# EQUILIBRIUM STATISTICAL MECHANICS OF ONE-DIMENSIONAL CLASSICAL LATTICE SYSTEMS

David Ruelle

I.H.E.S., 91440 Bures-sur-Yvette, France

Abstract. A broad description of the equilibrium statistical mechanics of classical one-dimensional lattice systems with exponentially decreasing interactions is given. We indicate unsolved problems, and mention applications to differentiable dynamical systems.

## 1. Introduction.

One-dimensional systems with short range interactions are not highly considered in statistical mechanics, because they have no phase transitions. This applies particularly to classical lattice spin systems. These are nevertheless the systems which I would like to discuss here. A reason for the renewed interest in them is that they occur naturally in connection with differentiable dynamical systems, and in particular in the study of the asymptotic behaviour of solutions of some types of differential equations. The spin systems which occur in this way have exponetially decreasing interactions. The basic facts about such systems were discovered by Araki [1] incidentally in his study of one-dimensional quantum systems. In what follows I shall give a broad description of the equilibrium statistical mechanics of classical lattice systems with exponentially decreasing interactions, indicate unsolved problems, and mention applications to differentiable dynamical systems.

## 2. Configuration space.

At each point $x$ of the lattice $\mathbb{Z}$ a finite number of "spin values" are allowed, forming a set $\Omega_o$. A matrix $t$ indexed by $\Omega_o \times \Omega_o$ and with entries 0 or 1 is also given. An allowed spin configuration on the interval $[k,\ell] \subset \mathbb{Z}$ is an element $\xi = (\xi_k, \xi_{k+1}, \ldots, \xi_\ell)$ of $(\Omega_o)^{[k,\ell]}$ such that

$$t_{\xi_k \xi_{k+1}} = \cdots = t_{\xi_{\ell-1}\xi_\ell} = 1$$

We denote by $\Omega[k,\ell]$ the set of these allowed configurations. The space of configurations on the whole lattice $\mathbb{Z}$ is

$$\Omega = \{\xi \in (\Omega_o)^{\mathbb{Z}} : t_{\xi_x \xi_{x+1}} = 1 \text{ for all } x \in \mathbb{Z}\} .$$

Using on $\Omega_o$ the discrete topology, and on $(\Omega_o)^{\mathbb{Z}}$ the product topology, we find that $\Omega$ is compact. If we define $\tau : \Omega \to \Omega$ by

$$(\tau\xi)_x = \xi_{x+1} \tag{1}$$

then $\tau$ is a homeomorphism of $\Omega$ ; (1) also defines a homeomorphism

$\tau : \Omega_{[k,\ell]} \mapsto \Omega_{[k-1,\ell-1]}$ for any finite interval $[k,\ell]$, or similarly for a semi-infinite interval $[k,+\infty)$.

It will be convenient to assume from now on that there exists $N > 0$ such that all entries of $t^n$ are $> 0$ for $n \geq N$. This amounts to requiring that the topological dynamical system $(\Omega,\tau)$ is topologically mixing.

3. Problem.

The system $(\Omega,\tau)$ is called a subshift of finite type, or a topological Markov chain. It is entirely determined by the number $|\Omega_o|$ of elements of $\Omega_o$ and by the $|\Omega_o| \times |\Omega_o|$ matrix $t$. Let the system $(\Omega',\tau')$ be similarly constructed from $\Omega'_o, t'$. We say that $(\Omega,\tau)$ and $(\Omega',\tau')$ are isomorphic if there exists a homeomorphism $h$ of $\Omega'$ on $\Omega$ such that $h\tau' = \tau h$. When do two square matrices $t,t'$ (in general of different orders) define isomorphic subshifts? This problem has been investigated by Williams [16], unfortunately his work is inconclusive [17], and the question remains open.

4. Thermodynamic limits.

An interaction $\Phi$ is a real function on the (allowed) configurations in finite intervals. We assume that it is invariant by the translation $\tau$. Given an interaction $\Phi$, an energy function

$$U^{\Phi}_{[a,b]} : \Omega_{[a,b]} \mapsto \mathbb{R}$$

is defined for each finite interval $[a,b]$ by

$$U^{\Phi}_{[a,b]}(\xi) = \Sigma_{k,\ell: a\leq k\leq \ell\leq b}\Phi(\xi|[k,\ell]) .$$

One writes then

$$Z^{\Phi}_{[a,b]} = \sum_{\xi\in\Omega_{[a,b]}} \exp[-U^{\Phi}_{[a,b]}(\xi)]$$

$$P^{\Phi}_{[a,b]} = \frac{1}{b-a} \log Z^{\Phi}_{[a,b]}$$

$$\sigma^{\Phi}_{[a,b]}\{\xi\} = (Z^{\Phi}_{[a,b]})^{-1} \exp[- U^{\Phi}_{[a,b]}(\xi)]$$

In particular $\sigma^{\Phi}_{[a,b]}$ is a measure on $\Omega_{[a,b]}$ . Suppose that

$$\|\Phi\| = \Sigma^{\infty}_{\ell=o}(\ell+1) \sup_{\xi\in\Omega_{[o,\ell]}} |\Phi(\xi)| < +\infty \ .$$

Then the following limit exists

$$P^{\Phi} = \lim_{b-a\to\infty} P^{\Phi}_{b-a}$$

(thermodynamic limit for the pressure - this would hold also without the factor $(\ell+1)$ in the definition of $\|\Phi\|$).

There is also a unique measure $\sigma$ on $\Omega$ such that for every finite interval $[k,\ell]$ ,

$$\lim_{a\to-\infty,\, b\to+\infty} \alpha_{[k,\ell],[a,b]}\, \sigma^{\Phi}_{[a,b]} = \alpha_{[k,\ell],\,\mathbb{Z}}\, \sigma \qquad (2)$$

where $\alpha_{[k,\ell],[a,b]} : \Omega_{[a,b]} \mapsto \Omega_{[k,\ell]}$ gives the restriction to $[k,\ell]$ of a configuration in $[a,b]$ when $[k,\ell] \subset [a,b]$ , and similarly for $\alpha_{[k,\ell],\,\mathbb{Z}}$ . The probability measure $\sigma$ is the unique Gibbs state for the interaction $\Phi$ (Dobrushin [5], Ruelle [7]) .

5. <u>Exponentially decreasing interactions</u>.

We say that $\Phi$ is exponentially decreasing if there exists $\theta \in (0,1)$ such that

$$\|\Phi\|_{\theta} = \sup_{[k,\ell]} \theta^{-(\ell-k)} \sup_{\xi\in\Omega[k,\ell]} |\Phi(\xi)| < +\infty \ .$$

For fixed $\theta$ these interactions form a Banach space $\mathbb{B}^{\theta}$ . Let $\mathbb{B}^{\theta}_{\mathbb{C}}$ be the corresponding complex Banach space, then one can show that

$$\Phi \mapsto P^{\Phi}$$

<u>extends to an analytic function on a neighborhood of</u> $\mathbb{B}^{\theta}$ in $\mathbb{B}^{\theta}_{\mathbb{C}}$ .

This can be proved by the transfer matrix method (Ruelle [7], Araki [1]).

We sketch the proof. For a continuous complex function $A$ on $\Omega_{[1,+\infty)}$, let

$$\mathrm{var}_n A = \sup\{|A(\xi)-A(\xi')| : \xi_x = \xi'_x \text{ for } 1 \le x \le n\}$$

$$\|A\|_\theta = \sup_{n\geq 0} (\theta^{-n}\, \mathrm{var}_n A) .$$

Those functions for which $\|A\|_\theta$ is finite form a Banach space $\mathfrak{F}^\theta_{\mathbb{C}>}$. An operator $\mathfrak{L}$ on $\mathfrak{F}^\theta_{\mathbb{C}>}$ is defined by

$$(\mathfrak{L}A)(\xi) = \sum_{\xi_o} A(\tau^{-1}(\xi_o,\xi)) \times \exp[-\sum_{\ell\geq 0} \Phi(\xi_o,\xi_1,\ldots,\xi_\ell)]$$

($\mathfrak{L}$ is the "transfer matrix").

For $\Phi \in \mathfrak{B}^\theta$, one can show that the number $\exp P^\Phi$ is a simple eigenvalue of $\mathfrak{L}$, and the rest of the spectrum of $\mathfrak{L}$ is contained in a cercle with radius $< \exp P^\Phi$ centered at the origin of $\mathbb{C}$. Since one can also show that $\Phi \to \mathfrak{L}$ is an entire analytic function on $\mathfrak{B}^\theta_{\mathbb{C}}$, it follows that $\Phi \to P^\Phi$ is analytic in a neighbourhood of $\mathfrak{B}^\theta$, as announced.

6. Problem.

Surprisingly, Dobrushin [6] has obtained analyticity properties of the map $\Phi \to P^\Phi$ without assuming exponential decrease of the interaction. His proof does not use the transfer matrix method, is not very transparent, and makes assumptions which are not very natural. Can one give a natural proof of Dobrushin's results ?

7. Exponential decay of correlations.

For a continuous real function $A$ on $\Omega$, let

$$\mathrm{var}_n A = \sup\{|A(\xi)-A(\xi')| : \xi_x = \xi'_x \text{ for } |x| \le n\}$$

$$\|A\|_\theta = \sup_{n\geq -1} (\theta^{-2n-1} \mathrm{var}_n A) .$$

Those functions for which $\|A\|_\theta$ is finite form a Banach space $\mathfrak{F}^\theta$ for the norm $\|A\|_\theta + \|A\|$. If $A \in \mathfrak{F}^\theta$ there is $\Phi \in \mathfrak{B}^\theta$ such that $\Phi|\Omega_{[k,\ell]} = 0$ when $\ell - k$ is odd, and

$$A(\xi) = \sum_{\ell\geq 0} \Phi(\xi|[-\ell,+\ell]) .$$

One writes then

$$P(A) = P^{\Phi}$$

and it can be checked that this definition does not depend on the particular choice of $\Phi$ . Also $A \to P(A)$ extends to an analytic function in a neighbourhood of $\mathfrak{F}^{\theta}$ in $\mathfrak{F}^{\theta}_{\mathbb{C}}$ (the complex Banach space corresponding to $\mathfrak{F}^{\theta}$ ).

Let $\sigma_A$ be the unique Gibbs state for the interaction $\Phi$ (it does not depend on the particular choice of $\Phi$ ). Sinai [15] has shown that if $A, A' \in \mathfrak{F}^{\theta}$, then $\sigma_A = \sigma_{A'}$ if and only if there exist $c \in \mathbb{R}$ and $C \in \mathfrak{F}^{\theta}$ such that

$$A' - A = c + C\circ\tau - C .$$

Let $A \in \mathfrak{F}^{\theta}$ , then there exist $a,b > 0$ such that, if $B_1, B_2 \in \mathfrak{F}^{\theta}$ ,

$$|\sigma_A(B_1\cdot(B_2\circ\tau^x)) - \sigma_A(B_1)\sigma_A(B_2)| \leq e^{a-b|x|}\|B_1\|_{\theta}\|B_2\|_{\theta}$$

(exponential decay of correlations). One can compute the derivatives of $P$ in terms of $\sigma_A$ . If $B_1,\dots,B_{\ell} \in \mathfrak{F}^{\theta}$ , let

$$D^{\ell}_A(B_1,\dots,B_{\ell}) = \frac{d^{\ell}}{ds_1\dots ds_{\ell}} P(A + \sum_i s_i B_i)\Big|_{s_1=\dots=s_{\ell}=0} .$$

Then

(a) $$D^1_A(B_1) = \sigma_A(B_1)$$

(b) $$D^2_A(B_1,B_2) = \sum_{x\in\mathbb{Z}} [\sigma(B_1\cdot(B_2\circ\tau^x)) - \sigma(B_1)\sigma(B_2)]$$

(c) $B_1 \mapsto D^2_A(B_1,B_1)$ is a positive semi-definite quadratic form on $\mathfrak{F}^{\theta}$. Its kernel is $\{c+C\circ\tau-C : c \in \mathbb{R},\ C \in \mathfrak{F}^{\theta}\}$ and is thus independent of $A$ . There is $R_A > 0$ such that $[D^2(B_1,B_1)]^{1/2} \leq R\|B_1\|_{\theta}$

(d) For all $p \in \mathbb{R}$ (mod $2\pi$) ,

$$\sum_{x\in\mathbb{Z}} e^{-ipx}[\sigma_A(B_1\cdot(B_2\circ\tau^x)) - \sigma_A(B_1)\sigma_A(B_2)] \geq 0 .$$

These results are of course not unexpected, and some of them should hold under much more general conditions. The proofs however are not as easy as one would imagine (see [10]).

## 8. $\zeta$-function.

Write $\Omega(m) = \{\xi \in \Omega : \tau^m \xi = \xi\}$ and let $A \in \mathcal{F}^\theta$. The power series

$$\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\xi \in \Omega(m)} \exp \sum_{k=0}^{m-1} A(\tau^k \xi)$$

converges for $|z| < e^{-P(A)}$. One can show that there exists $R > e^{-P(A)}$ such that

$$d_A(z) = \exp[-\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\xi \in \Omega(m)} \exp \sum_{k=0}^{m-1} A(\tau^k \xi)]$$

extends to an analytic function in $\{z: |z| < R\}$ with only one zero, this zero is simple and located at $e^{-P(A)}$.

It would be very interesting to increase the domain of analyticity (or meromorphy ?) of $d_A(z)$ because $1/d_A$ can be interpreted as a $\zeta$-function (see Bowen [3], section 5). In the case of a lattice gas with strictly exponential pair interaction one can show that $d_A$ is meromorphic in the entire complex plane.

## 9. Applications to differentiable dynamical systems.

In a remarkable paper, Sinai [15] has shown how to handle measure theoretical problems for a class of differentiable dynamical systems in terms of statistical mechanics of a one-dimensional lattice systems. Sinai treated Anosov diffeomorphisms and flows, using Markov partitions [13], [14]. As shown by Bowen, Markov partitions exist for the more general Axiom A diffeomorphisms [2] and flows [3]. This permits the extension of Sinai's ideas to these Axiom A diffeomorphisms (Ruelle [8] and flows (Bowen and Ruelle [4]). We cannot go here into all the necessary definitions, but mention a typical result (see [8]).

Theorem. Let $\Lambda$ be a $C^2$-Axiom A attractor for a diffeomorphism $f$. Then for almost every $x$ in a neighbourhood of $\Lambda$ (in the sense of smooth, or "Lebesgue" measure) the following limit exists and is independent of $x$

$$\text{weak} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \delta_{f^n x} = \mu .$$

Here $\delta_x$ is the unit mass at $x$ . If $f|\Lambda$ is mixing and $A,B$ are $C^1$ functions in a neighbourhood of $\Lambda$ ,

$$\mu(A.(B\circ f^n)) \quad - \quad \mu(A)\ \mu(B)$$

tends exponentially fast to zero when $n \to \infty$. (This last result is obtained from the exponential decay of correlations for Gibbs states).

A similar result holds for flows [4] (i.e. solutions of differential equations), but the exponential decrease of correlations has not been proved. Does it hold in general? The question is of some interest in relation to the problem of turbulence (see Ruelle and Takens [12], Ruelle [9]).

Let us also mention that the problem of $\zeta$-functions for flows would benefit from a better understanding of the question in Section 8.(See Bowen [3] Section 5). Finally, methods of statistical mechanics are also useful in the discussion of homology problems (see Ruelle and Sullivan [ 11] ).

## 10. References


[1] H. Araki. Gibbs states of a one-dimensional quantum lattice. Commun. math. Phys. 14, 120-157(1969).

[2] R. Bowen. Markov partitions for Axiom-A diffeomorphisms. Amer. J. Math. 92, 725-747(1970).

[3] R. Bowen. Symbolic dynamics for hyperbolic flows. Amer. J. Math. To appear.

[4] R. Bowen and D. Ruelle. The ergodic theory of Axiom-A flows. To appear.

[5] R.L. Dobrushin. The problem of uniqueness of a Gibbsian random field and the problem of phase transitions. Funkts. Analiz. Pril. 2 No. 4, 44-57(1968); English transl. Funct. Anal. Appl. 2, 302-313(1968).

[6] R. L. Dobrushin. Analyticity of correlation functions in one-dimensional classical systems with slowly decreasing potentials. Commun. math. Phys. 32, 269-289 (1973).

[7] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. Commun. math. Phys. 9, 267-278(1968).

[8] D. Ruelle. A measure associated with Axiom-A attractors. To appear.

[9] D. Ruelle. Turbulence and Axiom-A attractors. Notes from the school of mathematical Physics. Camerino (1974).

[10] D. Ruelle. Notes on classical statistical mechanics. To appear.

[11] D. Ruelle and D. Sullivan. Currents, flows, and diffeomorphisms. To appear.

[12] D. Ruelle and F. Takens. On the nature of turbulence. Commun. math. Phys. 20, 167-192(1971) and 23, 343-344(1971).

[13] Ia. G. Sinai. Markov partitions and C-diffeomorphisms. Funkts. Analiz Pril. 2 No. 1, 64-89(1968); English transl. Funct. Anal. Appl. 2, 61-82(1968).

[14] Ia. G. Sinai. Construction of Markov partitions. Funkts. Analiz Pril. 2 No. 3, 70-80(1968); English transl. Funct. Anal. Appl. 2, 245-253(1968).

[15] Ia. G. Sinai. Gibbsian measures in ergodic theory. Uspekhi mat. Nauk 27 No. 4, 21-64(1972); English transl. Russian math. Surveys 166, 21-69(1972).

[16] R.F. Williams. Classification of subshifts of finite type. Ann. of Math. 98, 120-153(1973).

[17] R.F. Williams. Errata to "Classification of subshifts of finite type". Ann. of Math. 99, 380-381(1974).