wedge F\wedge a\right\} \\ &= -\frac{1}{2g^2}\int d^5x\,\mathrm{tr}\left\{F_{MN}F^{MN} - \frac{\mu}{2}\epsilon^{MNPQR}F_{NP}F_{QR}a_M\right\}, \end{aligned} \tag{6.54}$$

where $x^M : M = 0,\ldots,4$ are spacetime coordinates, a is the background magnetic field and μ controls the coupling of a to the instanton density (3.28). We use a space-time splitting $x^M = (t, x_m)$ and assume that, in a suitable gauge, $a_t = 0$ and a_m is time-independent and ASD. The equations of motion following from (6.54) are

$$\begin{aligned} D_m F_{tm} + \frac{\mu}{4}\epsilon_{mnpq}F_{mn}f_{pq} &= 0 \\ D_t F_{tm} - D_n F_{nm} - \frac{\mu}{2}\epsilon_{mnpq}F_{tn}f_{pq} &= 0. \end{aligned} \tag{6.55}$$

A remarkable feature of these equations is that, by virtue of the anti-self-duality of f, a static instanton is still an exact solution. It therefore makes sense to employ the moduli space approximation. We make the same ansätze (3.30) as for the usual instanton problem and expand (6.54) to quadratic order in $\dot{X}^\mu$, finding

$$S_X = \int dt - \frac{1}{2}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \mathscr{A}_\mu\dot{X}^\mu, \tag{6.56}$$

where g is the moduli space metric (3.14) and

$$\mathscr{A}_\mu = -2\mu\int d^4x\,\mathrm{tr}\left\{\delta_\mu A_m F_{mn} a_n\right\} \tag{6.57}$$

is interpreted as a background magnetic field on moduli space, coupling to the free particle described by X.

To find a compact expression for the field strength, it is convenient to obtain an alternative expression for $\mathscr{A}$ by integrating (6.54) by parts

$$\mathscr{A}_\mu = -\mu \int d^4x \, \mathrm{tr}\left\{A_m \partial_\mu A_n\right\} f_{mn} + \mu \int_{S^1_\infty \times T^2} dS_m \epsilon_{mnpq} a_n \, \mathrm{tr}\left\{\Omega_\mu F_{pq} - A_p \partial_\mu A_q\right\}, \tag{6.58}$$

where $S^1_\infty \times T^2$ is the asymptotic spatial region, dS_m is the outward facing normal density and Ω_μ is the gauge-fixing parameter defined by (3.13). From this we define a field strength $\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu$. To get a handle on the resulting expression we rewrite partial derivatives $\partial_\mu A_n$ in terms of the zero modes $\delta_\mu A_n$ (3.13). Using integration by parts and (anti)-self-duality, we get

$$\mathscr{F}_{\mu\nu} = -2\mu \int d^4x \, \mathrm{tr}\left\{\delta_\mu A_m \delta_\nu A_n\right\} f_{mn} - 2\mu \int_{S^1_\infty \times T^2} a \wedge \mathrm{tr}\left\{\delta_\mu A \wedge \delta_\nu A\right\} + \text{extras},$$

where 'extras' consists of other contributions from the boundary at infinity with integrand proportional to F. Assuming that a grows no faster than linearly at infinity, these terms vanish since one can show that F is $O(r^{-2})$ as $r \to \infty$ [131]. Similarly, the remaining boundary term is gauge invariant and there is a gauge in which the integrand is $o(r^{-1})$ so this term vanishes.[25] We obtain our final expression for the field strength

$$\mathscr{F}_{\mu\nu} = -2\mu \int d^4x \, \mathrm{tr}\left\{\delta_\mu A_m \delta_\nu A_n\right\} f_{mn}. \tag{6.59}$$

Up to now we only needed anti-self-duality of f along with $f = da$ and sufficiently well-behaved asymptotics, so this method of lifting ASD static magnetic fields to instanton moduli space is quite general. If we specialise to the case (6.53) then we claim that (6.59) reproduces the target space holomorphic symplectic form (6.52). This follows directly from the standard formulae (3.14, 3.15) for lifting the hyper-Kähler structure of $\mathbb{R}^2 \times T^2_\tau$ to a canonical hyper-Kähler structure on moduli space.

Now that we have a sound understanding of the origin of the deformation (6.50), we briefly consider what we get from it. As already stated, we have a deformation of the spectral problem for L_0 which appears to be well-defined in that we expect a discrete spectrum with wavefunctions isolated away from singular points. This problem is in turn related by DLCQ to the calculation of scaling dimensions in $\mathcal{N} = 4$ SUSY Yang-Mills. These are known to exhibit an integrable structure [22], and we would like to understand this integrability in DLCQ. There is a lot to be

[25]Similar considerations lead to the same conclusion for instantons on $\mathbb{R}^4$.

done here, and some early considerations are found in [3], but for now we note the existence of an easy integrable limit. Observe that the parameter μ has dimensions of mass, so we can define dimensionless variables by transforming $a \mapsto a\mu^{-1}$. The potential in $\tilde{L}_0$ whose role is to localise the wavefunction away from singularities now reads

$$\mu \operatorname{Im} \tau_{IJ} \left(P^I - a^I\right) \left(\bar{P}^J - \bar{a}^J\right).$$

We see that, when $\mu \to \infty$, the localisation $P = a$ is absolute and is imposed as a constraint on the dynamics of states with finite energy. As a consequence, a^I becomes canonically conjugate to z_I so the reduced phase space is the holomorphic integrable system of [125] with phase space $\mathscr{J}$ and Poisson bracket given by $\eta = da^I \wedge dz_I$. This picture can be justified carefully using the Dirac bracket procedure with primary constraints $C^I = P^I - a^I$ (see appendix A.1). For example, one can check that $\left[a^I, z_J\right]_D = -\delta^I_J$. The integrable limit is described by the simple first-order Lagrangian

$$\mathscr{L} = a^I \dot{z}_I + \bar{a}^I \dot{\bar{z}}_I.$$

The relation of this integrability to $\mathcal{N} = 4$ SUSY Yang-Mills is unknown and left to future work.

7 Unitary Representations and Indices of $\mathfrak{osp}(4^*|4)$

In this section, based on yet to be published work, we begin a more in-depth analysis of the superconformal models of section 5 and their application to DLCQ with a careful analysis of the representation theory of $\mathfrak{osp}(4^*|4)$. In particular, we give a complete list of irreducible unitary lowest weight representations and use this to give BPS conditions. We define an index based on states annihilated by $E = \{q, s\}$ for a choice of supercharge q with $s = q^\dagger$. A thorough analysis of quantum mechanics on $\mathbb{C}^{2k}$ leads to a link between states contributing to the index and a form of Dolbeault cohomology, and the extension of this idea to general superconformal hyper-Kähler quantum mechanics provides a form of Hodge theory for E. Only BPS representations contain states with $E = 0$, and we characterise these states in terms of representations of the subalgebra $\mathfrak{su}(2|1) \subset \mathfrak{osp}(4^*|4)$ commuting with $\{q, s, E\}$. Combined with the branching rules for generic long representations at the unitarity threshold, this allows us to construct the superconformal index [53, 54] for $\mathfrak{osp}(4^*|4)$. We calculate the low lying BPS content in some simple examples, and comment on the connection between our formalism and those based on supersymmetric localisation [100, 126]. An explicit set of generators, root systems and conventions for $\mathfrak{osp}(4^*|4)$ which we use throughout can be found in appendix D.1.

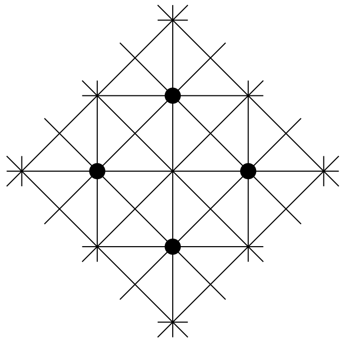
7.1 Representations

We begin with a recap of the unitary irreducible representations (*irreps*) of the bosonic subalgebra

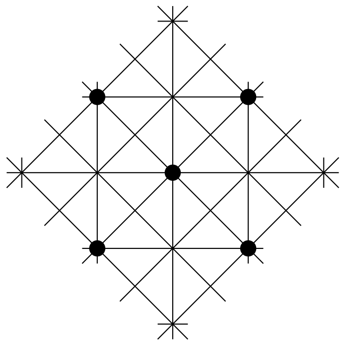
$$\mathfrak{g}_B = \mathfrak{su}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(4).$$

$\mathfrak{su}(2)$ and $\mathfrak{usp}(4)$ are the compact real forms of the simple Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{sp}(4)$, so their representation theory is standard (see [132] for the details). For $\mathfrak{su}(2)$, the lowest weight state has non-positive integer eigenvalue $-2j$ for the Cartan generator $2J_3$. The raising operator J_+ increases this eigenvalue by 2 and the highest weight is $2j$. The simplest non-trivial representation is the **2**, which has $j = 1/2$, and all higher values of j are obtained as symmetric tensor powers of the **2**.

For $\mathfrak{usp}(4)$ the lowest weight state v_λ is characterised by two integers $m \geq n \geq 0$ such that the Cartan generators M and N satisfy $Mv_\lambda = -mv_\lambda$ and $Nv_\lambda = -nv_\lambda$. We label these representations (m, n). The simplest representation is the $\mathbf{4} = (1, 0)$, which has weight diagram



with M increasing to the left and N upwards. The next simplest is the $\mathbf{5} = (0,1)$, obtained as a factor in $\Lambda^2(\mathbf{4}) = \mathbf{5} \oplus \mathbf{1}$. The $\mathbf{1}$ here corresponds to the invariant symplectic form Ω_{AB}. The weight diagram of the $\mathbf{5}$ is



Note that the $\mathbf{4}$ and $\mathbf{5}$ are respectively the spinor and vector representations of $\mathfrak{so}(5) \cong \mathfrak{usp}(4)$. The next simplest representations are $(m,0)$, which are the mth symmetric powers of the $\mathbf{4}$. Symmetric powers of the $\mathbf{5}$ are not irreps as ϵ_{ABCD} is an invariant tensor, or equivalently δ_{ij} is an invariant tensor for the vector of $\mathfrak{so}(5)$. To get the irrep $(0,n)$ take the nth symmetric power of the $\mathbf{5}$ and remove all traces with respect to δ_{ij}. Representations with general (m,n) are harder and need the technology of Young tableaux to give a useful description. Fortunately we won't need this, as we'll only need the lowest weight states. For these, it is sufficient to take tensor products of the lowest weight states for $(m,0)$ and $(0,n)$. A general weight diagram is a series of concentric octagons, unless $m = n$ or $n = 0$ at which point squares appear instead.

Turning to $\mathfrak{sl}(2;\mathbb{R}) \cong \mathfrak{so}(2,1) \cong \mathfrak{su}(1,1)$, we have a choice to make about Hermitian conjugation. Up to now we have used the basis $\{D, H, K\}$ of Hermitian operators satisfying (5.2), but this is awkward for representation theory. In particular, we saw in section 5 that D acts as a first-order differential operator on the Hilbert space $\Omega^*(M;\mathbb{C})$ and as such would be expected to have a continuous spectrum, whereas the usual expectation is that the dilatation operator has a discrete spectrum. To remedy this, consider the basis change

$$X \mapsto e^{-\mu K} e^{\frac{1}{2}\mu^{-1}H} X e^{-\frac{1}{2}\mu^{-1}H} e^{\mu K} := M^{-1} X M \qquad \forall X \in \mathfrak{so}(2,1) \tag{7.1}$$

with $\mu \in (0, \infty)$, under which

$$\begin{aligned} iD &\mapsto L_0 = \mu^{-1}\left(H + \mu^2 K\right) \\ H &\mapsto 2\mu L_- = \mu\left(\mu^{-1}H - \mu K - iD\right) \\ K &\mapsto -\frac{1}{2\mu}L_+ = -\frac{1}{4\mu}\left(\mu^{-1}H - \mu K + iD\right). \end{aligned} \tag{7.2}$$

These satisfy

$$L_0^\dagger = L_0, \qquad L_+^\dagger = L_-, \qquad [L_0, L_\pm] = 2L_\pm, \qquad [L_+, L_-] = -L_0,$$

which is a form of $\mathfrak{sl}(2;\mathbb{R})$ with conventions in agreement with appendix D.1. L_0 is represented by a second-order differential operator on Hilbert space, namely the Laplace operator with a regulating potential provided by K, so we expect it to have a discrete spectrum. We should be careful in interpreting the transformation (7.1), since as a similarity transformation $X \mapsto M^{-1}XM$ we would expect it to preserve spectra, but we find that the anti-Hermitian operator iD with continuous spectrum is mapped to the Hermitian operator L_0 with discrete spectrum. Moreover, the Hermitian operators H and K are mapped to the non-diagonalisable operators $L_\pm$! The solution to this apparent paradox is that (7.1) should be thought of as a map between distinct representations. For instance, normalisable states are mapped into non-normalisable ones, so the Hilbert space is changed. In conformal field theory a similar situation occurs in radial quantisation (see [12]). The reader might also be concerned that H, which is often thought of as a raising operator for dimension, is mapped to L_-, a lowering operator. There is no contradiction here, since the change of Hilbert spaces induced by (7.1) need not map lowest weight representations of $\{D,\ H,\ K\}$ to lowest weight representations of $\{L_0,\ L_\pm\}$, or vice-versa.

A similar transformation to the above is used in [133] to prove that positivity of the physical energy H implies that L_0 has non-negative real eigenvalues, which we refer to as *scaling dimensions*. To see this, observe that the unitary transformation $X \mapsto UXU^\dagger$ with $U = \exp\left[\frac{i\pi}{2}(H+K)\right]$ maps H to K. Thus if H has positive spectrum then so does K, from which it follows that $H + \mu^2 K$ also has positive spectrum since

$$\langle\psi| \left(H + \mu^2 K\right) |\psi\rangle = \langle\psi| H |\psi\rangle + \mu^2 \langle\psi| UHU^\dagger |\psi\rangle \geq 0 \qquad \forall \psi \in \mathcal{H}.$$

It follows that the unitary irreps of interest are lowest weight representations with lowest weight $\Delta \geq 0$, built from a vector v_Δ satisfying

$$L_0 v_\Delta = \Delta v_\Delta, \qquad L_- v_\Delta = 0, \qquad \|v_\Delta\| = 1$$

by repeated action of L_+. Inner products are determined via the commutation relations and the module is manifestly unitary for all $\Delta \geq 0$. For $\Delta > 0$ it is infinite-dimensional while for $\Delta = 0$ it is trivial.

Moving on to $\mathfrak{osp}(4^*|4)$, we denote the space of raising operators by $\mathfrak{n}^+$ and the lowering operators by $\mathfrak{n}^-$. These can be further subdivided into bosonic and fermionic parts $\mathfrak{n}_0^\pm$ and $\mathfrak{n}_1^\pm$ respectively, so that

$$\mathfrak{osp}(4^*|4) = \mathfrak{h} \oplus \mathfrak{n}_0^+ \oplus \mathfrak{n}_0^- \oplus \mathfrak{n}_1^+ \oplus \mathfrak{n}_1^- .$$

The above review can be summarised as follows. We work with lowest weight representations of the bosonic subalgebra $\mathfrak{g}_B$, with lowest weight

$$\lambda = \frac{\Delta}{2}(\epsilon_1 + \epsilon_2) - j(\epsilon_1 - \epsilon_2) - m\delta_1 - n\delta_2 \tag{7.3}$$

in the conventions for $\mathfrak{osp}(4^*|4)$ defined in appendix D.1. Here $(\Delta, -2j, -m, -n)$ are the eigenvalues of the Cartan generators $(L_0, 2J_3, M, N)$ on the lowest weight vector v_λ. Given any lowest weight vector, one can define a bosonic *Verma module*[26] $\mathcal{W}_\lambda = U(\mathfrak{n}_0^+)v_\lambda$, which is reducible and non-unitary in general. To obtain a unitary irrep, we impose $\Delta \geq 0$ and $(2j, m, n) \in \mathbb{N}^3$ with $m \geq n$, then take a quotient of $\mathcal{W}_\lambda$ by the resulting *null states*: states of zero norm in the Verma module. This restricts us to the usual finite-dimensional lowest weight representations of R-symmetry, but has no effect on $\mathfrak{so}(2,1)$. We denote the resulting module by W_λ.

We now wish to extend this analysis to classify the unitary irreducible representations of $\mathfrak{osp}(4^*|4)$. Any such representation must in particular be a unitary representation of $\mathfrak{g}_B$, and hence a sum of lowest weight representations W_λ. It follows that we can restrict ourselves to quotients of lowest weight Verma modules $\mathcal{V}_\lambda = U(\mathfrak{n}^+)v_\lambda$ for $\mathfrak{osp}(4^*|4)$. $\mathcal{V}_\lambda$ always contains $\mathcal{W}_\lambda$ as a $\mathfrak{g}_B$-submodule, and since we've chosen λ as detailed above, $\mathcal{V}_\lambda$ must contain null states. The question we need to answer is whether there are any new null states arising from fermionic degrees of freedom. This is related to the presence of BPS states v which are annihilated by one or more

[26]The universal enveloping algebra $U(\mathfrak{g})$ of a (super)algebra $\mathfrak{g}$, Verma modules, and several other results and constructions needed for the representation theory of Lie superalgbras are defined and proved carefully in appendix D.2. In the main text we largely restrict to heuristic definitions and do not give proofs.

supercharges, since v is BPS if and only if $\|Qv\|^2 = 0$ for at least one Q. We will start by examining the action of single supercharges on the lowest weight state, and more generally on *superconformal primaries*[27]. This will lead to inequalities giving necessary conditions for unitarity, whose saturation implies the existence of additional null states. We will then prove that these conditions are sufficient.

We need to be more precise about what we mean by unitarity of a lowest weight module. The real form $\mathfrak{osp}(4^*|4)$ of $\mathfrak{osp}(4|4)$ is determined by taking the real form

$$\mathfrak{g}_B = \mathfrak{sl}(2;\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(4)$$

for the bosonic subalgebra [49]. The conventions outlined in appendix D.1 fit with this real form, and in particular we may choose the convention

$$e_\alpha^\dagger = e_{-\alpha}, \qquad h^\dagger = h \qquad \forall \alpha \in \Phi^+,\ h \in \mathfrak{h} \tag{7.4}$$

for Hermitian conjugation, consistent with that used for the bosonic subalgebra earlier. We define an inner product $\langle -, - \rangle$ on $\mathcal{V}_\lambda$ by setting $\|v_\lambda\| = 1$, assuming that Hermitian conjugation as in (7.4) is with respect to this inner product, and using the commutation relations of $\mathfrak{osp}(4^*|4)$. $\mathcal{V}_\lambda$, or a quotient thereof, is a unitary representation if and only if this results in a genuine inner product.

The *Šapovalov form* F^λ on a Verma module $\mathcal{V}_\lambda$ is defined as a symmetric bilinear (as opposed to Hermitian) analogue of this candidate inner product, so if x and y are real elements of $U(\mathfrak{n}^+)$ then

$$F^\lambda(x, y) = \left\langle v_\lambda, x^\dagger y v_\lambda \right\rangle.$$

A careful definition is given in appendix D.2. Given the analogy with the inner product, it's no surprise that one can show that $\mathcal{V}_\lambda$ is reducible if and only if F^λ is degenerate. This can occur in two ways: via the already-discussed bosonic conditions, or via new fermionic conditions. These are known as *atypicality conditions*, and read

$$(\lambda - \rho, \alpha) = 0 \tag{7.5}$$

for some fermionic root α. ρ is the super Weyl vector (D.6), which for $\mathfrak{osp}(4^*|4)$ can be read off from (D.3) and (D.4) as

$$\rho = 2\delta_1 + \delta_2 - \epsilon_1 - 2\epsilon_2. \tag{7.6}$$

[27]Recall that a superconformal primary is usually defined as any state annihilated by all S's. Such states form a representation of R-symmetry.

In particular, one can show that if the atypicality condition for a fermionic root α is satisfied, then $\mathcal{V}_\lambda$ has a null state of weight $\alpha+\lambda$. We say that a state has *level* n if its scaling dimension is $n+\Delta$, where Δ is the dimension of the lowest weight state. We'll also say that a state is *naïve* level 1 if it is of the form $Q^{\pm}_{ia}v_\lambda$. The alternative is that a level 1 state is a sum of states involving the action of one supercharge and R-symmetries on v_λ, such as $(Q^-_{11}+Q^-_{21}J_+)v_\lambda$. The preceding discussion shows that atypicality implies the existence of null states at level 1.

We start our analysis proper with

Theorem

The Verma module $\mathcal{V}_\lambda$ with lowest weight λ as in (7.3) has extra null states arising from fermionic generators if and only if it is atypical.

Our first step is therefore to examine the atypicality conditions (7.5) for $\mathfrak{osp}(4^*|4)$, which read

fermionic root α	$(\lambda-\rho,\alpha)=0 \Rightarrow \Delta =$
$\epsilon_1-\delta_1$	$2(j+m+1)$
$\epsilon_1+\delta_1$	$2(j-m-3)$
$\epsilon_1-\delta_2$	$2(j+n)$
$\epsilon_1+\delta_2$	$2(j-n-2)$
$\epsilon_2-\delta_1$	$2(m-j)$
$\epsilon_2+\delta_1$	$-2(j+m+4)$
$\epsilon_2-\delta_2$	$2(n-j-1)$
$\epsilon_2+\delta_2$	$-2(j+n+3)$

(7.7)

If at least one of these eight conditions is satisfied then $\mathcal{V}_\lambda$ contains additional null states. Next we derive necessary conditions for unitarity.

We first consider the norms of naïve level 1 states. These are easy to calculate, and setting $\left\|Q^{\pm}_{ia}v_\lambda\right\|^2=q^{\pm}_{ia}$ gives

$$\begin{aligned}
q^+_{11}&=\frac{1}{2}\Delta-j+m & q^-_{11}&=\frac{1}{2}\Delta-j-m\\
q^+_{12}&=\frac{1}{2}\Delta-j+n & q^-_{12}&=\frac{1}{2}\Delta-j-n\\
q^+_{21}&=\frac{1}{2}\Delta+j+m & q^-_{21}&=\frac{1}{2}\Delta+j-m\\
q^+_{22}&=\frac{1}{2}\Delta+j+n & q^-_{22}&=\frac{1}{2}\Delta+j-n,
\end{aligned} \tag{7.8}$$

all of which must be non-negative for unitarity. Since $j\geq 0$ and $m\geq n\geq 0$, the strongest constraint thus imposed comes from q^-_{11}:

$$\Delta \geq 2(j+m). \tag{7.9}$$

If this is obeyed then all naïve level 1 norms are non-negative. When the bound is saturated we have $Q^-_{11}v_\lambda = 0$. For general values of the parameters j, m, n this is the only possible naïve level 1 vanishing, but for special values there are extra zero norm states

Condition	New zero norms	Total zero norms
$m = n$	q^-_{12}	2
$j = 0$	q^-_{21}	2
$j = 0, m = n$	q^-_{22}	4
$m = n = 0$	q^+_{11}, q^+_{12}	4
$j = m = n = 0$	q^+_{21}, q^+_{22}	8

The case $j = m = n = \Delta = 0$ is the trivial or *vacuum* representation.

These naïve level 1 norms are not the end of the story, as is made clear by looking at the atypicality conditions (7.7). The naïve bound (7.9) rules out most of these, but the condition

$$\Delta = 2(j+m+1)$$

corresponding to $\epsilon_1 - \delta_1$ is always available, as are a few others for special values of j, m, n. In particular, we expect to find a state of weight $\lambda + \epsilon_1 - \delta_1$ whose norm vanishes when $\Delta = 2(j+m+1)$. In fact, we have

$$\left\|(-2jQ^-_{11} + Q^-_{21}J_+)v_\lambda\right\|^2 = j(2j+1)\left[\Delta - 2(j+m+1)\right]. \tag{7.10}$$

If $j \neq 0$ this gives the strictly stronger unitarity bound

$$\Delta \geq 2(j+m+1). \tag{7.11}$$

One can also view the above vanishing as a linear combination of supercharges acting on distinct superconformal primary states, a situation which should be compared with the analyses in [107] and especially equation (4.24) of [134]. If $j = 0$ the state in (7.10) vanishes automatically, so the vanishing norm does not impose a constraint. However, we can compute

$$\left\|Q^-_{11}Q^-_{21}v_\lambda\right\|^2 = \left(\frac{\Delta}{2} - m - 1\right)\left(\frac{\Delta}{2} - m\right),$$

which rules out $2m < \Delta < 2(m+1)$. In this case the weaker bound (7.9) can still be saturated, so

$$\Delta = 2m, \; j = 0 \tag{7.12}$$

is possible.

The conditions (7.11) and (7.12) are necessary conditions for unitarity, but we'd also like to show sufficiency. This requires us to show that these bounds are strong enough that all possible states have non-negative norm, and that those with zero norm can always be removed by a quotient. We begin with the following:

Claim

The zero norm states of a Verma module form an invariant submodule.

In particular, this statement means that we may consistently remove all such states by a quotient. To prove it, we refer back to standard $\mathfrak{su}(2)$ representation theory. For any representation, unitary or otherwise, of $\mathfrak{su}(2)$, we have

$$J_- J_+ |m\rangle \propto |m\rangle \propto J_+ J_- |m\rangle \,,$$

where m are eigenvalues of J_3 and the proportionality factors vanish if and only if $|m\rangle$ is a top or bottom state. It follows that $\|J_\pm |m\rangle\|^2 \propto \||m\rangle\|^2$, so that if $|m\rangle$ has zero norm then so do $J_\pm |m\rangle$. The proportionality factors vanish exactly when we need them to, so that for instance if $|m\rangle$ is a top state then $\|J_- |m+1\rangle\|^2 = 0$ does not tell us that $|m\rangle$ has zero norm!

To extend this to the action of the compact semisimple R-symmetry group, consider the $\mathfrak{su}(2)$ subalgebra generated by $e_\alpha, e_{-\alpha} = e_\alpha^\dagger$ and $h_\alpha = [e_\alpha, e_{-\alpha}]$. Applying the same argument as above indicates that this subalgebra maps zero norm states to zero norm states, and hence so does the full algebra. Similar arguments to $\mathfrak{su}(2)$ apply to the conformal $\mathfrak{sl}(2;\mathbb{R})$ since the algebras are nearly identical. For fermions, consider the $\mathfrak{su}(2)$-like subalgebra $\left\langle Q,\ S = Q^\dagger,\ E = E^\dagger \right\rangle$ satisfying

$$\{Q, S\} = E, \qquad [E, Q] = Q, \qquad [E, S] = -S, \qquad Q^2 = S^2 = 0.$$

The structure here is analogous to $\mathfrak{su}(2)$ so the representation theory is also similar, the only real difference being that weight strings only have length ≤ 2 since $Q^2 = S^2 = 0$. In particular, this algebra also maps zero norm states to zero norm states, so this holds for all of $\mathfrak{g}$ as required.

The converse, that a state has zero norm if it lies in an invariant submodule, follows easily from the fact that the Šapovalov form vanishes on such a state. That is, we've shown that the states of zero norm comprise the unique maximal proper submodule of $\mathcal{V}_\lambda$, so can be consistently removed by a quotient to leave an irreducible module V_λ. We still need to show that the conditions (7.11) and (7.12) are strong enough to rule out negative norm states. In principle this requires a computation of the norm of every state in the Verma module, a task which looks at best unpleasant and at worst impossible. Fortunately, we can prove the following:

Claim

Suppose that the lowest weight λ is chosen such that the bosonic representation W_λ is unitary. Then V_λ is unitary if and only if $U(\mathfrak{n}_1^+)v_\lambda$ contains no states of negative norm.

By $U(\mathfrak{n}_1^+)$ we mean the span of all polynomials in fermionic raising operators, and the claim tells us that we need only check that the action of these purely fermionic words gives rise to no negative norm states. To prove it, note that by the *Poincaré-Birkhoff-Witt theorem* (see appendix D.2) we can write

$$U(\mathfrak{n}^+) = U(\mathfrak{n}_0^+)U(\mathfrak{n}_1^+).$$

Assume that all states in $U(\mathfrak{n}_1^+)v_\lambda$ have non-negative norm. If they have zero norm then they lie in an invariant submodule and we're done. If the norm is positive, observe that any such state lies in a finite direct sum of lowest weight representations for $\mathfrak{g}_B$. The finiteness of the sum follows since there are finitely many states in $U(\mathfrak{n}_1^+)$ and $\mathcal{V}_\lambda$ is a lowest weight module. The summands have non-negative weight for $\mathfrak{sl}(2;\mathbb{R})$ and integral weight for $\mathfrak{su}(2) \oplus \mathfrak{usp}(4)$, since all $Q^\pm_{ia}$ raise Δ and shift $2j, m, n$ by ± 1. By standard theory all such modules are unitary, which gives the result.

This is a big improvement since we now only need to check finitely many norms. This task can actually be accomplished, as demonstrated by the analysis of $\mathfrak{osp}(8^*|4)$ in [108]. However, there are still 2^8 states to check, so we make some more simplifications. First, we only show sufficiency for the case $\Delta \geq 2(j + m + 1)$. In section 7.2 we close this hole by giving explicit constructions of manifestly unitary representations with $\Delta = 2m,\ j = 0$. Now let $Q^{\mathbf{n}}$ denote a word in $U(\mathfrak{n}_1^+)$, where $\mathbf{n} = (n^\pm_{ia})$ is a multi-index with $n^\pm_{ia} = 1$ iff $Q^\pm_{ia}$ appears in the word, and $n^\pm_{ia} = 0$ otherwise. Thus

$$Q^{\mathbf{n}} = \prod_{\alpha \in \Phi_1^+} Q_\alpha^{n(\alpha)}.$$

For fixed $\mathbf{n}, j, m, n$ we can view the norm

$$\|Q^{\mathbf{n}} v_\lambda\|^2 = P_{\mathbf{n},j,m,n}(\Delta)$$

as a polynomial in Δ. We claim that $P \to +\infty$ as $\Delta \to +\infty$. To see this, write

$$\|Q^{\mathbf{n}} v_\lambda\|^2 = \left\langle v_\lambda, S^{-\mathbf{m}} S_{-\alpha} Q_\alpha Q^{\mathbf{m}} v_\lambda \right\rangle,$$

where Q_α is the leftmost supercharge appearing in $Q^{\mathbf{n}} = Q_\alpha Q^{\mathbf{m}}$ and $S^{-\mathbf{n}} = (Q^{\mathbf{n}})^\dagger$. Then we have

$$\|Q^{\mathbf{n}} v_\lambda\|^2 = h_\alpha(\eta_{\mathbf{m}} + \lambda)\|Q^{\mathbf{m}} v_\lambda\|^2 - \left\langle v_\lambda, S^{-\mathbf{m}} Q_\alpha S_{-\alpha} Q^{\mathbf{m}} v_\lambda \right\rangle,$$

where $\eta_{\mathbf{m}} + \lambda$ is the weight of $Q^{\mathbf{m}} v_\lambda$. In the first term we have the product of a factor $h_\alpha(\eta_{\mathbf{m}} + \lambda)$ which is linear in Δ with coefficient $+1/2$ and another norm polynomial. In the second term, $S_{-\alpha}$ can be commuted through $Q^{\mathbf{m}}$ to give lower-order words containing Q's and R-symmetry generators. Iterating, we find that the top degree term in $P_{\mathbf{n},j,m,n}$ occurs when all Q's are anticommuted with their conjugate S, producing a leading coefficient of $(\Delta/2)^{|\mathbf{n}|}$. In particular, $P \to +\infty$ as $\Delta \to +\infty$.

To complete the proof we identify the possible roots of $P_{\mathbf{n},j,m,n}$. By earlier work, we know that if $P(\Delta)$ vanishes for some Δ then $\mathcal{V}_\lambda$ is reducible and atypicality must be satisfied. Thus $\Delta = \Delta_A$, where Δ_A are the atypical values (7.7), and the only possible roots of P correspond to atypicality conditions. Since $\Delta = 2(j + m + 1)$ is the largest atypical Δ and P is positive for large Δ, we deduce that if $\Delta \geq 2(j + m + 1)$ then P is non-negative, hence proving sufficiency.

In summary, we have the following classification of unitary irreps

Theorem

Unitary irreducible representations of $\mathfrak{osp}(4^*|4)$ are lowest weight modules V_λ, consisting of quotients of the Verma module $\mathcal{V}_\lambda$ by the submodule of null states. We have the following types of representation:

- *Generic* or *long* representations, denoted $L(\Delta, j, m, n)$, with $\Delta > 2(j + m + 1)$.
- *Semishort* representations, denoted $SS(j, m, n)$, with $\Delta = 2(j + m + 1)$.
- *Short* representations, denoted $S(m, n)$, with $\Delta = 2m$, $j = 0$. Short representations further split into *1/4-BPS* representations with $m \neq n$ and *1/2-BPS* with $m = n$.

Short representations have zero norm states of the form $Q^{\pm}_{ia} v_\lambda$, so the lowest weight state is itself BPS. The BPS fraction indicates the number of supercharges $Q^{\pm}_{ia}$ annihilating the lowest weight state. In semishort representations we find zero norm states at level 1 of the form (7.10) as well as BPS states at level ≥ 1.

7.2 Quantum Mechanics on $\mathbb{C}^{2k}$

In this section we construct a large class of examples of the representations discussed in section 7.1 by applying the quantum mechanics of sections 2 and 5 to the flat hyper-Kähler space $\mathbb{C}^{2k}$. In particular, this construction will include all short representations, thereby making their unitarity manifest and filling a gap in our previous proofs. We'll use an index-like construction to identify the (semi)short representations, and observe that this formally computes the index of the Dolbeault operator $\bar{\partial}$ on $\mathbb{C}^{2k}$, a point which we return to in section 7.3.

To begin with we write out the generators of $\mathfrak{osp}(4^*|4)$ explicitly, referring to appendix E.3 for a list of the general forms. We use coordinates X_{Im} with $I = 1, \ldots, k$ and $m = 1, \ldots, 4$, so the X_{Im} for fixed I parameterise a block $\mathbb{C}^2 \cong \mathbb{R}^4$. We have

$$H = \frac{1}{2} P_{Im} P_{Im}, \qquad K = \frac{1}{2} X_{Im} X_{Im}, \qquad D = X_{Im} P_{Im} - 2ik,$$

where $P_{Im} = \Pi_{Im}$ is the momentum conjugate to X_{Im}. Then

$$L_0 = a^\dagger_{Im} a_{Im} + 2k, \qquad a_{Im} = \sqrt{\frac{\mu}{2}} \left(X_{Im} + i\mu^{-1} P_{Im} \right) \tag{7.13}$$

is a sum of $4m$ real decoupled harmonic oscillators with $\hbar = \omega = 1$ and mass μ. Similarly, we find

$$L_+ = -\frac{1}{2} a^\dagger_{Im} a^\dagger_{Im}, \qquad L_- = -\frac{1}{2} a_{Im} a_{Im}. \tag{7.14}$$

The basis change (7.1) also acts on supercharges as[28]

$$\begin{aligned} Q &\mapsto Q - i\mu S := \sqrt{\mu}\mathcal{S} \\ S &\mapsto -\frac{i}{2\mu} \left(Q + i\mu S \right) := -\frac{i}{2\sqrt{\mu}} \mathcal{Q}, \end{aligned}$$

independently of $SU(2) \times USP(4)$ R-symmetry labels. To write explicit expressions we choose complex structures

$$I^a_{Im,Jn} = -\bar{\eta}^a_{mn} \delta_{IJ},$$

where $\bar{\eta}^a$ are the 't Hooft symbols (see [67] for conventions). The full set of new supercharges is

$$\begin{aligned} \mathcal{Q} &= \sqrt{2\mu} \psi^\dagger_{Im} a^\dagger_{Im} & \tilde{\mathcal{Q}} &= \sqrt{2\mu} \psi_{Im} a^\dagger_{Im} \\ \mathcal{Q}^a &= \sqrt{2\mu} \psi^\dagger_{Im} \bar{\eta}^a_{mn} a^\dagger_{In} & \tilde{\mathcal{Q}}^a &= \sqrt{2\mu} \psi_{Im} \bar{\eta}^a_{mn} a^\dagger_{In}, \end{aligned} \tag{7.15}$$

and the superconformal charges are their adjoints

$$\begin{aligned} \mathcal{S} &= \sqrt{2\mu} \psi^\dagger_{Im} a_{Im} & \tilde{\mathcal{S}} &= \sqrt{2\mu} \psi_{Im} a_{Im} \\ \mathcal{S}^a &= \sqrt{2\mu} \psi^\dagger_{Im} \bar{\eta}^a_{mn} a_{In} & \tilde{\mathcal{S}}^a &= \sqrt{2\mu} \psi_{Im} \bar{\eta}^a_{mn} a_{In}. \end{aligned} \tag{7.16}$$

The generators of $SO(5)$ become

$$\begin{aligned} J^a_+ &= \frac{1}{2} \bar{\eta}^a_{mn} \psi^\dagger_{Im} \psi^\dagger_{In} & J^a_- &= \frac{1}{2} \bar{\eta}^a_{nm} \psi_{Im} \psi_{In} \\ R^a &= \frac{i}{2} \bar{\eta}^a_{mn} \psi^\dagger_{Im} \psi_{In} & J_3 &= \frac{1}{2} \psi^\dagger_{Im} \psi_{Im} - k, \end{aligned} \tag{7.17}$$

[28]Note that, in analogy with (7.2), Poincaré supercharges are mapped to lowering operators.

and those of $SU(2)$ are

$$T^a = i\bar{\eta}^a_{mn} a^\dagger_{Im} a_{In}. \tag{7.18}$$

To analyse the resulting representations it will be convenient to put the above generators into a form which makes more symmetry manifest. The first step is to switch to holomorphic coordinates with respect to I^3

$$Z_I = X_{I1} - iX_{I2}, \qquad W_I = X_{I3} + iX_{I4}.$$

These induce complex bases[29] for the fermions $\psi_{Im}, \psi^\dagger_{Im}$ and the harmonic oscillator ladder operators $a_{Im}, a^\dagger_{Im}$

$$\begin{aligned}
\zeta^\dagger_I &= \psi^\dagger_{I1} - i\psi^\dagger_{I2} & \qquad \chi^\dagger_I &= \psi^\dagger_{I3} + i\psi^\dagger_{I4} \\
\bar{\zeta}^\dagger_I &= \psi^\dagger_{I1} + i\psi^\dagger_{I2} & \qquad \bar{\chi}^\dagger_I &= \psi^\dagger_{I3} - i\psi^\dagger_{I4} \\
\alpha^\dagger_I &= a^\dagger_{I1} - ia^\dagger_{I2} & \qquad \beta^\dagger_I &= a^\dagger_{I3} + ia^\dagger_{I4} \\
\bar{\alpha}^\dagger_I &= a^\dagger_{I1} + ia^\dagger_{I2} & \qquad \bar{\beta}^\dagger_I &= a^\dagger_{I3} - ia^\dagger_{I4},
\end{aligned} \tag{7.19}$$

and similar for Hermitian conjugates. These satisfy nonzero commutation relations

$$\left[\alpha_I, \alpha^\dagger_J\right] = 2\delta_{IJ} = \left[\beta_I, \beta^\dagger_J\right], \qquad \left\{\chi_I, \chi^\dagger_I\right\} = 2\delta_{IJ} = \left\{\zeta_I, \zeta^\dagger_J\right\},$$

and similar for antiholomorphic generators.

In complete analogy with section 6.3, in particular (6.26), complex coordinates allow us to make $SO(5)$ R-symmetry manifest. To do so, define a multiplet

$$\psi^A_I = \frac{1}{\sqrt{2}}\left(\zeta_I, \bar{\chi}_I, \bar{\zeta}^\dagger_I, \chi^\dagger_I\right), \tag{7.20}$$

with $\bar{\psi}^{\bar{A}}_I = \left(\psi^A_I\right)^\dagger$. This satisfies

$$\left\{\psi^A_I, \bar{\psi}_{JB}\right\} = \delta^A_B \delta_{IJ}.$$

In this flat case we can also make $SU(2)$ R-symmetry manifest, by defining multiplets

$$\begin{aligned}
c^\alpha_I &= \left(\alpha_I, \bar{\beta}_I\right) & \qquad \bar{c}_{I\alpha} &= \left(\bar{\alpha}_I, \beta_I\right) \\
c^\dagger_{I\alpha} &= \left(\alpha^\dagger_I, \bar{\beta}^\dagger_I\right) & \qquad \bar{c}^{\dagger\alpha}_I &= \left(\bar{\alpha}^\dagger_I, \beta^\dagger_I\right),
\end{aligned} \tag{7.21}$$

which satisfy

$$\left[c^\alpha_I, c^{\dagger\beta}_J\right] = -2\delta_{IJ}\epsilon^{\alpha\beta} = -\left[\bar{c}^\alpha_I, \bar{c}^{\dagger\beta}_J\right]$$

[29]The similarity with the notation used in section 6 is no accident. The flat metric is a special case of the semi-flat metric (6.7) with $\tau_{IJ} = i\delta_{IJ}$.

with $\epsilon^{12} = \epsilon_{21} = 1$. We spare the reader the details of rewriting $\mathfrak{osp}(4^*|4)$ in this notation, and refer to appendix E.6 for a full list of generators and relations.

Next we construct the Hilbert space. We take a vacuum state $|0\rangle$ which satisfies

$$\bar{\psi}_I^A |0\rangle = c_I^\alpha |0\rangle = \bar{c}_I^\alpha |0\rangle = 0.$$

Using the dictionary (2.19) and definitions (7.19, 7.20), this corresponds to the differential form

$$dZ_1 \wedge \cdots \wedge dZ_k \wedge d\bar{W}_1 \wedge \cdots \wedge d\bar{W}_k e^{-\mu K}, \tag{7.22}$$

where $e^{-\mu K}$ is the ground state wavefunction for $4k$ harmonic oscillators. The forms have the effect of making the vacuum an $SO(5)$ singlet. A generic state is constructed by acting on $|0\rangle$ with arbitrary polynomials in $c_I^{\dagger\alpha}$, $\bar{c}_I^{\dagger\alpha}$, and ψ_I^A, subject to fermionic statistics. A state containing N bosonic excitations has scaling dimension

$$L_0 |N\rangle = (N + 2k) |N\rangle ,$$

where $2k$ is the harmonic oscillator ground state energy.

We can build R-symmetry irreps by suitable choices of creation operator polynomials. We refer back to section 7.1 for a refresher on building irreps of $SU(2)$ and $USP(4)$ using tensors. The simplest is $SU(2)$, for which we take a symmetric polynomial of the form

$$c_{I_1}^{\dagger(\alpha_1} \dots c_{I_l}^{\dagger\alpha_l} \bar{c}_{J_1}^{\dagger\beta_1} \dots \bar{c}_{J_m}^{\dagger\beta_m)} |0\rangle$$

for arbitrary index choices $I_1, \dots, I_l, J_1, \dots, J_m$. If $l + m = 2j$ then this tensor is a 'spin' j irrep. It has dimension $\Delta = 2(k + j)$ and is an $SO(5)$ singlet. One can use antisymmetric polynomials, which are $SU(2)$ singlets, to obtain states with larger dimension relative to their spin.

The basic irreps of $SO(5)$ are constructed as follows. The **4** is given by $\psi_I^A |0\rangle$ for any choice of I, while the **5** is $\Psi_{IJ}^{AB} |0\rangle$, where

$$\Psi_{IJ}^{AB} = \psi_I^{[A} \psi_J^{B]} + \frac{1}{4} \Omega^{AB} \Omega_{CD} \psi_I^C \psi_J^D \tag{7.23}$$

explicitly removes the singlet from $\Lambda^2(\mathbf{4})$. Representations with $n = 0$ are symmetric powers of the **4**, so

$$\psi_{I_1}^{(A_1} \dots \psi_{I_m}^{A_m)} |0\rangle$$

is the $(m, 0)$ representation for any choice of $I_1, \dots, I_m$ obeying fermionic statistics. In particular, the largest possible value of m is k. Next up are representations with $m = n$, for which we take symmetric polynomials in Ψ^{AB}_{IJ} with traces with respect to ϵ_{ABCD} removed. Again, the largest possible value of m and n is k. For a general representation with $m = n + p$, take the product of an (n, n) polynomial with a $(p, 0)$ polynomial. The representation so obtained will not be an irrep as there are various 'cross-traces' to be removed, but the lowest weight state is a straightforward product of those for the component representations. Note that all states formed by purely fermionic polynomials have $\Delta = 2k$.

We would like to pick out representations saturating the unitarity conditions (7.11) and (7.12). We begin by looking for states annihilated by $E = L_0 + 2J_3 + 2M$, corresponding to the naïve unitarity bound (7.9). We take the supercharge Q^{21} with conjugate $-S^{32}$, and find

$$-\left\{Q^{21}, S^{32}\right\} = L_0 + T^{12} + 2R^{23} = E,$$

where we chose $T^{12} = 2J_3$ and $R^{23} = M$. This is a matter of convention: we made this particular choice so that the final result takes a particular form, but any other choice is equivalent by applying a suitable R-symmetry transformation. With these choices and reading off commutators from (E.1), we find that the holomorphic creation operators $\alpha^\dagger_I$ and $\beta^\dagger_I$ are neutral under E, while antiholomorphic ones have charge +2. Similarly, ψ^2_I has charge -2, ψ^3_I has charge +2 and ψ^1_I, ψ^4_I are neutral. Since no fermion $SO(5)$ index can appear more than k times and the vacuum has E-charge $2k$, it follows that the minimal possible E-charge of a state is 0. The most general state with $E = 0$ takes the form

$$|\Psi\rangle = \psi^2_1 \dots \psi^2_k (\psi^1)^{\mathbf{p}} (\psi^4)^{\mathbf{q}} \left|\mathbf{m}, 0, \mathbf{n}, 0\right\rangle, \tag{7.24}$$

where $\mathbf{m}, \mathbf{n}, \mathbf{p}, \mathbf{q}$ are multi-indices valued in $\mathbb{N}$ for bosons and $\{0, 1\}$ for fermions, so for example

$$(\psi^1)^{\mathbf{p}} = (\psi^1_1)^{p(1)} \dots (\psi^1_k)^{p(k)}.$$

$|\mathbf{m}, \bar{\mathbf{m}}, \mathbf{n}, \bar{\mathbf{n}}\rangle$ is, up to normalisation, given by

$$\left|\mathbf{m}, \bar{\mathbf{m}}, \mathbf{n}, \bar{\mathbf{n}}\right\rangle = \alpha^{\dagger\mathbf{m}} \bar{\alpha}^{\dagger\bar{\mathbf{m}}} \beta^{\dagger\mathbf{n}} \bar{\beta}^{\dagger\bar{\mathbf{n}}} \left|0\right\rangle,$$

so that $E = 0$ states are formed by holomorphic polynomials in the bosonic creation operators. Indeed, if we unpack the differential geometric form of (7.24) completely, using (7.20) and (7.22), then we find that

$$\{\text{states with } E = 0\} \cong \mathbb{C}\left[Z, W, dZ, dW\right] e^{-\mu K}. \tag{7.25}$$

That is, a general state with $E = 0$ corresponds to a unique holomorphic polynomial differential form multiplied by the harmonic oscillator ground state factor for normalisability.

When $\mathbf{m} = \mathbf{n} = 0$ it's easy to see that (7.24) is a lowest weight state. This corresponds to the case of short 1/4-BPS representations, since $|\Psi\rangle$ also has $j = 0$ and hence $\Delta = 2m$. With a choice of convention for N, the 1/2-BPS representations with $m = n$ are obtained by taking $p(I) = 1,\ q(I) = 0\ \forall I$. These lowest weights always have $m = k$, so by varying k we cover all possible values of (m, n). This proves the sufficiency of the BPS scaling dimension (7.12) for unitarity.

Semishort representations are a little more complicated. In fact, we'll shortly give a general argument to show that states with $E = 0$ in a semishort representation are never lowest weight states. Instead, there is a unique supercharge Q satisfying $[E, Q] = -2Q$, and the simplest $E = 0$ state is Qv_λ and has weight $\epsilon_2 - \delta_1 + \lambda$. We'll also see that it isn't quite possible to tell which (semi)short representation an $E = 0$ state belongs to just by reading off its Cartan charges. We conjecture that our construction actually contains all semishort representations, but we do not currently have a proof.

It's interesting to examine the isomorphism (7.25) in more detail. Since the spectrum of E is discrete, there's no obstruction to applying the Hodge theoretic arguments of appendix B to find that

$$\{\text{states with } E = 0\} \cong \left\{Q^{21}\text{-cohomology classes}\right\} \cong \left\{S^{32}\text{-cohomology classes}\right\}.$$

It's natural to ask what this cohomology is. Observe that $\mathbb{C}\left[Z, W, dZ, dW\right]$ is the same thing as the $\bar{\partial}$-Dolbeault cohomology of polynomial forms on $\mathbb{C}^{2k}$. In the following section we'll argue that this is almost true for a general hyper-Kähler target. For now we observe, as a consequence of the commutation relations (E.1), that S^{32} acts on a general state

$$(\psi^1)^{\mathbf{p}}(\psi^2)^{\mathbf{q}}(\psi^3)^{\mathbf{r}}(\psi^4)^{\mathbf{s}}\,|\mathbf{m}, \bar{\mathbf{m}}, \mathbf{n}, \bar{\mathbf{n}}\rangle$$

exactly as $\bar{\partial}$ acts on the form

$$Z^{\mathbf{m}}\bar{Z}^{\bar{\mathbf{m}}}W^{\mathbf{n}}\bar{W}^{\bar{\mathbf{n}}}dZ^{k-\mathbf{p}} \wedge d\bar{Z}^{\mathbf{r}} \wedge dW^{\mathbf{s}} \wedge d\bar{W}^{k-\mathbf{q}},$$

where $(k - \mathbf{q})\,(I) = 1 - \mathbf{q}(I)$. Thus S^{32}-cohomology is indeed isomorphic to Dolbeault cohomology.

7.3 Indices and Dolbeault Cohomology

In this section we examine the extent to which results from flat space quantum mechanics generalise to curved hyper-Kähler targets. Clearly it's too much to hope to figure out the full spectrum of L_0 on a curved target, but we'll see that many considerations about (semi)short representations and indices do carry over. In particular, we'll use a generalisation of E to construct an index which gets contributions only from (semi)short representations, and give a full characterisation of the submodules of states with $E = 0$. The relation to Dolbeault cohomology also carries through, and provides a form of Hodge theory for the Dolbeault operator on a class of non-compact hyper-Kähler manifolds.

Once again we consider an index. Since we are aiming to reproduce the fact that the lowest weight states of a short representation contribute to this index, we want to pick a supercharge q and conjugate $s = q^\dagger$ such that

$$\{q, s\} = E = \frac{1}{2}(L_0 + 2J_3 + 2M) = h_1 + h_3. \tag{7.26}$$

In the notation of appendix D.1, the appropriate choice is Q^-_{11}. We have

$$E(\lambda) := (h_1 + h_3)(\lambda) = \frac{1}{2}(\Delta - 2j - 2m) = \begin{cases} 0 & \text{short} \\ 1 & \text{semishort} \\ > 1 & \text{long.} \end{cases}$$

It's useful to grade $\mathfrak{osp}(4^*|4)$ by E-eigenvalue. We say that $X \in \mathfrak{osp}(4^*|4)$ has *E-grade a*, and write $X \in \mathfrak{g}_a$, if

$$[h_1 + h_3, X] = aX.$$

This gives a decomposition

$$\mathfrak{osp}(4^*|4) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where in particular $\mathfrak{g}_0$ is the subalgebra commuting with E, also known as the *little group*. More precisely,

$$\begin{aligned} \mathfrak{g}_0 &= \left\langle v_{\pm 2\delta_2}, E, J_3, M, N, q, s, Q^\pm_{22}, S^\pm_{22} \right\rangle \\ \mathfrak{g}_1 &= \left\langle v_{\delta_1 \pm \delta_2}, J_+, L_+, Q^\pm_{12}, Q^+_{21}, S^-_{21} \right\rangle \\ \mathfrak{g}_{-1} &= \left\langle v_{-\delta_1 \pm \delta_2}, J_-, L_-, S^\pm_{12}, S^+_{21}, Q^-_{21} \right\rangle \\ \mathfrak{g}_2 &= \left\langle v_{2\delta_1}, Q^+_{11} \right\rangle \\ \mathfrak{g}_{-2} &= \left\langle v_{-2\delta_1}, S^+_{11} \right\rangle. \end{aligned} \tag{7.27}$$

There are two important consequences of this list, the first of which is a proper identification of the little group. It has an ideal $I = \langle q, s, E \rangle$ such that

$$\frac{\mathfrak{g}_0}{I} \cong \mathfrak{u}(2|1).$$

$\mathfrak{u}(2|1)$ has bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)_{J_3} \oplus \mathfrak{u}(1)_M$, where $\mathfrak{su}(2) = \langle v_{\pm 2\delta_2}, N \rangle$. It follows from unitarity that if a state has $E = 0$ then q and s also act trivially on it, so that states with $E = 0$ form representations of $\mathfrak{u}(2|1)$. For the superconformal index of section 7.4 it will be useful to restrict $\mathfrak{u}(2|1)$ to the subalgebra commuting with q and s as well as E. The only effect of this is to break $U(1)^2$ to the diagonal $U(1)$ generated by $M + 2J_3$, resulting in an $\mathfrak{su}(2|1)$ algebra.

The second important consequence of (7.27) is a simple classification of the representations containing $E = 0$ states, and how these states are obtained. Recall that the irrep V_λ is a quotient of the Verma module $\mathcal{V}_\lambda = U(\mathfrak{n}^+)v_\lambda$, and we can use the Poincaré-Birkhoff-Witt theorem to order any word in $U(\mathfrak{n}^+)$ with E-grade increasing right to left. That is,

$$U(\mathfrak{n}^+) = U(\mathfrak{g}_2^+)U(\mathfrak{g}_1^+)U(\mathfrak{g}_0^+)U(\mathfrak{g}_{-1}^+)U(\mathfrak{g}_{-2}^+).$$

Unitarity implies that E is positive semidefinite, so any state with $E = 0$ is annihilated by any operator with negative E-grade. Furthermore, no state with $E = 0$ can be obtained from a word containing operators with positive E-grade, else they must have raised a non-existent state with $E < 0$. Therefore, to have any hope of obtaining a state with $E = 0$ we must take a representation with $E(\lambda) \in \mathbb{N}$, act with some operators to lower E to zero, then apply arbitrary elements of $\mathfrak{u}(2|1)$. Crucially, there is a unique element of $\mathfrak{n}^+$ with negative grade, namely $Q_{21}^- \in \mathfrak{g}_{-1}$, and this operator squares to 0, so our remaining options are highly restricted.

We now analyse the short, semishort and long cases in turn. In the short case the lowest weight state has $E = 0$, so the set of states with $E = 0$ is the lowest weight irrep of $\mathfrak{u}(2|1)$ with lowest weight λ. In the semishort case we have $E(\lambda) = 1$, so we act with Q_{21}^- to obtain the $E = 0$ state

$$w_\lambda = Q_{21}^- v_\lambda.$$

It's straightforward to check that w_λ is a lowest weight state for $\mathfrak{u}(2|1)$, so the $E = 0$ states of a semishort representation form the lowest weight irrep of $\mathfrak{u}(2|1)$ with lowest weight $\lambda + \epsilon_2 - \delta_1$. For the long case we either have $E(\lambda) \in \mathbb{N}$ or not. If not then we cannot possibly reach an $E = 0$ state since all operators have integer grade. If $E(\lambda) \in \mathbb{N}$ we are still stuck, since $E(\lambda) \geq 2$ and Q_{21}^- lowers E by 1 and can be applied at most once. Hence no long representations contain $E = 0$ states. To sum up, we have

Theorem

With E as in (7.26) and $\mathfrak{u}(2|1) = \mathfrak{g}_0/I$, the $E = 0$ content V_0 of the unitary irrep V_λ of $\mathfrak{osp}(4^*|4)$ is:

- If V_λ is short then V_0 is the lowest weight irrep of $\mathfrak{u}(2|1)$ with lowest weight vector v_λ.
- If V_λ is semishort then V_0 is the lowest weight irrep of $\mathfrak{u}(2|1)$ with lowest weight vector $w_\lambda = Q_{21}^- v_\lambda$.
- If V_λ is long then $V_0 = \emptyset$.

In light of the above, a natural task is to classify the representations of $\mathfrak{su}(2|1)$ which occur[30]. Fortunately, $\mathfrak{su}(2|1)$ has rather easier representation theory than $\mathfrak{osp}(4^*|4)$ as its rank is smaller and its bosonic subalgebra is compact. Indeed, if the lowest weight is integral then an irrep splits into at most four finite-dimensional representations of $\mathfrak{su}(2)$, since we can use Poincaré-Birkhoff-Witt to consider first the action of the two supercharges $Q_{22}^\pm$. Schematically

$$V_0 = \mathfrak{su}(2)(u_\lambda) + \mathfrak{su}(2)(Q_{22}^- u_\lambda) + \mathfrak{su}(2)(Q_{22}^+ u_\lambda) + \mathfrak{su}(2)(Q_{22}^+ Q_{22}^- u_\lambda), \tag{7.28}$$

where $u_\lambda = v_\lambda$ in the short case and w_λ in the semishort. The remaining problem is twofold. First, we must check whether any of the above summands vanish and whether the sum is direct. Second, we need to calculate the $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ quantum numbers of the summands. We need the latter information if we are to formulate the most general possible index for E.

First we write down the lowest weight for V_0 in the short and semishort cases

$$\begin{aligned} \lambda &= m(\epsilon_1 + \epsilon_2 - \delta_1) - n\delta_2 && \text{(short)} \\ \lambda + \epsilon_2 - \delta_1 &= (m+1)\,\epsilon_1 + (2j + m + 2)\,\epsilon_2 - (m+1)\delta_1 - n\delta_2 && \text{(semishort)}. \end{aligned} \tag{7.29}$$

Next we check the vanishing of the summands (7.28). The only naïve vanishings we find, corresponding to one of $Q_{22}^\pm u_\lambda,\ Q_{22}^+ Q_{22}^- u_\lambda$ vanishing, are for the 1/2-BPS case which we already knew about from (7.8). However, we must take extra care since $Q_{22}^+ u_\lambda$ and $v_{2\delta_2} Q_{22}^- u_\lambda$, which appear to be parts of distinct $\mathfrak{su}(2)$ submodules, have the same weight and may not be independent. Calculating the norm of a general linear combination $\left(\alpha Q_{22}^+ + \beta v_{2\delta_2} Q_{22}^-\right) u_\lambda$, we find an additional vanishing for $\alpha = \beta = 1,\ n = 0$ in both the short and semishort cases. This follows since $Q_{22}^+ + v_{2\delta_2} Q_{22}^- = Q_{22}^- v_{2\delta_2}$ and $v_{2\delta_2} u_\lambda = 0$ when $n = 0$. Consequently, the submodules $\mathfrak{su}(2)(Q_{22}^\pm u_\lambda)$ coincide in this case.

Next we need the quantum numbers of the $\mathfrak{su}(2)$ summands. The $\mathfrak{u}(1)$ charge is easy to read off from weights, while for $\mathfrak{su}(2)$ we use the quadratic Casimir

$$C_2 = h_4(h_4 - 2) + 4 v_{2\delta_2} v_{-2\delta_2} = n(n+2).$$

[30] We restrict to $\mathfrak{su}(2|1)$ for brevity, and because it is all we need for the superconformal index. The extra $\mathfrak{u}(1)$ is straightforward.

It is a simple matter to calculate the value of C_2 on various submodules, for instance

$$C_2 Q^-_{22} u_\lambda = \left[C_2, Q^-_{22}\right] u_\lambda + n(n+2) Q^-_{22} u_\lambda = (n+1)(n+3) Q^-_{22} u_\lambda,$$

where the second equality follows since u_λ is a lowest weight state for $\mathfrak{su}(2|1)$. It follows that the module $\mathfrak{su}(2)(Q^-_{22}u_\lambda)$ has $\mathfrak{su}(2)$ quantum number $n' = n+1$. $Q^+_{22}Q^-_{22}u_\lambda$ is similarly straightforward and has $n' = n$. A little more care is required with $Q^+_{22}u_\lambda$ since this turns out not to be an eigenstate of C_2. Instead, we find

$$C_2 \left(Q^+_{22} u_\lambda + \frac{1}{n+1} v_{2\delta_2} Q^-_{22} u_\lambda \right) = (n-1)(n+1) \left(Q^+_{22} u_\lambda + \frac{1}{n+1} v_{2\delta_2} Q^-_{22} u_\lambda \right)$$

so that $n' = n-1$ for the submodule generated from this state. In the case $n = 0$ we've seen that this state vanishes, while in the 1/2-BPS case it reduces to $Q^+_{22}v_\lambda$.

In summary, we've shown that the $E = 0$ content V_0 of an irrep V_λ decomposes under $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ as

$$\begin{array}{ll}
(m,m)\oplus(m-1,m+1) & S(m,m) \\
(0,m)\oplus(1,m+1)\oplus(0,m+2) & S(m,0) \\
(n,m)\oplus(n+1,m+1)\oplus(n-1,m+1)\oplus(n,m+2) & S(m,n): 0<n<m \\
(0,m+2j+2)\oplus(1,m+2j+3)\oplus(0,m+2j+4) & SS(j,m,0) \\
(n,m+2j+2)\oplus(n+1,m+2j+3)\oplus(n-1,m+2j+3)\oplus & \\
\qquad\qquad\qquad\oplus(n,m+2j+4) & SS(j,m,n): n>0,
\end{array} \tag{7.30}$$

where the first entry of a pair $(x,y)\in\mathbb{Z}^2$ is the $\mathfrak{su}(2)$ quantum number and the second is the $\mathfrak{u}(1)$ charge under

$$Z = -(M + 2J_3). \tag{7.31}$$

Notice that there is considerable overlap between cases, so it is not possible to determine which $\mathfrak{osp}(4^*|4)$ representation a given $E = 0$ state belongs to by reading off its quantum numbers.

Having analysed the $E = 0$ content of representations of $\mathfrak{osp}(4^*|4)$, there are two natural questions: what indices can we build from this, and what can they tell us about hyper-Kähler quantum mechanics? To address the latter, we attempt to identify the form of cohomology corresponding to q. In section 7.2 we saw that in the flat case we had the Dolbeault cohomology of polynomial forms on $\mathbb{C}^{2k}$, so we'd expect to generalise this here. To do so, consider the supercharge of section 2 which, in the differential form representation, corresponds to the Dolbeault operator $\bar\partial$. Under the basis change (7.1) this maps to

$$s = \frac{1}{2\sqrt{\mu}}\left(Q - iQ^I - i\mu S - \mu S^I\right) = \frac{1}{\sqrt{\mu}}\left(\bar\partial + \mu\bar\partial K\wedge\right),$$

and with $q = s^\dagger$ we have

$$E = \{q, s\} = L_0 - T^I + 2(J_3 - R^I), \tag{7.32}$$

which corresponds to (7.26) with an appropriate choice of conventions. But if β is any p-form and $\alpha = \beta \exp(-\mu K)$ then

$$s\alpha = \frac{1}{\sqrt{\mu}}(\bar{\partial}\beta)e^{-\mu K},$$

which generalises the identification between harmonic oscillator eigenstates and polynomials of section 7.2. The key point is that s acts exactly as if it were the Dolbeault operator, provided we carry around an overall exponential factor.

It's no surprise that s-cohomology turns out to reproduce $\bar{\partial}$-cohomology, since s is obtained from $\bar{\partial}$ by the similarity transformation (7.1). Indeed, if $s = M\bar{\partial}M^{-1}$ and $s\alpha = 0$ then $\bar{\partial}M\alpha = 0$, and if $\alpha = s\gamma$ then $M\alpha = \bar{\partial}(M\gamma)$. However, one might ask what happened to the factor $\exp(\frac{1}{2}\mu^{-1}H)$ in this conjugation. Of course, the resolution here is simply that conjugation of a supercharge by H has no effect, since supercharges commute with the Hamiltonian.

That the H-conjugation is nevertheless essential becomes clear when we try to interpret this cohomology. In the representation of s, states in the Hilbert space are forms β for which $\beta \exp(-\mu K)$ is L^2-normalisable, whereas in the representation of $\bar{\partial}$ they are ordinary L^2-normalisable forms. It's clear that the Dolbeault cohomologies on the two spaces are completely unrelated. Indeed, in the flat space case there are no L^2-normalisable holomorphic forms whatsoever, whereas with an exponential suppression factor all polynomials are L^2. This tallies with the fact that the dilatation operator D, being a first-order differential operator, has a continuous spectrum when acting on ordinary forms and is therefore quite unlikely to have well-defined Hodge theory, whereas L_0 is second order and, provided that K is smooth and grows 'at infinity', has a discrete spectrum and (presumably) well-behaved Hodge theory.

Two further aspects of interpretation are worth noting. The first comes from noticing that s takes the form of the $(0,1)$ part of a flat connection on a real line bundle $\mathcal{L} \to M$, with connection 1-form $A = \mu dK$. Moreover, we can view $\xi = \exp(-\mu K)$ as defining a section[31], and hence a metric on $\mathcal{L}$ via $G(\xi, \xi) = \exp(-2\mu K)$. Then a differential form $\beta \exp(-\mu K)$ can be viewed as an $\mathcal{L}$-valued form, and for any two such forms there is a natural L^2 inner product

$$(\alpha, \beta) = \int_M \alpha \wedge *\bar{\beta}\, G(\xi, \xi).$$

[31] Only locally since in general K can shift by the real part of a holomorphic function, giving transition functions $\xi \mapsto \xi' = \exp(\phi + \bar{\phi})\xi$.

States of the Hilbert space can then be viewed as L^2-normalisable $\mathcal{L}$-valued forms. The natural notion of cohomology is L^2 Dolbeault cohomology with values in $\mathcal{L}$, which seems to have well-defined Hodge theory (see [135]). It's worth noting that the bundle $\mathcal{L}$ is not holomorphic, so this will differ slightly from the usual notion of bundle-valued cohomology in Kähler geometry (see [60]), but enough key features survive that it ought to go through unchanged.

A more physical interpretation comes by observing that this cohomology theory can be obtained from a supersymmetric quantum mechanical model whose Hamiltonian is E upon canonical quantisation. Such a model is a simple deformation of our basic σ-model (2.7). We start by finding the explicit form of (7.32), using the formulae in appendix E. After some routine algebra, and working classically to remove some artifacts of quantum ordering, we find

$$\begin{aligned} E = &\frac{1}{2\mu} g^{\mu\nu}\Pi_\mu\Pi_\nu + \frac{1}{4\mu}R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\nu}\psi^\rho\psi^\sigma + \frac{\mu}{2}D^\mu D_\mu \\ &- I^\mu{}_\nu D^\nu \Pi_\mu + i\psi^{\dagger\mu}\psi^\nu\nabla_\mu D^I_\nu + g_{\mu\nu}\psi^{\dagger\mu}\psi^\nu. \end{aligned} \tag{7.33}$$

This can be simplified by noting

$$\nabla_\mu D^I_\nu = \nabla_{[\mu} D^I_{\nu]} = \frac{1}{2}(dd^I K)_{\mu\nu} = -\omega^I_{\mu\nu},$$

where we used the fact that D^I is an isometry, D is closed, and K is a Kähler potential. We can return to the Lagrangian formulation by Legendre transform

$$\mathscr{L}' = P_\mu \dot{X}^\mu - E, \qquad \dot{X}^\mu = \frac{\partial E}{\partial P_\mu},$$

with P_μ related to Π_μ by (2.15). We find

$$\begin{aligned} \mathscr{L}' = &\mu\left(\frac{1}{2}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \omega^I_{\mu\nu}D^\nu\dot{X}^\mu\right) - \frac{1}{4\mu}R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\nu}\psi^\rho\psi^\sigma \\ &+ ig_{\mu\nu}\psi^{\dagger\mu}\left(\frac{D\psi^\nu}{Dt} + i\psi^\nu - I^\nu{}_\rho\psi^\rho\right). \end{aligned} \tag{7.34}$$

Physically, we have a supersymmetrised coupling of the original σ-model to the magnetic field

$$A_\mu = \omega^I_{\mu\nu}D^\nu.$$

This model has $\mathcal{N} = (1,1)$ supersymmetry generated by $\{q, s, E\}$, with $U(1)$ R-symmetry (7.31). The little group $\mathfrak{su}(2|1)$ is also linearly realised in this model, taking the form of an exotic $\mathcal{N} = (2,2)$ supersymmetry with central charges whose interpretation is somewhat mysterious. The important thing is that the Witten index of this model, as defined in section 2.3, should compute the index of the bundle-valued L^2 Dolbeault complex discussed above. Adding appropriate chemical potentials to (7.34) should lead to an alternative interpretation of the superconformal index, the subject of the next section.

7.4 The Superconformal Index of $\mathfrak{osp}(4^*|4)$

Finally, we can put together everything we've learned about the representations of $\mathfrak{osp}(4^*|4)$ to construct the superconformal index. First introduced in [53] in the context of four-dimensional superconformal field theory, and extended to three, five, and six dimensions in [54], this is an extension of the idea of the Witten index discussed in section 2.3 to the superconformal case. The idea is that short and semishort representations saturate unitarity bounds determined by operators of the form $E = \{q, s\}$, so a Witten index of the form $\operatorname{tr}\left[(-1)^F e^{-\beta E}\right]$ will be independent of β and receive contributions only from short and semishort representations. More information can be obtained by including terms in the trace which weight states according to their quantum numbers under the little group $\mathfrak{su}(2|1)$ of $\{q, s, E\}$. The objective is to obtain a calculable object which contains as much protected information about (semi)short representations in a given theory as possible.

Since (semi)short representations contain fewer states than a long representation and have dimensions saturating a bound, their existence is often insensitive to continuous changes in the parameters of a theory. The exception to this rule is when a long representation's dimension is continuously lowered to the unitarity bound, at which point it splits into a semishort representation with manifestly positive norm and a collection of null states. The null states form their own representation of $\mathfrak{osp}(4^*|4)$, which cannot be long as it contains fewer states than a long representation. Therefore, it is possible for a long representation to split into a sum of (semi)short representations upon hitting the unitarity bound, and conversely such a sum of (semi)short representations can coalesce into a long representation and move away from the bound.[32]

The preceding discussion indicates that, in general, the number of (semi)short representations of any given type in a theory is not protected. Therefore, an index must be designed so that it counts sums of representations which can pair into a long representation the same way as it counts a long representation. That is, they must not contribute. We can write a general index as

$$I_\alpha = \sum_{R \in \mathscr{R}} \alpha(R) N(R),$$

where $\mathscr{R}$ is the set of possible (semi)short representations, $N(R)$ is the number of representations

[32] One might wonder how null states, which are usually thought of as unphysical, can join into a physical representation. The point is that null states are only null if assumed to be part of the larger representation already. If they are part of a separate representation then there is no inconsistency in assigning them a positive norm. In other words, the 'null' states are genuine physical states.

of a given type R present, and $\alpha(R)$ are coefficients chosen such that, for any set of representations $R_1, \ldots, R_k$ which can coalesce into a long representation,

$$\alpha(R_1) + \ldots \alpha(R_k) = 0.$$

Aside from these constraints, the α are arbitrary and the set of all possible indices therefore forms a vector space.

A good starting point is to write down a basis for this vector space, for which we need to know how long representations decompose as they hit the unitarity bound $\Delta = 2(j+m+1)$. Our results here are preliminary in that we do not at present have a proof of these decompositions. Such a proof probably requires characters for $\mathfrak{osp}(4^*|4)$ irreps and there is no general formula available in the infinite-dimensional case, though a number of special cases have been worked out for field theoretic superconformal algebras [54,89,136–140]. Nevertheless, we proceed as follows. To simplify notation, we write Δ_{SS} for the semishort scaling dimension $\Delta = 2(j+m+1)$ and Δ_S for the short dimension $\Delta = 2m$.

Consider a long representation $L(\Delta_{SS} + \epsilon, j, m, n)$ with $j > 0$ for the moment. $0 < \epsilon << 1$ indicates that the representation is just above the unitarity bound, and we want to know what happens as $\epsilon \to 0$. When $\epsilon = 0$ the semishort bound is attained and we know from section 7.1 that there is a null state

$$x_{\lambda'} = \left(-2jQ^-_{11} + Q^-_{21}J_+\right) v_\lambda.$$

Since there is no null state which can be reached from v_λ via a shorter string of simple roots, this must be the lowest weight state of a null representation with lowest weight $\lambda' = \lambda + \epsilon_1 - \delta_1$. The question we are unable to answer at present is whether the null representation is irreducible. This seems to be the case in known examples whenever only one atypicality condition is met. Assuming this is true here, the remaining issue is to classify the null representation. To do so, observe that

$$E(\lambda') = E(\lambda) + E(\epsilon_1) - E(\delta_1) = E(\lambda) = 1,$$

so the representation is semishort. One might then wonder whether this representation in turn has null states, but fortunately

$$\left(-2jQ^-_{11} + Q^-_{21}J_+\right)^2 = 0$$

so that the candidate null states are not present in the long representation anyway, and can be ignored. Finally, we can check that λ' translates into quantum numbers

$$j' = j - \frac{1}{2}, \qquad m' = m + 1, \qquad n' = n,$$

so we obtain

$$L(\Delta_{SS}+\epsilon, j, m, n) \to SS(j,m,n) \oplus SS(j-\frac{1}{2}, m+1, n) \qquad (\epsilon \to 0,\ j>0). \tag{7.35}$$

The case $j=0$ can be handled analogously, but this time the state $x_{\lambda'}$ vanishes in the long representation. Indeed, there are no new null states at level 1, and the lowest weight null state turns out to be

$$y_{\lambda''} = Q^-_{11} Q^-_{21} v_\lambda$$

with $\lambda'' = \lambda + \epsilon_1 + \epsilon_2 - 2\delta_1$. Consequently, $E(\lambda'') = 0$ and the null representation is short, with quantum numbers

$$j'' = 0, \qquad m'' = m+2, \qquad n'' = n.$$

The decomposition is

$$L(\Delta_{SS}+\epsilon, 0, m, n) \to SS(0,m,n) \oplus S(m+2, n) \qquad (\epsilon \to 0). \tag{7.36}$$

An interesting feature of these decompositions is that the short representations for which $m-n \le 1$ are absolutely protected and cannot combine into long representations. In particular, this includes the 1/2-BPS representations.

More generally, we obtain the constraints

$$\begin{aligned} 0 &= \alpha(SS(0,m,n)) + \alpha(S(m+2,n)) & \forall m \ge n \ge 0 \\ 0 &= \alpha(SS(j,m,n) + \alpha(SS(j-\frac{1}{2}, m+1, n)) & \forall m \ge n \ge 0,\ j>0 \end{aligned} \tag{7.37}$$

such that I_α is an index. Solving these gives a basis of indices

$$I^{m,n} = \begin{cases} N(S(m,n)) & (m-n \le 1) \\ N(S(m,n)) + \sum_{k=0}^{m-n-2} (-1)^{k+1} N(SS(\frac{k}{2}, m-2-k, n)) & (m-n>1). \end{cases} \tag{7.38}$$

Notice in particular that each $I^{m,n}$ is uniquely determined by its action on short representations. Given any index

$$I = \sum_{m,n} c_{mn} I^{m,n},$$

we can therefore calculate the coefficients c_{mn} by evaluating I on short representations.

We wish to do this for the superconformal index. Recall that the subalgebra commuting with $\{q, s, E\}$ is $\mathfrak{su}(2|1)$, with Cartan subalgebra spanned by $N \in \mathfrak{su}(2)$ and $Z = -(M+2J_3) \in \mathfrak{u}(1)$. The superconformal index is

$$I(a,b) = \operatorname{tr}\left[(-1)^F e^{-\beta E} a^Z b^N\right]. \tag{7.39}$$

As for the Witten index in section 2.3, one argues that $I(a,b)$ is independent of β as the bosonic and fermionic contributions from states with $E > 0$ cancel, and will generically be independent of continuous variations in the parameters of a theory. In view of our discussion of coalescence of (semi)short representations, this will only be true if $I(a,b)$ really is an index, which we verify below.

Write $I_R(a,b)$ for the index evaluated on a particular representation R, and R_0 for the $E=0$ content of R. Then

$$I_R(a,b) = \mathrm{tr}_{R_0}\left[(-1)^F a^Z b^N\right].$$

We fix a convention so that $(-1)^F$ acts on the lowest weight state of R_0 as the constant $(-1)^{F_R}$, and require that the action of supercharges shifts F by ± 1. A more precise convention will follow. $I_R(a,b)$ is proportional to a supercharacter of $\mathfrak{su}(2|1)$, which can be expanded in characters of $\mathfrak{su}(2)$. Denoting the 'spin' $m/2$ character of $\mathfrak{su}(2)$ by $\chi_m = \chi_m(b)$ and using (7.30), we find

$$(-1)^{F_R} I_R(a,b) = \begin{cases} a^m\left(\chi_m - a\chi_{m-1}\right) & S(m,m) \\ a^m\left((1+a^2)\chi_n - a(\chi_{n+1}+\chi_{n-1})\right) & S(m,n) : n < m \\ a^{m+2j+2}\left((1+a^2)\chi_n - a(\chi_{n+1}+\chi_{n-1})\right) & SS(j,m,n), \end{cases} \tag{7.40}$$

where we used $\chi_0 = 1$ and $\chi_{-1} = 0$.

This allows us to check that $I(a,b)$ really is an index. That is, we need to verify that the contributions (7.40) satisfy the constraints (7.37). It's easy to see that they do provided that F_R differs by 1 (mod 2) for any two pairable representations. We assume that, up to an irrelevant overall shift, F is in the Cartan subalgebra of $\mathfrak{osp}(4^*|4)$. In order that it shifts by ± 1 under all supercharges, we must have one of

$$F = h_1 \pm h_2, \qquad F = h_3 \pm h_4.$$

It's important to note that non-trivial linear combinations of these will not work. We need this to be consistent with the natural notion of fermion number for quantum mechanics on $\mathbb{C}^{2k}$, which rules out $h_1 \pm h_2$ since fermion creation operators commute with these in flat space. Either choice $h_3 \pm h_4$ would be consistent with flat space since ψ_I^A transforms in the $\mathbf{4}$ of $SO(5)$, and moreover each choice gives the required relative sign between pairable representations. We therefore choose

$$F = h_3 + h_4 = M + N.$$

This choice generalises nicely to curved target spaces since the $SO(5)$ action is local and algebraic, thus effectively independent of curved structure. Furthermore, it makes sense in the context of instanton quantum mechanics since in five-dimensional MSYM the fermions transform in the **4**, so shift F by ± 1, while the scalars are in the **5** and shift F by ± 2 or 0. Using (7.40) together with the fact that the $I^{m,n}$ are determined by their action on short representations, we find

$$I(a,b) = \sum_{m=0}^{\infty} a^m \left[(\chi_m - a\chi_{m-1})\, I^{m,m} + \sum_{n=0}^{m-1} (-1)^{m-n} \left((1+a^2)\chi_n - a(\chi_{n+1} + \chi_{n-1}) \right) I^{m,n} \right]. \tag{7.41}$$

If the value of $I(a,b)$ is known then one can extract the value of the basic indices $I^{m,n}$ using orthogonality of characters of $\mathfrak{su}(2)$. Explicitly, the $\mathfrak{su}(2)$ characters are

$$\chi_n(e^{i\theta}) = \mathrm{tr}_n\, e^{i\theta N} = \frac{\sin\left[(n+1)\theta\right]}{\sin\theta} \tag{7.42}$$

and obey the orthogonality relation

$$\langle \chi_m, \chi_n \rangle = \frac{2}{\pi} \int_0^{\pi} \sin^2\theta \chi_n(e^{i\theta}) \chi_m(e^{i\theta}) d\theta = \delta_{mn}.$$

Thus if $I(a,b) = \sum_n c_n(a)\chi_n(b)$ then

$$c_n(a) = \left\langle I(a, e^{i\theta}), \chi_n(e^{i\theta}) \right\rangle. \tag{7.43}$$

Comparing coefficients of a^k against (7.41) allows extraction of $I^{m,n}$. Since we do not have an orthogonality relation for supercharacters of $\mathfrak{su}(2|1)$, it is not immediately apparent that this process is well-defined. In particular, there could be nonzero values of the $I^{m,n}$ for which $I(a,b) = 0$. Fortunately, one can prove inductively that the functions occurring in (7.41) are linearly independent, so this cannot occur. We conclude with some examples where the index can be calculated.

Example 1: Quantum Mechanics on $\mathbb{C}^2$

We'll work through this example in detail, both because it is simple enough to see explicitly what is going on and because quantum mechanics on the instanton moduli space $\mathcal{M}_{k,N}$ has this as a universal decoupled sector corresponding to the instanton centre. It is easy to calculate the index using the representation of $\mathfrak{osp}(4^*|4)$ discussed in section 7.2. The fundamental variables are the four bosons $\alpha^\dagger, \bar{\alpha}^\dagger, \beta^\dagger, \bar{\beta}^\dagger$ and the fermions $\psi^A : A = 1,2,3,4$. These are simultaneous eigenvectors of E, Z, N, and F so the trace (7.39) splits as a product of contributions from each variable, along with the contribution of the harmonic oscillator vacuum $|0\rangle$. The vacuum contribution is

$$\langle 0| (-1)^F e^{-\beta E} a^Z b^N |0\rangle = e^{-\beta}$$

since the vacuum has $E = 1$. The fermion contributions are

$$I_F(a,b) = \prod_A \mathrm{tr}_A \left[(-1)^F e^{-\beta E} a^Z b^N\right] = (1-b^{-1})(1-ae^{\beta})(1-a^{-1}e^{-\beta})(1-b),$$

where tr_A is the contribution of ψ^A. The boson contributions are

$$I_B(a,b) = \left(\mathrm{tr}_{\alpha^\dagger}\right)^2 \left(\mathrm{tr}_{\bar{\alpha}^\dagger}\right)^2,$$

since $\alpha^\dagger$ and $\beta^\dagger$ have the same charges. This gives

$$I_B(a,b) = \left(\sum_{r=0}^{\infty} a^r\right)^2 \left(\sum_{s=0}^{\infty} a^{-s} e^{-\beta s}\right)^2 = \left(\frac{1}{1-a}\right)^2 \left(\frac{1}{1-a^{-1}e^{-\beta}}\right)^2,$$

where in the last equality we assumed $|a| < 1$ and $|ae^{\beta}| > 1$. Putting everything together, we find

$$I(a,b) = \frac{a}{b}\left(\frac{1-b}{1-a}\right)^2. \tag{7.44}$$

It's a pleasing check to see that this is independent of β. This answer is simple enough that we can easily extract a few $I^{m,n}$. Expanding (7.44) gives

$$I(a,b) = (\chi_1(b) - 2)(a + 2a^2 + 3a^3 + \dots),$$

and equating coefficients with (7.41) gives an infinite set of equations for $I^{m,n}$ which can in principle be solved iteratively. Working to $O(a^2)$ gives

$$I^{0,0} = 0, \qquad I^{1,0} = 2, \qquad I^{2,0} = -3, \qquad I^{1,1} = 1, \qquad I^{2,1} = 0, \qquad I^{2,2} = 0.$$

These can be used to deduce something about the state content using the definitions (7.38). $I^{0,0} = 0$ implies that there is no vacuum representation, while $I^{2,1} = 0$ means there are no short representations with $(\Delta, m, n) = (4,2,1)$, and $I^{2,2} = 0$ means there are no 1/2-BPS representations with $(\Delta, m) = (4,2)$. $I^{1,0} = 2$ says there are exactly two 1/4-BPS representations with $(\Delta, m, n) = (2,1,0)$, while $I^{1,1} = 1$ means there is exactly one 1/2-BPS representation with $(\Delta, m) = (2,1)$. $I^{2,0} = -3$ is less trivial since

$$I^{2,0} = N(S(2,0)) - N(SS(0,0,0)),$$

so we can only deduce that there are three more semishort representations with $(\Delta, j, m, n) = (2,0,0,0)$ than there are 1/4-BPS representations with $(\Delta, m, n) = (4,2,0)$.

Since quantum mechanics on $\mathbb{C}^2$ is so simple, we can compare these results with answers obtained by brute force construction of representations. We know that all states in this quantum mechanics have $\Delta \geq 2$, so there cannot be a vacuum representation. We also know that $m \leq 1$ due to Fermi statistics, so $N(S(2,2)) = N(S(2,1)) = N(S(2,0)) = 0$. The remaining representation types under consideration, $S(1,0), S(1,1)$, and $SS(0,0,0)$, all have $\Delta = 2$ so we disallow bosonic excitations. The two $S(1,0)$ representations correspond to the two ways of obtaining the **4** of $SO(5)$

$$\psi^A |0\rangle , \qquad \epsilon^A{}_{BCD}\psi^B\psi^C\psi^D |0\rangle ,$$

while the single $S(1,1)$ is the unique choice of **5**

$$\Psi^{AB} |0\rangle ,$$

with Ψ^{AB} as in (7.23). Finally, the three required $SS(0,0,0)$ representations correspond to the three $SO(5)$ singlets

$$|0\rangle , \qquad \Omega_{AB}\psi^A\psi^B |0\rangle , \qquad \epsilon_{ABCD}\psi^A\psi^B\psi^C\psi^D |0\rangle .$$

Thus the index is in perfect agreement with explicit constructions, at least up to the order we computed. This order is sufficient to get all the short representations (since $n \leq m \leq 1$), but to get semishorts with large j one must go to arbitrarily high order.

Example 2: Quantum Mechanics on $\mathbb{C}^{2k}$

Computing this index is a simple matter since all operators in the quantum mechanics decompose into sums corresponding to $\mathbb{C}^2$ 'blocks', and the Hilbert space is the kth tensor power of that of $\mathbb{C}^2$. For example, L_0 is a sum of decoupled harmonic oscillator Hamiltonians. It follows that

$$\begin{aligned} I_k(a,b) &= \mathrm{tr}\left[(-1)^F e^{-\beta E} a^Z b^N\right] \\ &= \mathrm{tr}\left[(-1)^{F_1+\cdots+F_k} e^{-\beta(E_1+\cdots+E_k)} a^{Z_1+\cdots+Z_k} b^{N_1+\cdots+N_k}\right] = (I_1(a,b))^k . \end{aligned}$$

Substituting in (7.44), we find

$$I_k(a,b) = \left[\frac{a}{b}\left(\frac{1-b}{1-a}\right)^2\right]^k . \tag{7.45}$$

In principle one can go through the same procedure, calculating coefficients of $\mathfrak{su}(2)$ characters and extracting the elementary indices $I^{m,n}$, but we won't do this explicitly.

Example 3: One $SU(2)$ Instanton

$\mathcal{M}_{1,2}$ is the non-trivial manifold $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2$, so we do not have an explicit construction of the

Hilbert space. As such, a direct computation of the index looks difficult. Fortunately, supersymmetric localisation can be used to carry out the computation [100]. The relevant formula in that paper is equation (2.45), which must be changed to our conventions by setting

$$e^{-i\gamma_2} = b, \qquad e^{-i\gamma_R} = a, \qquad \gamma_1 = 0, \qquad \mu_{12} = 0.$$

Doing this gives

$$I_{k=1,N=2}(a,b) = \frac{a}{b}\left(\frac{1-b}{1-a}\right)^2\left(1+\frac{b+b^{-1}}{a+a^{-1}}\right).$$

Observe that this expression contains a factor of the $\mathbb{C}^2$ result (7.44), which reflects the universal factor I_{com} given by equation (2.46) of [100]. It's a simple matter to expand this expression in $SU(2)$ characters and equate coefficients as before. Working to $O(a^3)$ gives

$$\begin{aligned} &I^{0,0} = 0, \qquad I^{1,0} = 2, \qquad I^{1,1} = 1, \qquad I^{2,2} = 1, \qquad I^{2,0} = -2 \\ &I^{2,1} = 2, \qquad I^{3,0} = 0, \qquad I^{3,1} = -2, \qquad I^{3,2} = 0, \qquad I^{3,3} = 0. \end{aligned}$$

As the difference $m-n$ increases, the amount of information that can be extracted from $I^{m,n}$ diminishes as it is in alternating sum of more terms. For example, $I^{3,0} = 0$ tells us that

$$N(S(3,0)) - N(SS(0,1,0)) + N(SS(\frac{1}{2},0,0)) = 0.$$

We can restrict this a little since $N(S(3,0))$ should equal zero. This follows because the fermionic $SO(5)$ action is purely algebraic, so can be understood in the tangent space to a single point of the target manifold M. Thus we can choose a basis at that point such that the $SO(5)$ action is indistinguishable from the flat space case, and fermionic statistics lead to the general bound

$$m \leq \frac{1}{4}\dim_{\mathbb{R}} M. \tag{7.46}$$

In particular, $\mathcal{M}_{k,N}$ should admit R-charges no larger than kN, which is 2 in this case. However, even in flat space we can't do much better than this for large $m-n$ as the task of identifying which representation a given $E=0$ state belongs to is rather involved. This makes counting semishort representations a serious challenge.

In view of the above examples, two important questions arise. The first is whether the index is calculable in any generality. We've seen that even a $\mathbb{Z}_2$ quotient of flat space seems to require a powerful technique such as localisation to be accessible, and one might wonder how far this can be pushed. The localisation computation of [100] works for any instanton moduli space $\mathcal{M}_{k,N}$, and relies on a link between the superconformal index and the partition functions of [126]. The discussion of section 7.3 may give a more satisfying first-principles approach to establishing this link.

The second important question is whether we can do any better than the index. As we saw, beyond the first few elementary indices it is difficult to interpret the results too precisely due to the amount of redundancy for indices with large $m-n$. One possible improvement, which is in fact crucial to the localisation computation, is to exploit global symmetries. In any model with global symmetries, the Hilbert space will split as a direct sum of irreducible representations of those symmetries. For instance, with a single $U(1)$ symmetry generator X we have

$$\mathcal{H} = \bigoplus_x \mathcal{H}_x,$$

where the states of $\mathcal{H}_x$ have charge x under X. Since X is global it commutes with $\mathfrak{osp}(4^*|4)$, so each $\mathcal{H}_x$ itself decomposes into a sum of representations of $\mathfrak{osp}(4^*|4)$. One can then define a *refined superconformal index*

$$\begin{aligned} I_{\mathcal{H}}(a,b,c) &= \mathrm{tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta E} a^{M+2J_3} b^N c^X\right] \\ &= \sum_x c^x \, \mathrm{tr}_{\mathcal{H}_x} \left[(-1)^F e^{-\beta E} a^{M+2J_3} b^N\right] = \sum_x c^x I_x(a,b), \end{aligned} \tag{7.47}$$

where $I_x(a,b)$ is the contribution of $\mathcal{H}_x$ and is itself a superconformal index. This idea generalises straightforwardly to an arbitrary global symmetry algebra $\mathfrak{g}$, with the refined superconformal index splitting as a sum of superconformal indices multiplying characters of $\mathfrak{g}$. One can always set $c=1$ to recover the original index, but the refined index contains more information and it may be easier to extract meaningful results by examining the I_x individually. In the case of quantum mechanics on $\mathcal{M}_{k,N}$, there are global symmetries given by the 'other' $SU(2)$ subgroup of the rotational $SO(4)$, as well as the $SU(N)$ action. These induce isometries of $\mathcal{M}_{k,N}$, and more generally any group of isometries of target space commuting with $\mathfrak{osp}(4^*|4)$ will give rise to a global symmetry with which we can refine the index.

8 Conclusion

8.1 Summary of Results

We conclude with a summary of our main results and a discussion of some of the open problems in this area. We refer back to the relevant sections for notational conventions. We studied supersymmetric quantum mechanical σ-models of the generic form

$$S = \int dt \, \frac{1}{2} g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu + i g_{\mu\nu}(X) \psi^{\dagger\mu} \frac{D}{Dt} \psi^\nu - \frac{1}{4} R_{\mu\nu\rho\sigma}(X) \psi^{\dagger\mu} \psi^{\dagger\nu} \psi^\rho \psi^\sigma,$$

where g is the metric on some target manifold M. In section 5 we proved

Theorem

1. Let (M, g) be a Riemannian manifold admitting a *closed homothety*, that is a vector field D satisfying

$$\mathcal{L}_D g = 2g, \qquad \mathcal{L}_D K = 2K, \qquad D_\mu = \partial_\mu K$$

   for some function K on M. Then the σ-model with target space M admits an $\mathfrak{su}(1,1|1)$ superconformal symmetry.

2. Let (M, g) as above be Kähler such that D is holomorphic and K is a Kähler potential. Then the σ-model has $\mathfrak{u}(1,1|2)$ symmetry.

3. Let (M, g) be hyper-Kähler such that D is triholomorphic and K is a hyper-Kähler potential. Then the σ-model has $\mathfrak{osp}(4^*|4)$ symmetry.

The proof of this result relied upon the link between wavefunctions of the σ-model and the exterior algebra on M. In particular, our results and arguments were at every stage manifestly geometric in character. These constructions gave us an easy proof that the moduli space $\mathcal{M}_{k,N}$ of k $SU(N)$ Yang-Mills instantons on $\mathbb{R}^4$ admits $\mathfrak{osp}(4^*|4)$ invariance, as is required by the DLCQ model of the (2,0) theory.

In section 6 we considered models of the above type formulated on a semi-flat torus bundle $\mathcal{J}$ over a scale-invariant special Kähler manifold, with metric

$$ds^2 = \operatorname{Im} \tau_{IJ} da^I d\bar{a}^J + \left(\operatorname{Im} \tau^{-1}\right)^{IJ} \delta z_I \delta \bar{z}_J.$$

We gave an explicit construction of the supersymmetry algebra of this model, and showed that, in the sector with zero units of momentum around the fibres, the symmetry enhances to the superconformal algebra $\mathfrak{su}(1,1|4)$. This enhancement is as required by the DLCQ model of $\mathcal{N} = 4$

SUSY Yang-Mills. The truncation to zero fibre momentum resulted in a novel type of σ-model, with $\mathcal{N} = (4,4)$ supersymmetry, $SO(6) \times U(1)$ R-symmetry, a scale-invariant special Kähler target, and an unusual chiral 4-fermion term

$$-\frac{1}{12}\,\mathrm{Re}\left(\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}\right)$$

in its Lagrangian. Scale-invariant special Kähler manifolds arise naturally from the Coulomb branches of $\mathcal{N} = 2$ superconformal gauge theories in four dimensions, so there is a large class of these models provided by known physical results. Going beyond zero fibre momentum, we showed that a mild deformation of the original σ-model, which can be interpreted as a worldline magnetic field coupling to the holomorphic symplectic form on $\mathcal{J}$, preserves an $\mathfrak{osp}(1,1|4)$ 'superconformal' algebra. We proved that this deformation can be interpreted in the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills as a spacetime magnetic field coupling to instanton number density, and provided evidence that it improves the behaviour of the spectral problem for the light-cone dilatation operator.

In section 7 we turned to the representation theory of $\mathfrak{osp}(4^*|4)$. We gave a complete classification of all unitary irreducible lowest weight representations, with necessary and sufficient conditions for unitarity, resulting in

Theorem

Unitary irreducible representations of $\mathfrak{osp}(4^*|4)$ are lowest weight modules V_λ obtained as quotients of lowest weight Verma modules $\mathcal{V}_\lambda$ by their null submodules. We have the following types:

- *Generic* or *long* representations with $\Delta > 2(j + m + 1)$.
- *Semishort* representations with $\Delta = 2(j + m + 1)$.
- *Short* representations with $\Delta = 2m,\ j = 0$. Short representations further split into *1/4-BPS* representations with $m \neq n$ and *1/2-BPS* with $m = n$.

We then studied the superconformal index. We classified the states in a given representation which contribute to an index defined by the supercharge q, its conjugate s and $E = \{q, s\}$, and found that short and semishort representations contribute via states in an irreducible representation of the little group $\mathfrak{su}(2|1) \subset \mathfrak{osp}(4^*|4)$ commuting with $\{q, s, E\}$. We decomposed these representations into $\mathfrak{su}(2)$ submodules, allowing a calculation of how each representation contributes to the superconformal index. Together with a description of the branching rules as long representations hit the unitarity bound, this led to a formula for the superconformal index in terms of characters of $\mathfrak{su}(2|1)$:

$$I(a,b)=\sum_{m=0}^{\infty}a^m\left[(\chi_m-a\chi_{m-1})\,I^{m,m}+\sum_{n=0}^{m-1}(-1)^{m-n}\left((1+a^2)\chi_n-a(\chi_{n+1}+\chi_{n-1})\right)I^{m,n}\right].$$

As a link back to earlier geometric ideas, we proved that states contributing to the superconformal index are in one-to-one correspondence with elements of the L^2 Dolbeault cohomology of the target manifold with values in the real line bundle with metric $e^{-2\mu K}$. We gave a deformed version of our general σ-model whose Witten index computes the index of this Dolbeault complex.

8.2 Future Directions and Open Problems

The geometric approach to superconformal quantum mechanics laid out in section 5 suggests a geometric interpretation for quantities such as the superconformal index of section 7.4. In the case of $\mathfrak{osp}(4^*|4)$, we gave one such interpretation in terms of L^2 Dolbeault cohomology valued in a line bundle in section 7.3. In fact, our construction only used the Kähler structure with respect to a preferred complex structure, so should also apply to systems with $\mathfrak{u}(1,1|2)$ invariance. However, this prescription seems quite far removed from the interpretation of scale transformations in terms of a homothetic vector field. Another possibility which we consider worth investigating is that the superconformal index can be interpreted in terms of equivariant cohomology (see [141] for a review) with respect to the action of the homothety D. This intuition stems from the fact that ordinary indices of non-compact spaces, such as the Witten index, are difficult to make sense of, whereas the superconformal index appears to be well-defined. The homothety in some sense 'squashes' a non-compact space down to a compact one, and making rigorous sense of this may well lead to an interpretation in terms of equivariant cohomology. This would, however, be somewhat non-standard, as the usual formulation of equivariant cohomology for differential forms works for Hamiltonian actions of compact Lie groups on symplectic manifolds. One would need a generalisation of this setup to make sense of the above ideas.

A related issue concerns the identification of our superconformal index, defined algebraically in terms of sums of protected representations, with other indices obtained via supersymmetric localisation [100–102]. In particular, the connection between the index of [100] and the partition function of [126] is somewhat mysterious. In section 7.3 we gave an interpretation of the superconformal index in terms of a deformed σ-model (7.34), which should admit additional supersymmetry-preserving deformations allowing a continuous interpolation between the various models and interpretations. In a similar vein, [40] analysed BPS states of the (2,0) theory DLCQ via cohomology with compact support. It would be interesting to incorporate this into our picture, perhaps by 'undoing' the similarity transformation (7.1) or analysing the $\mu \to \infty$ limit.

In section 6 we analysed superconformal algebras on special Kähler manifolds, relying crucially on a choice of special coordinate system a^I. A subtlety which we suppressed is that these coordinate systems are local in a very precise sense. A generic special Kähler manifold, and in particular one arising from the Coulomb branch of a four-dimensional $\mathcal{N} = 2$ gauge theory, has singular submanifolds of complex codimension one around which the special coordinates a^I and their duals a_I^D undergo electromagnetic $Sp(2g;\mathbb{Z})$ duality transformations. These will induce transformations on the generators of $\mathfrak{su}(1,1|4)$, and it will be important to determine what impact this lack of single-valuedness has on our analysis. One possible means to evade this issue is to work on the universal covering space of the Coulomb branch. By definition this is also special Kähler, and any given sheet of the cover corresponds to a choice of duality frame for the original target manifold. This solution seems unsatisfactory. At the very least, one would expect to significantly overcount states in the quantum mechanics without some form of quotient procedure to return to the original target. It is possible that the regularisation defined in section 6.5 could help with this issue by localising wavefunctions away from the singular submanifolds, but this is certainly an issue which needs more careful analysis.

A related issue concerns the lack of a manifestly geometric formulation of the $\mathfrak{su}(1,1|4)$-invariant σ-model. In the standard σ-model (2.7), the fermions can be thought of as sections of the cotangent bundle of the target manifold, but in the case of $\mathfrak{su}(1,1|4)$ there are too many fermions for this. One possibility is that they should naturally be valued in an $Sp(2g;\mathbb{Z})$ bundle such as those discussed in [117]. This might have the desirable side effect of revealing that the σ-model is invariant under $Sp(2g;\mathbb{Z})$. Such a symmetry exists and can easily be made manifest in the purely bosonic model with special Kähler target, but it is far from clear whether it extends to the supersymmetric case. In the case of the DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills, the target manifold has a decoupled factor corresponding to instanton centre of mass, preserved by a subgroup $SL(2;\mathbb{Z}) \subset Sp(2g;\mathbb{Z})$. This symmetry should be present for all values of k, thus also as $k \to \infty$, and it is tempting to conjecture that it corresponds to the $SL(2;\mathbb{Z})$ duality of $\mathcal{N} = 4$ SUSY Yang-Mills.

More broadly, the question of what exactly these models can tell us about $\mathcal{N} = 4$ SUSY Yang-Mills and the (2,0) theory remains to be answered. For the (2,0) theory, a natural hope is to go beyond indices and give a complete description of the spectrum of BPS states, for which at time of writing only partial results are known [54, 89, 100–106]. This would require an analysis of the branching rules for representations of $\mathfrak{osp}(8^*|4)$ into sums of irreducible representations of the DLCQ subalgebra $\mathfrak{osp}(4^*|4)$. In particular, the relation between BPS shortening conditions in the field theory and in quantum mechanics would need to be understood. One would also need to characterise the BPS states in the quantum mechanics more precisely than the rather crude index methods we employed in section 7.3. The correct approach here may be to use the ADHM construction along with methods of constrained quantisation to push the flat space results of section 7.2 through to instanton moduli space. It may also be possible to exploit an analogue of the integrable limit discussed for the $\mathfrak{su}(1,1|4)$ case in section 6.5. It remains to be understood how much detail these approaches can provide for the spectrum of D.

In the $\mathcal{N}=4$ SUSY Yang-Mills case, the spectrum of 1/2 and 1/4-BPS states is already known and the full spectrum of the dilatation operator is quite well understood, at least in the planar limit, thanks to large-N integrability [22]. However, the origin of this integrability is far from clear, relying on increasingly miraculous properties of spin chains with longer distance interactions at each loop order. Since integrability is so much better understood in finite-dimensional systems, it's reasonable to hope that DLCQ can shed some light on the origin of planar integrability. Realistically, one would hope to find that the DLCQ model of section 6 is integrable at large N. We saw in section 6.5 that our model, or rather the version regularised by a magnetic field, has an integrable limit at finite N. It remains to be seen whether this bears any relation to planar integrability. Even if it does not, the class of models we construct based on Coulomb branches of $\mathcal{N}=2$ superconformal field theories represent a large class of integrable models whose analysis is itself an interesting problem. Some early work on a related problem is [142]. There is also a strong analogy between the magnetic deformation of section 6.5 and the supersymmetric model of section 7.3 whose Witten index is the superconformal index of $\mathfrak{osp}(4^*|4)$. This suggests that the magnetic deformation is particularly useful for analysing BPS states, and the integrable limit may be very helpful in that context. An important feature of this limit is that it can be obtained via the symmetry transformation (7.1), and as such preserves not only the superconformal index but the full spectrum of the theory.

More speculatively still, our results may be of value to other research programmes. The conformal bootstrap has recently been used to derive interesting results about the (2,0) theory [90]. Generally speaking, the bootstrap is more constraining if it has more 'initial data', and a complete characterisation of the BPS states would be an improvement in this respect. Moreover, recent results show that a sector of the (2,0) theory exhibits a chiral algebra known as $\mathcal{W}$-symmetry [89].

It may be that such results are accessible in DLCQ. Indeed, there is evidence emerging that subsectors of instanton moduli space quantum mechanics admit current algebra symmetries [143, 144]. Understanding how this plays out for the full instanton moduli space is an exciting question. In particular, there are hints of Yangian invariance [145]. This may also provide a smoking gun for integrable structures, potentially relating to that of $\mathcal{N} = 4$. In addition, the main bulk of this thesis concerns the construction of quantum mechanical models with superconformal invariance. The AdS/CFT correspondence suggests that these should be dual to (quantum) gravitational theories on spacetimes with asymptotically AdS_2 factors, and it would be interesting to identify these spacetimes.

Finally, note that the DLCQ models we've studied all apply to theories with maximal supersymmetry. One possible extension therefore is to maximally supersymmetric theories in other dimensions, such as that describing the worldvolume of coincident M2-branes. One might even hope to be able to extend these ideas to theories with less supersymmetry. This is clearly a very difficult task, as giving any useful description of the DLCQ models would be much harder. This is in part down to the fact that the physics of vacuum moduli spaces for four-supercharge theories is much more complicated than for eight. However, if it could be done then it might have the potential to shed tremendous light on non-perturbative field theory.

A Constrained Quantisation

In sections 2.1, 6.3 and 6.5, we find ourselves quantising systems subject to constraints. In such systems it is often inconsistent to impose canonical commutation relations, and this appendix describes some methods to deal with this. We will encounter two broad cases:

- 'Artificial' constraints $C(Q, P) = 0$.
- First-order Lagrangians.

An artificial constraint externally imposes the fact that motion most occur within some submanifold of phase space defined by the vanishing of k functions $C_r(Q, P)$ of the positions Q^a and momenta P_a. A simple example of this is a free particle moving on the surface of a sphere. As written here, this is a very general type of constraint which is handled by a procedure due to Dirac [56]. A first-order Lagrangian is a special case of this, and we'll describe a (sometimes) simpler method due to Faddeev and Jackiw [57] to arrive at Dirac's result.

A.1 The Dirac Procedure

When quantising a classical Hamiltonian system, it is conventional to impose canonical commutation relations on the fundamental variables, which amounts to multiplying the canonical Poisson brackets

$$\{Q^a, P_b\}_P = \delta^a_b \tag{A.1}$$

by i. However, if the system is subject to constraints $C_r(Q, P) = 0$ then this procedure is need not be consistent. In particular, the constraints should translate into operators on Hilbert space represented by zero, so their commutator with any function of the fundamental variables should also vanish, but this may not be true for canonical quantisation.

We now describe a general procedure for dealing with this, laid out by Dirac in [56]. We start with a set of *primary* constraints which we impose on the system. These include obvious 'artificial' constraints $C_r(Q, P) = 0$, such as forcing a particle to move on the surface of a sphere by setting $\sum_a (Q^a)^2 = 1$, as well as less obvious ones imposed by choosing a Lagrangian for which the map between velocities $\dot{Q}^a$ and canonical momenta $\partial\mathscr{L}/\partial\dot{Q}^a$ is not invertible. An example of the latter which we cover in more detail later is a Lagrangian which is first-order in time derivatives

$$\mathscr{L} = f_a(Q)\dot{Q}^a + g(Q),$$

for which the definition

$$P_a = \frac{\partial \mathscr{L}}{\partial \dot{Q}^a} = f_a(Q)$$

takes the form of a constraint. Some common examples of this phenomenon include standard fermion Lagrangians and non-relativistic bosonic field theories. Given a set of primary constraints, there may be *secondary* constraints which arise from demanding that the time derivatives of all constraints vanish on the constraint surface. In favourable circumstances the tower of constraints so defined will eventually terminate and leave a finite set of constraints $C_r(Q,P)$ which are consistent with time evolution. Once such a set is obtained, the distinction between primary and secondary constraints becomes unimportant. Indeed, a set of constraints consistent with evolution is all that is needed in classical mechanics.

However, the quantum requirement that constraints are represented as zero on Hilbert space imposes extra consistency conditions on commutation relations, and it is often necessary to modify the canonical prescription to get a sensible answer. To do this, we further divide the constraints according to

Definition

A constraint is called *first class* if its Poisson bracket with all other constraints vanishes on the constraint surface. It is called *second class* otherwise.

One can show [56] that first class constraints are associated to redundancies in our description of the system, and may be eliminated by a suitable choice of gauge. Furthermore, after removing all first class constraints the matrix

$$C_{rs} = \{C_r, C_s\}_P$$

has nonzero determinant even on the constraint surface. Like the Poisson bracket, the matrix C is *antisupersymmetric*: it is symmetric in swapping two fermionic constraints and antisymmetric otherwise. In particular, there is always an even number of bosonic second class constraints. For second class constraints, Dirac defines a modified bracket

$$\{f,g\}_D = \{f,g\}_P - \{f,C_r\}_P\,(C^{-1})^{rs}\,\{C_s,g\}_p\,, \tag{A.2}$$

which satisfies the same basic properties as the Poisson bracket and, crucially, has the property that the Dirac bracket of any function with a constraint vanishes. To complete quantisation, one applies the canonical quantisation prescription with the Dirac bracket in place of the Poisson bracket.

Note that, while the properties of the Dirac bracket seem to be what we need to make quantisation consistent, it is not obvious that this is the correct prescription. A significant piece of evidence in its favour is the following result [146]

Theorem

Let $\{Q^a, P_b : a, b = 1, \dots, m+n\}$ be canonical coordinates for a Hamiltonian system with second class constraints $C_r(Q,P) = 0$ $(r = 1, \dots, 2n)$. Then there is a canonical transformation to variables $\left\{q^i, p_j : i, j = 1, \dots, n\right\}$ and $\{\mathcal{Q}^\alpha, \mathcal{P}_\beta : \alpha, \beta = 1, \dots, m\}$ such that the constraints take the form $q^r = p_s = 0$. Furthermore, the Dirac bracket of two arbitrary functions of Q and P agrees with their canonical Poisson bracket calculated in terms of the unconstrained variables $\mathcal{Q}^\alpha$ and $\mathcal{P}_\beta$.

This tells us more or less everything we need. Given any system with second class constraints, it is possible to find canonical variables such that the constraints become straightforward and the Dirac bracket coincides with the naïve prescription for commutation relations.

A.2 The Faddeev-Jackiw Method

We now turn to a prescription for dealing with the special case of a first-order Lagrangian, which is often simpler than Dirac's analysis and also more intuitive [57]. For instance, it is strange to think of the free fermion $\mathscr{L} = i\psi^\dagger\dot{\psi}$ as constrained. Instead, we usually just declare $\psi^\dagger$ to be canonically conjugate to ψ and impose appropriate anticommutation relations. The Faddeev-Jackiw method will, in particular, show that this intuition is justfied. The starting observation is

Claim

Given a Lagrangian $\mathscr{L}(Q, \dot{Q})$ containing terms of both first and second order in time derivatives, there is a quantum-equivalent Lagrangian $\tilde{\mathscr{L}}(\xi, \dot{\xi})$ which is entirely first-order.

To see this, assume that $\mathscr{L}$ takes the explicit form

$$\mathscr{L} = \frac{1}{2} M_{ab}(Q,q)\dot{Q}^a\dot{Q}^b + f_a(Q,q)\dot{Q}^a + g_i(Q,q)\dot{q}^i - V(Q,q)$$

with configuration space parameterised by the Q^a and q^i. The notation emphasises which variables appear with quadratic time derivatives, so M_{ab} is a positive-definite symmetric matrix. Both f and g are vectors some of whose components may be zero. We define the usual canonical momenta for the quadratic variables

$$P_a = \frac{\partial \mathscr{L}}{\partial \dot{Q}^a} = M_{ab}\dot{Q}^b + f_a, \tag{A.3}$$

and use the Legendre transform to obtain the Hamiltonian

$$H(P, Q, q) = \frac{1}{2} M^{ab}(P_a - f_a)(P_b - f_b) + V.$$

Putting this together, we see that the alternative Lagrangian

$$\tilde{\mathscr{L}} = P_a \dot{Q}^a + g_i \dot{q}^i - H$$

is equal to $\mathscr{L}$ if we use the definition (A.3) of P.

The trick now is to view $\tilde{\mathscr{L}}$ as a first-order Lagrangian for an enlarged configuration space where the P_a are independent variables. It is easy to check that the equations of motion of $\tilde{\mathscr{L}}$ coincide with those of $\mathscr{L}$. In particular, the Euler-Lagrange equations for P_a reproduce the canonical formula (A.3). In fact, we can go further and observe that P appears without time derivatives and quadratically, hence is an auxiliary variable. We can therefore replace P with its equation of motion in $\tilde{\mathscr{L}}$ and find that this is equivalent to doing the path integral over P, which proves equivalence to $\mathscr{L}$ even at the quantum level.

Having gone through all this setup, Faddeev and Jackiw give a simple prescription for commutation relations from a first-order system. A general first-order Lagrangian takes the form

$$\mathscr{L} = a_i(\xi)\dot{\xi}^i - V(\xi) = a_i \dot{\xi}^i - H(\xi),$$

but we assume for simplicity that a_i is linear in the coordinates. That is,

$$a_i(\xi) = \frac{1}{2} \xi^j \omega_{ji}$$

for an invertible antisupersymmetric matrix ω. The method can be generalised to deal with both nonlinearity and degeneracy, but we won't need this. The equations of motion following from $\mathscr{L}$ are

$$\omega_{ij} \dot{\xi}^j = -\frac{\partial H}{\partial \xi^i},$$

which correspond to *unconstrained* dynamics by virtue of the invertibility of ω. Faddeev and Jackiw define brackets $\{f, g\}_{FJ}$ such that the Lagrangian equations of motion are reproduced by brackets with the Hamiltonian

$$\left\{\xi^i, \xi^j\right\}_{FJ} = \omega^{ij} \quad \Rightarrow \quad \dot{\xi}^i = \omega^{ij} \frac{\partial H}{\partial \xi^j} = \left\{H, \xi^i\right\}_{FJ}. \tag{A.4}$$

These satisfy all the same axioms as the Poisson bracket, and one can check that they agree with the Dirac bracket in this context [57].

B Hodge Theory

In this appendix we review the Hodge decomposition theorem and Hodge theory, which relates the de Rham cohomology of a compact Riemannian manifold to its harmonic forms. We also briefly describe the extension to Dolbeault cohomology for compact Kähler manifolds, and give examples to illustrate the subtlety of the theory in the non-compact case. We will not give rigorous proofs of any results, but we give a 'proof' relying on properties of self-adjoint operators which are too strong to be valid in de Rham-Hodge theory. However, the proof will apply in a similar context relevant to the index theory of $\mathfrak{osp}(4^*|4)$ in section 7. Results from this section not otherwise referenced can be found in [55, 60].

The *Hodge decomposition theorem* is

Theorem

Let (M, g) be a compact orientable Riemannian manifold without boundary. Then $\Omega^r(M)$ has an orthogonal decomposition

$$\Omega^r(M) = d\Omega^{r-1}(M) \oplus \delta\Omega^{r+1}(M) \oplus \mathscr{H}^r(M) \tag{B.1}$$

with respect to the L^2 norm (2.2). $\mathscr{H}^r(M)$ denotes the space of harmonic forms of degree r on M.

In particular, neither exact nor coexact forms can be harmonic. On the other hand, we can express the Laplacian as

$$(d + \delta)^2 = \Delta = (d - \delta)(d - \delta)^\dagger,$$

so that forms are harmonic if and only if they are both closed and coclosed. This suggests a link to cohomology, for which we recall

Definition

The *de Rham cohomology groups* of a manifold M are

$$H^r_{dR}(M) = \frac{\{\text{closed forms of degree } r\}}{\{\text{exact forms of degree } r\}}.$$

We denote the class of a closed form α by $[\alpha]$. De Rham's theorem says that $H^r_{dR}(M) \cong H^r(M)$, the ordinary (singular) cohomology, so we often drop the subscript [147]. Furthermore, one can prove that the cohomology of compact manifolds is finite-dimensional. From the definition and the Hodge decomposition theorem, one obtains *Hodge's theorem*

Theorem

Let M be a compact orientable Riemannian manifold. Then every cohomology class has a unique harmonic representative. That is,

$$H^r(M) \cong \mathscr{H}^r(M).$$

The ideas of Hodge theory generalise to a wide range of cohomology theories. We now give a 'proof' of the two theorems above. The rigorous version for de Rham cohomology requires elliptic PDE theory and can be found in [148]. The arguments we give are illustrative, and valid in the case of the index theory of $\mathfrak{osp}(4^*|4)$ discussed in section 7. Our setup is that we have a trio of operators $\left\{Q,\ S = Q^\dagger,\ H = H^\dagger\right\}$ on a Hilbert space $\mathcal{H}$, satisfying

$$Q^2 = S^2 = 0, \qquad \{Q, S\} = H.$$

We begin by showing

Claim

$|\psi\rangle$ is in the image of H if and only if it is orthogonal to the kernel of H.

To prove this, we note that since H is Hermitian we can work in a basis of orthonormal eigenstates

$$H\,|n\rangle = E_n\,|n\rangle\,.$$

Thus, for $|\phi\rangle \in \ker H$ and $|\psi\rangle = H\,|\chi\rangle$, we have

$$|\phi\rangle = \sum_{n:E_n=0} \alpha_n\,|n\rangle\,, \qquad |\psi\rangle = \sum_n \gamma_n E_n\,|n\rangle := \sum_n \beta_n\,|n\rangle\,,$$

so that $\beta_n = 0$ whenever $E_n = 0$ and $\langle\psi|\,\phi\rangle = 0$ by orthonormality of eigenstates. Conversely, if $\langle\psi|\,\phi\rangle = 0\ \forall\,|\phi\rangle : H\,|\phi\rangle = 0$, then $\beta_n = 0$ whenever $E_n = 0$. It follows that we can write

$$|\psi\rangle = H \sum_n \frac{\beta_n}{E_n}\,|n\rangle := H\,|\chi\rangle\,,$$

which proves the claim. Now consider orthogonal projection P onto the kernel of H

$$P\,|\psi\rangle = P \sum_n \beta_n\,|n\rangle = \sum_{n:E_n=0} \beta_n\,|n\rangle\,.$$

$|\psi\rangle - P\,|\psi\rangle$ is orthogonal to the kernel of H, so by the claim we have

$$|\psi\rangle - P\,|\psi\rangle = H\,|\chi\rangle\,.$$

Rearranging gives

$$|\psi\rangle = Q\left(S\left|\chi\right\rangle\right) + S\left(Q\left|\chi\right\rangle\right) + P\left|\psi\right\rangle.$$

We also have

$$\left(Q\left|\alpha\right\rangle\right)^{\dagger} S\left|\beta\right\rangle = 0 \qquad \forall\left|\alpha\right\rangle, \left|\beta\right\rangle \in \mathcal{H},$$

so the expression above is orthogonal. This proves the Hodge decomposition theorem if the construction is unique. The orthogonal projection P is unique, hence so is the harmonic part of the orthogonal decomposition. Now if we shift $QS\left|\chi\right\rangle \mapsto QS\left|\chi\right\rangle + Q\left|\alpha\right\rangle$ then we must also shift $SQ\left|\chi\right\rangle \mapsto SQ\left|\chi\right\rangle + S\left|\beta\right\rangle$ with $S\left|\beta\right\rangle = -Q\left|\alpha\right\rangle$ to compensate. But then

$$\left\|Q\left|\alpha\right\rangle\right\|^{2} = \left\langle\alpha\right| SQ\left|\alpha\right\rangle = -\left\langle\alpha\right| S^{2}\left|\beta\right\rangle = 0$$

so the decomposition is unchanged and we have the result.

To prove the isomorphism between Q-cohomology and the kernel of H, note that since $(Q+S)^2 = (Q-S)(S-Q) = H$, all states in the kernel of H are Q-closed. By Hodge decomposition, any such state has no Q-exact part, so every state in the kernel of H represents a Q-cohomology class. Conversely, suppose that $\left|\psi\right\rangle$ represents a class. Then

$$\left(S\left|\alpha\right\rangle\right)^{\dagger}\left|\psi\right\rangle = \left\langle\alpha\right| Q\left|\psi\right\rangle = 0 \qquad \forall\left|\alpha\right\rangle,$$

so by Hodge decomposition

$$\left|\psi\right\rangle = Q\left|\beta\right\rangle + \left|\gamma\right\rangle$$

for unique $\left|\gamma\right\rangle \in \ker H$. Thus $\left|\gamma\right\rangle = P\left|\psi\right\rangle$ represents the same class as $\left|\psi\right\rangle$, is harmonic, and is the unique such state. This establishes the required isomorphism.

We now describe the extension of de Rham Hodge theory to the case of compact Kähler geometry. Here one can refine the exterior derivative to the Dolbeault operators, suggesting

Definition

Let M be a complex manifold. The *$\bar{\partial}$-Dolbeault cohomology groups* of M are

$$H_{\bar{\partial}}^{p,q}(M) = \frac{\left\{\bar{\partial}\text{-closed forms of bidegree } (p,q)\right\}}{\left\{\bar{\partial}\text{-exact forms of bidegree } (p,q)\right\}}.$$

There is an analagous definition for ∂-Dolbeault cohomology.

The three Laplacians Δ, Δ_{∂} and $\Delta_{\bar{\partial}}$ on a Kähler manifold are equivalent, so the Laplace operator preserves bidegree and the Hodge theorem can be refined to:

Theorem

Let M be a compact Kähler manifold without boundary. Then we have an isomorphism

$$H^{p,q}_{\bar{\partial}}(M) \cong \mathscr{H}^{p,q}(M).$$

Thus the Dolbeault cohomology of a compact Kähler manifold is a refinement of the de Rham cohomology

$$H^r_{dR}(M) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar{\partial}}(M),$$

and the ∂ and $\bar{\partial}$-Dolbeault cohomologies are isomorphic.

We now give some simple examples to show how these results fail without compactness. The simplest possible non-compact Riemannian manifold is $\mathbb{R}$ with the standard Euclidean metric. A form on $\mathbb{R}$ is harmonic if and only if it is constant, so $\dim \mathscr{H}^0 = \dim \mathscr{H}^1 = 1$. On the other hand, it follows from the Poincaré lemma that the de Rham cohomology consists of constant functions at degree 0 but is trivial at degree 1. As such, Hodge's theorem fails in this case. The simplest possible Kähler manifold is $\mathbb{C}$ with the Euclidean metric. In this instance, the de Rham cohomology is again $\mathbb{R}$ at degree 0 and trivial for higher degrees, whereas the Dolbeault cohomology consists of all holomorphic forms so is much larger. One can attempt to salvage a general statement of Hodge theory using L^2 cohomology, which consists of closed L^2-normalisable forms modulo derivatives of L^2-normalisable forms, and a fairly robust statement arises in this way (see [135]). Our discussion of the superconformal index for $\mathfrak{osp}(4^*|4)$ in section 7.3 could be viewed as deforming the Laplacian in such a way that the L^2 cohomology is significantly enlarged.

C The Hyper-Kähler Quotient

In this appendix we review the hyper-Kähler quotient construction, which is a method for obtaining a hyper-Kähler manifold from a larger one carrying a 'nice' group action. We begin with some basics of symplectic geometry and the quotient of a manifold by a group action. This procedure is flawed in that it need not produce a manifold with the same 'nice' structure as the parent. Marsden-Weinstein reduction [149] is a technique to solve this problem for Hamiltonian actions on symplectic manifolds, producing another symplectic manifold, and can be generalised to the hyper-Kähler case [42]. We refer the reader to [55] for standard differential geometry not otherwise referenced. In section 3.2 we show that the ADHM construction of instantons is an example of a hyper-Kähler quotient, and this structure leads to a simple proof of $\mathfrak{osp}(4^*|4)$-invariance of instanton quantum mechanics in 5.4.

C.1 Ordinary Quotients

Given the action of a group G on some set X, we say that two elements $x, y \in X$ are equivalent if they lie in the same orbit, that is $\exists g \in G : g.x = y$. The set of all inequivalent orbits is called the quotient of X by G, written X/G. One can try to apply this idea to a manifold carrying the action of a Lie group, but in general the resulting space will not be a manifold. It may have singularities (so fail to be smooth) and may even fail the Hausdorff property. A simple example of the latter is the quotient of $\mathbb{C}$ by the group $\mathbb{C}^*$ of scale and phase transformations $z \mapsto \lambda z : \lambda \in \mathbb{C}^*$. Here there are exactly two orbits: the point 0 and everything else, and the resulting quotient cannot be Hausdorff since the nonzero orbit contains points which started out arbitrarily close to 0. This quotient fails because the group action allows disjoint orbits whose limit points coincide, so treats points which 'ought to be the same' differently.

To fix this, we require

Definition

- A group action is *free* if the only group element with fixed points is the identity. That is, $\forall x \in X \; g.x = x \Rightarrow g = 1$.
- A group action is *proper* if, under the map $(g, x) \mapsto (g.x, x)$, the preimage of any compact set is compact.

Given these properties, we have the *manifold quotient theorem* [150]:

Theorem

Let G be a Lie group acting smoothly, freely and properly on a manifold M. Then the quotient space M/G has a unique manifold structure such that the canonical projection $M \to M/G$ is smooth and has surjective derivative. In particular, $\dim M/G = \dim M - \dim G$.

In the example above, the action is neither free nor proper since 0 is a fixed point for all group elements and the orbit of a closed disc centred on 0 is $\mathbb{C}$. Even when the theorem holds it doesn't necessarily do what we want in that it need not respect any additional structure on M. The following sections describe how to solve this problem for symplectic and (hyper-)Kähler structures.

C.2 Symplectic Reduction

Recall that a manifold M is *symplectic* if it admits a closed, non-degenerate 2-form ω. Non-degeneracy here means that if $\omega(X, Y) = 0 \ \forall Y \in \mathrm{Vect}(M)$ then $X = 0$. In particular, the manifold must be even-dimensional. This even-dimensionality is already a problem for the group quotient, since if the Lie group is odd-dimensional then so will the quotient manifold be. To solve this, we need to restrict to a special kind of group action.

A *Hamiltonian* vector field on M is a vector field V_f satisfying

$$i_{V_f}\omega = -df$$

for some function f. In particular, Cartan's formula gives $\mathcal{L}_{V_f}\omega = 0$. This notion can be extended to group actions by considering the vector fields induced by the flow along an action. We have a map $\mathfrak{g} = \mathrm{Lie}(G) \to \mathrm{Vect}(M)$, $v \mapsto X_v$, and

Definition

The action of a Lie group G on a symplectic manifold M is *Hamiltonian* if $\forall v \in \mathfrak{g}$, X_v is Hamiltonian.

Given such an action, we associate a *Hamiltonian function* μ^v to each $v \in \mathfrak{g}$ via

$$i_{X_v}\omega = d\mu^v. \tag{C.1}$$

This only defines μ^v up to a constant, but for a semisimple Lie group the ambiguity can be fixed by demanding that the Hamiltonians are equivariant

$$X_v\left(\mu^w\right) = \mu^{[v,w]} \qquad \forall v, w \in \mathfrak{g}.$$

Once this is done, the collection of Hamiltonian functions is known as a *moment map* for the action of G. When G has abelian factors there is still some ambiguity in the moment map, but there will always be a 'natural' choice in examples we consider. We may view the collection of moment maps as a map $\mu : M \to \mathfrak{g}^*$ taking $p \in M$ to the element of $\mathfrak{g}^*$ whose pairing with $v \in \mathfrak{g}$ produces $\mu^v(p)$. We have

Definition

The *symplectic* or *Marsden-Weinstein* quotient of a symplectic manifold by the Hamiltonian action of a Lie group is [149]

$$\tilde{M} = \frac{\mu^{-1}(\xi)}{G}$$

for some $\xi \in \mathfrak{g}^*$.

The level set $\mu^{-1}(\xi)$ is the set of points $p \in M$ such that $\mu^v(p) = \xi(v)$, and we take the ordinary group quotient of this level set. Note that if ξ is a regular value for μ, so $d\mu$ has full rank, then $\mu^{-1}(\xi)$ is a submanifold of M of dimension $\dim M - \dim G$. If the action of G on the level set is free and proper then $\tilde{M}$ is a manifold of dimension $\dim M - 2\dim G$, but more is true [149]:

Theorem

Let G be a Lie group with a Hamiltonian action on a symplectic manifold M. Let μ be a moment map and $\xi \in \mathfrak{g}^*$ be a regular value for μ such that G acts freely and properly on $\mu^{-1}(\xi)$. Let $\pi : \mu^{-1}(\xi) \to \tilde{M}$ be the canonical projection and $i : \mu^{-1}(\xi) \to M$ the canonical embedding. Then $\tilde{M}$ has a unique symplectic structure $\tilde{\omega}$ such that

$$\pi^*\tilde{\omega} = i^*\omega.$$

We postpone a more detailed description of the induced symplectic structure for the hyper-Kähler case. For now, we note that the various regularity assumptions in the theorem can fail in mild ways and still leave a sensible quotient, for instance a manifold with singular points. This is the case for the instanton moduli space.

C.3 Hyper-Kähler Quotient

We begin with the Kähler quotient, which is a very mild extension of symplectic reduction. Recall that a Kähler manifold is in particular a symplectic manifold, whose symplectic form is related to the metric by $\omega(X, Y) = g(X, IY)$. Thus the considerations of the previous section apply and we can construct a new symplectic manifold $\tilde{M}$ from a Kähler manifold with a Hamiltonian group action. If we want the resulting quotient to be Kähler then the group action must also be isometric, or equivalently the induced vector fields must be holomorphic. In fact, by the Kähler property any two of isometric, Hamiltonian and holomorphic imply the third.

In this case it is easy to describe the Kähler structure of the quotient explicitly. First, we observe that IX_v span the set of vectors normal to $\mu^{-1}(\xi)$ since

$$g(Y, IX_v) = \omega(Y, X_v) = -(i_{X_v}\omega)(Y) = -d\mu^v(Y) = 0,$$

where the final equality follows since $Y \in T\mu^{-1}(\xi) = \ker d\mu|_{\mu^{-1}(\xi)}$. Second, if the action of G on $\mu^{-1}(\xi)$ is free then we can view $\mu^{-1}(\xi)$ as a principal G-bundle over $\tilde{M}$ whose vertical subspace is spanned by X_v. Thus we can define a unique horizontal lift $X \in T\mu^{-1}(\xi) \subset TM$ of a vector $\tilde{X} \in T\tilde{M}$ by demanding

$$\pi_* X = \tilde{X}, \qquad g(X, X_v) = g(X, IX_v) = 0 \qquad \forall v \in \mathfrak{g}. \tag{C.2}$$

This lift allows us to define a metric, symplectic form and complex structure $(\tilde{g}, \tilde{\omega}, \tilde{I})$ on $\tilde{M}$. We set

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y) \tag{C.3}$$

and similarly for $\tilde{\omega}$ and $\tilde{I}$. Since the group action preserves all three tensors, the above is well-defined. In particular, observe that if $p, q \in \mu^{-1}(\xi)$ lie in the the same G-orbit, so $q = g.p$, then the lifts of $\tilde{X}$ at p and q are related by $X(q) = g_* X(p)$. Furthermore, the structure so defined is Kähler and obeys the defining property of the Marsden-Weinstein symplectic structure.

The extension of this construction to the hyper-Kähler case is due to [42]. In this instance, the group action must be isometric and triholomorphic, so all three complex structures I^a are preserved. This implies that the action is Hamiltonian with respect to each Kähler form ω^a, so there is a triplet of moment maps $\vec{\mu} : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$. Defining $\tilde{M} = \vec{\mu}^{-1}(\vec{\xi})/G$, we obtain

Theorem

Let G be a Lie group acting smoothly, isometrically and triholomorphically on a hyper-Kähler manifold M. Let $\vec{\xi}$ be a regular value for $\vec{\mu}$ such that G acts freely and properly on $\vec{\mu}^{-1}(\vec{\xi})$. Then $\tilde{M}$ is a hyper-Kähler manifold of dimension $\dim M - 4\dim G$.

The hyper-Kähler structure works analogously to the Kähler case. In particular, $\vec{\mu}^{-1}(\vec{\xi}) \subset M$ is a principal G-bundle over $\tilde{M}$ and the vectors $I^a X_v$ span its normal bundle. Horizontal lifts of vectors

$X \in T\tilde{M}$ work as in (C.2) and the hyper-Kähler structure is induced by (C.3). We remark again that this whole procedure can still make sense with mild relaxations of the regularity assumptions in the theorem, provided that one is willing to accept singularities in the quotient manifold. In section 3.2 we work through the example of instanton moduli space.

D Lie Superalgebras

In this appendix we collect the mathematical details of Lie superalgebras which we use in the main text, especially for the representation theory of $\mathfrak{osp}(4^*|4)$ in section 7. We begin with a brief review of basic material, emphasising similarities to and differences from the theory of Lie algebras. In particular, notions such as (semi)simplicity and root spaces exist, and there is a classification theorem for simple superalgebras. We also give the standard matrix realisations of the $\mathfrak{sl}$ and $\mathfrak{osp}$ series superalgebras. In the second part we discuss the classification of irreducible representations, using Verma modules and the Šapovalov form as our main tools. The author learned superalgebra theory from [151] and results not otherwise referenced can be found there. Standard terminology from Lie algebra theory can be found in [114].

D.1 Basics and Matrix Superalgebras

The results of this section come from [47]. We start from the beginning with

Definition

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

- $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$.
- If $x \in \mathfrak{g}_a,\ y \in \mathfrak{g}_b$, then $[x, y] = (-1)^{ab+1} [y, x]$.
- If $x \in \mathfrak{g}_a,\ y \in \mathfrak{g}_b,\ z \in \mathfrak{g}_c$ then $(-1)^{ac} [a, [b, c]] + (-1)^{ba} [b, [c, a]] + (-1)^{cb} [c, [a, b]] = 0$.

The third condition is called the *(super)Jacobi identity.* $x \in \mathfrak{g}_0$ is called *even* or *bosonic* while $y \in \mathfrak{g}_1$ is called *odd* or *fermionic.* We call a bilinear map satisfying the second condition *antisupersymmetric.*

In physical applications we usually form Lie superalgebras from associative superalgebras by taking the anticommutator of two fermionic elements and the commutator otherwise. An associative superalgebra is a $\mathbb{Z}_2$-graded associative algebra in which multiplication respects the grading in analogy with the first condition above. We'll see many examples of this when discussing matrix superalgebras. From now on, unless confusing, we will refer to Lie superalgebras just as superalgebras.

The notion of (semi)simplicity translates directly from the Lie algebra case, so a superalgebra is said to be *simple* if it has no non-trivial ideals. Note that the fermionic part of a superalgebra always forms a representation of the bosonic part, and a superalgebra is said to be *classical simple* if this representation is completely reducible. In analogy with the Lie algebra case, we have a classification theorem [47, 152, 153]:

Theorem

Let $\mathfrak{g}$ be a finite-dimensional classical simple Lie superalgebra. Then either $\mathfrak{g}$ is a simple Lie algebra or it is isomorphic to one of

$$\begin{array}{llll} \mathfrak{sl}(m|n) & (m > n \geq 1) & \mathfrak{p}(n) & (n \geq 2) \\ \mathfrak{psl}(n|n) & (n \geq 2) & \mathfrak{q}(n) & (n \geq 2) \\ \mathfrak{osp}(m|n) & (m, n > 0) & G(3) & \\ D(2,1;\alpha) & (\alpha \in \mathbb{R} \setminus \{0, -1\}) & F(4). & \end{array}$$

The series $\mathfrak{sl}$, $\mathfrak{psl}$ and $\mathfrak{osp}$ are ubiquitous in this text and easily described in terms of matrices, which we do shortly. The remainder will not feature and we will neglect them henceforth. The theorem above is valid for complex superalgebras. An analogous result for real superalgebras was given in [49] but is more involved so we state it piecemeal as needed. The key fact is that the real forms of a complex superalgebra are determined by the real forms of its bosonic subalgebra.

An important object in Lie algebra theory is the Killing form, which is the unique $\mathfrak{g}$-invariant, symmetric, non-degenerate bilinear form on $\mathfrak{g}$. The Killing form can be defined analogously for superalgebras, but unfortunately is almost always degenerate so is not such a useful object. However, for each of the classical simple superalgebras it is possible to come up with a bilinear form which is supersymmetric, non-degenerate and $\mathfrak{g}$-invariant, and to prove that such a form is unique up to an overall constant. It is this form which is therefore used in studying the structure theory. We will construct this form explicitly for the matrix superalgebras.

The notions of Cartan subalgebra and root space decomposition are important in the structure theory of Lie algebras. For the most part these generalise to superalgebras, but we will need to be careful in places. The Cartan subalgebra $\mathfrak{h}$ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined as the Cartan subalgebra of the bosonic part $\mathfrak{g}_0$, and one can decompose the superalgebra into root spaces $\mathfrak{g}_\alpha : \alpha \in \mathfrak{h}^*$ satisfying

$$[h, v_\alpha] = \alpha(h) v_\alpha \qquad \forall h \in \mathfrak{h},\ v_\alpha \in \mathfrak{g}_\alpha.$$

That is,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where Φ is the space of roots. One can also define positive and negative roots, which we denote Φ^+ and Φ^- respectively. Furthermore, one can check that root spaces are either purely bosonic or purely fermionic, so we can decompose further in terms of even and odd positive/negative roots $\Phi_0^\pm$ and $\Phi_1^\pm$. We also denote $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$. There are two important distinctions with the theory of Lie algebras at this point.

Firstly, recall that for a Lie algebra all Cartan subalgebras and choices of positive roots are equivalent via an inner automorphism of the Lie algebra. For a superalgebra, while all choices of Cartan are still equivalent, there are inequivalent choices of positive roots. In particular, given a choice of positive roots one correspondingly has a choice of simple roots and a Dynkin diagram, and the existence of inequivalent choices means that there is more than one possible Dynkin diagram for a given simple superalgebra. While this degeneracy is an important aspect of superalgebra theory, we won't discuss it further as it turns out that representation theory is in some sense 'covariant' with respect to these choices. Furthermore, when discussing superconformal algebras there is a physically motivated 'natural' choice of positive roots which we use exclusively.

For the second distinction, recall that the root spaces of a Lie algebra are such that if α is a root then the only other allowed multiple is $-\alpha$. This is not quite true for superalgebras, where it is possible to have a fermionic root α such that 2α is a bosonic root. This occurs if and only if the fermionic root has nonzero 'length' with respect to the invariant bilinear form. Such a root is called *non-isotropic*, while roots of length zero are called *isotropic*. In spite of this complication, the remainder of the standard structure theory of root spaces for Lie algebras goes through unchanged, and in particular one can find an analogue of a Chevalley-Serre basis. Rather than prove this abstractly, we give these bases explicitly for the matrix superalgebras.

We now describe the matrix superalgebras in the $\mathfrak{sl}$, $\mathfrak{psl}$ and $\mathfrak{osp}$ series. Begin with $\mathfrak{gl}(m|n)$, defined as the set of all $(m+n) \times (m+n)$ complex block matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{D.1}$$

where A and D are bosonic and B and C are fermionic, with bracket given by (anti)commutators as appropriate. A is $m \times m$, B is $m \times n$, C is $n \times m$ and D is $n \times n$. The bosonic subalgebra is $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. We define a standard basis for $\mathfrak{gl}(m|n)$ using elementary matrices. Let $i, j = 1, \ldots, m$ and $a, b = 1, \ldots, n$, and take $\{E_{ij},\ E_{ia},\ E_{jb},\ E_{ab}\}$ to be elementary matrix bases for A, B, C and D respectively. That is, we have

$$M = A_{ij}E_{ij} + B_{ib}E_{ib} + C_{aj}E_{aj} + D_{ab}E_{ab}.$$

With $A, B = 1, \ldots, m+n$, these obey

$$[E_{AB}, E_{CD}] = \delta_{BC} E_{AD} - \delta_{DA} E_{CB}, \tag{D.2}$$

from which it follows that the fermions transform in the $(\mathbf{m}, \bar{\mathbf{n}}) \oplus (\bar{\mathbf{m}}, \mathbf{n})$ of the bosonic subalgebra. A natural choice of Cartan subalgebra consists of the diagonal matrices E_{AA}. If we define the *supertrace*

$$\operatorname{str} M = \operatorname{tr} A - \operatorname{tr} B,$$

then we get a supersymmetric invariant bilinear form on $\mathfrak{gl}(m|n)$ via

$$(M, N) = \operatorname{str} MN.$$

In particular, we find that $(E_{ii}, E_{jj}) = \delta_{ij}$ and $(E_{aa}, E_{bb}) = -\delta_{ab}$. This product can be used to induce a non-degenerate inner product on $\mathfrak{h}^*$. We define the elementary roots ϵ_i and δ_b via

$$\epsilon_i(E_{jj}) = \delta_{ij}, \qquad \delta_a(E_{bb}) = \delta_{ab}, \qquad \epsilon_i(E_{aa}) = \delta_a(E_{ii}) = 0,$$

so that

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \qquad (\delta_a, \delta_b) = -\delta_{ab}, \qquad (\epsilon_i, \delta_a) = 0.$$

We get a root space decomposition

root vector	E_{ij}	E_{ib}	E_{aj}	E_{ab}
root	$\epsilon_i - \epsilon_j$	$\epsilon_i - \delta_b$	$\epsilon_a - \epsilon_j$	$\epsilon_a - \epsilon_b$

,

so that

$$\begin{aligned} \Phi_0 &= \{\epsilon_i - \epsilon_j : i, j = 1, \ldots, m\} \cup \{\delta_a - \delta_b : a, b = 1, \ldots, n\} \\ \Phi_1 &= \{\pm(\epsilon_i \pm \delta_b) : i = 1, \ldots, m,\ b = 1, \ldots, n\}. \end{aligned}$$

Observe that all bosonic roots have length ± 2 while all fermionic roots are isotropic.

$\mathfrak{gl}(m|n)$ is not simple since the bracket of any two matrices has supertrace zero. The set of all matrices with supertrace zero is denoted $\mathfrak{sl}(m|n)$ and is simple for $m \neq n$. When $m = n$ the set $\mathbb{C}\mathbb{1}$ of scalar multiples of the identity is also central, so forms an ideal which must be quotiented out to obtain the simple algebra $\mathfrak{psl}(n|n) \cong \mathfrak{sl}(n|n)/\mathbb{C}\mathbb{1}$. The Cartan subalgebra of $\mathfrak{sl}(m|n)$ is

$$\mathfrak{h} = \{H_i = E_{ii} - E_{i+1,i+1} : i = 1, \ldots, m-1\} \cup \{H_a = E_{aa} - E_{a+1,a+1} : a = 1, \ldots, n-1\} \cup \{H\}$$

with $H = n \sum_i E_{ii} + m \sum_a E_{aa}$, and the roots are as for $\mathfrak{gl}(m|n)$. The supertrace inner product also restricts nicely to $\mathfrak{sl}(m|n)$. To go to $\mathfrak{psl}(n|n)$ we drop H from the Cartan subalgebra and again keep roots the same. Observe that the supertrace form is well-defined on cosets $M + \mathbb{C}\mathbb{1}$, since if $M, N \in \mathfrak{sl}(n|n)$ then

$$\operatorname{str}[(M + \mu\mathbb{1})(N + \lambda\mathbb{1})] = \operatorname{str}(MN) + \lambda \operatorname{str} M + \mu \operatorname{str} N + \mu\lambda \operatorname{str} \mathbb{1} = \operatorname{str}(MN).$$

The two abelian factors removed to go from $\mathfrak{gl}(n|n)$ to $\mathfrak{psl}(n|n)$ have rather different character. The '$\mathfrak{s}$' quotient removes non-central elements which are not generated by brackets, so requires only a straightforward restriction, while the '$\mathfrak{p}$' quotient removes central elements which are generated by anticommutators, and requires a formal quotient. This should be compared with the construction of $\mathfrak{u}(1,1|2)$ in section 5.2, where there are two $\mathfrak{u}(1)$ factors: an R-symmetry and a global symmetry. The former, denoted R^I, is non-central and not generated by brackets, so corresponds to the '$\mathfrak{s}$' quotient. The latter, denoted by D^I, is central and generated, so is the '$\mathfrak{p}$' quotient. This situation is also familiar from $\mathcal{N} = 4$ supersymmetric Yang-Mills, which has superconformal algebra $\mathfrak{psu}(2,2|4)$ (see [154]).

We now move on to $\mathfrak{osp}(m|n)$. In fact, since this is all we need in the main text, we only cover $\mathfrak{osp}(2m|2n)$. The case of $\mathfrak{osp}(2m+1|2n)$ can be handled analogously, but requires separate treatment since the odd and even orthogonal algebras have different properties. $\mathfrak{osp}(2m|2n)$ is the subalgebra of $\mathfrak{gl}(2m|2n)$ which preserves the bilinear form

$$J_{2m|2n} = \begin{pmatrix} \mathcal{I}_{2m} & 0 \\ 0 & \Omega_{2n} \end{pmatrix},$$

where $\mathcal{I}_{2m}$ and Ω_{2n} are invariant forms for $\mathfrak{so}(2m)$ and $\mathfrak{sp}(2m)$ respectively:

$$\mathcal{I}_{2m} = \begin{pmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix}, \qquad \Omega_{2n} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

Writing M as a block matrix as in (D.1), preservation of J means that

$$\mathcal{I}A + A^t\mathcal{I} = 0, \qquad D^t\Omega + \Omega D = 0, \qquad C^t\Omega + \mathcal{I}B = 0,$$

so in particular the bosonic subalgebra is

$$\mathfrak{g}_B = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n).$$

The supertrace form again restricts nicely to $\mathfrak{osp}(2m|2n)$. The Cartan subalgebra is

$$\begin{aligned} \mathfrak{h} &= \mathfrak{h}_{\mathfrak{so}(2m)} \oplus \mathfrak{h}_{\mathfrak{sp}(2n)} \\ &= \{h_i = E_{ii} - E_{i+m,i+m} : i = 1, \ldots, m\} \oplus \{h_a = E_{aa} - E_{a+n,a+n} : a = 1, \ldots, n\}, \end{aligned}$$

and the root space is spanned by $\epsilon_i : i = 1, \ldots, m$ and $\delta_a : a = 1, \ldots, n$. The bosonic positive roots are as for $\mathfrak{g}_B$ (see [132])

$$\Phi_0^+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq m\} \cup \{\delta_a \pm \delta_b : 1 \leq a < b \leq n\} \cup \{2\delta_a : 1 \leq a \leq n\},$$

while a choice of fermionic positive roots is

$$\Phi_1^+ = \{\epsilon_i \pm \delta_a : i = 1, \ldots, m,\ a = 1, \ldots, n\}.$$

Again all fermionic roots are isotropic, while the bosonic roots have length ± 2 or -4. The fermions fit into the representation $(\mathbf{2m}, \mathbf{2n})$ of the bosonic subalgebra.

We'll be more explicit for $\mathfrak{osp}(4|4)$ as we need it for representation theory in section 7. $\mathfrak{so}(4)$ decomposes into two copies of $\mathfrak{sl}(2)$, and the real form $\mathfrak{osp}(4^*|4)$ is such that the first of these becomes $\mathfrak{sl}(2;\mathbb{R}) \cong \mathfrak{so}(2,1) \cong \mathfrak{su}(1,1)$ while the second is $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, and $\mathfrak{sp}(4) \mapsto \mathfrak{usp}(4)$. We have

$$\mathfrak{sl}(2)_1 = \{L_0 = h_1 + h_2,\ L_+ = E_{32} - E_{41},\ L_- = E_{23} - E_{14}\}$$

and

$$\mathfrak{sl}(2)_+ = \{2J_3 = h_1 - h_2,\ J_+ = E_{12} - E_{43},\ J_- = E_{21} - E_{34}\},$$

which obey the standard $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ algebra

$$[L_0, L_\pm] = \pm 2L_\pm, \qquad [L_+, L_-] = -L_0, \qquad [2J_3, J_\pm] = \pm 2J_\pm, \qquad [J_+, J_-] = 2J_3.$$

The positive roots are $\epsilon_1 + \epsilon_2$ for $\mathfrak{sl}(2)_1$ and $\epsilon_1 - \epsilon_2$ for $\mathfrak{sl}(2)_2$, with corresponding root vectors L_+ and J_+ respectively. The $\mathfrak{sp}(4)$ subalgebra is

$$\begin{aligned}
\mathfrak{h} &= \{M = h_3,\ N = h_4\} \\
\mathfrak{g}^+ &= \{v_{2\delta_1} = E_{57},\ v_{2\delta_2} = E_{68},\ v_{\delta_1+\delta_2} = E_{58} + E_{67},\ v_{\delta_1-\delta_2} = E_{56} - E_{87}\} \\
\mathfrak{g}^- &= \{v_{-2\delta_1} = E_{75},\ v_{-2\delta_2} = E_{86},\ v_{-\delta_1-\delta_2} = E_{76} + E_{85},\ v_{-\delta_1+\delta_2} = E_{65} - E_{78}\},
\end{aligned}$$

with positive roots $\{2\delta_1,\ 2\delta_2,\ \delta_1 \pm \delta_2\}$. In summary, the positive bosonic roots of $\mathfrak{osp}(4|4)$ are

$$\Phi_0^+ = \{\epsilon_1 \pm \epsilon_2,\ \delta_1 \pm \delta_2,\ 2\delta_1,\ 2\delta_2\}. \tag{D.3}$$

As for fermions, there are a total of 16 in the $(\mathbf{4}, \mathbf{4})$ of $\mathfrak{so}(4) \oplus \mathfrak{sp}(4)$. In view of applications to superconformal algebras, we denote these

$$\begin{aligned}
Q_{ia}^+ &= E_{i,a+2} - E_{a,i+2} & \qquad Q_{ia}^- &= E_{ia} + E_{a+2,i+2} \\
S_{ia}^+ &= E_{i+2,a} + E_{a+2,i} & \qquad S_{ia}^- &= E_{ai} - E_{i+2,a+2}.
\end{aligned}$$

Here $i, a \in \{1, 2\}$ and Q^+_{ia} is the root vector for $\epsilon_i + \delta_a$, Q^-_{ia} for $\epsilon_i - \delta_a$, S^+_{ia} for $-(\epsilon_i + \delta_a)$ and S^-_{ia} for $-\epsilon_i + \delta_a$. The Q's are chosen to be positive roots, which is equivalent to making a fermionic root positive if and only if it has positive eigenvalue under L_0, and in applications will correspond to positive scaling dimension. To sum up, the positive fermionic roots are

$$\Phi^+_1 = \{\epsilon_i \pm \delta_a : i, a = 1, 2\} . \tag{D.4}$$

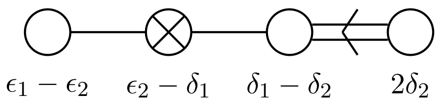
Commutation relations can be read off from the explicit matrix expressions using (D.2). In particular, $\{h_i + h_a,\ Q^+_{ia},\ S^+_{ia}\}$ and $\{h_i - h_a,\ Q^-_{ia},\ S^-_{ia}\}$ form canonically normalised $\mathfrak{sl}(2)$-like subalgebras. With choices of positive roots as above, we have simple roots

$$\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \delta_1 - \delta_2,\ \alpha_3 = 2\delta_2,\ \beta = \epsilon_2 - \delta_1\} ,$$

where the α's are bosonic and β is fermionic. The positive roots can be written as sums of simple roots

$$\begin{aligned}
\Phi^+_0 =& \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_2 + \alpha_3,\ 2\alpha_2 + \alpha_3,\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\beta\} \\
\Phi^+_1 =& \{\beta,\ \alpha_1 + \beta,\ \alpha_2 + \beta,\ \alpha_1 + \alpha_2 + \beta,\ \alpha_2 + \alpha_3 + \beta, \\
&\quad 2\alpha_2 + \alpha_3 + \beta,\ \alpha_1 + \alpha_2 + \alpha_3 + \beta,\ \alpha_1 + 2\alpha_2 + \alpha_3 + \beta\} .
\end{aligned}$$

With this choice of positive roots, the Dynkin diagram of $\mathfrak{osp}(4|4)$ is



where the crossed node indicates an isotropic fermionic root.

D.2 Irreducible Representations

In this section we give a sketch of the theory required to arrive at a useable characterisation of reducible representations of superconformal algebras. We begin with some definitions and theorems which are exact parallels of the Lie algebra case[33].

Definition

Let $\mathfrak{g}$ be a Lie superalgebra. The *Universal Enveloping Algebra* (*UEA*) of $\mathfrak{g}$ is

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y \pm y \otimes x - [x, y]\ \ \forall x, y \in \mathfrak{g} \rangle ,$$

where $T(\mathfrak{g})$ is the tensor algebra on $\mathfrak{g}$ and $\pm$ is - unless x and y are both fermionic.

[33] Many of the papers cited in this section prove results for finite-dimensional Lie algebras or Kac-Moody algebras which generalise straightforwardly to superalgebras. See [151] for the details.

That is, we take all possible words in $\mathfrak{g}$ and quotient by the commutation relations of the algebra. We then have the *Poincaré-Birkhoff-Witt (PBW)* theorem:

Theorem

Let $\mathfrak{g}$ be a Lie superalgebra with basis $x_1, \ldots, x_m$. Then the set of elements of the form

$$x_1^{a_1} \ldots x_m^{a_m},$$

where $a_m \in \mathbb{N}$ if $x_m \in \mathfrak{g}_0$ and $a_m \in \{0, 1\}$ if $x_m \in \mathfrak{g}_1$, is a basis for $U(\mathfrak{g})$.

Recall that a vector v_λ in a representation of $\mathfrak{g}$ is a *lowest weight vector of weight* $\lambda \in \mathfrak{h}^*$ if

$$\mathfrak{n}^- v_\lambda = 0, \qquad h v_\lambda = \lambda(h) v_\lambda \qquad \forall h \in \mathfrak{h}.$$

We then have [155]

Definition

Let v_λ be a lowest weight vector for $\mathfrak{g}$ of weight λ. The *Verma module* for $\mathfrak{g}$ of lowest weight λ is

$$\mathcal{V}_\lambda = U(\mathfrak{g}) v_\lambda = U(\mathfrak{n}^+) v_\lambda.$$

For $\mathfrak{g} = \mathfrak{osp}(4^*|4)$ we will always work with lowest weight representations. Our starting point is [156]

Theorem

Every irreducible lowest weight representation of $\mathfrak{g}$ is the quotient of a Verma module by its unique maximal proper submodule.

Our task now is to characterise which Verma modules are irreducible and which require a quotient.

First observe that the PBW theorem allows us to 'normal order' an element v of the UEA. We achieve this by moving all lowering operators to the right and raising operators to the left, resulting in a sum of words of the form XYZ with $X \in U(\mathfrak{n}^+)$, $Y \in U(\mathfrak{h})$, $Z \in U(\mathfrak{n}^-)$. The *Harish-Chandra projection* of v, denoted $\zeta(v)$, extracts all resulting words containing only Cartan generators [157]. This gives us an initial characterisation of reducible Verma modules:

Theorem

$\mathcal{V}_\lambda$ is reducible if and only if there exists $y \in U(\mathfrak{g})$ such that $y v_\lambda \neq 0$ and $\zeta(x^t y) v_\lambda = 0$ $\forall x \in U(\mathfrak{g})$, where t is the antiautomorphism of $\mathfrak{g}$ satisfying $(e_\alpha)^t = e_{-\alpha}$ and $h^t = h$ for all $\alpha \in \Phi^+$ and $h \in \mathfrak{h}$.

To see this, observe that $\zeta(x^t y)v_\lambda = 0$ if and only if $x^t y v_\lambda \in \bigoplus_{\mu \neq \lambda} V^\mu$, where V^μ are weight spaces with weight μ. Since this holds for all x, we see that yv_λ lies in a proper submodule of $\mathcal{V}_\lambda$. To refine this characterisation we introduce [158]

Definition

The *Šapovalov form* of a Verma module $\mathcal{V}_\lambda$ is the symmetric bilinear form

$$F^\lambda(x, y) = \zeta(x^t y)(\lambda) \in \mathbb{C}, \tag{D.5}$$

satisfying

$$F^\lambda(zx, y) = F^\lambda(x, z^t y).$$

Notice that the definition of the Šapovalov form is very similar to the usual procedure for defining an inner product on a Verma module for a real algebra. The only distinction is that t is linear, while the Hermitian conjugate is antilinear. However, if we pick a unitarity condition such that the adjoint of a root vector e_α is $e_{-\alpha}$ and Cartan generators are Hermitian then the two coincide for real polynomials in the generators.

Since a Verma module is a sum of weight spaces $V_{\lambda+\eta}$ with η in the positive root lattice $\mathbb{N}\Phi^+$, we can define the *restricted Šapovalov form* F^λ_η to be F^λ restricted to $x, y \in U(\mathfrak{n}^+)^\eta$, the set of elements in the UEA with weight η. In particular, the Šapovalov form vanishes if x and y have different weights. Now F^λ_η acts on a finite-dimensional space so we can define its determinant, and we have a second characterisation for reducibility:

Theorem

The Verma module $\mathcal{V}_\lambda$ is irreducible if and only if $\det F^\lambda_\eta \neq 0 \ \forall \eta \in \mathbb{N}\Phi^+$.

To see this, observe that vanishing of the Šapovalov determinant $\det F^\lambda_\eta$ for some η is equivalent to the existence of y as in the previous theorem. This condition can be further refined by giving an explicit formula for the determinant. To state it, we need the *super Weyl vector*

$$\rho = \frac{1}{2}\left(\sum_{\alpha \in \Phi_0^+} \alpha - \sum_{\alpha \in \Phi_1^+} \alpha\right), \tag{D.6}$$

as well as

Definition

Let $\eta \in \mathbb{N}\Phi^+$. A *partition* of η is a sum

$$\eta = \sum_{\alpha \in \Phi^+} \pi(\alpha)\alpha,$$

where $\pi : \Phi^+ \to \mathbb{N}$ takes values in $\{0, 1\}$ for fermionic roots. That is, π defines a word in the UEA with weight η via

$$e_\eta = \prod_{\alpha \in \Phi^+} e_\alpha^{\pi(\alpha)}.$$

We set $p(\eta)$ to be the number of partitions of η, and for fermionic α we set $p_\alpha(\eta)$ to be the number of partitions of η such that $\pi(\alpha) = 0$.

Finally one obtains [159, 160]

Theorem

Let $\mathfrak{g}$ be a superalgebra all of whose fermionic roots are isotropic, and $\mathcal{V}_\lambda$ a Verma module for $\mathfrak{g}$. Then the Šapovalov determinant is $\det F_\eta^\lambda = AB$, where

$$\begin{aligned} A &= \prod_{\alpha \in \Phi_0^+} \prod_{r=1}^{\infty} \left(\lambda - \rho - \frac{r}{2}\alpha, \alpha \right)^{p(\eta - r\alpha)} \\ B &= \prod_{\alpha \in \Phi_1^+} (\lambda - \rho, \alpha)^{p_\alpha(\eta - \alpha)} . \end{aligned} \tag{D.7}$$

There is an extra factor for algebras with non-isotropic fermions, but this will not concern us. The proof is long, technical, and omitted. Reducibility comes down to analysing the factors in this determinant. It's easy to check in examples that A corresponds to the usual conditions for a bosonic Verma module to be reducible. Since we'll always assume that our lowest weight has been chosen appropriately for the bosons, we give this no further thought. More interesting is B, which inspires

Definition

A Verma module $\mathcal{V}_\lambda$ is *atypical* if there is a fermionic root α such that

$$(\lambda - \rho, \alpha) = 0. \tag{D.8}$$

It is *typical* otherwise.

Say that a Verma module is *minimally reducible* if its maximal proper submodule contains only those states required by the vanishing of A, and *more reducible* otherwise. That is, the maximal proper submodule of a minimally reducible Verma module can be deduced purely from the action of bosonic generators, while more reducible Verma modules have 'extra reducibility' coming from fermionic generators. This leads to our working characterisation of reducible Verma modules [159]

Theorem

A Verma module $\mathcal{V}_\lambda$ is more reducible if and only if it is atypical.

This reduces the question to a set of simple linear conditions, which can be checked explicitly given a presentation of the root system of $\mathfrak{g}$ and a choice of lowest weight. We'll see in section 7.1 that there's an intimate relation between atypicality and unitarity conditions, as suggested by the analogy between the Šapovalov form and the inner product. Note that if the atypicality condition corresponding to the root α is satisfied then $p_\alpha(\alpha) = p(0) = 1$, the empty partition. Thus $\det F^\lambda_\alpha$ vanishes, so $\mathcal{V}_\lambda$ contains a state of weight $\lambda + \alpha$ lying in a proper submodule, as claimed without proof in the text.

E A Compendium of Superalgebras

Throughout this document we've given explicit constructions of a large collection of superalgebras. This appendix contains complete listings of all their generators and relations. We don't display vanishing commutation relations or those which can be obtained by Hermitian conjugation. The standard matrix constructions of these superalgebras are found in appendix D.1.

E.1 $\mathfrak{su}(1,1|1)$

This is a simple superalgebra with bosonic part

$$\mathfrak{g}_B = \mathfrak{su}(1,1) \oplus \mathfrak{u}(1)$$

and fermions in the $\mathbf{2}_+ \oplus \mathbf{2}_-$. It was constructed in section 5.1. The generators are

$$\begin{aligned}
2H = \Delta = -R_{\mu\nu\rho\sigma}\psi^{\dagger\mu}\psi^{\dagger\rho}\psi^{\sigma}\psi^{\nu} - i\Gamma^{\mu}_{\nu\rho}\psi^{\dagger\rho}\Pi_{\mu}\psi^{\nu} + i\Pi^{\dagger}_{\nu}\psi^{\dagger\mu}\Gamma^{\nu}_{\mu\rho}\psi^{\rho} + \Pi^{\dagger}_{\mu}g^{\mu\nu}\Pi_{\nu} - \partial_{\mu}\Gamma^{\rho}_{\nu\rho}\psi^{\dagger\mu}\psi^{\nu} \\
D = -i\mathcal{L}_D + i\left(r - \frac{m}{2}\right) = D^{\mu}\Pi_{\mu} - \frac{im}{2} \qquad\qquad K = \frac{1}{2}\|D\|^2 \\
J_3 = \frac{1}{2}\left(r - \frac{m}{2}\right) = \frac{1}{2}\left(\psi^{\dagger\mu}\psi_{\mu} - \frac{m}{2}\right) \\
Q = d = i\psi^{\dagger\mu}\Pi_{\mu} \qquad\qquad Q^{\dagger} = \delta = -i\Pi^{\dagger}_{\mu}\psi^{\mu} \\
S = idK\wedge = iD_{\mu}\psi^{\dagger\mu} \qquad\qquad S^{\dagger} = -ii_D = -iD^{\mu}\psi_{\mu},
\end{aligned}$$

where r is the degree of a form and m is the real dimension of the target space. All bosonic generators are self-adjoint, the fundamental variables $X^{\mu}, \Pi_{\mu}, \psi^{\mu}, \psi^{\dagger\mu}$ satisfy (2.16) and (2.17), and the vector field D satisfies (5.1) . The commutation relations are

$$\begin{aligned}
&[D,H] = 2iH \qquad [D,K] = -2iK \qquad [H,K] = -iD \\
&[D,Q] = iQ \qquad [D,S] = -iS \qquad [H,S] = -iQ \qquad [K,Q] = iS \\
&[J_3,Q] = \frac{1}{2}Q \qquad [J_3,S] = \frac{1}{2}S \\
&\left\{Q,Q^{\dagger}\right\} = 2H \qquad \left\{S,S^{\dagger}\right\} = 2K \qquad \left\{Q,S^{\dagger}\right\} = D - 2iJ_3.
\end{aligned}$$

E.2 $\mathfrak{u}(1,1|2)$

This superalgebra is not simple. The bosonic part is

$$\mathfrak{g}_B = \mathfrak{su}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1),$$

where the $\mathfrak{u}(1)$ factors are respectively the trace, which is central, and the supertrace, which is not. The fermion representations are $(\mathbf{2}\otimes\mathbf{2})_+ \oplus (\mathbf{2}\otimes\mathbf{2})_-$. This was constructed in section 5.2. The generators include all those of $\mathfrak{su}(1,1|1)$ as well as

$$\begin{aligned}
J_+^I &= \omega^I\wedge = \frac{1}{2}\omega^I_{\mu\nu}\psi^{\dagger\mu}\psi^{\dagger\nu} & J_-^I &= \left(\omega^I\wedge\right)^\dagger = \frac{1}{2}\omega_{\nu\mu}\psi^\mu\psi^\nu \\
R^I &= \frac{1}{2}(p-q) = \frac{i}{2}\omega^I_{\mu\nu}\psi^{\dagger\mu}\psi^\nu & D^I &= -i\mathcal{L}_{D^I} = D^{I\mu}\Pi_\mu + i\omega^I_{\mu\nu}\psi^{\dagger\mu}\psi^\nu \\
Q^I &= i\left(\bar\partial-\partial\right) = -i\psi^{\dagger\mu}I^\nu{}_\mu\Pi_\nu & Q^{\dagger I} &= i\left(\partial^\dagger-\bar\partial^\dagger\right) = i\Pi^\dagger_\nu I^\nu{}_\mu\psi^\mu \\
S^I &= \left(\partial-\bar\partial\right)K\wedge = i\psi^{\dagger\mu}D^I_\mu & S^{\dagger I} &= -ii_{D^I} = -iD^I_\mu\psi^\mu,
\end{aligned}$$

where (p,q) is the bidegree of a form. Again all bosons are self-adjoint. D is holomorphic and $D^I = ID$ is a holomorphic isometry. The commutation relations are as for $\mathfrak{su}(1,1|1)$ together with

$$\begin{array}{llll}
\left[J_+^I, J_-^I\right] = 2J_3 & \left[J_3, J_\pm^I\right] = \pm J_\pm^I & & \\
\left[D, Q^I\right] = iQ^I & \left[D, S^I\right] = -iS^I & \left[H, S^I\right] = -iQ^I & \left[K, Q^I\right] = iS^I \\
\left[J_+^I, Q^\dagger\right] = -Q^I & \left[J_+^I, Q^{\dagger I}\right] = Q & \left[J_3, Q^I\right] = \frac{1}{2}Q^I & \\
\left[J_+^I, S^\dagger\right] = -S^I & \left[J_+^I, S^{\dagger I}\right] = S & \left[J_3, S^I\right] = \frac{1}{2}S^I & \\
\left[R^I, Q\right] = \frac{i}{2}Q^I & \left[R^I, Q^I\right] = -\frac{i}{2}Q & \left[R^I, S\right] = \frac{i}{2}S^I & \left[R^I, S^I\right] = -\frac{i}{2}S \\
\left\{Q^I, Q^{\dagger I}\right\} = 2H & \left\{S^I, S^{\dagger I}\right\} = 2K & \left\{Q, S^I\right\} = -2iJ_+^I & \left\{Q^I, S\right\} = 2iJ_+^I \\
\left\{Q, S^{\dagger I}\right\} = D^I & \left\{Q^I, S^\dagger\right\} = -D^I & \left\{Q^I, S^{\dagger I}\right\} = D - 2iJ_3. &
\end{array}$$

E.3 $\mathfrak{osp}(4^*|4)$: General Case

This is a simple superalgebra with bosonic part

$$\mathfrak{g}_B = \mathfrak{su}(1,1)\oplus\mathfrak{su}(2)\oplus\mathfrak{usp}(4)$$

and fermion representation $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{4}$. It was constructed in section 5.3. The generators are those of $\mathfrak{su}(1,1|1)$ together with three copies of those of $\mathfrak{u}(1,1|2)$ which require a complex structure, one for each of the structures $I^a : a = 1,2,3$. The exception is the redefinition

$$T^a = D^a - 2R^a.$$

The commutation relations are as for $\mathfrak{su}(1,1|1)$ and $\mathfrak{u}(1,1|2)$, together with

$$
\begin{aligned}
\left[T^a, T^b\right] &= 2i\epsilon^{abc}T^c & \left[J_+^a, J_-^b\right] &= 2\left(\delta^{ab}J_3 + i\epsilon^{abc}R^c\right) \\
\left[R^a, J_+^b\right] &= i\epsilon^{abc}J_+^c & \left[R^a, R^b\right] &= i\epsilon^{abc}R^c \\
\left[T^a, Q^b\right] &= i\left(\delta^{ab}Q + \epsilon^{abc}Q^c\right) & \left[T^a, S^b\right] &= i\left(\delta^{ab}S + \epsilon^{abc}S^c\right) \\
\left[R^a, Q^b\right] &= \frac{i}{2}\left(-\delta^{ab}Q + \epsilon^{abc}Q^c\right) & \left[R^a, S^b\right] &= \frac{i}{2}\left(-\delta^{ab}S + \epsilon^{abc}S^c\right) \\
\left[J_+^a, Q^{\dagger b}\right] &= \delta^{ab}Q - \epsilon^{abc}Q^c & \left[J_+^a, S^{\dagger b}\right] &= \delta^{ab}S - \epsilon^{abc}S^c \\
\left\{Q^a, Q^{\dagger b}\right\} &= 2\delta^{ab}H & \left\{S^a, S^{\dagger b}\right\} &= 2\delta^{ab}K \\
\left\{Q^a, S^b\right\} &= -2i\epsilon^{abc}J_+^c & \left\{Q^a, S^{\dagger b}\right\} &= \delta^{ab}\left(D - 2iJ_3\right) - \epsilon^{abc}\left(T^c - 2R^c\right).
\end{aligned}
$$

E.4 $\mathfrak{su}(1,1|4)$

This algebra was constructed in section 6.4. It is a simple superalgebra with bosonic part

$$
\mathfrak{g}_B = \mathfrak{su}(1,1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(4)
$$

and fermion representations $(\mathbf{2},\mathbf{4})_- \oplus (\mathbf{2},\bar{\mathbf{4}})_+$, where $\pm$ are $U(1)$ charges. The generators are

$$
\begin{aligned}
H &= \left(\operatorname{Im}\tau^{-1}\right)^{IJ}\Pi_I\bar{\Pi}_J + \frac{1}{2}r_{IJKL}\psi^{IA}\bar{\psi}^J_A\psi^{KB}\bar{\psi}^L_B & D &= a^I\Pi_I + \bar{a}^I\bar{\Pi}_I \\
&\quad + \frac{1}{12}\operatorname{Re}\left(\epsilon_{ABCD}G_{IJKL}\psi^{IA}\psi^{JB}\psi^{KC}\psi^{LD}\right) & K &= \operatorname{Im}\left(a_I^D\bar{a}^I\right) \\
R^{A\bar{B}} &= i\operatorname{Im}\tau_{IJ}\left(\psi^{IA}\bar{\psi}^{J\bar{B}} - \frac{1}{4}\delta^{A\bar{B}}\psi^{IC}\bar{\psi}^J_C\right) & \mathcal{R} &= i\left(a^I\Pi_I - \bar{a}^I\bar{\Pi}_I\right) + \frac{1}{2}\operatorname{Im}\tau_{IJ}\psi^{IA}\bar{\psi}^J_A \\
Q^A &= \psi^{IA}\Pi_I + \frac{1}{12}\epsilon^A{}_{\bar{B}\bar{C}\bar{D}}\bar{\mathcal{F}}_{IJK}\bar{\psi}^{I\bar{B}}\bar{\psi}^{J\bar{C}}\bar{\psi}^{K\bar{D}} & \bar{Q}^{\bar{A}} &= \left(Q^A\right)^\dagger \\
S^A &= \operatorname{Im}\tau_{IJ}\bar{a}^J\psi^{IA} & \bar{S}^{\bar{A}} &= \left(S^A\right)^\dagger
\end{aligned}
$$

with all bosons self-adjoint. The commutation relations are

$$
\begin{aligned}
\left[R^{A\bar{B}}, R^{C\bar{D}}\right] &= i\left(\delta^{C\bar{B}}R^{A\bar{D}} - \delta^{A\bar{D}}R^{C\bar{B}}\right) & [D,H] &= 2iH & [D,K] &= -2iK \\
\left[R^{A\bar{B}}, Q^C\right] &= i\left(\delta^{C\bar{B}}Q^A - \frac{1}{4}\delta^{A\bar{B}}Q^C\right) & [H,K] &= -iD & \left[D,Q^A\right] &= iQ^A \\
\left[R^{A\bar{B}}, S^C\right] &= i\left(\delta^{C\bar{B}}S^A - \frac{1}{4}\delta^{A\bar{B}}S^C\right) & \left[D,S^A\right] &= -iS^A & \left[\mathcal{R},Q^A\right] &= -\frac{1}{2}Q^A \\
\left[K,Q^A\right] &= iS^A & \left[\mathcal{R},S^A\right] &= -\frac{1}{2}S^A & \left[H,S^A\right] &= -iQ^A \\
\left\{Q^A, \bar{S}^{\bar{B}}\right\} &= \frac{1}{2}\delta^{A\bar{B}}\left(D - i\mathcal{R}\right) - R^{A\bar{B}} & \left\{Q^A, \bar{Q}^{\bar{B}}\right\} &= \delta^{A\bar{B}}H & \left\{S^A, \bar{S}^{\bar{B}}\right\} &= \delta^{A\bar{B}}K.
\end{aligned}
$$

E.5 $\mathfrak{osp}(1,1|4)$

This is a deformed subalgebra of $\mathfrak{su}(1,1|4)$ constructed in section 6.5. It is a simple superalgebra with bosonic part

$$\mathfrak{g}_B = \mathfrak{so}(1,1) \oplus \mathfrak{usp}(4)$$

and eight real fermions in the $\mathbf{4}_+ \oplus \mathbf{4}_-$, where $\pm$ are $\mathfrak{so}(1,1)$ charges. The generators are

$$\begin{aligned}
\tilde{L}_0 &= \mu^{-1} \Big[\left(\operatorname{Im}\tau^{-1}\right)^{IJ} \Pi_I \bar{\Pi}_J + \frac{1}{2} r_{IJKL} \psi^{IA} \bar{\psi}^J_A \psi^{KB} \bar{\psi}^L_B + \frac{1}{12} \operatorname{Re}\left(\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}\right) \\
&\quad + \operatorname{Im}\tau_{IJ} \left(P^I - \mu a^I\right)\left(\bar{P}^J - \mu \bar{a}^J\right) + \frac{1}{2} \operatorname{Re}\left(\mathcal{F}_{IJK} \Omega_{AB} \psi^{JA} \psi^{KB} P^I\right) \Big] \\
q^A &= \mu^{-1/2} \Big[\psi^{IA} \Pi_I + \frac{1}{12} \epsilon^A_{\bar{B}\bar{C}\bar{D}} \bar{\mathcal{F}}_{IJK} \bar{\psi}^{I\bar{B}} \bar{\psi}^{J\bar{C}} \bar{\psi}^{K\bar{D}} + \operatorname{Im}\tau_{IJ} \Omega^A_{\bar{B}} \psi^{J\bar{B}} \left(P^I - \mu a^I\right) \Big] \\
&\quad + i\Omega^A_{\bar{B}} \times (\text{hermitian conjugate}) \\
\mathbb{R}^{AB} &= i \operatorname{Im}\tau_{IJ} \left(\Omega^B_{\bar{B}} \psi^{IA} \bar{\psi}^{J\bar{B}} + \Omega^A_{\bar{A}} \psi^{IB} \bar{\psi}^{J\bar{A}} \right).
\end{aligned}$$

The nonzero commutation relations are

$$\begin{aligned}
\left\{ q^A, \bar{q}^{\bar{B}} \right\} &= 2\delta^{A\bar{B}} \tilde{L}_0 - 2i\Omega^{\bar{B}}_B \mathbb{R}^{AB} \qquad\qquad \left[\tilde{L}_0, q^A\right] = q^A \\
\left[\mathbb{R}^{AB}, \mathbb{R}^{CD}\right] &= i\left(\Omega^{BC}\mathbb{R}^{AD} + \Omega^{AC}\mathbb{R}^{BD} + \Omega^{AD}\mathbb{R}^{BC} + \Omega^{BD}\mathbb{R}^{AC}\right).
\end{aligned}$$

E.6 $\mathfrak{osp}(4^*|4)$: Flat Case

This algebra was constructed in section 7.2. It is a rewriting of $\mathfrak{osp}(4^*|4)$ in the special case of a flat target space $\mathbb{C}^{2k}$, with all R-symmetries made manifest. The generators are

$$\begin{aligned}
L_0 &= \frac{1}{2}\left(c^\dagger_{I\alpha} c^\alpha_I + \bar{c}^{\dagger\alpha}_I \bar{c}_{I\alpha}\right) + 2k & L_+ &= -\frac{1}{2} c^\dagger_{I\alpha} \bar{c}^{\dagger\alpha}_I & L_- &= -\frac{1}{2} c^\alpha_I \bar{c}_{I\alpha} \\
T^{\alpha\beta} &= c^{\dagger(\alpha}_I c^{\beta)}_I - \bar{c}^{\dagger(\alpha}_I \bar{c}^{\beta)}_I & R^{AB} &= 2\psi^{(A}_I \bar{\psi}^{B)}_I \\
Q^{A\alpha} &= i\left(\bar{\psi}^A_I \bar{c}^{\dagger\alpha}_I + \psi^A_I c^{\dagger\alpha}_I\right) & S^{A\alpha} &= -i\left(\psi^A_I \bar{c}^\alpha_I + \bar{\psi}^A_I c^\alpha_I\right),
\end{aligned}$$

where $T^{\alpha\beta}$ and R^{AB} generate $SU(2)$ and $USP(4)$ respectively. The nonzero commutation relations are

$$\begin{aligned}
[L_0, L_\pm] &= \pm 2L_\pm & \left[T^{\alpha\beta}, T^{\gamma\delta}\right] &= \epsilon^{\gamma\beta}T^{\alpha\delta} + \epsilon^{\gamma\alpha}T^{\beta\delta} + \epsilon^{\delta\beta}T^{\alpha\gamma} + \epsilon^{\delta\alpha}T^{\beta\gamma} \\
[L_+, L_-] &= -L_0 & \left[R^{AB}, R^{CD}\right] &= \Omega^{CB}R^{AD} + \Omega^{CA}R^{BD} + \Omega^{DA}R^{BC} + \Omega^{DB}R^{AC} \\
\left[L_0, Q^{A\alpha}\right] &= Q^{A\alpha} & \left[T^{\alpha\beta}, Q^{A\gamma}\right] &= \epsilon^{\gamma\alpha}Q^{A\beta} + \epsilon^{\gamma\beta}Q^{A\alpha} \\
\left[L_-, Q^{A\alpha}\right] &= S^{A\alpha} & \left[R^{AB}, Q^{C\alpha}\right] &= \Omega^{CB}Q^{A\alpha} + \Omega^{CA}Q^{B\alpha} \\
\left\{Q^{A\alpha}, Q^{B\beta}\right\} &= 2\epsilon^{\alpha\beta}\Omega^{AB}L_+ & \left\{Q^{A\alpha}, S^{B\beta}\right\} &= \epsilon^{\alpha\beta}\Omega^{AB}L_0 + \Omega^{AB}T^{\alpha\beta} - 2\epsilon^{\alpha\beta}R^{AB}.
\end{aligned}$$

For representation theory it's also useful to know the action of $\mathfrak{osp}(4^*|4)$ on the fundamental variables. The nonzero commutation relations are

$$\begin{aligned}
&[L_0, c_I^\alpha] = -c_I^\alpha \qquad && [L_0, \bar{c}_I^\alpha] = -\bar{c}_I^\alpha \qquad && \left[T^{\alpha\beta}, c_I^\gamma\right] = \epsilon^{\gamma\alpha} c_I^\beta + \epsilon^{\gamma\beta} c_I^\alpha \\
&[L_+, c_I^\alpha] = \bar{c}_I^{\dagger\alpha} \qquad && [L_+, \bar{c}_I^\alpha] = c_I^{\dagger\alpha} \qquad && \left[T^{\alpha\beta}, \bar{c}_I^\gamma\right] = \epsilon^{\gamma\alpha} \bar{\gamma}_I^\beta + \epsilon^{\gamma\beta} \bar{c}_I^\alpha \\
& && && \left[R^{AB}, \psi_I^C\right] = \Omega^{CB} \psi_I^A + \Omega^{CA} \psi_I^B .
\end{aligned} \tag{E.1}$$

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