Algebraic Quantum Field Theory, Microlocal Analysis and Quantum Energy Inequalities

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Contents

1 Introduction

1.1 Negative Energy

Considering the theory of relativity [41], it is a consequence of causality that no matter or information may travel at speeds faster than the speed of light. This is classically enforced by the stipulation of the 'dominant energy condition', which in particular entails that all macroscopic matter should have a positive energy density, when measured by any timelike observers. Such pointwise energy conditions are an important feature of general relativity, being integral for example to the proof of singularity theorems (e.g.[34]).

It is a long known and curious feature of quantum field theory, however, that negative energy densities are allowed to exist under some conditions [9]. This violates any of the classical pointwise energy conditions of relativity theory. Although such a negative energy density has not been directly measured, there is strong experimental evidence of their existence, most notably from the Casimir effect [6].

In response to this it was proposed that energy densities in quantum fields may obey time averaged versions of the classical energy conditions [45], for example the 'averaged weak energy condition' proposes that

$$\int_{-\infty}^{\infty} T_{\mu\nu} v^\mu v^\nu \, d\tau \geq 0, \tag{1.1}$$

where the average is taken over a timelike geodesic with tangent vector v^μ, τ is proper time and $T_{\mu\nu}$ is the stress-energy tensor. It has indeed been shown that this condition is obeyed by free scalar fields in Minkowski spacetime [30]. However, this condition too can be violated in flat spacetime with boundaries [6] or compact spatial dimensions [30]. It is also violated in certain curved spacetimes [46]. An example of the construction of a 'squeezed vacuum state' which has a negative expected energy density can be seen in section 5, giving proof of the theoretical existence of negative energy densities.

Much interest has been taken in the possible macroscopic effects that could result if arbitrarily negative quantum energy densities are allowed to exist for extended periods of time. It has been shown that the effects could include several exotic and surprising results. For example, Alcubierre [1] proposed a hypothetical warp drive which would allow someone from Earth to travel to distant galaxies and back in less then the time taken for a pulse of light to complete the same journey. Other proposed phenomena include traversable wormholes, violations of the second law of thermodynamics and the possible construction of time machines. An expanded account of these proposals can be found in an article by Ford and Roman [23].

This report is concerned with the derivation of bounds one can find in quantum field theory on the magnitude and duration of negative energy densities, which severely limit the possibility that most of these proposed constructions are physically realistic. For example, it has been shown that the production of the warp drive of Alcubierre would require a wildly unrealistic amount of energy [38].

1.2 Quantum Inequalities

The bounds in question which will be focused on here have become known as 'quantum inequalities', and those discussed here in particular are often called quantum weak energy inequalities. The idea of quantum inequalities was first put forward by Ford [20], where he used thermodynamical considerations to derive a lower bound not on the energy density but on the energy flux. Following from this more work by Ford, in collaboration with Roman, led to further bounds being derived on the magnitude and duration of negative energy densities [21][22].

These types of lower bounds involve integrating the expected energy density against a 'smearing function' which is smooth, vanishes at infinity and integrates to unity. The resulting bounds in four-dimensional flat spacetime are of the form

$$\int_{-\infty}^{\infty} f(t)\langle T_{00}\rangle_{\psi}\, dt \geq -\frac{C}{t_0^4}, \tag{1.2}$$

where f is the sampling function in question, t_0 is a choice of 'sampling time', C is a constant depending on f and $\langle T_{00}\rangle_{\psi}$ is the expected energy density in the state ψ. By looking at the form of this bound it is clear that in the limit $t_0 \to \infty$ this reduces to something resembling the averaged weak energy condition and in the limit $t_0 \to 0$ it appears that the energy density can be arbitrarily negative. This implies that quantum inequalities restrict the magnitude of negative energy over extended periods of time. It would also seem to suggest that if the existence of some negative energy occurs it must be eventually cancelled out by at least an equal amount of positive energy appearing in the same region in the future.

The inequalities derived by Ford and Roman employed the use of a specific sampling function. Other bounds derived by Fewster, with Eveson [17], and Flanagan [19] gave bounds valid for arbitrary smooth and compactly supported sampling functions. Allowing for functions of compact support has the benefit of allowing for a more local analysis. The bound given by Flanagan is valid only in the specific case of a massless scalar field in two dimensions, and is claimed to be optimal. Further work generalised some of these results to be valid in static curved spacetimes for static observers

[36][11]. The result of [11] was then generalised further by Fewster [14] to give a bound valid for all timelike observers in any globally hyperbolic spacetime. This will be the main focus of this report.

More on the progress of quantum inequalities can be found in summaries by Roman [42] and Fewster [15], and references therein. This report is concerned only with quantum inequalities in scalar fields, but it should be noted that similar bounds have also been shown to exist for Dirac [12][47] and Electromagnetic [22][35] fields.

1.3 Aims

The main interest of this report is the derivation of the general worldline quantum inequality given by Fewster [14]. The aim is to give an account of the approach and methods required for the proof of this inequality, showing how this is more rigorous and general than previous derivations. This will involve some exploration into an algebraic approach to quantum field theory and a branch of mathematical analysis known as 'microlocal analysis'. The inequality considered reduces to those in [11] and [17] in the cases of static and Minkowski spacetimes. However this result has the advantage of being valid in any globally hyperbolic curved spacetime of dimension $n \geq 2$, which is the largest class of spacetimes such that the Klein-Gordon field is well posed [49]. This inequality also has the advantage over those derived by Ford and Roman that it is valid for arbitrary smooth and compactly supported test functions, which means it can be used to study more localised effects.

In static curved spacetimes there exists a prescription to construct a Klein-Gordon quantum field theory on a preferred Hilbert space representation, by decomposing the field by means of mode functions. This is the usual approach to quantum field theory, as is well covered in [3]. This approach is analogous to the usual approach to quantum field theory in flat spacetime [4][44][24]. This is also the prescription adopted in the derivation of previous quantum inequalities in curved spacetimes [11][36].

However, in a general curved spacetime such a prescription does not exist [49]. Therefore to prove a quantum inequality in such a general setting requires the adoption of a different approach to quantum field theory. This is achieved by taking an algebraic approach, in which all states can be considered on an equal footing, without the initial need to construct a Hilbert space. A comprehensive review of this approach in flat spacetime is given by Haag [25]. Wald [49] gives an introduction to the subject in curved spacetimes, as well as detailing why this approach is needed for a rigorous theory.

The algebraic formulation required to prove this inequality for the Klein-Gordon field will be described in section 2. Following this, some results from

the theory of distributions and microlocal analysis will be described. These techniques are required to perform a rigorous analysis on the algebraically formulated quantum field theory, in order to derive a lower bound for the time-smeared expected energy density. Several of these ideas will be explained without a complete accompanying proof, due to considerations of length and mathematical difficulty. In all cases the proofs can be found in the text by Hörmander [27], which provides the most complete treatment of this subject. The Hadamard condition [50] will then be stated in terms of microlocal analysis [40]. This is a condition that it is assumed that states should obey in order to be considered physical.

In section 3 it will be shown how to formulate a quantum energy density in this setting, using a point-splitting technique [32]. It will then be demonstrated how some of the techniques of microlocal analysis described in the previous section can be applied to the resulting object to analyse some of its properties.

The focus in section 4 will then shift back to the motivating topic of quantum inequalities, where it will be shown how these techniques are applied to derive the actual inequality, which is of the form

$$\int d\tau \, (g(\tau))^2 \langle : T : \rangle_\omega (\tau, \tau') \geq - \int_0^\infty \frac{d\alpha}{\pi} \, [(g,g) \langle T \rangle_{\omega_0}]^\wedge (-\alpha, \alpha). \tag{1.3}$$

In the above $\langle : T : \rangle_\omega$ denotes the expected energy for a state ω, normal ordered with respect to an arbitrary state ω_0; g is an arbitrary test function, and $\wedge$ denotes the Fourier transform, which is defined in section 2.

After showing how the inequality is derived it will be compared to related results in section 5, in particular the optimal bound given by Flanagan [19]. A sketch of how this bound is proved will be given, before the result is compared with the reduced form of the inequality above. These will also be compared with the corresponding bound of Ford and Roman [21].

2 Setting and Tools

2.1 Spacetime

A general, n-dimensional, curved spacetime can be represented by a Lorentzian manifold (M, g), defined through the specification of a Lorentz metric g_{ab} on a differentiable, n-dimensional manifold M [41]. A manifold is a topological space which has the same local structure as $\mathbb{R}^n$. A definition and discussion of some of the properties of manifolds can be found in the opening chapter of [28], and some elaboration will be given in the following sections of this

report. The convention will be taken that a Lorentzian metric is one with signature $+-\cdots-$. Note that some authors use a different sign convention, for example in [49].

Central to previous derivations of quantum inequalities in curved spacetime was the additional condition that the spacetimes in question be static [11][36][37]. The result derived here is valid for more general curved spacetimes, being restricted only to spacetimes which are 'globally hyperbolic'. Note that to impose this condition it is required that a spacetime be time-orientable. A causal curve is defined as a curve for which the tangent vector is either null or timelike at all points, and a Cauchy surface is defined as any subset of M which is intersected by every inextensible causal curve exactly once. A spacetime is then said to be globally hyperbolic if it admits a Cauchy surface. This condition ensures that causality holds on M, in the sense that no closed timelike curves exist.

2.2 Quantum Field Theory in Curved Spacetime

The theory of quantum fields in curved spacetime should be thought of as a semi-classical theory, in which the effects of gravity are incorporated by formulating quantum field theory against the classical backdrop of general relativity. The inequality we are interested in concerns the Klein-Gordon scalar field. The classical Klein-Gordon field equation in curved spacetime is given by

$$(g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2 + \xi R)\phi = 0, \tag{2.1}$$

where $g^{\alpha\beta}$ is the metric tensor, ∇_α is the derivative operator with respect to the metric (see e.g. [41]), $m > 0$ is the mass of the field ϕ and the term ξR represents the coupling between the scalar and gravitational fields. The ξ in this term is a constant and R is the Ricci scalar, representing curvature. In this derivation, however, only the simplest case shall be considered, in which $\xi = 0$. This is known as minimal coupling. Inequalities have also been found for the non-minimally coupled case [18].

This field can be quantised in a standard manner [3], which runs analogously to the case of Minkowski spacetime [44]. This involves constructing a Hilbert space on which one can represent possible states of the quantum system. One then represents observables as operators which act upon these states. A quantum theory can then be represented by $(\mathscr{H}, V_i)$ where $\mathscr{H}$ is a Hilbert space and the V_i are a collection of operators representing the theory.

We say that two constructions $(\mathscr{H}, V_i)$ and $(\mathscr{H}', V_i')$ are unitarily equivalent if a unitary map $U : \mathscr{H} \to \mathscr{H}'$ exists such that $U^{-1}V_i'U = V_i$ for each

i. In this case each state ψ in $\mathscr{H}$ will be physically equivalent to the state $U\psi$ in $\mathscr{H}'$, meaning that the two constructions yield physically equivalent quantum theories.

Wald shows [49] how for the Klein-Gordon field, choosing a Hilbert space representation is equivalent to the specification of an inner product μ on the space of solutions, which satisfies certain conditions. It is also shown how different choices of μ can give rise to theories which are unitarily inequivalent. This is not an issue in Minkowski spacetime, where a preferred Hilbert space can be naturally chosen as the choice of 'positive frequency solutions' [44].

Indeed, as also shown in [49], one can also choose a preferred Hilbert space representation for those curved spacetimes which are static. This is because of the fact that static spacetimes admit a timelike Killing vector, with a prescription for choosing a space of positive frequency solutions with respect to the corresponding 'Killing time'. In this case the usual prescription for quantisation can be applied, as detailed in [3], and using this setting quantum inequalities such as that in [11] can be derived for static spacetimes only.

In a general curved spacetime however, considering what has been said above, it appears that by attempting to follow the usual approach to quantisation one is led into ambiguity. Luckily, these apparent problems can be averted by taking an algebraic approach to quantum field theory.

2.3 Advanced Minus Retarded Solution

This section will describe how to construct the unique advanced minus retarded solution to the Klein-Gordon equation, which will be an important construction when we come to investigate the algebraic relations obeyed by the observables on the field. Firstly, we define set of test functions [27] on (M, g) to be the set of all complex valued, smooth and compactly supported functions on M, denoted by $C_0^\infty(M)$. A function is said to be smooth if it is continuous and partial derivatives of the function exist to all orders and are also continuous. The support of a function is defined to be the closure of the set on which the function is not identically equal to zero. If the support is a compact set then the function is said to be compactly supported.

Now, as our spacetime is time-orientable and has a well defined causal structure due to global hyperbolicity it is possible to define retarded and advanced solutions $E^\pm f$ to the Klein-Gordon equation with source f, for each f in $C_0^\infty(M)$, as follows:

$$(g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)(E^\pm f) = f, \tag{2.2}$$

where the advanced solution $E^- f = 0$ outside of the causal past of the

support of f, and the retarded solution $E^+ f = 0$ outside of the causal future of the support of f.

Now if we define the 'advanced minus retarded solution' E by $E = E^- - E^+$ we have that Ef is a solution to the source free Klein-Gordon equation for each f in $C_0^\infty(M)$, hence we have created in E a linear map from the space of test functions into the space of solutions. A similar definition is given in [49], with opposite signs for the solutions due to the different convention used for the signature of the metric, and in [8] where instead the retarded minus advanced solution is used as E.

2.4 Algebraic Formulation

By formulating a quantum field theory algebraically, the aim is to try to understand the algebraic relations obeyed by the operators on the field separately from the difficulty of forming a specific Hilbert space representation for the theory. This can be viewed, in a way, as the reverse of the usual quantisation procedure.

Usually, a quantum theory is specified by first representing states as vectors, or more generally 'density matrices', on a Hilbert space. Observables are then represented as operators on these states. In the algebraic construction however, we construct an algebra on a set of objects representing the observables and states then act as maps from these observables to the real numbers, in correspondence to taking expectation values.

This allows all states to be considered on an equal footing, without introducing a preferred representation. An extensive introduction to this approach in Minkowski spacetime is given by Haag [25]. We will take the approach of constructing a $*$-algebra on the set of 'smeared field operators', labelled by $\{\phi(f) : f \in C_0^\infty(M)\}$. There are other ways of constructing an algebra on the Klein-Gordon field, such as using the Weyl algebra outlined in [49] or the $*$-algebra described in [2], which is similar to algebra used here. We use the construction briefly outlined in [14].

A $*$-algebra is defined over a $*$-ring. An associative ring R is a $*$- ring if it is equipped with an operation $* : R \to R$ satisfying the properties:

- $(a + b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $(a^*)^* = a$
- $1^* = 1$

for all a, b in R, where 1 denotes the identity. A $*$-algebra is then defined as a $*$-ring which is an associative algebra over R, inheriting the $*$ operation from R.

Our algebra will be constructed over the complex numbers, $\mathbb{C}$. The relations we want the 'smeared field operators' to obey will now be explored. This is done using ideas from references [49][8][14][2][50], adapted to suit our conventions and uses. Classical observables correspond to functions on the phase space for the Klein-Gordon field, where points in phase space are given by pairs of functions θ and π on a spacelike Cauchy surface Σ_{t_0}. Each point in phase space then corresponds to a unique solution to the Klein-Gordon equation, and we can define a natural bilinear form $\Omega : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ on the space of real solutions, $\mathcal{S}$ given by

$$\Omega(\varphi_1, \varphi_2) = \int_{\Sigma_{t_0}} \theta_1(x)\pi_2(x) - \pi_1(x)\theta_2(x) \tag{2.3}$$

where φ_1 and φ_2 are real solutions corresponding to phase space points (θ_1, π_1) and (θ_2, π_2) respectively. Here, Ω is clearly antisymmetric and is also 'nondegenerate', meaning that $\Omega(\varphi_1, \varphi_2) = 0$ for all φ_1 in $\mathcal{S}$ if and only if $\varphi_2 = 0$.

Through this construction we can define a linear map $\Omega(\varphi, \cdot) : \mathcal{S} \to \mathbb{R}$ for each φ in $\mathcal{S}$. Since these are linear functions on the space of solutions we can think of them as representing observables, and we can identify them with the smeared fields, through $\phi(f) = \Omega(Ef, \cdot)$ where E is the advanced minus retarded solution. This is possible because of the fact that every solution φ in $\mathcal{S}$ can be written in the form $\varphi = Ef$ for some f in $C_0^\infty(M)$. A proof of this in the case of flat spacetimes is given in [49]; for a proof adapted to the case of globally hyperbolic curved spacetimes see appendix A.

By making this identification we have a convenient way to derive the relations that we want the smeared fields $\phi(f)$ to obey, by exploring the properties of Ω. We also have the correspondence $\Omega(Ef, \varphi) = \int dV_g f\varphi$ for any solution φ (see Lemma A.1 in [8]), where V_g is the volume element associated with the metric. This shows why, in a Hilbert space representation of a quantum theory where φ becomes a field operator, we interpret $\phi(f)$ as the field 'smeared out' by f over its support.

We can now use Ω to identify the properties we expect the smeared fields to satisfy. Firstly, Ω is clearly linear. Also, as we permit complex smearing

functions, we have

$$\begin{aligned}
\Omega(Ef,\varphi)^* &= \left(\int dV_g\, f\varphi\right)^* \\
&= \left(\int dV_g\, (\mathrm{Re}(f) + i\mathrm{Im}(f))\varphi\right)^* \\
&= \int dV_g\, (\mathrm{Re}(f) - i\mathrm{Im}(f))\varphi \\
&= \int dV_g\, \bar{f}\varphi \\
&= \Omega(E\bar{f},\varphi), \qquad (2.4)
\end{aligned}$$

for any φ in $\mathcal{S}$ and f in $C_0^\infty(M)$, where $\bar{f}$ denotes complex conjugation and $*$ denotes Hermitian conjugation. We also have $\Omega((g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)f,\cdot) = 0$ for any f in $C_0^\infty(M)$. To see this let φ be any solution in $\mathcal{S}$ and let $h = g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)f$, for any f in $C_0^\infty(M)$. Then note that $E^-h = E^+h = f$, so that $Eh = 0$. Hence we have

$$\begin{aligned}
\Omega(Eh,\varphi) &= \Omega(0,\varphi) \\
&= 0. \qquad (2.5)
\end{aligned}$$

Now, if we take φ_1 and φ_2 in $\mathcal{S}$ and denote $\Omega(\varphi_1,\cdot)$ and $\Omega(\varphi_2,\cdot)$ by $F(\theta,\pi)$ and $G(\theta,\pi)$ respectively, we then have the Poisson bracket relation:

$$\{F,G\} = \int_{\Sigma_{t_0}} \frac{\delta F}{\delta\theta}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\theta}, \qquad (2.6)$$

where θ and π are phase space functions. Then to calculate, for example $\frac{\delta F}{\delta\theta}$, we see from equation 2.3 that

$$\begin{aligned}
\frac{\delta F}{\delta\theta}(x) &= \frac{\delta}{\delta\theta}(x)\int_{\Sigma_{t_0}} \theta_1(x')\pi(x') - \pi_1(x')\theta(x') \\
&= -\int_{\Sigma_{t_0}} \pi_1(x')\frac{\delta\theta(x')}{\delta\theta(x)} \\
&= -\int_{\Sigma_{t_0}} \pi_1(x')\delta(x'-x) \\
&= -\pi_1(x). \qquad (2.7)
\end{aligned}$$

Similarly we have

$$\frac{\delta F}{\delta\pi} = \theta_1, \quad \frac{\delta G}{\delta\theta} = -\pi_2, \quad \frac{\delta G}{\delta\pi} = \theta_2. \qquad (2.8)$$

Then putting these into equation 2.6 we find

$$\begin{aligned} \{F, G\}(\theta, \phi) &= \int_{\Sigma_{t_0}} \theta_1 \pi_2 - \pi_1 \theta_2 \\ &= \Omega(\varphi_1, \varphi_2). \end{aligned} \tag{2.9}$$

Now if we write φ_1 and φ_2 in the form Ef and Eg respectively, we have

$$\begin{aligned} \{\Omega(Ef, \cdot), \Omega(Eg, \cdot)\} &= \Omega(Ef, Eg) \\ &= \int dV_g \, fEg \\ &= E(f, g), \end{aligned} \tag{2.10}$$

where we now write $E(f, g) \equiv \int dV_g \, fEg$ for all f and g in $C_0^\infty(M)$. Then in a quantum theory we could expect the $\phi(f)$ to obey a corresponding canonical commutator relation.

Gathering these results, we are now in a position to state the relations we wish to impose on the objects $\phi(f)$ as follows:

Q1 Hermiticity: $\phi(f)^* = \phi(\bar{f})$ for all f in $C_0^\infty(M)$

Q2 Complex linearity: $\phi(f + \lambda g) = \phi(f) + \lambda\phi(g)$ for all f, g in $C_0^\infty(M)$ and all λ in $\mathbb{C}$

Q3 Field equation: $\phi((g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)f) = 0$ for all f in $C_0^\infty(M)$

Q4 Canonical commutation relations (CCR): $[\phi(f), \phi(g)] = iE(f, g)\mathbb{1}$ for all f and g in $C_0^\infty(M)$,

where $\mathbb{1}$ denotes the identity. In particular it is the relation **Q4** that makes sure that the resulting objects represent a quantum theory rather than classical observables. This also means that although we can not write the objects $\phi(f)$ in an explicit form, we do have an explicit representation for the commutator of any two such objects.

We will now construct an algebra to incorporate these relations. This is done in the same way as in [14], by first taking the natural free, unital $*$-algebra over $\mathbb{C}$ generated by the set $\{\phi(f) : f \in C_0^\infty(M)\}$. To elaborate on these terms, a unital algebra is an algebra containing an identity, which is satisfied here by $\mathbb{1}$. By a free algebra we mean one consisting of all finite polynomials of the objects $\phi(f)$, the $\phi(f)^*$ and $\mathbb{1}$. We call this algebra $\mathfrak{A}$.

The relations Q1-Q4 are then enforced by using them to define a quotient of $\mathfrak{A}$. To do this we first construct a $*$-ideal, consisting of all elements of the form ABC where A and C are any members of $\mathfrak{A}$ and B takes the form of any one of the following:

1. $\phi(\lambda_1 f + \lambda_2 g) - \lambda_1\phi(f) - \lambda_2\phi(g)$
2. $\phi(f)^* - \phi(\bar{f})$
3. $\phi((g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)f)$
4. $\phi(f)\phi(g) - \phi(g)\phi(f) - iE(f,g)\mathbb{1}$

for all f and g in $C_0^\infty(M)$, λ_1, λ_2 in $\mathbb{C}$. We call the resulting subset $\mathfrak{I}$. Then for any A in $\mathfrak{A}$ and B in $\mathfrak{I}$ we have that both AB and BA are in $\mathfrak{I}$, so $\mathfrak{I}$ satisfies the definition of an ideal.

Now, for the properties Q1-Q4 to be satisfied, we would like for any element of $\mathfrak{I}$ to act as zero. This is exactly what we achieve by taking the quotient $\mathfrak{A}/\mathfrak{I}$. This is done by defining the equivalence relation $A \sim B$ if and only if $A - B$ is in $\mathfrak{I}$, for all A and B in $\mathfrak{A}$. The quotient $\mathfrak{A}/\mathfrak{I}$ then consists of the resulting equivalence classes $[A] = A + \mathfrak{I}$ for each A in $\mathfrak{A}$. We suppress the equivalence class notation and denote the quotient algebra by $\mathfrak{A}(M,g)$. This is now our desired construction, ensuring that the relations Q1-Q4 are satisfied.

Having constructed a satisfactory algebra on the objects representing observables in quantum field theory, the next step is to define states. We define these as linear maps, or functionals, $\omega : \mathfrak{A}(M,g) \to \mathbb{C}$ from the algebra onto the underlying space $\mathbb{C}$. The states are then required to satisfy the normalisation condition $\omega(\mathbb{1}) = 1$ and also the positivity condition $\omega(A^*A) \geq 0$ for all A in $\mathfrak{A}(M,g)$. As previously mentioned, the action of a state on an observable can be thought of as corresponding to taking an expectation value, so this condition corresponds to the positivity of expectation values for products of operators with their adjoints.

Having defined an algebra of observables on the Klein-Gordon field and a notion of states, the next step will be to consider which states are considered to be allowable in the theory. An important feature in the analysis of quantum field theories is the n-point functions of states. The n-point function for a state ω is defined by

$$\omega_n(f_1, \ldots, f_n) = \omega(\phi(f_1)\cdots\phi(f_n)) \tag{2.11}$$

for all $f_1, \ldots, f_n$ in $C_0^\infty(M)$. In particular for the purposes of this report we will want to consider the properties of a states two-point function:

$$\omega_2(f,g) = \omega(\phi(f)\phi(g)), \tag{2.12}$$

for all f and g in $C_0^\infty(M)$.

The proof of the quantum inequality in section 4 will require the use of the fact that every two-point function shares a common anti-symmetric part.

To see that this is true, let f and g be in $C_0^\infty(M)$ and note that from the commutation relation **Q4** we have that

$$\frac{1}{2}\phi(f)\phi(g) = \frac{1}{2}\phi(g)\phi(f) + \frac{i}{2}E(f,g)\mathbb{1}, \tag{2.13}$$

so that adding $\frac{1}{2}\phi(f)\phi(g)$ to each side gives

$$\omega_2(f,g) = \frac{1}{2}(\omega_2(f,g) + \omega_2(g,f)) + \frac{i}{2}E(f,g)\mathbb{1}. \tag{2.14}$$

Further, since

$$E(f,g) = \Omega(Ef, Eg) \tag{2.15}$$

it follows from the anti-symmetry of Ω that the imaginary part of 2.14 is anti-symmetric, so clearly every two-point function has a real symmetric part and a common anti-symmetric part.

2.5 Distributions

The aim of the following material will be to give conditions on which states will be considered to be physically 'allowable'. These will be those states which have what is called a globally Hadamard two-point function. We will start by making the restriction to states for which the two-point function is a distribution, which will be defined shortly. Some results from microlocal analysis will then be given. Microlocal analysis is a tool used to describe the singularity structure of distributions. The definitive text on these techniques is by Hörmander [27]. The material presented here is mainly drawn from ideas in that reference along with [13].

Firstly, a distribution on $\mathbb{R}^n$ is defined as a linear map $u : C_0^\infty(\mathbb{R}^n) \to \mathbb{C}$ such that u is linear and satisfies the continuity condition that for every compact subset $K \subset \mathbb{R}^n$ there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_K |D^\alpha \phi|, \tag{2.16}$$

for all ϕ in $C_0^\infty(K)$, where D^α denotes some partial differential operator of order $|\alpha|$. The set of all distributions is then denoted by $\mathscr{D}'(\mathbb{R}^n)$.

We can now define some simple operations on distributions. We can take linear combinations by defining

$$(\lambda_1 u + \lambda_2 v)(f) = \lambda_1 u(f) + \lambda_2 v(f), \tag{2.17}$$

for all λ_1 and λ_2 in $\mathbb{C}$, u and v in $\mathscr{D}'(\mathbb{R}^n)$ and f in $C_0^\infty(\mathbb{R}^n)$. This satisfies the continuity condition, since if u is in $\mathscr{D}'(\mathbb{R}^n)$ and satisfies 2.16 for some C_1 and k_1, and if λ is a constant, then

$$|\lambda u(\phi)| \leq C_1' \sum_{|\alpha| \leq k_1} \sup_K |D^\alpha \phi|, \tag{2.18}$$

where $C_1' = |\lambda| C_1$. Also, if v is another distribution in $\mathscr{D}'(\mathbb{R}^n)$, satisfying 2.16 for some C_2 and k_2, we have that

$$\begin{aligned} |u(\phi) + v(\phi)| &\leq |u(\phi)| + |v(\phi)| \\ &\leq (C_1 + C_2) \sum_{|\alpha| \leq k} \sup_K |D^\alpha \phi|, \end{aligned} \tag{2.19}$$

where $k = \min\{k_1, k_2\}$. Therefore we have that $\lambda_1 u + \lambda_2 v$ is in $\mathscr{D}'(\mathbb{R}^n)$.

Also, for any u in $\mathscr{D}'(\mathbb{R}^n)$ we can define multiplication by a smooth function χ in $C^\infty(\mathbb{R})$ by

$$(\chi u)(f) = u(\chi f), \tag{2.20}$$

for all f in $C_0^\infty(\mathbb{R}^n)$. Then we have that χf is in $C_0^\infty(\mathbb{R}^n)$ so that 2.16 is trivially satisfied and χu is in $\mathscr{D}'(\mathbb{R}^n)$.

We can also define the derivative of u with respect to a differential operator D^α of order $|\alpha|$ by

$$(D^\alpha u)(f) = (-1)^{|\alpha|} u(D^\alpha f), \tag{2.21}$$

for all f in $C_0^\infty(\mathbb{R}^n)$. This definition is designed to ensure that integration by parts holds true. It can again be shown that 2.16 is satisfied for this definition.

In particular, we can construct a distribution associated with any locally integrable function on $\mathbb{R}^n$. To see this, let F be locally integrable on $\mathbb{R}^n$. That is, F is integrable over any compact subset of $\mathbb{R}^n$. The distribution associated with F is then defined by

$$F(f) = \int F(x) f(x)\, d^n x, \tag{2.22}$$

for any f in $C_0^\infty(\mathbb{R}^n)$ and where, in following a common abuse of notation, the distribution associated with F has also been denoted by F. A notable feature of this construction is that while the function F may not be differentiable, the distribution F is always infinitely differentiable, in the distributional sense defined above. This can easily be seen by the fact that one can simply integrate by parts and use the fact that any compactly supported f vanishes at infinity.

In order to carry out the analysis needed for our purposes this definition must be extended to one for distributions on a more general manifold. In order to achieve this some elaboration will be needed on the structure of manifolds. As mentioned in 2.1, n-dimensional manifolds look locally like $\mathbb{R}^n$. More specifically, each point x in an n-dimensional manifold has a neighbourhood X_κ which is homeomorphic to an open subset of $\mathbb{R}^n$ [28]. We then call the pair (X_κ, κ) a 'chart', where κ denotes the homeomorphism.

If a collection of charts comprises a covering of X, that is $X \subset \cup X_\kappa$, then this collection is called an 'atlas'. In the case where two charts (X_κ, κ) and $(X_{\kappa'}, \kappa')$ are overlapping, so that $X_\kappa \cap X_{\kappa'} \neq \emptyset$, we can construct what is known as a 'transition function' $\vartheta : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'})$ between open subsets of $\mathbb{R}^n$. The transition function is required to be smooth and is defined by $\vartheta = \kappa' \circ \kappa^{-1}$. An object defined on overlapping charts will then have two different 'chart expressions'. We then say that a manifold X is 'smooth' if there is a C^∞ structure on X. That is, a collection $\mathcal{F}$ of homeomorphisms κ from open sets $X_\kappa \subset X$ onto open subsets of $\mathbb{R}^n$ satisfying the properties:

1. $\vartheta : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'})$ is smooth, for all κ, κ' in $\mathcal{F}$.
2. $\cup X_\kappa = X$.
3. Any homeomorphism κ' from an open subset $X' \subset X$ to an open subset of $\mathbb{R}^n$, such that the map $\kappa' \circ \kappa^{-1} : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'})$ and its inverse are smooth for all κ in $\mathcal{F}$, is also in $\mathcal{F}$.

Now using this structure we can define distributions on a smooth manifold. From this point if (X_κ, κ) is a chart on an n-dimensional manifold X the notation $\tilde{X}_\kappa$ will be used to denote the image of X_κ in $\mathbb{R}^n$, under the action of κ. Now to define a distribution, let X be a smooth manifold. Then for any chart κ on X let u_κ be a distribution on $\tilde{X}$. Then if for every κ and κ' we have that

$$u_{\kappa'} = u_\kappa \circ (\kappa \circ \kappa'^{-1}) \text{ on } \kappa'(X_\kappa \cap X_{\kappa'}), \tag{2.23}$$

we say that the family of local representatives u_κ defines a distribution u on X. The u referred to is such that $u_\kappa = u \circ \kappa^{-1}$ for each κ. It can be shown (Theorem 6.3.4 in [27]) that this definition is a consistent one, and agrees with the definition in $\mathbb{R}^n$.

Now in order to produce an example of a distribution on a manifold X analogous to that given in $\mathbb{R}^n$ we require the existence of a smooth density ρ_X on X. We can define a density through the way that its expressions on different charts depend on the transition function ϑ [16]. A density ρ with

chart expressions ρ_κ and $\rho_{\kappa'}$ satisfies the transformation law:

$$\rho_{\kappa'}(\vartheta(x))|\mathrm{det}D\vartheta(x)| = \rho_\kappa(x), \tag{2.24}$$

for all x in $U \cap U'$, where $D\vartheta(x)$ denotes the derivative of ϑ at x and det denotes the determinant.

This is important for constructing distributions on a manifold because of the integration properties of densities. It follows from the transformation law that for any density ρ supported on $U \cap U'$ we have

$$\int_{\kappa(U)} \rho_\kappa(x)\, d^n x = \int_{\kappa'(U)} \rho_{\kappa'}(x')\, d^n x', \tag{2.25}$$

so that for a density ρ supported on U we can make the definition:

$$\int_U \rho = \int_{\kappa(U)} \rho_\kappa(x)\, d^n x. \tag{2.26}$$

Then if X is a smooth manifold with a smooth density ρ_X we can define a distribution on X for any smooth function F by

$$F(f) = \int \rho(x) f(x) F(x)\, d^n x, \tag{2.27}$$

for any f in $C_0^\infty(X)$.

In particular we can always make this definition on a spacetime (M, g) using the natural density defined on M by $|\mathrm{det}g|^{\frac{1}{2}}$.

2.6 Wave-front Sets

Now that a definition of distributions has been given, we can begin to look at the techniques used to analyse their singularity structure. This will be done through looking at the 'wave-front set'. Again the techniques presented here can be found detailed in [27] and the material in [13] has aided this presentation. The conventions taken will follow those in [14].

The main motivation here for considering the wave-front set is because of the relation to the Hadamard condition which we will look to impose on states. The techniques discussed here are also important for other reasons. For example, the product of two distributions is not always well defined, and the wave-front set provides information on when distributions can be multiplied together. These techniques can also be applied invaluably to the analysis of partial differential equations. To begin with the wave-front set will be defined for distributions on $\mathbb{R}^n$, then this definition will be related to smooth manifolds and the Hadamard condition will be stated.

We want to know where a distribution is singular, so it makes sense that we begin by defining where a distribution is supported. Since distributions act on a space of functions, we can not achieve this by considering the 'value' of distributions at points on $\mathbb{R}^n$. However, it makes more sense to say that a distribution u vanishes on a neighbourhood if $u(f) = 0$ for all f supported on that neighbourhood. With this in mind we define the support of a distribution u by:

$$\operatorname{supp} u = \mathbb{R}^n \backslash \{x \in \mathbb{R}^n : \exists \text{ a neighbourhood } X \text{ of } x, \text{ s.t. } u(f) = 0 \; \forall \; f \in C_0^\infty(X)\}. \tag{2.28}$$

This way we can still interpret the support of a distribution u being the smallest space of all closed subsets where u does not vanish, which is exactly analogous to how the support of a function is defined. Note that from this and 2.20 it follows that for any f in $C_0^\infty(\mathbb{R}^n)$ and u in $\mathscr{D}'(\mathbb{R}^n)$ we have $\operatorname{supp} fu \subset \operatorname{supp} f$.

Now to try to understand where distributions have singular behaviour we must first give a definition of what it means for a distribution to be 'smooth'. We do this by appealing to the properties of functions, and defining a distribution u to be smooth at any point having a neighbourhood where we can identify u with a smooth function. Specifically, given a distribution u in $\mathscr{D}'(\mathbb{R}^n)$, we say that u is smooth at a point x in $\mathbb{R}^n$ if there is a neighbourhood X of x and a smooth function F in $C^\infty(X)$ such that

$$u(f) = F(f) = \int F(x) f(x) \, d^n x, \tag{2.29}$$

for all f in $C_0^\infty(X)$. Following this definition we can define the 'singular support' of a distribution u, denoted $\operatorname{sing\,supp} u$, to be the complement of the set of points on which u is smooth. Since u is zero outside of the support of u this definition means that u is smooth everywhere outside of its singular support, and therefore all of the singularities are contained within the singular support.

We can gain further understanding about the singular structure of distributions by studying the Fourier transform. We will define the Fourier transform for distributions which are of compact support. First we use the notation $e_k(x) = e^{ik\cdot x}$. For a compactly supported distribution u in $\mathscr{D}'(\mathbb{R}^n)$ we then define its Fourier transform by

$$\hat{u}(k) = u(e_k), \tag{2.30}$$

where this notation is really a shorthand for $u(\chi e_k)$ for some χ in $C_0^\infty(\mathbb{R}^n)$ which is equal to unity everywhere on the support of u. For this definition the Fourier transform yields an analytic function of k.

Again we can appeal to the example of the distribution generated by a smooth function, now assumed to have compact support. Then if F in $\mathscr{D}'(\mathbb{R}^n)$ is the distribution corresponding to the function F, now assumed to be in $C_0^\infty(\mathbb{R}^n)$, its Fourier transform is given by

$$\hat{F}(k) = \int F(x) e^{ik\cdot x}\, d^n x, \tag{2.31}$$

and it is clear that this corresponds to the Fourier transform of F when viewed as a function (taking the convention that the form above does in fact define the Fourier transform for a function).

The link between Fourier transforms and singularities is that for compactly supported distributions u there is a correspondence between u being smooth and the Fourier transform of u being 'rapidly decreasing'. We define a cone $V \subset \mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ such that for any k in V we have that λk is in V, for every $\lambda \geq 0$. We then define $\hat{u}(k)$ to be of rapid decrease in V if there exists some constant C_N in $\mathbb{R}$ such that

$$|\hat{u}(k)| \leq C_N |1 + |k||^{-N}, \tag{2.32}$$

for each N in $\mathbb{N}$, where $|k|$ denotes the Euclidean norm of k. If a vector k in $\mathbb{R}^n$ has a conical neighbourhood such that the above definition is satisfied then we call k a direction of rapid decay for u.

It is a well known result in classical Fourier analysis that the Fourier transform of a test function decays rapidly in all directions as $k \to \infty$ [43]. We can then apply this result to compact distributions if they are generated by a smooth function. There is also an even stronger result from Hörmander [27] (Theorem 7.3.1) that the converse holds. To be precise, if u is a distribution in $\mathscr{D}'(\mathbb{R}^n)$ with compact support and if its Fourier transform $\hat{u}$ decays rapidly in all directions, then u can be identified with some function F in $C_0^\infty(\mathbb{R}^n)$. The slightly technical proof of this result will not be given here.

This relationship between smoothness and the rapid decay of the Fourier transform lets us make the interpretation that all of the singularities of a compactly supported distribution u correspond to those 'frequencies' for which the Fourier transform is not rapidly decaying. We can then define the 'singular directions' associated with the singularities of a distribution in a similar manner to how the singular support is constructed. That is, for any compactly supported distribution u in $\mathscr{D}'(\mathbb{R}^n)$ we define the set of singular directions $\Sigma(u)$ of u to be the complement in $\mathbb{R}^n \backslash 0$ of the set of directions in which $\hat{u}$ decays rapidly.

The set of singular directions provides us with a global notion of the directions that are associated with the occurrence of singularities. For the task

of defining the wave-front set, a more local version of this will be required. Crucial to extracting such information is that it can also be shown that for any f in $C_0^\infty(\mathbb{R}^n)$ and u in $\mathscr{D}'(\mathbb{R}^n)$:

$$\Sigma(fu) \subset \Sigma(u). \tag{2.33}$$

This means that no new singular directions can be created by multiplying a compact distribution by a test function. This property allows us to localise the information about singular directions to the extent that we can find the singular directions relating to a single point. Localisation can be achieved by multiplying a distribution u by a test function f. Then since the support of fu is no larger than the support of f we can take the limit of the support of f to a single point $\{x\}$ to find information about the singular directions associated with that point. This is what is achieved by defining the set of singular directions of u in $\mathscr{D}'(\mathbb{R}^n)$ for a point x in $\mathbb{R}^n$ by

$$\Sigma_x(u) = \bigcap_{\substack{f \in C_0^\infty(\mathbb{R}^n) \\ f(x) \neq 0}} \Sigma(fu). \tag{2.34}$$

In this definition it has not been necessary to impose that u is of compact support, as it is always multiplied by compactly supported functions. A consequence of this definition is that we can now see that a distribution u is only smooth at a point x if $\Sigma_x(u) = \emptyset$.

This is finally all of the information needed to define the wave-front set, which is essentially a compact way of expressing this information. For a distribution u in $\mathscr{D}'(\mathbb{R}^n)$ we define the wave-front set of u by

$$\mathrm{WF}(u) = \{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) : k \in \Sigma_x(u)\} \tag{2.35}$$

We can see that this now contains all of the information we could expect to gain about the singularities of u; their positions x along with a corresponding momentum k.

A key property of the wave-front set that is important to the proof of the inequality of interest is that the wave-front set does not expand under the action of partial differential operators with smooth coefficients. The slightly lengthy proof of this result, including some elaboration on the similar proof given in [27], is detailed in appendix B. Of course to analyse states on a curved spacetime the definition of the wave-front set will have to be related to smooth manifolds, for which we will need to study its transformation properties.

2.7 More Properties of the Wave-Front Set and the Hadamard Condition

We will now relate the wave-front set to smooth manifolds. For this we will continue to use material that can be found in [27]. The Hadamard condition [49][50] will then be stated, in terms of the wave-front set [40][29][5]. In order to consistently define the wave-front set for smooth manifolds we need to understand how it translates onto different coordinate systems. In preparation, we first look at some of the structures relating to manifolds. Again, for an elaboration on these structures refer to Jöst [28]. As before, definitions will be given in $\mathbb{R}^n$ before relating these to general n-dimensional manifolds.

To begin with we define the tangent space at a point x_0 in an open subset $X \subset \mathbb{R}^n$ as the space $\{x_0\} \times E_{x_0}$, where E_{x_0} is the set of all tangent vectors at the point x_0. This is denoted by $T_{x_0}(X)$. If one thinks of tangent vectors as directional derivatives, the space E_{x_0} is defined as the n-dimensional vector space spanned by the basis of partial derivatives with respect to the coordinates of $\mathbb{R}^n$, evaluated at x_0. Then if we have another open subset $X' \subset \mathbb{R}^n$ and a differentiable map $f : X \to X'$ it is possible to define a linear map, or 'derivative', $Df(x_o) : T_{x_0}(X) \to T_{f(x_0)}(X')$, induced by f, for any x_0 in X. The action on a vector v in E is then given by:

$$Df(x_0) : v = v^i \frac{\partial}{\partial x^i} \mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j}, \tag{2.36}$$

where the x^i are coordinates in $\mathbb{R}^n$. In particular the v transform as vectors under coordinate changes.

We then define the 'tangent bundle' $T(X)$ of an open subset $X \subset \mathbb{R}^n$ to be the collection of all the tangent spaces of X. That is;

$$T(X) = \bigcup_{x \in X} \{x\} \times E_{x_0}. \tag{2.37}$$

We can then define induced maps between tangent bundles in the same way as for tangent spaces. The definition of the tangent space at a point p in an n-dimensional, differentiable manifold M requires a little more work. First let (X_κ, κ) be a chart in M with p in X_κ. We then say that the tangent space $T_p(X_\kappa)$ has the chart representation $T_{\kappa(p)}(\tilde{X_\kappa})$ for the chart κ. Then if $(X_{\kappa'}, \kappa')$ is another chart containing p, we have that the transition map ϑ induces a vector space isomorphism:

$$D\vartheta(\kappa(p)) : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'}), \tag{2.38}$$

where $\vartheta = \kappa' \circ \kappa^{-1}$. For a point p in M we now consider the set

$$\{(\kappa, v) : (\kappa, X_\kappa) \text{ a chart with } p \in X_\kappa, v \in T_{\kappa(p)}(\tilde{X_\kappa})\}. \tag{2.39}$$

Then for members of this set we define an equivalence relation by $(\kappa, v) \sim (\kappa', v')$ if and only if $v' = D(\kappa' \circ \kappa^{-1})v$, and the resulting space of equivalence classes is what we call the tangent space to M at p. We then define the tangent bundle in the same way as in the case of $\mathbb{R}^n$.

The tangent bundle is a particular example from a wider class of constructions known as 'vector bundles', where in this case the 'fibers' are tangent spaces. More details can be found in [28]. What we are really interested in is actually the cotangent bundle, as it will turn out that the wave-front set is a subset of this particular construction. Firstly, we define the cotangent space at any point x in a manifold M to be the dual space to the tangent space. By the dual space we mean the space of all linear maps from $T_x(M)$ to M. Therefore we have that the elements of the cotangent space are covectors, and we denote the resulting space by $T_x^*(M)$. We then define the cotangent bundle to be the disjoint union over M of cotangent spaces, that is

$$T^*(M) = \bigcup_{x \in M} \{x\} \times T_x^*(M). \tag{2.40}$$

As mentioned, the wave-front set is a subset of the cotangent bundle, so where we referred earlier to the vector k we really should have been calling it a covector. In particular this tells us how elements of the wave-front set behave under transformations, and so the wave-front set of a distribution on a manifold can be defined in terms of charts. Then if (X_κ, κ) is a chart in X, using the transformation properties of covectors we can define $\kappa^*\mathrm{WF}(u \circ \kappa^{-1})$ by

$$\kappa^*\mathrm{WF}(u \circ \kappa^{-1}) = \{(x, (D\kappa(x))^t k) : (\kappa(x), k) \in \mathrm{WF}(u \circ \kappa^{-1})\}, \tag{2.41}$$

where $(D\kappa(x))^t$ denotes the transpose of the derivative of κ. Then κ^* 'pulls back' the wave-front set of $u \circ \kappa^{-1}$ in $\mathbb{R}^n$ onto X. Then we set $\mathrm{WF}(u) = \kappa^*\mathrm{WF}(u \circ \kappa^{-1})$ to give a definition which is independent of the choice of chart.

The wave-front set can now be used to state the Hadamard condition on the two-point functions of states. The Hadamard condition essentially allows one to appropriate those states which can be thought of as 'physical'. It arises from considerations of how to define the stress-energy tensor for quantum fields in curved spacetime [48]. It has already been stated that states are distributions, and that products of distributions are not guaranteed

to be well defined. Indeed, the two point function $\omega_2(\phi(x), \phi(x'))$ is not well defined at the point $x' = x$ [49], which is why the Hadamard condition is required.

After some work on the subject and under several considerations, the Hadamard condition was derived explicitly [50] in a more complicated form than that which will be given here. It has since been shown, originally by Radzikowski, that this is equivalent to a condition on the wave-front sets of two-point functions of states [40][5]. If x and x' are two points separated by a null geodesic γ, where k is cotangent to γ at x, k' is cotangent to γ at x' and k' is the parallel transport of k along γ, then we denote this relation by $(x, k) \sim (x', k')$. Now we can state the Hadamard condition simply by saying that a state ω is Hadamard if and only if

$$\mathrm{WF}(\omega_2) = \{(x, k; x', -k') : (x, k) \sim (x', k'),\ k \text{ is future pointing}\}. \quad (2.42)$$

Where $x' = x$ we take the convention that $(x, k) \sim (x, k')$ if and only if $k = k'$ is null. On general distributions, this condition is known as the 'microlocal spectral condition'. We will use the result, as outlined in [14], that the two-point functions of any Hadamard states share a common singular part.

One advantage to this formulation over the original exposition of the Hadamard condition is that the statement itself is more compact. Also with the condition in this form, techniques of microlocal analysis can be used to derive results in quantum field theory, such as quantum inequalities.

2.8 Some Results Concerning Pullbacks

We have already seen one example of a pullback κ^* on the wavefront set in the previous section. In the following sections we will need to use some technical results on the properties of pullbacks on manifolds, which will be given here. If φ is a smooth map between manifolds $\varphi : X \to Y$ then we can define an induced map φ^* from $\mathscr{D}'(Y)$ to $\mathscr{D}'(X)$ called a pullback of distributions. Under certain conditions this can always defined uniquely in such a way that it is of the form $\varphi^* u = u \circ \varphi$ for any u in $\mathscr{D}'(Y)$ where u is associated with a continuous function. We will need to use this result in the following section. Specifically, define the set of normals for φ to be

$$N_\varphi = \{(\varphi(x), k) \in T^*(X) : (D\varphi(x))^t = 0\}, \quad (2.43)$$

for all x in X. Then if u is in $\mathscr{D}'(Y)$ and $N_\varphi \cap \mathrm{WF}(u) = \emptyset$ the pullback can be uniquely defined, and takes the form given above if u is associated with a continuous function. This is the result of theorem 2.5.11' in [26]. This theorem also says that

$$\mathrm{WF}(\varphi^* u) \subset \varphi^* \mathrm{WF}(u) = \{(x, (D\varphi(x))^t k) : (\varphi(x), k) \in \mathrm{WF}(u)\}. \quad (2.44)$$

A distribution u on $\mathscr{D}'(X\times X)$ is said to be of 'positive type' if $u(\bar{f},f)\geq 0$ for every f in $C_0^\infty(X)$. In particular the two-point function ω_2 is a distribution of positive type, following from the positivity condition on states mentioned in 2.4. Now let $\gamma : X \to Y$ be smooth and let $\varphi : X \times X \to Y \times Y$ be given by $\varphi(\cdot,\cdot) = (\gamma(\cdot),\gamma(\cdot))$. We have that if X and Y are endowed with smooth positive densities ρ_X and ρ_Y then for any u in $\mathscr{D}'(Y \times Y)$ of positive type with $N_\varphi \cap \mathrm{WF}(u) = \emptyset$, $\varphi^* u$ is also of positive type in $\mathscr{D}'(X \times X)$. This is the result from theorem 2.2 in [14] and will also be needed in the next section. The proof of this result requires the use of what is called the 'Hörmander pseudo-topology', the details of which will not be gone into here.

3 Energy Density

In Minkowski spacetime, or indeed static curved spacetimes, we can quantise the stress-energy tensor with little effort as it is simply a case of substituting expressions for the quantum field operators into the expression for the classical stress-energy tensor. The resulting divergent expression for energy density can then be used to define a finite quantity through a normal ordering procedure to subtract the expectation value in the vacuum state. This is the prescription used in previous derivations of quantum inequalities [11][36][37].

In the present case, neither of these methods is available. Without an explicit expression of ϕ to substitute into the expression for T_{ab} we will have to find another way to quantise the energy density. Once this is achieved, there is no preferred vacuum state in this formulation, so this will not be a valid way to achieve a finite expression. The classical stress-energy tensor corresponding to the Klein-Gordon field is given by

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \tfrac{1}{2}g_{ab}g^{cd}\nabla_c\phi\nabla_d\phi + \tfrac{1}{2}m^2\phi^2 g_{ab}. \tag{3.1}$$

We seek to define a quantised version of the energy density T_{00}. This will be achieved by following the same procedure as in [14].

To begin with we need to give the definition of another example of a vector bundle, called the 'normal bundle'. First, if N is a submanifold of a manifold M equipped with a metric h, $\dim N < \dim M$, then we define the normal space at x in N [28] by

$$T_x^\perp = \{v \in T_x(M) : \langle v, w\rangle = 0, \forall w \in T_x(N)\}, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product induced by h [31]. The normal bundle is then the disjoint union of these spaces over all x in N. We can now define a 'tubular neighbourhood' [33].

First let $N \subset M$ be a submanifold with $\dim N < \dim M$, and denote the normal bundle of N by $T^{\perp}(N)$. Then an open set $\Omega \subset M$ containing N is called a tubular neighbourhood of N in M if there is an open neighbourhood Z of the zero section of $T^{\perp}(N)$ and a diffeomorphism $f : Z \to \Omega$ such that $f(0_x) = x$ for any zero vector 0_x in $T^{\perp}(N)$ corresponding to x in N.

In (M, g) we let γ be some smooth, timelike curve and let Γ be a tubular neighbourhood of γ. We can now define an orthonormal frame in Γ, $\{v_0^a, \ldots, v_{n-1}^a\}$, where the v_μ^a are mutually orthonormal with respect to the scalar product, so that $g^{ab} = \eta^{\mu\nu} v_\mu^a v_\nu^b$. Here η is the usual Minkowski metric, or the matrix $diag[1, -1, \cdots, -1]$. We choose this frame such that on γ, v_0^a is equal to the four-velocity $\dot{\gamma}(\tau) \equiv u(\tau)$ of the trajectory. The classical energy density as measured by an observer along γ is then given by $u^a(\tau) u^b(\tau) T_{ab}(\gamma(\tau))$, which will be denoted by $T(\tau)$.

Now looking at the middle term of 3.1 on γ we can substitute in the expression for g^{ab} to get:

$$\begin{aligned} -\tfrac{1}{2} g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi &= -\tfrac{1}{2} g_{ab} \eta^{cd} v_c^e v_d^f \nabla_e \phi \nabla_f \phi \\ &= -\tfrac{1}{2} g_{ab} v_0^c v_0^d \nabla_c \phi \nabla_d \phi + \tfrac{1}{2} \sum_{\mu=1}^{n-1} g_{ab} v_\mu^c v_\mu^d \nabla_c \phi \nabla_d \phi. \end{aligned} \tag{3.3}$$

We also have that on γ;

$$\begin{aligned} u^a u^b g_{ab} &= v_0^a v_0^b \eta_{\mu\nu} v_a^\mu v_b^\nu \\ &= \eta_{\mu\nu} \delta_0^\mu \delta_0^\nu \\ &= 1. \end{aligned} \tag{3.4}$$

Now putting this together we can write $T(\tau)$:

$$\begin{aligned} T(\tau) &= v_0^a v_0^b \nabla_a \phi \nabla_b \phi - \tfrac{1}{2} v_0^a v_0^b \nabla_a \phi \nabla_b \phi + \tfrac{1}{2} \left(\sum_{\mu=1}^{n-1} v_\mu^a v_\mu^b \nabla_a \phi \nabla_b \phi \right) + \tfrac{1}{2} m^2 \phi^2 \\ &= \tfrac{1}{2} \left(\sum_{\mu=0}^{n-1} v_\mu^a v_\mu^b \right) \nabla_a \phi \nabla_b \phi + \tfrac{1}{2} m^2 \phi^2. \end{aligned} \tag{3.5}$$

We now employ what is known as a point splitting technique to write this as a bi-scalar field, allowing us to quantise using the two-point function. We write the expression

$$T(\tau, \tau') = \tfrac{1}{2} \left(\sum_{\mu=0}^{n-1} v_\mu^a(\gamma(\tau)) v_\mu^{b'}(\gamma(\tau')) \right) \nabla_a \phi|_{\gamma(\tau)} \nabla_{b'} \phi|_{\gamma(\tau')} + \tfrac{1}{2} m^2 \phi(\gamma(\tau)) \phi(\gamma(\tau')), \tag{3.6}$$

where we can clearly obtain $T(\tau)$ by taking $\tau' = \tau$ in the above. Here we have used point-splitting only on the energy density. Similar techniques can be employed to try to obtain a quantised expression for the stress-energy tensor, which are discussed in [32].

The energy density has now been written in a form which can easily be quantised. First let ω be any state on $\mathfrak{A}(M, g)$ which obeys the Hadamard condition. We then define the bidistribution $\langle T \rangle_\omega$ by

$$\langle T \rangle_\omega = \tfrac{1}{2} \sum_{\mu=0}^{n-1} \varphi^* \left((v_\mu^a \nabla_a, v_\mu^{b'} \nabla_{b'}) \omega_2 \right) + \tfrac{1}{2} m^2 \varphi^* \omega_2. \tag{3.7}$$

Here we have that φ^* denotes the pullback from $M \times M$ to $\mathbb{R}^2$ induced by the smooth function $\varphi : \mathbb{R}^2 \to M \times M$ defined as $\varphi(\tau, \tau') = (\gamma(\tau), \gamma(\tau'))$. Therefore we have that $\langle T \rangle_\omega$ is defined on $\mathbb{R}^2$. To check whether the pullback φ^* is well defined it is necessary to find the set of normals for φ. We start by calculating the Jacobian:

$$D\varphi(\tau, \tau') = \begin{bmatrix} \frac{\partial \gamma^1}{\partial \tau} & \frac{\partial \gamma^1}{\partial \tau'} \\ \vdots & \vdots \\ \frac{\partial \gamma^{n-1}}{\partial \tau} & \frac{\partial \gamma^{n-1}}{\partial \tau'} \end{bmatrix}, \tag{3.8}$$

so that we have

$$\begin{aligned} (D\varphi(\tau, \tau'))^t (k, k') &= \begin{bmatrix} \frac{\partial \gamma^1}{\partial \tau} & \cdots & \frac{\partial \gamma^{n-1}}{\partial \tau} \\ \frac{\partial \gamma^1}{\partial \tau'} & \cdots & \frac{\partial \gamma^{n-1}}{\partial \tau'} \end{bmatrix} \begin{bmatrix} k_1 & k'_1 \\ \vdots & \vdots \\ k_{n-1} & k'_{n-1} \end{bmatrix} \\ &= (\dot{\gamma}^a(\tau) k_a, \dot{\gamma}^{b'}(\tau') k'_{b'}), \end{aligned} \tag{3.9}$$

where (k, k') is in $T^*_{(\gamma(\tau), \gamma(\tau'))}(M \times M)$. That is, $(D\varphi(\tau, \tau'))^t$ is a linear map onto $\mathbb{R}^2$ such that

$$(D\varphi(\tau, \tau'))^t (k, k') = (u^a(\tau) k_a, u^{b'}(\tau') k'_{b'}). \tag{3.10}$$

Then from this equation it follows that the set of normals for φ is given by

$$N_\varphi = \{ (\gamma(\tau), k; \gamma(\tau'), k') : k_a u^a(\tau) = k'_{b'} u^{b'}(\tau') = 0 \}. \tag{3.11}$$

Now it would clearly be the case that $\langle T \rangle_\omega$ would not be well defined were the pullback $\varphi^* \omega_2$ not well defined.

We have that the pullback is defined well if $N_\varphi \cap \mathrm{WF}(\omega_2) = \emptyset$. If a point $(x, k; x', k')$ is in this intersection we have that $x = \gamma(\tau)$ and $x' = \gamma(\tau')$ for

some τ and τ'. We then have that k and k' are only in $\mathrm{WF}(\omega_2)$ if they are both null. Furthermore, we require that $k_a u^a(\tau) = 0$ and $k'_{b'} u^{b'}(\tau') = 0$, where $u^a(\tau)$ and $u^{b'}(\tau')$ are non-zero timelike velocities. Since this condition cannot be fulfilled by any (non-zero) null vectors [41], it follows that the intersection is indeed empty, and so $\varphi^*(\omega_2)$ is well defined. What's more, due to the fact that wave-front sets cannot be added to under the action of partial differential operators, it can also be concluded from this argument that the other terms in $\langle T \rangle_\omega$ are well defined.

From comments in the previous sections and the fact that ω_2 is of positive type, it follows that $\varphi^*\omega_2$ is also of positive type. The other terms in the expression for the quantised energy density are of the form

$$\varphi^*((v^a_\mu \nabla_a, v^{b'}_\mu \nabla_{b'})\omega_2). \tag{3.12}$$

Then since

$$(v^a_\mu \nabla_a, v^{b'}_\mu \nabla_{b'})\omega_2(\bar{f}, f) = \omega_2(\overline{\nabla_a(v^a_\mu f)}, \nabla_{b'}(v^{b'}_\mu f)) \tag{3.13}$$

and this is positive because of ω_2 being of positive type, it follows that all of the terms of the quantised energy density are of positive type. Therefore $\langle T \rangle_\omega$ is of positive type.

To prove the inequality in the next section it will also be necessary to know about the behaviour of the product of the Fourier transform of $\langle T \rangle_\omega$ with a smooth function. Everything needed can be found from the wave-front set. Firstly, it follows from equation 2.44 that if $(\tau, \xi; \tau', -\xi')$ is in $\mathrm{WF}(\varphi^*\omega_2)$ then $(\tau, \xi; \tau', -\xi')$ is in $\mathrm{WF}(\omega_2)$, so

$$\begin{aligned} (\xi, -\xi) &= (D\varphi(\tau, \tau'))^t (k, -k') \\ &= (u^a(\tau) k_a, -u^{b'}(\tau') k'_{b'}). \end{aligned} \tag{3.14}$$

Also from equations 2.42 and 2.44 it follows that

$$(\gamma(\tau), k) \sim (\gamma(\tau'), -k'), \tag{3.15}$$

where k and k' are future pointing. It then follows from the fact that k and k' are required to be future pointing and the velocities u_a are future pointing that ξ must be positive. This is as much as is required.

From proposition 8.1.3. in [27] the projection of the wave-front set onto the second variable gives the set of singular directions. If we let g be any function in $C^\infty_0(\mathbb{R}^n)$ it then follows from the non-expansion of the wave-front set under multiplication by smooth functions that

$$\mathrm{WF}(g\langle T \rangle_\omega) \subset \mathrm{WF}(\langle T \rangle_\omega). \tag{3.16}$$

Hence, by the preceding comments

$$\Sigma(g\langle T\rangle_\omega) \subset \Sigma(\langle T\rangle_\omega) \subset \{(\xi, -\xi') : \xi, \xi' > 0\}. \tag{3.17}$$

If we then denote the Fourier transform of $g\langle T\rangle_\omega$ by

$$[g\langle T\rangle_\omega]^\wedge(-\alpha, \alpha), \tag{3.18}$$

it then follows from 3.17 that this decays rapidly in all directions as $\alpha \to +\infty$.

4 Proof of the Inequality

The techniques and results that have been described thus far can now be put to use in the context of the main motivation of this report; that is, the proof of a quantum inequality. The aim of this section is to describe how to prove that the inequality

$$\int d\tau\, (g(\tau))^2 \langle : T : \rangle_\omega(\tau, \tau') \geq -\int_0^\infty \frac{d\alpha}{\pi}\, [(g,g)\langle T\rangle_{\omega_0}]^\wedge(-\alpha, \alpha) \tag{4.1}$$

holds for all real g in $C_0^\infty(\mathbb{R})$. In the above, $\langle : T : \rangle_\omega$ is the normal-ordered energy density in the state ω with respect to the state ω_0, where both are Hadamard.

In the usual Hilbert space formulation of quantum field theory normal ordering is conventionally employed to define a finite expected energy density for any state ψ by subtracting the expected energy density for the vacuum state [3], so one defines

$$\langle : T_{00} : \rangle_\psi = \langle \psi | T_{00} | \psi \rangle - \langle 0 | T_{00} | 0 \rangle. \tag{4.2}$$

Obviously in the present formulation there is no state which can be picked out as a preferred vacuum state, so to overcome this problem we take the logical step of employing normal ordering of the energy density for a state ω with respect to an arbitrary state ω_0 to achieve a finite expression for the energy density; that is

$$\langle : T : \rangle_\omega = \langle T\rangle_\omega - \langle T\rangle_{\omega_0}. \tag{4.3}$$

To prove the result it is required to show three things. It has to be shown that $\langle : T : \rangle_\omega$ is smooth and that the inequality holds. Then it must be shown that the integral on the right converges to a finite limit, so that the result is more than trivial. To begin with it follows quite straightforwardly that $\langle : T : \rangle_\omega$ is smooth. It was mentioned in 2.7 that all Hadamard two-point

functions have the same singular structure, so it follows that $\omega_2 - \omega_2^{(0)}$ is smooth, and then from the definition of the energy density we have that $\langle : T : \rangle_\omega$ is smooth.

Now it will be shown that the inequality of the above form can be derived. To begin with we can write the LHS in terms of two variables by introducing the delta distribution into the integrand. We can then express this by integrating over another variable, so we have

$$\begin{aligned} \int d\tau \, (g(\tau))^2 \langle : T : \rangle_\omega(\tau, \tau) &= \int d\tau d\tau' \, g(\tau) g(\tau') \langle : T : \rangle_\omega(\tau, \tau') \delta(\tau - \tau') \qquad (4.4) \\ &= \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \int d\tau d\tau' \, g(\tau) g(\tau') e^{-i\alpha(\tau - \tau')} \langle : T : \rangle_\omega(\tau, \tau'). \end{aligned}$$

Then writing $g_\alpha(\tau) = g(\tau) e^{i\alpha\tau}$ we can write this as

$$\int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \, \langle : T : \rangle_\omega(g_{-\alpha}, g_\alpha). \qquad (4.5)$$

Using the fact that all two-point functions are formed of a common anti-symmetric part together with a real symmetric part, as shown in 2.4, it follows that $\langle : T : \rangle_\omega$ is symmetric; hence the integrand is symmetric in α. This means that the integral can be written as

$$\int_{0}^{\infty} \frac{d\alpha}{\pi} \, \langle : T : \rangle_\omega(g_{-\alpha}, g_\alpha). \qquad (4.6)$$

Now since g is real, it follows that $g_{-\alpha} = \bar{g_\alpha}$. Then we can use the fact that $\langle T \rangle_\omega$ is of positive type to write this as an inequality so that we have

$$\int d\tau \, (g(\tau))^2 \langle : T : \rangle_\omega(\tau, \tau') \geq - \int_{0}^{\infty} \frac{d\alpha}{\pi} \, \langle T \rangle_{\omega_0}(\bar{g_\alpha}, g_\alpha). \qquad (4.7)$$

This can be compared to the derivation of the inequality seen in [11], where the inequality in that case stems from the fact that the product of an operator with its adjoint is positive. This gives an idea of how a more sophisticated approach was required to prove an inequality in more general curved spacetimes.

It now remains to show that the integral on the right hand side converges, and in that case this is indeed a valid quantum inequality. This can be achieved by a simple re-writing of the integrand:

$$\begin{aligned} \langle T \rangle_{\omega_0}(\bar{g_\alpha}, g_\alpha) &= \langle T \rangle_{\omega_0}((g, g) e_{(-\alpha, \alpha)}) \\ &= [(g, g) \langle T \rangle_{\omega_0}](e_{(-\alpha, \alpha)}). \qquad (4.8) \end{aligned}$$

Then by the definition of the Fourier transform for distributions this can now be written as

$$[(g,g)\langle T\rangle_{\omega_0}]^{\wedge}(-\alpha,\alpha). \tag{4.9}$$

It now follows that this does indeed converge to a finite limit when integrated over α from 0 to ∞, as a result of the comment at the end of the previous section. This completes the proof that the inequality 4.1 is valid.

5 Other Results: An Optimal Bound

5.1 Sketch of the Proof

It should be noted that the bound in the previous section is not shown to be an optimal one. Furthermore, as mentioned in [14], the bound may depend on the choice of frame used to define the split-point energy density. There is a case where a Klein-Gordon quantum inequality has been shown to be a sharp bound, proved by Flanagan [19] for the case of a massless two-dimensional field in Minkowski spacetime. This is also a bound valid for arbitrary smooth and compactly supported test functions. Unfortunately the proof uses features specifically related to the two-dimensional massless field, which means that a generalisation of the bound given to higher dimensional cases is not possible. A sketch of the proof will be given, which should illustrate this point and show how other methods can be used to derive quantum inequalities.

First let $E_D(g)$ and $E_F(g)$ denote respectively the energy density and energy flux, smeared by the test function g with respect to time. The proof in [19] actually places a bound on the energy flux, as it is shown that in two dimensions $\min\langle E_D(g)\rangle = 2\min\langle E_F(g)\rangle$, where the minimum is taken over all states. Another feature of two-dimensional spacetimes is that there is only two spatial directions, and therefore the quantum theory can be split into the left moving and right moving components, by introducing the coordinates $u = t + x$ and $v = t - x$. The field operator $\hat{\phi}(x,t)$ can then be written as $\hat{\phi}^R(v) + \hat{\phi}^L(u)$, where these represent the right and left moving sectors of the theory, respectively. It is already clear at this stage that this proof cannot be replicated in higher dimensions.

As the energy flux is only positive in two dimensions for right moving excitations it follows that

$$\min\langle E_D(g)\rangle = 2\min\langle E^R(g)\rangle, \tag{5.1}$$

where $E^R(g)$ denotes the smearing of the right moving energy flux by g. This means that by finding a bound on the time smeared right moving energy flux

one also has a bound on the time smeared energy density (and indeed the space smeared energy density, by symmetry in two-dimensional Minkowski spacetime). The smeared temporal average of the right moving energy flux is then given explicitly by

$$E^R(g) = \int dv\, g(v) T_{vv}(v), \tag{5.2}$$

where of course T_{ab} is the (normal ordered) stress-energy tensor. Here an idea of the proof using Hilbert space representations of the theory is given, but it should be noted that this is made rigorous in [19] by generalising the same argument in an algebraic formulation.

The right moving field operator can be expanded in the usual way [44] as

$$\hat{\phi}^R(v) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}} [e^{-i\omega v} \hat{a}_\omega + e^{i\omega v} \hat{a}_\omega^\dagger], \tag{5.3}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators. Now a coordinate transformation $v \to V$ can be defined by $V = f(v)$ where f is a monotonically increasing bijection from $\mathbb{R}$ onto itself. An expansion of the right moving field can then be given in terms of this new coordinate, though still having its argument in terms of the v coordinate, by

$$\hat{\phi}^R(v) = \hat{\phi}^R(f^{-1}(V)) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}} [e^{-i\omega V} \hat{b}_\omega + e^{i\omega V} \hat{b}_\omega^\dagger]. \tag{5.4}$$

It follows that since f is bijective, the creation and annihilation operators used in the two expansions will be members of the same algebra, and can therefore be written as linear combinations of one another. Assume, then, that there is a unitary operator U such that

$$U \hat{a}_\omega U^\dagger = \hat{b}_\omega. \tag{5.5}$$

A key point in the proof is that such an operator will not exist in all cases, and this is where the argument is made more rigorous in an algebraic setting. However, the main structure of the proof in that case is essentially the same, so in this sketch we assume that such a U exists. We then have that $U^\dagger \hat{\phi}^R(v) U = \hat{\phi}(f(v))$ by the expansion given in terms of V.

The next part of the proof uses the Hadamard condition in the form given in [50]. Using this condition the right moving component of the normal ordered stress-energy tensor can be written in a split-point form as

$$\lim_{v' \to v} \partial_{v'} \partial_v [\hat{\phi}^R(v') \hat{\phi}^R(v) - H(v - v')]. \tag{5.6}$$

This follows from the fact that in the u and v coordinates the non-vanishing terms of the stress-energy tensor are $T_{vv} = (\partial_v \hat{\phi}^R)^2$ and $T_{uu} = (\partial_u \hat{\phi}^L)^2$. Then on normal ordering of these terms, H is the distribution that is subtracted, where H has the same singular structure as any of the states in the theory. Here H is given by

$$H(\Delta v) = -\frac{1}{4\pi}[\ln|\Delta v| + \pi i \Theta(-\Delta v)], \tag{5.7}$$

where Θ denotes the Heaviside step function.

It follows that

$$U^\dagger T_{vv} U = \lim_{v' \to v} \partial_{v'} \partial v [\hat{\phi}^R(f(v'))\hat{\phi}^R(f(v)) - H(v - v')], \tag{5.8}$$

and writing $\frac{\partial}{\partial v} = \frac{dV}{dv}\frac{\partial}{\partial V} = V'(v)\partial_V$, we have that this is equal to

$$\begin{aligned}
& \lim_{v' \to v} V'(v)^2 \partial_V \partial_{V'} \hat{\phi}^R(V')\hat{\phi}^R(V) - \partial_{v'}\partial_v H(v - v') \\
= & \lim_{v' \to v} V'(v)^2 \partial_V \partial_{V'} [\hat{\phi}^R(V')\hat{\phi}^R(V) - H(V - V')] + \partial_{v'}\partial_v H(V - V') - \partial_{v'}\partial_v H(v - v') \\
= & \; V'(v)^2 : (\partial_V \hat{\phi}^R(V))^2 : - \Delta(v) \\
= & \; V'(v)^2 T_{vv}(V) - \Delta(v),
\end{aligned} \tag{5.9}$$

where $\Delta(v)$ now denotes $\lim_{v' \to v} \partial_v \partial_{v'}[H(v - v') - H(f(v) - f(v'))]$, and $: \cdot :$ denotes normal ordering. It can then be calculated directly from the expression given for H that $\Delta(v)$ can be written as

$$\Delta(v) = -\frac{1}{12\pi}\sqrt{V'(v)}\left(\frac{1}{\sqrt{V'(v)}}\right)'', \tag{5.10}$$

where dash denotes the derivative with respect to v.

Now set the coordinate V to be such that $g(v)'(V) = 1$ for some test function g. Then if we integrate the expression on the right of equation 5.9 against g we find that

$$U^\dagger E^R(g) U = \int dV \; T_{VV}(V) - \int dv \; g(v)\Delta(v), \tag{5.11}$$

as $dv = \frac{dV}{dv} dv$. We then have that the first term on the right hand side is just the Hamiltonian H^R, so that

$$U^\dagger E^R(g) U = H^R - I, \tag{5.12}$$

where $I = -\frac{1}{12\pi}\int dv\,\sqrt{g(v)}(\sqrt{g(v)})''$, following from 5.10. Then integrating by parts it is found that

$$\begin{aligned} I &= -\frac{1}{12\pi}\left\{\frac{1}{2}\left[\frac{g'(v)}{\sqrt{g(v)}}\left(\sqrt{g(v)}\right)''\right]_{-\infty}^{\infty} - \frac{1}{2}\int dv\,\frac{g'(v)}{\sqrt{g(v)}}\left(\sqrt{g(v)}\right)'\right\} \\ &= -\frac{1}{12\pi}\left\{-\frac{1}{4}\int dv\,\frac{g'(v)^2}{g(v)}\right\} \\ &= \frac{1}{48\pi}\int dv\,\frac{g'(v)^2}{g(v)}, \end{aligned} \tag{5.13}$$

where it is assumed that g', since g is a test function, vanishes at infinity.

This argument has now provided a lower bound, since we know that the Hamiltonian is a positive operator, and its least possible eigenvalue is therefore zero. Hence this shows that

$$\min\langle E^R(g)\rangle \geq -I. \tag{5.14}$$

Importantly in this argument it can also be shown that this lower bound is attained. If we let $|0_V\rangle$ be the vacuum state in the V coordinate, given by $|0_V\rangle = U|0\rangle$ then looking back to equation 5.9 it is clear that this state annihilates the first term on the right hand side, and so it follows that

$$\langle E^R(g)\rangle_{0_V} = -I. \tag{5.15}$$

Hence it can be concluded that the bound is indeed sharp. The state $|0_V\rangle$ provides an example of a theoretically existing state which admits a negative value for its expected energy density. Then using the relationship given by 5.1, it follows that this proves the existence of the sharp quantum inequality

$$\min\langle E_D(g)\rangle = -\frac{1}{24\pi}\int dt\,\frac{g'(t)^2}{g(t)}. \tag{5.16}$$

5.2 Comparison

It should be noted that the bound 4.1 as proved here is not totally guaranteed to be valid for a massless, two-dimensional field (see footnote 1 in [14]). However, it is quite reasonable to assume and can be proved that it is valid, so we make this assumption and will look at how it compares in this case with the optimum bound. In particular, we will employ the test function used by Ford and Roman in their quantum inequalities [21][22] to see how both of the bounds considered here compare to those inequalities as well.

It is shown in the introduction of [14] how the inequality 4.1 reduces to the bound derived in [11] in the case of a static observer in a static spacetime. From this point we denote the weighted energy density by ρ, that is

$$\rho = \int_{-\infty}^{\infty} dt \, \langle : T_{\mu\nu} u^\mu u^\nu : \rangle g(t), \tag{5.17}$$

for a smearing function g. Then in a static spacetime for a static observer the inequality 4.1 reduces to

$$\rho \geq -\frac{1}{\pi} \int_0^\infty d\omega \sum_\lambda \left(\frac{\omega_\lambda^2}{|g_{tt}|} + \frac{1}{4} \nabla^i \nabla_i \right) |U_\lambda|^2 |\widehat{g^{1/2}}(\omega + \omega_\lambda)|^2. \tag{5.18}$$

In the above, the scalar field ϕ is assumed to be able to be expanded by a set of mode functions $f_\lambda(t, \underline{x}) = U_\lambda(\underline{x}) e^{-i\omega_\lambda t}$. Then $\phi = \sum_\lambda (a_\lambda f_\lambda + a_\lambda^\dagger f_\lambda^*)$, where a and $a^\dagger$ are the creation and annihilation operators. This is the 'usual' formulation of quantum field theory, as comprehensively covered in [3].

In a massless scalar field theory in $(n+1)$-dimensional Minkowski spacetime, which is the case we are interested in for this comparison, the field can be expanded in terms of modes $U_{\underline{k}}$, where

$$U_{\underline{k}}(\underline{x}) = \frac{1}{[(2\pi)^n 2|\underline{k}|]^{1/2}} e^{i\underline{k}.\underline{x}}, \tag{5.19}$$

with the momentum vectors k_i such that $-\infty < k_i < \infty$ [44][4][24]. We then have that

$$\begin{aligned} |U_{\underline{k}}|^2 &= U_{\underline{k}} \overline{U_{\underline{k}}} \\ &= \frac{1}{(2\pi)^n 2|\underline{k}|} \end{aligned} \tag{5.20}$$

and $g_{tt} = 1$. Then substituting these into 5.18 yields:

$$\rho \geq -\frac{1}{2\pi} \int_0^\infty d\omega \int \frac{d^n \underline{k}}{(2\pi)^n} \, |\underline{k}| (\widehat{g^{1/2}}(\omega + |\underline{k}|))^2. \tag{5.21}$$

We then have that the only term in the integrand dependent on $\underline{k}$ is $|\underline{k}|$, the length of $\underline{k}$. Therefore writing $|\underline{k}| = r$ we can evaluate the integral over $n-1$ angular directions, to obtain

$$\rho \geq -\frac{C_n}{2\pi} \int_0^\infty d\omega \int_0^\infty dr \, r^n (\widehat{g^{1/2}}(\omega + r))^{1/2}, \tag{5.22}$$

where C_n is the area of the unit $(n-1)$-sphere divided by 2π:

$$C_n = \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \Gamma(\frac{1}{2} n)}. \tag{5.23}$$

This can then be written in a simpler form by making the change of variables $u = \omega + r$ and $v = r$ to give

$$\begin{aligned} \rho &\geq -\frac{C_n}{2\pi}\int_0^\infty du\,(\widehat{g^{1/2}}(u))^{1/2}\int_0^u dv\,v^n \\ &= -\frac{C_n}{2\pi(n+1)}\int_0^\infty du\,(\widehat{g^{1/2}}(u))^{1/2}u^{n+1}. \end{aligned} \tag{5.24}$$

Now substituting in the value $n = 1$, as this is the case we are interested in here, we have

$$\begin{aligned} \rho &\geq -\frac{1}{4\pi^2}\int_0^\infty du\,(\widehat{g^{1/2}}(u))^{1/2} \\ &= -\frac{1}{8\pi^2}\int_{-\infty}^\infty du\,(\widehat{g^{1/2}}(u))^{1/2} \end{aligned} \tag{5.25}$$

where the convention has been taken that $C_1 = \frac{1}{\pi}$. Then the symmetry in u of the integrand has been used to write this as an integral over all of $\mathbb{R}$. By differentiating the Fourier transform, in the conventions used here, it can be seen that

$$\widehat{g^{1/2}}'(u) = -iu\int_0^\infty g^{1/2}(t)e^{-iut}\,dt, \tag{5.26}$$

so that

$$(\widehat{g^{1/2}}'(u))^2 = u^2\widehat{g^{1/2}}(u). \tag{5.27}$$

Then with the application of Parseval's theorem [39], again under these conventions, it follows that

$$2\pi\int_{-\infty}^\infty dt\,(g^{1/2\prime}(t))^2 = \int_{-\infty}^\infty du\,u^2(\widehat{g^{1/2}}(u)), \tag{5.28}$$

from which it follows that

$$\begin{aligned} \rho &\geq -\frac{1}{4\pi}\int_\infty^\infty dt\,(g^{1/2\prime}(t))^2 \\ &= -\frac{1}{16\pi}\int_{-\infty}^\infty dt\,\frac{g'(t)^2}{g(t)}. \end{aligned} \tag{5.29}$$

This is now in the same form as the bound 5.16, so a direct comparison can be made. It is seen that this bound is a factor of $\frac{3}{2}$ weaker than the optimal bound for a massless field in two-dimensional flat spacetime.

We can now put the Lorentzian sampling function employed by Ford and Roman into either of these bounds to see how both compare to the result given in [21]. The Lorentzian function is given by

$$g(t) = \frac{t_0}{\pi(t^2 + t_0^2)}, \tag{5.30}$$

For a sampling time of t_0. Putting this into 5.29 gives

$$\rho \geq -\frac{1}{16\pi}\int_{-\infty}^{\infty} dt\, \frac{4t^2 t_0}{\pi(t^2+t_0^2)^3}, \tag{5.31}$$

and upon evaluating the integral it is found that

$$\rho \geq -\frac{1}{32\pi t_0^2}. \tag{5.32}$$

It can be seen that in comparison with the bound derived by Ford and Roman [21] this result is stronger by a factor of four when the same sampling function is employed. By extension, the optimal bound given by Flanagan is a factor of six stronger than that in [21] in the case of this sampling function.

6 Concluding Remarks

This report focused on exploring the techniques which were required to derive the inequality proved by Fewster [14], which was a more general version of a previous bound [11]. The argument explained here resulted in an inequality that not only held in a wider class of spacetimes and for all timelike observers, it also used a more mathematically rigorous environment to derive this bound. The result is not only an inequality which is more general, it is also valid for a well defined and large class of states; all of those which are globally Hadamard. For the derivation given in [11] it was reasonable to assume that the result was valid for a large class of states, but nothing in that derivation provided a specific condition on which states these are.

Generally when it comes to physical theories, one tends to associate making a result more mathematically rigorous with adding extra conditions, therefore narrowing the scope of the result in question. This particular result is an interesting example that this does not necessarily have to be the case, as here the argument has at the same time been made not only more rigorous but also more general.

It was mentioned in the introduction that classical energy conditions are an important feature of proofs of the singularity theorems; recent research relating to quantum inequalities involves using quantum inequality style bounds to prove analogous results to the classical singularity theorems that are compatible with quantum theory [10].

A Appendix A

It will be shown here that any solution φ to the Klein-Gordon equation can be written in the form $\varphi = Ef$ for some f in $C_0^\infty(M)$, where E represents the advanced minus retarded solution detailed in 2.3. (cf. Lemma 3.2.1 in [49])

Proof. First impose on M a 'slicing' by spacelike Cauchy surfaces, Σ_t, labelled by some 'time' parameter t (see prop.2.6 in [7]). Then let φ be a real solution to the Klein Gordon equation and χ be some smooth function on spacetime such that $\chi = 0$ for $t < 0$ and $\chi = 1$ for $t > 1$, and define the function f by:

$$f = -(g^{\alpha\beta}\nabla_\alpha\nabla_\beta + m^2)(\chi\varphi). \tag{A.1}$$

We see that f is compactly supported on the region $0 \leq t \leq 1$, and f is smooth because we have imposed that χ is smooth. Also, we can write the advanced and retarded solutions $E^- = (1-\chi)\varphi$ and $E^+ = -\chi\varphi$. We can see that these are solutions to the Klein-Gordon equation with source f and have support satisfying the correct definitions. Then we have $(E^- - E^+)(f) = \varphi$, so $Ef = \varphi$. □

B Appendix B

Here it will be shown that the wave-front set of any distribution u in $\mathscr{D}'(\mathbb{R}^n)$ is not expanded by the action upon u of any partial differential operator with smooth coefficients.

Proof. To see this let u be in $\mathscr{D}'(\mathbb{R}^n)$. Then take functions χ and χ_1 in C_0^∞ such that χ is equal to one near to x and χ_1 is equal to one on a neighbourhood of the support of χ, where x is a point in $\mathbb{R}^n$. Then since $\chi(x) \neq 0$, it follows from the definition in 2.34 that

$$\Sigma_x(D^\alpha u) \subset \Sigma(\chi D^\alpha u). \tag{B.1}$$

Further, from the fact that χ_1 is equal to one everywhere on the support of χ, hence also of $\chi D^\alpha u$, we have

$$\begin{aligned} \Sigma(\chi D^\alpha u) &= \Sigma(\chi D^\alpha \chi_1 u) \\ &\subset \Sigma(D^\alpha \chi_1 u). \end{aligned} \tag{B.2}$$

The second line follows from the non-expansion of the set of singular directions under the action of smooth functions, using the fact that χ is smooth. Also, if $k \notin \Sigma\chi_1 u$ then $k \notin \Sigma D^\alpha \chi_1 u$, since for each j we have

$$\widehat{\partial_j^{\alpha_j}\chi_1 u}(k) = (ik_j)^{\alpha_j}\widehat{\chi_1 u}(k), \tag{B.3}$$

by lemma 7.1.3 in [27]. Hence we have that

$$\Sigma D^{\alpha}\chi_1 u \subset \Sigma\chi_1 u. \tag{B.4}$$

It is shown in proposition 8.1.3 in [27] that, as one would expect, the projection of the wave-front set onto the second variable is the set of singular directions. Therefore to complete the proof we just take the limit of the support of χ_1 to $\{x\}$ and this gives us that

$$\Sigma_x D^{\alpha} u \subset \Sigma_x u, \tag{B.5}$$

for all x in $\mathbb{R}^n$. As the set of singular directions is also not expanded under the action of smooth functions, it follows that the wave-front set cannot be expanded under the action of any partial differential operators with smooth, possibly non-constant, coefficients. □

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