Published by IOP Publishing for SISSA
Received: *September 27, 2008*
Accepted: *October 16, 2008*
Published: *October 27, 2008*

Integrability of the holomorphic anomaly equations

Babak Haghighat, Albrecht Klemm and Marco Rauch

Physikalisches Institut der Universität Bonn and Bethe Center for Theoretical Physics, Nußallee 12, 53115 Bonn, Germany
E-mail: babak@th.physik.uni-bonn.de, aklemm@th.physik.uni-bonn.de, rauch@th.physik.uni-bonn.de

Abstract: We show that modularity and the gap condition make the holomorphic anomaly equation completely integrable for non-compact Calabi-Yau manifolds. This leads to a very efficient formalism to solve the topological string on these geometries in terms of almost holomorphic modular forms. The formalism provides in particular holomorphic expansions everywhere in moduli space including large radius points, the conifold loci, Seiberg-Witten points and the orbifold points. It can be also viewed as a very efficient method to solve higher genus closed string amplitudes in the $\frac{1}{N^2}$ expansion of matrix models with more then one cut.

Keywords: Topological Strings, Anomalies in Field and String Theories.

JHEP10(2008)097

Contents

1. Introduction

String theory on non-compact Calabi-Yau geometries is relevant for the construction of 4d supersymmetric theories decoupled from gravity and provides simple examples for important concepts of string theory in nontrivial geometrical backgrounds, as e.g. the behavior of the amplitudes under topology change of the background geometry. Exploring the topological sector has been especially fruitful in providing examples of large N-dualities connecting topological string theory on these backgrounds to 3d Chern-Simons theory and matrix models. If the geometric background has a non-trivial space time duality symmetry group, which is the case if the local mirror geometry involves a Riemann surface of at least genus one, the situation is as follows. Large N-dualities or localization principles apply to certain holomorphic limits of the topological string amplitudes and lead to local holomorphic expansion of the latter at special points in the moduli space of the theory. Typically at large radius these come in closed formulas involving infinite sums or products over partitions coming from joining topological vertices or from Nekrasov localization formulas. The expressions lead to formal, i.e. non-convergent expansions, in the string coupling whose coefficients have finite radius of convergence in the moduli parameter. However, since these limits break the invariance of the amplitudes under the space duality group this fundamental symmetry property of the theory is obscured.

In this article we show that a simple bootstrap approach using extensively the full space time modular invariance, the holomorphic anomaly equation and a local analysis of the gap condition at the nodes is highly efficient in reconstructing modular invariant, non-holomorphic string amplitudes for local Calabi-Yau spaces to all genus. They are polynomials in generators of the modular groups, which are globally defined in the moduli space of the theory. As a consequence the amplitudes are globally defined and holomorphic limits can be easily obtained everywhere in the moduli space. The approach extends to $N = 2$ gauge theories and matrix models.

The paper is organized as follows. In section 2 we recall the local Calabi-Yau A-model geometries and how local mirror symmetry leads to a B-model geometry that is governed by a family of Riemann surfaces Σ_g with a canonical meromorphic differential. We derive the Picard-Fuchs equations for the periods and their solutions and thereby solve the genus zero sector.

In section 3 we discuss the formalism of direct integration for local Calabi-Yau spaces. The space-time modular group of Σ_g is a finite index subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$. The invariance of the closed topological string amplitudes F_g under Γ and the holomorphic anomaly equation implies that the F_g are elements in the ring of almost holomorphic modular functions of Γ. The latter is generated by a finite number of holomorphic and non-holomorphic generators. The relevant ones are constructed from the genus zero and genus one sector, i.e. ultimatively from the solutions of the Picard-Fuchs equations. The covariant derivative closes on these generators by (rigid) special geometry. The holomorphic anomaly equation can then be algebraically integrated w.r.t. the non-holomorphic modular generators. This leaves a holomorphic modular ambiguity, which is fixed by the gap conditions at the conifold discriminant.

JHEP10(2008)097

In section 4 we exemplify the formalism and show that the topological string on a local Calabi-Yau geometry, which is the canonical line bundle over $\mathbb{P}^2$, is completely and very efficiently solved by our bootstrap approach. We also show how the generators, which we can construct in all cases from the solutions of the Picard-Fuchs equations relate in this case to classical modular functions on the $\Gamma_0(3) \subset \mathrm{SL}(2,\mathbb{Z})$ curve. We solve the theory to genus 105 and present some of the holomorphic data at conifold, large structure point and orbifold point.

In sections 5 and 6 we extend this formalism to multi moduli examples. We show for the canonical bundle over $\mathbb{F}_0$ and $\mathbb{F}_1$, which have two parameters, how the gap condition at the conifold is again sufficient to fix all boundary conditions. In these cases the unknowns in the holomorphic ambiguity grow in leading order with $(cg)^2$ much faster then in the one moduli case. However, this is compensated by the fact that gap condition holds for all normal directions to the conifold discriminant in the complex two dimensional moduli space.

In section 7 we discuss relations of the results to $N = 2$ Seiberg-Witten theory and general matrix models for which the spectral curve is a family of Riemann surfaces with $g > 0$ and to open string amplitudes.

The appendix A reviews the necessary facts from the theory of modular functions. We try to give well known mathematical concepts a physical interpretation, which might shed some light on the relation between the holomorphic and the modular anomaly.

2. Local mirror symmetry

The term local mirror symmetry refers to mirror symmetry for non-compact Calabi-Yau manifolds. Examples for the A-model geometry are the canonical line bundle $\mathbb{K}_S = \mathcal{O}(-K_S) \to S$ over a Fano surface[1] S. The compact part of B-model geometry is in this case given by a family of elliptic curves and a meromorphic differential. Using toric geometry as below an infinite set of examples of non-compact three-folds can be constructed. They have a partial overlap with the $\mathbb{K}_S$ cases namely $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{P}^2$ and blow-ups thereof $S = \mathbb{BP}^2_1, \mathbb{BP}^2_2, \mathbb{BP}^2_3$. The mirror geometry are Riemann surfaces with a meromorphic differential, whose genus is given by the number of closed meshes in the degeneration locus in the base of symplectic fibration, where two S^1's degenerate. For early applications of local mirror symmetry to BPS state counting and geometric engineering of gauge theories see [35] and [31] respectively. For a systematic formulation see [12, 24, 25]. Below we give a very short review of the techniques.

2.1 The local A-model

The A-model geometry of a non-compact toric variety is given by a quotient

$$M = (\mathbb{C}^{k+3} - Z)/G, \tag{2.1}$$

[1]Simpler examples involve line bundles over a complex curve such as $\mathcal{O}(2(g-2)+k) \oplus \mathcal{O}(-k) \to \mathcal{C}_g$ [10] or manifolds M, which are given by a toric tree diagrams of the degeneration locus that correspond to genus 0 mirror curves.

where $G = (\mathbb{C}^*)^k$ [14]. On the homogeneous coordinates $x_i \in \mathbb{C}$ the group G acts like $x_i \to \mu_\alpha^{Q_i^\alpha} x_i$, $\alpha = 1, \ldots, k$ with $\mu_\alpha \in \mathbb{C}^*$, $Q_i^\alpha \in \mathbb{Z}$. Here Z is the Stanley-Reisner ideal, which has to be chosen so that the above quotient M exists as a variety.[2] The standard example is $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/(\mathbb{C}^*)$, with $Q_i^1 = 1$, $i = 1, \ldots, n$. We denote generically by $\mathcal{S}$ the compact part of M.

As explained in [44] M can also be viewed as the vacuum field configuration of a 2d gauged linear $(2,2)$ supersymmetric σ model. The coordinates $x_i \in \mathbb{C}$, $i = 1, \ldots, k+3$ are the vacuum expectation values of chiral superfields transforming as $x_i \to e^{iQ_i^\alpha \epsilon_\alpha} x_i$, $Q_i^\alpha \in \mathbb{Z}$, $\epsilon_\alpha \in \mathbb{R}$, $\alpha = 1, \ldots, k$ under the gauge group $\mathrm{U}(1)^k$. The vacuum field configuration are the equivalence classes under the gauge group, which fulfill in addition the D-term constraints

$$D^\alpha = \sum_{i=1}^{k+3} Q_i^\alpha |x_i|^2 = r^\alpha, \quad \alpha = 1, \ldots, k \ . \tag{2.2}$$

The r^α are the Kähler parameters $r^\alpha = \int_{C_\alpha} \omega$, where ω is the Kähler form and C_α are curves spanning the Mori cone, which is dual to the Kähler cone. $r^\alpha \in \mathbb{R}_+$ defines the Kähler cone. For M to be well defined, field configurations for which the dimensionality of the gauge orbits drop have to be excluded. This corresponds to the choice of Z. In string theory r^α is complexified to $T^\alpha = r^\alpha + i\theta^\alpha$ with $\theta^\alpha = \int_{C_\alpha} B$, where B is the NS B-field, while in the gauged linear σ-model the θ^α are the θ-angles of the $\mathrm{U}(1)^k$ gauge group.

One can always describe M by a completely triangulated fan. In this case the Q_i^α are linear relations between the points spanning the fan. A basis of such relations, which corresponds to a Mori cone can be constructed from a complete triangulation of the fan. Z likewise follows combinatorially from the triangulation, see the examples.[3]

The Calabi-Yau condition $c_1(TM) = 0$ holds if and only if[4]

$$\sum_{i=1}^{k+3} Q_i^\alpha = 0, \qquad \alpha = 1, \ldots, k. \tag{2.3}$$

Note from (2.2) that negative Q_i lead to non-compact directions in M, so that by (2.3) all toric Calabi-Yau manifolds M are necessarily non-compact. To summarize, toric non-compact A-model geometries will be defined by suitably chosen charge vectors $Q_i^\alpha \in \mathbb{Z}$.

We now come to invariants calculated by the A-model amplitudes. We consider maps $f : \mathcal{C}_g \to M$ from a genus g curve $\mathcal{C}_g$, whose image curve is in the class $\beta \in H_2(\mathcal{S}, \mathbb{Z})$. Now let as in [34]

$$r_\beta^g = \int_{\overline{\mathcal{M}}(\beta,\mathcal{S})} c_{\mathrm{vir}}(U_\beta)\,, \tag{2.4}$$

[2]We assume that M is simplicial, or that a simplicial subdivision in coordinate patches exists.

[3]Often there are many possible triangulation, which correspond to different phases of the model [44, 5], e.g. Kähler cones connected by flopping a $\mathbb{P}^1$. The union of the cones define by all triangulations is called the secondary fan.

[4]Physically these are the conditions that the chiral $\mathrm{U}(1)_A$ anomaly cancels in the gauged linear σ-model [44].

with U_β the bundle whose fiber over $(\mathcal{C}, f) \in \overline{\mathcal{M}(\beta, \mathcal{S})}$ is $H^1(\mathcal{C}_g, f^*M)$, be the Gromov-Witten invariant. The classical task in the closed topological A-model is to calculate the generating function

$$\mathcal{F} = \log(\mathcal{Z}) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g(Q) = \frac{c(T)}{\lambda^2} + l(T) + \sum_{g=0}^{\infty} \sum_{\beta} \lambda^{2g-2} r_\beta^g Q^\beta \,, \tag{2.5}$$

with $Q^\beta = \exp(2\pi i \sum_{i=1}^{b_2(S)} \beta_i T_i)$, $\beta_i \in \mathbb{Z}_+$, involving all closed string Gromov-Witten invariants as well as classical intersection numbers of the harmonic $(1,1)$-forms $\frac{1}{3!} T^\alpha T^\beta T^\gamma \int_M \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma$ in the cubic $c(T)$ and $\frac{1}{24} T^\alpha \int_M c_2 \wedge \omega_\alpha$ in the linear $l(T)$ term. The generating function $\mathcal{F}$ can be reexpressed as one

$$\mathcal{F} = \frac{c(T)}{\lambda^2} + l(T) + \sum_{g=0}^{\infty} \sum_{\beta \in H_2(S,\mathbb{Z})} \sum_{m=1}^{\infty} n_\beta^g \frac{1}{m} \left(2 \sin \frac{m\lambda}{2} \right)^{2g-2} Q^{\beta m} \tag{2.6}$$

for the BPS or Gopakumar-Vafa invariants $n_\beta^g \in \mathbb{Z}$ or with $q_\lambda = e^{i\lambda}$ the holomorphic partition function

$$\mathcal{Z} = \sum_{\beta, k \in \mathbb{Z}} \tilde{n}_\beta^k (-q_\lambda)^k Q^\beta = \prod_\beta \left[\left(\prod_{r=1}^{\infty} (1 - q_\lambda^r q^\beta)^{r n_\beta^0} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1 - q_\lambda^{g-l-1} Q^\beta)^{(-1)^{g+r} \binom{2g-2}{l} n_\beta^g} \right] \tag{2.7}$$

becomes the generating function for the Donaldson-Thomas invariants[5] $\tilde{n}_\beta^k \in \mathbb{Z}$.

2.2 The local B-model

In the following we will describe the non-compact mirror W following [24, 31, 6]. Let $w^+, w^- \in \mathbb{C}$ and $x_i =: e^{y_i} \in \mathbb{C}^*$, $i = 1, \ldots, k+3$ are homogeneous coordinates,[6] i.e. equivalence classes subject to the $\mathbb{C}^*$ action

$$x_i \mapsto \lambda x_i, \quad i = 1, \ldots, k+3, \quad \lambda \in \mathbb{C}^* \,. \tag{2.8}$$

The mirror W is defined from the charge vectors Q_i^α by the exponentiated D-term constraints

$$(-1)^{Q_0^\alpha} \prod_{i=1}^{k+3} x_i^{Q_i^\alpha} = z_\alpha, \quad \alpha = 1, \ldots, k \,. \tag{2.9}$$

and the general equation

$$w^+ w^- = H = \sum_{i=1}^{k+3} x_i \,. \tag{2.10}$$

The Calabi-Yau condition (2.3) ensures the compatibility of (2.9) with (2.8). Using the latter two equations to eliminate variables x_i in (2.10) H can be parameterized by two variables $x = \exp(u), y = \exp(v) \in \mathbb{C}^*$ and the defining equations of the mirror geometry W becomes

$$w^+ w^- = H(x, y; z_\alpha), \tag{2.11}$$

[5] Here we dropped the classical terms.

[6] The x_i here should not be identified with the x_i, which describe the A model in the previous section.

which is a conic bundle over $\mathbb{C}^* \times \mathbb{C}^*$, where the conic fiber degenerates to two lines over the family of Riemann surfaces with punctures

$$\Sigma(z) := \{H(x, y; z^\alpha) = 0\} \subset \mathbb{C}^* \times \mathbb{C}^* , \tag{2.12}$$

parameterized by the complex parameters z^α. To establish that W is a non-compact Calabi-Yau manifold note that

$$\Omega = \frac{dH dx dy}{H x y} \tag{2.13}$$

is a regularizable no-where vanishing holomorphic volume form on W. Its periods are regularizable in the sense that H, y can be integrated out to yield a meromorphic one-form on Σ

$$\lambda = \frac{\log(y)\mathrm{d}x}{x} , \tag{2.14}$$

whose periods clearly exist. They are annihilated by the linear differential operators

$$D_\alpha = \prod_{Q_i^\alpha > 0} \partial_{x_i}^{Q_i^\alpha} - \prod_{Q_i^\alpha < 0} \partial_{x_i}^{-Q_i^\alpha} . \tag{2.15}$$

The redundancy in the parameterization of the complex structure is removed using the relations (2.9) and the scaling relation (2.8). To do that it is convenient to write the differential operator (2.15) in terms of logarithmic derivatives $\theta_i := x_i \partial_{x_i}$ and transform to logarithmic derivatives $\Theta_\alpha := z_\alpha \partial_{z_\alpha}$ using $\theta_i = Q_i^\alpha \Theta_\alpha$.

As it is well known the solutions to (2.15) are constructed by the Frobenius method [12], i.e. defining

$$w_0(\underline{z}, \underline{\rho}) = \sum_{\underline{n^\alpha}} \frac{1}{\prod_i \Gamma[Q_i^\alpha(n^\alpha + \rho^\alpha) + 1]} ((-1)^{Q_0^\alpha} z^\alpha)^{n^\alpha} , \tag{2.16}$$

then

$$X^0 = w_0(\underline{z}, \underline{0}) = 1, \qquad T^\alpha = \frac{\partial}{2\pi i \partial \rho^\alpha} w_0(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \tag{2.17}$$

are solutions. Note that the flat coordinates T^α approximate $T^\alpha \sim \log(z^\alpha)$ in the limit $z^\alpha \to 0$. Higher derivatives

$$X^{(\alpha_{i_1} \cdots \alpha_{i_n})} = \frac{1}{(2\pi i)^n} \frac{\partial}{\partial \rho^{\alpha_{i_1}}} \cdots \frac{\partial}{\partial \rho^{\alpha_{i_n}}} w_0(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \tag{2.18}$$

also obey the recursion imposed by (2.15), i.e. they fulfill (2.15) up to finitely many terms. However, a unique, up to addition of previous solutions, linear combinations of the $X^{\alpha_{i_1} \cdots \alpha_{i_2}}$ is actually the last solution of the Picard-Fuchs system. This solution encodes the genus zero Gromov-Witten invariants. It is a derivative of the holomorphic prepotential $\mathcal{F}_0$ and the triple intersection $C_{ijk} = \partial_{T_i} \partial_{T_j} \partial_{T_k} \mathcal{F}_0$ can be constructed from it, see the examples for more details. We will turn to generating functions for the higher genus amplitudes in the next section.

JHEP10(2008)097

3. Integrability of the holomorphic anomaly equation

This section is to review the recent results of [22, 4] on the polynomial recursive solution of the holomorphic anomaly equation of [8] and to set our conventions. This recursive solution is a generalization of the pioneering work of Yamaguchi and Yau [41] who observed that the non-holomorphic dependence of the topological free energy function of the quintic can be expressed by a finite number of generators. Our main focus is the local geometry, hence we will mainly explain how the formalism simplifies in the non-compact case.

3.1 Direct integration in local Calabi-Yau geometries

One of the main tasks in topological string theory is to compute the free energies F_g appearing in the topological string partition function $Z = \exp(\sum \lambda^{2g-2} F_g)$. We will assume that the genus zero sector has been determined from the solutions to the Picard-Fuchs equations discussed in section 2.2. The genus one amplitude is associated to the Ray-Singer torsion of the Calabi-Yau space [8]. It fulfills a special holomorphic anomaly equation, which is integrated to [7][7]

$$F_1 = \frac{1}{2} \log \left[\exp \left[K \left(3 + h^{1,1} - \frac{\chi}{12} \right) \right] \det G^{-1}_{i\bar{j}} |f_1|^2 \right] . \qquad (3.1)$$

While the exponential of the real Kähler potential $\exp(K) \sim X^0 \to 1$ in the holomorphic limit in the non-compact models [34], the F_1 is non-holomorphic due to the Kähler metric $G_{i\bar{j}}$ on the complex structure moduli space. f_1 is the holomorphic ambiguity in this integration and it can be argued to be a power of the discriminant loci of Σ [7, 21], i.e. $f = \prod_i \Delta_i^{a_i} \prod_{i=1}^{h^{2,1}} z_i^{b_i}$. The parameters, a_i, b_i, can be solved from the limiting behavior of F_1 near singularities, $\lim_{z_i \to 0} F_1 = -\frac{1}{24} \sum_{i=1}^{h^{2,1}} t_i \int_M c_2 J_i$ as well as the universal behavior at conifold singularities $a_{\rm con} = -\frac{1}{12}$.

As was shown in [8] F_g is for $g > 1$ a non-holomorphic section of a line bundle $\mathcal{L}^{2-2g}$ which fulfills a recursive differential equation

$$\bar{\partial}_{\bar{\imath}} F_g = \frac{1}{2} \bar{C}^{jk}_{\bar{\imath}} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_{g-r} D_k F_r \right) , \qquad (g > 1) \qquad (3.2)$$

called the holomorphic anomaly equation. The covariant derivatives contain the connection $\partial_i K = K_i$ of $\mathcal{L}$ and the Christoffel symbols Γ^i_{jk} of the Kähler metric. The recursive nature is due to the fact that Riemann surfaces with marked points split at the boundary of moduli space, $\mathcal{M}$, into either pairs of lower genus surfaces or surfaces with fewer marked points.

The key input for the direct integration procedure is the special geometry integration condition

$$\bar{\partial}_{\bar{\imath}} \Gamma^k_{ij} = \delta^k_i G_{j\bar{\imath}} + \delta^k_j G_{i\bar{\imath}} - C_{ijl} \bar{C}^{kl}_{\bar{\imath}} \ . \qquad (3.3)$$

Here C_{ijl} are the holomorphic Yukawa couplings which transform as $\mathrm{Sym}^3(T\mathcal{M}) \otimes \mathcal{L}^{-2}$ and $\bar{C}^{kl}_{\bar{\imath}} = e^{2K} G^{k\bar{k}} G^{l\bar{l}} \bar{C}_{\bar{\imath}\bar{k}\bar{l}}$. (3.3) implies that the propagator S^{ij}, which is defined by

[7] In the following we denote the non-holomorphic quantities by straight characters F_g and the holomorphic limits by calligraphic characters $\mathcal{F}^f_1$, with a label f of the patch, where the limit is taken.

$\bar{\partial}_{\bar{k}} S^{ij} = \bar{C}^{ij}_{\bar{k}}$, can be solved from the integrated version of (3.3) [8]

$$\Gamma^k_{ij} = \delta^k_i \partial_j K + \delta^k_j \partial_i K - C_{ijl} S^{kl} + \tilde{f}^k_{ij}\,, \tag{3.4}$$

up to the holomorphic ambiguity $\tilde{f}^k_{ij}$. Taking the anti holomorphic derivative, using (3.3) and $\partial_{\bar{j}} S^k = S^k_{\bar{j}}$ it follows that

$$\bar{\partial}_{\bar{k}}(D_i S^{kl}) = \bar{\partial}_{\bar{k}}(\delta^k_i S^l + \delta^l_i S^k - C_{inm} S^{km} S^{ln})\ , \tag{3.5}$$

and so

$$D_i S^{kl} = \delta^k_i S^l + \delta^l_i S^k - C_{inm} S^{km} S^{ln} + f^{kl}_i\ . \tag{3.6}$$

In the local case one has the following simplifications.[8] The Kähler connection in D_i becomes trivial, and the S^l, (as well as the S, see [8]) vanish, i.e. the above equation becomes simply

$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f^{kl}_i. \tag{3.7}$$

Also, the Kähler connection $\partial_j K$ in (3.4) drops out, so the S^{ij} are solved from

$$\Gamma^k_{ij} = -C_{ijl} S^{kl} + \tilde{f}^k_{ij}\ . \tag{3.8}$$

Note that this is an over-determined system in the multi moduli case which requires a suitable choice of the ambiguity $\tilde{f}^k_{ij}$. This choice is simplified by the fact [1] that $\partial_i F_1$ can be expressed through the propagator as

$$\partial_i F_1 = \frac{1}{2} C_{ijk} S^{jk} + A_i, \tag{3.9}$$

with an ambiguity A_i, which can be determined by the ansatz $A_i = \partial_i(\tilde{a}_j \log \Delta_j + \tilde{b}_j \log z_j)$.

Once the S^{ij} are obtained and the ambiguities in (3.7), (3.8) have been fixed, the direct integration of (3.2) is quite simple. Everything on the right hand side of the holomorphic anomaly equation (3.2) can be rewritten in terms of the generators S^{ij} and holomorphic functions. If we further express the anti-holomorphic derivative of F_g as

$$\bar{\partial}_{\bar{i}} F_g = \bar{C}^{jk}_{\bar{i}} \frac{\partial F_g}{\partial S^{jk}}, \tag{3.10}$$

and assume linear independence of $\bar{C}^{jk}_{\bar{i}}$, (3.2) can be written as

$$\frac{\partial F_g}{\partial S^{jk}} = \frac{1}{2}\left(D_j \partial_k F_{g-1} + \sum_{r=1}^{g-1} \partial_j F_{g-r} \partial_k F_r \right). \tag{3.11}$$

This equation can easily be integrated w.r.t. S^{ij} and it can be shown that F_g is a polynomial in S^{jk} of degree $3g-3$.

[8]In the global case on needs further the closing of covariant derivatives of S^i and S with $\partial_{\bar{i}} S = G_{\bar{i}j} S^j$. This has been discussed in [41, 22] and particular nicely in [4].

3.2 Fixing the ambiguity

Due to the equation (3.11) the iteration in the genus is in principle quite easy on the B-model side and the topological invariants of the A-model geometry can be extracted without effort. However, the issue is fixing the holomorphic ambiguity f_g arising after each integration step w.r.t. the S^{ij}. Modularity, regularity at the orbifold point and at the large radius point, as well as the leading behavior at the conifold singularities [21] imply the following ansatz for f_g

$$f_g = \sum_i \frac{A_g^i}{\Delta_i^{2g-2}}, \tag{3.12}$$

where A_g^i is a polynomial in z of degree $(2g-2)\cdot\deg\Delta_i$ and the sum runs over all irreducible components of the discriminant locus. Note that the moduli space $\mathcal{M}(\Sigma)$ allows a compactification, which includes only the ordinary double point discriminants or conifolds at complex codimension one loci in the moduli space. A_g^i are polynomials in the monodromy invariant variables z_i, $i = 1,\ldots,n$ of the model. Their degree is bounded by regularity of the F_g in the limit that these variables tend to infinity by the degree of the Δ_i. In general this implies a growth of the unknowns roughly with $(c_i g)^n$, where c_i depends on the degrees of Δ_i. However, if we approach a conifold singularity we also find in the multi parameter case a gap. It is of the form

$$\mathcal{F}_g^c = \frac{c^{g-1}B_{2g}}{2g(2g-2)t_c^{2g-2}} + \mathcal{O}(t_c^0)\ . \tag{3.13}$$

where we approach a conifold in the limit $t_c \to 0$, with t_c a flat coordinate normal to the singularity[9] (see figure 3). The coefficients of the sub-leading powers of t_c depend generically on the further $n-1$ directions, which are tangential to the discriminant locus. For a generic choice of coordinates these coefficients are (infinite) series in the tangential $n-1$ variables. However, demanding the vanishing of these coefficients is an over-determined system and it is not easy to count the independent conditions. But in local models where the geometry of the B-model is completely encoded in a Riemann surface of genus $g > 0$ we find that the gap condition is sufficient to determine all parameters in the ambiguity except for the one, which corresponds to the constant term in F_g. The latter can be determined by the known constant map contribution to $\mathcal{F}_g$ at the point of large radius in moduli space

$$\mathcal{F}_g = \frac{\chi B_{2g-2}B_{2g}}{4g(2g-2)(2g-2)!} + \mathcal{O}(Q). \tag{3.14}$$

Therefore we find that the holomorphic anomaly equations are completely integrable for local Calabi-Yau spaces. Our claim that this is true in general is motivated by the fact that the only type of degeneration of a Riemann surface in complex codimension one is the nodal degeneration and the leading local behavior of the F_g at this singularity is always governed by the gap structure and in particular the argument for the existence of the gap [30] does not depend on the direction nor on the particular point at which the conifold locus is approached.

[9] c is an undetermined constant, which can be absorbed by rescaling the variable t_c.

JHEP10(2008)097

4. $\mathbb{K}_{\mathbb{P}^2} = \mathcal{O}(-3) \to \mathbb{P}^2$

The toric data of $\mathbb{K}_{\mathbb{P}^2}$ is summarized in the following matrix

$$(V|Q) = \left(\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right) \tag{4.1}$$

The A-model is described from these data as follows. The generators of the toric fan $\mathbb{F}$ v_i, $i = 0, \ldots, 3$ are the rows of V, while the columns of Q are the charge vectors, which are the coefficients of linear relations among the v_i. To each v_i we associate homogeneous coordinates x_i. There is an unique complete triangulation of $\mathbb{F}$ into simplexes given by $\mathcal{T} = \{\{v_0, v_1, v_2\}, \{v_0, v_1, v_3\}, \{v_0, v_2, v_3\}\}$. The Stanley-Reisner ideal Z is generated by intersection of divisors $D_i = \{x_i = 0\}$, whose associated points are not on a common simplex in $\mathcal{T}$, i.e. by $Z = \{x_1 = x_2 = x_3 = 0\}$. The $(x_1 : x_2 : x_3)$ are hence the homogeneous coordinates of $\mathbb{P}^2$. The three $\mathbb{C}^3$ patches that cover the 3-fold $\mathbb{K}_{\mathbb{P}^2}$ are specified by the scaling in (2.1) as $(l_1 = x_0 x_1^3; u_1 = x_2/x_1, v_1 = x_3/x_1)$, $(l_2 = x_0 x_2^3; u_2 = x_1/x_2, v_2 = x_3/x_2)$ and $(l_3 = x_0 x_3^3; u_3 = x_1/x_3, v_3 = x_2/x_3)$ with the obvious transition functions.

The B-model geometry is defined by the one parameter family of Riemann surfaces $\Sigma(z)$

$$H(x, y; z) = x + 1 - z\frac{x^3}{y} + y = 0\ . \tag{4.2}$$

Here we set $x_1 = 1$ in (2.10) by the scaling relation (2.8) and eliminated x_2 using (2.9) in favor of $x := x_0$ and $y := x_3$.

4.1 Global properties of the moduli space of $\Sigma(z)$

After writing (4.2) in Weierstrass form in $\mathbb{P}^2$ we find the j-function of the elliptic family $\Sigma(z)$

$$j = -\frac{(1 + 24\, z)^3}{z^3\, (1 + 27\, z)}. \tag{4.3}$$

Its moduli space for the complex structure parameter z is $\mathcal{M}(\Sigma(z)) = \mathbb{P}^1 \setminus \{z = 0, z = -\frac{1}{27}, \frac{1}{z} = 0\}$. The critical points of j are referred to as large radius point, conifold points and orbifold point,[10] respectively.

Following the description after (2.15) we find

$$\mathcal{D} = \Theta^3 + 3z(3\Theta - 2)(3\Theta - 1)\Theta = \mathcal{L}\Theta, \tag{4.4}$$

here $\mathcal{L}$ is the Picard-Fuchs equation for the periods over the holomorphic differential $\omega = \frac{\mathrm{d}x}{y}$. From this follows that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\lambda = \omega + \text{exact}, \tag{4.5}$$

[10] By using a multi covering variable $\psi = -\frac{1}{3z^{\frac{1}{3}}}$ one gets three symmetric conifold points at $\psi^3 = 1$ and no orbifold point.

JHEP10(2008)097

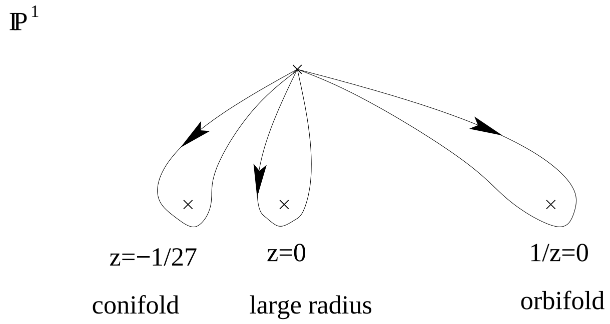

Figure 1: Definition of the monodromies in $\mathcal{M}(\Sigma(z)) = \mathbb{P}^1 \setminus \{z=0, z=-\frac{1}{27}, \frac{1}{z}=0\}$.

where λ is the meromorphic differential. This meromorphic differential λ has a pole with non-vanishing residue and we denote the cycle around this pole γ, while $a, b \in H_1(\Sigma, \mathbb{Z})$ are a basis for the integral cycles on Σ. On $\hat{\Pi} = (\int_b \lambda, \int_a \lambda, \int_\gamma \lambda)^T$ the monodromy acts by

$$M_{z=0} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad M_{z=-\frac{1}{27}} = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\frac{1}{z}=0} = M^{-1}_{z=-\frac{1}{27}} M^{-1}_{z=0}\,, \tag{4.6}$$

as can be seen explicitly by analytic continuation of the periods into the three patches near the singular points (4.8), (4.17) as well as (4.23), (4.24). It follows from the monodromy invariance of z and (4.5), that the upper left (2×2) block in the above matrices acting on $\hat{\Pi}$ represents also the monodromy action on the $\Pi = (\int_b \omega, \int_a \omega)^T$. The later generates

$$\Gamma^0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, b \equiv 0 \bmod 3 \right\}\,. \tag{4.7}$$

4.2 Periods and genus zero and one amplitudes in all patches

We review now the construction of the holomorphic prepotential encoding the genus zero amplitude and the an-holomorphic Ray-Singer torsion encoding the genus one amplitude in the patches near the three singular points described above. In each patch we introduce appropriate flat coordinates, distinguished by the monodromies around the critical points. Once the flat coordinate is chosen one can consider a holomorphic limit of the amplitudes for $g > 0$. This yields holomorphic generating functions for certain topological invariants, depending on the point in moduli space. Notably the Gromov-Witten invariants near $z = 0$ and the orbifold Gromov-Witten invariants near $\frac{1}{z} = 0$. The most useful structure for the integrability comes from the gap in the expansion at the conifold.

4.2.1 Expansion near the large radius point

The solutions near $z = 0$ are according to (2.17), (2.18) given[11] by $\omega_0(z, 0) = 1$, $X^{(1)} = \frac{1}{2\pi i}(\log(z) + \sigma_1(z))$ and $X^{(1,1)} = \frac{1}{(2\pi i)^2}(\log(z)^2 + 2\sigma_1 \log(z) + \sigma_2(z))$, where the first orders

[11] We also note that the system (4.4) is related to the Meijer G-functions and $T = -\frac{1}{2\pi i \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} G^{3\,3}_{2\,2}\left(\begin{smallmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{smallmatrix}\middle| 27z\right)$.

are $\sigma_1 = -6\,z + 45\,z^2 + \mathcal{O}(z^3)$ and $\sigma_2 = -18\,z + \frac{423\,z^2}{2} + \mathcal{O}(\tilde{z}^3)$. The actual integral basis of periods is given by the linear combinations

$$\hat{\Pi} = \begin{pmatrix} T_D \\ T \\ 1 \end{pmatrix} = \begin{pmatrix} -9\partial_T \mathcal{F}_0 \\ T \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}X^{(1,1)} - \frac{3}{2}T + \frac{3}{4} \\ X^{(1)} \\ 1 \end{pmatrix} \tag{4.8}$$

In order to express $\mathcal{F}_0$ in terms of the flat coordinate T, we introduce the monodromy invariant quantity $Q = e^{2\pi i T}$ and invert. This yields the large radius mirror map

$$z(Q) = Q + 6\,Q^2 + 9\,Q^3 + 56\,Q^4 - 300\,Q^5 + \cdots \tag{4.9}$$

The normalization $T_D = -9\partial_T \mathcal{F}_0$ is such that $\mathcal{F}_0$ is the generating function for the genus zero Gromov-Witten invariants of $\mathcal{O}(-3) \to \mathbb{P}^2$ in the normalization that reproduces the A-model results obtained first by localization [34], see table B.1 for the BPS invariants

$$\mathcal{F}_0 = -\frac{T^3}{18} + \frac{T^2}{12} - \frac{T}{12} + 3\,Q - \frac{45\,Q^2}{8} + \frac{244\,Q^3}{9} - \frac{12333\,Q^4}{64} + \frac{211878\,Q^5}{125} + \cdots \tag{4.10}$$

The normalization of the Yukawa coupling, with which we get this expansion is

$$C_{zzz} = -\frac{1}{3}\frac{1}{z^3(1+27z)} \ . \tag{4.11}$$

The Yukawa coupling transforms as $\mathrm{Sym}^3(T\mathcal{M}) \otimes \mathcal{L}^{-2}$, where the Kähler connection, i.e. the line bundle $\mathcal{L}$ is trivial in the local case. From the special Kähler relations in flat coordinates we get

$$\left(\frac{\partial}{\partial T}\right)^3 \mathcal{F}_0 = C_{TTT} = \left(\frac{\partial z}{\partial T}\right)^3 C_{zzz} \ . \tag{4.12}$$

Note that (4.11) is modular invariant and valid in all $\mathcal{M}(\Sigma)$. The expression (4.12) on the other hand requires a choice of the flat coordinate T, which is only canonical near $z = 0$. One can view T as the coordinate and $P_T = \partial_T \mathcal{F}_0$ as the dual momentum and show that $Z = \exp(F)$ transforms as a wavefunction under a change of polarization, i.e. when a different choice (related by a linear transformation) for coordinates and momenta is made [43, 3].

Using the standard definition of the modular parameter of the family of elliptic curves $\tau = \frac{\int_b \omega}{\int_a \omega}$, (4.5) and (4.8) we find

$$\tau = \frac{\frac{\partial T_D}{\partial z}}{\frac{\partial T}{\partial z}} = -9\frac{\partial^3 \mathcal{F}_0}{\partial^3 T} \ . \tag{4.13}$$

The resulting relation $z(q)$, with $q = \exp(2\pi i \tau)$ has to be compatible with (4.3). Indeed inserting $z(q)$ into (4.3) yields the standard expansion of the elliptic j-function (A.11). Using $z(q)$ we can express the non-holomorphic genus one potential as

$$F_1 = -\log\left(\sqrt{\tau_2}\,\eta(q)\bar{\eta}(\bar{q})\right) - \frac{1}{24}\log\left(1 + \frac{1}{27z}\right) \ . \tag{4.14}$$

Both the Dedekind η function as well as $1+\frac{1}{27z}$ are powers of the discriminant of Σ. The former transforms with weight $\frac{1}{2}$ that is canceled by that of τ_2 (A.2). We note that both forms of F_1 (3.1) and (4.14) are manifestly modular invariant.

Using $\det G^{-1}_{i\bar{j}} \to C \det \frac{\partial z_i}{\partial T_j}$ in the holomorphic limit $\bar{T} \to \infty$ or equivalently $\tau \to i\infty$ one gets up to irrelevant constants the holomorphic expression

$$\mathcal{F}_1 = \frac{1}{2}\log\left(\frac{\partial z}{\partial T}\right) - \frac{1}{12}\log\left(z^7\left(1+\frac{1}{27z}\right)\right) . \tag{4.15}$$

This expression is not modular invariant and depends on the choice of our special coordinate. It does give however the generating function for GW invariants at genus one

$$\mathcal{F}_1 = \frac{T}{12} + \frac{Q}{4} - \frac{3\,Q^2}{8} - \frac{23\,Q^3}{3} + \frac{3437\,Q^4}{16} - \frac{43107\,Q^5}{10} + \cdots \tag{4.16}$$

in accordance with [34], see table B.1 for the BPS invariants.

4.2.2 Expansion near the conifold

To obtain the closed variables at the conifold we solve the Picard-Fuchs equation after the variable transformation $z = \frac{\Delta - 1}{27}$. The basis of periods at large radius (4.8) has the following expansion at the conifold point

$$\begin{aligned}\Pi &= \begin{pmatrix} a\,t_c \\ 3\,a\,t_{cD} \\ 1 \end{pmatrix} = \begin{pmatrix} a\,t_c \\ 3\,a\,\partial_{t_c}\,\mathcal{F}^c_0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a(\Delta + \frac{11\,\Delta^2}{18} + \frac{109\,\Delta^3}{243} + \mathcal{O}(\Delta^4)) \\ a\left(a_0 + a_1 t_c - \frac{1}{2\pi i}(t_c \log(\Delta) + \frac{7\,\Delta^2}{12} + \frac{877\,\Delta^3}{1458} + \mathcal{O}(\Delta^4))\right) \\ 1 \end{pmatrix} ,\end{aligned} \tag{4.17}$$

where $a = -\frac{\sqrt{3}}{2\pi}$, $a_0 = -\frac{\pi}{3} - 1.678699904i = \frac{1}{i\sqrt{3}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} G^{3\,3}_{2\,2}\left(\begin{smallmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{smallmatrix}\middle| -1\right)$ and $a_1 = \frac{3\log(3)+1}{2\pi i}$.

The natural local flat coordinate at the conifold is t_c and with the conifold mirror map

$$\Delta = t_c - \frac{11 t_c^2}{18} + \frac{145 t_c^3}{486} - \frac{6733 t_c^4}{52488} + \mathcal{O}\left(t_c^5\right) \tag{4.18}$$

the genus zero prepotential becomes

$$\mathcal{F}^c_0 = c_0 + \frac{a_0}{3} t_c + \left(\frac{a_1}{6} - \frac{1}{12}\right) t_c^2 + t_c^2 \frac{\log(t_c)}{6} - \frac{t_c^3}{324} + \frac{t_c^4}{69984} + \frac{7\,t_c^5}{2361960} - \frac{529\,t_c^6}{1700611200} + \mathcal{O}(t_c^7) . \tag{4.19}$$

Note that we rescaled t_c by a in order to avoid non rational numbers in this expansion and the extra factor 3 in (4.17) is so that $\partial^3_{t_c}\mathcal{F}^c_0 = \left(\frac{\partial z}{\partial t_c}\right)^3 C_{zzz}(t_c)$ We can also find the holomorphic limit of the genus one prepotential as

$$\mathcal{F}^c_1 = \frac{1}{2}\log\left(\frac{\partial z}{\partial t_c}\right) - \frac{1}{12}\log\left(z^7\left(1+\frac{1}{27z}\right)\right) \tag{4.20}$$

and expand it as

$$\mathcal{F}^c_1 = c'_0 - \frac{\log(t_c)}{12} + \frac{5\,t_c}{216} - \frac{t_c^2}{23328} - \frac{5\,t_c^3}{157464} + \frac{283\,t_c^4}{75582720} - \frac{43\,t_c^5}{153055008} + \frac{4517\,t_c^6}{385698620160} + \mathcal{O}(t_c^7) . \tag{4.21}$$

4.2.3 Coordinates and amplitudes at the orbifold

At the orbifold point, the model admits an exact field theory description as an orbifold of three complex bosons $\mathbb{C}^3/\mathbb{Z}_3$. After transforming the Picard-Fuchs equation to the $\psi = -\frac{1}{3z^{\frac{1}{3}}}$ coordinate we find the following local expansion of a basis of solutions $(1, B_1, B_2)$ with

$$B_k = (-1)^{\frac{k}{3}+k+1}\frac{(3\psi)^k}{k}\sum_{n=0}\frac{\left[\frac{k}{3}\right]_n^3}{\prod_{i=1}^3\left[\frac{k+i}{3}\right]_n}\psi^{3n}\,, \tag{4.22}$$

where $[a]_n = a(a+1)\dots(a+n+1)$ is the Pochhammer symbol. We define orbifold periods, which diagonalize the $\mathbb{Z}_3$ orbifold monodromy action

$$\Pi_{\text{orb}} = \begin{pmatrix}\sigma_D\\ \sigma\\ 1\end{pmatrix} = \begin{pmatrix}-3\partial_\sigma\mathcal{F}_0^{\text{orb}}\\ \sigma\\ 1\end{pmatrix} = \begin{pmatrix}B_2\\ B_1\\ 1\end{pmatrix}\,, \tag{4.23}$$

i.e. $(B_2, B_1, 1) \mapsto (\exp\left(\frac{4\pi i}{3}\right)B_2, \exp\left(\frac{2\pi i}{3}\right)B_2, 1)$ under $\psi \mapsto \exp\left(\frac{2\pi i}{3}\right)\psi$. Note, that this is not the basis at large radius, but rather connected to it by the transformation $\Pi = M\Pi_{\text{orb}}$ with

$$M = \begin{pmatrix}-\frac{3}{1-\alpha}A & \frac{3\alpha}{1-\alpha}B & 1\\ A & B & 0\\ 0 & 0 & 1\end{pmatrix}\,. \tag{4.24}$$

Here we introduced

$$A := \frac{i\Gamma\left(\frac{2}{3}\right)}{2\pi\Gamma^2\left(\frac{1}{3}\right)}\,,\quad B := \frac{i\Gamma\left(\frac{1}{3}\right)}{2\pi\Gamma^2\left(\frac{2}{3}\right)}\,,\quad \alpha := \exp\left(\frac{2\pi i}{3}\right)\,. \tag{4.25}$$

We normalize the flat coordinate σ and $\mathcal{F}_0^{\text{orb}}$ to match the orbifold Gromov-Witten invariants of [13] in the orbifold prepotential

$$\mathcal{F}_0^{\text{orb}} = \frac{\sigma^3}{18} - \frac{\sigma^6}{19440} + \frac{\sigma^9}{3265920} - \frac{38497\,\sigma^{12}}{2571324134400} + \cdots \tag{4.26}$$

and the special geometry relation $\partial_\sigma^3\mathcal{F}_0^{\text{orb}} = \left(\frac{\partial z}{\partial\sigma}\right)^3 C_{zzz}(\sigma)$, which implies the orbifold mirror map

$$\frac{\psi}{\alpha^2} = \frac{\sigma}{3} + \frac{\sigma^4}{1944} - \frac{29\sigma^7}{11022480} + \mathcal{O}(\sigma^{10})\,. \tag{4.27}$$

The expansion for the holomorphic limit of the Ray-Singer Torsion reads

$$\mathcal{F}_1^{\text{orb}} = \frac{1}{2}\log\left(\frac{\partial z}{\partial\sigma}\right) - \frac{1}{12}\log\left(z^7\left(1+\frac{1}{27z}\right)\right) = c_0 + \frac{\sigma^6}{174960} - \frac{\sigma^9}{6298560} + \frac{13007\,\sigma^{12}}{3142729497600} + \cdots\,. \tag{4.28}$$

4.3 Direct integration for $\mathbb{K}_{\mathbb{P}^2}$

Let us now discuss the direct integration for the non-compact $\mathbb{K}_{\mathbb{P}^2}$ geometry. Here we have only one propagator, which we denote in the z variables by S^{zz}. The propagator has a holomorphic ambiguity, which we may choose by imposing in (3.9) the vanishing of A_z

$$S^{zz} = \frac{2}{C_{zzz}}\partial_z F_1\,. \tag{4.29}$$

This implies the following ambiguity factors in (3.4)

$$\Gamma^z_{zz} = -C_{zzz}S^{zz} - \frac{7+216z}{6z\Delta} \tag{4.30}$$

and in (3.7)

$$D_z S^{zz} = -C_{zzz}S^{zz}S^{zz} - \frac{z}{12\Delta} \; . \tag{4.31}$$

The right hand side of equation (3.11) is easily evaluated using the connection Γ^z_{zz} and yields e.g. for $g=2$ using (4.29), (4.30) and (4.31)

$$\partial_{S^{zz}} F_2 = C^2_{zzz}\left(\frac{5(S^{zz})^2}{8} - \frac{3z^2 S^{zz}}{8} + \frac{z^4}{16}\right) , \tag{4.32}$$

which integrates to

$$F_2 = C^2_{zzz}\left(\frac{5(S^{zz})^3}{24} - \frac{3z^2(S^{zz})^2}{16} + \frac{z^4 S^{zz}}{16} + \frac{z^6(729z^2+162z-11)}{1920}\right) \; . \tag{4.33}$$

The integration constant f_g of the S^{zz} integration ($f_2 = \frac{729z^2+162z-11}{1920(1+27z)^2}$ in (4.33)) can be fixed from the boundary behavior of $\mathcal{F}_g$. Since z is a global parameter, we only need to know the holomorphic limit of S^{zz} in terms of the flat coordinates $t_f \in \{T, t_c, \sigma\}$ near large radius, conifold and orbifold point

$$S^{zz}_f = \frac{2}{C_{zzz}}\partial_z \mathcal{F}^f_1 = \frac{2}{C_{zzz}}\partial_z\left(\frac{1}{2}\log\left(\frac{\partial z}{\partial t_f}\right) - \frac{1}{12}\log\left(z^7\left(1+\frac{1}{27z}\right)\right)\right) \tag{4.34}$$

in order to evaluate $\mathcal{F}_g$ in the local coordinates in all patches.

The conditions on the local expansion are similar as in the compact case in [30], namely the gap condition at the conifold, regularity at orbifold and the constant map contribution at infinity. The difference is that in the non-compact case these conditions are sufficient to fix the kernel of (3.11) completely. The argument is as follows. The maximal pole at the conifold is $\frac{1}{\Delta^{2g-2}(z)}$ and there is no pole at the orbifold nor at infinity. Modularity implies that the possible numerator of the ambiguity is a polynomial in the modular invariant z. Since F_g is finite at the orbifold at $\frac{1}{z}=0$ the denominator degree of z cannot exceed $2g-2$, i.e. the ambiguity has to be of the form $\frac{p_{2g-2}(z)}{\Delta^{2g-2}}$. $2g-2$ of the $2g-1$ coefficients of $p_{2g-2}(z)$ follow from the gap condition

$$\mathcal{F}_g = \frac{3^{g-1}B_{2g}}{2g(2g-2)t_c^{2g-2}} + \mathcal{O}(t_c^0), \tag{4.35}$$

here t_c is the unique vanishing period at the conifold given in (4.17). One additional condition follows from constant map contribution at infinity

$$\mathcal{F}_g = \frac{3B_{2g-2}B_{2g}}{4g(2g-2)(2g-2)!} + \mathcal{O}(Q) \; . \tag{4.36}$$

With this boundary information the model is completely integrable. The integration step can be further simplified. As all F_g are of the form $F_g = C^{2g-2}_{zzz}P_g =$

$C_{zzz}^{2g-2}\sum_{i=0}^{3g-3}(S^{zz})^i f_g^i(z)$, it is natural to rewrite (3.11) for the P_g. To do this denote $\delta_z = \frac{1}{C_{zzz}}\partial_z$, so that e.g. $\delta_z S^{zz} = (S^{zz})^2 - z^2(7+216z)S^{zz} + \frac{z^4}{4}$, and define the derivative δ on a weight k function g_k as $\delta g_k = (\delta_z + 3kz^2(1+36z))g_k$. The weights are $[P_g] = 6g-6$ and $[\delta P_g] = 6g-3$ and (3.11) reads

$$\partial_{S^{zz}} P_g = \frac{1}{2}\left(\left(\delta - \frac{\Gamma^z_{zz}}{C_{zzz}}\right)\delta P_{g-1} + \sum_{r=1}^{g-1}\delta P_{g-r}\delta P_r\right) . \tag{4.37}$$

In this form the equation is most easily integrated to very high genus (up to genus 80 in a few hours on a modern PC).

4.4 Modular expressions for the F_g on $\mathbb{K}_{\mathbb{P}^2}$

The aim of this section is to relate the expression for F_g obtained in the previous section to classical modular forms. Some results in this direction have been obtained in [3] for a related family of elliptic curves $\tilde{\Sigma}(\tilde{z})$

$$\sum_{i=1}^3 x_i^3 + \tilde{z}^{-\frac{1}{3}}\prod_{i=1}^3 x_i = 0\,, \tag{4.38}$$

which comes from the Landau-Ginzburg model, whose infrared limit is the exact field theory $\mathbb{C}^3/\mathbb{Z}_3$ mentioned in the section 4.2.3.

In order to understand the relation between the curves let us calculate the j-function of (4.38)

$$\tilde{j} = \frac{(216\tilde{z}-1)^3}{\tilde{z}(1+27\tilde{z})^3} . \tag{4.39}$$

$\tilde{j}$ is transformed into (4.3) when we identify

$$\tilde{z} = -\frac{1}{27}(1+27z) \tag{4.40}$$

which exchanges the large radius point and the conifold point of $\tilde{\Sigma}(\tilde{z})$ and $\Sigma(z)$. Such reparametrization symmetries are ubiquitous in $N=2$ supersymmetric theories, e.g. in Seiberg-Witten theory [33], and the associated curves Σ and $\tilde{\Sigma}$ are called isogenous. It can be checked that periods of $\tilde{\Sigma}(\tilde{z})$ fulfill the same Picard-Fuchs equation (4.4) as the ones of $\Sigma(z)$ with the argument z replaced by $\tilde{z}$. In fact the periods of the curves are related by a rescaling so that their modular parameter is rescaled by a factor 3

$$\tau = 3\tilde{\tau}\,, \tag{4.41}$$

as can be seen by comparing the $\tilde{z}(\tilde{q})$ and $z(q)$ expansions that follow from (4.39) and (4.3).

In [3] quantities in the parameterization of the curve (4.38) have been related to θ-constants that generate modular forms of $\Gamma_0(3)$[12]

$$a := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{1}{6}\end{bmatrix}, \quad b := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{1}{2}\end{bmatrix}, \quad c := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{5}{6}\end{bmatrix}, \quad d := \theta^3\begin{bmatrix}\frac{1}{2}\\ \frac{1}{6}\end{bmatrix}, \tag{4.42}$$

[12] Because [3] worked with (4.38) all modular quantities below are understood to have the argument $\tilde{\tau}$.

which all have weight 3/2 and satisfy with $\alpha = \exp\left(\frac{2\pi i}{3}\right)$ the relations [20]

$$c = b - a, \qquad d = a + \alpha b, \qquad \eta^{12} = \frac{i}{3^{3/2}} abcd \; . \tag{4.43}$$

Following the observation in [3] $\tilde{\psi} = -\frac{1}{\tilde{z}^{1/3}} = \alpha^2 \left(\frac{a-c-d}{d}\right)$ and (4.43) we get

$$\tilde{z} = -\frac{1}{3^3}\frac{d^4 + \eta^{12}}{d^4} \tag{4.44}$$

and

$$\frac{\partial T}{\partial \tilde{\psi}} = -\alpha\sqrt{3}\frac{d}{\eta}\,, \tag{4.45}$$

For this curve one finds the genus one amplitude

$$F_1 = -\log(\sqrt{\bar{\tau}_2}\eta(\tilde{\tau})\bar{\eta}(\bar{\tilde{\tau}})) + \frac{1}{24}\log\left(1 + \frac{1}{27\tilde{z}}\right) = -\frac{1}{2}\log\left(\tilde{\tau}_2 \theta^{\frac{1}{2}}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{6}\end{bmatrix}\eta^{\frac{1}{2}}\bar{\eta}\right) \tag{4.46}$$

Note that (4.46) can be transformed into (4.14) by applying (4.40) and (4.41). A small calculation using (4.46), (A.13) and (4.45) gives the propagator in terms of standard modular expressions

$$S^{\tilde{\psi}\tilde{\psi}} = \left(\frac{\partial\tilde{\psi}}{\partial\tilde{z}}\right)^2\left(S^{zz} - \frac{\tilde{z}^2}{4}\right) = \frac{1}{12}\left(\frac{\eta}{d}\right)^2 \hat{E}_2(\tilde{\tau}) \; . \tag{4.47}$$

This and (4.44) allows to rewrite all F_g in terms of theta functions and $\hat{E}_2$. With $F_g = X^{g-1}\hat{P}_g$, where $X = \frac{d^2}{2^9 3^6 \eta^{18}}$ is a weight -3 form, we get e.g.

$$\hat{P}_2 = 5\hat{E}_2^3 + \frac{\alpha}{\eta^2}\left(\frac{d^4+27\eta^{12}}{d}\right)^{\frac{2}{3}}\hat{E}_2^2 - \frac{\alpha^2}{3\eta^4}\left(\frac{d^4+27\eta^{12}}{d}\right)^{\frac{4}{3}}\hat{E}_2 - \frac{(d^4-27\eta^{12})(d^4+33\eta^{12})}{15d^2\eta^2} \; . \tag{4.48}$$

Since $\hat{E}_2, d, \eta$ close under derivatives $d_{\tilde{\tau}} d = \frac{E_2 d}{8} + \frac{d^3}{108\eta^2(-\tilde{z})^{\frac{2}{3}}}$ $(d_{\tilde{\tau}}\tilde{z} = -3^3\frac{\eta^{10}}{d^2}(-\tilde{z})^{\frac{4}{3}})$, it is obviously possible to set up the direct integration in terms of the modular expression. We leave this to the reader.

4.5 The higher genus results for $\mathbb{K}_{\mathbb{P}^2}$

At the large radius point we recorded some Gopakumar-Vafa invariants in appendix B. The results agree with the literature as far as they are known. Both w.r.t. to the genus as well as to the degree the method outlined here is the most effective one to get these generating functions. An excellent check on this data is provided already by the formulas $n_d^{g(d)} = (-1)^{\frac{d(d+3)}{2}}\frac{(d+1)(d+2)}{2}$ and $n_d^{g(d)-1} = -(-1)^{\frac{d(d+3)}{2}}\binom{d}{2}(d^2+d-3)$ for the highest genus $g(d) = \frac{(d-1)(d-2)}{2}$ and the next to highest genus BPS invariant in each degree d, which were derived in [32]. In fact we checked that the spaces in [32], which model the moduli space of the D_2-D_0 brane system with D_2 brane charge d are smooth for D_2 branes wrapping holomorphic curves of genus $g(d)-\delta$ with up to $\delta = d-1$ nodes. As a consequence the formula (4.15) of [32] applies for $n_d^{g(d)-\delta}$, with $e(\mathcal{C}^{(p)}) = e(\mathbb{P}^{(d(d+3)/2-p)} e(\mathrm{Hilb}^p\mathbb{P}^2)$ for

JHEP10(2008)097

$g\backslash d$	0	1	2	3	4
0		$\frac{1}{3}$	$\frac{-1}{3^3}$	$\frac{1}{3^2}$	$\frac{-1093}{3^6}$
1		0	$\frac{1}{3^5}$	$\frac{-14}{3^5}$	$\frac{13007}{3^8}$
2	$\frac{1}{2^7 3^3 5}$	$\frac{1}{2^4 3^5 5}$	$\frac{-13}{2^4 3^6}$	$\frac{20693}{2^4 3^8 5}$	$\frac{-12803923}{2^4 3^{10} 5}$
3	$\frac{-1}{2^9 3^5 5\cdot 7}$	$\frac{-31}{2^5 3^7 5\cdot 7}$	$\frac{11569}{2^5 3^9 5\cdot 7}$	$\frac{-2429003}{2^5 3^{10} 5\cdot 7}$	$\frac{871749323}{2^5 3^{11} 5\cdot 7}$
4	$\frac{-311}{2^{11} 3^8 5^2 7}$	$\frac{313}{2^7 3^9 5^2}$	$\frac{-1889}{2^8 3^9}$	$\frac{115647179}{2^6 3^{13} 5^2}$	$\frac{-29321809247}{2^8 3^{12} 5^2}$
5	$\frac{24559}{2^{14} 3^9 5^2 7\cdot 11}$	$\frac{-519961}{2^9 3^{11} 5^2 7\cdot 11}$	$\frac{196898123}{2^9 3^{12} 5^2 7\cdot 11}$	$\frac{-339157983781}{2^9 3^{14} 5^2 7\cdot 11}$	$\frac{78658947782147}{2^9 3^{16} 5^2 7}$
6	$\frac{-49922143}{2^{14} 3^{11} 5^3 7^2 11\cdot 13}$	$\frac{14609730607}{2^{12} 3^{13} 5^3 7^2 11\cdot 13}$	$\frac{-258703053013}{2^{10} 3^{15} 5\cdot 7^2 11\cdot 13}$	$\frac{2453678654644313}{2^{12} 3^{14} 5^3 7^2 11\cdot 13}$	$\frac{-4001577419369601803}{2^{11} 3^{18} 5^3 7^2 11\cdot 13}$
7	$\frac{1341390269}{2^{16} 3^{13} 5^3 7^2 11\cdot 13}$	$\frac{-1122101011}{2^{13} 3^{14} 5^3 7\cdot 11}$	$\frac{2196793414201}{2^{11} 3^{17} 5^3 7\cdot 11}$	$\frac{-2127526097369539}{2^{13} 3^{18} 5^2 7\cdot 11}$	$\frac{26373375124439869913}{2^{12} 3^{20} 5^3 7\cdot 11}$
8	$\frac{-1701146456533}{2^{19} 3^{15} 5^3 7^2 11\cdot 13\cdot 17}$	$\frac{1424424798274897}{2^{15} 3^{17} 5^4 7^2 11\cdot 13\cdot 17}$	$\frac{-80699319730594681}{2^{15} 3^{19} 5^3 7^2 11\cdot 17}$	$\frac{347152749067185797 6969}{2^{16} 3^{20} 5^3 7^2 11\cdot 13\cdot 17}$	$\frac{-114258620434929543630324227}{2^{16} 3^{22} 5^4 7^2 11\cdot 13\cdot 17}$

Table 1: Low genus orbifold Gromov-Witten invariants $N_{g,d}$

$\delta = 0, \ldots, d-1$, yielding 120 non-trivial checks for the BPS numbers in appendix B. We also expect that the relatively simple recursive nature of the procedure described here will allow to study high genus asymptotics of BPS states.

The $\mathcal{F}_g^c$ near the conifold are expected to correspond to a perturbation of the $c=1$ string at selfdual radius, which has been established as a dual description of the topological string at the conifold [21], but the details of the identification of the perturbation parameters are not completely clarified [16]. The most notable structure is the gap in the $\mathcal{F}_g^c$ expansion at higher genus. We display a few low genus $\mathcal{F}_g^c$

$$
\begin{aligned}
\mathcal{F}_2^c &= \frac{1}{80\,t_c^2} - \frac{1}{51840} - \frac{t_c}{19440} + \frac{3187 t_c^2}{377913600} - \frac{239\,t_c^3}{255091680} + \mathcal{O}(t_c^4)\\
\mathcal{F}_3^c &= \frac{1}{112\,t_c^4} - \frac{1}{117573120} - \frac{t_c}{1469664} + \frac{23855\,t_c^2}{179992689408} - \frac{557\,t_c^3}{24794911296} + \mathcal{O}(t_c^4)\\
\mathcal{F}_4^c &= \frac{3}{160\,t_c^6} - \frac{1}{63489484800} - \frac{7\,t_c}{377913600} + \frac{6830569\,t_c^2}{1190155742208000} - \frac{1561279\,t_c^3}{1205032688985600} + \mathcal{O}(t_c^4)\\
\mathcal{F}_5^c &= \frac{27}{352\,t_c^8} - \frac{1}{16761223987200} - \frac{809\,t_c}{942818849280} + \frac{118418785\,t_c^2}{326612060022657024} - \frac{113975899\,t_c^3}{1002105184160424960} + \mathcal{O}(t_c^4)\\
\mathcal{F}_6^c &= \frac{18657}{36400\,t_c^{10}} - \frac{691}{1853204730144768000} - \frac{1276277\,t_c}{21059144660736000} + \frac{279842720162009\,t_c^2}{905283603276270446592 0000} + \mathcal{O}(t_c^3)\\
\mathcal{F}_7^c &= \frac{81}{16\,t_c^{12}} - \frac{691}{200146110855634944000} - \frac{7943\,t_c}{1309171316428800} + \frac{27776712091\,t_c^2}{7792369912031464488960} + \mathcal{O}(t_c^3)\\
\mathcal{F}_8^c &= \frac{2636793}{38080\,t_c^{14}} - \frac{3617}{816596132290990571520 00} - \frac{25034924437\,t_c}{306223549609121464320 00} + \mathcal{O}(t_c^2)
\end{aligned}
\tag{4.49}
$$

If we denote as in [3] the generating function

$$
\mathcal{F}_g^{\rm orb} = \frac{1}{(3k)!} N_{g,k} \sigma^{3k}\,, \tag{4.50}
$$

we can read of the orbifold Gromov-Witten invariants, see [3, 9], from our results, as in the table below.[13] Some of the results beyond $g=0$ have been confirmed in [9].

5. $\mathbb{K}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$

We are considering the non-compact Calabi-Yau geometry $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$, i.e. the canonical line bundle over the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This local model can be obtained from the compact elliptic fibration over $\mathbb{F}_0$ with fiber $X_6(1,2,3)$. The three complexified Kähler volumes have the corresponding Mori cone generators $(-6;3,2,1,0,0,0,0)$,

[13] It corrects some misprints in [3, 9].

$(0;0,0,-2,1,0,1,0)$, $(0;0,0,-2,0,1,0,1)$. Roughly, in the local limit the volume of the elliptic fiber is send to infinity. The B-model mirror description of the local geometry is encoded in a Riemann surface with a meromorphic differential as pointed out before.

According to [27] and using the above mentioned charge vectors, one can derive a Picard-Fuchs system governing the periods of the global mirror geometry. They are given by

$$\begin{aligned}
\mathcal{D}_1 &= \Theta_1(\Theta_1 - 2\Theta_2 - 2\Theta_3) - 18z_1(1+6\Theta_1)(5+6\Theta_1) \\
\mathcal{D}_2 &= \Theta_2^2 + z_2(1-\Theta_1+2\Theta_2+2\Theta_3)(\Theta_1-2\Theta_2-2\Theta_3) \\
\mathcal{D}_3 &= \Theta_3^2 + z_3(1-\Theta_1+2\Theta_2+2\Theta_3)(\Theta_1-2\Theta_2-2\Theta_3),
\end{aligned} \tag{5.1}$$

where we denote the logarithmic derivative by $\Theta_i = z_i\frac{\partial}{\partial z_i}$. z_1 is the complex structure parameter dual to the Kähler parameter of the elliptic fiber $t_{\rm F}$. The local limit is obtained by sending this parameter to zero, $z_1 \to 0$.

Now let us turn to the non-compact geometry. The toric data of local $\mathbb{F}_0$ is summarized in the following matrix, V denoting the vectors which span the fan and Q denoting the charge vectors.

$$(V|Q) = \left(\begin{array}{ccc|cc}
0 & 0 & 1 & -2 & -2 \\
1 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1
\end{array}\right) \tag{5.2}$$

From there we conclude the following quantities as was explained in section 2.2. $C^{(0)}_{ijk}$ denote the classical triple intersection numbers. They, as well as $\int_M c_2 J_i$, were computed using toric geometry.

$$\begin{aligned}
&a) && Q^1 = (-2,1,0,1,0),\ Q^2 = (-2,0,1,0,1) \\
&b) && Z = \{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\} \\
&c) && M = (\mathbb{C}^5[x_0,\ldots,x_4] \setminus Z)/(\mathbb{C}^*)^2 \\
&d) && H(x,y) = y^2 - x^3 - (1-4z_1-4z_2)x^2 - 16z_1z_2x \\
&e) && \mathcal{D}_1 = \Theta_1^2 - 2z_1(\Theta_1+\Theta_2)(1+2\Theta_1+2\Theta_2) \\
& && \mathcal{D}_2 = \Theta_2^2 - 2z_2(\Theta_1+\Theta_2)(1+2\Theta_1+2\Theta_2) \\
& && \Delta = 1 - 8(z_1+z_2) + 16(z_1-z_2)^2 \\
&f) && C^{(0)}_{111} = \frac{1}{4},\ C^{(0)}_{112} = -\frac{1}{4},\ C^{(0)}_{122} = -\frac{1}{4},\ C^{(0)}_{222} = \frac{1}{4} \\
&g) && \int_M c_2 J_1 = \int_M c_2 J_2 = -1.
\end{aligned} \tag{5.3}$$

$H(x,y)=0$ defines a family of elliptic curves $\Sigma(z_1,z_2)$ whose j-function is given by

$$j(z_1,z_2) = \frac{((1-4z_1-4z_2)^2 - 48z_1z_2)^3}{z_1^2z_2^2(1-8(z_1+z_2)+16(z_1-z_2)^2)}. \tag{5.4}$$

JHEP10(2008)097

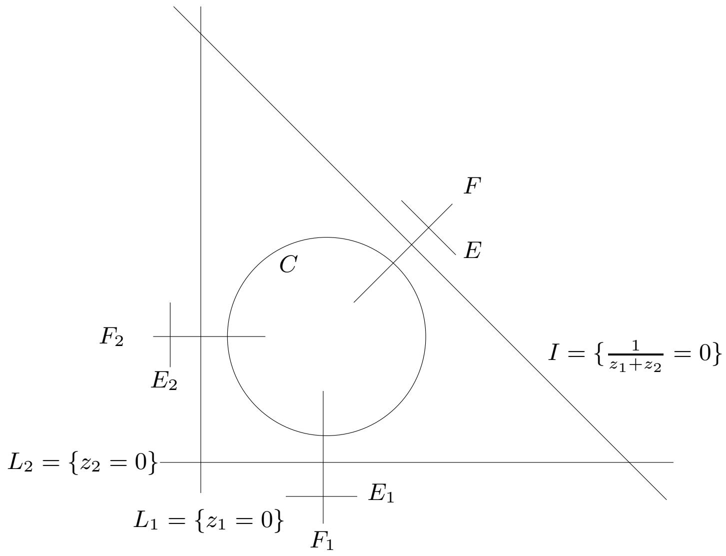

Figure 2: Resolved Moduli Space of $\mathbb{F}_0$

5.1 Review of the moduli space $\mathcal{M}$

The moduli space, $\mathcal{M}$, of the local Calabi-Yau $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ is spanned by two Kähler moduli controlling the sizes of the two $\mathbb{P}^1$'s. The B-model mirror description of this geometry can be expressed through a Riemann surface together with a meromorphic differential. The meromorphic differential is the reduction of the holomorphic three-form of the mirror geometry to a one-form living on a Riemann surface as described in section 2.2. In our particular case we get a genus one Riemann surface with two non-trivial cycles. Apart from these the meromorphic differential has a residue arising from integration over a certain trivial cycle. Together these periods parameterize the two complex structure moduli which are mirror to the two Kähler moduli of the original model. The period integrals satisfy two linear differential equations of order two, given by the Picard-Fuchs operators. It is well known that these periods can at worse have logarithmic singularities. The singular locus in the moduli space can be obtained by calculating the discriminant of the Picard-Fuchs system (5.3). This yields

$$z_1 z_2 \left(1 - 8(z_1 + z_2) + 16(z_1 - z_2)^2\right) =: z_1 z_2 \Delta = 0. \tag{5.5}$$

One sees that the singular locus splits into three irreducible components given by the divisors $z_1 = 0$, $z_2 = 0$ and $\Delta = 0$. The moduli z_1, z_2 are compactified to $\mathbb{P}^2$.

At the large complex structure point $L_1 \cap L_2$, two of the periods, $t_1 = \log(z_1) + \mathcal{O}(z)$ and $t_2 = \log(z_2) + \mathcal{O}(z)$, give the classical large Kähler volumes of the two $\mathbb{P}^1$. As C touches L_1 at $z_2 = \frac{1}{4}$, L_2 at $z_1 = \frac{1}{4}$ and I at $u = \frac{z_1}{z_1+z_2} = \frac{1}{2}$ and all intersections are with contact order two, the Picard-Fuchs system cannot be solved around these points in moduli space. Therefore, the moduli space has to be blown up around these points so that all divisors have normal crossings. This is done by introducing two new divisors at each of these points which is depicted in figure 2. More details about this moduli space can be found in [1].

For us the most relevant points are $I \cap F$ which is a $\mathbb{Z}_2$ orbifold point admitting a matrix model expansion, and the conifold locus C, relevant for fixing the holomorphic ambiguity of the free energy functions.

5.2 Solving the topological string on local $\mathbb{F}_0$ at large radius

By the method of Frobenius one can calculate the periods eliminated by the Picard-Fuchs system. As the charge vectors are chosen such that they span the Mori cone, the periods are calculated at the large radius point of the moduli space $\mathcal{M}(M)$. It is well known that the regular solution for this local model is simply $\omega_0(\underline{z}, 0) = 1$. Therefore the mirror map is equal to the single logarithmic solution and given by

$$\begin{aligned} 2\pi i T_1(z_1, z_2) &= \log z_1 + 2(z_1+z_2) + 3(z_1^2+4z_1z_2+z_2^2) + \frac{20}{3}(z_1^3+9z_1^2z_2+9z_1z_2^2+z_2^3) + \mathcal{O}(z^4) \\ 2\pi i T_2(z_1, z_2) &= \log z_2 + 2(z_1+z_2) + 3(z_1^2+4z_1z_2+z_2^2) + \frac{20}{3}(z_1^3+9z_1^2z_2+9z_1z_2^2+z_2^3) + \mathcal{O}(z^4). \end{aligned} \tag{5.6}$$

By inverting the above series we arrive at ($Q_i = e^{2\pi i T_i}$)

$$\begin{aligned} z_1(Q_1, Q_2) &= Q_1 - 2(Q_1^2+Q_1Q_2) + 3(Q_1^3+Q_1Q_2^2) - 4(Q_1^4+Q_1^3Q_2+Q_1^2Q_2^2+Q_1Q_2^3) + \mathcal{O}(Q^5) \\ z_2(Q_1, Q_2) &= Q_2 - 2(Q_1Q_2+Q_2^2) + 3(Q_1^2Q_2+Q_2^3) - 4(Q_1^3Q_2+Q_1^2Q_2^2+Q_1Q_2^3+Q_2^4) + \mathcal{O}(Q^5). \end{aligned} \tag{5.7}$$

We observe that the following combination does not receive any instanton corrections which can be easily derived from the Picard-Fuchs system

$$\frac{z_1}{z_2} = \frac{Q_1}{Q_2} = e^{2\pi i (T_1 - T_2)} =: Q_1^x, \tag{5.8}$$

or in other words, the mirror map can be brought in trigonal form by means of the coordinate choice, $x_1 = \frac{z_1}{z_2}$ and $x_2 = z_2$, as well as $Q_2^x = Q_2$. We have

$$\begin{aligned} x_1(Q_1^x, Q_2^x) &= Q_1^x, \\ x_2(Q_1^x, Q_2^x) &= Q_2^x - 2Q_2^{x\,2} + Q_1^x Q_2^{x\,2} + 3Q_2^{x\,3} + \mathcal{O}(Q^4). \end{aligned} \tag{5.9}$$

The next step is to determine the Yukawa couplings. Four independent combinations are

$$\begin{aligned} C_{111} &= \frac{(1-4z_2)^2 - 16z_1(1+z_1)}{4z_1^3\Delta}, & C_{112} &= \frac{16z_1^2 - (1-4z_1)^2}{4z_1^2z_2\Delta}, \\ C_{122} &= \frac{16z_2^2 - (1-4z_2)^2}{4z_1z_2^2\Delta}, & C_{222} &= \frac{(1-4z_1)^2 - 16z_1(1+z_2)}{4z_2^3\Delta}. \end{aligned} \tag{5.10}$$

The numerator is fixed by the help of the known classical triple intersection numbers as well as the genus zero GV invariants, whereas the denominator is fixed by the Picard-Fuchs system. Note, that the Yukawa couplings are of the well-known structure, i.e. a rational function in the z_i's multiplied by the inverse of the discriminant. Here we note, that in local models the choice of the classical data is crucial for the success of direct integration. This is due to the fact, that one can obtain the right GV invariants for different choices of $C^{(0)}$ and $\int c_2 J$. However, if one does not use consistent data, higher genus calculations become wrong or even impossible. In contrast, the dependence on some Euler number drops out completely, as it does not effect the GV invariants. In this work we simply set χ to zero.

Using the ansatz (3.1) for the free energy function of genus one and the classical data $\int c_2 J_i$ as well as the known genus one GV invariants we are able to fix the holomorphic ambiguity at genus one, f_1. The result as well as the expansion at large radius in the holomorphic limit $\overline{T} \to 0$ reads as follows

$$\begin{aligned} F_1 &= \log\left(\Delta^{-\frac{1}{12}}(z_1 z_2)^{-\frac{13}{24}}(\det(G_{i\bar{j}}))^{-\frac{1}{2}}\right), \\ \mathcal{F}_1(T_1,T_2) &= -\frac{1}{24}\log(Q_1Q_2) - \frac{1}{6}(Q_1+Q_2) - \frac{1}{12}(Q_1^2 + 4Q_1Q_2 + Q_2^2) + \mathcal{O}(Q^3). \end{aligned} \tag{5.11}$$

In order to perform the method of direct integration, we have to calculate the propagator and express all quantities which carry non-holomorphic information through our propagators. As a first step the holomorphic ambiguity, $\tilde{f}$, in (3.8) can be fixed by the choice

$$\begin{aligned} &\tilde{f}^1_{11} = -\frac{1}{z_1}, && \tilde{f}^1_{12} = -\frac{1}{4z_2}, && \tilde{f}^1_{22} = 0, \\ &\tilde{f}^2_{11} = 0, && \tilde{f}^2_{12} = -\frac{1}{4z_1}, && \tilde{f}^2_{22} = -\frac{1}{z_2}, \end{aligned} \tag{5.12}$$

where all other combinations follow by symmetry. We note that the propagator has only one independent component for we can write

$$S^{ij} = \begin{pmatrix} S(z_1,z_2) & \frac{z_2}{z_1} S(z_1,z_2) \\ \frac{z_2}{z_1} S(z_1,z_2) & \frac{z_2^2}{z_1^2} S(z_1,z_2) \end{pmatrix} \tag{5.13}$$

where $S(z_1,z_2) = \frac{1}{2}z_1^2 - 2z_1^3 - 2z_1^2 z_2 - 8z_1^3 z_2 - 32 z_1^4 z_2 + \mathcal{O}(z^6)$. This is due to the fact, that the mirror geometry is solely determined by the elliptic curve $\Sigma(z_1,z_2)$, which has only one relevant elliptic parameter τ. The dependence on a second parameter is due to a non-vanishing residue of the meromorphic differential on $\Sigma(z_1,z_2)$.

Often it is convenient and also more natural to perform the calculations in the coordinates x_1, x_2, in which some Christoffel symbols are rational

$$\Gamma^1_{11} = \frac{1}{x_1}, \quad \Gamma^1_{12} = 0, \quad \Gamma^1_{22} = 0.$$

Noting, that from the tensorial transformation law of the propagator and the relation (3.8) the ambiguity of the propagator $\tilde{f}$ has to transform as $\tilde{f}^i_{jk}(x) = \frac{\partial x_i}{\partial z_l}(\frac{\partial^2 z_l}{\partial x_j \partial x_k}) + \frac{\partial x_i}{\partial z_l}\frac{\partial z_m}{\partial x_j}\frac{\partial z_n}{\partial x_k}\tilde{f}^l_{mn}(x(z))$. We obtain

$$\tilde{f}^1_{11} = -\frac{1}{x_1}, \; \tilde{f}^2_{12} = -\frac{1}{4x_1}, \; \tilde{f}^2_{22} = -\frac{3}{2x_2}, \tag{5.14}$$

where all other combinations are either 0 or follow by symmetry. As $\Gamma^1_{ij} = -\tilde{f}^1_{ij}$ we observe that the propagator takes the following simple form $S^{11} = S^{12} = S^{21} = 0$ and $S^{22} = \frac{x_2^2}{2} - 2x_2^3 - 2x_1 x_2^3 + \mathcal{O}(x^5)$.

In addition, we fix the holomorphic ambiguity of the covariant derivative of S^{ij}, (3.7), and obtain

$$\begin{aligned} f_1^{11} &= -\tfrac{1}{8}z_1(1+4z_1-4z_2),\ f_1^{12} = -\tfrac{1}{8}z_2(1+4z_1-4z_2),\ f_1^{22} = -\tfrac{z_2^2}{8z_1}(1+4z_1-4z_2), \\ f_2^{11} &= -\tfrac{z_1^2}{8z_2}(1+4z_2-4z_1),\ f_2^{12} = -\tfrac{1}{8}z_1(1+4z_2-4z_1),\ f_2^{22} = -\tfrac{1}{8}z_2(1+4z_2-4z_1), \end{aligned} \tag{5.15}$$

where all other combinations follow by symmetry. Further we can express the covariant derivative of F_1 through the generator S (3.9) by

$$D_iF_1 = \frac{1}{2}C_{ijk}S^{jk} - \frac{1}{12}\Delta^{-1}\partial_i\Delta + \frac{7}{24z_i}. \tag{5.16}$$

Note, that in contrast to an one parameter model like in section 4 the holomorphic ambiguity $A_i = \partial_i(\tilde{a}_j \log\Delta_j + \tilde{b}_j \log z_j)$ in (5.16) cannot be set to zero. More generally, in the local models we are considering here the geometry of the B-model is encoded in a Riemann surface of genus one whose moduli space admits only one quasimodular form of weight 2, namely the second Eisenstein series. Therefore and from the discussions in the case of local $\mathbb{P}^2$ in the previous section we expect there to be a coordinate system in which the propagator is proportional to the second Eisenstein series. The relevant coordinate system is given by the x-coordinates in which it is allowed to set all but one component of the propagator to zero and subsequently one can use (3.9) and (3.1) to solve for this non-zero component. Now, in the multi-parameter case this gives, for each direction of the derivative of F_1 w.r.t. z_i, $h^{2,1}$ equations on $\tilde{a}_j, \tilde{b}_j$. In this and the following example, we are lucky as these constraints fix the parameters completely. In addition one arrives at a series expansion for the non-vanishing component of S^{ij}. This can be used to fix all ambiguities in the model as rational functions of the z_i with poles only at the singular divisors of the Picard-Fuchs system.

Now, all input to perform direct integration is provided and applying this method we are able to determine F_g for genus g up to four. Using that local $\mathbb{F}_0$ has a discriminant with $\deg\Delta = 2$ and we can further reduce the number of coefficients in A_g due to symmetry in z_1 and z_2, one can easily calculate, that at genus g there are $(2g-1)^2$ unknowns in the holomorphic ambiguity. Therefore genus four corresponds to fixing 49 coefficients in the holomorphic ambiguity $f_g = \frac{A_g}{\Delta^{2g-2}}$. They are determined by the gap condition at the conifold locus and the known constant map contributions. We will further comment on this in the next section.

Let's present at least the genus two results. The free energy is given by

$$\begin{aligned} F_2 =& \frac{5}{24z_1^6\Delta^2}S^3 + \frac{-13+48z_1^2+z_1(40-96z_2)+40z_2+48z_2^2}{48z_1^4\Delta^2}S^2 \\ &+ \frac{384z_1^3+z_1^2(80-384z_2)+(1-4z_2)^2(17+24z_2)-16z_1(7-46z_2+24z_2^2)}{144z_1^2\Delta^2}S+f_2, \end{aligned} \tag{5.17}$$

where the ambiguity $f_2 = \frac{A_2}{\Delta^2}$ is fixed by the following choice

$$A_2 = -\frac{1}{1440}(25-258z_1+696z_1^2+416z_1^3-2688z_1^4-258z_2+2768z_1z_2-6560z_1^2z_2-1536z_1^3z_2 \\ +696z_2^2-6560z_1z_2^2+8448z_1^2z_2^2+416z_2^3-1536z_1z_2^3-2688z_2^4). \tag{5.18}$$

The solution around the conifold is described in the next section. The GV invariants can be found in the appendix B. They are in accord with [2] as far as they have been computed.

5.3 Solving the topological string on local $\mathbb{F}_0$ at the conifold locus

Our next task is to solve the Picard-Fuchs equations around the conifold locus. In order to do that we choose some convenient point on the locus and define variables which are good coordinates around this point. In our case we choose the point to be $z_1 = \frac{1}{16}, z_2 = \frac{1}{16}$. As one can easily check inserting these numbers into the discriminant yields zero. To find the right variables we have to be careful as their gradients at the relevant point must not be colinear. The following choice will do the job

$$z_{c,1} = 1 - \frac{z_1}{z_2}, \quad z_{c,2} = 1 - \frac{z_2}{\frac{1}{8} - z_1}. \tag{5.19}$$

We transform the Picard-Fuchs system to the above coordinates and find the following polynomial solutions

$$\begin{aligned} \omega_0^c &= 1, \\ \omega_1^c &= -\log(1 - z_{c,1}), \\ \omega_2^c &= z_{c,2} + \frac{1}{16}(2z_{c,1}^2 + 8z_{c,1}z_{c,2} + 13z_{c,2}^2) + \mathcal{O}(z_c^3). \end{aligned} \tag{5.20}$$

As mirror coordinates we take $t_{c,1} := \omega_1^c$ and $t_{c,2} := \omega_2^c$. Inverting these series gives the following mirror map

$$\begin{aligned} z_{c,1}(t_{c,1}, t_{c,2}) &= 1 - e^{-t_{c,1}}, \\ z_{c,2}(t_{c,1}, t_{c,2}) &= t_{c,2} - \frac{1}{16}(t_{c,1}^2 + 8t_{c,1}t_{c,2} + 13t_{c,2}^2) + \mathcal{O}(t_c^5). \end{aligned} \tag{5.21}$$

The divisor $\{z_{c,1} = 0\}$ is normal to the conifold locus at $(z_1, z_2) = \left(\frac{1}{16}, \frac{1}{16}\right) = p_{\text{con}}$ whereas $\{z_{c,2} = 0\}$ is tangential (see figure 3). Therefore $z_{c,1}$ parameterizes the tangential direction to the conifold locus at p_{con} in moduli space and $z_{c,2}$ the normal one. Hence we expect the flat mirror coordinate $t_{c,2}$ to be controlling the size of the shrinking cycle at p_{con}, thus $t_{c,2}$ should appear in inverse powers in the expansion of the free energies.

Transforming the Yukawa couplings, the Christoffel symbols and the holomorphic ambiguities $\tilde{f}$ to the conifold coordinates we obtain the propagator around this locus. In the choice of our coordinates (5.19) the propagator takes the following simple form $S^{11} = S^{12} = S^{21} = 0$ and

$$S^{22} = \frac{1}{2}t_{c,2} + \frac{1}{1536}(24t_{c,1}^2 t_{c,2} + t_{c,2}^3) + \mathcal{O}(t_c^4).$$

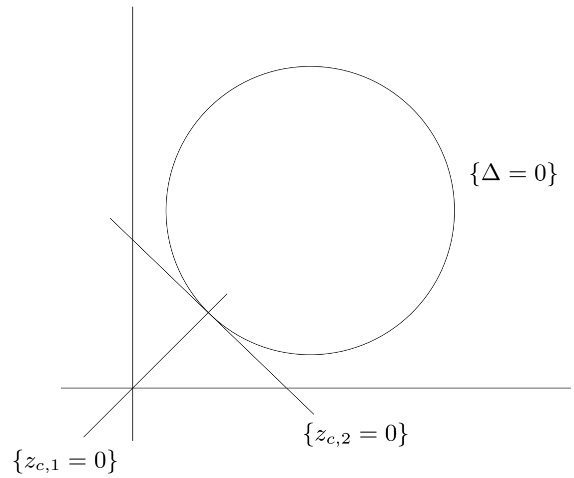

Figure 3: Conifold coordinates

Assuming the gap condition holds, we are able to fix all but one coefficients of the holomorphic ambiguity. Expanding the free energies at the large radius point in moduli space the constant map contribution fixes the last unknown, i.e. we observe that the gap condition yields at genus two 8 out of 9 unknowns, at genus three 24 out of 25 unknowns, etc. Our results up to genus four are given below (rescaling: $t_{c,2} \to 2t_{c,2}$)

$$\begin{aligned}
\mathcal{F}^c_2 &= -\frac{1}{240t^2_{c,2}} - \frac{1}{1152} + \frac{53t_{c,2}}{122880} + \frac{t^2_{c,1}}{61440} - \frac{2221t^2_{c,2}}{14745600} + \mathcal{O}(t^3_c) \\
\mathcal{F}^c_3 &= \frac{1}{1008t^4_{c,2}} + \frac{23}{5806080} + \frac{407t_{c,2}}{198180864} - \frac{t^2_{c,1}}{3096576} - \frac{258485t^2_{c,2}}{49941577728} + \mathcal{O}(t^3_c) \\
\mathcal{F}^c_4 &= -\frac{1}{1440t^6_{c,2}} - \frac{19}{278691840} + \frac{114773t_{c,2}}{362387865600} + \mathcal{O}(t^2_c).
\end{aligned} \tag{5.22}$$

5.4 Solving the topological string on local $\mathbb{F}_0$ at the orbifold point

As we have noted already there exists an orbifold point in the moduli space $\mathcal{M}$ at which we can compare our results with the known matrix model expansions.

At this point we expand the periods in the local variables

$$z_{o,1} = 1 - \frac{z_1}{z_2}, \quad z_{o,2} = \frac{1}{\sqrt{z_2}\left(1 - \frac{z_1}{z_2}\right)}. \tag{5.23}$$

Transforming the Picard-Fuchs system to these coordinates and solving it, we obtain the following set of periods

$$\begin{aligned}
\omega^o_0 &= 1, \\
\omega^o_1 &= -\log(1 - z_{o,1}), \\
\omega^o_2 &= z_{o,1}z_{o,2} + \frac{1}{4}z^2_{o,1}z_{o,2} + \frac{9}{64}z^3_{o,1}z_{o,2} + \mathcal{O}(z^5_o), \\
F^{(0)}_{\omega^o_2} &= \omega^o_2 \log(z_{o,1}) + \frac{1}{2}z^2_{o,1}z_{o,2} + \frac{21}{64}z^3_{o,1}z_{o,2} + \mathcal{O}({z_o}^5).
\end{aligned} \tag{5.24}$$

We define the mirror map to be given by the first two periods

$$t_{o,1} := \omega_1^o, \quad t_{o,2} := \omega_2^o, \tag{5.25}$$

and will express the B-model correlators in terms of these coordinates. In order to invert the mirror map and find the function $z_o(t_o)$, we have to consider the two series $\tilde{t}_{o,1} = t_{o,1} = z_{o,1} + 1 + \mathcal{O}(z_o{}^2)$ and $\tilde{t}_{o,2} = \frac{t_{o,2}}{t_{o,1}} = z_{o,2} + \mathcal{O}(z_o^2)$. Inverting these we obtain

$$\begin{aligned} z_{o,1}(\tilde{t}_{o,1}) &= 1 - e^{-\tilde{t}_{o,1}}, \\ z_{o,2}(\tilde{t}_{o,1}, \tilde{t}_{o,2}) &= \tilde{t}_{o,2} + \frac{1}{4}\tilde{t}_{o,1}\tilde{t}_{o,2} + \frac{1}{192}\tilde{t}_{o,1}^2\tilde{t}_{o,2} - \frac{1}{256}\tilde{t}_{o,1}^3\tilde{t}_{o,2} + \mathcal{O}(\tilde{t_o}^5), \end{aligned} \tag{5.26}$$

which together form the mirror map at the orbifold point in moduli space.

Transforming the Yukawa couplings, the Christoffel symbols and the holomorphic ambiguities $\tilde{f}$ to the orbifold coordinates we obtain the propagator around this locus. In the choice of our coordinates (5.23) the propagator takes the following simple form $S^{11} = S^{12} = S^{21} = 0$ and

$$S^{22} = \frac{1}{16}(t_{o,2}^2 - t_{o,1}^2) + \frac{1}{6144}(t_{o,1}^4 - 6t_{o,1}^2 t_{o,2}^2 + 5t_{o,2}^4) + \mathcal{O}(t_o^5).$$

In order to match the matrix model expansion one has to choose appropriate coordinates. As explained in [1] the right variables S_1, S_2 that match the 't Hooft parameters on the matrix model side are given by

$$S_1 = \frac{1}{4}(t_{o,1} + t_{o,2}),\ S_2 = \frac{1}{4}(t_{o,1} - t_{o,2}). \tag{5.27}$$

In addition the overall normalization of the all genus partition function $\mathcal{F} = \sum_g g_s^{2g-2}\mathcal{F}_g$ has to be determined. By comparing to the matrix model one gets, that the string coupling on the topological side, $g_s^{\rm top}$, is related to the coupling on the matrix model side, $\hat{g}_s$, by the identification $g_s^{\rm top} = 2i\hat{g}_s$. Using these expressions we find

$$\begin{aligned} \mathcal{F}_2^{\rm orb} &= -\frac{1}{240}\left(\frac{1}{S_1^2} + \frac{1}{S_2^2}\right) + \frac{1}{360} - \frac{1}{57600}(S_1^2 + 60S_1S_2 + S_2^2) + \mathcal{O}(S^4) \\ \mathcal{F}_3^{\rm orb} &= \frac{1}{1008}\left(\frac{1}{S_1^4} + \frac{1}{S_2^4}\right) + \frac{1}{22680} + \frac{1}{34836480}(S_1^2 - 252S_1S_2 + S_2^2) + \mathcal{O}(S^4) \\ \mathcal{F}_4^{\rm orb} &= -\frac{1}{1440}\left(\frac{1}{S_1^6} + \frac{1}{S_2^6}\right) + \frac{1}{340200} - \frac{1}{82944000}(S_1^2 + 102S_1S_2 + S_2^2) + \mathcal{O}(S^4). \end{aligned} \tag{5.28}$$

The genus two results are in accord with [1], genus three corrects the misprints in this article and genus four is a prediction on the matrix model.

5.5 Relation to the family of elliptic curves

At the beginning of this section we pointed out, that $H(x, y) = 0$ defines a family of elliptic curves $\Sigma(z_1, z_2)$ whose j-function is given by

$$j(z_1, z_2) = \frac{((1 - 4z_1 - 4z_2)^2 - 48z_1z_2)^3}{z_1^2 z_2^2(1 - 8(z_1 + z_2) + 16(z_1 - z_2)^2)}. \tag{5.29}$$

Using the usual j-function description (A.11) one can establish a relation between the elliptic parameter $q = e^{2\pi i\tau}$ and the complex structure variables z_1 and z_2 which reads

$$q = z_1^2 z_2^2 + 16 z_1^3 z_2^2 + 160 z_1^4 z_2^2 + 16 z_1^2 z_2^3 + 400 z_1^3 z_2^3 + 160 z_1^2 z_2^4 + \mathcal{O}(z^7). \tag{5.30}$$

We observe that

$$\tau = 4\partial_{t_{x,2}}\partial_{t_{x,2}}\mathcal{F}_0, \quad \partial_{t_{x,2}}\tau = -4C_{t_{x,2}t_{x,2}t_{x,2}}, \tag{5.31}$$

where $t_{x,i}$ is obtained from $Q_i^x = e^{2\pi i t_{x,i}}$, which hints at that the not instanton corrected parameter x_1 or Q_1^x, respectively, is merely an auxiliary parameter. [3] work with an isogenous description of $\Sigma(z_1, z_2)$. They use the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ given by the map

$$([x_0 : x_1], [x'_0 : x'_1]) \mapsto [X_0 : X_1 : X_2 : X_3] = [x_0 x'_0, x_1 x'_0, x_0 x'_1, x_1 x'_1], \tag{5.32}$$

where $[x_0 : x_1]$ and $[x'_0 : x'_1]$ are homogeneous coordinates of the $\mathbb{P}^1$'s and $X_0, \ldots, X_3$ are homogeneous coordinates of $\mathbb{P}^3$. Then $\tilde{\Sigma}(\tilde{z}_1, \tilde{z}_2)$ is given by the complete intersection of $\mathbb{P}^1 \times \mathbb{P}^1$, defined by $X_0X_3 - X_1X_2$, with the hypersurface given by $X_0^2 + \tilde{z}_1 X_1^2 + X_2^2 + \tilde{z}_2 X_3^2 + X_0 X_3$. Its j-function reads

$$\tilde{\jmath}(\tilde{z}_1, \tilde{z}_2) = \frac{((1 - 4\tilde{z}_1 - 4\tilde{z}_2)^2 + 192\tilde{z}_1\tilde{z}_2)^3}{\tilde{z}_1\tilde{z}_2(1 - 8(\tilde{z}_1 + \tilde{z}_2) + 16(\tilde{z}_1 - \tilde{z}_2)^2)^2}. \tag{5.33}$$

Defining $\tilde{q} = e^{2\pi i\tilde{\tau}}$ we can calculate that $\tilde{\tau} = \partial_{t_{x,2}}\partial_{t_{x,2}}\mathcal{F}_0$, i.e. their modular parameters are related by a simple rescaling by a factor of 4

$$\tau = 4\tilde{\tau}. \tag{5.34}$$

This transfers to a rescaling of the periods of the elliptic curve, similar to the discussion in section 4.4.

With this input it is possible to write the full non-holomorphic F_1 as

$$F_1 = -\log\sqrt{\tilde{\tau}_2}\,\eta(\tilde{\tau})\bar{\eta}(\bar{\tilde{\tau}}) \tag{5.35}$$

6. $\mathbb{K}_{\mathbb{F}_1} = \mathcal{O}(-2,-3) \to \mathbb{F}_1$

We are considering the non-compact Calabi-Yau geometry $\mathcal{O}(-2,-3) \to \mathbb{F}_1$, i.e. the canonical line bundle over the Hirzebruch surface $\mathbb{F}_1 = \mathbb{BP}_1^2$, where $\mathbb{BP}_1^2$ denotes the first del Pezzo surface, i.e. $\mathbb{P}^2$ with one blow up. This local model can be obtained again from the compact elliptic fibration over $\mathbb{F}_1$ with fiber $X_6(1,2,3)$. The three complexified Kähler volumes have the corresponding Mori cone generators $(-6; 3, 2, 1, 0, 0, 0, 0), (0; 0, 0, -1, 1, -1, 1, 0), (0; 0, 0, -2, 0, 1, 0, 1)$.

A Picard-Fuchs system governing the periods of the global mirror geometry is given by

$$\begin{aligned}
\mathcal{D}_1 &= \Theta_1(\Theta_1 - 2\Theta_2 - \Theta_3) - 18z_1(1 + 6\Theta_1)(5 + 6\Theta_1) \\
\mathcal{D}_2 &= \Theta_2(\Theta_2 - \Theta_3) - z_2(-1 + \Theta_1 - 2\Theta_2 - \Theta_3)(\Theta_1 - 2\Theta_2 - \Theta_3) \\
\mathcal{D}_3 &= \Theta_3^2 - z_3(\Theta_1 - 2\Theta_2 - \Theta_3)(\Theta_2 - \Theta_3).
\end{aligned} \tag{6.1}$$

Now let us turn to the non-compact geometry. The toric data of local $\mathbb{F}_1$ is summarized in the following matrix

$$(V|Q) = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & -2 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right). \tag{6.2}$$

From there we conclude the following quantities[14]

$$\begin{aligned}
&a) && Q^1 = (-2,1,0,1,0),\ Q^2 = (-1,0,1,-1,1) \\
&b) && Z = \{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\} \\
&c) && M = (\mathbb{C}^5[x_0,\ldots,x_4] \setminus Z)/(\mathbb{C}^*)^2 \\
&d) && H(x,y) = y^2 - x^3 - (1-4z_1)x^2 + 8z_1z_2x - 16z_1^2z_2^2 \\
&e) && \mathcal{D}_1 = \Theta_1(\Theta_1 - \Theta_2) - z_1(2\Theta_1 + \Theta_2)(1 + 2\Theta_1 + 2\Theta_2) \\
& && \mathcal{D}_2 = \Theta_2^2 - z_2(\Theta_2 - \Theta_1)(2\Theta_1 + \Theta_2) \\
& && \Delta = (1-4z_1)^2 - z_2(1 - 36z_1 + 27z_1z_2) \\
&f) && C^{(0)}_{111} = -\frac{1}{3},\ C^{(0)}_{112} = -\frac{1}{3},\ C^{(0)}_{122} = -\frac{1}{3},\ C^{(0)}_{222} = \frac{2}{3} \\
&g) && \int_M c_2J_1 = -2,\ \int_M c_2J_2 = 0.
\end{aligned} \tag{6.3}$$

$H(x,y) = 0$ defines a family of elliptic curves $\Sigma(z_1,z_2)$ whose j-function is given by

$$j(z_1,z_2) = \frac{((1-4z_1)^2 + 24z_1z_2)^3}{z_1^3z_2^2((1-4z_1)^2 - z_2(1-36z_1+27z_1z_2))}. \tag{6.4}$$

6.1 Solving the topological string on local $\mathbb{F}_1$ at large radius

The mirror map at the point of large radius is given by

$$\begin{aligned}
2\pi i T_1(z_1,z_2) &= \log z_1 + 2z_1 + 3z_1^2 - 4z_1z_2 + \frac{20}{3}z_1^3 + 24z_1^2z_2 + O(z^4) \\
2\pi i T_2(z_1,z_2) &= \log z_2 + z_1 + \frac{3}{2}z_1^2 - 2z_1z_2 + \frac{10}{3}z_1^3 + -12z_1^2z_2 + \mathcal{O}(z^4).
\end{aligned} \tag{6.5}$$

Inverting the series we obtain for $Q_i = e^{2\pi i T_i}$

$$\begin{aligned}
z_1(Q_1,Q_2) &= Q_1 - 2Q_1^2 + 3Q_1^3 + 4Q_1^2Q_2 - 4(Q_1^4 + Q_1^3Q_2) + \mathcal{O}(Q^5) \\
z_2(Q_1,Q_2) &= Q_2 - Q_1Q_2 + Q_1^2Q_2 + 2Q_1Q_2^2 - Q_1^3Q_2 + \mathcal{O}(Q^5).
\end{aligned} \tag{6.6}$$

Now, one realizes again that there is a relation between the Q coordinates:

$$\frac{Q_1}{Q_2^2} = \frac{z_1}{z_2^2} = e^{2\pi i(T_1 - 2T_2)} =: Q_1^x. \tag{6.7}$$

[14] Using toric geometry it is only possible to determine an one-parameter family of classical intersection numbers $C^{(0)}_{ijk}$, resulting in an one-parameter family for $\int_M c_2J_i$. Their correct values are fixed by a limiting procedure of local $\mathbb{F}_1 = \mathbb{BP}_1^2$ to local $\mathbb{P}^2$ which is described below.

JHEP10(2008)097

Defining further $Q_2^x := Q_2$ and $x_1 = \frac{z_1}{z_2^2}$ as well as $x_2 = z_2$ one finds that

$$
\begin{aligned}
x_1(Q_1^x, Q_2^x) &= Q_1^x, \\
x_2(Q_1^x, Q_2^x) &= Q_2^x - Q_1^x Q_2^{x\,3} + 2Q_1^x Q_2^{x\,4} + \mathcal{O}(Q^6).
\end{aligned} \tag{6.8}
$$

The Yukawa couplings can be fixed through the relation $\partial_{T_i}\partial_{T_j}\partial_{T_k}\mathcal{F}_0 = C_{T_iT_jT_k}$ and the known genus zero GV invariants up to a dependence on one unfixed parameter. This unfixed parameter can be determined by the fact that there exists a limit of local $\mathbb{F}_1$ to local $\mathbb{P}^2$, as $\mathbb{F}_1 = \mathbb{BP}_1^2$. This blow-down limit can be seen by comparing the two j-functions (6.4), (4.3) and turns out to be

$$
z_1 \to 0, \text{ with } z_1 z_2 = z \text{ fixed.}
$$

We obtain the following Yukawa couplings

$$
\begin{aligned}
C_{111} &= \frac{-1 - 4z_1^2 + z_2 - z_1(7 - 6z_2)}{3z_1^3\Delta}, \qquad & C_{112} &= \frac{-1 + 8z_1^2 + z_2 + z_1(2 - 3z_2)}{3z_1^2 z_2\Delta}, \\
C_{122} &= \frac{z_2(1 - 12z_1) - (1 - 4z_1)^2}{3z_1 z_2^2\Delta}, \qquad & C_{222} &= \frac{2(1 - 4z_1)^2 + z_2(1 - 60z_1)}{3z_2^3\Delta}.
\end{aligned} \tag{6.9}
$$

The next step is to determine the propagators of local $\mathbb{F}_1$. This is best done in x coordinates, where one finds again that some Christoffel symbols are either trivial or have a rational form

$$
\Gamma_{11}^1 = -\frac{1}{x_1}, \quad \Gamma_{12}^1 = 0, \quad \Gamma_{22}^1 = 0. \tag{6.10}
$$

Choosing $\tilde{f}_{11}^1 = -\frac{1}{x_1}$, $\tilde{f}_{12}^1 = 0$, $\tilde{f}_{21}^1 = 0$, $\tilde{f}_{22}^1 = 0$, one finds from (3.8) that S^{11}, S^{12} are immediately zero. Demanding symmetry we are able to fix all ambiguities $\tilde{f}_{jk}^i$ by the choice

$$
\begin{aligned}
\tilde{f}_{11}^1 &= -\frac{1}{x_1}, \quad \tilde{f}_{11}^2 = -\frac{x_2}{12x_1^2\Delta_x}(1 - x_2 - 12x_1x_2^2 + 49x_1x_2^3 - 36x_1x_2^4 + 32x_1^2x_2^4 - 12x_1^2x_2^5), \\
\tilde{f}_{12}^2 &= -\frac{1}{12x_1\Delta_x}(3 - 3x_2 - 32x_1x_2^2 + 144x_1x_2^3 - 108x_1x_2^4 + 80x_1^2x_2^4), \\
\tilde{f}_{22}^2 &= -\frac{1}{12x_2\Delta_x}(20 - 21x_2 - 176x_1x_2^2 + 828x_1x_2^3 - 648x_1x_2^4 + 384x_1^2x_2^4),
\end{aligned} \tag{6.11}
$$

where Δ_x denotes the discriminant in x coordinates and all other combinations of $\tilde{f}_{jk}^i$ are either zero or follow by symmetry. This singles out one non-vanishing propagator only, given by $S^{22}(x_1, x_2) = \frac{x_2^2}{12} - \frac{1}{3}x_1x_2^4 + x_1x_2^5 + 4x_1^2x_2^7 + \mathcal{O}(x^{10})$. After tensor transforming to z coordinates we obtain

$$
S^{ij} = \begin{pmatrix} S(z_1, z_2) & \frac{z_2}{2z_1} S(z_1, z_2) \\ \frac{z_2}{2z_1} S(z_1, z_2) & \frac{z_2^2}{4z_1^2} S(z_1, z_2) \end{pmatrix}, \tag{6.12}
$$

where $S(z_1, z_2) = \frac{z_1^2}{3} - \frac{4z_1^3}{3} + 4z_1^3 z_2 + 16z_1^4 z_2 + \mathcal{O}(z^6)$. This again has a similar form as in the case of local $\mathbb{F}_0$.

In addition, we fix the holomorphic ambiguity of the covariant derivative of S^{ij}, (3.7), and obtain, that in x coordinates there are two non-zero contributions only, given by

$$\begin{aligned} f_1^{22} &= -\frac{x_2^2}{144x_1\Delta_x}(3-3x_2+4x_1x_2^2)(1-8x_1x_2^2+24x_1x_2^3+16x_1^2x_2^4), \\ f_2^{22} &= -\frac{x_2}{144\Delta_x}(8-9x_2)(1-8x_1x_2^2+24x_1x_2^3+16x_1^2x_2^4). \end{aligned} \tag{6.13}$$

The f_i^{jk} in z coordinates are again obtained after tensor transformation.

Further we can express the covariant derivative of F_1 through the generator S by

$$D_iF_1 = \frac{1}{2}C_{ijk}S^{jk} + A_i. \tag{6.14}$$

As the free energy function of genus one is given by

$$\begin{aligned} F_1 &= \log\left(\Delta^{-\frac{1}{12}} z_1^{-\frac{7}{12}} z_2^{-\frac{1}{2}} \det(G_{i\bar{j}})^{-\frac{1}{2}}\right), \\ \mathcal{F}_1(T_1,T_2) &= -\frac{1}{12}\log(Q_1) - \frac{1}{12}(2Q_1+Q_2) - \frac{1}{24}(2Q_1^2+6Q_1Q_2+Q_2^2) + \mathcal{O}(Q^3), \end{aligned} \tag{6.15}$$

we find that $A_i = \partial_i A$ and

$$A = -\frac{1}{24}\log\Delta + \frac{1}{24}\log z_1 + \frac{1}{12}\log z_2. \tag{6.16}$$

Now, we are prepared to perform the direct integration procedure. Demanding the gap at the conifold and using further the known constant map contributions we are able to fix the ambiguities up to genus three. In this more general two parameter model with one discriminant component of degree three the number of coefficients in A_g is

$$\binom{(2g-2)\deg\Delta+2}{2} = 10 - 27g + 18g^2, \tag{6.17}$$

i.e. at genus three we have to fix 91 coefficients in the holomorphic ambiguity.

The invariants can be found in the appendix B. The solutions around the conifold locus are described in the next section.

6.2 Solving the topological string on local $\mathbb{F}_1$ at the conifold locus

In order to apply the gap condition in this example, we have to transform and solve the Picard-Fuchs system at a specific point on the conifold locus. We make the choice $z_1 = 2$, $z_2 = -\frac{1}{2}$. Again we define two variables which vanish at this point

$$z_{c,1} = 1 - \frac{z_2}{-\frac{1}{4}(z_1-2)-\frac{1}{2}}, \quad z_{c,2} = 1 - \frac{z_2}{4(z_1-2)-\frac{1}{2}}. \tag{6.18}$$

$z_{c,1}$ is a coordinate normal to the conifold divisor and $z_{c,2}$ describes a tangential direction. Transforming the Picard-Fuchs system to these coordinates we find the following set of periods:

$$\omega_0^c = 1,$$

$$\omega_1^c = z_{c,1} + \frac{6773 z_{c,1}^2}{14450} - \frac{58 z_{c,1} z_{c,2}}{7225} - \frac{z_{c,2}^2}{1445} + \mathcal{O}(z_c^3),$$
$$\omega_2^c = z_{c,2} + \frac{10858 z_{c,1}^2}{7225} + \frac{2871 z_{c,2}^2}{2890} - \frac{4886 z_{c,1} z_{c,2}}{7225} + \mathcal{O}(z_c^3). \tag{6.19}$$

Next, we can express the $z_{c,i}$ through the mirror coordinates $t_{c,1} := \omega_1^c$ and $t_{c,2} := \omega_2^c$ by inverting the above series

$$z_{c,1}(t_{c,1}, t_{c,2}) = t_{c,1} - \frac{6773 t_{c,1}^2}{14450} + \frac{58 t_{c,1} t_{c,2}}{7225} + \frac{t_{c,2}^2}{1445} + \mathcal{O}(t_c^3),$$
$$z_{c,2}(t_{c,1}, t_{c,2}) = t_{c,2} - \frac{10858 t_{c,1}^2}{7225} + \frac{4886 t_{c,1} t_{c,2}}{7225} - \frac{2871 t_{c,2}^2}{2890} + \mathcal{O}(t_c^3). \tag{6.20}$$

Transforming the Yukawa couplings, the Christoffel symbols and the holomorphic ambiguities $\tilde{f}$ to the conifold coordinates we obtain the propagator around this locus. In the choice of our coordinates the propagator takes the following form

$$S^{11} = \frac{5}{12} - \frac{2 t_{c,1}}{25} - \frac{337 t_{c,1}^2}{10625} - \frac{4 t_{c,1} t_{c,2}}{2125} + \mathcal{O}(t_c^3),$$
$$S^{12} = -\frac{55}{4} + \frac{66 t_{c,1}}{25} + \frac{11121 t_{c,1}^2}{10625} + \frac{132 t_{c,1} t_{c,2}}{2125} + \mathcal{O}(t_c^3), \tag{6.21}$$
$$S^{22} = \frac{1815}{4} - \frac{2178 t_{c,1}}{25} - \frac{366993 t_{c,1}^2}{10625} - \frac{4356 t_{c,1} t_{c,2}}{2125} + \mathcal{O}(t_c^3).$$

Again the gap condition in combination with the known leading behavior at the large radius point suffices to fix all coefficients in the holomorphic ambiguity. From the conifold alone we get at genus two 27 out of 28 unknowns and at genus three 90 out of 91 unknowns. Our results read

$$\mathcal{F}_2^c = \frac{1}{48 t_{c,1}^2} + \frac{1567}{9000000} + \frac{98333}{1593750000} t_{c,1} - \frac{123}{10625000} t_{c,2} + \mathcal{O}(t_c^2)$$
$$\mathcal{F}_3^c = \frac{25}{1008 t_{c,1}^4} + \frac{480217}{283500000000} + \frac{106245283 t_{c,1}}{17929687500000} + \frac{69949 t_{c,2}}{167343750000} + \mathcal{O}(t_c^2). \tag{6.22}$$

6.3 Relation to the family of elliptic curves

Starting point is again the j-function of $\Sigma(z_1, z_2)$ which we will repeat here

$$j(z_1, z_2) = \frac{((1-4z_1)^2 + 24 z_1 z_2)^3}{z_1^3 z_2^2 ((1-4z_1)^2 - z_2(1 - 36 z_1 + 27 z_1 z_2))}. \tag{6.23}$$

Using again the usual j-function description (A.11) one can establish a relation between the elliptic parameter $q = e^{2\pi i \tau}$ and the complex structure variables z_1 and z_2 which reads

$$q = z_1^3 z_2^2 + 16 z_1^4 z_2^2 + 160 z_1^5 z_2^2 - z_1^3 z_2^3 - 60 z_1^4 z_2^3 + \mathcal{O}(z^8). \tag{6.24}$$

We observe that

$$\tau = \partial_{t_{x,2}} \partial_{t_{x,2}} F_0, \quad \partial_{t_{x,2}} \tau = -C_{t_{x,2} t_{x,2} t_{x,2}}, \tag{6.25}$$

where $t_{x,i}$ is obtained from $Q_i^x = e^{2\pi i t_{x,i}}$, which hints at that the not instanton corrected parameter x_1 or Q_1^x, respectively, is merely an auxiliary parameter. As in the previous cases it is possible to write the full non-holomorphic F_1 as

$$F_1 = -\log \sqrt{\tau_2}\eta(\tau)\bar{\eta}(\bar{\tau}) + A, \tag{6.26}$$

where A is given by (6.16).

7. Summary and further directions

In this article we find convincing evidence that closed topological string theories on non-compact Calabi-Yau spaces whose mirror can be reduced to Riemann surfaces is completely integrable using the holomorphic anomaly equation and the gap at the divisors at which a single cycle vanishes. The physical argument for the gap from the local form of the effective action in the presence of a single black hole hypermultiplet state that becomes massless at the nodal singularity [30] applies also after the decompactification limit. The massless hypermultiplet is now a dyonic hypermultiplet of a rigid 4d theory. This extends in particular to the geometric engineering limits, which leads to $N = 2$ supersymmetric gauge theories in $4d$. Indeed the gap was found in simple Seiberg-Witten theories [28] and it made the holomorphic anomaly equations integrable in these cases.

Generally there are two sorts of parameters associated to the geometry (Σ_g, λ). There are r parameters, which are given by periods over $H^1(\Sigma_g)$. The monodromy acts on them and T duality requires that their occurrence in higher genus amplitudes is organized in terms of almost holomorphic modular forms, which correspond to non-trivial components of the propagators S^{ij}. Further there might be m parameters encoding the non-vanishing residua of the meromorphic form λ. The monodromy acts trivially on them. In mathematics they are referred to as isomonodromic deformations. We find that they occur in rational expressions in the amplitudes.

In Seiberg-Witten theory the r parameters correspond to the number of U(1) vector multiplets in the Coulomb phase, while m parameters are the masses of perturbative hypermultiplets. Similar del Pezzo surfaces with $1 + m$ Kähler parameters have genus one mirror curves and we could identify the one parameter that corresponds to an integral over $H^1(\Sigma_1)$ and the m residue parameters by choosing a parameterization in which we have only one non-trivial propagator. In all cases we found by a local analysis of the gap condition near the discriminant components with single vanishing cycle that there are sufficiently many conditions to solve the theory. For Seiberg-Witten theories with matter fields this has been established in [29].

In recent years strong relations between topological string theory on local Calabi-Yau manifolds and matrix models and other integrable structures such as Chern-Simons theory have been discovered. These developments have been excellently reviewed in [37, 40].

In particular [11, 15] show that rigid special geometry, which is essential in making the ring of the propagators close under derivatives (section 3), is an intrinsic property of the multi cut matrix model if the filling fractions are considered as parameters. Further it was argued in [18] that the method of solving the recursive loop equation using the Bergman kernel and the kernel differentials of [17] can be made modular by adding a non-holomorphic modular completion to the Bergmann kernel. It was further shown in [18] that this completion makes the formalism of [17] compatible with the holomorphic anomaly equation. The modular property has not yet been derived within the matrix model. In fact the analysis of [18] is inspired by the way modularity is realized in the higher genus expansion of topological string theory on non-compact Calabi-Yau and Seiberg-Witten theory [3, 28], where T or S duality is an intrinsic property. In any case it is clear that the matrix model correlation functions in the $\frac{1}{N^2}$ expansions fulfill the holomorphic anomaly equations. Moreover [38] applies the formalism of [17] to local mirror curves and successfully checks expansions of closed and open low genus amplitudes large against A-model calculations. This leads to the expectation that the F_g for many multi-cut matrix models are solvable using the modular properties of the spectral curve and the gap condition.

To summarize we have good evidence that the holomorphic anomaly equation and the gap conditions solve the closed amplitudes for the following cases: non-compact Calabi-Yau with mirror curves, Seiberg-Witten theories and for many multi cut matrix models. What makes the claim plausible in general is that the Riemann surfaces have in the co-dimension one locus in the moduli space just one type of degeneration, the nodal degeneration, which exhibits as local property the gap behavior. E.g. $\mathrm{SU}(N)$ theories can be degenerated to $\mathrm{SU}(N_1)\times\ldots\times\mathrm{SU}(N_k)$ theories, with $\sum_{i=1}^k N_i = N$ by stretching higher genus components of the curve apart. Such operations can not affect the local leading behavior of F_g near the pinching cycles and for $N_i = 2$ the gap is established [28].

Due to a more extensive use of the symmetry the method outlined here is more efficient then any other to calculate the F_g for high g and provides global expressions instead of local expansions. Combined with numerical analysis of asymptotic expansions this has applications in investigations of non-perturbative completions of topological string theory [39, 19]. Understanding the role of holomorphicity and modularity, which are the basis of our approach, could give decisive hints for such completions.

One might further speculate that the approach extends to open strings. The open string version of the holomorphic anomaly equation in the presence of open string moduli has yet some problems [18].[15] The open string variables are not subject to modular transformations and in this sense similar to the m residue parameters. But in the open case we have so far not understood how to provide enough boundary conditions to make the holomorphic anomaly approach completely integrable. For the open string on compact Calabi-Yau spaces without open string moduli no particular structure has been found at the boundary of the closed string moduli space [42].

Extracting the full constraints from the local analysis of the multi parameter gap condition is also relevant to multi parameter global Calabi-Yau spaces and could lead to integrability of these systems. Different then in the one parameter cases where the situation has been analyzed in [30, 26, 23] one can employ here further known limits such as the large base limit in K3 fibrations, in which formulas for the all genus generating functions of GW invariants have been mathematically rigorously established in [36].

[15] We thank Marcos Mariño for a discussion on the issue.

Acknowledgments

We like to thank Vincent Bouchard, Minxin Huang, Thomas Grimm, Rainald Flume, Marcos Mariño, Sara Pasquetti, Piotr Sulkowski and S.T. Yau for discussions.

The work of B.H. and M.R. is supported by the German Excellence Initiative via the graduate school BCGS. A.K. is supported by DOE grant DE-FG02-95ER40896.

A. Modular anomaly versus holomorphic anomaly

Physically the amplitudes F_g of the topological string are invariant under the space-time modular group Γ of the target space. This is the most important restriction on these functions. The nicest case is when the B-model geometry is a family of elliptic curves. Then Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and the classical theory of modular forms applies. We will recapitulate below the relevant aspects of $\mathrm{SL}(2,\mathbb{Z})$ almost holomorphic modular forms. This gives some insight in the interplay between the breaking of the modularity and the breaking of holomorphicity. The different modular forms that we need for the general families of elliptic curves, i.e. general two cut matrix models, follow from the Picard-Fuchs equations. The relation between the Picard-Fuchs equations and modular forms is again a classical subject, which has been beautifully reviewed in [45].

A.1 $\mathrm{PSL}(2,\mathbb{Z})$ modular forms

We define $q := e^{2\pi i\tau}$, with $\tau \in \mathbb{H}_+ = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) = \frac{1}{2i}(\tau - \bar\tau) > 0\}$ and the projective action $\mathrm{PSL}(2,\mathbb{Z})$ of $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, ad - bc = 1,\ a,b,c,d \in \mathbb{Z} \right\}$ on $\mathbb{H}_+$ by

$$\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \tag{A.1}$$

for $\gamma \in \Gamma_1$. It follows that

$$\frac{1}{\mathrm{Im}(\tau_\gamma)} = \frac{(c\tau + d)^2}{\mathrm{Im}(\tau)} - 2ic(c\tau + d) = \frac{|c\tau + d|^2}{\mathrm{Im}(\tau)}\ . \tag{A.2}$$

Modular forms of Γ_1 transform as

$$f_k(\tau_\gamma) = (c\tau + d)^k f_k(\tau) \tag{A.3}$$

with weight $k \in \mathbb{Z}$ for all $\tau \in \mathbb{H}_+$ and $\gamma \in \Gamma_1$, are meromorphic for $\tau \in \mathbb{H}_+$ and grow like $\mathcal{O}(e^{C\mathrm{Im}(\tau)})$ for $\mathrm{Im}(\tau) \to \infty$ and $\mathcal{O}(e^{C/\mathrm{Im}(\tau)})$ for $\mathrm{Im}(\tau) \to 0$ with $C > 0$. A strategy to build modular forms of weight k is to sum over orbits of Γ_1

$$G_k = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}\ . \tag{A.4}$$

It is easy to see that this expression transforms like (A.3), converges absolutely for $k > 2$ and vanishes for k odd. In the standard definition of the Eisenstein series E_k the sum runs over coprime (m,n), which yields a proportionality $G_k(\tau) = \zeta(k)E_k(\tau)$, where $\zeta(k) = \sum_{n\geq 1}\frac{1}{n^k}$. One shows ([45]) the central fact that E_4, E_6 (or G_4,G_6 of course) generate freely the graded (by k) ring of modular forms $\mathcal{M}_*(\Gamma_1)$.

Still one may spot two shortcomings. Firstly the ring $\mathcal{M}_*(\Gamma_1)$ does not close under any differentiation and secondly there should be a modular form for weight 2. These facts are related as $d_\tau = \frac{d}{2\pi i d\tau}$ has weight 2. The second is remedied by an ϵ regularization in the sum $G_{2,\epsilon} = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^k|m\tau+n|^\epsilon}$ after which it is possible to define $G_2 = \lim_{\epsilon\to 0} G_{2,\epsilon}$. Then all G_k, $k\in 2\mathbb{Z}$, $k\geq 2$ have a Fourier expansion[16] in $q = \exp(2\pi i\tau)$

$$G_k(\tau) = \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right), \tag{A.5}$$

with $\sigma_k(n) = \sum_{p|n} p^k$ the sum of kth powers of positive divisors of n and $\sum_{k=0}^{\infty}\frac{B_k x^k}{k!} = \frac{x}{e^x-1}$ defining the Bernoulli numbers B_k, e.g. $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}$ etc.

Very much like in QFT the regularization introduces an anomaly in the symmetry transformation so that E_2 transforms

$$E_2(\tau_\gamma) = (c\tau+d)^2E_2(\tau) - \frac{6ic}{\pi}(c\tau+d) \tag{A.6}$$

with an inhomogeneous term.

At least (E_2, E_4, E_6) form a ring, the ring of quasi modular holomorphic forms $\mathcal{M}^!$, which closes under differentiation, i.e.

$$d_\tau E_2 = \frac{1}{12}(E_2^2 - E_4),\quad d_\tau E_4 = \frac{1}{3}(E_2E_4 - E_6),\quad d_\tau E_6 = \frac{1}{2}(E_2E_6 - E_4^2)\ . \tag{A.7}$$

Using (A.2) and (A.6) we see that the inhomogeneous terms in (A.2), (A.6) cancel so that

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\mathrm{Im}(\tau)} \tag{A.8}$$

transforms like a modular form of weight 2, albeit not a holomorphic one. $(\hat{E}_2, E_4, E_6)$ form the ring of almost holomorphic modular forms of Γ_1. The latter closes under the Maass derivative, which acts on forms of weight k by

$$D_\tau f_k = \left(d_\tau - \frac{k}{4\pi\mathrm{Im}(\tau)}\right)f_k \tag{A.9}$$

and maps $D_\tau : \mathcal{M}^!_k \to \mathcal{M}^!_{k+2}$. Note that the equations (A.7) hold with d_τ replaced by D_τ and $E_2(\tau)$ replaced by $\hat{E}_2(\tau)$. This Maass derivative corresponds to the covariant derivative that appears in topological string theory (3.2).

From the physical point of view there seems the following story behind these well known mathematical facts. The holomorphic propagator, which can be made proportional to E_2, see (4.47) needs some regularization, which breaks T duality. The latter is restored

[16]Note that the Eisenstein series start with coefficient 1.

by adding the non-holomorphic term (A.8). The modular anomaly and the holomorphic anomaly are in this sense counterparts, which cannot both be realized at least perturbatively. T-duality is physically better motivated. Attempts in the literature, e.g. in an interesting paper [19], to define a holomorphic and modular non-perturbative completion by summing over orbits seem to make sense only if absolute convergence in the moduli is established, which is hard.

F_1 is an index, which is finite for smooth compact spaces. It diverges therefore only from singular configurations, that occur if e.g. the discriminant of the elliptic curve given below for the Weierstrass form $y^2 = 4x^3 - 3xE_4 + E_6$

$$\Delta(\tau) = \eta^{24}(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{1}{1728}(E_4^3(\tau) - E_6^2(\tau))\,, \tag{A.10}$$

vanishes. Note that the j for this curve is

$$j = 1728\frac{E_4^2}{E_4^3 - E_6^2} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \mathcal{O}(q^3)\ . \tag{A.11}$$

It follows from (A.3) that $\eta(\tau_\gamma) = (c\tau + d)^{\frac{1}{2}}\eta(\tau)$ transforms with weight $\frac{1}{2}$ and from (A.7) that

$$d_\tau \log(\eta(\tau)) = \frac{1}{24}E_2(\tau). \tag{A.12}$$

Further from (A.2) we see that $\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2$ is an almost holomorphic modular invariant and from (A.7), (A.8), (A.10) that

$$d_\tau \log(\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2) = \frac{1}{24}\hat{E}_2(\tau). \tag{A.13}$$

We need also the theta functions of general characteristic

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+a)\tau(n+a) + 2\pi i\sum_i (z+b)n\right)\ . \tag{A.14}$$

B. Gopakumar-Vafa invariants of local Calabi-Yau manifolds

$g\backslash d$	1	2	3	4	5	6	7	8	9	10	11	12	13
0	3	-6	27	-192	1695	-17064	188454	-2228160	27748899	-360012150	4827935937	-66537713520	938273463465
1	0	0	-10	231	-4452	80948	-1438086	25301295	-443384578	7760515332	-135854179422	2380305803719	-41756224045650
2	0	0	0	-102	5430	-194022	5784837	-155322234	3894455457	-93050366010	2145146041119	-48109281322212	1055620386953940
3	0	0	0	15	-3672	290853	-15363990	649358826	-23769907110	786400843911	-24130293606924	698473748830878	-19298221675559646
4	0	0	0	0	1386	-290400	29056614	-2003386626	109496290149	-5094944994204	210503102300868	-7935125096754762	278055282896359878
5	0	0	0	0	-270	196857	-40492272	4741754985	-396521732268	26383404443193	-1485630816648252	73613315148586317	-3295843339183602162
6	0	0	0	0	21	-90390	42297741	-8802201084	1156156082181	-111935744536416	8698748079113310	-572001241783007370	32970159716836634586
7	0	0	0	0	0	27538	-33388020	12991744968	-2756768768616	395499033672279	-42968546119317066	3786284014554551293	-283123099266200799858
8	0	0	0	0	0	-5310	19956294	-15382690248	5434042220973	-1177301126712306	181202644392392127	-21609631514881755756	2112545679539410950111
9	0	0	0	0	0	585	-9001908	14696175789	-8925467876838	2978210177817558	-658244675887405242	107311593188998164015	-13822514517126743782638
10	0	0	0	0	0	-28	3035271	-11368277886	12289618988434	-6445913624274390	2074294284130247058	-466990545532708577390	79879064190633923380059
11	0	0	0	0	0	0	-751218	7130565654	-14251504205448	12001782164043306	-5702866358492557440	1791208287019324701495	-410078597629344199822644
12	0	0	0	0	0	0	132201	-3624105918	13968129299517	-19310842755095748	13744538465609779287	-6085017394087513680618	1879279054884558476271255
13	0	0	0	0	0	0	-15636	1487970738	-11600960414160	26952467292328782	-29157942375100015002	18384612378910358924791	-7719669723503111567547498
14	0	0	0	0	0	0	1113	-490564242	8178041540439	-32736035592797946	54641056077839878893	-49578782776769125835658	28526676358086583457401470
15	0	0	0	0	0	0	-36	128595720	-4896802729542	34693175820656421	-90735478019244786786	119723947998685791289164	-95133281572651610511963924
16	0	0	0	0	0	0	0	-26398788	2489687953666	-32151370513161966	133885726253316075984	-259634731498425150837576	287135121651412378735811628
17	0	0	0	0	0	0	0	4146627	-1073258752968	26099440805196660	-175976406401479949154	506961721474582218552270	-786399027397491244523992902
18	0	0	0	0	0	0	0	-480636	391168899747	-18580932613650624	206477591201198965488	-893407075206205808615238	1959017333330728105822648251
19	0	0	0	0	0	0	0	38703	-120003463932	11609627766170547	-216671841840838260606	1424048002136300951108030	-4448639278908209789290494420
20	0	0	0	0	0	0	0	-1932	30788199027	-6367395873587820	203674311322868998065	-2057099617415644933602618	9227698060582367238347571297
21	0	0	0	0	0	0	0	45	-6546191256	3064262549419899	-171730940091766865658	2697839037217627321703085	-17516854338718408479048652494
22	0	0	0	0	0	0	0	0	1138978170	-1292593922494452	130015073789764141299	-3217397468483821476968358	30484235431876601864618838477
23	0	0	0	0	0	0	0	0	-159318126	477101143946277	-88451172530198637924	3494176460021369389735746	-48714141405866403558298334202
24	0	0	0	0	0	0	0	0	17465232	-153692555590206	54098277648908454123	-3460084190968494003073062	71589014392836043739746597686
25	0	0	0	0	0	0	0	0	-1444132	43057471189239	-29751302949160261398	3127576636374963802648718	-96883378729032302906983199856
26	0	0	0	0	0	0	0	0	84636	-10441089412308	14709694749741501501	-2582938330708242629937150	120896635270154811844637720853
27	0	0	0	0	0	0	0	0	-3132	2177999212647	-6535189635435373326	1950461493734929553600580	-139265452548367336541395204974
28	0	0	0	0	0	0	0	0	55	-387688567518	2606677300588276035	-1347524558332336039964082	148248962783129110225181956473
29	0	0	0	0	0	0	0	0	0	58269383541	-932238829973577348	852109374825775079556606	-145971211921687755538330192746
30	0	0	0	0	0	0	0	0	0	-7292193288	298408032566091294	-493309207337589509893062	133055268914412223044065820018
31	0	0	0	0	0	0	0	0	0	745600245	-85297647759486510	261477149328500781917776	-112357587854133668267639057304
32	0	0	0	0	0	0	0	0	0	-60650490	21708810999461607	-126876156355185161374314	87952573421916830793908406099
33	0	0	0	0	0	0	0	0	0	3773652	-4901354114590566	56339101711825399890960	-63854998146538947089287681014
34	0	0	0	0	0	0	0	0	0	-168606	977233475777499	-22881258328195868502320	43014954675567051362685843069
35	0	0	0	0	0	0	0	0	0	4815	-171090302865948	8492649924309368930964	-26893867445735937777389156538
36	0	0	0	0	0	0	0	0	0	-66	26117674453665	-2877665040430021956492	15609149489150170649459123934
37	0	0	0	0	0	0	0	0	0	0	-3445690553358	888968505074075552261	-8410678555930907126997555630
38	0	0	0	0	0	0	0	0	0	0	388460380746	-249952226921825722236	4207181054847947125893653841
39	0	0	0	0	0	0	0	0	0	0	-36878620320	63836429603183934921	-1953390408100284549295950018
40	0	0	0	0	0	0	0	0	0	0	2891025822	-14772524364719546808	841584918442722082197039960
41	0	0	0	0	0	0	0	0	0	0	-182125500	3088415413809592461	-336303963530686998053325696
42	0	0	0	0	0	0	0	0	0	0	8859513	-581271967556317272	124578181981904234839792755
43	0	0	0	0	0	0	0	0	0	0	-312270	98073062075574517	-42747487172239308320629266
44	0	0	0	0	0	0	0	0	0	0	7095	-14758388168491098	13575203399517277381780818
45	0	0	0	0	0	0	0	0	0	0	-78	1968679573589997	-3985442773959057781888308
46	0	0	0	0	0	0	0	0	0	0	0	-231043750764510	1080285938069626293744591
47	0	0	0	0	0	0	0	0	0	0	0	23635158339861	-269941588355351530486098
48	0	0	0	0	0	0	0	0	0	0	0	-2082988758060	62071685247348448583484
49	0	0	0	0	0	0	0	0	0	0	0	155790863415	-13107037881479259880974
50	0	0	0	0	0	0	0	0	0	0	0	-9693024822	2535413161347832616322
51	0	0	0	0	0	0	0	0	0	0	0	488072208	-448021340092704131004
52	0	0	0	0	0	0	0	0	0	0	0	-19105426	72081314665875044232
53	0	0	0	0	0	0	0	0	0	0	0	545391	-10518282775104442896
54	0	0	0	0	0	0	0	0	0	0	0	-10098	1385776784546520000
55	0	0	0	0	0	0	0	0	0	0	0	91	-163957628794736484
56	0	0	0	0	0	0	0	0	0	0	0	0	17308773135965754
57	0	0	0	0	0	0	0	0	0	0	0	0	-1617775223270352
58	0	0	0	0	0	0	0	0	0	0	0	0	132598956698970
59	0	0	0	0	0	0	0	0	0	0	0	0	-9417757882494
60	0	0	0	0	0	0	0	0	0	0	0	0	570827232216
61	0	0	0	0	0	0	0	0	0	0	0	0	-28937028858
62	0	0	0	0	0	0	0	0	0	0	0	0	1193305917
63	0	0	0	0	0	0	0	0	0	0	0	0	-38446296
64	0	0	0	0	0	0	0	0	0	0	0	0	907638
65	0	0	0	0	0	0	0	0	0	0	0	0	-13962
66	0	0	0	0	0	0	0	0	0	0	0	0	105

Table 2: BPS numbers n_d^g of local $\mathbb{K}_{\mathbb{P}^2}$

$g \backslash d$	14	15	16
0	-13491638200194	197287568723655	-2927443754647296
1	733512068799924	-12903696488738656	227321059950010137
2	-22755110195405850	483361869975894765	-10141562289815822472
3	513289541565539286	-13226687073790872894	331823525571283260201
4	-9179532480730484952	288379973286696180135	-8687442794831017531164
5	135875843241729533613	-5230662528295888702200	190039036692844531945431
6	-1707886552705077581628	80979854504456065293006	-3563867463166417898028954
7	18542695412600660315361	-1088520963699453440916068	58282787779310795828265801
8	-176025683917043985316470	12859768243573241278178232	-841384783491737401244384802
9	1474526726447064266472180	-134757205470641562616231254	10821001300311246341021564538
10	-10977980889990531917192040	1261570911839587494149142842	-124874019649583355894388594860
11	73064685775641172550245287	-10612768147751995597000536768	1300522099010448259785204366045
12	-436819481534388188001943032	80609689831056406245089196399	-12282259406284392700728072598380
13	2355513273192090467243746317	-555076023119960500416799982130	105610021148696431180837019630313
14	-11497443929941810386360760836	3477350942096950595265747641307	-829670278849892181781849621379508
15	50957816625388139826624170478	-19879905201401719017710643014418	5973078056543810630750683669041915
16	-205652305240430396439317210640	104002483609117190050345885357888	-39514272034437878290981964700219558
17	757662182424151902292508060760	-499131780135424315237215856091700	240783170189880817347584410692217926
18	-2554172799784278307930353343650	2202490064957497348139065195375923	-1354480673392411243817417189322344568
19	7895754493218057420092021506113	-8954639504409724407149809525209876	7048138037612506058374426122985053006
20	-22427232411211780101200732425704	33609442420420363371558079371477897	-33989638359010910511829498976232173802
21	58641982305309178343956442671761	-116665374813978856919164270653965004	152177429687299045634195140398757281504
22	-141401208897408849912677957634426	375171975689925627795105702569276418	-633577305407951231393255913015258866196
23	314936293736322225639438406290906	-1119499786780134169488432381035973630	2456788969008030141112263866239290403350
24	-648914458538905816592788630323204	3104428455059919596365239612226837893	-8885705515250117005267140410756908248060
25	1238729525477721787558439282945913	-8011735505069380039747056004195148158	30017610322127336868268653017585124628509
26	-2193714849283185866296047977715732	19268553790789301424928605404355434727	-94840749609277896052768672110491634949896
27	3608707818428642570478628800173229	-43242261816789265741007295810748532346	280605057707017406909127007390466194394946
28	-5520883353409528317073211897355024	90663853802203137933306954643315253691	-778393014388590417238764984731393282887020
29	7863819793987212697365737198659608	-177798466610751232952320725159422435082	2026759853430165190102329620422860687585627
30	-10439407430615613012302526201362232	326485907378338211346720861587117970456	-4958833886852841391769716597979254624540758
31	12928610180341150034322836488952889	-561942547080213731914225348713239621740	11412550296076971807748473068781297451522425
32	-14950160344272093791582580675043770	907472899819981949341378110154809715179	-24731051682639273759928788064570383223587450
33	16155141195563338657676309047193424	-1376221833275547070832128586054735372508	50509111549196111473773779016577737498441600
34	-16325577147375427860973939656206328	1961693839492212348975823973760605178136	-97309453221413773043941215167796843939911898
35	15438699952211792461501868440272006	-2630351797529897275169582942504929641060	176998827962979419006632617735024028247509860
36	-13670950561519930299442634525960712	3320209617080744793536230682508683729616	-304207058486939035761632027745346701554392134
37	11341331254037410590805413778181826	-3948147707280064838262852891535570186660	494410094613580637022824637004569132261581337
38	-8818779760352979981093230774298768	4425711236123134040037125307810442778760	-760401137180328056132510682398249514096499842
39	6429899593697638377309898113128922	-4679494553303078534463778068219617270016	1107482602759799563288386448416485515421284059
40	-4397374036769751008463322908782910	4669652936832854394263463248975498878290	-1528462115056218114570979335500644097095534590
41	2821554211274109523679503985207847	-4400106254768736151728313032685539961182	2000160725563917132179905076326003637180892066
42	-1698912467948949129560869039953552	3916855379279461279221307993825696427649	-2483251708332362892319927253873288086259403006
43	960046155315203117418840808804773	-3295267219632260845527037781198777751564	2926577962952883339749137031574000966373245705
44	-509179555726352503519141063593888	2621111844486986451254604074365108363518	-3275713142668992052529309428415969231063023464
45	253451113819739835364074059798592	-1971812573451411885627749313392558182050	3483904701814212061038593283228132682075549932
46	-118389947168479890750632497410792	1403312933377004028862981565442973543528	-3522371316635229491247268462169458134757105464
47	51886065603954874404443957042232	-945053007338240357264153578599020565678	3386820173441323644547878165608589144769825167
48	-21329621619955768840911025552404	602357958352568601380177713664341213368	-3098153949612082982692311364945230932330343462
49	8221566820070900740353505791441	-363424394221025691054817552283652902222	2697240439767053396463974972685005158785053058
50	-2970069311765455481972622777384	207576540278226299713443515310430317627	-2235529915625030927303840038293674618900390568
51	1005025072505687381818254012408	-112245893221491292810656209179463439420	1764453889147386373603244995693702112542853817
52	-318343714588601446289902619514	57463599462589050262475996892474119014	-1326540709770816642857568011527748818110546378
53	94315841627219273729130903345	-27850048295787497770823375266491469404	950187625360653365442292542829662379272587472
54	-26112384771570368923969170006	12776978787594387340553429759603391024	-648576794650189013320411455658959525512117310
55	6748788027550952195964742023	-5547978740904665931141578010944949534	421937425133907991848569862888256746117511476
56	-1626294421921639428910251198	2279592049200707616750142850971442961	-261653904373506954462272350191930719663472024
57	364895789433184980319614999	-886097485370467196508941396264934954	154683254836945777002236036266996682330483757
58	-76112034189381985020175470	325737599996001371774781437345699514	-87182031774437129103221179319923458818515422
59	14732503609736930453484630	-113200548053500475981259876192593970	46848433642117480552518593768368041920562717
60	-2640916850239964173599120	37172879137830513571751814110923425	-24002155036758457181631085461714939355806198
61	437401366115589567105201	-11528486374505944039801174613228858	11724132067252252961121481996868612000842980
62	-66757682295093850108074	3374617424077804322055975007580667	-5459588433471115640268244139046059464312500
63	9360475152166271210124	-931717941771724520241451824642690	2423516939099795213736698102760577855684629
64	-1201572318798328545552	242445583199669177802171529887963	-1025372499957341193769521527796082730973324
65	140636199781905104400	-59406436347336103284960618838086	413421387850156758263534038143024966922470
66	-14937821508912805788	13693511614437447866182451591241	-158814139265015970443647097358315457975566
67	1431881996665071882	-2966056690095091215596316365650	58111480154269905095003702190283748687415
68	-123052419492491526	602964061459997826813897760425	-20248102099048586010853896930342213253380
69	9405610862204928	-114881804402322846237903518958	6715987562335571702895084161471595142398
70	-633262403070492	20482739123073799677521468811	-2119686272246933874479296048592102581734
71	37104943421451	-3411539050097625960596686398	636327301275593340564500756630892960888
72	-1863092101590	529779487751768057098206870	-181603685243514533206257308227728006336
73	78555244146	-76537521430219395497397970	49245538517730604027400707548844254645
74	-2704922562	10261688845601881130576166	-12680710012534023754662301107764099988
75	73042542	-1273262385518182537205076	3098576636136322332765443327522683569
76	-1450566	145745369442728219604564	-717958438984338052148208054892633914
77	18837	-15334685649693749837484	157614910429442975046611813101623345
78	-120	1476882153683304214572	-32753807023600751443254718262391294
79	0	-129570236001093540324	6436618186938177743660110084929618
80	0	10296551220074653518	-1194828125495051995774017119062608
81	0	-736193973365226018	209254854442982989200792104245359
82	0	46980006025877057	-34528964162911686764252717214276
83	0	-2649703493070342	5360226360838025518747643332527
84	0	130483368718983	-781552787017954063881049904586
85	0	-5523954774108	106835163248905121474925193965
86	0	196982534997	-13663623902520673876538375880
87	0	-5753758530	1631266628080478241806291013
88	0	132187057	-181335628942097862187587378
89	0	-2239776	18715106642693346773644794
90	0	24885	-1787472318324401713866702
91	0	-136	157403590327004713054215
92	0	0	-12725416274500452565074
93	0	0	939888468084608425683
94	0	0	-63057203296464493164
95	0	0	3816835413000842085
96	0	0	-206756601273744390
97	0	0	9924692846551290
98	0	0	-417022184399886
99	0	0	15101577327810
100	0	0	-461770564404
101	0	0	11593366485
102	0	0	-229464288
103	0	0	3357255
104	0	0	-32280
105	0	0	153

Table 3: BPS numbers n_d^g of local $\mathbb{K}_{\mathbb{P}^2}$

	d_1	0	1	2	3	4	5	6
d_2								
0			-2	0	0	0	0	0
1		-2	-4	-6	-8	-10	-12	-14
2		0	-6	-32	-110	-288	-644	-1280
3		0	-8	-110	-756	-3556	-13072	-40338
4		0	-10	-288	-3556	-27264	-153324	-690400
5		0	-12	-644	-13072	-153324	-1252040	-7877210
6		0	-14	-1280	-40338	-690400	-7877210	-67008672

Table 4: Instanton numbers $n^{g=0}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_0}$

	d_1	0	1	2	3	4	5	6
d_2								
0			0	0	0	0	0	0
1		0	0	0	0	0	0	0
2		0	0	9	68	300	988	2698
3		0	0	68	1016	7792	41376	172124
4		0	0	300	7792	95313	760764	4552692
5		0	0	988	41376	760764	8695048	71859628
6		0	0	2698	172124	4552692	71859628	795165949

Table 5: Genus one GV invariants $n^{g=1}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_0}$

	d_1	0	1	2	3	4	5	6
d_2								
0			0	0	0	0	0	0
1		0	0	0	0	0	0	0
2		0	0	0	-12	-116	-628	-2488
3		0	0	-12	-580	-8042	-64624	-371980
4		0	0	-116	-8042	-167936	-1964440	-15913228
5		0	0	-628	-64624	-1964440	-32242268	-355307838
6		0	0	-2488	-371980	-15913228	-355307838	-5182075136

Table 6: Genus two GV invariants $n^{g=2}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_0}$

JHEP10(2008)097

d_2 \ d_1	0	1	2	3	4	5	6
0		0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	15	176	1130
3	0	0	0	156	4680	60840	501440
4	0	0	15	4680	184056	3288688	36882969
5	0	0	176	60840	3288688	80072160	1198255524
6	0	0	1130	501440	36882969	1198255524	23409326968

Table 7: Genus three GV invariants $n^{g=3}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_0}$

d_2 \ d_1	0	1	2	3	4	5	6
0		0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	-18	-248
3	0	0	0	-16	-1560	-36408	-450438
4	0	0	0	-1560	-133464	-3839632	-61250176
5	0	0	-18	-36408	-3839632	-144085372	-2989287812
6	0	0	-248	-450438	-61250176	-2989287812	-79635105296

Table 8: Genus four GV invariants $n^{g=4}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_0}$

d_2 \ d_1	0	1	2	3	4	5	6	7
0		-2	0	0	0	0	0	0
1	1	3	5	7	9	11	13	15
2	0	0	-6	-32	-110	-288	-644	- 1280
3	0	0	0	27	286	1651	6885	23188
4	0	0	0	0	-192	-3038	-25216	-146718
5	0	0	0	0	0	1695	35870	392084
6	0	0	0	0	0	0	-17064	-454880
7	0	0	0	0	0	0	0	188454

Table 9: Instanton numbers $n^{g=0}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_1}$

	d_1	0	1	2	3	4	5	6	7
d_2									
0			0	0	0	0	0	0	0
1		0	0	0	0	0	0	0	0
2		0	0	0	9	68	300	988	2698
3		0	0	0	-10	-288	-2938	-18470	-86156
4		0	0	0	0	231	6984	90131	736788
5		0	0	0	0	0	-4452	-152622	-2388864
6		0	0	0	0	0	0	80948	3164814
7		0	0	0	0	0	0	0	-1438086

Table 10: Genus one GV invariants $n^{g=1}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_1}$

	d_1	0	1	2	3	4	5	6	7
d_2									
0			0	0	0	0	0	0	0
1		0	0	0	0	0	0	0	0
2		0	0	0	0	-12	-116	-628	-2488
3		0	0	0	0	108	2353	23910	160055
4		0	0	0	0	-102	-7506	-161760	-1921520
5		0	0	0	0	0	5430	329544	7667739
6		0	0	0	0	0	0	-194022	-11643066
7		0	0	0	0	0	0	0	5784837

Table 11: Genus two GV invariants $n^{g=2}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_1}$

	d_1	0	1	2	3	4	5	6	7
d_2									
0			0	0	0	0	0	0	0
1		0	0	0	0	0	0	0	0
2		0	0	0	0	0	15	176	1130
3		0	0	0	0	-14	-992	-18118	-182546
4		0	0	0	0	15	4519	179995	3243067
5		0	0	0	0	0	-3672	-447502	-16230032
6		0	0	0	0	0	0	290853	28382022
7		0	0	0	0	0	0	0	-15363990

Table 12: Genus three GV invariants $n^{g=3}_{d_1 d_2}$ of local $\mathbb{K}_{\mathbb{F}_1}$

JHEP10(2008)097

References

[1] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, *JHEP* **02** (2004) 010 [hep-th/0211098].

[2] M. Aganagic, M. Mariño and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, *Commun. Math. Phys.* **247** (2004) 467 [hep-th/0206164].

[3] M. Aganagic, V. Bouchard and A. Klemm, *Topological strings and (almost) modular forms*, *Commun. Math. Phys.* **277** (2008) 771 [hep-th/0607100].

[4] M. Alim and J.D. Lange, *Polynomial structure of the (open) topological string partition function*, *JHEP* **10** (2007) 045 [arXiv:0708.2886].

[5] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Space-time topology change and stringy geometry*, *J. Math. Phys.* **35** (1994) 5321.

[6] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *J. Alg. Geom.* **3** (1994) 493.

[7] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[9] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Remodeling the B-model*, arXiv:0709.1453.

[10] J. Bryan and R. Pandharipande, *Curves in Calabi-Yau* 3*-folds and topological quantum field theory*, math.AG/0306316.

[11] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *Complex geometry of matrix models*, hep-th/0506075.

[12] T.M. Chiang, A. Klemm, S.-T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, *Adv. Theor. Math. Phys.* **3** (1999) 495 [hep-th/9903053].

[13] T. Coates, A. Corti, H. Iritani and H.H. Tseng, *Computing genus-zero twisted Gromov-Witten inavariants*, math.AG/0702234v3.

[14] D.A. Cox, *The homogeneous coordinate ring of a toric variety, revised version*, alg-geom/9210008.

[15] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[16] R. Dijkgraaf and C. Vafa, $N = 1$ *supersymmetry, deconstruction and bosonic gauge theories*, hep-th/0302011.

[17] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, math-ph/0702045.

[18] B. Eynard, M. Mariño and N. Orantin, *Holomorphic anomaly and matrix models*, *JHEP* **06** (2007) 058 [hep-th/0702110].

[19] B. Eynard, *Large-N expansion of convergent matrix integrals, holomorphic anomalies and background independence*, arXiv:0802.1788.

JHEP10(2008)097

[20] H. Farkas and I. Kra, *Theta constants, Riemann surfaces and the modular group*, American Mathematical Society, U.S.A. (2001).

[21] D. Ghoshal and C. Vafa, *$C = 1$ string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[22] T.W. Grimm, A. Klemm, M. Mariño and M. Weiss, *Direct integration of the topological string*, *JHEP* **08** (2007) 058 [hep-th/0702187].

[23] B. Haghighat and A. Klemm, *Topological strings on Grassmannian Calabi-Yau manifolds*, arXiv:0802.2908.

[24] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[25] K. Hori et al., *Mirror symmetry*, American Mathematical Society, Providence U.S.A. (2003).

[26] S. Hosono and Y. Konishi, *Higher genus Gromov-Witten invariants of the Grassmannian and the Pfaffian Calabi-Yau threefolds*, arXiv:0704.2928.

[27] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **167** (1995) 301 [hep-th/9308122].

[28] M.-X. Huang and A. Klemm, *Holomorphic anomaly in gauge theories and matrix models*, *JHEP* **09** (2007) 054 [hep-th/0605195].

[29] M.-X. Huang and A. Klemm, *Modularity versus holomorphicity in gauge theory and matrix models*, to appear.

[30] M.-X. Huang, A. Klemm and S. Quackenbush, *Topological string theory on compact Calabi-Yau: modularity and boundary conditions*, hep-th/0612125.

[31] S.H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys.* **B 497** (1997) 173 [hep-th/9609239].

[32] S.H. Katz, A. Klemm and C. Vafa, *M-theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999) 1445 [hep-th/9910181].

[33] A. Klemm, W. Lerche and S. Theisen, *Nonperturbative effective actions of $N = 2$ supersymmetric gauge theories*, *Int. J. Mod. Phys.* **A 11** (1996) 1929 [hep-th/9505150].

[34] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus*, hep-th/9906046.

[35] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional non-critical strings*, hep-th/9607139.

[36] A. Klemm, D. Maulik, R. Pandharipande and E. Scheidegger, *Noether-Lefschetz theory and the Yau-Zaslow conjecture*, arXiv:0807.2477.

[37] M. Mariño, *Chern-Simons theory, matrix models and topological strings*, Clarendon, Oxford U.K. (2005).

[38] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, *JHEP* **03** (2008) 060 [hep-th/0612127].

[39] M. Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings*, arXiv:0805.3033.

[40] M. Mariño, *Matrix models and topological strings*, in proceedings of the *Les Houches 2004, applications of random matrices in physics*, Springer, U.S.A. (2006), pg. 319.

[41] S. Yamaguchi and S.-T. Yau, *Topological string partition functions as polynomials*, *JHEP* **07** (2004) 047 [hep-th/0406078].

[42] J. Walcher, *Extended holomorphic anomaly and loop amplitudes in open topological string*, arXiv:0705.4098.

[43] E. Witten, *Quantum background independence in string theory*, hep-th/9306122.

[44] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[45] D. Zagier, *Elliptic modular forms and their applications*, in *The* 1-2-3 *of modular forms*, Springer, U.S.A. (2008).