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This thesis is devoted to a better understanding of the deep interconnection between **fluid dynamics** and **gravitational physics** by means of **holography**, a recent paradigm stating that any theory of gravity can be equivalently described by a quantum field theory in one dimension less.

Central to the discussion is the **effective action** formulation of fluid dynamics, **holography beyond** the best understood example of **Anti-de Sitter** spacetime and the role of the **membrane paradigm** as an approximation scheme to neglect the interior of a black hole.

A Holographic Journey From Fluids to Black Holes



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A HOLOGRAPHIC JOURNEY
FROM FLUIDS
TO BLACK HOLES

This work has been accomplished at the Institute for Theoretical Physics (ITFA) of the University of Amsterdam (UvA) and is part of the research programme of the Foundation for Fundamental Research on Matter (FOM), which is part of the Netherlands Organisation for Scientific Research (NWO).

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A HOLOGRAPHIC JOURNEY FROM FLUIDS TO BLACK HOLES

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ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

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prof. dr. D.C. van den Boom

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PUBLICATIONS

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- [1] J. de Boer, M. P. Heller and N. Pinzani-Fokeeva,
“*Effective actions for relativistic fluids from holography*“,
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Presented in Chapter 3.

- [2] N. Pinzani-Fokeeva and M. Taylor,
“*Towards a general fluid/gravity correspondence*“,
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Presented in Chapter 4 and Chapter 5.

- [3] J. de Boer, M. P. Heller and N. Pinzani-Fokeeva,
“*Testing the membrane paradigm with holography*“,
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Presented in Chapter 5.

OTHER PUBLICATIONS BY THE AUTHOR:

- [4] F. Bigazzi, A. L. Cotrone, D. Musso, N. Pinzani-Fokeeva, D. Seminara
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The author participated to all the conceptual discussions in all the publications. In [1] the author contributed to Section 2, to Section 4 in collaboration with M. P. Heller and to Section 5. In [2] the author contributed to parts of Section II, to Section IIIA and IIIB, to Section IV except IVB, to Section V and to Section VI. In [3] the author contributed to an earlier calculation on similar grounds of Section II, partly to the comment [17] and to Section III.

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1

PRELUDE

Fluids

The water in the canals of Amsterdam, the air circulating in the atmosphere, and for example our morning coffee, have a common property: they are all *fluids*. By definition, they tend to assume the form of the vessel they are contained in, contrary to solid systems which have a defined form. Hence, differently from common language, in physics the term fluid refers to both liquid and gas phases of matter.

The theoretical model able to capture the behavior of a general fluid is *fluid dynamics*, also referred to as *hydrodynamics*. The power of this formalism is due to its ability in effectively describing a fluid as a collective medium, neglecting the contributions of all the individual particles. For example, in order to study how ocean currents spread out, it is certainly not necessary to know how all the water molecules move in water, which would be quite a complicated if not impossible task to perform. For this reason fluid dynamics turns out to be a very satisfactory and useful description currently used in a wide range of physical applications, from weather forecasts to the study of air interacting with solid surfaces, such as the wings of an airplane.

Although hydrodynamics is a very general description, valid for any fluid, clearly not every fluid behaves in the same way, for example, honey is more reluctant to motion than water. This means that the fluid dynamical description must be supplemented with some additional intrinsic parameters, different for every fluid,

and depending on how specifically the individual particles interact with each other, for example the *viscosity*. The latter measures how *dissipative* the fluid is, that is how much energy is lost in heat when moving parts of the fluid. In other words, it describes how resistant the fluid is to motion or to movements of an object in its interior. Honey is an example of a very viscous fluid moving in general very slowly, while water, on the contrary, has a very low value of the viscosity which manifests in its ability to flow very well. The viscosity, and the other intrinsic parameters, can either be measured in experiments or theoretically derived from the microscopic particle description (if any), which governs the behavior of the individual particles of a fluid. There are cases in which such a computation can be performed and other cases, when the particles are very interacting, or *strongly coupled*, in which computations can be very difficult, if not impossible. Most of our current theoretical tools are in fact better adapted to the opposite regime where systems are *weakly coupled*, that is almost non interacting.

Besides the common types of fluids mentioned above, fluid behavior is present in more exotic places such as at the Relativistic Heavy Ion Collider (RHIC) in Brookhaven, US and at the Large Hadron Collider (LHC) in Geneva, Switzerland. In these two laboratories there have been ongoing experiments which, among others, collide two oppositely accelerated beams of atoms at very high energy. The energy is so high that upon impact, the atoms dissolve in their elementary constituents, the so-called quarks and gluons. For a very short time, before cooling down and recombining into atoms, these particles behave collectively as a strongly coupled fluid at a very high temperature and high density, which goes under the name of *quark-gluon plasma*¹. The environment created in these experiments reproduces the characteristics of the universe shortly after the Big Bang. At that time it was too hot for the atoms to form, and the quark-gluon plasma was dominating the scene. Hence, analyzing the behavior of the quark-gluon plasma provides a window into the universe in its very early stages and helps to explain why the universe is the way we observe it. However, in order to make any prediction to be compared with experimental data, one needs the actual values of the intrinsic properties, which, as we said, are necessary inputs of the hydrodynamic description of a fluid. Given that the quark-gluon plasma is a strongly coupled fluid, the computation of even the viscosity coefficient turns out to be a very difficult task.

¹A plasma is another phase of matter which behaves as a fluid. The difference between a common liquid and a plasma is that the plasma is made of charged particles while the liquid is made of neutral ones. The particles in the quark-gluon plasma carry charges under the quantum-chromodynamic force, the force responsible of keeping together the nucleus of an atom.

Fluids in gravity

Yet another place where fluids can be found is in the context of gravity and in particular within black holes. The key property of these objects, believed to be present in the center of every galaxy of our universe, is that they attract everything with such a strong gravitational force that even light cannot escape after crossing a certain surface, the so-called *event horizon*. It turns out that this surface is *dynamical*, changing over time. In fact if we were to throw something into a black hole, its event horizon and the external space, or to say it better *spacetime*², would start to oscillate in a similar way as if we were to throw a stone into water. Very suggestively, given that the event horizon acts as a surface of no-return where energy is lost, black holes are naturally dissipative objects. This peculiar behavior resembles very much fluid dynamics and has led to many developments to make these statements more precise.

These ideas came about for the first time in the 1980's in work which culminated in the formulation of the so-called *membrane paradigm*. In this model a surface, or *membrane*, covers completely any black hole whose interior is reflected in simple physical properties of the membrane itself. In particular, in one of the many formulations of the paradigm, this membrane behaves as a fluid, the *membrane fluid*. It turns out, however, that this fluid has a negative viscosity, which is a weird and seemingly unphysical property since it translates in the ability of the fluid to produce energy along the flow rather than losing it. Hence, even though the membrane paradigm provides a great simplification for studying black holes as we can see them, for example, from earth where the interior of any black hole is always inaccessible, perhaps the fluid interpretation of such a membrane might be misleading. In fact a better way in which fluid behavior appears in gravity is given in the context of holography.

Holography, developed in the mid 1990's, is the idea that any gravitational theory, a theory of a dynamical spacetime or geometry, can be equivalently described by a theory of particles in one dimension less. Pictorially the additional dimension and gravity should be thought of as a hologram, mathematically encoded on some lower-dimensional surface in the language of a theory of particles where gravity is absent. In other words, holography establishes an intriguing equivalence between two very different theories which should be thought of as being the same. One only needs to know how to decode the hologram to go, for example, from the geometrical description to a particle description. Most importantly and surprisingly, it turns out that this equivalence is realized in a very crucial way, that is, when one theory

²Spacetime in Einstein's theory of gravity can be thought of as a grid or fabric. The presence of a mass distorts spacetime and the presence of a perturbation which changes over time can create propagating waves of the spacetime itself: gravitational waves.

is weakly coupled the other is strongly coupled and vice versa, and the two theories are said to be *dual* to each other. This statement has very profound consequences because, when it is convenient, one can use the weakly coupled side of the duality to tackle quantities which in the strongly coupled theory one would not have been able to compute.

This very suggestive idea of holography so far has been made very precise only for a specific class of spacetimes, those which are negatively curved³ called *anti-de Sitter*. By precise we mean that there is now a *holographic dictionary*, a recipe capable to explicitly translate quantities expressed in the language of geometry to quantities defined in the language of the particle theory and vice versa. For example, a fixed empty anti-de Sitter spacetime corresponds to have no particles in the dual theory. A black hole in anti-de Sitter is dual to a set of particles which are at equilibrium at a certain temperature. Following this analogy, slightly perturbing the geometry of this black hole beautifully corresponds to a collective behavior of the particles around the equilibrium configuration in the dual theory, and they effectively behave as a fluid. It turns out this fluid is very closely related to the strongly coupled quark-gluon plasma discussed before. Very impressively now, its intrinsic properties, such as the viscosity, can be computed very easily by analyzing the geometry of the black holes. Hence, not only holography shows how fluid behavior can be realized in gravity, but it can also be used as a theoretical tool to deal with strongly coupled theories of particles by means of their gravitational counterparts.

Motivation and main results

In this thesis we are going on a journey between the concepts presented above. We are interested both in what gravity can tell us about fluid dynamics itself, as well as how fluid dynamics can help us in understanding better holography.

For example, we said that hydrodynamics is a valid description encompassing the dynamical behavior of any fluid, but it turns out not to be unique. There is one conventional formulation which is very old and well-established, and more recent formulations, which are motivated by the desire of rewriting the theory more sys-

³An example of a negatively curved spacetime is the saddle surface, as opposed to a positively curved spacetime of which an example is the sphere.

tematically, in a language which possibly requires less principles. However, these novel approaches are less studied and it is not clear how reliable they are. In this thesis we help clarifying this point by means of holography. Using a gravitational theory, more precisely a perturbed black hole in anti-de Sitter spacetime, we show that fluid behavior appears as well, precisely in one of these non-conventional formulations. We are therefore able to confirm, at least within our example, that certain features of these formulations are robust.

In this thesis we also find yet another evidence of fluid behavior from a slightly different way of implementing holography. Usually in holography the theory of particles is encoded on a lower dimensional surface which is located at the border of the anti-de Sitter spacetime. Here we consider this surface to be somewhere in the interior instead, without specifying the actual background geometry but keeping it general up to a certain extent. In this way we are able to generalize the holographic dictionary, in the fluid regime, to spacetimes which are not necessarily anti-de Sitter but also, for example, flat. We also push this surface closer to the event horizon of a black hole to see whether the fluid holographically encoded on this surface behaves similarly to the membrane fluid introduced before. We show that the two fluids are different in many ways, clarifying some long-standing issues with the fluid interpretation of the membrane paradigm.

However, the membrane paradigm does not need to be totally dismissed since it has yet another more general definition. In this case the interior of a black hole is also completely neglected, and one only retains the key information that anything hovering outside the event horizon will eventually fall in and not come back. In this thesis we question whether this approximation to black holes works and we show that it does up to some special cases. The exception lies in those perturbations of the black hole geometry which are not captured by hydrodynamics. To be more precise when perturbing a black hole before the onset of the situation where fluid dynamics works very well, there is another very complicated configuration where not only there are propagating waves but also other disturbances which decay very fast. Holography is powerful enough that is capable to capture this behavior as well, whilst the membrane paradigm turns out to be slightly more restrictive.

2

INTRODUCTION AND OVERVIEW

ON DIFFERENT FORMULATIONS OF FLUID DYNAMICS AND THE SEARCH FOR DISSIPATION WITHIN DYNAMICAL HORIZONS

This Chapter is devoted to give a more detailed and self-contained overview of the concepts which will be relevant for this thesis. We start with a brief introduction to fluid dynamics and proceed our journey moving to the gravitational setting through the membrane paradigm and holography. A summary of the notation can be found at the end of this manuscript.

2.1 Fluid dynamics

Fluid dynamics is the theory which describes the collective behavior of an interacting quantum field theory. Consider a system at thermal equilibrium at a temperature T and allow fluctuations. If these fluctuations are sufficiently long-wavelength such that their gradients are small with respect to the background temperature, then one can always divide the system in domains in which the temperature is locally roughly constant $T(x)$, and the system is said to be at *local thermal equilibrium*. Fluid dynamics is the theory that governs the interactions between these domains named *fluid particles*, defined to be infinitesimal with respect to the size of the system but big enough to contain a large amount of microscopic constituents whose dynamics is neglected. Hence, more precisely, hydrodynamics is a valid description if there is a separation of scales between the wavelength λ of the perturbations and the so-called mean free path l_{mfp} , *i.e.* the scale measuring

the distance between two subsequent collisions of the microscopic constituents

$$\frac{l_{\text{mfp}}}{\lambda} \ll 1. \quad (2.1)$$

In the following we give a brief overview on the different formulations of relativistic fluid dynamics for the case of an uncharged fluid in $d+1$ dimensions for simplicity¹. We start with the well-known conventional description based on a conserved stress-energy tensor, its conservation equations and a local entropy current. Subsequently we discuss some newer formulations in which equations of motion are obtained by means of a variational principle. The general aspects of fluid dynamics are going to be used extensively throughout this thesis in the context of fluids in gravity, in particular the novel formulations are going to be implemented in Chapter 3.

2.1.1 Conventional formulation

A fluid is a system at local thermal equilibrium and one can always define local variables associated to the fluid particles. For an uncharged fluid they can be chosen to be, for example, the temperature $T(x)$ and the fluid velocity $u_a(x)$, normalized to $u_a u^a = -1$. The system is characterized by a symmetric stress-energy tensor T_{ab} and its conservation equations

$$\nabla_a T^{ab} = 0. \quad (2.2)$$

The crucial assumption in fluid dynamics is in choosing the stress tensor to be expressed in terms of the fluid variables through the so-called *constitutive relations*. In this way the amount of independent degrees of freedom for the stress tensor reduces drastically from $(d+1)(d+2)/2$ to only $d+1$, and therefore eq. (2.2) are a closed system, see *e.g.* [5] for a review.

Hydrodynamics is an *effective theory* where the fluid variables are assumed to depend slowly on space and time. The constitutive relations can be generically provided in a derivative expansion of the fluid variables, therefore the stress tensor can be assumed to be given

$$T_{ab} = T_{ab}^{(0)} + T_{ab}^{(1)} + T_{ab}^{(2)} + \dots, \quad (2.3)$$

where each term contains more and more derivatives, becoming increasingly irrelevant according to the expansion parameter set in eq. (2.1). At leading order, for example, we simply have

$$T_{ab}^{(0)} = P(\gamma_{ab} + u_a u_b) + \epsilon u_a u_b, \quad (2.4)$$

¹Generalizations to charged fluids are straightforward.

where γ_{ab} is the background fluid metric, $\epsilon(T)$ is the energy density and $P(T)$ is the pressure. The latter is determined by an equation of state $P = P(\epsilon)$, characteristic of the fluid under consideration.

First order hydrodynamics

To determine the form of the stress tensor at first order, define all the symmetric tensor structures constructed out of derivatives $\nabla_a T$ and $\nabla_a u_b$ compatible with the symmetries of the system. Then select the ones which are on-shell independent, *i.e.* inequivalent after imposing the equations of motion (2.2) with the leading order expansion (2.4). In this way, for example, one can trade temperature derivatives for derivatives of the velocity field. The latter can be recasted into shear, vorticity, acceleration, and expansion² respectively as

$$\sigma_{ab} = h_{(a}^c h_{b)}^d \nabla_c u_d; \quad \omega_{ab} = \nabla_{[a} u_{b)}; \quad a_a = u^c \nabla_c u_a; \quad \theta = \nabla_c u^c, \quad (2.5)$$

where we have defined $h_{ab} = \gamma_{ab} + u_a u_b$ to be the projector operator along the directions transverse to u_a . Then, the most general constitutive relation for the stress tensor at first order is given by a linear combination of symmetric tensor structures constructed out of (2.5), the metric γ_{ab} , and the fluid velocity u_a as in

$$T_{ab}^{(1)} = -2\eta \sigma_{ab} - \zeta \theta h_{ab}. \quad (2.6)$$

The undetermined coefficients $\eta(T)$ and $\zeta(T)$ are the so-called *shear* and *bulk viscosity* respectively and they incorporate the dissipative nature of a physical fluid. These values must be provided experimentally or derived from a microscopic theory if it is available, see *e.g.* [6]. Notice that the term ω_{ab} is not allowed since it is antisymmetric. A term of the form $u_{(a} a_{b)}$ would be perfectly fine instead, but it turns out not to be relevant since it can be canceled by field redefinitions. In fact the fluid variables T and u_a are only well defined at equilibrium. Out of equilibrium there is always an ambiguity in shifting $T \rightarrow T + \delta T$ and $u^a \rightarrow u^a + \delta u^a$ by choosing δT and δu^a to be proportional to the derivatives of the fluid variables. Usually one chooses to work in a particular reference frame where the fluid velocity becomes unambiguously defined. A standard choice is the *Landau frame* defined as

$$T_{ab}^{(n)} u^b = 0 \quad \text{with} \quad n \geq 1, \quad (2.7)$$

where the velocity field is aligned with the energy flow $T_{ab} u^b = -\epsilon u_a$. However, one could also choose to work with other reference frames or even with a frame invariant formulation, see *e.g.* [7]. No frame is preferred by nature reflecting the

²We refer the reader to the end of this thesis for a summary on the notation.

fact that only the value of the stress-energy tensor T_{ab} matters and not how it is expressed in terms of u_a and T .

One can proceed analogously at higher orders in derivatives by defining the on-shell inequivalent tensor structures build up from derivatives of the fluid variables. To each independent tensor structure entering in the constitutive relation of the stress tensor one associates an arbitrary function: the *transport coefficient*, as we did in (2.6) where the transport coefficients are exactly the shear and bulk viscosity. It turns out that based on symmetry grounds alone, $T_{ab}^{(2)}$ for an uncharged fluid can be parametrized by 15 such independent transport coefficients, see *e.g.* [8].

The entropy current

Another physical requirement that has to be implemented in the fluid dynamics approach is the existence of a local *entropy current* J_a . In the same way as the entropy increases when a system evolves from one global equilibrium configuration to another, in fluid dynamics we should also require the local entropy current to increase. This statement translates into a local form of the second law of thermodynamics

$$\nabla_a J^a \geq 0, \quad (2.8)$$

which we will generically refer to as the *entropy constraint*. This equation can be used to restrict the number of independent transport coefficients. For example, at first order this constraint gives conditions on the values of the shear and bulk viscosity

$$\eta \geq 0; \quad \zeta \geq 0. \quad (2.9)$$

It has been shown that at second order there are no more sign-definite conditions on the transport coefficients similar to (2.9), see *e.g.* [8]. Nevertheless, there are 5 relations among the 15 independent coefficients. This comes from the fact that the divergence of the entropy current would admit negative terms which must be set to zero to satisfy the entropy constraint (2.8). These relations reduce the stress tensor at second order to effectively a 10 parameter family of solutions. This procedure, although systematic and straightforward, can become technically quite involved beyond the first order in a derivative expansion. It is natural to seek for a principle which would allow to reformulate fluid dynamics without the requirement of the existence of a local entropy current.

Non relativistic fluid dynamics

Let us briefly comment on the regime where fluid velocities are small compared to the speed of light $|\vec{v}| \ll 1$. In this case conservation equations (2.2) with the

stress energy tensor up to first order (2.6) become the continuity equation for the mass density ρ

$$\partial_t \rho + \rho \nabla_k v^k + v^i \nabla_i \rho = 0, \quad (2.10)$$

and the well-known Navier-Stokes equations

$$\left(\partial_t v_i + v_j \nabla_j v_i \right) \rho = -\nabla_i P + \nabla_j \Pi_i^j, \quad (2.11)$$

with

$$\Pi_{ij} = \eta \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{d} q_{ij} \nabla_k v^k \right) + \zeta q_{ij} \nabla_k v^k, \quad (2.12)$$

where q_{ij} is the space metric. This non relativistic limit is obtained by assuming the usual relativistic velocity parametrization $u^a = \gamma(1, \vec{v})$ with $\gamma = (1 - v_i v^i)^{-1/2}$, a non relativistic equation of state for which $P \ll \epsilon$ and requiring the energy density to be dominated by mass density $\epsilon \sim \rho$.

2.1.2 Effective action formulation

Hydrodynamics is an effective theory valid at low energies and as such it would be desirable to understand it from first principles following the rules of an effective field theory. This entails in postulating an effective action as a functional of certain local fields compatible with required spacetime and internal symmetries, and the Euler-Lagrangian equations should carry the same information as the conservation equations of the stress-energy tensor (2.2). The complication here is that the system is at finite temperature and out of equilibrium, and the usual quantum field theory path integral adapted to describe pure initial and final states cannot be directly applied here. Given that a fluid dynamical system is at local thermal equilibrium with small deviations from the global thermal equilibrium configuration, it is widely believed that a complete treatment should be given in terms of the Schwinger-Keldysh formalism [9, 10], which is well suited to describe finite temperature systems in a time-varying setup as the dissipative fluids under consideration where initial and final states are mixed. Some progress has been made in this direction [11, 12, 13, 14, 15, 16], see also [17, 18] for cases including thermodynamical fluctuations. However, it is fair to say that a complete systematic understanding of a variational principle formulation of fluid dynamics is still absent.

In this section we are going to restrict to the case of relativistic *perfect fluids*, those which do not convert kinetic and potential energy to heat and therefore are *dissipationless*, *i.e.* do not exhibit entropy production

$$\nabla_a J^a = 0. \quad (2.13)$$

While it is questionable whether this restriction encompasses *any* physical system at all, let us for the moment be agnostic about this issue and take it as a useful starting point towards a complete and systematic variational principle formulation of fluid dynamics including dissipation.

The degrees of freedom and the symmetries

An effective action approach to perfect fluid dynamics has been initiated in [19] and revisited more recently in [20, 21, 22], see also [23, 24, 25]. The Lagrangian of perfect fluids in $d + 1$ dimensions can be given in terms of d scalar fields

$$\phi^I = \phi^I(t, \vec{x}) \quad \text{where} \quad I = 1, \dots, d. \quad (2.14)$$

Perhaps the most natural interpretation of such scalars is that of a map at fixed lab-frame time t between space coordinates $\vec{x}$ labeling the lab-frame (Eulerian frame) and the internal coordinates ϕ^I . The latter label the comoving (Lagrangian) frame, *i.e.* $\phi^I(t, \vec{x})$ describes which volume element ϕ^I is seen by a fixed observer at position $\vec{x}$ when the lab-frame time t is varied. The internal parametrization of the fluid elements is not unique, there is always an obvious freedom of shifting or rotating the fluid elements

$$\phi^I \rightarrow \phi^I + c^I \quad \text{and} \quad \phi^I \rightarrow R_J^I \phi^J. \quad (2.15)$$

It turns out, however, that the description of perfect fluids requires a much larger symmetry group: invariance under all reparametrizations that do not compress or dilute fluid cells. This is expressed by demanding invariance of the action under the *volume-preserving diffeomorphisms* in the space of ϕ^I fields:

$$\phi^I \rightarrow \xi^I(\phi^J) \quad \text{with} \quad \det \left(\frac{\partial \xi^I}{\partial \phi^J} \right) = 1. \quad (2.16)$$

In particular this invariance is what encodes the physical requirement that a fluid does not resist to non compressional deformations, *i.e.* shear stresses. In the case of a jelly, a solid with a continuous rotational invariance, the internal symmetry would be reduced to rotational and translational invariance only (2.15), such that a jelly responds to shear stresses as well, see *e.g.* [26]. In the case of solids, one instead imposes relevant discrete rotational and translational invariance. Let

us emphasize that volume-preserving diffeomorphisms invariance is not the same as requiring a fluid to be incompressible. Incompressibility is the regime of fluid dynamics in which $\nabla \cdot \vec{v} \sim 0$, and the mass density is constant along the fluid flow as can be seen from (2.10). The difference between a generic and an incompressible fluid is that the latter does not feature compression (sound) waves, while the requirement of volume-preserving diffeomorphisms invariance holds for both type of fluids as long as they are perfect, non dissipative.

The leading order effective action

The effective action for a relativistic perfect fluid is given by

$$S = \int d^{d+1}x \sqrt{-\gamma} F(s), \quad (2.17)$$

where

$$s = s_0 \sqrt{\det \partial_a \phi^I \partial^a \phi^J}, \quad (2.18)$$

s_0 is a suitable normalization constant and γ is the determinant of the background metric γ_{ab} . The argument s of the yet unspecified scalar function F is proportional to the unique invariant of spacetime and internal symmetries that can be constructed out of the fields ϕ^I and the background metric γ_{ab} restricting to the lowest possible number of derivatives. The combination in the square root of (2.18) is dimensionless given that the fields ϕ^I are the comoving coordinates and carry the length dimension.

In order to make contact with the developments of the previous section, let us derive the conserved energy-momentum tensor. By varying the action (2.17) with respect to the background metric γ_{ab} , it takes the form

$$T_{ab} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = -s F'(s) B_{IJ}^{-1} \partial_a \phi^I \partial_b \phi^J + F(s) \gamma_{ab} \quad \text{with} \quad B^{IJ} = \partial_a \phi^I \partial^a \phi^J \quad (2.19)$$

and becomes the energy-momentum tensor of a perfect fluid (2.4) upon identifying the energy density and the pressure with

$$\epsilon(s) = -F(s); \quad P = -F'(s)s + F(s). \quad (2.20)$$

The fluid velocity u_a can be derived assuming it is unit-normalized $u_a u^a = -1$ and that the scalar fields ϕ^I are the comoving coordinates, that is their derivatives must be orthogonal to the fluid flow: $u^a \partial_a \phi^I = 0$. These requirements select the unique velocity field to be

$$u^a = J^a / s \quad \text{where} \quad s = \sqrt{-J_a J^a} \quad (2.21)$$

and

$$J^a = s_0 {}^*(d\phi^1 \wedge \cdots \wedge d\phi^d) = \frac{1}{d!} s_0 \epsilon^{a b_1 \dots b_d} \epsilon_{I_1 \dots I_d} \partial_{b_1} \phi^{I_1} \dots \partial_{b_d} \phi^{I_d} \quad (2.22)$$

which is an *identically* conserved current, as the spacetime hodge dual of the volume form in the configuration space. With the identifications (2.20) and (2.21) the scalar s in (2.18) assumes the interpretation of being the entropy density of the fluid and the vector (2.22) as the identically (off-shell) conserved entropy current. The physics of these fluids is therefore intrinsically non dissipative.

The equations of motion for ϕ^I derived from the effective action (2.17) turn out to be the conservation of the energy-momentum tensor (2.4) projected transversally to the flow

$$(\gamma^{ab} + u^a u^b) \nabla^c T_{cb} = 0. \quad (2.23)$$

The remaining component of the conservation equation $u^b \nabla^a T_{ab} = 0$, which incorporates the conservation of energy, is implied by the conservation of the entropy current (2.13).

Higher orders in derivatives

The action (2.17) receives corrections carrying higher number of derivatives of ϕ^I fields

$$S^{(0)} + S^{(1)} + S^{(2)} + \dots \quad (2.24)$$

These corrections can be obtained order by order in a derivative expansion by constructing all the possible scalars allowed by volume-preserving diffeomorphisms symmetry as in [23], see also [25]. However, as it has been shown in [23], at second order in a derivative expansion the number of independent transport coefficients derived by such an effective action turns out to be less than the ones derived from the conventional formalism, restricted to configurations with no entropy production (2.13). This contrasting result brings us to question whether volume-preserving diffeomorphisms invariance should be taken as a fundamental symmetry for perfect fluids after all. Or whether maybe the conventional formalism is incomplete and one needs to consider additional physical constraints. Let us also mention here that the gravitational calculation in Ref. [27] indicates that including dissipation requires relaxing the volume-preserving diffeomorphism invariance as the exact symmetry at subleading orders in the gradient expansion. On physical grounds, this can be understood by the presence of shear viscosity which implies that the fluid responds now nontrivially to shear stresses as well. This provides an excellent motivation for exploring possible generalizations of the action (2.17) as well as gravitational embeddings of this problem which we will pursue in Chapter 3.

The linearized expansion

In the rest of the current section we assume a flat background metric $\gamma_{ab} = \eta_{ab}$. A natural way to fix the fluid's parametrization is requiring that in equilibrium and on a given time slice the fluid elements are aligned with the spatial coordinates

$$\phi^I(\vec{x}, t) = \delta_a^I x^a. \quad (2.25)$$

This configuration spontaneously breaks the spacetime Poincaré symmetry and the global subgroup of the internal symmetry (2.15-2.16) down to diagonal rotations and spatial translations. In the presence of broken global symmetries *Goldstone bosons* arise. Although there should be one Goldstone boson per broken generator, it turns out that in the presence of spacetime symmetries not all the Goldstones are independent, see *e.g.* [28]. It has been shown in [29], by means of the so-called coset construction, that the only independent Goldstone bosons correspond to the breaking of the space and internal translations down to the diagonal combination thereof. At the linearized level, such Goldstones are realized as perturbations on top of the equilibrium configuration (2.25)

$$\phi^i(\vec{x}, t) = x^i + \pi^i(t, \vec{x}), \quad i = 1, \dots, d. \quad (2.26)$$

In the formula above, we do not distinguish the internal and coordinate indices anymore since the Goldstone bosons transform under the diagonal combination.

Let us now consider linearizing the fluid's action (2.17) in the Goldstone fields (5.21). The Goldstones can be classified according to their orientation with respect to the propagation direction being longitudinal or transverse

$$\vec{\pi} = \vec{\pi}^L + \vec{\pi}^T \quad \text{with} \quad \vec{\nabla} \times \vec{\pi}^L = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{\pi}^T = 0. \quad (2.27)$$

Linearization of the velocity field and the entropy density give

$$u^t = -1 - \frac{1}{2}(\partial_t \vec{\pi})^2 + \dots, \quad \vec{u} = \partial_t \vec{\pi} + \dots \quad (2.28)$$

$$s = s_0 + s_0 \nabla \cdot \vec{\pi} - \frac{1}{2}s_0(\partial_t \vec{\pi})^2 + \dots \quad (2.29)$$

and the effective action (2.17) becomes

$$S^{(0)} = \int d^{d+1}x \left\{ F(s_0) - \frac{1}{2}F'(s_0)s_0 \left((\partial_t \vec{\pi}^T)^2 + (\partial_t \vec{\pi}^L)^2 - c_s^2 (\nabla \cdot \vec{\pi}^L)^2 \right) + \dots \right\}. \quad (2.30)$$

The equations of motion, equivalent to the conservation (2.23) of the energy-momentum tensor (2.2), give the following leading order dispersion relations

$$\pi^L : \quad \omega_L = \pm c_s k, \quad (2.31)$$

$$\pi^T : \quad \omega_T = 0. \quad (2.32)$$

The longitudinal Goldstone describes a sound wave. Its group velocity c_s is given by

$$c_s^2 = \frac{F''(s_0)}{F'(s_0)} s_0 = \frac{P'(s_0)}{\epsilon'(s_0)} = \frac{\partial P}{\partial \epsilon}. \quad (2.33)$$

For a conformal fluid for example we have

$$F(s) \sim s^{d+1/d}, \quad (2.34)$$

and the speed of sound takes the familiar form

$$c_s = \frac{1}{\sqrt{d}}. \quad (2.35)$$

The transverse Goldstones do not propagate instead, because they have a trivial dispersion relation (2.32). This is a direct consequence of the volume-preserving diffeomorphism invariance of the action (2.17), which at a linearized level acts as follows

$$\vec{\pi}(t, \vec{x}) \rightarrow \vec{\pi}(t, \vec{x}) + \vec{\xi}(\vec{x}) \quad \text{with} \quad \nabla \cdot \vec{\xi} = 0 \quad (2.36)$$

and does not allow the gradient terms of the form $\nabla \times \vec{\pi}$ to appear in the action (energy).

2.1.3 The equilibrium partition function formalism

Let us to conclude this section mentioning yet another alternative formulation of fluid dynamics which will also be relevant in Chapter 3. This formalism has been recently proposed in [30, 31] and it relies on an even more restrictive class of fluids: not only they are dissipationless following (2.13), but also *hydrostatic*, that is time-independent. The fundamental quantity in this construction is the Euclidean generating functional $W = -\ln Z$, where Z is the partition function as a function of time-independent sources. For an uncharged fluid and at leading order in a derivative expansion, it is defined as an integral over d spatial dimensions and a compactified time direction with period T_0

$$W = \int d^{d+1}x \sqrt{\gamma} P(T). \quad (2.37)$$

The integrand is a function of the temperature T through the pressure $P(T)$, and the background metric γ_{ab} sourcing the stress-energy tensor, assumed to have

a timelike Killing vector such that it is time-independent. The latter can be computed as usual by

$$T_{ab} = -\frac{2}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma^{ab}}, \quad (2.38)$$

and at leading order reproduces (2.4) with the identifications

$$T = \frac{T_0}{\sqrt{\gamma_{tt}}}; \quad u^a = \frac{1}{\sqrt{\gamma_{tt}}} (1, \vec{0}); \quad \epsilon = T s - P; \quad s = P'(T), \quad (2.39)$$

where γ_{tt} is the local redshift factor.

This formalism is very useful in characterizing hydrodynamics without postulating the existence of an entropy current. Higher derivative contributions to (2.37) can be constructed by adding all the independent scalars that can be build out of the derivatives of the background metric. The new stress tensor can be readily derived using (2.38). By requiring this quantity to be compatible with the more general stress tensor provided in the conventional formalism, gives relations among transport coefficients, which turn out to be the same as the ones coming from the entropy constraint (2.8) as discussed around (2.9). This means that as soon as the first order hydrodynamic transport has been analyzed, one can trade the existence of a local entropy current with the requirement of compatibility with hydrostatic equilibrium. In fact, as shown with some rigor in [32, 33], sign-definite relations of the type (2.9), which would not be captured by the partition function formalism, appear only at first order.

Notice that the partition function is given in terms of the pressure and the effective action of Section 2.1.2 as a functional of the energy density. Naturally these two formalisms should be related by a Legendre transform with respect to the entropy density $P(T) = T s - \epsilon(s)$. It has been shown in [34] that this is exactly the case, and in particular, together with a companion paper [35], the authors show that there exists a more general Lagrangian formalism which encompasses all the cases discussed above, and also able to incorporate all the possible transport coefficients, even those which are not reproduced by the effective action formalism. To understand the relations among the various approaches to fluid dynamics we set up a precise gravitational holographic dual for effective actions and partition functions in Chapter 3. With such a description at hand one might be able to understand in detail the separation between dissipative and dissipationless transport in fluids and find a path towards a complete formulation of a variational principle for general (dissipative) fluid dynamics.

2.2 Fluids in gravity

It has been appreciated for quite a while that gravity, and in particular black hole physics, can be related to fluid dynamics. First hints of this behavior appear in the context of the membrane paradigm. Even if this is an old-standing topic, it has many formulations that seemingly not everyone has agreed upon. In this Section we provide a general historical introduction on this subject hopefully clarifying some confusions. These concepts will be relevant in Chapter 3 where the membrane paradigm will act as a useful approximation for the black hole interior and in Chapter 5 where we will explore the limits of validity of such an approximation scheme. We then continue our journey moving to the holographic interpretation of gravity where fluid dynamics arises quite naturally. Holography will be relevant throughout this thesis and in particular we will use it for embedding fluid effective actions in gravity in Chapter 3 and for constructing general hydrodynamic solutions to Einstein gravity in the interior of spacetime in Chapter 4.

2.2.1 The membrane paradigm

Black holes are very complicated objects which have fascinated physicists ever since a few years after the formulation of general relativity. They form presumably after the collapse of very massive stars and continue to grow by acquiring mass from surrounding matter. It is nowadays widely accepted that at the center of every galaxy sits a supermassive black hole. However, due to the fact that black holes do not emit any type of radiation apart from the very weak Hawking one [36], they have so far escaped any direct detection. Current astronomical research is focused on indirect detections based on black hole interactions with their surroundings, see *e.g.* [37] for a review. Theoretical models are necessary to understand how those interactions come about and what to look at when searching for black hole signals in the sky. Typical examples are the study of gravitational waves created from colliding black holes [38] or gamma ray bursts obtained from hot regions of the accretion disk of spinning black holes in the presence of a companion star, see also [39]. Now, using the full machinery of general relativity to model these systems can become quite nasty and it is often useful, if not strictly necessary, to rely on a certain degree of approximation.

The simplest way to model a black hole is to neglect whatever is behind the event horizon. From the point of view of an asymptotic observer who can never reach the interior of a black hole this is after all everything that is needed, right? Well, if we were simply to truncate all computations at the event horizon we would arrive at nonphysical conclusions. Say there is an electric charge right outside the black

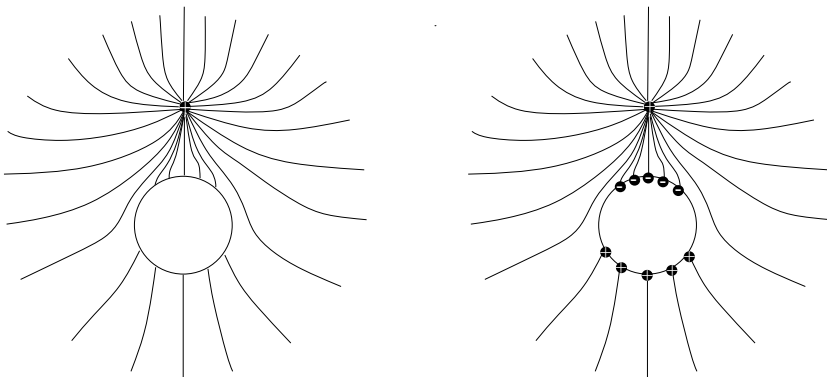


Figure 2.1: Electric field lines created by a charged particle outside the event horizon of a black hole. If we were to naively neglect the interior of a black hole, Gauss law would not be satisfied (left). The assumption of an induced charge on the event horizon restores effectively the validity of the Gauss law (right).

hole horizon, its electric field lines would intersect the horizon and stop inevitably violating the Gauss law. The correct thing to do is to require additional boundary conditions, such as the presence of induced charges on the horizon as suggested by R. Hanni and R. Ruffini in [40] in a way that the total charge is vanishing, see Figure 2.2. Analogously, R. Znajek in [41] and independently T. Damour in [42] introduced the concept of horizon current in order to complete all circuits entering and leaving the horizon, which now behaves as if it had finite conductivity. A few years later in [43], see also [44] for a review, T. Damour also showed how the horizon surface of a black hole in Einstein gravity behaves as a fluid bubble obeying dynamical Navier-Stokes equations with finite shear and bulk viscosity.

The set of all these electromagnetic and mechanical analogies for black hole physics form comprehensively what goes under the name of the *Membrane Paradigm*. The interior of a black hole is inaccessible to an external observer and can be effectively replaced by a membrane endowed with simple physical properties, providing convenient mental pictures useful for an astrophysical point of view.

This membrane was initially thought to be located at the event horizon. However, problems can arise due to the peculiar nature of the horizon, which is a null globally defined hypersurface and it is hardly accessible to an external observer. Hence, for practical purposes, it has been suggested by K. Thorne and collaborators in [45, 46] to move the membrane to a stretched horizon when necessary. The latter is a timelike hypersurface, a very small distance away from the horizon itself, invoked during explicit calculations and sent towards the event horizon at the end.

Applications of the membrane paradigm in the realm of astrophysics mainly use the electromagnetic properties of the membrane. Examples are the study of the magnetosphere of a black hole surrounded by a magnetized accretion disk [45] and jets emitted by a spinning black hole [47], see [48] for many more applications and further references. The membrane paradigm has also been applied in the context of holography as we will discuss at the end of this Chapter.

Let us summarize what we have learned so far in a generic statement:

The membrane paradigm: *as a way to replace the interior of a black hole with a membrane living on the horizon/stretched horizon endowed with simple physical properties.*

In the reminder part of this Section we are going to enter in more details. We first define the electromagnetic and gravitational membrane as they appeared historically and make a precise distinction with what we call the membrane fluid, yet another interpretation of the membrane paradigm. Subsequently we give a more modern and compact definition of the membrane as a boundary condition and show how it can be coupled to the external spacetime.

The electromagnetic membrane

Let us for illustration derive the electromagnetic properties of the membrane on a stretched horizon in 3+1 dimensions following closely the original formulation of [45]. We will show how the induced charge, the electric current and the horizon conductivity arise on the membrane as a consequence of the horizon being a regular place.

Fields on a stretched horizon can be probed by fiducial observers (FIDOs). They are confined to the timelike hypersurface, have a constant acceleration and behave singularly when the stretched horizon is pushed towards the horizon. Observers which exhibit regular behavior instead are the freely falling (FFOs) ones. It turns out that FIDOs and FFOs are in relative motion with respect to each other along the normal direction $\vec{n}$ to the stretched horizon at approximatively the speed of light $\vec{\beta} = \beta \vec{n}$, see [48]. The closer the stretched horizon is to the horizon, the faster the two types of observers move as $\beta = 1 - \mathcal{O}(\alpha^2)$ and $\alpha = \sqrt{g_{tt}}$ is the redshift parameter such that $\alpha \rightarrow 0$ when the horizon is approached.

The electric $\vec{E}^{\text{FFO}}$ and magnetic $\vec{B}^{\text{FFO}}$ fields as measured by a FFO are regular on the horizon, hence they are of order $\mathcal{O}(1)$. The corresponding electromagnetic fields $\vec{E}$ and $\vec{B}$ as seen by a FIDO can be obtained by a simple Lorentz transfor-

mation with a Lorentz factor $\gamma = \mathcal{O}(\alpha^{-1})$

$$\begin{aligned}\vec{E}_\perp &= \vec{E}_\perp^{\text{FFO}}, & \vec{E}_\parallel &= \gamma (\vec{E}_\parallel^{\text{FFO}} + \vec{n} \wedge \vec{B}_\parallel^{\text{FFO}}) + \mathcal{O}(\alpha), \\ \vec{B}_\perp &= \vec{B}_\perp^{\text{FFO}}, & \vec{B}_\parallel &= \gamma (\vec{B}_\parallel^{\text{FFO}} - \vec{n} \wedge \vec{E}_\parallel^{\text{FFO}}) + \mathcal{O}(\alpha),\end{aligned}\tag{2.40}$$

where $\perp$ and $\parallel$ denote normal and tangential components to the stretched horizon respectively. Notice that the tangential components are singular at the horizon. These expressions can be rewritten in a compact form

$$\vec{E}_\parallel = \vec{n} \wedge \vec{B}_\parallel + \mathcal{O}(\alpha), \quad \vec{B}_\parallel = -\vec{n} \wedge \vec{E}_\parallel + \mathcal{O}(\alpha)\tag{2.41}$$

and have a simple physical interpretation. In the near-horizon limit when $\alpha \rightarrow 0$, local observers on the stretched horizon see the tangential electromagnetic field to behave as an ingoing plane wave.

One can define the horizon charge density q to terminate the normal component of the electric field in order to satisfy Gauss's law, and the horizon current density $\vec{J}_\parallel$ to complete Ampere's law

$$q = \frac{1}{4\pi} E_\perp, \quad 4\pi \vec{J}_\parallel \wedge \vec{n} = \vec{B}_\parallel.\tag{2.42}$$

Using (2.42) into (2.41) we obtain Ohm's law on the horizon

$$\vec{J}_\parallel = \sigma \vec{E}_\parallel \quad \text{with} \quad \sigma = \frac{1}{4\pi},\tag{2.43}$$

where σ is the horizon conductivity. Equations (2.42) together with (2.43) form the celebrated *electromagnetic membrane*.

The gravitational membrane

We now turn our attention to the case of gravitational interactions of the black hole with surrounding matter. As the electromagnetic tensor field $F_{\mu\nu}$ in $3+1$ dimensions has been split into an electric field $E_m = F_{m0}$ and a magnetic field component $B_m = \epsilon_{mnp} F^{np}$, gravitational perturbations parametrized by the Weyl tensor $C_{\mu\nu\rho\sigma}$ can be analogously split according to [46] into a gravitoelectric field $\mathcal{E}$ and a gravitomagnetic field $\mathcal{B}$

$$\mathcal{E}_{mn} = C_{m0n0}, \quad \mathcal{B}_{mn} = \frac{1}{2} \epsilon_{0mpq} C^{pq}{}_{n0},\tag{2.44}$$

where ϵ is the Levi-Civita tensor in $3+1$ dimensions. Similar arguments developed in the previous section apply to normal-normal components $\mathcal{E}_{\perp\perp}$, $\mathcal{B}_{\perp\perp}$, normal-tangential components $\mathcal{E}_{\perp\parallel}$, $\mathcal{B}_{\perp\parallel}$ and tangential-traceless components $\mathcal{E}_{\parallel\parallel}$, $\mathcal{B}_{\parallel\parallel}$

of the gravitoelectromagnetic fields. In fact requiring FFOs to be well behaved at the horizon selects the ingoing plane wave behavior for the normal-tangential and the tangential-tangential components

$$\begin{aligned}\vec{\mathcal{E}}_{\perp||} &= \vec{n} \wedge \vec{\mathcal{B}}_{\perp||} + \mathcal{O}(\alpha), & \vec{\mathcal{B}}_{\perp||} &= -\vec{n} \wedge \vec{\mathcal{E}}_{\perp||} + \mathcal{O}(\alpha), \\ (\mathcal{E}_{||||})_{mn} &= \epsilon_{mpq} n^p \mathcal{B}_{||||}^{qn} + \mathcal{O}(\alpha), & (\mathcal{B}_{||||})_{mn} &= -\epsilon_{mpq} n^p \mathcal{E}_{||||}^{qn} + \mathcal{O}(\alpha),\end{aligned}\tag{2.45}$$

analogously to (2.41).

Despite such a straightforward generalization to incorporate gravitational perturbations, the interpretation of this *gravitational membrane* in terms of simple physical properties is now somehow obscured. There is no natural way to see the gravitational membrane's properties in terms of charges, currents etc.

The membrane fluid

The gravitational properties of the membrane are better known through the developments of T. Damour in [43, 44]. He showed that Einstein equations in $d + 2$ dimensions describing the evolution of the horizon of a black hole can be recasted into nonrelativistic dissipative Navier-Stokes equations³

$$\mathcal{L}_l \mathcal{P}_i + \theta \mathcal{P}_i = -\nabla_i P + 2\eta \nabla_j \sigma_i^j + \zeta \nabla_i \theta - f_i, \tag{2.46}$$

where σ_{ij} is the shear tensor and θ is the volume expansion with associated shear η and bulk viscosity ζ respectively

$$\eta = 1/2; \quad \zeta = -\frac{(d-1)}{d}. \tag{2.47}$$

The momentum surface density is $\mathcal{P}_i$, P is the fluid pressure, f_i is an external force per unit surface area, ∇_i is the covariant derivative in d spatial dimensions and $\mathcal{L}_l$ is the Lie derivative along the time direction represented by the vector l which is tangent and normal to the event horizon, which is a null hypersurface. Navier-Stokes equations (2.46) characterize what we call the *Membrane Fluid* with specific universal transport properties (2.47), among which an unphysical negative bulk viscosity (2.47), see the condition (2.9) coming from the entropy current constraint.

³These equations can be mapped to (2.11) with simple identifications of the shear tensor, the volume expansion, the momentum density, and the the substantive derivative (derivative taken along the path moving with velocity $\vec{v}$)

$$\sigma_{ij} = \frac{1}{2} \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{d} q_{ij} \nabla_k v^k \right); \quad \theta = \nabla_k v^k; \quad \mathcal{P}_i = \rho v_i; \quad \mathcal{L}_l = \partial_t + v^i \nabla_i,$$

and f_i is an additional external force per unit area.

Although the fluid interpretation of the membrane paradigm is very suggestive, as we shall see explicitly in Chapter 5, eq. (2.46) are problematic in many ways. They are just a rewriting of a subset of Einstein equations in terms of certain suitable geometrical quantities and never include any information on the regularity property of the horizon as opposed to *e.g.* eq. (2.45). Moreover, as we will see explicitly in Chapter 5, these membrane fluid equations are part of a bigger system of coupled equations which comprehensively describe the evolution of the null horizon surface, but do not have a hydrodynamic interpretation. This means, in particular, that the evolution of the part of the spacetime external to the horizon determines the dynamics of the horizon itself. A recent interesting generalization of the membrane paradigm where horizon dynamics decouples and can be considered as an isolated system has been given in [49, 50] in the large- d limit of gravity⁴. All these subtleties bring us to question whether there is a better, compact and comprehensive formulation of the membrane paradigm which encompasses the necessary properties of the black hole interior, that is the horizon being a regular place, and is useful for applications. We will address this issue in the following subsection.

The membrane paradigm as a boundary condition

In the last few sections we have observed that the main physical ingredient for the formulation of the membrane paradigm is the regularity condition for the FFOs on the horizon surface, which translates into an ingoing plane wave behavior for the fields when approaching the horizon, as in (2.41) and (2.45). Here we take this physical boundary condition to be the definition of the membrane paradigm following the modern approach of N. Iqbal and H. Liu in [52].

Consider for illustration a probe scalar field in a black hole background. For any nonextremal black hole, the near-horizon expansion of a scalar field $\phi(\omega, k, u)$ given in Fourier space is

$$\begin{aligned} \phi = e^{-i\omega t + i\vec{k} \cdot \vec{x}} & \left\{ c_{\text{out}}(1-u)^{i\omega/4\pi T} \left(1 + \alpha_1(1-u) + \dots \right) + \right. \\ & \left. c_{\text{in}}(1-u)^{-i\omega/4\pi T} \left(1 + \beta_1(1-u) + \dots \right) \right\}, \end{aligned} \quad (2.48)$$

where T is the temperature of the black hole, u is the radial coordinate and $u = 1$ is the rescaled horizon radius⁵. The universal leading terms $(1-u)^{\pm i\omega/4\pi T}$ follow

⁴This peculiar limit has been initiated in [51]. The hope is that many things simplify similarly to the large- N_c limit for $SU(N_c)$ gauge theories.

⁵One could for example derive such an expansion from the metric of a black hole in anti-de

from the fact that the near-horizon region of any finite-temperature black hole is Rindler⁶. The values of the coefficients α_i and β_j of the subleading terms are non-universal and depend on the number of dimensions, the mass of the field and its momentum.

We are interested in imposing ingoing boundary conditions at the event horizon, which correspond to regularity conditions for the FFOs. These are typically selected by requiring $c_{\text{out}} = 0$ (or $c_{\text{in}} = 0$ for purely outgoing boundary conditions). A suggestive way of rewriting these boundary conditions is in terms of a relation between the radial and time derivative (in Fourier space) of the field at the horizon

$$4\pi T (1-u) \frac{\partial_u \phi}{i\omega \phi} \Big|_{u=1} = \sigma, \quad (2.50)$$

where $\sigma = 1$ corresponds to purely ingoing and $\sigma = -1$ to purely outgoing modes. We keep σ general and refer to it as the *membrane coupling*. This statement can be extended to a stretched horizon by keeping σ fixed and equal to 1 and viewing eq. (2.50) not as the response of the event horizon, a null surface residing at $u = 1$, but rather of a timelike membrane located at $u = u_\delta = 1 - \delta$ with $\delta \ll 1$. One might have thought that for sufficiently small δ the membrane paradigm always *effectively* imposes ingoing boundary conditions on the event horizon. Surprisingly, as we shall see in Chapter 5, this turns out not to be the case.

To summarize let us fix the definition of the membrane paradigm which we will use extensively throughout this thesis:

The membrane paradigm:

as an ingoing-like boundary condition for scalar combinations of the fields taken on the horizon/stretched horizon

$$4\pi T (1-u) \frac{\partial_u \phi}{i\omega \phi} \Big|_{u=u_\delta} = \sigma \quad (2.51)$$

Sitter spacetime in $d + 2$ dimensions

$$ds^2 = \frac{du^2}{4u^2 f(u)} - \frac{(4\pi T/(d+1))^2}{u} f(u) dt^2 + \frac{(4\pi T/(d+1))^2}{u} d\vec{x}^2, \quad (2.49)$$

where $f(u) = 1 - u^{(d+1)/2}$, the horizon is at $u = 1$ and we set the anti-de Sitter radius to unity.

⁶The Rindler spacetime [53] is a patch of Minkowski spacetime, a reference frame of constantly accelerated observers, experiencing a thermal bath.

Coupling the membrane to outer spacetime

So far we have showed how the membrane paradigm effectively encodes the contribution of the interior of a black hole. We want to couple this information to the exterior of the spacetime and transfer it to the infinity, where an asymptotic observer sits. Combining [54, 27], consider a black hole in the presence of matter which we generically represent by a scalar field ϕ . Divide the full spacetime into a contribution coming from the interior and the exterior of the black hole by a surface Σ located at the horizon or at a stretched horizon. The action separates accordingly

$$S = S_{\text{in}} + S_{\text{out}}. \quad (2.52)$$

The variational principle in $d+2$ dimensions generates field equations in the interior and exterior of spacetime with two additional $(d+1)$ -dimensional boundary terms

$$\delta S = (\text{e.o.m.})_{\text{in}} + \int_{\Sigma} d^{d+1}x \Pi_{\text{in}} \delta\phi^{\Sigma} + (\text{e.o.m.})_{\text{out}} + \int_{\Sigma} d^{d+1}x \Pi_{\text{out}} \delta\phi^{\Sigma}, \quad (2.53)$$

where

$$\Pi_{\text{in}} = \frac{\delta S_{\text{in}}}{\delta\phi^{\Sigma}} \quad \text{and} \quad \Pi_{\text{out}} = \frac{\delta S_{\text{out}}}{\delta\phi^{\Sigma}} \quad (2.54)$$

are the conjugate momenta of the generic field ϕ with respect to the normal to the hypersurface Σ according to a radial slicing of the spacetime. Now, Dirichlet boundary conditions $\delta\phi = 0$ can only be assumed on the singularity and at infinity of the spacetime⁷ but not on the intermediate surface Σ on which the field $\delta\phi^{\Sigma}$ is arbitrary. In fact on the intermediate surface the necessary boundary condition is a continuity equation, since the field $\delta\phi^{\Sigma}$ is fictitious and has to be integrated out

$$\frac{\delta S}{\delta\phi^{\Sigma}} = \frac{\delta S_{\text{in}}}{\delta\phi^{\Sigma}} + \frac{\delta S_{\text{out}}}{\delta\phi^{\Sigma}} = \Pi_{\text{in}} + \Pi_{\text{out}} = 0. \quad (2.55)$$

Hence, in order to correctly take into account the evolution of the field in the exterior of the black hole one needs a suitable boundary condition for the conjugate momentum Π_{out} , identified with Π_{in} on Σ . The latter could be in principle computed evaluating S_{in} . However, this action refers to the part of the spacetime containing the event horizon of the black hole which is a dissipative object at finite temperature. As such, the correct formalism to take these effects into account is the Schwinger-Keldysh formalism [9, 10] which is a quite involved procedure. In a gravitational setting this has been implemented, for example, in [55] with a double-sided eternal black hole in anti-de Sitter spacetime, a black hole with two

⁷In the case of black holes in AdS spacetime there is a conformal boundary at infinity on which it is possible to assume nontrivial Dirichlet boundary conditions. Surface terms will be necessary to assure the variational problem to be well defined at infinity as we will discuss later in the context of AdS/CFT.

asymptotic boundaries on which one can define two copies of the dual field theory acting as the doubled degrees of freedom in the Schwinger-Keldysh formalism. The membrane paradigm comes as a simplification since it replaces the full dynamical interior region with a simple membrane response

$$\Pi_{\text{in}} = \frac{\delta S_{\text{in}}}{\delta \phi^\Sigma} = i\omega \sigma \phi, \quad (2.56)$$

which can be easily obtained from (2.50), (2.55) and knowing that

$$\Pi_{\text{out}} = \frac{\delta S_{\text{out}}}{\delta \phi^\Sigma} = \frac{\delta \mathcal{L}}{\delta \partial_u \phi} \Big|_\Sigma = -\sqrt{-g} g^{uu} \partial_u \phi \Big|_\Sigma = -4\pi T(1-u) \partial_u \phi + \mathcal{O}(1-u), \quad (2.57)$$

where generically the kinetic term is of the form $\mathcal{L} \sim -1/2\sqrt{-g} g^{uu} \partial_u \phi \partial_u \phi + \dots$ and in the last equality of (2.57) we have performed a near-horizon expansion.

The upshot of this discussion is that the membrane paradigm (2.51) can be reinterpreted as a universal response function (2.56) characterizing nonextremal horizons and yet having a particularly simple form which can be coupled to the exterior of spacetime and transferred to infinity by requiring (2.55) as we shall see explicitly in Chapter 3 and 5.

2.2.2 Holographic fluid dynamics

Another way to realize fluid dynamics in gravity is through the low energy limit of a subset of black holes in the context of holographic gravity, which is the proposal that any gravitational theory can be described by a quantum field theory in one dimension less with no gravity. In this Section we first provide a small introduction and overview to holography itself, then we illustrate its low energy regime, that is fluid/gravity duality. At the end we show how fluid effective actions can be in principle obtained from gravity and how the membrane paradigm can be implemented within holography.

The holographic principle

A holographic interpretation of gravity was firstly given within the so-called *holographic principle* stated by G. 't Hooft in [56] and by L. Susskind in [57], see also

[58] for a review. This principle can be understood by inspecting the necessary properties that any putative theory of quantum gravity should have. Even without knowing the precise nature of a unified theory of gravity with quantum mechanics, we can question how the fundamental degrees of freedom scale when changing the volume V of space they are contained in. Let us perform a gedanken experiment. Imagine to throw matter in such a volume of space until a black hole of the same size is formed. Black holes are known to be thermodynamical objects carrying entropy [59, 60, 36] which grows as the area A of the event horizon

$$S = \frac{A}{4G_N}, \quad (2.58)$$

where G_N is the gravitational Newton constant. Given the generalized version of the second law of thermodynamics for black holes [61], the amount of entropy which was present in the volume of space to start with must have been no greater than the amount of entropy of a black hole of the same size. Now, given that the entropy is related to the number of degrees of freedom⁸ $S \sim N$, we have to conclude that the fundamental gravitational degrees of freedom in a volume of space must scale as the area A enclosing the volume. This is a surprising result since for a local quantum field theory the degree of complexity grows as the volume V instead⁹. If we are willing to retain locality as our defining principle, a theory of quantum gravity could still be described by a local quantum field theory but at the price of one dimension. This is precisely the main statement of *holography*: all the information concerning gravitational physics can be encoded on a certain lower dimensional hypersurface of the spacetime in terms of a local quantum field theory where gravity is absent.

The AdS/CFT correspondence

A concrete realization of holography appeared in 1997 when J. M. Maldacena conjectured in [62] an unexpected equivalence or *duality* between a string theory in an anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) in one lower dimension¹⁰, the so-called *AdS/CFT correspondence*. Soon after, many more examples followed proposing equivalences between other types of string theories

⁸Define $N = \log \Omega$, where $\Omega = \dim(\mathcal{H})$ is the number of states of a quantum mechanical system, *i.e.* the dimensionality of the Hilbert space $\mathcal{H}$. For example, for a system of 3 spins there are $\Omega = 2^3$ states and $N = 3 \log 2$ degrees of freedom.

⁹A local quantum field theory can be thought of as a lattice of quantum oscillators. Each oscillator has generically ∞ states but in the case the theory admits an ultra-violet cutoff, we might assume there is a finite amount of states n . Hence, in a generic volume V the number of states is $\Omega \sim n^V$.

¹⁰The precise equivalence is between type IIB String Theory in $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N_c)$ supersymmetric Yang-Mills (SYM) in four dimensions, which is a conformal field theory.

on different backgrounds and (non) conformal field theories, see *e.g.* [63, 64, 65] for comprehensive reviews on these so-called gauge/gravity dualities. The amount of evidence that these conjectures hold is overwhelming and AdS/CFT is probably the most striking and significant outcome of string theory that earned [62] 10.000+ citations.

Interestingly enough, these correspondences, which from now on we will generically refer to as AdS/CFT, realize a *strong/weak coupling duality*. This is a statement about the regime in which the two theories are expected to overlap. When one theory is strongly coupled, the other one is in the weakly coupled regime and vice versa. For example, the strong coupling (planar)¹¹ limit of the field theory can be accessed via a classical gravitational dual, which is generically a simple setup. This feature is at the basis of the great popularity of AdS/CFT since the correspondence can be used as a nonperturbative tool to explore otherwise inaccessible regimes of certain field theories by means of their gravity duals. In addition, differently from other nonperturbative approaches, AdS/CFT is well suited to analyze not only thermodynamics but also physics out of equilibrium.

Applications rely on two possibilities. The first one is a *top down* approach where one deals with the low energy supergravity limit of a certain specific string theory configuration. Here the dual field theory is mostly known and there are few parameters that one can play with. The second approach is *bottom up* which does not have any specific string theory embedding in mind. Typically one chooses to work with *classical gravity* where fields and their interactions are chosen based on some required generic properties that the dual field theory should have. Both the approaches are limited in their applicability to realistic field theories, for example to quantum chromo-dynamics (QCD). However, even if only toy models can be analyzed, AdS/CFT has provided many qualitative and quantitative insights into condensed matter and high energy physics, such as high-temperature superconductors¹², cold atoms at unitarity, quark-gluon plasma, etc, see [66, 67, 68, 69] for extensive reviews.

Hence, to summarize, on one side AdS/CFT, or more generically holography, seems to be a key ingredient for the formulation of a theory of quantum gravity. On the other side it represents a useful theoretical tool for investigating the strongly coupled regime of certain quantum field theories admitting gravitational holographic duals. This provides an excellent motivation for studying holography and extending the realm to which it can be applied. If holography is true, one should be able to translate all the generic features of any quantum field theory, such as the entan-

¹¹Large- N_c .

¹²See *e.g.* [4] for an application relevant for unbalanced superconductors developed by the author.

lement entropy or the Wilsonian renormalization group flow, to the dual gravity side. Fluid behavior is yet another example where the low energy regime of a field theory should be mapped to the long-wavelength gravitational perturbations of a class of black holes. This picture has been beautifully realized in what goes under the name of fluid/gravity duality and represents exactly the way fluid behavior arises in gravitational physics. In the remainder part of this Section we show some aspects of the holographic dictionary focusing on those which will be relevant for this thesis.

Basic holographic dictionary

Let us here enter in more detail of the correspondence and show how quantities on one side can be translated to the other one by means of the *holographic dictionary*. We concentrate on the classical gravity limit of the correspondence and in particular on *asymptotically (locally)* AdS spacetimes in $d + 2$ dimensions for which the holographic interpretation is best understood, see *e.g.* [70] for a review. These spacetimes are solutions to Einstein gravity with a negative cosmological constant possibly coupled to additional matter

$$S = \frac{1}{2k_{(d+2)}^2} \int dr d^{d+1}x \sqrt{-g} (R - 2\Lambda) + S_{\text{matter}}, \quad \text{with} \quad \Lambda = -\frac{d(d+1)}{2L^2}, \quad (2.59)$$

where $2k_{(d+2)}^2 = 16\pi G_N^{(d+2)}$, L is the parameter measuring the curvature of AdS spacetime and g is the determinant of the bulk metric $g_{\mu\nu}$. At large radius, this solution asymptotes to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{r^2}{L^2} \gamma_{ab} dx^a dx^b + \frac{L^2}{r^2} dr^2 + \dots \quad \text{for} \quad r \rightarrow \infty. \quad (2.60)$$

This would be an AdS spacetime in the so-called Poincaré patch if $\gamma_{ab} = \eta_{ab}$, *i.e.* the maximally symmetric solution to vacuum Einstein gravity with a negative cosmological constant. For a generic metric γ_{ab} instead, the spacetime (2.60) is only *locally* (asymptotically) AdS. An important feature of these geometries is that they admit a conformal boundary at infinity which can be seen by fixing an horizon radius $r = r_c$ and then sending $r_c \rightarrow \infty$. The metric γ_{ab} on this boundary is an arbitrary Dirichlet value that needs to be externally provided.

The same is valid for any other *bulk field* evolving in the interior, the *bulk*, of AdS. The scalar field ϕ or the gauge field A_μ , for example, must be supplemented with Dirichlet boundary conditions on the boundary at infinity¹³ $\phi^{(0)}$, $A_a^{(0)}$. Following the standard prescription given in [71, 72], to every bulk field ϕ , A_μ , $g_{\mu\nu}$ it

¹³This property reflects the fact that AdS spacetime is not globally hyperbolic, hence additional data on the boundary must be supplied to define the evolution of fields in the bulk.

is possible to associate an operator of the boundary field theory $\mathcal{O}$, A_a , T_{ab} by identifying the Dirichlet boundary condition $\phi^{(0)}$, $A_a^{(0)}$, γ_{ab} with the source of the corresponding dual operator¹⁴. For example, the boundary metric value γ_{ab} of the bulk metric $g_{\mu\nu}$ is the source associated to the stress-energy tensor T_{ab} of the dual field theory.

The fundamental assumption of the holographic correspondence relies in identifying the on-shell action (2.59) with the generating functional of the connected correlators of the dual field theory as a function of the sources

$$S_{\text{on-shell}}[\phi^{(0)}, A_a^{(0)}, \gamma_{ab}, \dots] = W[\phi^{(0)}, A_a^{(0)}, \gamma_{ab}, \dots], \quad (2.61)$$

where $W = -\ln Z$ and Z is the partition function¹⁵. Euclidean correlation functions can be easily derived as usual by performing subsequent derivations of the generating functional with respect to the sources.

However, one needs to be careful because of the presence of a boundary in AdS. In order for the variational problem to be well defined, one needs to supply the Hilbert-Einstein action (2.59) with a Gibbons-Hawking [73, 74] term

$$S_{\text{GH}} = \frac{1}{2k_{(d+2)}^2} \int d^{d+1}x \sqrt{-\gamma} 2K, \quad (2.62)$$

where K is the trace of the extrinsic trace K_{ab} at the boundary. In addition, given that the boundary is at infinity, one needs to add boundary counterterms to compensate large volume, *i.e.* large r , divergences of the on-shell action. These boundary terms can be found by applying the method of holographic renormalization [75, 70, 76, 77], which is equivalent to the renormalization procedure in field theory to remove the ultra-violet divergences. After holographically renormalizing the action, one obtains finite correlation functions.

For example, for the gravitational sector alone the corresponding counterterm reads

$$S_{\text{ct}} = -\frac{1}{2k_{(d+2)}^2} \int d^{d+1}x \sqrt{-\tilde{\gamma}} \frac{2d}{L} + \dots \quad (2.63)$$

and ellipses denote terms containing contributions of the Ricci scalar and tensors of the boundary hypersurface which vanish when $\tilde{\gamma}_{ab}$ is flat. Since the boundary is at infinity, algorithmically one first computes counterterms S_{ct} on a finite cutoff hypersurface r_c with the rescaled induced metric $\tilde{\gamma}_{ab} = r_c^2/L^2 \gamma_{ab}$ and then performs

¹⁴Alternative identifications of bulk and boundary fields are possible.

¹⁵Notice that this is valid for the case of the classical gravity regime. The same expression can be obtained as a saddle point approximation of a more general relation between the partition functions of the putative string theory embedding of gravity and of the dual field theory.

the limit $r_c \rightarrow \infty$. The stress-energy tensor can be derived [78, 79, 80, 81] from $S_{\text{ren}} = S + S_{\text{GH}} + S_{\text{ct}}$

$$\langle T_{ab} \rangle = - \lim_{r_c \rightarrow \infty} \left[\left(\frac{r_c}{L} \right)^{d-1} \frac{2}{\sqrt{-\tilde{\gamma}}} \frac{\delta S_{\text{ren}}}{\delta \tilde{\gamma}^{ab}} \right] = \lim_{r_c \rightarrow \infty} \left[\left(\frac{r_c}{L} \right)^{d-1} (T_{ab}^{\text{BY}} + T_{ab}^{\text{ct}}) \right], \quad (2.64)$$

where T_{ab}^{BY} is the Brown-York stress tensor [82] and T_{ab}^{ct} is the counterterm contribution

$$T_{ab}^{\text{BY}} = 2(K\tilde{\gamma}_{ab} - K_{ab}), \quad T_{ab}^{\text{ct}} = -\frac{2d}{L}\tilde{\gamma}_{ab} + \dots,$$

where we have imposed $2k_{(d+2)}^2 = 1$ for simplicity.

Fluid/gravity duality

Fluid/gravity duality can be thought of as the low energy regime of the AdS/CFT correspondence, where long wavelength perturbations of $(d+2)$ -dimensional gravitational solutions can be mapped to the low energy fluid regime of the dual $(d+1)$ -dimensional strongly coupled field theory. It was developed first in the pioneering works of [83, 84, 85] and nonlinearly generalized in [86, 87], see [88, 89] for reviews. Let us here illustrate the main findings for the simplest case of holography in AdS spacetime.

Thermodynamics. According to holography, different asymptotically (locally) AdS spacetimes correspond to different states of the dual conformal field theory. Pure AdS spacetime represents a vacuum CFT and a black hole in AdS realizes a CFT at finite temperature. Restricting ourselves to the case of a dual field theory in Minkowski spacetime $\gamma_{ab} = \eta_{ab}$, for reasons which will be clearer in the following, the global thermal equilibrium configuration is represented by a *planar* black hole, or black brane, in AdS:

$$ds^2 = \frac{r^2}{L^2} (-f(r)dt^2 + dx_i dx^i) + \frac{L^2}{f(r)r^2} dr^2 \quad \text{with} \quad f(r) = 1 - \left(\frac{r_H}{r} \right)^{d+1} \quad (2.65)$$

where r_H is the location of the horizon radius. Clearly AdS asymptotics (2.60) with flat Dirichlet boundary conditions is respected. Notice also that the topology of the horizon is that of a brane rather than that of a hole as it is for Schwarzschild black holes. The Hawking temperature associated to the black brane (2.65) is proportional to the horizon radius

$$T = \frac{(d+1)r_H}{4\pi L^2}. \quad (2.66)$$

The equilibrium configuration properties can be obtained from the holographic definition of the stress-energy tensor (2.64). This gives the ideal fluid stress tensor

(2.4) restricted to static configurations, that is with $u^a = (1, \vec{0})$, with pressure and energy density given by

$$P = \frac{1}{L} \left(\frac{r_H}{L} \right)^{d+1}; \quad \epsilon = \frac{d}{L} \left(\frac{r_H}{L} \right)^{d+1}. \quad (2.67)$$

These quantities satisfy the conformal equation of state $\epsilon = dP$ giving a traceless stress-energy tensor $T_a^a = 0$, which is compatible with the fact that the dual field theory is conformal.

Hydrodynamics. Fluid behavior is achieved by allowing the fluid variables T and u_a to fluctuate. On the gravity side this can be implemented, for example, by allowing fluctuations of the horizon radius¹⁶ $r_H(x)$, given its relation to the temperature through (2.66). These fluctuations must be sufficiently long wavelength in order for a hydrodynamic description to hold. Hence, we must require gradients in the parameters to be small with respect to the temperature

$$\frac{\partial_a r_H / L}{T} \ll 1, \quad (2.68)$$

in the same way as it was in (2.1), where the mean free path is now proportional to the unique microscopic scale $l_{\text{mfp}} \sim 1/T$ set by conformal invariance. Notice that had we had an additional scale, such as the curvature R of the field theory background, we would have needed to consider a dimensionless RT expansion parameter. In order for the gradient expansion to be valid we would have to require $RT \gg 1$, that is variations of the curvature must be smaller than the scale set by the temperature. This translates into the requirement that the geometry should be locally flat and this is why we have chosen to work with planar black holes.

Once the horizon is taken to be a dynamical object, the metric (2.65) is no longer a solution of the Einstein equations of motion. A near equilibrated solution to the Einstein equations must be constructed order by order in a derivative expansion. In other words the equilibrium metric (2.65)¹⁷ with spacetime-dependent parameters must be supplemented by additional terms $g_{\mu\nu}^{(1)} + g_{\mu\nu}^{(2)} + \dots$ containing more and more derivatives $\partial_a r_H$. Using repeatedly the prescription given in (2.64), one can find the dual fluid stress tensor (2.3) at each order in a hydrodynamic expansion. For example, at first order we can reproduce (2.6) in the Landau frame with a

¹⁶Using a diffeomorphism it is possible to generalize the metric (2.65) to a $(d+1)$ -family of solutions with d parameters associated to the fluid velocity u^a . Dynamical configurations with u_a promoted to $u^a(x)$ can be studied as well.

¹⁷In order to avoid problems with regularity at the horizon it is common choice to switch to Eddington-Finkelstein [90, 91] coordinates.

specific form of the shear and bulk viscosity

$$\eta = \left(\frac{r_H}{L}\right)^d; \quad \zeta = 0; \quad \frac{\eta}{s} = \frac{1}{4\pi}, \quad (2.69)$$

where s is the entropy density, obtained from (2.58) with $s = S/V$. The dual field theory, and consequently its fluid regime, is conformal and as such the stress tensor should be traceless, which is assured by a vanishing bulk viscosity given that $T_a^a \sim \zeta$.

The last expression in (2.69) is the celebrated shear viscosity over entropy ratio which is of order $\mathcal{O}(1)$, hence *universal* for every field theory admitting an Einstein gravity dual, that is independent of the details of the model under consideration, see [83, 92, 88, 93, 52]. This result has played a prominent role in the AdS/CFT correspondence and has had valuable consequences for a broader physics community. In 4+1 dimensions the values (2.69) correspond specifically to the shear and bulk viscosity of the low energy $\mathcal{N} = 4$ SYM at strong coupling. Even if this theory is a conformal field theory which has little to do with real world quantum chromo-dynamics (QCD), the finite temperature and hydrodynamic regime resemble instead the properties of the deconfined phase of QCD at finite temperature and finite chemical potential, namely the quark-gluon plasma (QGP). The AdS/CFT correspondence provided useful insights and helped in clarifying the nature of this QGP. By dimensional analysis one would have in fact expected $\eta/s \sim f(\lambda, N_c)/4\pi$, dependent on the dimensionless parameters of the dual field theory, and at weak coupling there have been estimates of this quantity showing that it diverges [94, 95]. This divergence is a sign of the fact that in the weak coupling limit the $\mathcal{N} = 4$ SYM behaves as a gas where dissipative effects are strong. Only after holography entered the scene, the high energy community started to appreciate that the QGP behaves as a strongly coupled nearly ideal fluid instead, as suggested by the smallness of the η/s ratio. Recent experimental findings in [96] have set the value of the QGP shear viscosity over entropy ratio to be $\eta/s \leq 2.5/4\pi$, which is remarkably close to (2.69).

Fluid/gravity duality on a finite cutoff

So far we have discussed holography in AdS spacetimes. However, if holography is the correct paradigm for gravity one should be able to extend the holographic description to more general gravitational theories with different asymptotics. Recent non-asymptotically AdS holographic-type constructions have been given for Lifshitz [97] and Schroedinger spacetimes [98, 99], which are relevant for applications to non relativistic condensed matter systems. But constructing a holographic duality, for example, for asymptotically flat spacetimes has been

a long standing challenge. Different approaches in this direction can be found in [100, 101, 102, 103, 104], but there are strong indications that the dual field theory should be non local anyway, see also [105, 106]. Another case of study is the de Sitter/CFT correspondence [107, 108] relevant for mimicking our universe where the spacetime has positive cosmological constant.

The fluid regime can be thought of as a simplifying starting point in exploring different asymptotics. In this case instead of searching for a comprehensive dictionary between all the possible bulk fields and boundary data valid at all scales, one can start by considering the stress-energy tensor alone in its low energy regime. An interesting approach of this type can be found in the so called fluid/gravity duality on a finite cutoff. This entails in locating a hard cutoff, a hypersurface at a fixed radial position, in the interior of spacetime and considering only physics between the horizon of the black hole and the cutoff itself. In this way the asymptotics, the region near the boundary, is completely neglected. By assuming the same conventional holographic dictionary on a finite cutoff, one can as well apply a fluid/gravity duality reasoning, provided one gives a consistent prescription for the dual stress tensor. The natural candidate is the conserved Brown-York stress tensor (2.65) although some ambiguities are involved. We will show how the latter can be partially fixed in Chapter 4.

Early developments of this scheme can be found in [109, 110, 111] for the case of vacuum Einstein gravity. A systematic construction of the metric to all orders in the hydrodynamic expansion was provided first in a non relativistic setting in [111] and generalized to relativistic setups in [112, 113]. The thermal state corresponds here to the Rindler spacetime which has an horizon determined by accelerated observers and the dual fluid, the so-called *Rindler fluid*, lives on a finite cutoff with flat background metric and has its own specific (pathological) fluid properties. At this point one can try to generalize the procedure to spacetimes with different asymptotics and general bulk stress tensor with matter, see also [114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124] for related work. We will consider this scenario in detail in Chapter 4.

Analyzing fluid/gravity duality on a finite cutoff is an interesting problem on its own. The finite cutoff, which is in no particular place of spacetime, can be freely moved from the asymptotic boundary to the horizon. Hence, it is particularly interesting exploiting the simple case of fluid/gravity duality on a finite cutoff in AdS, see *e.g.* [125, 126], and understand what is its near-horizon limit which, not surprisingly, turns out to be Rindler fluid dynamics. We have encountered in Section 2.2.1 yet another fluid behavior on the horizon, the membrane fluid. We will show in Chapter 5 what are the substantial differences between the membrane fluid and a holographic fluid on a finite cutoff pushed towards the horizon.

Retarded Green's function in holography

Let us here mention that so far we have considered a holographic dictionary for Euclidean correlators. However, it should be possible to holographically compute also correlators defined in Minkowski spacetime, useful for real time and finite temperature situations which will be relevant in Chapter 5. A prescription has been first proposed for the simplest case of the retarded Green's function in [127], once this is known one can use standard relations to obtain the other Minkowski correlators, see *e.g.* [128]. This prescription has been later justified in [55] and generalized to generic n -point functions in [129, 130].

Holographic retarded Green's function. For the purposes of this thesis we will only be interested in the simple prescription of [127] which is inspired by linear response theory. Consider a system at equilibrium and apply a perturbation $\phi^{(0)}$. From a field theory point of view this is a source which modifies the Lagrangian as $\delta\mathcal{L} \sim \mathcal{O}\phi^{(0)}$. If the disturbance $\phi^{(0)}$ is sufficiently small in amplitude, the system will react modifying the expectation value of the corresponding operator in a simple linear way

$$\langle \mathcal{O}(\omega, \vec{k}) \rangle = -G_R(\omega, \vec{k})\phi^{(0)}(\omega, \vec{k}) \quad (2.70)$$

by means of the retarded Green's function G_R of the operator $\mathcal{O}$, which in coordinate space is given by

$$G_R(x, y) = -i\theta(x^0 - y^0)\langle [\mathcal{O}(x), \mathcal{O}(y)] \rangle \quad (2.71)$$

The reason one employs specifically a retarded Green's function here and not any other real time correlator can be understood from causality considerations: modifications to the system need to happen only after the source has been applied. At this point one can readily define a holographic prescription for the retarded Green's function by inverting eq. (2.70) and using the holographic dictionary for the one-point function

$$G_R(\omega, \vec{k}) = -\frac{1}{\phi^{(0)}(\omega, \vec{k})} \frac{\delta S_{\text{ren}}}{\delta \phi^{(0)}(\omega, \vec{k})}, \quad (2.72)$$

where S_{ren} is the renormalized gravitational on-shell action. Hence, in practice, in order to compute the retarded Green's function for a certain observable $\mathcal{O}$ from a holographic point of view, one needs to perturb the bulk with the corresponding linearized bulk field perturbation ϕ whose boundary value is $\phi^{(0)}$. Given that Einstein equations are a set of second order differential equations one needs a second boundary condition in the interior of the spacetime. For the prescription above to work one specifically requires an ingoing boundary condition on the horizon.

Once the solution is known one computes the (renormalized) on-shell action and eventually applies (2.72).

Quasinormal modes. Retarded correlators are particularly interesting since they encode nontrivial physics in the structure of their poles. From these poles one can extract dispersion relations $\omega = \omega(\vec{k})$ obeyed by the respective perturbations. In fact after a source is applied, the system generically tends to come back to equilibrium configuration and the dispersion relation contains the information on how the system will eventually equilibrate.

What is the interpretation of these poles from a gravitational point of view? As it has been shown in [131, 127] they can be identified with the so-called black hole *quasinormal modes*, see [132] for a review. These modes are defined as perturbations of black hole solutions which are precisely ingoing at the horizon and vanishing at asymptotic infinity. Every black hole has its own set of quasinormal modes which have been studied in the realm of stability of black hole solutions, in fact for a stable configuration after perturbing a little a black hole, these fluctuations should not grow big destroying the black hole background. The fact that quasinormal modes are related to poles of the retarded Green's function in holography can be understood by staring a little at the eq. (2.72). It is easy to see that poles are exactly the places where the source $\phi^{(0)}$ or equivalently the asymptotic Dirichlet boundary condition vanishes, which is the defining property of quasinormal modes.

The structure of quasinormal modes is quite rich. They exhibit oscillatory behavior, hence they are *normal*, and also damped behavior due to the presence of an horizon surface, this is why they are *quasinormal*. Generically they organize in an infinite tower of short-lived *gapped* quasinormal modes which die off very quickly, and long-lived *hydrodynamic* modes which encode the hydrodynamic response, that is $\omega(k) \rightarrow 0$ for $k \rightarrow 0$ in the low energy regime $k, \omega \ll T$.

For example for a massless scalar field ϕ in a (planar) black hole in AdS_3 , *i.e.* the non rotating Banados-Teitelboim-Zanelli (BTZ) black hole [133, 127], the quasinormal modes are given by

$$\omega = \pm k - i 4\pi T (n + 1) \quad \text{for } n = 0, 1, 2, \dots \quad (2.73)$$

These can be exactly matched to the poles of the retarded Green's function of the dual $(1 + 1)$ -dimensional CFT at finite temperature, as derived in [127].

Hydrodynamic quasinormal modes. Let us now consider in field theory the response to linearized metric perturbations $\delta h_{ab}^{(0)}$ sourcing a stress-energy tensor T_{ab} . By choosing for concreteness a convenient direction in momentum space such that $\vec{k} = (k, 0, \dots)$, one can classify the perturbations according to their

transformation properties with respect to the residual symmetry $SO(d-1)$ of the plane transverse to the direction of propagation. In general one has scalar, vector and tensor modes, but it turns out that in the low energy regime tensor modes do not feature any hydrodynamic behavior, hence we will not consider them anymore. By inspecting the pole structure of the retarded correlator $G_R \sim \theta(t)\langle [T T] \rangle$ in the hydrodynamic regime $\omega, k \ll T$, one can then extract the corresponding dispersion relation.

The vector sector, or the *shear* or *transverse channel*, encodes diffusion in the transverse direction and the dispersion relation in a hydrodynamic expansion is purely imaginary

$$\omega_T = -i \frac{\eta}{\epsilon + P} k^2 - i \left(\frac{\eta}{\epsilon + P} \right)^2 \tau k^4 + \mathcal{O}(k^5). \quad (2.74)$$

Here ϵ and P are as usual the energy density and pressure, η is the shear viscosity and τ is the relaxation time, a coefficient which in the conventional formalism would be encoded in the second order expansion of the stress-energy tensor $T_{ab}^{(2)}$, see *e.g.* [134]. Notice that in the case we restrict ourselves to a dissipationless response such a dispersion relation is trivially vanishing, meaning that in this case there is no dynamics in the transverse sector. This is the linearized manifestation of the volume-preserving diffeomorphisms invariance for perfect fluids in the effective action formalism as discussed previously, see (2.32).

The dispersion relation for the scalar sector, or *longitudinal channel*, encodes instead the information on the perturbations along the longitudinal direction of the propagation. For example for a conformal fluid for which $\zeta = 0$ it is given by

$$\begin{aligned} \omega_L &= \pm c_s k - i \Gamma k^2 \pm \frac{\Gamma}{c_s} \left(c_s^2 \tau - \frac{\Gamma}{2} \right) k^3 + \mathcal{O}(k^4), \\ \text{with } \Gamma &= \frac{1}{\epsilon + P} \frac{d-1}{d} \eta, \end{aligned} \quad (2.75)$$

where c_s is the sound velocity defined as $c_s^2 = \partial P / \partial \epsilon$, see *e.g.* [134].

Holographically the same results can be obtained by analyzing the corresponding linear bulk field perturbations $\delta h_{\mu\nu}$ whose Dirichlet boundary conditions are $\delta h_{ab}^{(0)}$, and computing the poles of the retarded Green's function following the prescription (2.72). For a planar black-hole in AdS_5 and restricting to the hydrodynamic regime, one can reproduce the dispersion relations (2.74-2.75) with the values of the thermodynamic and transport properties found above (2.67) and (2.69) where the relaxation time given by

$$\tau = \frac{2 - \ln 2}{2\pi T}, \quad (2.76)$$

as derived to first order in [84, 84, 135] and to second order in [134].

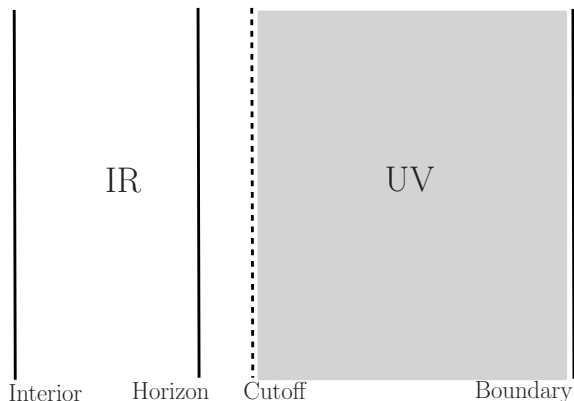


Figure 2.2

Holographic Wilsonian effective action

An interesting generic feature of the AdS/CFT correspondence is the so-called *UV/IR connection* [136]. The radial dimension extending in the interior of the spacetime is naturally related to the energy scale of the dual field theory. It turns out that large volumes in gravity, hence the infra-red (IR) physics, are dual to the ultra-violet (UV) regime of the field theory and going into the interior of the spacetime is mapped to the IR behavior of the dual field theory. In other words, one should think of holography as a map between a gravitational theory and a field theory taken together with its renormalization group flow.

This idea has been initially explored in [75, 137] and more recently revisited in [138, 139] building the concept of a holographic Wilsonian effective action, see also [27, 140, 141]. Let us illustrate these findings by considering a cutoff hypersurface Σ in the interior of a spacetime, at a fixed radial position, which divides the spacetime into two regions. Borrowing the field theory language, we dub the region between the interior of spacetime and the cutoff, possibly containing an horizon, as the *IR region* and the spacetime between the cutoff and the asymptotic boundary as the *UV region*, schematically represented in Figure 2.2. The action also naturally separates accordingly in two pieces

$$S = S^{\text{IR}} + S^{\text{UV}}. \quad (2.77)$$

We have discussed a similar separation in the context of the membrane paradigm in Section 2.2.1. The difference here is that in general there is no need to place the cutoff on the horizon or a stretched horizon, but it can be located anywhere in the interior of the spacetime. These two regions can be analyzed separately imposing Dirichlet boundary conditions on the cutoff Σ , and, for the solutions to

be smoothly connected to each other, requiring a continuity equation equivalent to (2.55) which we rewrite here for completeness

$$\frac{\delta S}{\delta \phi^\Sigma} = \frac{\delta S^{\text{IR}}}{\delta \phi^\Sigma} + \frac{\delta S^{\text{UV}}}{\delta \phi^\Sigma} = 0. \quad (2.78)$$

The UV region is related to the high energy degrees of freedom of the dual field theory, which can be generically integrated-out in favor of a Wilsonian effective action [142]. From a gravitational point of view, such a Wilsonian effective action can be identified with S^{UV} evaluated on-shell between the two boundaries on which one has imposed two Dirichlet boundary conditions. In this way the UV spacetime is effectively "integrated-out" and the bulk UV action is replaced with a boundary term. Given that the radial position of the cutoff surface Σ is supposedly related to the energy scale, moving the surface into the interior of the spacetime corresponds for S^{UV} to perform the low energy limit of the Wilsonian effective action. This procedure has been applied to electromagnetic fields in an AdS black brane background in [138, 139, 27]. In Chapter 3 we will show how to extend it to the case of gravitational fields, and the outcome will be the perfect fluid effective action discussed in Section 2.1.2.

Dissipative effects can be naturally studied by coupling the UV region to the IR which contains the horizon surface. As we already discussed, the evaluation of S^{IR} is not a simple task since it contains the contribution of the interior of the black hole and in principle one needs to consider the full Schwinger-Keldysh formalism. However, the membrane paradigm is an excellent approximation scheme which allows us to neglect all these complications and replace the full dynamical IR sector with a simple membrane response (2.56) as in [27, 139]. We will see in Chapter 3 how this coupling correctly reproduces the hydrodynamic modes (2.74) and (2.75) at infinity.

2.3 Preview of the main results

The aim of this introduction has been in illustrating the main concepts that are going to be used throughout this thesis, emphasizing the questions which will be discussed in the following.

We started our journey with fluid dynamics showing how this behavior could

be implemented in different ways. The effective action formalism, for example, was particularly interesting since it involved a huge reparametrization symmetry, the volume-preserving diffeomorphisms invariance. One context in which such a symmetry can be tested is holography by computing explicitly a holographic Wilsonian effective action.

Proceeding our journey we showed interest in developing holography beyond asymptotically AdS spacetimes, which motivated a fluid/gravity duality approach on a finite cutoff. Moving the cutoff deeper in the interior of spacetime provides a holographic fluid interpretation near the horizon and we wondered whether this is equivalent to the membrane fluid.

Finally, we also showed how, among the many interpretations of the membrane paradigm, there is one that better suits applications to holography. We motivated the importance of a simple membrane response to replace the complicated near horizon dynamics and we also questioned whether such an approximation is always allowed.

Let us here outline the structure of this thesis together with a summary of the *main results*.

In Chapter 3:

- we *derive* the effective action for relativistic, conformal, perfect fluids from holography. At least at leading order in a hydrodynamic expansion the volume-preserving diffeomorphisms symmetry is reproduced,
- we show that at subleading order there are intrinsic *divergences* which are only removed when dissipative effects are taken into account. This seems to suggest, at least within our holographic setup, that it is not possible to decouple the dissipationless sector alone,
- we holographically identify the *Goldstone fields* ϕ^I of the effective action approach to fluid dynamics with spacelike geodesics in the bulk.

In Chapter 4:

- we identify a (partially) unambiguous *holographic prescription* for the dual hydrodynamic stress-energy tensor on a finite cutoff,
- we construct, to a certain extent, fluid/gravity duality on a finite cutoff for Einstein gravity with a general bulk stress tensor.

In Chapter 5:

- we show that the membrane paradigm as a boundary condition is a *good* approximation scheme with the only exception being in the computation of the short-lived massive quasinormal modes,
- we show that a holographic fluid behavior on a finite cutoff taken near the horizon is a more complete fluid description than the membrane fluid.

3

EFFECTIVE ACTIONS FOR FLUIDS FROM HOLOGRAPHY

ON INTEGRATING OUT SPACETIME GEOMETRY FOR THE SAKE OF AN EFFECTIVE
FIELD THEORY OF LOW ENERGY GOLDSTONE DEGREES OF FREEDOM

A recent reformulation of perfect fluid dynamics relies on an effective action description, which is constrained by volume-preserving diffeomorphisms invariance. However, it is not clear whether such a symmetry is sufficient for a complete description of non dissipative transport. In this Chapter, based on [1], we are going to explicitly derive the low energy effective action for relativistic, perfect, conformal, strongly coupled fluids from gravity. This provides an example where the above symmetry holds and our results can be interpreted as yet another derivation of fluid/gravity duality.

3.1 Introduction

As already illustrated in Section 2.2.2 of the Introduction, the key step in deriving an effective action from gravity is to consider the holographic Wilsonian effective field theory approach. High energy field theory degrees of freedom are encoded in the UV region of the spacetime between a finite cutoff and the asymptotic boundary of the spacetime. Integrating-out these degrees of freedom corresponds to “integrate-out” the UV geometry. This procedure entails in finding the solution for fields in the UV region with two Dirichlet boundary conditions on the two boundaries, that is, one needs to solve what we call a *double-Dirichlet problem*. Once the solution is known, the UV action can be evaluated on-shell and interpreted as the Wilsonian effective action for the dual field theory.

Applying this reasoning to the derivation of an effective action for (conformal) fluids corresponds to consider gravitational perturbations in an AdS spacetime in the presence of a planar horizon. The double-Dirichlet problem in the UV fixes in this case two independent metrics, one is G_{AB} on a finite cutoff and the other is γ_{ab} on the asymptotic boundary. The effective action is a function of the two Dirichlet boundary data, invariant under diffeomorphism $\times$ diffeomorphism symmetry. However, in order for the action to be local one needs additional fields ϕ^A , bifundamental between the two metrics¹, which allow to pull back the metric on one boundary to the other through for example $\partial_a \phi^A \partial_b \phi^B G_{AB}$. These scalar fields can also be interpreted as Goldstone bosons if we consider the two boundary metrics to be Minkowski. The effective action in this case is invariant under Poincaré $\times$ Poincaré symmetry, but any solution of the gravitational field equations will connect the two boundaries and break the symmetry spontaneously to a diagonal Poincaré subgroup and thereby give rise to a set of Goldstone bosons². Though this argument clearly does not apply if we start with two arbitrary metrics on the boundaries, we will nonetheless stick to the term Goldstone bosons.

The final necessary step is to push the cutoff hypersurface towards the horizon where the metric degenerates. This feature will be crucial since it will trade the generic diffeomorphisms invariance on the cutoff hypersurface for the volume-preserving diffeomorphism internal symmetry of the Goldstones, giving rise to the effective action for perfect (conformal) fluids as discussed in Section 2.1.2, generalized to an arbitrary metric in the configuration space of the Goldstones which we will discuss in Section 3.2. We find it however rather intriguing why this should be the case. Why is the low energy dynamics of planar black holes in AdS compatible with fluid dynamics after all? Why is it not resembling, for example, jelly dynamics, which responds nontrivially to shear stresses too and does not have the above mentioned internal symmetry? Unfortunately we have not been able to answer those questions from first principles within our gravitational embedding.

In this Chapter we construct the effective action for perfect conformal fluids in two ways. First in Section 3.3 we work in a fully nonlinear setting at leading order in a derivative expansion. Subsequently in Section 3.4 we work with linearized metric perturbations which allow to extend the reasoning to second order in derivatives where we find more intriguing results. Although the effective action is still compatible with the volume-preserving diffeomorphism symmetry, we find, perhaps not surprisingly, that the answer is in general divergent, but remains finite when we restrict to stationary configurations. In fact by sending the cutoff all the way

¹In this perspective, the effective action can also be reinterpreted as a bigravity theory as in [143].

²Completely analogous reasoning, albeit applied to gauge fields, explains the emergence of pions in the Sakai-Sugimoto model of holographic QCD [144].

to the horizon amounts in imposing Dirichlet boundary conditions on the horizon itself which in general is not a physically reasonable thing to do.

To cure these divergences, in Section 3.5.1 we consider a setup where we couple the UV effective theory to a dissipative IR system, which is supposed to describe the near-horizon physics and in particular encode the one-way nature of the event horizon. As discussed in Section 2.2.1 we can replace the full dynamical IR sector with a simple membrane paradigm boundary condition a small distance away from the horizon and then take the membrane to the horizon. Technically, the coupling to the membrane simply modifies the IR boundary condition in such a way that it imposes ingoing, as opposed to Dirichlet, boundary conditions on the horizon. Hence, by including the contribution of the dissipative sector we are lead to a finite answer. This means, at least within our holographic setup, that it is not possible to separate the dissipative sector from the dissipationless one. This is perhaps not such a surprising result since the field theory dual to a planar black hole in AdS is intrinsically dissipative, and trying to isolate a dissipationless sector might not be a physically allowed thing to do.

In Section 3.5.2 we restrict ourselves to stationary configurations and we describe the coupling of the UV effective action to a different, non dissipative, IR sector. The latter is given as a simple functional of the metric and it captures the contribution of the tip of the Euclidean cigar which describes the Euclidean black hole. If we then extremize the sum of this IR sector with our double-Dirichlet UV effective action, we automatically obtain the lowest order contribution to the equilibrium partition function discussed in Section 2.1.2. The extremalization procedure turns out to be equivalent to a Legendre transform which transforms the energy density into the pressure. This is in perfect agreement with the fact that the action for a fluid in terms of Goldstone bosons is given by the energy while the equilibrium partition function is given by the pressure.

Finally in Section 3.6 we start questioning the role of the entropy current in the effective action formalism for perfect fluids, which do not feature entropy production by definition. It is natural to ask whether such a conserved entropy current originates in any way as the Noether current of some symmetry of the effective action. We will show that there exists indeed a nontrivial transformation on the Goldstones which, if assumed to be a symmetry of the effective action, correctly reproduces the entropy current as the Noether current of this would-be symmetry.

3.2 Generalized effective action formulation

Before diving directly into an effective action construction from gravity, let us here illustrate two generalizations of the perfect fluid action (2.17), considered also in [27], which will be relevant for us.

General configuration space metric

First of all notice that there is no fundamental reason for having a trivial metric on the configuration space of the Goldstones ϕ^i as it was in Section 2.1.2. Let us therefore consider a generic, albeit non-dynamical, background metric $G_{ij}(\vec{\phi})$ which couples to the Goldstones in the effective action

$$S = \int d^{d+1}x \sqrt{-\gamma} F(s), \quad (3.1)$$

through the entropy density

$$s = s_0 \sqrt{\det(\partial_a \phi^i \partial_b \phi^j \gamma^{ab} G_{jk})}. \quad (3.2)$$

Such a rewriting is motivated precisely by the fact that in our gravitational setup the action will generically depend on two metrics: one on the asymptotic boundary, which is going to represent the background fluid metric, and another on the cutoff (sent towards the horizon) which is going to be the configuration space metric for the Goldstones.

The entropy current (2.22) depends now on the configuration space metric via the Levi-Civita tensor

$$\epsilon_{i_1 \dots i_{d-1}} \rightarrow \sqrt{\det G_{ij}} \epsilon_{i_1 \dots i_{d-1}}^{(0)}, \quad (3.3)$$

where $\epsilon^{(0)}$ is the (flat space) Levi-Civita symbol. Nevertheless, it is still identically conserved

$$\nabla_a J^a = -\frac{1}{2} J^a (\partial_a G_{ij}) G^{ij} = -\frac{1}{2} J^a (\partial_a \phi^k) (\partial_k G_{ij}) G^{ij} = 0, \quad (3.4)$$

as ϕ^i fields are the comoving coordinates and hence $J^a \partial_a \phi^i = s \cdot (u^a \partial_a \phi^i) = 0$.

The timelike Goldstone

The second generalization follows from considering systems parametrized by $d+1$, instead of d , scalar fields

$$\phi^A(x) = \delta_a^A x^a + \pi^A(x), \quad (3.5)$$

where now $\phi^A = (\phi^t, \vec{\phi})$ and ϕ^t is what we call the *timelike Goldstone*. This additional Goldstone is motivated by a gravitational embedding in mind with two hypersurfaces characterized by $(d+1)$ -dimensional metrics. The bifundamental fields realizing the pull-back of one metric to the other should be in fact combined in a $(d+1)$ -dimensional vector ϕ^A .

The addition of this Goldstone modifies the parametrization of the configuration space metric. Given that the latter will be the degenerate metric of the horizon, a convenient parametrization can be given in terms of the so-called Galilean metric which writes

$$ds^2 = G_{AB} d\phi^A d\phi^B = G_{ij} (d\phi^i - v^i d\phi^t) (d\phi^j - v^j d\phi^t), \quad (3.6)$$

together with the following null vector

$$n^A = \frac{1}{\gamma} (1, v^i) \quad \text{with} \quad G_{AB} n^B = 0 \quad \text{and} \quad \gamma = (1 - G_{ij} v^i v^j)^{-1/2}, \quad (3.7)$$

and the variables G_{ij} and $\vec{v}$ depend on the internal spacetime coordinates ϕ^A .

The entropy current (2.22) can be generalized to be the spacetime Hodge dual of the volume form of the coordinates e^i

$$J^a = \frac{1}{d!} s_0 \epsilon^{a b_1 \dots b_d} \epsilon_{i_1 \dots i_d} e_{b_1}^{i_1} \dots e_{b_d}^{i_d}, \quad (3.8)$$

where the combinations e_a^i are defined as

$$e_a^i = \partial_a \phi^i - v^i \partial_a \phi^t. \quad (3.9)$$

The indices a, b , etc. are raised with the spacetime inverse metric γ^{ab} and the Latin indices i, j , etc. are lowered and raised with the configuration space metric G_{ij} and its inverse. Writing explicitly the dependence of the Levi Civita tensors on the relevant metrics, the generalized entropy current takes the form

$$J_a = \frac{1}{d!} s_0 \sqrt{\det G_{ij}} \sqrt{-\gamma} \gamma^{b_1 c_1} \dots \gamma^{b_d c_d} \epsilon_{a c_1 \dots c_d}^{(0)} \epsilon_{i_1 \dots i_d}^{(0)} e_{b_1}^{i_1} \dots e_{b_d}^{i_d}, \quad (3.10)$$

and the corresponding entropy density is

$$s = s_0 \sqrt{\det(e_a^i e^{aj} G_{jk})}. \quad (3.11)$$

Note that a priori the current (3.8) is *not* conserved off-shell for generic G_{ij} and $\vec{v}$.

Linearized expansion

Let us now demonstrate that both generalizations of the effective action for fluids illustrated above lead to the correct equations of relativistic fluid mechanics. We will show this at linear level for clarity but the same reasoning can be applied at nonlinear level.

Consider for concreteness the following linearized expansion of the configuration spacetime metric

$$G_{ij} = \delta_{ij} + H_{ij}(\phi^A); \quad v_i = -H_{ti}(\phi^A), \quad (3.12)$$

where H_{ij} and H_{ti} are small perturbations. Without any loss of generality we can restrict the perturbations H_{AB} to depend only on *e.g.* (ϕ^t, ϕ^x) , which in Fourier space corresponds to a choice of the direction of propagation of the perturbations being $\vec{k} = (k_x, 0, \dots, 0)$. This choice allows us to divide the perturbations according to their transformation properties in the remaining transverse $O(d-1)$ plane. There are three distinguished sectors

$$\begin{aligned} \text{scalar:} & \quad H_{xx}, \quad H, \quad H_{tx}, \quad H_{tt} \\ \text{vector:} & \quad H_{x\alpha}, \quad H_{t\alpha} \\ \text{tensor:} & \quad H_{\alpha\beta} - \frac{1}{(d-1)}\delta_{\alpha\beta}H, \end{aligned} \quad (3.13)$$

where $H = \sum_{\alpha} H_{\alpha\alpha}$ is the trace in the transverse direction $\alpha = 1, \dots, d-1$. We will not consider the tensor sector in the following since it can be shown that it is not linked to any hydrodynamic behavior. Inserting the scalar and vector sectors expansions into (3.11) and using the linearized Goldstone expansions (3.5), the leading order effective action (3.1) takes the form

$$\begin{aligned} S^{(0)} = \int d^{d+1}x & \left(F(s_0) + \frac{1}{2}s_0 F'(s_0)H_{ii} + s_0 F'(s_0)\delta_x \pi^x + \right. \\ & - \frac{1}{2}s_0 F'(s_0) \left(H_{tx}^2 + \frac{1}{4}H_{xx}^2 - \frac{1}{2}H_{xx}H - \frac{1}{4}c_s^2 H_{ii}^2 \right) + \\ & + s_0 F'(s_0) \left(\frac{1}{2}\dot{H}_{ii} - \partial_x H_{tx} \right) \pi^t + \\ & - \frac{1}{2}s_0 F'(s_0) \left((\dot{\pi}^x)^2 - c_s^2 (\delta_x \pi^x)^2 - 2\pi^x \dot{H}_{tx} + c_s^2 \pi^x \partial_x H_{ii} \right) + \\ & \left. - \frac{1}{2}s_0 F'(s_0) \sum_{\alpha} \left((\dot{\pi}^{\alpha})^2 + H_{t\alpha}^2 + H_{x\alpha}^2 - 2\pi^{\alpha} \dot{H}_{t\alpha} \right) \right), \end{aligned} \quad (3.14)$$

where $H_{ii} = H_{xx} + H$ and c_s is the speed of sound defined in (2.33). This explicit expansion will be the one that we will derive in Section 3.4 from a linearized gravitational setting.

Notice that the linearized timelike Goldstone π^t appears here as a Lagrange multiplier and the corresponding equation of motion ensures now the on-shell conservation of the entropy current

$$\nabla_a J^a = -\frac{1}{2}\dot{H}_{ii} + \partial_x H_{tx} + \dots \Big|_{\text{on-shell}} = 0. \quad (3.15)$$

The equations of motion for the transverse and longitudinal Goldstones are now respectively

$$\partial_t^2 \pi^\alpha + \partial_t H_{t\alpha} = 0, \quad \partial_t^2 \pi^x - c_s^2 \partial_x^2 \pi^x + \partial_t H_{tx} - \frac{1}{2} c_s^2 \partial_x H_{ii} = 0 \quad (3.16)$$

and correspond to the conservation equations (2.23) of the perfect fluid stress-energy tensor (2.4) with the velocity and the entropy density *redefined* in the following way

$$u^t = -1 - \frac{1}{2}(H_{t\bar{x}} + \partial_t \bar{\pi})^2 + \dots, \quad \vec{u} = H_{t\bar{x}} + \partial_t \bar{\pi} + \dots \quad (3.17)$$

$$s = s_0 + \frac{1}{2} s_0 H_{ii} + s_0 \nabla \cdot \vec{\pi} + \dots \quad (3.18)$$

These expressions can be obtained from the linearization of (3.11) and using $u^a = J^a/s$, where J^a given in (3.10). Hence we showed explicitly that the above generalizations of the perfect fluid effective action of Section 2.1.2 reproduce the correct hydrodynamic equations. It can be shown that the same is true at nonlinear level. This analysis indicates that the standard action for relativistic perfect fluids is a particular instance of the more general action obtained by making the Lagrangian (3.1) depend on s defined by Eq. (3.11), rather than by Eq. (2.18). Actions of these types were encountered previously in Ref. [27] and in the following sections we will derive such an action using holography. Notice that a key ingredient in this derivation is the degenerate nature of the configuration space metric. Had we worked with a general non-degenerate metric instead, the timelike Goldstone would be dynamical and not only a Lagrange multiplier.

3.3 Nonlinear effective action from gravity

In this Section we derive the low energy leading order effective action for relativistic, conformal, perfect fluids from holography at full nonlinear level. As a first step

we solve the double-Dirichlet problem between a finite cutoff in the interior of the spacetime, which we dub as the *IR brane*, and another hypersurface, which we dub as the *UV brane*, also at a finite radial coordinate but closer to asymptotic infinity. With such a solution at hand we compute the (partially) on-shell UV action³ in that region deriving in this intermediate step what we call the *double-Dirichlet effective action*. Subsequently we stretch the UV region by sending the IR brane to the horizon and the UV brane to infinity, thereby deriving the effective action for perfect fluids (3.1).

3.3.1 The double-Dirichlet effective action

The effective action between two generic branes in the interior of the spacetime should generically depend on the IR and UV brane metrics, G_{AB} and γ_{ab} respectively, and on the Goldstones ϕ^A

$$S_{\text{eff}}(\gamma_{ab}, G_{AB}, \phi^A), \quad (3.19)$$

and should be invariant under diffeomorphisms on the two hypersurfaces, therefore explicit dependences on the positions of the branes, say u_1 and u_2 , are not allowed. Moreover, given that this construction relies on two slices in no particular place in the interior of the spacetime, the effective action should be invariant under $\gamma \leftrightarrow G$, up to possible field redefinitions. The Goldstone modes $\phi^A(x)$ are the bifundamental fields between the two branes and can be used to rewrite the effective action (3.19) as a *local* theory on one of the branes

$$S_{\text{eff}} = \int d^{d+1}x \sqrt{-\gamma} F(\gamma_{ab}, h_{ab}), \quad (3.20)$$

where h_{ab} is the pull-back of the IR metric G_{AB} to the UV brane

$$h_{ab} = G_{AB} \frac{\partial \phi^A}{\partial x^a} \frac{\partial \phi^B}{\partial x^b} \quad (3.21)$$

and $F(\gamma_{ab}, h_{ab})$ is a scalar quantity built from the two tensors γ and h .

Let us now restrict the form of (3.20) by analyzing what are the scalar structures that are allowed. In principle, the action (3.20) contains derivative terms of the Goldstone fields and of the curvatures built from γ and h . However, here we are interested in the low energy regime only. Hence recalling that Goldstones have mass dimension -1, we accept terms with arbitrary number of Goldstones with no more than one derivative applied to each of these fields. This is the analogue

³Not all Einstein equations are used to compute this action. Constraint equations are left unsolved and turn out to be equations of motion for the Goldstones connecting the two branes.

of the hydrodynamic gradient expansion and is also the reason why we assume the effective action depends only on the metrics and the Goldstones. Given the diffeomorphisms $\times$ diffeomorphisms invariance, the independent scalar structures that we can construct out of γ_{ab} and h_{ab} , without using derivatives, are the traces $\text{Tr } M$, $\text{Tr } M^2, \dots$ of the matrix $M \equiv h^{ac}\gamma_{cb}$ where h^{ac} is the inverse of h_{ac} . In $d+1$ dimensions there are only $d+1$ independent traces corresponding to the amount of eigenvalues of the matrix M . Therefore the effective action (3.19) depends on $d+1$ scalars

$$F(\gamma_{ab}, h_{ab}) = F[M] = F[\text{Tr } M, \dots, \text{Tr } M^{d+1}]. \quad (3.22)$$

Equivalently, we could have chosen to work with traces of M^{-1} , but will find traces of M to be more convenient.

Geometric interpretation of the Goldstone bosons

Let us here show how the Goldstone bosons arise more precisely. Suppose there is a solution to the double-Dirichlet problem in Einstein gravity, that is we know what is the bulk geometry with two Dirichlet boundary conditions on the two branes. By applying a diffeomorphism $x^\mu \rightarrow y^\mu(u, x^a)$, the solution can be brought to the radial Arnowitt-Deser-Misner (ADM) [145] form

$$ds^2 = g_{\mu\nu} dy^\mu dy^\nu = dU^2 + 2 A_a(y) dy^a dU + g_{ab}(y, U) dy^a dy^b, \quad (3.23)$$

with new bulk spacetime coordinates $y^\mu = (U, y^a)$ with $y^a = (t, \vec{y})$. The original radial slices located say at u_1 and u_2 are now at $U = U_{1,2}(y^\mu)$ for some functions $U_{1,2}(y^\mu)$ and lines of $y^a = \text{const}$ are spatial geodesics. By undoing the change of coordinates that put the metric in the radial ADM form (3.23) with $y^\mu \rightarrow x^\mu(U, y^a)$, we can identify the Goldstone bosons with the map from $x^\mu(U_2(y^\mu), y^a)$ to $x^\mu(U_1(y^\mu), y^a)$ along the spacelike geodesics. By construction, these Goldstone modes are covariant and transform correctly under diffeomorphisms of the metrics on the two slices. We could also imagine alternative definitions based on spacelike or null geodesics which make a prescribed angle with one of the two boundaries, but we expect these to be related through a field redefinition to the previous construction.

The derivation

We now specify to the case of Einstein gravity with a negative cosmological constant

$$S = \frac{1}{2k_{(d+2)}^2} \int du d^{d+1}x \sqrt{-g} (R - 2\Lambda), \quad (3.24)$$

where g is the determinant of the spacetime metric $g_{\mu\nu}$. Let us from now on set $2k_{(d+2)}^2 = 1$ for simplicity. Einstein field equations are simply given by

$$R_{\mu\nu} = \frac{2}{d} \Lambda g_{\mu\nu}. \quad (3.25)$$

The most general metric ansatz where Goldstones are given by spatial geodesics was given above in (3.23). Let us however restrict to a subset of solutions where the UV and IR metric are constant. Because of this restriction we can even take the full metric to be independent on the field theory coordinates y^a . Moreover we can always perform a shift $y^a \rightarrow y^a - U A^a$ to get rid of A_a in the metric. Finally, replacing U with u , and y with x , the metric ansatz that we are going to adopt is simply

$$ds^2 = du^2 + g_{ab}(u) dx^a dx^b. \quad (3.26)$$

Moreover since we have anticipated above that the effective action involves eigenvalues of the matrix M through (3.22), we can probe this feature with diagonal metrics on the two boundaries. Hence we can parametrize $g_{ab}(u)$ as follows

$$g_{ab}(u) dx^a dx^b = e^{\psi_a(u)} \delta_{ab} dx^a dx^b = -e^{\psi_t(u)} dt^2 + e^{\psi_i(u)} dx_i dx^i. \quad (3.27)$$

Equations of motion and solutions. The Ricci scalar on the ansatz (3.26) as a function of the single variable g_{ab} can then be evaluated to be

$$R = -\text{Tr} (g^{-1} \partial_u^2 g) + \frac{3}{4} \text{Tr} (g^{-1} \partial_u g g^{-1} \partial_u g) - \frac{1}{4} (\text{Tr}(g^{-1} \partial_u g))^2, \quad (3.28)$$

with the Ricci tensor given by

$$R_{uu} = -\frac{1}{2} \text{Tr} (g^{-1} \partial_u^2 g) + \frac{1}{4} \text{Tr} (g^{-1} \partial_u g g^{-1} \partial_u g), \quad (3.29)$$

$$R_{ab} = -\frac{1}{2} \partial_u^2 g + \frac{1}{2} \partial_u g g^{-1} \partial_u g - \frac{1}{4} \text{Tr} (g^{-1} \partial_u g) \partial_u g, \quad (3.30)$$

where g^{-1} is the inverse of g_{ab} , and we have suppressed the indices for simplicity. If we multiply (3.30) by g^{-1} , take the trace and use Einstein equations (3.25), we get

$$-\frac{1}{2} \partial_u \text{Tr}(g^{-1} \partial_u g) - \frac{1}{4} (\text{Tr}(g^{-1} \partial_u g))^2 = 2\Lambda \frac{d+1}{d}, \quad (3.31)$$

which is a first order equation for the combination $\text{Tr}(g^{-1} \partial_u g)$. Assuming a negative cosmological constant parametrized as follows

$$\ell^2 \equiv -2\Lambda \frac{d+1}{d}, \quad (3.32)$$

the solution to (3.31) is simply

$$\text{Tr} (g^{-1} \partial_u g) = 2 \ell \coth \ell(u - u_0), \quad (3.33)$$

where u_0 is an integration constant. Since $\text{Tr}(g^{-1}\partial_u g) = \partial_u(\log \det g)$ identically, we can solve for $\det g$ which gives

$$\det g = C \sinh^2 \ell(u - u_0) \quad (3.34)$$

with some integration constant C . Now, plugging the metric ansatz (3.26) together with (3.27) into Einstein equations (3.25) and using (3.34) gives

$$\frac{1}{2}\partial_u^2 \psi_a + \frac{\ell}{2} \tanh \ell(u - u_0) \partial_u \psi_a = \frac{\ell^2}{d+1}, \quad (3.35)$$

which can be readily solved giving

$$\psi_a = A_a + B_a \log \left(\tanh \frac{\ell}{2}(u - u_0) \right) + \frac{2}{d+1} \log(\sinh \ell(u - u_0)) \quad (3.36)$$

where A_a and B_a are some integration constants. The constants A_a do not obey any constraint and B_a must satisfy

$$\sum_a B_a = 0. \quad (3.37)$$

After analyzing the remaining uu component of the Einstein equations (3.25) with (3.29) given by

$$R_{uu} = \sum_a \left(-\frac{1}{2} \partial_u^2 \psi_a - \frac{1}{4} \partial_u \psi_a \partial_u \psi_a \right), \quad (3.38)$$

we get one more constraint

$$\sum_a B_a^2 = 4 \frac{d}{d+1}. \quad (3.39)$$

Let us count the amount of integration constants. We have the freedom to choose the values of $u = u_1$ and $u = u_2$ where the IR and UV brane live⁴. The total number of variables is therefore $d+1$ from A_a , $(d-1)$ from B_a , plus u_0, u_1, u_2 . However, shifting u_0, u_1, u_2 simultaneously by a constant does not change the solution, so there are only two independent variables among u_0, u_1, u_2 . The total number of free variables is therefore $2(d+1)$, which is precisely the right number to determine a double-Dirichlet solution.

Now, the solution is completely determined after imposing the Dirichlet boundary conditions on the IR and UV branes located in u_1 and u_2 respectively. We can parametrize their metrics simply by $\psi_a^1 = \psi_a(u_1)$ and $\psi_a^2 = \psi_a(u_2)$. We expect the effective action to only depend on the ratio of the IR and UV metric, as that is what appears in the matrix M we used above. Therefore the interesting quantities

⁴This is reminiscent of the situation encountered when calculating transition amplitudes in quantum gravity [146].

are the combinations $e^{\psi_a^2 - \psi_a^1}$ which form the eigenvalues of M . Indeed, the shift variables A_a do not appear in the solutions in a very profound way and do not affect the ratio of the IR and UV metric. In other words, they effectively decouple, as expected. Therefore we are left with the following system of equations where we have set $u_0 = 0$ for simplicity

$$\begin{aligned}\psi_a^2 - \psi_a^1 &= \frac{2}{d+1} \log \left(\frac{\sinh \ell u_2}{\sinh \ell u_1} \right) + B_a \log \left(\frac{\tanh \frac{\ell}{2} u_2}{\tanh \frac{\ell}{2} u_1} \right), \\ \sum_a B_a &= 0, \\ \sum_a B_a^2 &= 4 \frac{d}{d+1},\end{aligned}\tag{3.40}$$

which we need to solve. We can easily solve the first equation for B_a and are then left with two equations for u_1 and u_2 which are not particularly easy to solve.

The on-shell action. We are now ready to evaluate the on-shell action. The Hilbert-Einstein contribution (3.24) because the Ricci scalar is constant⁵ reads

$$S = V \frac{4\Lambda}{d} \int_{u_1}^{u_2} du \exp \left(\sum_a \psi_a(u)/2 \right),\tag{3.41}$$

where V is the volume in the $(t, \vec{y})$ directions. Such action can be easily evaluated using our solution (3.36) and takes the final form

$$S = -\frac{2V\ell}{d+1} e^{\sum_a A_a/2} (\cosh \ell u_2 - \cosh \ell u_1).\tag{3.42}$$

Besides the Hilbert-Einstein term, in order for the variational principle to be well defined on the two boundaries, one needs to include the usual Gibbons-Hawking term (2.62) evaluated on both the UV as well as the IR boundary. Where γ is the determinant of the metric on the selected slice and K is its extrinsic trace. Such contribution is of the form

$$S_{GH} = - \int d^{d+1}x \sqrt{-g} \text{Tr} (g^{-1} \partial_u g),\tag{3.43}$$

and on *e.g.* u_2 it evaluates to

$$S_{GH} = -2V\ell e^{\sum_a A_a/2} \cosh \ell u_2.\tag{3.44}$$

⁵This is certainly true when both the constraints and the dynamical components of the Einstein equations are imposed. Here, we do not want to impose the constraints associated with the choice of the shift vector A_a in (3.23). However, in the radial gauge, the relevant off-diagonal contributions from the equations of motion to the Ricci scalar vanish, as the inverse metric is diagonal. This is the reason why also in our setup the Ricci scalar is constant.

The final result, combining all three contributions, thus reads

$$S_{\text{total}} = -\frac{2V(d+2)\ell}{d+1} e^{\sum_a A_a/2} (\cosh \ell u_2 - \cosh \ell u_1). \quad (3.45)$$

Let us express the just found generic gravitational effective action in terms of the eigenvalues of the matrix M . From (3.40) we obtain that

$$\log \left(\frac{\sinh \ell u_2}{\sinh \ell u_1} \right) = \frac{1}{2} \sum_a (\psi_a^2 - \psi_a^1) = \frac{1}{2} \text{Tr}(\log M), \quad (3.46)$$

$$\begin{aligned} \left(\log \left(\frac{\tanh \frac{\ell}{2} u_1}{\tanh \frac{\ell}{2} u_2} \right) \right)^2 &= \frac{d+1}{4d} \left(\sum_a (\psi_a^2 - \psi_a^1)^2 - \frac{1}{d+1} \left(\sum_a (\psi_a^2 - \psi_a^1) \right)^2 \right) = \\ &= \frac{d+1}{4d} \left(\text{Tr}(\log M)^2 - \frac{1}{d+1} (\text{Tr}(\log M))^2 \right) \end{aligned} \quad (3.47)$$

and equations (3.45), (3.46) and (3.47) are the final set of equations we would like to solve. To compare with the effective action which we introduced in (3.20) with (3.22) we need to insert the value of $\sqrt{-\gamma}$ at the UV brane. We read off that

$$F[M] = -\frac{2(d+2)\ell}{d+1} \frac{\cosh \ell u_2 - \cosh \ell u_1}{\sinh \ell u_2}. \quad (3.48)$$

The eigenvalues of M are $\exp(\psi_a^2 - \psi_a^1)$, *i.e.* the eigenvalues of the UV metric times the ones of the inverse IR metric. We have therefore succeeded in writing $F[M]$ in terms of the eigenvalues of the matrix M : one first needs to solve for u_1 and u_2 in terms of the eigenvalues using equations (3.46) and (3.47) and substitute those in (3.48) to get the expression of $F[M]$ in terms of its eigenvalues. One conclusion we can already draw is that

$$F[M] \equiv F[\text{Tr}(\log M), \text{Tr}(\log M)^2] = F[t_1, t_2], \quad (3.49)$$

since those are the only combinations of eigenvalues that appear. Similar results appeared simultaneously in [147]. Hence, the double-Dirichlet effective action is a function with two arguments where the relation between $t_1 \equiv \text{Tr}(\log M)$, $t_2 \equiv \text{Tr}(\log M)^2$ and u_1, u_2 is given by (3.46) and (3.47). It appears difficult to obtain the solution for u_1 and u_2 in any compact form, so eq. (3.48) is as close as it gets to finding an explicit expression for the gravitational double-Dirichlet effective action.

3.3.2 Fluid effective action

Now that we have derived the double-Dirichlet effective action between two arbitrary branes in AdS, let us make contact with the fluid effective action as in (3.1).

We want to send the IR brane to the horizon where the metric degenerates. This can be achieved by requiring $e^{\psi_t^1} \rightarrow 0$. Inspecting (3.40), this limit in our setup translates to require $u_1 \rightarrow 0$ while keeping u_2 finite. At the same time, $t_1 \rightarrow +\infty$ and $t_2 \rightarrow +\infty$ by looking at (3.46) and (3.47). Hence in the near-horizon limit $u_1 \rightarrow 0$ we find that

$$\frac{t_1}{2} + \left[\frac{d+1}{4d} \left(t_2 - \frac{t_1^2}{d+1} \right) \right]^{\frac{1}{2}} = 2 \log \cosh \frac{\ell u_2}{2} \quad (3.50)$$

and the effective action (3.48) reduces to

$$F[t_1, t_2] = -\frac{2(d+2)\ell}{d+1} \tanh \frac{\ell u_2}{2}. \quad (3.51)$$

Taking the expressions (3.46) and (3.47) it is easy to see that for in the near-horizon limit where $\psi_t^1 \rightarrow -\infty$ we have

$$\frac{t_1}{2} + \left[\frac{d+1}{4d} \left(t_2 - \frac{t_1^2}{d+1} \right) \right]^{\frac{1}{2}} = \frac{d+1}{2d} \sum_i (\psi_i^2 - \psi_i^1) = \frac{d+1}{2d} \text{Tr}' \log M \quad (3.52)$$

where Tr' is defined as the trace with the degenerate eigenvalue ψ_t removed. We can now solve for u_2 combining (3.50) and (3.52) and plug it into the effective action (3.51). The result is

$$F[M] = -\frac{2(d+2)\ell}{d+1} \left[1 - \exp \left(-\frac{d+1}{2d} \text{Tr}' \log M \right) \right]^{\frac{1}{2}}, \quad (3.53)$$

which is a very concrete effective action. Notice that it only depends on $\text{Tr}' \log M = \log \det' M$ and, therefore, it is invariant under the volume-preserving diffeomorphisms.

Finally, let us take the limit where the UV brane is taken to the boundary of AdS. Then the exponential in equation (3.53) becomes very small and we can approximate

$$F[M] \approx -\frac{2(d+2)\ell}{d+1} \left[1 - \frac{1}{2} \exp \left(-\frac{d+1}{2d} \text{Tr}' \log M \right) \right]. \quad (3.54)$$

The first term is a constant, so it can be canceled by a local counterterm of the form (2.63). The effective action therefore becomes

$$F[M] = \frac{(d+2)\ell}{d+1} (\det' M)^{-\frac{d+1}{2d}}, \quad (3.55)$$

which is exactly the power that we need to describe a conformal fluid in $d+1$ spacetime dimensions as in (2.34). More explicitly, assume the IR metric to be of the form

$$G_{AB} d\phi^A d\phi^B = G_{tt} d\phi^t d\phi^t + G_{ij} d\phi^i d\phi^j. \quad (3.56)$$

In the near-horizon limit, with $G_{tt} \rightarrow 0$, one of the eigenvalues of M will blow up, or equivalently, one of the eigenvalues of M^{-1} will go to zero. It is then very easy to see that

$$\det 'M^{-1} = \det (\partial_a \phi^i \partial_b \phi^j \gamma^{ab} G_{jk}) = (s/s_0)^2 \quad (3.57)$$

and therefore our action (3.55) is indeed of the type (3.1) with the entropy density given by (3.2). Notice that under conformal rescalings of the UV metric $\gamma_{ab} \rightarrow \Omega \gamma_{ab}$ the determinant scales as $\sqrt{-\gamma} \rightarrow \Omega^{(d+1)/2} \sqrt{-\gamma}$ and $\det 'M \rightarrow \Omega^d \det 'M$. Hence, combining (3.55) together with the determinant as in (3.20), makes the effective action invariant under rescalings of the UV metric and finite. We have therefore found a direct derivation of the effective action for ideal conformal fluids from holography.

3.4 Linearized effective action from gravity

We will now move to the explicit construction of the linearized effective action for the (3+1)-dimensional conformal fluid⁶. We will focus our attention on the regime where deviations from equilibrium are not only long-wavelength, but also small in their amplitude. This will allow us to extend the analysis from the previous Section to higher orders in the low momentum/frequency expansion. The relevant gravity action is

$$S = \frac{1}{2k_5^2} \int du d^4x \sqrt{-g} (R - 2\Lambda) \quad (3.58)$$

and the corresponding equations of motion are

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \quad (3.59)$$

The black brane geometry dual to the plasma state of $\mathcal{N} = 4$ super Yang-Mills takes the following form⁷

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{(\pi T L)^2}{u} (-f(u) dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2 du^2}{4u^2 f(u)}, \quad (3.60)$$

⁶This specific fluid corresponds to the low energy regime of $\mathcal{N} = 4$ SYM in the large- N_c limit and at strong coupling

⁷We could have taken the equivalent form (2.65), but we find this one more convenient.

where u is the radial coordinate extending from $u = 0$ (UV boundary) to $u = 1$ (the event horizon), the emblackening factor reads $f(u) = 1 - u^2$, T is the Hawking temperature and L is the curvature radius of the vacuum AdS_5 . In this convention, $\Lambda = -\frac{6}{L^2}$.

In studying small perturbations $\delta h_{\mu\nu}(t, x, u)$ of the black brane background (3.60) it will be convenient to work in the Fourier space

$$\delta h_{\mu\nu}(t, x, u) = \int \frac{d\omega dk}{(2\pi)^2} \delta h_{\mu\nu}(\omega, k, u) e^{-i\omega t + ikx} \quad (3.61)$$

and to further define

$$H_{ab} := |g^{ac}| \delta h_{cb}; \quad \partial_u H_{uu} := \frac{4u\sqrt{f(u)}}{L^2} \delta h_{uu}; \quad H_{au} := 2\pi T |g^{ac}| \delta h_{cu}, \quad (3.62)$$

where g^{ab} is the inverse of the black brane metric (4.90) restricted to $a, b, \dots$ indices. For definiteness, we aligned the momentum along the x -direction. The perturbations (3.62) are classified according to their transformation properties with respect to residual rotations $\mathcal{O}(2)$ in the plane transverse to their momentum, see *e.g.* [135]. This gives rise to the scalar, vector and tensor channels, which, by construction, decouple from each other. Given that the tensor channel does not support the hydrodynamic (gapless) excitations, the corresponding modes are not going to contribute to the hydrodynamic effective action and we will neglect them. Hence, we are only going to consider the scalar and vector modes

$$\begin{aligned} \text{Scalar (sound channel): } & H_{tt}, \quad H_{xt}, \quad H_{ii}, \quad H_{aa}, \quad H_{tu}, \quad H_{xu}, \quad H_{uu}, \\ \text{Vector (shear channel): } & H_{\alpha t}, \quad H_{\alpha x}, \quad H_{\alpha u}, \quad \text{with } \alpha = y, z. \end{aligned} \quad (3.63)$$

For the future convenience, the formulas above utilized the following notation:

$$H_{ii} = H_{xx} + H_{yy} + H_{zz} \quad \text{and} \quad H_{aa} = H_{xx} - H_{yy} - H_{zz}. \quad (3.64)$$

Notice that our analysis here keeps arbitrary values of the lapse and shift variables, as opposed to the previous Section. This will allow us to be very explicit about the emergence of the Goldstone bosons on the gravity side.

3.4.1 Shear channel

The shear channel equations of motion are $E_{\alpha t}$ and $E_{\alpha x}$ ($\alpha = y, z$) and take the form

$$H''_{\alpha t} - \frac{1}{u} H'_{\alpha t} - \tilde{k}^2 \frac{1}{uf} H_{\alpha t} - \tilde{k} \tilde{\omega} \frac{1}{uf} H_{\alpha x} + i \tilde{\omega} H'_{\alpha u} - i \tilde{\omega} \frac{1}{u} H_{\alpha u} = 0, \quad (3.65)$$

$$H''_{\alpha x} - \frac{(1+u^2)}{uf} H'_{\alpha x} + \tilde{\omega}^2 \frac{1}{uf^2} H_{\alpha x} + \tilde{\omega} \tilde{k} \frac{1}{uf^2} H_{\alpha t} - i \tilde{k} H'_{\alpha u} + i \tilde{k} \frac{(1+u^2)}{uf} H_{\alpha u} = 0, \quad (3.66)$$

where we defined the dimensionless frequency and momentum

$$\tilde{\omega} = \frac{\omega}{2\pi T} \quad \text{and} \quad \tilde{k} = \frac{k}{2\pi T}. \quad (3.67)$$

The equations (3.65) and (3.66) need to be supplemented with the constraint $E_{\alpha u}$

$$\tilde{k} H'_{\alpha x} + \frac{\tilde{\omega}}{f} H'_{\alpha t} + i \frac{(\tilde{\omega}^2 - f \tilde{k}^2)}{f} H_{\alpha u} = 0. \quad (3.68)$$

Since we are interested in the low energy dynamics of the linearized perturbations, we will search for solutions in a perturbative derivative expansion. The fields $H_{\alpha u}$ naturally appear with a field theory derivative and they have to be retained at the same order as the other fields $H_{\alpha t}$ and $H_{\alpha x}$. We implement the gradient expansion by redefining $\omega \rightarrow \lambda \omega$, $k \rightarrow \lambda k$, rescaling the fields $H_{\alpha u} \rightarrow 1/\lambda H_{\alpha u}$ and searching for solutions in a power series of the bookkeeping parameter $\lambda \ll 1$

$$H_{ab} = H_{ab}^{(0)} + \lambda H_{ab}^{(1)} + \lambda^2 H_{ab}^{(2)} + \dots \quad (3.69)$$

At this point, one usually fixes a gauge (typically, the radial gauge $H_{\mu u} = 0$) and solves the full set of equations (3.65-3.68) in the small- λ expansion. In each transversal direction α , the relevant equations are a set of two coupled second order ordinary differential equations and one first order equation. The total number of the integration constants per transverse direction is then three. They are usually fixed by setting two Dirichlet boundary conditions in the UV and imposing the ingoing boundary condition on the horizon. However, as in the previous Section, we want to solve here a double Dirichlet problem, namely we want to impose the Dirichlet boundary conditions not only in the UV but also on some IR brane. We are going then to solve (3.65) and (3.66) and leave the constraint (3.68) unsolved. At leading order in the hydrodynamic expansion (3.69) equations (3.65-3.66) are then

$$H_{\alpha t}^{(0)''} - \frac{1}{u} H_{\alpha t}^{(0)'} + i \tilde{\omega} H'_{\alpha u} - i \tilde{\omega} \frac{1}{u} H_{\alpha u} = 0, \quad (3.70)$$

$$H_{\alpha x}^{(0)''} - \frac{(1+u^2)}{u f} H_{\alpha x}^{(0)'} - i \tilde{k} H'_{\alpha u} + i \tilde{k} \frac{(1+u^2)}{u f} H_{\alpha u} = 0. \quad (3.71)$$

The solution with Dirichlet boundary conditions H_{ab}^B in the UV ($u = 0$) and Dirichlet boundary conditions H_{ab}^δ at some $u = u_\delta$ is not unique since it depends on the arbitrary gauge choice encoded in the fields $H_{\alpha u}$

$$H_{\alpha t}^{(0)}(u) = H_{\alpha t}^B - \frac{u^2}{u_\delta^2} \Delta H_{\alpha t} - i \tilde{\omega} \int_0^u H_{\alpha u}(w) dw, \quad (3.72)$$

$$H_{\alpha x}^{(0)}(u) = H_{\alpha x}^B - \frac{\log f}{\log f_\delta} \Delta H_{\alpha x} + i \tilde{k} \int_0^u H_{\alpha u}(w) dw. \quad (3.73)$$

In the formula above, $f_\delta = f(u_\delta)$ and we have also defined the following bulk diffeomorphisms invariant combinations

$$\begin{aligned}\Delta H_{\alpha t} &= H_{\alpha t}^B - H_{\alpha t}^\delta - i\tilde{\omega} \pi_\alpha, \\ \Delta H_{\alpha x} &= H_{\alpha x}^B - H_{\alpha x}^\delta + i\tilde{k} \pi_\alpha,\end{aligned}\tag{3.74}$$

with π_α defined as a following Wilson line-like object

$$\pi_\alpha = \int_0^{u_\delta} H_{\alpha u}(u) du.\tag{3.75}$$

The transverse Goldstones

The Wilson line-like objects defined in (3.75) are the (linearized) Goldstone bosons of certain spontaneously broken symmetries. In fact, one can easily see that the combinations (3.75) are invariant under those bulk diffeomorphisms, which involve diagonal combinations of the diffeomorphisms on the two boundaries, and transform nontrivially otherwise. The gauge symmetry of reparametrizing the two Dirichlet boundary conditions $\text{Diffs}_4 \times \text{Diffs}_4$ is broken down to the diagonal combination $\text{diag}(\text{Diffs}_4)$ by the classical solution (3.72-3.73) and the Goldstones (3.75) can be associated to the spontaneous breaking of the global symmetry subgroup

$$\text{Poincaré}_4 \times \text{Poincaré}_4 \rightarrow \text{diag}(\text{Transl}_4 + \text{Rot}_3).\tag{3.76}$$

In the formula above, the Lorentz group is broken completely as the two boundaries are characterized by different speed of light and only the diagonal combination of spacetime translations and rotations survive.

If we work instead in a specific gauge, e.g. the radial gauge, the Goldstones (3.75) arise as non-trivial boundary conditions to be imposed on the second boundary. For instance we can perform a bulk diffeomorphism $x^\mu \rightarrow x^\mu + \xi^\mu$ in order to transform the metric (3.60) with its perturbations $H_{\mu\nu}$ to a form where the new metric perturbation satisfy the condition $\tilde{H}_{\mu U} = 0$ in the new bulk coordinates $y^\mu = (y^a, U)$. Such diffeomorphism, in the lowest order in the derivative expansion, is

$$\xi_\alpha^{(0)}(u) = \frac{1}{u} C_\alpha - \frac{1}{u} \int_0^u H_{\alpha u}(w) dw,\tag{3.77}$$

where $C_\alpha = C_\alpha(\tilde{\omega}, \tilde{k})$ does not depend on the radial direction and can be set to zero. The bulk metric perturbations change to

$$\tilde{H}_{\alpha t}^{(0)}(u) = H_{\alpha t}^{(0)}(u) + i\tilde{\omega} \int_0^u H_{\alpha u}(w) dw,\tag{3.78}$$

$$\tilde{H}_{\alpha x}^{(0)}(u) = H_{\alpha x}^{(0)}(u) - i\tilde{k} \int_0^u H_{\alpha u}(w) dw\tag{3.79}$$

and the boundary values transform accordingly

$$\tilde{H}_{t\alpha}^\delta = H_{t\alpha}^\delta + i\tilde{\omega}\pi_\alpha, \quad (3.80)$$

$$\tilde{H}_{x\alpha}^\delta = H_{x\alpha}^\delta - i\tilde{k}\pi_\alpha. \quad (3.81)$$

Notice that in the radial gauge the metric is of the form

$$ds^2 = dU^2 + 2A_a(y^a)dy^a dU + g_{ab}(y^a, U)dy^a dy^b \quad (3.82)$$

and the lines of constant y^a are spatial geodesics with affine parameter U . As described in previously in Section 3.3, the Goldstone bosons (3.75) correspond to a map $x^a(y^a, 0) \rightarrow x^a(y^a, u_\delta)$ from the conformal boundary to the IR brane following suitable spatial geodesics.

The transverse effective action

Now that we have the solution of the double Dirichlet problem, we are ready to compute the partially on-shell action between the IR and the UV brane. In order to make the variational problem well-defined we need to include the Gibbons-Hawking term on both of the boundaries (as in the previous Section) and a counterterm in the UV

$$S_\delta = S_{HE}|_0^{u_\delta} + S_{GH}|_{u_\delta} - S_{GH}|_{u=0} - S_{ct}|_{u=0}, \quad (3.83)$$

where S_{HE} is given in (3.58) and

$$S_{GH} = \frac{1}{k_5^2} \int d^4x \sqrt{-\gamma} K; \quad S_{ct} = \frac{L}{2k_5^2} \int d^4x \sqrt{-\gamma} \left(\frac{6}{L^2} + {}^4R \right), \quad (3.84)$$

In the formulas above, γ is the determinant of the induced metric on the timelike hypersurface, K is the trace of the extrinsic curvature tensor and 4R is the Ricci scalar on the (3+1)-dimensional timelike hypersurface, which will only contribute in the second order of the derivative expansion. We will now set the action (3.83) *partially* on-shell by using the solutions (3.72-3.73). The background contribution takes the form

$$S_{\text{const}} = P_0 V_4 \left(3 - \frac{6}{u_\delta^2} \right) \quad \text{with} \quad P_0 = \frac{\pi^4 T^4 L^3}{8k_5^2}, \quad (3.85)$$

where V_4 is the four-dimensional volume term and P_0 is the thermodynamic pressure. The contribution of the perturbation is given by

$$\begin{aligned} S_T = & -P_0 V_2 \int \frac{dk d\omega}{(2\pi)^2} \sum_\alpha \left(\frac{3}{2} (H_{\alpha t}^B)^2 + \frac{1}{2} (H_{\alpha x}^B)^2 + \right. \\ & \left. + \frac{3}{u_\delta^2} (H_{\alpha t}^\delta)^2 - \frac{(2+f_\delta)}{u_\delta^2} (H_{\alpha x}^\delta)^2 - \frac{2}{u_\delta^2} (\Delta H_{\alpha t})^2 - \frac{2}{\log f_\delta} (\Delta H_{\alpha x})^2 \right), \end{aligned} \quad (3.86)$$

where V_2 is the two-dimensional transverse volume term and we have omitted the arguments of the fields for which we use the convention

$$A B = \frac{1}{2} (A(\tilde{\omega}, \tilde{k}) B(-\tilde{\omega}, -\tilde{k}) + A(-\tilde{\omega}, -\tilde{k}) B(\tilde{\omega}, \tilde{k})). \quad (3.87)$$

The equations of motion for the Goldstone fields, as derived from the effective action (3.86), correspond to the constraint equations (3.68) and represent conservation of the energy-momentum tensor in the dual field theory. Imposing vanishing double Dirichlet boundary conditions, the effective action will depend only on the Goldstone degrees of freedom

$$S_{\pi^T} = P_0 V_2 \int \frac{dk d\omega}{(2\pi)^2} \sum_{\alpha} 2 \left(\frac{\tilde{\omega}^2}{u_{\delta}^2} + \frac{\tilde{k}^2}{\log f_{\delta}} \right) \pi_{\alpha}^2. \quad (3.88)$$

The linear dispersion relation is immediately derived

$$\tilde{\omega}_T = \pm c_T \tilde{k}, \quad \text{with} \quad c_T = \frac{u_{\delta}}{\sqrt{-\log f_{\delta}}}, \quad (3.89)$$

and depends on the position u_{δ} of the IR brane suggesting that on a finite cutoff u_{δ} the volume preserving diffeomorphisms is broken. This is very much in line with the analysis presented in the previous Section.

In the near horizon limit $u_{\delta} \rightarrow 1$ the background on-shell action

$$S_{\text{const}}|_{\mathcal{H}} = -3 P_0 V_4, \quad (3.90)$$

represents the energy density times the four-volume of a holographic conformal fluid. The transverse effective action

$$\begin{aligned} S_T|_{\mathcal{H}} = P_0 V_2 \int \frac{dk d\omega}{(2\pi)^2} \sum_{\alpha} & \left(\frac{1}{2} (H_{\alpha t}^B)^2 - \frac{1}{2} (H_{\alpha x}^B)^2 - 4 H_{\alpha t}^B H_{\alpha t}^{\delta} + 2 (H_{\alpha x}^{\delta})^2 + \right. \\ & \left. - (H_{\alpha t}^{\delta})^2 + 2 i \tilde{\omega} ((H_{\alpha t}^B - H_{\alpha t}^{\delta}) \pi_{\alpha} - \pi_{\alpha} (H_{\alpha t}^B - H_{\alpha t}^{\delta})) + 2 \tilde{\omega}^2 \pi_{\alpha}^2 \right) \end{aligned} \quad (3.91)$$

turns out to be equivalent to the Fourier transform of the transverse sector in Eq. (3.14) derived in Section 3.2 when the boundary metric expansion is included. In order to demonstrate it, one needs to redefine $\pi_{\alpha} \rightarrow -\pi_{\alpha}$, impose the conformal fluid equation of state $F(s) = -s^{4/3}$ and set s_0 to $s_0^{4/3} \equiv 3 P_0$. Furthermore, one also needs to add the contribution $+3 (H_{\alpha t}^{\delta})^2$ coming from the difference between the near horizon form of the metric (3.60) with linear perturbations and the Galilean form of the horizon metric (3.6), where in the tt -component the first nontrivial term is second order in an amplitude expansion. Notice also that in the near horizon limit the transverse velocity $c_T \rightarrow 0$ and the trivial shear waves dispersion relation (2.32) is recovered.

At higher orders of the hydrodynamic expansion, the effective action contains divergent terms as the stretched horizon approaches the position of the event horizon. However, the resulting dispersion relation for the Goldstones is trivial

$$\tilde{\omega}_T = \mathcal{O}(1 - u_\delta) + \mathcal{O}(\tilde{k}^4), \quad (3.92)$$

and does not retain any of the aforementioned undesired features if one is careful in taking the near horizon limit at each order of the hydrodynamic expansion. The reason for it is simply that the two limits do not commute.

To recap, we demonstrated here that up to the second order of hydrodynamic gradient expansion, the shear mode does not propagate provided one ignores the dissipative effects. This hints towards the volume-preserving diffeomorphisms invariance being the symmetry of the effective action for holographic fluids at least up to the second order of the gradient expansion.

3.4.2 Sound channel

All the manipulations of the previous Section can be repeated pretty much straightforwardly also for the sound channel perturbations. The dynamical Einstein's equations ($E_{ab} = 0$) in the leading order of the gradient expansion take the form

$$\begin{aligned} H_{xt}^{(0)''} - \frac{1}{u} H_{xt}^{(0)'} - i\tilde{k} f H'_{tu} + i\tilde{k} \frac{(1+3u^2)}{u} H_{tu} + i\tilde{\omega} H'_{xu} - i\tilde{\omega} \frac{1}{u} H_{xu} &= 0, \\ H_{aa}^{(0)''} - \frac{(1+u^2)}{uf} H_{aa}^{(0)'} - 2i\tilde{k} H'_{xu} + 2i\tilde{k} \frac{(1+u^2)}{uf} H_{xu} &= 0, \\ H_{ii}^{(0)''} - \frac{1}{uf} H_{ii}^{(0)'} - 2i\tilde{k} H'_{xu} + 2i\tilde{k} \frac{1}{uf} H_{xu} + \frac{3}{2} \sqrt{f} H''_{uu} - \frac{3}{2} \frac{(1+2u^2)}{u\sqrt{f}} H'_{uu} &= 0, \\ H_{tt}^{(0)''} - \frac{(1+2u^2)}{uf} H_{tt}^{(0)'} - \frac{2}{3} H_{ii}^{(0)''} + \frac{2}{3} \frac{(1+u^2)}{uf} H_{ii}^{(0)'} - 2i\tilde{\omega} \frac{(1+2u^2)}{uf} H_{tu} + \\ + 2i\tilde{\omega} H'_{tu} + \frac{4}{3} i\tilde{k} H'_{xu} - i\tilde{k} \frac{4(1+u^2)}{3uf} H_{xu} + \\ - \frac{(3-u^2)}{2\sqrt{f}} H''_{uu} + (1+u^2) \frac{(3-2u^2)}{2uf^{3/2}} H'_{uu} &= 0, \end{aligned} \quad (3.93)$$

and the constraint equations ($E_{\mu u} = 0$) read

$$\begin{aligned} i\tilde{k} \left(H_{xt}^{(0)'} + \frac{2u}{f} H_{xt}^{(0)} \right) + i\tilde{\omega} \left(H_{ii}^{(0)'} + \frac{u}{f} H_{ii}^{(0)} + \frac{3}{2} \sqrt{f} H'_{uu} \right) + \tilde{k}^2 f H_{tu} + \tilde{k} \tilde{\omega} H_{xu} &= 0, \\ i\tilde{k} \left(H_{tt}^{(0)'} - \frac{u}{f} H_{tt}^{(0)} + \frac{2}{3} (H_{aa}^{(0)'} - H_{ii}^{(0)'}) - \frac{(3-u^2)}{2\sqrt{f}} H'_{uu} \right) + \frac{i\tilde{\omega} H'_{xt}}{f} - \tilde{k} \tilde{\omega} H_{tu} - \frac{\tilde{\omega}^2 H_{xu}}{f} &= 0, \\ H_{tt}^{(0)'} - \frac{(3-u^2)}{3f} H_{ii}^{(0)'} + 2i\tilde{\omega} H_{tu} + i\tilde{k} \frac{2(3-u^2)}{3f} H_{xu} - \frac{2}{\sqrt{f}} H'_{uu} &= 0. \end{aligned} \quad (3.94)$$

The solutions to eq. (3.93) with double-Dirichlet boundary conditions depend on, basically freely-specifiable, values of H_{tu} , H_{xu} and H_{uu}

$$\begin{aligned}
 H_{xt}^{(0)}(u) &= H_{xt}^B - \frac{u^2}{u_\delta^2} \Delta H_{tx} + i\tilde{k} f \int_0^u H_{tu}(w) dw - i\tilde{\omega} \int_0^u H_{xu}(w) dw, \\
 H_{ii}^{(0)}(u) &= H_{ii}^B - \frac{1 - \sqrt{f}}{1 - \sqrt{f_\delta}} \Delta H_{ii} + 2i\tilde{k} \int_0^u H_{tu}(w) dw - \frac{3}{2} f H_{uu}(u), \\
 H_{aa}^{(0)}(u) &= H_{aa}^B - \frac{\log f}{\log f_\delta} \Delta H_{aa} + 2i\tilde{k} \int_0^u H_{xu}(w) dw, \\
 H_{tt}^{(0)}(u) &= H_{tt}^B - \frac{\sqrt{f_\delta}(1 - \sqrt{f})}{\sqrt{f}(1 - \sqrt{f_\delta})} \Delta H_{tt} + \frac{1}{3} \frac{(1 - \sqrt{f})(\sqrt{f_\delta} - \sqrt{f})}{\sqrt{f}(1 - \sqrt{f_\delta})} \Delta H_{ii} + \\
 &\quad - 2i\tilde{\omega} \int_0^u H_{tu}(w) dw + \frac{1 + u^2}{2\sqrt{f}} H_{uu}(u),
 \end{aligned} \tag{3.95}$$

where we have defined the following bulk diffeomorphisms invariant combinations

$$\begin{aligned}
 \Delta H_{xt} &= H_{xt}^B - H_{xt}^\delta + i\tilde{k} f_\delta \pi_t - i\tilde{\omega} \pi_x, \\
 \Delta H_{ii} &= H_{ii}^B - H_{ii}^\delta + 2i\tilde{k} \pi_x - \frac{3}{2} \sqrt{f_\delta} H_{uu}(u_\delta), \\
 \Delta H_{aa} &= H_{aa}^B - H_{aa}^\delta + 2i\tilde{k} \pi_x, \\
 \Delta H_{tt} &= H_{tt}^B - H_{tt}^\delta - 2i\tilde{\omega} \pi_t + \frac{1 + u_\delta^2}{2\sqrt{f_\delta}} H_{uu}(u_\delta).
 \end{aligned} \tag{3.96}$$

In complete analogy with the previous Section, we also defined the following (linearized) Goldstones

$$\pi_t = \int_0^{u_\delta} H_{tu}(u) du \quad \text{and} \quad \pi_x = \int_0^{u_\delta} H_{xu}(u) du. \tag{3.97}$$

Notice that the contribution H_{uu} appears here with no derivatives. This metric component is in fact non-dynamical and it is associated to the parametrization of the position of the IR brane u_δ .

The longitudinal effective action

The on-shell action (3.83) up to second order in an amplitude expansion in the sound channel with vanishing double Dirichlet boundary conditions is

$$\begin{aligned}
 S_{\pi^L} &= P_0 V_2 \int \frac{dk d\omega}{(2\pi)^2} \left(\frac{f_\delta}{u_\delta^2} (2\tilde{k}^2 - 3\tilde{\omega}^2) \pi_t^2 - 2\tilde{\omega} \tilde{k} \frac{f_\delta}{u_\delta^2} \pi_t \pi_x + \right. \\
 &\quad \left. + \frac{(8\tilde{k}^2 u_\delta^2 - (\tilde{k}^2(1 + u_\delta^2) - 6\tilde{\omega}^2) \log f_\delta)}{3u_\delta^2 \log f_\delta} \pi_x^2 \right),
 \end{aligned} \tag{3.98}$$

where we followed the same convention as in Eq. (3.87). Notice that the contribution of H_{uu} was integrated out. If the IR brane is kept at an arbitrary radial position u_δ , both Goldstones π_t and π_x are dynamical with coupled equations of motion

$$\frac{f_\delta}{u_\delta^2} (2\tilde{k}^2 - 3\tilde{\omega}^2) \pi_t - \tilde{\omega} \tilde{k} \frac{f_\delta}{u_\delta^2} \pi_x = 0, \quad (3.99)$$

$$\tilde{\omega} \tilde{k} \frac{f_\delta}{u_\delta^2} \pi_t + \frac{(8\tilde{k}^2 u_\delta^2 - (\tilde{k}^2(1 + u_\delta^2) - 6\tilde{\omega}^2) \log f_\delta)}{3u_\delta^2 \log f_\delta} \pi_x = 0. \quad (3.100)$$

As previously, these equations correspond to the constraint equations, here the first two equations of (3.94), and hence follow from the conservation of the dual energy-momentum tensor. We can now solve Eq. (3.99) for the dispersion relations. We obtain two modes, which decouple in the vicinity of the event horizon and correspond *then* to the independent oscillations of π_t and π_x :

$$\pi_t : \quad \tilde{\omega} = \pm \sqrt{\frac{2}{3}} \tilde{k} + \mathcal{O}(\tilde{k}^3) \quad (3.101)$$

$$\pi_x : \quad \tilde{\omega}_L = \pm \frac{1}{\sqrt{3}} \tilde{k} + \mathcal{O}(1 - u_\delta) + \mathcal{O}(\tilde{k}^3). \quad (3.102)$$

Notice that the longitudinal Goldstone π_x has the standard dispersion relation for sound waves, see Eq. (2.35). The other mode is not present in relativistic hydrodynamics. In fact, it is easily seen from the effective action point of view (3.98) that in the near-horizon limit $u_\delta \rightarrow 1$ all π_t contributions vanish and only the longitudinal mode π_x survives. Hence, although the dispersion relation (3.101) is finite on the horizon, it is associated with unphysical mode and has to be discarded⁸.

Going to higher order in the hydrodynamic gradient expansion at the level of the effective action is technically quite demanding. Despite that, it is still possible to solve the double Dirichlet problem and investigate the constraint equations, which we did up to the second order in a derivative expansion. Proceeding in this way, we derived the correction to the dispersion relation for the longitudinal sector

$$\tilde{\omega}_L = \pm \frac{1}{\sqrt{3}} \tilde{k} \pm \left(\frac{2}{3\sqrt{3}} + \frac{\log(1 - u_\delta)}{18\sqrt{3}} - \frac{5 \log 2}{18\sqrt{3}} \right) \tilde{k}^3 + \mathcal{O}(1 - u_\delta) + \mathcal{O}(\tilde{k}^4). \quad (3.103)$$

Notice that although such dispersion relation is purely real and, hence, dissipationless, it diverges in the near-horizon limit. We expect the corresponding divergence to appear in the effective action, although we did not check this explicitly. It is

⁸Another way to see this is by looking at the residue of the resulting two-point function for the dual energy-momentum tensor. The residue related to the pole (3.101) vanishes in the near-horizon limit and as a result the corresponding mode disappears. We thank Dam T. Son for pointing this out.

hard to interpret this divergence unequivocally. Perhaps the most straightforward interpretation is that beyond the leading order in the gradient expansion keeping the vanishing Dirichlet boundary conditions on the event horizon is unphysical. A more speculative interpretation is that at the level of the holographic correspondence it is simply not possible to split the fluid into the dissipative and dissipationless part. Perhaps this is not such surprising since we are dealing with the low energy limit of $\mathcal{N} = 4$ SYM which is intrinsically dissipative, similar conclusions were given in [147]. We finish this Section by pointing out that the divergent contributions to (3.103) are intrinsically associated with the ω -dependence. Hence, it is natural to expect that in the Euclidean setting in thermal equilibrium such divergences are absent and that the action functional for fluids (also beyond the leading order in the gradient expansion) is well-defined.

3.5 Coupling to an IR sector

So far we dealt only with the part of the spacetime between some IR and UV branes, ultimately sending one of the cutoffs to the UV boundary and trying to send the other to the event horizon. However we never included the very important property of the horizon being a surface of no return, *i.e.* we never included the dynamical contributions of the part of the spacetime between the horizon and the IR brane. Having an intermediate cutoff u_δ naturally splits the spacetime into a UV and IR sector and, as a consequence, the bulk action also splits into two parts as in eq. (2.77), which explicitly is

$$S = S^{IR} + S^{UV} = \frac{1}{2k_5^2} \int_{u_\delta}^1 du d^4x \sqrt{-g} (R - 2\Lambda) + \frac{1}{2k_5^2} \int_0^{u_\delta} du d^4x \sqrt{-g} (R - 2\Lambda). \quad (3.104)$$

The (partially) on-shell UV part of the action computed in a derivative expansion is what acquires the interpretation of the effective action for dissipationless hydrodynamical excitations, at least at the leading order. In order to couple such action to the IR sector, one needs to integrate out the IR fields (2.78) on the finite cutoff $H_{\mu\nu}^\delta$

$$\frac{\delta S}{\delta H_{\mu\nu}^\delta} = \frac{\delta S^{IR}}{\delta H_{\mu\nu}^\delta} + \frac{\delta S^{UV}}{\delta H_{\mu\nu}^\delta} = 0. \quad (3.105)$$

Hence setting Dirichlet boundary on the IR brane has to be understood as a useful intermediate step.

In the remaining part of this Section we are going to focus on two different ways to couple the UV sector to the IR. First, we use a membrane paradigm approximation and derive the usual damped dispersion relation for the sound waves, without any trace of the divergence discussed in the previous Section (see Eq. (3.103)). Secondly, we focus on static configuration and employ the coupling to a regular Rindler-type dynamical sector, providing the first derivation of the hydrodynamic partition function from holography.

3.5.1 Membrane paradigm coupling

In order to recover dissipation, it is clearly necessary to include the horizon contribution and ultimately recovering the ingoing boundary condition. As already mentioned in Section 2.2.1 of the Introduction, instead of retaining the full dynamical IR sector we can equivalently use the membrane paradigm boundary condition on a finite cutoff u_δ (2.51) which here explicitly writes

$$2(1-u) \frac{Z'(u)}{i\tilde{\omega} Z(u)} \Big|_{u_\delta} = \sigma \quad \text{with} \quad \sigma = 1, \quad (3.106)$$

where Z is the relevant linearized gauge invariant gravitational perturbation. Here we will not question whether this approximation is valid in general, as far as hydrodynamic transport is concerned it seems to work very well. As we will see in Chapter 5, the membrane paradigm only fails to reproduce the gapped quasinormal modes which are not considered here. The nonlocal gauge-invariant combinations as defined in literature [135] are

$$\begin{aligned} Z_\alpha^T &= \tilde{k} H_{t\alpha} + \tilde{\omega} H_{x\alpha}, \\ Z^L &= 2\tilde{k}^2 f H_{tt} + 4\tilde{\omega} \tilde{k} H_{xt} + 2\tilde{\omega}^2 H_{xx} + H_{aa} (\tilde{k}^2(1+u^2) - \tilde{\omega}^2), \end{aligned} \quad (3.107)$$

respectively in the shear and sound channels. Keeping now the IR Dirichlet boundary conditions $H_{\mu\nu}^\delta$ non-vanishing, solving the constraint equations with respect to the Goldstones and using Eq. (3.106) gives the dispersion relations for the shear

and sound modes

$$\tilde{\omega}_T = -\frac{i}{2}\sigma \tilde{k}^2 - \frac{i}{8}\sigma (2 + (1 - \sigma^2) \log(1 - u_\delta) - (1 + \sigma^2) \log 2) \tilde{k}^4 + \mathcal{O}(1 - u_\delta),$$

$$\tilde{\omega}_L = \pm \sqrt{\frac{1}{3}} \tilde{k} - \frac{i}{3}\sigma \tilde{k}^2 + \tag{3.108}$$

$$\pm \left(\frac{1}{2\sqrt{3}} - \frac{\log 2}{3\sqrt{3}} + (1 - \sigma^2) \frac{(1 + \log 2 + \log(1 - u_\delta))}{6\sqrt{3}} \right) \tilde{k}^3 + \mathcal{O}(1 - u_\delta), \tag{3.109}$$

as functions of the membrane coupling σ . Notice that decoupling the membrane by setting the membrane coupling $\sigma = 0$ gives a dissipationless dispersion relation which, however, does not coincide with Eq. (3.103). There is a simple explanation to this. From Eq. (3.106) it follows that imposing $\sigma = 0$ corresponds to setting Neumann rather than Dirichlet boundary condition on the IR brane. This result demonstrates that also for a different set of boundary conditions we do get the divergent terms in the dispersion relation for sound waves. Several boundary conditions could in principle give different dissipationless effective actions and dispersion relations, as long as we make sure there is no net flux through the IR brane. The divergent logarithmic term is removed when the ingoing boundary conditions ($\sigma = 1$) are imposed, reproducing the correct dispersion relations as given in Section 2.2.2 in (2.74) and (2.75) with the values of the shear and bulk viscosity given in (2.69) and the relaxation time being (2.76). This complements our discussion from the previous Section on the division of holographic fluids into dissipative and non-dissipative contribution.

3.5.2 Euclidean IR sector coupling

The equilibrium partition function discussed in 2.1.3 can be computed holographically by evaluating the on-shell action on solutions to Einstein's equations in the Euclidean signature with arbitrary boundary metrics and a regular boundary condition at the tip of the cigar. Our setup can be viewed as an intermediate step to obtain the same result. In fact, this can be achieved by coupling the effective action in the static limit $\omega \rightarrow 0$ to an Euclidean IR sector which takes care of the near horizon (regular) region of the spacetime. Notice that in the conventional derivation of the thermodynamic partition function from gravity there is no Gibbons-Hawking term in the IR, while in the effective action formalism we had to retain such term in (3.83) since it was non-vanishing in the near horizon limit. It is then natural to expect that the IR sector is proportional to such a contribution and we will show in the following that this is, in fact, the case.

Consider a regular cigar-shaped geometry, which near the horizon of a black hole

looks like the tip of a cigar times the horizon geometry

$$ds^2 = \frac{\beta_{IR}^2 G_{tt}}{(2\pi)^2 r_0^2} \left(dr^2 + \frac{(2\pi)^2}{\beta_{IR}^2} r^2 (d\phi^t)^2 \right) + G_{ij} (d\phi^i - v^i d\phi^t)(d\phi^j - v^j d\phi^t), \quad (3.110)$$

where we assumed Euclidean time has periodicity β_{IR} . Setting $r = r_0$ we recover the Euclidean metric on the IR brane

$$ds^2 = G_{AB} d\phi^A d\phi^B = G_{tt} d\phi^t d\phi^t + G_{ij} (d\phi^i - v^i d\phi^t)(d\phi^j - v^j d\phi^t). \quad (3.111)$$

The geometry (3.110) does not solve Einstein equations, but since we are working at the leading order in derivatives this does not matter. Moreover, we will assume that r_0 is very small with $G_{tt} \sim r_0^2$. We denoted the inverse temperature by β_{IR} to emphasize that this is the temperature as seen by the IR metric, which is not necessarily the same as the temperature defined by the UV metric.

The on-shell value of the Euclidean action that covers the near horizon region $0 \leq r \leq r_0$ contains in principle two contributions

$$S^{IR} = S_{HE} \Big|_{r_0}^0 + S_{GH} \Big|_{r_0}. \quad (3.112)$$

The bulk Einstein-Hilbert action scales as $S_{HE} \sim \mathcal{O}(r_0)$ since the integration domain shrinks to zero. The Gibbons-Hawking term turns out to be independent of r_0 and equal to

$$S_{GH} = \int d^{d+1} \phi^A \frac{\sqrt{\det G_{ij}}}{\beta_{IR}}. \quad (3.113)$$

To proceed, we make a change of coordinates $(\phi^i - v^i \phi^t) \rightarrow \phi^i$ which we can always undo later. Since we are working at the lowest order in derivatives we can assume the v^i to be constant, and the change of coordinates therefore removes the $d\phi^t d\phi^i$ cross terms from the metric. We can then rewrite (3.113) as

$$S_{GH} = \int d^{d+1} \phi^A \frac{\sqrt{\det G_{AB}}}{\beta_{IR} \sqrt{G_{tt}}} = \int d^{d+1} x \frac{\sqrt{\det h}}{\beta_{IR} \sqrt{G_{tt}}}, \quad (3.114)$$

where h is defined in (3.21). If we denote

$$\Sigma^{AB} = \frac{\partial \phi^A}{\partial x^a} \frac{\partial \phi^B}{\partial x^b} \gamma^{ab},$$

then we can use the fact that G_{AB} is block diagonal to deduce the following identity

$$\det(\Sigma^{AB} G_{BC}) = \frac{\det(\Sigma^{ij} G_{jk}) \sqrt{G_{tt}}}{(\Sigma^{-1})_{tt}}. \quad (3.115)$$

If we insert this identity in (3.114) we obtain

$$S_{GH} = \int d^{d+1} x \sqrt{\gamma} \det \left(\frac{\partial \phi^i}{\partial x^a} \frac{\partial \phi^j}{\partial x^b} \gamma^{ab} G_{jk} \right)^{1/2} \frac{1}{\beta_{IR} \sigma}, \quad (3.116)$$

where

$$\sigma^2 = (\Sigma^{-1})_{tt} = \gamma_{ab} \frac{\partial x^a}{\partial \phi^t} \frac{\partial x^b}{\partial \phi^t}. \quad (3.117)$$

The quantity σ has a simple interpretation: it is the norm of the vector field $\frac{\partial}{\partial \phi^t}$ pulled back to the UV boundary. Therefore, $\beta_{IR} \sigma$ is the proper length of the Euclidean time circle as perceived on the UV boundary. We will therefore take

$$\beta_{UV} = \sigma \beta_{IR} \quad (3.118)$$

as our definition of the inverse UV temperature. With this definition, we now see that

$$S_{GH} = \int d^{d+1}x \sqrt{\gamma} \det \left(\frac{\partial \phi^i}{\partial x^a} \frac{\partial \phi^j}{\partial x^b} \gamma^{ab} G_{jk} \right)^{1/2} \frac{1}{\beta_{UV}} = \int d^{d+1}x \sqrt{\gamma} \frac{s}{\beta_{UV}}, \quad (3.119)$$

where in the last line we reinstate the v^i -dependence by undoing the coordinate transformation $(\phi^i - v^i \phi^t) \rightarrow \phi^i$ to recover precisely the entropy density as defined in Eq. (3.11). Hence, to summarize, we have just shown that the relevant contribution of the IR sector in the near horizon limit is given by the Gibbons-Hawking term (3.119). Most importantly, it is of the form $S^{IR} \sim Ts V_d$, where s is the entropy density, T is the temperature of the fluid and V_d is the spacetime volume.

Now, as promised, we couple the IR action (3.119) to the UV effective action derived in Section 3.4 in the static limit $\omega \rightarrow 0$. The coupling is realized by integrating out IR data as required in (3.105), which effectively sets the Goldstones on-shell. Notice also that since $S^{UV} \sim -\epsilon V_d$ where ϵ is the energy density, we are actually performing a Legendre transform of the energy density with respect to the entropy density which gives the pressure $P = Ts - \epsilon$ as a function of T . With arbitrary background metric configurations and using the notation of Section 3.4 the final result is

$$\begin{aligned} S = & P_0 V_4 + P_0 V_3 \int dx \left(\frac{3}{2} H_{tt}^B + \frac{1}{2} H_{ii}^B \right) + \\ & + P_0 V_3 \int dx \left(\frac{15}{8} (H_{tt}^B)^2 + \frac{1}{2} (H_{xt}^B)^2 - \frac{1}{8} (H_{xx}^B)^2 + \frac{3}{4} H_{tt}^B H_{ii}^B + \frac{1}{2} H_{xx}^B H_{yy}^B + \right. \\ & \left. + \frac{1}{2} \sum_{\alpha} ((H_{\alpha t}^B)^2 - (H_{\alpha x}^B)^2) \right). \end{aligned} \quad (3.120)$$

As expected such expression corresponds to the equilibrium partition function described in (2.37) where the background fluid metric in 3+1 dimensions is a linearized perturbation around the flat metric and only shear and sound channels are taken into account

$$ds^2 = -(1 - H_{tt}^B(x)) dt^2 + 2 H_{it}^B(x) dx^i dt + (\delta_{ij} + H_{ij}^B(x)) dx^i dx^j. \quad (3.121)$$

The pressure for a conformal fluid is $P(T) = cT^{d+1}$, the temperature is $T = T_0/\sqrt{1 - H_{tt}^B}$ and the constant c is fixed to match the equilibrium pressure in $3+1$ dimensions: $P_0 = cT_0^4$.

3.6 The entropy current as a Noether current

In this Section we want to explore the role that the conserved entropy current $J^a = su^a$ plays in our setup. It turns out that the entropy current is related to a symmetry as in [35, 34]. To describe this symmetry, we put $v^i = 0$ for simplicity and first define an IR stress tensor

$$T_{IR}^{AB} = -\frac{2}{\sqrt{-G}} \frac{\delta S}{\delta G_{AB}} \det \left(\frac{\partial x^a}{\partial \phi^A} \right). \quad (3.122)$$

The extra determinant has been put in because we want the IR stress tensor to be defined with respect to the measure $d^{d+1}\phi^A$ and not with respect to $d^{d+1}x$. Just as we do in fluids in Landau frame, we can look for a unit timelike eigenvector u_{IR}^M of T_{IR} which obeys

$$T_{IR}^{AB} (u_{IR})_B = -\rho_{IR} u_{IR}^A. \quad (3.123)$$

We can in principle find the eigenvalue ρ_{IR} using the explicit form of the near-horizon metric (3.110), and using the fact that the derivative of the effective action with respect to a boundary metric is proportional to the conjugate momentum, or radial derivative, of that metric; however, we do not need the explicit form of ρ_{IR} in our analysis below. We now claim that whenever

$$\phi^A \rightarrow \phi^A + \frac{u_{IR}^A}{\rho_{IR}} \quad (3.124)$$

is a symmetry of the action, the corresponding conserved current is precisely the entropy current.

To show this, we first observe that for our action, which was of the type

$$S = \int d^{d+1}x \sqrt{-\gamma} F[\gamma_{ab}, h_{ab}], \quad (3.125)$$

with h given in (3.21). The covariantly conserved Noether current for a transformation of the type (3.124) is

$$j^a = 2 \frac{\delta F}{\delta h_{ab}} G_{AB} \frac{\partial \phi^A}{\partial x^b} \frac{u_{IR}^B}{\rho_{IR}} = 2 \frac{\delta F}{\delta G_{AB}} \frac{\partial x^a}{\partial \phi^A} G_{BC} \frac{u_{IR}^C}{\rho_{IR}}. \quad (3.126)$$

Using the definition of the IR stress tensor in (3.122) and the eigenvalue equation (3.123) this becomes

$$j^a = \frac{\sqrt{-G}}{\sqrt{-\gamma}} \det \left(\frac{\partial \phi^A}{\partial x^a} \right) \frac{\partial x^a}{\partial \phi^B} u_{IR}^B. \quad (3.127)$$

We now perform a near-horizon limit specializing to the case where $u_{IR}^A = \delta^{At} / \sqrt{-G_{tt}}$ is a vector purely in the ϕ^t -direction. The conserved current is then

$$j^a = \frac{\sqrt{-G}}{\sqrt{-\gamma}} \det \left(\frac{\partial \phi^A}{\partial x^a} \right) \frac{\sigma}{\sqrt{-G_{tt}}} u_{UV}^a, \quad (3.128)$$

where we introduced the unit vector

$$u_{UV}^a = \frac{1}{\sigma} \frac{\partial x^a}{\partial \phi^t}, \quad (3.129)$$

which can be thought of as the suitably normalized pull back of the IR vector u_{IR} . If we look back at our analysis of the IR effective action, in particular at (3.114) and (3.119), we see that one way to write the entropy density s is as

$$s = \frac{\sqrt{-G}}{\sqrt{-\gamma}} \det \left(\frac{\partial \phi^A}{\partial x^a} \right) \frac{\sigma}{\sqrt{-G_{tt}}} \quad (3.130)$$

and therefore

$$j^a = s u_{UV}^a, \quad (3.131)$$

which is indeed the same as the entropy current.

Strictly speaking, we are not quite done at this point, because we should also show that u_{UV}^a is the fluid velocity. This can be demonstrated as follows. Because the function F in (3.125) must be a scalar and does not involve derivatives, it must be a function of traces of products of h_{ab} and γ^{ab} . This implies in particular that it obeys the equation

$$\frac{\partial F}{\partial \gamma_{ac}} \gamma_{cb} + \frac{\partial F}{\partial h_{ac}} h_{cb} = 0. \quad (3.132)$$

It is not difficult to see that this equation implies that if u_{IR}^A is an eigenvector of T_{IR}^{AB} , then

$$u_{UV}^a = \frac{\partial x^a}{\partial \phi^A} u_{IR}^A \quad (3.133)$$

is automatically an eigenvector of T_{UV}^{ab} , the stress tensor obtained by varying the action with respect to γ_{ab} . Therefore, the vector u_{UV}^a appearing in (3.131) is automatically an eigenvector of the UV stress tensor and therefore precisely equal to the fluid velocity in Landau frame. To summarize we have shown that the Noether current associated to the symmetry (3.124), with u_{IR}^A the unit eigenvector

of the IR stress tensor T_{IR}^{AB} defined in (3.122) with eigenvalue ρ_{IR} as defined in (3.123) is precisely the entropy current of the system.

It is interesting that our system appears to have two temperatures, two stress tensors, and two fluid velocities, defined with respect the IR and UV boundary respectively as in [35, 34]. This is perhaps an automatic consequence of our setup where the two boundaries appeared on equal footing. In the limit where the IR boundary becomes very close to the horizon of a black hole, the IR fluid physics becomes quite simple, as it is governed by the universal near-horizon Rindler region. These simple properties are then propagated to the UV boundary with the help of the Goldstone bosons. In particular, the entropy, which in the near-horizon region is very simple and proportional to the area of the horizon, becomes somewhat more involved when described in terms of the UV variables⁹. We have also explained how the entropy current can be associated to a symmetry which is purely based on the IR variables. This symmetry corresponds to some type of invariance of the IR dynamics as one flows along with the IR fluid velocity, with a suitable normalization. It would clearly be very interesting to explore these connections in more detail and extend them to the case where higher derivative corrections are included in the effective action. Finally, we note that the entropy current is conserved on-shell, but once we take the limit where the IR boundary coincides with the horizon the variable ϕ^t decouples from the theory and the entropy current (which remains finite in this limit) becomes conserved off-shell as well.

3.7 Discussion and Outlook

In this Chapter we used the ideas of the holographic Wilsonian effective action approach illustrated in Section 2.2.2 to compute the effective action for (conformal) perfect fluids from gravity. Our results can also be viewed as an alternative derivation of the fluid/gravity duality at least up to the leading order in a derivative expansion. In particular we showed how to identify the field theory Goldstones with geometric quantities in gravity and how the derived effective action exhibits the volume-preserving diffeomorphism invariance, even if we do not understand

⁹See [148, 149] for earlier constructions of hydrodynamic entropy currents from gravity. These results were directly motivated by area theorems.

exactly the geometric origin of this symmetry. At subleading orders in a derivative expansion we have not been able to decouple the dissipationless sector from the dissipative one because of certain logarithmic divergences in the dispersion relations. We interpret this issue as a pathology arising when taking (incorrectly) the Dirichlet boundary conditions all the way to the horizon. Moreover, our gravitational setup is specifically dual to $\mathcal{N} = 4$ super Yang Mills, which at low energies behaves as a dissipative plasma and it should not be that surprising that when attempting to decouple the dissipationless sector, something goes wrong. On the other hand, it is still intriguing to see whether there exists a holographic dual of a dissipationless fluid, possibly involving a geometry with no black hole event horizon.

The gravitational embedding of perfect fluid dynamics is a natural simple starting point for the derivation of a complete effective action for fluids featuring dissipative effects, and in this Chapter we have seen how to incorporate those effects in our setup by means of a simple membrane coupling. The necessary step to go beyond this approximation scheme would be an extension of our work to two-sided eternal AdS black holes as in [55], which would naturally give rise to the doubled set of degrees of freedom one needs in the Schwinger-Keldysh formalism to describe real time, finite temperature phenomena. It would also be interesting to understand the relation between the above mentioned approach and the recent field theory developments of [35, 34], where yet another general master Lagrangian for fluid dynamics is proposed containing a double set of degrees of freedom. The authors of [35, 34] suggest in particular that an eventual effective action description for fluids should contain an additional $U(1)$ symmetry as well. The Noether current associated to this symmetry is related to what they call the the adiabaticity equation, an off-shell generalization of the on-shell entropy current conservation. We find it interesting to relate their results with our entropy current findings in Section 3.6.

In this Chapter we showed how to couple the derived effective action to different IR sectors. Besides the ones discussed here, there are other boundary conditions one often encounters, such as the near-horizon AdS₂ boundary conditions for near extremal black branes which feature prominently in various AdS/CMT applications, see *e.g.* [150]. These strongly coupled IR boundary conditions would, in combination with our effective action, lead to a gravitational version of semi-holography [151] which would be clearly interesting to explore further.

The double-Dirichlet problem can alternatively be interpreted as the transition amplitude of gravity in radial quantization from a Hartle-Hawking [146] point of view. It would be interesting to develop this picture in more detail, and also consider the analogue problem in de Sitter space, where it could shed further light on the relation between de Sitter correlation functions and Euclidean partition

functions. It is also tempting to use the effective action to find a description of spacetimes with a hole, making contact with the ideas developed in [152, 153]. Other interesting developments would be the study of terms higher order in the fields and/or derivatives in both gravity and in the effective actions; and possible generalizations to other systems such as solids, superfluids, etc, see *e.g.* [29]. Moreover it would be interesting to understand the relation to bigravity theories which also rely on two metrics, see *e.g.* [154]. Also it would be interesting to understand the relation of our entropy current to Wald entropy [155]. Finally, hydrodynamic effective actions appeared recently in a model of dense nuclear matter [156] and it would be very interesting to pursue this connection further.

4

HOLOGRAPHIC FLUIDS ON FINITE CUTOFFS

ON HOLOGRAPHY IN THE INTERIOR OF SPACETIME AND FLUID BEHAVIOR
BEYOND ANTI-DE SITTER ASYMPTOTICS

Extensions of holography to non asymptotically AdS spacetimes are often hard to come by. Restricting to the hydrodynamic regime and considering a holographic-like dictionary on a timelike slice in the interior of the spacetime, rather than on the asymptotic boundary, allows us to extend at least the fluid/gravity duality approach on a finite cutoff and therefore to spacetimes which are not necessarily asymptotically AdS. In this Chapter, based on [2], we are going to show how this approach can be implemented in general, giving a prescription for the dual stress-energy tensor on a finite cutoff.

4.1 Introduction

In order to study the fluid/gravity duality on a slice in the interior of a generic spacetime we consider a $(d+2)$ -dimensional spacetime with a general bulk stress-energy with an event horizon. For example, for the case of a negative cosmological constant it could be a black hole and for a vanishing cosmological constant it could be Rindler spacetime. We assume that such a manifold admits a radial foliation with timelike $(d+1)$ -dimensional hypersurfaces Σ_c at constant $r = r_c$ as in Figure 4.1. We restrict to the IR part of the spacetime between the horizon and the finite cutoff and assume there exists a dual field theory on Σ_c in a holographic sense. In the low energy limit such field theory should behave as a fluid with an r_c -dependent background metric $\gamma_{ab}(r_c)$ which can be viewed as a running coupling

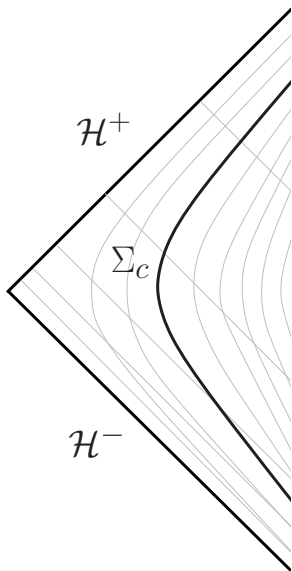


Figure 4.1: The spacetime has past and future horizons $\mathcal{H}^-$ and $\mathcal{H}^+$ respectively. The dual fluid lives on a timelike surface Σ_c . Lines of constant t and constant r in the Eddington-Finkelstein coordinate system are shown.

at an energy scale identified with r_c .

In order to claim there is a dual field theory on Σ_c one needs to be able to unambiguously define how to compute *e.g.* field theory quantities from the bulk point of view. In the low energy limit the quantity of interest is the conserved¹ stress-energy tensor that satisfies a thermodynamic relation with the temperature and entropy density. As many authors have pointed out, the Brown-York stress energy tensor defined in (2.65) is a natural candidate for the fluid energy momentum tensor on Σ_c as it is conserved. In Section 4.2 we give a precise derivation for such prescription highlighting the fact that, when the hypersurface is only conformally flat, as is indeed the case for AdS/CFT, the holographic stress tensor cannot be the Brown-York stress energy tensor but rather should be conformal to it.

In Section 4.3 we proceed by identifying the most general seed equilibrium metric solution with Dirichlet flat boundary conditions on a finite cutoff hypersurface Σ_c . We show how thermodynamic properties arise by reading off the expression for the dual stress tensor in a fluid form. In particular we argue how different choices of coordinate systems, although agreeing at thermodynamic equilibrium, give different results when performing a hydrodynamic expansion.

In section 4.4 we provide a general set up for hydrodynamics by promoting the

¹Strictly speaking the fluid stress energy tensor should satisfy an appropriate conservation equation but is not always conserved. The generalization to cases in which it is not conserved because of *e.g.* sources for currents is straightforward.

defining parameters of the equilibrium bulk solutions to be position dependent. Working in a relativistic gradient expansion, we work out the hydrodynamic equations of motion. We show that the conservation of the dual fluid is associated with the integrability of the bulk equations and we derive a general expression for the first order dissipative corrections to the fluid stress energy tensor.

As an example of our formalism we revisit the case of a finite cutoff in a planar AdS black hole spacetime in Section 4.5. We compute the first order hydrodynamic metric corresponding to a Dirichlet boundary condition on the hypersurface Σ_c . As this hypersurface is taken towards the conformal boundary we recover the usual fluid/gravity results summarized in Section 2.2.2. Given our solution with a Dirichlet boundary condition on Σ_c , we work out the asymptotic expansion of the metric in the neighborhood of the conformal boundary. This metric remains asymptotically locally AdS but the background metric for the dual CFT is no longer conformally flat. In fact having Dirichlet boundary conditions on Σ_c corresponds to have some mixed boundary conditions on the asymptotic infinity and the dual conformal field theory at infinity is deformed. We give the precise form for this background metric, up to first order in derivatives, thereby identifying the precise deformation of the dual CFT captured by the Dirichlet condition in the bulk. We observe in Section 4.5 and again in 4.6, that when the cutoff hypersurface is pushed towards the horizon it gives the so-called Rindler fluid dynamics, at least at first order, studied previously in the literature.

Finally in section 4.6 we consider various examples of bulk stress energy tensors, including cosmological constant and gauge fields. Using our examples, we observe that the existence of a flat timelike hypersurface is (as one would expect) highly non-trivial: in the absence of negative bulk curvature this requirement forces us into scaling regions of black holes with spherical horizon topologies.

4.2 Dual stress-energy tensor on a finite cutoff

We will now derive a prescription for the stress-energy tensor on a finite cutoff. The proof makes use of a Hamiltonian description of the bulk dynamics, following the same approach as in the Hamiltonian method of holographic renormalization [76]. One difference relative to the latter is that we will work with a finite cutoff

and we will not need to look in detail at the renormalization since on-shell actions are now finite. A second difference relative to Hamiltonian holographic renormalization is that we do not consider generic bulk solutions with given asymptotics; instead we restrict to the hydrodynamic regime in which the bulk solution is near to an equilibrium solution with horizon. We do not need to assume a negative cosmological constant or indeed any specific form for the bulk stress energy tensor.

General case

Let us consider a $(d + 2)$ -dimensional manifold which can be radially foliated by $(d + 1)$ -dimensional timelike hypersurfaces Σ_c of constant $r = r_c$. The metric can be parametrized according to the radial Arnowitt-Deser-Missner (ADM) decomposition [145]

$$ds^2 = (N^2 + N_a N^a) dr^2 + 2 N_a dx^a dr + \gamma_{ab} dx^a dx^b, \quad (4.1)$$

where N and N_a are the lapse and shift function and $x^a = (t, x^i)$ are the coordinates relative to the hypersurface Σ_c endowed with metric γ_{ab} . The action for Einstein gravity coupled to an arbitrary matter Lagrangian $\mathcal{L}_m$ which may or may not include a cosmological constant can be written as

$$S = -\frac{1}{2k_{(d+2)}^2} \int d^{d+2}x \sqrt{-\gamma} N \left({}^{d+1}R + K^2 - K_{ab} K^{ab} - \mathcal{L}_m \right), \quad (4.2)$$

where ${}^{d+1}R$ is the curvature of Σ_c , K_{ab} together with $K = K_{ab} \gamma^{ab}$ are the extrinsic curvature and its trace and γ is the determinant of γ_{ab} . We will set from now on $2k_{(d+2)}^2 = 1$ for simplicity. Note that in this rewriting of the bulk Einstein-Hilbert action one obtains a boundary term which precisely cancels the Gibbons-Hawking contribution (2.62), the variational problem for this action is then well defined for given boundary data γ_{ab} . The canonical momentum π^{ab} conjugate to γ_{ab} can be easily derived from the Lagrangian L defined via $S = \int dr L$

$$\pi^{ab} = \frac{\delta L}{\delta(\partial_r \gamma_{ab})} = -\frac{\sqrt{-\gamma}}{2} T^{\text{BY} ab}, \quad (4.3)$$

and it is proportional to the Brown-York stress tensor (2.65) defined on the hypersurface Σ_c . We rewrite it here for completeness

$$T_{ab}^{\text{BY}} = 2(K\gamma_{ab} - K_{ab}). \quad (4.4)$$

Let us now recall the Hamilton-Jacobi formalism of mechanics to express the momenta on any given hypersurface as variations of the on-shell action with respect

to the induced values of the fields on this surface, namely

$$\pi^{ab}(r_c) = \frac{\delta S_{\text{onshell}}}{\delta \gamma_{ab}(r_c)}. \quad (4.5)$$

Note that r_c is arbitrary, with the relation holding for any r_c provided that the radial coordinate is well-defined. The crucial step now is to assume the usual holographic dictionary of the form given in [71, 72], and explained in Section 2.2.2, to hold on a finite cutoff as well. Namely we identify the gravitational on-shell action S_{onshell} with the dual field theory generating functional $W[\gamma_{ab}(r_c)]$ similarly to (2.61) and interpret the background metric $\gamma_{ab}(r_c)$ as the source of the stress-energy tensor T_{ab} of the (putative) dual cutoff field theory which can be computed as

$$\langle T^{ab} \rangle = -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{onshell}}}{\delta \gamma_{ab}(r_c)} = T^{\text{BY} ab}, \quad (4.6)$$

where in the last equality we used the Hamilton-Jacobi relation (4.5) and the canonical momentum definition (4.3). From now on we will omit the brackets in the stress tensor.

Let us now discuss some subtleties associated with the prescription (4.6). Whilst the variational problem is well-defined for the action (4.2) given the metric on the bounding hypersurface, the variational problem would be equally well posed if one added to the action boundary terms $S_B[\gamma]$ depending only on quantities intrinsic to the induced geometry. The corresponding dual stress-energy tensor would then become

$$T^{ab} \rightarrow T^{ab} - \frac{2}{\sqrt{-\gamma}} \frac{\delta S_B[\gamma]}{\delta \gamma_{ab}}. \quad (4.7)$$

In usual AdS/CFT such boundary terms are uniquely defined when one takes the conformal class of the metric to be fixed as one takes the boundary to infinity. These terms are equivalent to the counterterms needed to obtain finite renormalized quantities, but at finite r_c there is no natural way to fix such ambiguity with a generic bulk solution. In addition to this, another source of ambiguity is associated to rescalings of the metric γ_{ab} on the hypersurface. If the boundary metric for the dual theory is redefined through a conformal factor Ω to $\tilde{\gamma}_{ab} = \Omega^2 \gamma_{ab}$ then the stress tensor associated to the conformal structure γ_{ab} is given by

$$T_{ab}^\gamma \equiv -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{onshell}}}{\delta \gamma^{ab}(r_c)} = \Omega^{d-1} T_{ab}^{\tilde{\gamma}}. \quad (4.8)$$

All this discussion points out that there is no unique way to holographically define a stress tensor associated to a general finite cutoff field theory. However, as we will show in the following section, if we restrict to hydrodynamic regime with the

induced metric being (conformally) flat, we will be able to partially remove the above mentioned ambiguities.

In the hydrodynamic regime

In the hydrodynamic regime the dual conserved stress-energy tensor at lowest order in a derivative expansion should acquire a perfect fluid form as in (2.4)

$$T_{ab}^F = P(\gamma_{ab} + u_a u_b) + \epsilon u_a u_b, \quad (4.9)$$

where P is the pressure, ϵ the energy density and u_a the fluid velocity. The Brown-York stress tensor (4.4) can be recasted in this form but as we just saw it is not uniquely defined. Any linear combination with covariant tensors built from the intrinsic metric γ_{ab} on the hypersurface Σ_c and its curvature as in (4.7) or any rescaling as in (4.8) would generate an equally conserved tensor.

When the induced metric is intrinsically flat $\gamma_{ab} = \eta_{ab}$ the ambiguities can be parametrized by two constants only

$$T_{ab}^F = C_1(r_c) T_{ab}^{\text{BY}} + C_2(r_c) \eta_{ab}. \quad (4.10)$$

As noticed in previous works [110, 111, 112], the parameter $C_2(r_c)$ causes only a shift in the pressure and energy density

$$P \rightarrow P + C_2(r_c), \quad \epsilon \rightarrow \epsilon - C_2(r_c), \quad (4.11)$$

which however does not affect the thermodynamic combination

$$(P + \epsilon) = sT + \dots, \quad (4.12)$$

where s is the entropy density, T the temperature and the ellipses denote additional contributions from charges etc.

To understand the role of the prefactor $C_1(r_c)$ one should consider the effect of a conformal transformation on the hypersurface metric $\tilde{\gamma}_{ab} = \Omega^2 \gamma_{ab}$, which is equivalent to a rescaling of the coordinates as $\tilde{x}^a = \Omega^{-1} x^a$. Under such a rescaling, it is easy to see that the thermodynamic variables become

$$\tilde{P} = \Omega^2 P; \quad \tilde{\epsilon} = \Omega^2 \epsilon; \quad \tilde{s} = \Omega^d s; \quad \tilde{T} = \Omega T. \quad (4.13)$$

As a consequence, if the thermodynamic relation (4.12) is satisfied when the induced metric on Σ_c is Minkowski then it is not satisfied for any non-trivial conformal factor Ω unless an appropriate prefactor $C_1(r_c)$ is included which scales homogeneously. Hence, in the case the induced metric is conformally flat $\gamma_{ab} = \Omega^2 \eta_{ab}$,

the fluid stress energy tensor T_{ab}^F should be related to the Brown-York stress energy tensor T_{ab}^{BY} defined on γ_{ab} by

$$T_{ab}^F = \Omega^{d-1} T_{ab}^{\text{BY}} + \dots . \quad (4.14)$$

This expression closely resembles the expression for the renormalized stress energy tensor in AdS/CFT given in (2.64). The choice of $C_1(r_c)$ defines in fact the holographic dictionary for the fluid. For $\Omega = 1$ the value of $C_1(r_c)$ which is consistent with the thermodynamic relation is precisely $C_1(r_c) = 1$.

Hydrodynamic dual stress-energy tensor on a finite cutoff:

It is proportional to the Brown-York stress-energy tensor associated to Σ_c . For the case of a (conformally) flat metric it is

$$T_{ab}^F = \Omega^{d-1} T_{ab}^{\text{BY}} + C_2 \gamma_{ab} \quad \text{with} \quad \gamma_{ab} = \Omega^2 \eta_{ab} \quad (4.15)$$

$$\text{where} \quad T_{ab}^{\text{BY}} = 2(K\gamma_{ab} - K_{ab}),$$

and C_2 is an ambiguity in the definition of the pressure and energy density which does not affect the thermodynamic relation or hydrodynamics.

4.3 Thermodynamic equilibrium

Let us here extract the general thermodynamic properties of a fluid defined on a finite cutoff timelike hypersurface Σ_c with flat Dirichlet boundary conditions.

4.3.1 The general seed metric ansatz

We begin by considering a generic static solution. Once a static solution is constructed one can always boost it to find a general stationary solution which corresponds to fluids with nonzero velocity. A convenient homogeneous and isotropic metric ansatz in Eddington-Finkelstein type coordinates [90, 91] is of the form

$$ds^2 = 2 dt dr - f(r) dt^2 + g(r) dx^i dx_i. \quad (4.16)$$

We assume the existence of an horizon at r_H such that $f(r_H) = 0$ and the above metric represents a general black brane solution at finite temperature. The reason we work in these coordinates rather than in Schwarzschild type ones as in (2.65) is because we want a manifestly regular metric on the horizon. On a hypersurface Σ_c of constant r_c the induced metric is

$$ds^2|_{\Sigma_c} = \gamma_{ab} dx^a dx^b = -f(r_c) dt^2 + g(r_c) dx_i dx^i, \quad (4.17)$$

which is assumed to be nondegenerate by requiring $f(r_c) \neq 0$, $g(r_c) \neq 0$. After rescaling the coordinates $t \rightarrow \sqrt{f(r_c)} t$ and $x^i \rightarrow \sqrt{g(r_c)} x^i$ the metric (4.16) can always be written in a way that the induced metric on Σ_c is in a manifestly flat form

$$ds^2 = \frac{2 dt dr}{\sqrt{f(r_c)}} - \frac{f(r)}{f(r_c)} dt^2 + \frac{g(r)}{g(r_c)} dx_i dx^i. \quad (4.18)$$

The induced flat metric on Σ_c is invariant under boost transformations. Hence we can obtain a more general (stationary) family of solutions by performing a Lorentz transformation on the spacetime coordinates

$$t \rightarrow \gamma t - \gamma \vec{v} \cdot \vec{x}; \quad \vec{x} \rightarrow \vec{x} - \gamma t \vec{v} + (\gamma - 1) \frac{\vec{x} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}. \quad (4.19)$$

Defining the four velocity $u^a = \gamma(1, v^i)$ and $\gamma = (1 - v^i v_i)^{-1/2}$ as usual, the resulting metric is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\frac{2}{\lambda} u_a dx^a dr + G(r) h_{ab} dx^a dx^b - F(r) u_a u_b dx^a dx^b, \quad (4.20)$$

where $h_{ab} = \eta_{ab} + u_a u_b$ and we have redefined the quantities

$$G(r) \equiv \frac{g(r)}{g(r_c)}; \quad F(r) \equiv \frac{f(r)}{f(r_c)}; \quad \lambda \equiv \sqrt{f(r_c)}, \quad (4.21)$$

such that $G(r_c) = F(r_c) = 1$. The inverse metric $g^{\mu\nu}$ is then given by

$$g^{rr} = \lambda^2 F(r); \quad g^{ra} = \lambda u^a; \quad g^{ab} = \frac{h^{ab}}{G(r)}. \quad (4.22)$$

The metric (4.20) is the most general (stationary) homogeneous and isotropic equilibrium metric ansatz with (flat) Dirichlet boundary conditions on a finite cutoff r_c that we are going to use throughout this Chapter. It represents a black brane solution with Hawking temperature

$$T = \lambda \frac{F'(r_H)}{4\pi}, \quad (4.23)$$

and entropy density at the horizon

$$s = 4\pi(G(r_H))^{d/2}. \quad (4.24)$$

The specific form of (4.20) depends on the bulk stress tensor $\mathcal{T}_{\mu\nu}$ through Einstein equations

$$G_{\mu\nu} = \mathcal{T}_{\mu\nu} \quad (4.25)$$

which are explicitly given by

$$G'' = \frac{1}{2} \frac{G'^2}{G} - \frac{2G}{d} \mathcal{T}_{rr}, \quad (4.26)$$

$$F'G' = -\frac{1}{2}(d-1) \frac{FG'^2}{G} + \frac{4GF}{d} \mathcal{T}_{rr} + \frac{4G}{d\lambda} \mathcal{T}_{ra} u^a, \quad (4.27)$$

$$F'' = \frac{2}{d\lambda^2 G} \mathcal{T}_{ab} h^{ab} + d(d-1) \frac{FG'^2}{4G^2} - 2 \frac{(d-1)}{d} F \mathcal{T}_{rr} - \frac{4(d-1)}{d\lambda} \mathcal{T}_{ra} u^a, \quad (4.28)$$

together with a constraint on the bulk stress tensor such that the metric takes the required stationary form

$$\mathcal{T}_{ar} F \lambda + \mathcal{T}_{ab} u^b = 0. \quad (4.29)$$

One can understand this latter constraint as follows. Taking the static limit of the metric (4.20), a bulk stress energy tensor compatible with the symmetries must be characterized by three scalar functions as

$$\begin{aligned} \mathcal{T}_{\mu\nu} dx^\mu dx^\nu &= \mathcal{T}^r dr^2 + \mathcal{T}^t dt \left(dt - \frac{2}{\lambda F} dr \right) + \mathcal{T}^i dx^i dx_i = \\ &= \left(\mathcal{T}^r - \frac{\mathcal{T}^t}{\lambda^2 F^2} \right) dr^2 + \mathcal{T}^t dt_s^2 + \mathcal{T}^i dx^i dx_i, \end{aligned} \quad (4.30)$$

where t_s is the Schwarzschild time, such that $dt_s = dt - dr/\lambda F$. Conservation of the bulk stress energy tensor implies that only two out of these three functions are independent. Under a boost the form of the stress energy tensor becomes

$$\mathcal{T}_{\mu\nu} dx^\mu dx^\nu = \mathcal{T}^r dr^2 + \frac{2}{\lambda F} \mathcal{T}^t u_a dr dx^a + (\mathcal{T}^t u_a u_b + \mathcal{T}^i h_{ab}) dx^a dx^b. \quad (4.31)$$

Thus we recover the constraint (4.29) given that

$$\mathcal{T}_{ab} = \mathcal{T}^i h_{ab} + \mathcal{T}^t u_a u_b; \quad \mathcal{T}_{ar} = \frac{\mathcal{T}^t}{\lambda F}. \quad (4.32)$$

4.3.2 Ideal fluid stress-energy tensor

Following the prescription (4.15) for a flat induced metric on Σ_c , in order to compute the dual fluid stress tensor it suffices to evaluate the Brown-York stress tensor on Σ_c . Such quantity can be recasted in a perfect fluid form

$$T_{ab}^F = P h_{ab} + \epsilon u_a u_b, \quad \text{with} \quad h_{ab} = \eta_{ab} + u_a u_b, \quad (4.33)$$

where the fluid velocity corresponds to the parameter u_a in (4.20) and the pressure P together with the energy density ϵ are characterized by the values of the gradients of the metric functions and λ

$$P = \lambda ((d-1)G'(r_c) + F'(r_c)); \quad (4.34)$$

$$\epsilon = -\lambda d G'(r_c). \quad (4.35)$$

There is an apparent redundancy in these expressions, as the two thermodynamic quantities are expressed in terms of three metric parameters. However, recall that λ characterizes the rescaling of the time coordinate on the hypersurface Σ_c , see eq. (4.21). By choosing the time coordinate to be adapted to this hypersurface one can always take $\lambda = 1$ but then the time Killing vector at asymptotic infinity will not have its usual normalization. Therefore λ measures the relative normalization of the Killing vector. Hence the fluid parameters (P, ϵ, u_a) are related to the metric parameters $(F'(r_c), G'(r_c), u_a)$ in (4.20). Depending on the bulk stress energy tensor and matter present, the fluid may have other parameters. For example, if there is a bulk gauge field then the ansatz for the gauge field consistent with stationarity would be of the form

$$A_\mu dx^\mu = (\mu(r) + (r - r_c)\rho(r)) u_a dx^a. \quad (4.36)$$

Then one would regard $\mu(r_c)$ as a boundary condition, characterizing the chemical potential in the field theory, and $\rho(r_c)$ as characterizing the charge density in the fluid.

Hamiltonian constraint. Not all the fluid parameters above are independent. They are related by the Hamiltonian constraint, namely a specific component of the Einstein equations which in radial slicing can be expressed as

$$K^2 - K_{ab}K^{ab} = {}^{d+1}R + 2\mathcal{T}_{\mu\nu}n^\mu n^\nu, \quad (4.37)$$

with n^μ the unit normal vector to Σ_c given by

$$n^\mu \partial_\mu = \lambda F \partial_r + u^a \partial_a; \quad n_\mu dx^\mu = \frac{1}{\lambda} dr. \quad (4.38)$$

Using the definition (4.4) it is easy to show that Eq. (4.37) can be written as a quadratic constraint of the Brown-York stress tensor

$$dT_{ab}^{\text{BY}} T^{\text{BY} ab} - (T^{\text{BY}})^2 = -8 d \lambda^{-2} \mathcal{T}^{rr}. \quad (4.39)$$

For the case at hand, where the induced metric on Σ_c is flat, the Brown-York stress tensor is directly related to the fluid one according to (4.15), hence using the ideal fluid expression (4.33) into the constraint (4.39) we have the relation

$$\left(1 - \frac{1}{d}\right) \epsilon = -P \pm \sqrt{P^2 - 8 \frac{(d-1)}{d \lambda^2} \mathcal{T}^{rr}}, \quad (4.40)$$

which effectively defines an equation of state for the dual fluid. Since the equation (4.39) is quadratic, one can always find two possible solutions for given data $\mathcal{T}^{rr}$ on Σ_c . For example, for a bulk stress tensor given by a negative cosmological constant only, the equation of state implies a relation between the pressure and energy density of the dual fluid. Now, given that a solution exists for a certain sign in (4.40) determining a specific pressure P and energy density ϵ , a corresponding solution for the equation of state with the opposite sign is obtained by replacing

$$\epsilon \rightarrow -\epsilon; \quad P \rightarrow -P. \quad (4.41)$$

Now the switch in signs in the energy density and pressure can be achieved by switching the direction of the normal to the hypersurfaces $n^\mu \rightarrow -n^\mu$, which corresponds to changing the sign of the extrinsic curvature. Physically, however, the opposite sign solution (4.41) gives a negative value for $(\epsilon + P)$ and therefore the thermodynamic relation $(\epsilon + P) = sT$ could only be satisfied by a negative temperature. Therefore the second equation of state never gives physically meaningful solutions.

Momentum constraint. The conservation of the Brown-York stress tensor is incorporated in the momentum constraint equations, another subset of Einstein equations which in radial slicing can be written as

$$\nabla_c K_a^c - \nabla_a K = -\frac{1}{2} \nabla_b T_a^{\text{BY} b} = \mathcal{T}_{a\mu} n^\mu, \quad (4.42)$$

where ∇_a is the covariant derivative in the induced geometry γ_{ab} . Requiring that the Brown-York stress-energy tensor is conserved on Σ_c implies a constraint on the bulk stress tensor $\mathcal{T}_{a\mu} n^\mu = 0$ which is indeed satisfied due to Eq. (4.29).

Clearly if the fluid parameters are constants, the fluid stress tensor (4.33) is identically conserved. Allowing them to vary with respect to field theory coordinates, conservation equations $\partial_a T^{F ab} = 0$ are

$$(P + \epsilon) \partial_a u^a = -D\epsilon; \quad (4.43)$$

$$(P + \epsilon) a_c = -D_c^\perp P, \quad (4.44)$$

where $D_a^\perp \equiv h_a^b \partial_b$, $D \equiv u^a \partial_a$ and the acceleration $a_c = Du_c$.

4.3.3 Other choices of coordinate systems

Let us now discuss some subtleties associated to different choices of coordinates. For example consider the original coordinates in (4.16) in which the induced metric on Σ_c (4.17) is only conformally flat

$$ds^2 \Big|_{\Sigma_c} = \Omega^2 (dx_i dx_i - c_l^2 dt^2) \quad \text{with} \quad \Omega = g(r_c)^{1/2}, \quad (4.45)$$

where the effective speed of light is $c_l^2 = f(r_c)/g(r_c)$. If one works with the original coordinates as in (4.16), rather than the rescaled ones as in (4.18), then the boost should preserve the induced metric (4.45). This implies that the boost must use c_l as the effective speed of light, resulting in

$$ds^2 = -2 \frac{U_a}{c_l} dx^a dr + g(r) (dx^i dx_i - c_l^2 dt^2) + (U_a dx^a)^2 \left(g(r) - \frac{f(r)}{c_l^2} \right) \quad (4.46)$$

with the new four velocity defined as

$$U_a = \gamma_l \left(c_l, \frac{v_i}{c_l} \right); \quad \gamma_l^2 = \left(1 - \frac{v_i v^i}{c^2} \right)^{-1}. \quad (4.47)$$

With this form of the metric the entropy density and temperature are rescaled with respect to (4.24) and (4.23) with appropriate factors of $\Omega = g(r_c)^{1/2}$ as anticipated in (4.13)

$$s = 4\pi (g(r_H))^{d/2} \equiv 4\pi (G(r_H))^{d/2} (g(r_c))^{d/2}; \quad (4.48)$$

$$T = \frac{1}{4\pi c_l} f'(r_H) \equiv \frac{1}{4\pi} F'(r_H) (g(r_c))^{1/2}. \quad (4.49)$$

The fluid tensor associated with a hypersurface Σ_c in the metric in (4.46) according to the prescription (4.15) is now

$$T_{ab}^F = g(r_c)^{d-1} T_{ab}^{\text{BY}} = P (\gamma_{ab} + g(r_c) U_a U_b) + \epsilon g(r_c) U_a U_b, \quad (4.50)$$

where the pressure and energy density are related to (4.34-4.35) by a rescaling as in (4.13) and a redefinition due to the prescription (4.15)

$$P^F = \lambda g(r_c)^{(d+1)/2} ((d-1)G'(r_c) + F'(r_c)); \quad (4.51)$$

$$\epsilon^F = -\lambda d g(r_c)^{(d+1)/2} G'(r_c). \quad (4.52)$$

In equilibrium the two forms of the metric (4.20) and (4.46) differ from each other by trivial rescalings of the coordinates and the choice of a coordinate system is

merely one of computational convenience. Once one goes beyond equilibrium into the hydrodynamic regime, however, the two forms of the metric are no longer equivalent and different choices really correspond to distinct boundary conditions on the hypersurface Σ_c . The reason is that the metric functions $f(r)$ and $g(r)$ depend on the thermodynamic quantities: the temperature, charge etc. In the hydrodynamic regime the latter are promoted to be spatially dependent, and therefore both the conformal factor and the effective speed of the light c_l in (4.46) become spatially dependent. Hence it is only sensible to keep the induced metric on Σ_c fixed as (4.45) when one extends to the hydrodynamic regime if the conformal factor $g(r_c)$ and the speed of light c_l^2 are independent of the fluid parameters. In fact if the conformal factor depends on the fluid parameters then implicitly the background metric on Σ_c is only flat to leading order in the hydrodynamic expansion. Moreover the fluid stress energy tensor defined in (4.15) would also only be conserved to leading order in gradients.

4.4 Near equilibrium hydrodynamic solutions

In this section we will promote the general seed equilibrium metric to allow for hydrodynamic configurations by allowing the fluid parameters to become slowly varying. We will show a general algorithm to compute such family of near equilibrated solutions in Einstein gravity with a flat Dirichlet boundary condition on Σ_c .

4.4.1 General hydrodynamic equations

In order to enter in the hydrodynamic regime let us first as promised promote the thermodynamic parameters in (4.20) to become slowly varying functions of the rescaled coordinates x^a which, contrary to the conventional fluid/gravity duality picture summarized in Section 2.2.2 of the Introduction, satisfy a Dirichlet boundary condition on Σ_c rather than on the asymptotic infinity. The seed metric becomes then

$$\begin{aligned} ds^2 &= g_{\mu\nu}^{(0)} dx^\mu dx^\nu = \\ &= -2\lambda(x)^{-1} u_a(x) dx^a dr + \left(G(r, x) h_{ab} - F(r, x) u_a(x) u_b(x) \right) dx^a dx^b, \end{aligned} \tag{4.53}$$

with $h_{ab} = \eta_{ab} + u_a u_b$ and

$$\partial_a u^b(x) \sim \mathcal{O}(\epsilon); \quad \partial_a \lambda \sim \mathcal{O}(\epsilon); \quad \partial_a F(r, x) \sim \mathcal{O}(\epsilon); \quad \partial_a G(r, x) \sim \mathcal{O}(\epsilon), \quad (4.54)$$

with ϵ small and the subscript $^{(0)}$ denoting leading order quantities in ϵ expansion. By construction this generalization preserves the induced metric on Σ_c and satisfies the Einstein plus matter equations to leading order in ϵ with the fluid parameters as given above. However, to any subleading order $n \geq 1$ one will need to correct the leading order metric contribution $g_{\mu\nu}^{(0)}$ given in (4.53) by terms $g_{\mu\nu}^{(n)}$

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} + \cdots + g_{\mu\nu}^{(n)} + \cdots \quad (4.55)$$

such that Einstein equations are still satisfied at order n . Supposing that one weights derivatives such that $\partial_r \sim 1$ and $\partial_a \sim \epsilon$, then each such metric correction $g_{\mu\nu}^{(n)}$ will contain n derivatives ∂_a of the fluid parameters. The new metric (4.55) describes then a near equilibrium gravitational solution which can be constructed order by order in a derivative expansion.

Let us describe the general algorithm to obtain (4.55). Suppose we have derived the necessary metric component $g_{\mu\nu}^{(n-1)}$ which assures Einstein equations are satisfied up to order $n-1$. Define $\hat{R}_{\mu\nu}^{(n)}$ as the part of the Ricci tensor containing partial derivatives ∂_a of $g_{\mu\nu}^{(n-1)}$ and $\delta R_{\mu\nu}^{(n)}$ as the part of the Ricci tensor related to radial derivatives ∂_r of the new, yet to be found, component $g_{\mu\nu}^{(n)}$. Einstein equations (4.25) at order n are

$$\hat{R}_{\mu\nu}^{(n)} + \delta R_{\mu\nu}^{(n)} - \frac{1}{2} \sum_{k=0}^n g_{\mu\nu}^{(k)} \hat{R}^{(n-k)} - \frac{1}{2} g_{\mu\nu}^{(0)} \delta R^{(n)} = \mathcal{T}_{\mu\nu}^{(n)}, \quad (4.56)$$

where the Ricci scalar tensors at each order are defined as

$$\hat{R}^{(n)} = \sum_{k=0}^n \hat{R}_{\rho\sigma}^{(k)} g^{(n-k)\rho\sigma}; \quad \delta R^{(n)} = \delta R_{\rho\sigma}^{(n)} g^{(0)\rho\sigma}, \quad (4.57)$$

and the inverse metric is defined in a way to assure that the trace of the metric is fixed to $\text{Tr}(g) = d+2$, for example at first order it is

$$g^{(1)\mu\nu} = -g^{(0)\mu\rho} g^{(0)\nu\sigma} g_{\rho\sigma}^{(1)}. \quad (4.58)$$

Equations (4.56) can be then rewritten in a more compact form

$$\hat{R}_{\mu\nu}^{(n)} + \delta R_{\mu\nu}^{(n)} = \bar{\mathcal{T}}_{\mu\nu}^{(n)}, \quad (4.59)$$

where the bulk stress tensor is redefined to

$$\bar{\mathcal{T}}_{\mu\nu}^{(n)} = \mathcal{T}_{\mu\nu}^{(n)} - \frac{1}{d} \sum_{k=0}^n g_{\mu\nu}^{(k)} \mathcal{T}^{(n-k)}; \quad \mathcal{T}^{(n)} = \sum_{k=0}^n \mathcal{T}_{\rho\sigma}^{(k)} g^{(n-k)\rho\sigma}. \quad (4.60)$$

Now, equations (4.59) can be solved by writing explicitly the dependence on the unknown metric component $g_{\mu\nu}^{(n)}$ contained in the linearized Ricci tensor

$$\delta R_{\mu\nu}^{(n)} = -\nabla_{\mu}^{(0)} \delta \Gamma_{\rho\nu}^{(n)\rho} + \nabla_{\rho}^{(0)} \delta \Gamma_{\mu\nu}^{(n)\rho}, \quad (4.61)$$

through the linearized Christoffel symbols

$$\delta \Gamma_{\mu\nu}^{(n)\rho} = \frac{1}{2} g^{(0)\rho\lambda} \left(\nabla_{\mu}^{(0)} g_{\lambda\nu}^{(n)} + \nabla_{\nu}^{(0)} g_{\lambda\mu}^{(n)} - \nabla_{\lambda}^{(0)} g_{\mu\nu}^{(n)} \right), \quad (4.62)$$

where $\nabla_{\mu}^{(0)}$ is the covariant derivative associated to the leading order metric (4.53). It is useful to restrict to radial gauge $g_{r\mu}^{(n)} = 0$ for $n \geq 1$ so that the lines at constant x^a are bulk radial null geodesics and the metric keeps the Eddington-Finkelstein form to all orders, which is useful in order to avoid coordinate singularities at the horizon. The generic metric correction $g_{ab}^{(n)}$ can then be decomposed on a basis of two linearly independent scalars, one vector and a traceless symmetric tensor

$$g_{ab}^{(n)} = \alpha^{(n)} u_a u_b + 2 u_{(a} \beta_{b)}^{(n)} + \tilde{\gamma}_{ab}^{(n)} + \frac{1}{d} \gamma^{(n)} h_{ab} \quad \text{for } n \geq 1, \quad (4.63)$$

with

$$u^a \beta_a^{(n)} = u^a \tilde{\gamma}_{ab}^{(n)} = \tilde{\gamma}_{ab}^{(n)} h^{ab} = 0. \quad (4.64)$$

Combining all the expressions given above for the general metric ansatz given in (4.53), the independent equations (4.59) are given by

$$\frac{1}{2} \partial_r^2 \left(\frac{1}{G} \gamma^{(n)} \right) + \frac{1}{2} \frac{G'}{G} \partial_r \left(\frac{1}{G} \gamma^{(n)} \right) + \bar{T}_{rr}^{(n)} = 0; \quad (4.65)$$

$$\partial_r^2 \beta_a^{(n)} + \frac{(d-2)}{2} \frac{G'}{G} \partial_r \beta_a^{(n)} - \left(\frac{(d-2)}{2} \frac{G'^2}{G^2} + \frac{G''}{G} \right) \beta_a^{(n)} - \frac{2}{\lambda} (\hat{R}_{rb}^{(n)} - \bar{T}_{rb}^{(n)}) h_a^b = 0; \quad (4.66)$$

$$\begin{aligned} & \frac{1}{G^{d/2}} \partial_r \left(G^{d/2} G' \alpha^{(n)} \right) - \frac{G'^2}{G} \alpha^{(n)} - \frac{1}{d G^{d/2}} \partial_r \left(G^{d/2} F \partial_r \gamma^{(n)} \right) + \\ & + \frac{2G'F}{dG} \partial_r \gamma^{(n)} - \frac{FG'^2}{dG^2} \gamma^{(n)} - \frac{FG'}{2} \partial_r \left(\frac{1}{G} \gamma^{(n)} \right) + \frac{2}{d\lambda^2} \left(\hat{R}_{ab}^{(n)} - \bar{T}_{ab}^{(n)} \right) h^{ab} = 0; \end{aligned} \quad (4.67)$$

$$\begin{aligned} & \frac{\lambda^2}{2} F \partial_r^2 \tilde{\gamma}_{ab}^{(n)} + \lambda^2 \left(\frac{1}{4} (d-4) \frac{FG'}{G} + \frac{1}{2} F' \right) \partial_r \tilde{\gamma}_{ab}^{(n)} + \frac{\lambda^2}{2} \frac{FG'^2}{G^2} \tilde{\gamma}_{ab}^{(n)} + \\ & - \hat{R}_{cd}^{(n)} h_a^c h_b^d + \frac{1}{d} \hat{R}_{cd}^{(n)} h^{cd} h_{ab} + \left(\bar{T}_{cd}^{(n)} h_a^c h_b^d - \frac{1}{d} \bar{T}_{cd}^{(n)} h^{cd} h_{ab} \right) = 0; \end{aligned} \quad (4.68)$$

which are respectively the (rr) equation, the h^{ab} projection of the (ra) equation, the h^{ab} trace of the (ab) equation and the projection $(h_a^c h_b^d - 1/d h^{cd} h_{ab})$ of the (ab) equation. Details of the derivation can be found in Appendix A.

Hence, given a bulk stress tensor $\bar{T}_{\mu\nu}^{(n)}$ and knowing the structure of $\hat{R}_{\mu\nu}^{(n)}$, which as we saw comes from derivatives of $g_{\mu\nu}^{(n-1)}$, equations (4.65-4.68) are ready to be

solved for the unknown variables $\alpha^{(n)}$, $\beta_a^{(n)}$, $\tilde{\gamma}_{ab}^{(n)}$ and $\gamma^{(n)}$. As we will see in the next section, in order to completely specify the solution additional constraints are needed. These will be provided by requiring the hydrodynamic fluid stress tensor to be in a specific frame of reference such as Landau frame.

4.4.2 The fluid stress tensor

The Brown-York stress-energy tensor at order n can be split in two contributions

$$T_{ab}^{\text{BY}(n)} = \hat{T}_{ab}^{\text{BY}(n)} + \delta T_{ab}^{\text{BY}(n)}, \quad (4.69)$$

where

$$\hat{T}_{ab}^{\text{BY}(n)} = 2 \left(\hat{K}^{(n)} \eta_{ab} - \hat{K}_{ab}^{(n)} \right); \quad \delta T_{ab}^{\text{BY}(n)} = 2 \left(\delta K_{ab}^{(n)} \eta_{ab} - \delta K_{ab}^{(n)} \right), \quad (4.70)$$

since the extrinsic curvature itself can be separated into two pieces: $\hat{K}_{ab}^{(n)}$ corresponding to contributions at order n arising from spacetime derivatives ∂_a of $g_{\mu\nu}^{(n-1)}$ and $\delta K_{ab}^{(n)}$ coming from radial derivatives of $g_{\mu\nu}^{(n)}$

$$K_{ab}^{(n)} = \frac{1}{2} (\mathcal{L}_n g_{ab})^{(n)} \Big|_{\Sigma_c} = \hat{K}_{ab}^{(n)} + \delta K_{ab}^{(n)}, \quad (4.71)$$

where $\mathcal{L}_n$ is the Lie derivative along the normal n^μ defined in (4.38) and the traces are defined as

$$\hat{K}^{(n)} = \hat{K}_{ab}^{(n)} \eta^{ab}; \quad \delta K^{(n)} = \delta K_{ab}^{(n)} \eta^{ab}. \quad (4.72)$$

The extrinsic curvature is given explicitly by

$$\hat{K}_{ab}^{(1)} = \sigma_{ab} + \frac{1}{d} \theta h_{ab} - u_{(a} a_{b)} - u_{(a} \partial_{b)} \ln \lambda, \quad (4.73)$$

$$\hat{K}_{ab}^{(n)} = \frac{1}{2} D g_{ab}^{(n-1)} \Big|_{\Sigma_c} \quad \text{with} \quad n > 1, \quad (4.74)$$

$$\begin{aligned} \delta K_{ab}^{(n)} &= \frac{1}{2} \lambda \partial_r g_{ab}^{(n)} \Big|_{\Sigma_c} = \frac{1}{2} \lambda \left(\alpha^{(n)'}(r_c) u_a u_b + \right. \\ &\quad \left. + 2 \beta_{(a}^{(n)'}(r_c) u_{b)} + \tilde{\gamma}_{ab}^{(n)'}(r_c) + \frac{1}{d} \gamma^{(n)'}(r_c) h_{ab} \right), \end{aligned} \quad (4.75)$$

where we have introduced a shorthand notation for the velocity derivatives

$$\theta = \partial_c u^c; \quad a_a = D u_a; \quad \mathcal{K}_{ab} = h_{(a}^c h_{b)}^d \partial_c u_d; \quad \sigma_{ab} = \mathcal{K}_{ab} - \frac{1}{d} \theta h_{ab}, \quad (4.76)$$

and $D = u^c \partial_c$. The above formulae can be used to compute *e.g.*

$$\begin{aligned} \hat{T}_{ab}^{\text{BY}(1)} &= -2\theta u_a u_b + 2h_{ab} \left(\frac{(d-1)}{d} \theta - D \ln \lambda \right) + \\ &\quad -2\sigma_{ab} + 2u_{(a} a_{b)} + 2u_{(a} D_{b)}^\perp \ln \lambda, \end{aligned} \quad (4.77)$$

$$\begin{aligned} \lambda^{-1} \delta T_{ab}^{\text{BY}(n)} &= \left(-u_a u_b \gamma^{(n)'}(r_c) - 2\beta_{(a}^{(n)'}(r_c) u_{b)} + \right. \\ &\quad \left. + h_{ab} \left(-\alpha^{(n)'}(r_c) + \frac{(d-1)}{d} \gamma^{(n)'}(r_c) \right) - \tilde{\gamma}_{ab}^{(n)'}(r_c) \right). \end{aligned} \quad (4.78)$$

We will work in Landau frame for the fluid stress tensor

$$T_{ab}^{F(n)} u^a = 0 \quad \text{for} \quad n \geq 1, \quad (4.79)$$

which, using (4.69) and the expression (4.78), gives two constraints order by order for $n \geq 1$ according to the two independent projections of (4.79)

$$\lambda \gamma^{(n)'}(r_c) - \hat{T}_{ab}^{\text{BY}(n)} u^a u^b = 0; \quad \lambda \beta_c^{(n)'}(r_c) + \hat{T}_{ab}^{\text{BY}(n)} u^a h_c^b = 0. \quad (4.80)$$

At first order using (4.83) we have for example the relations

$$\lambda \gamma^{(1)'}(r_c) = -2\theta, \quad (4.81)$$

$$\lambda \beta_a^{(1)'}(r_c) = a_a + D_a^\perp \ln \lambda, \quad (4.82)$$

giving the general first order correction to the Brown-York stress tensor in Landau frame

$$T_{ab}^{\text{BY}(1)} = -h_{ab} \left(2D \ln \lambda + \lambda \alpha^{(1)'}(r_c) \right) - 2\sigma_{ab} - \lambda \tilde{\gamma}_{ab}^{(1)'}(r_c). \quad (4.83)$$

Note that this result holds generally, regardless of the structure of the bulk stress energy tensor.

In total we have four classes of differential equations (4.65-4.68), three second order and one first order, together with two constraints (4.80). This set of equations allows us to find solutions for the four classes of unknown variables $\alpha^{(n)}$, $\beta_a^{(n)}$, $\tilde{\gamma}_{ab}^{(n)}$ and $\gamma^{(n)}$ order by order after imposing four Dirichlet boundary conditions on Σ_c , namely $\alpha^{(n)}(r_c) = \beta_a^{(n)}(r_c) = \tilde{\gamma}_{ab}^{(n)}(r_c) = \gamma(r_c)^{(n)} = 0$, and one regularity condition on the horizon.

By inspecting the Bianchi identities as it is shown in Appendix B, the conservation of the Brown-York stress tensor at order n is assured as long as the same constraint (4.29) is satisfied, *i.e.* $(\mathcal{T}_{\mu\nu} n^\nu)^{(n)} = 0$ at generic order n .

4.5 Negative cosmological constant background

After setting the stage in Sections 4.3 and 4.4, let us consider the simplest example, *i.e.* the familiar case in which the bulk stress-energy tensor is simply a negative cosmological constant

$$\mathcal{T}_{\mu\nu} = -\Lambda g_{\mu\nu} \quad \text{with} \quad \Lambda = -\frac{d(d+1)}{2L^2}. \quad (4.84)$$

4.5.1 Solutions at thermodynamic equilibrium

The leading order Einstein equation (4.26) for the variable $g(r)$ using the redefinitions (4.21) gives the general solution

$$g(r) = (c_1 r + c_2)^2. \quad (4.85)$$

Vanishing c_1 is generically only consistent with the other Einstein equations when the bulk stress-energy tensor is zero. Hence assuming $c_1 \neq 0$ one can absorb both constants into a redefinition of the origin and scale of the radial coordinate, $(c_1 r + c_2) \rightarrow r/L$. Integrating the remaining Einstein equations (4.27) one obtains the solution for $f(r)$

$$f(r) = \frac{r^2}{L^2} + \frac{c_3}{r^{d-1}}, \quad (4.86)$$

Let us in the following analyze the possible type of fluids that can be described within this class of asymptotically AdS solutions to Einstein gravity.

Vacuum AdS solution. This simple solution is achieved for $c_3 = 0$ in (4.86) which placed into (4.18) gives the vacuum AdS metric in Eddington-Finkelstein rescaled coordinates

$$ds^2 = -\frac{r^2}{r_c^2} dt^2 + \frac{r^2}{r_c^2} dx_i dx^i + 2\frac{L}{r_c} dt dr. \quad (4.87)$$

The energy density and the pressure can be computed from (4.34-4.35) with $G(r) = F(r) = r^2/r_c^2$ and $\lambda = r_c/L$ giving

$$\epsilon = -\frac{2d}{L}; \quad P = \frac{2d}{L}. \quad (4.88)$$

The combination $(\epsilon + P) = Ts$ is invariant and identically vanishing reflecting the fact that the AdS spacetime metric (4.87) has a trivial horizon geometry and thus one cannot associate to it a non-zero entropy. The arbitrariness in the definition of the fluid stress-energy tensor (4.15) can be used to redefine and set the pressure

and energy density to zero. This can be obtained by shifting the dual stress tensor by

$$T_{ab}^F \rightarrow T_{ab}^F - \frac{2d}{L} \eta_{ab}. \quad (4.89)$$

The AdS black brane solution. Let us now consider the case $c_3 \neq 0$. For $c_3 > 0$ the solution with (4.86) represents the well-known unphysical negative mass black brane, with naked singularity at $r = 0$, and we shall not consider it further here. Conversely setting $c_3 L^2 = -r_H^{d+1}$, the resulting geometry (4.18) is that of a (positive mass) AdS black brane in ingoing rescaled Eddington-Finkelstein coordinates

$$ds^2 = -\frac{r^2}{r_c^2} h(r) dt^2 + \frac{r^2}{r_c^2} dx^i dx_i + 2 \frac{L}{r_c \sqrt{h_c}} dt dr \quad \text{with} \quad h(r) = 1 - \left(\frac{r_H}{r} \right)^{d+1}, \quad (4.90)$$

and $h_c \equiv h(r_c)$. After boosting, the metric (4.90) can be brought into the form (4.20) with

$$G(r) = \frac{r^2}{r_c^2}; \quad F(r) = \frac{r^2}{r_c^2} \frac{h(r)}{h_c}; \quad \lambda = \frac{r_c}{L} \sqrt{h_c}, \quad (4.91)$$

and the thermodynamic properties are easily obtained from (4.34-4.35) to be

$$\epsilon = -\frac{2d}{L} \sqrt{h_c}, \quad (4.92)$$

$$P = \frac{1}{L \sqrt{h_c}} (2d h_c + r_c h'(r_c)). \quad (4.93)$$

Notice that the energy density (4.92) is negative, but as already mentioned one can use the ambiguity in the definition of the Brown-York stress-energy tensor (4.15) in order to shift it to a positive value and, as we have seen, the thermodynamic relation

$$(\epsilon + P) = \frac{d+1}{L \sqrt{h_c}} \frac{r_H^{d+1}}{r_c^{d+1}} \quad (4.94)$$

is independent of this ambiguity. The Hawking temperature (4.23) and the horizon entropy density (4.24) are in fact given by

$$T_H = \frac{(d+1)}{4\pi r_c L} \frac{r_H}{\sqrt{h_c}}; \quad s = 4\pi \frac{r_H^d}{r_c^d}, \quad (4.95)$$

and thus the thermodynamic relation (4.94) is indeed satisfied.

Conformal boundary fluid. The previous AdS black brane solution (4.90) in the limiting case for which the timelike hypersurface approaches the asymptotic infinity $r_c \rightarrow \infty$ should reproduce the well-known case of thermodynamics of a conformal fluid in asymptotically AdS black brane. However, according to conventional AdS/CFT, the metric at infinity is only conformally flat $\gamma_{ab} = r_c^2/L^2 \eta_{ab}$

and the dual field theory coordinates are defined with respect to η_{ab} . Hence, before taking the limit in the thermodynamic relations derived in the previous section one has to consider the rescaled definitions (4.51) and (4.52) instead with $g(r_c) = r_c^2/L^2$

$$\epsilon = - \left(\frac{r_c}{L} \right)^{d+1} \frac{2d}{L} \sqrt{h_c}, \quad (4.96)$$

$$P = \frac{1}{L \sqrt{h_c}} \left(\frac{r_c}{L} \right)^{d+1} (2d h_c + r_c h'(r_c)), \quad (4.97)$$

together with the entropy density and Hawking temperature as redefined in (4.48) and (4.49)

$$s = 4\pi \frac{r_H^d}{L^d}; \quad T_H = \frac{(d+1)}{4\pi L^2} \frac{r_H}{\sqrt{h_c}}. \quad (4.98)$$

This is yet not completely satisfying since even if the temperature and entropy density are finite in the limit $r_c \rightarrow \infty$, the pressure (4.97) and energy density (4.96) are clearly diverging. Again we can use the ambiguity in the definition of the dual stress tensor (4.15) to fix such pathology. The necessary term corresponds to the shift (4.89) that is needed to set the vacuum AdS metric (4.87) to have trivial thermodynamics, with η_{ab} replaced by $\gamma_{ab} = r_c^{d+1}/L^{d+1} \eta_{ab}$. In other words to get finite results for the AdS black brane solution with Dirichlet boundary conditions on the conformal structure at infinity we need to subtract the vacuum AdS background, which is equivalent to perform holographic renormalization with a counterterm of the form (2.63). After such redefinition of the pressure (4.97) and energy density (4.96) and sending $r_c \rightarrow \infty$ we get indeed the usual thermodynamic properties, see also (2.67), of the dual conformal fluid at infinity

$$\epsilon = \frac{d}{L} \left(\frac{r_H}{L} \right)^{d+1}; \quad P = \frac{1}{L} \left(\frac{r_H}{L} \right)^{d+1}; \quad T_H = \frac{(d+1)r_H}{4\pi L^2}; \quad s = 4\pi \frac{r_H^d}{L^d}. \quad (4.99)$$

Near-horizon fluid. Let us here consider the other limiting behavior, namely the case when the cutoff hypersurface Σ_c approaches the horizon $r_c \rightarrow r_H$ instead. In this case the energy density (4.92) vanishes and the pressure (4.93) diverges as

$$\epsilon = \mathcal{O}(r_c - r_H)^{\frac{1}{2}}, \quad (4.100)$$

$$P = \sqrt{(d+1)} \frac{\sqrt{r_H}}{L} \frac{1}{\sqrt{r_c - r_H}} + \mathcal{O}(r_c - r_H)^{\frac{1}{2}}, \quad (4.101)$$

mimicking the same behavior of the Rindler fluid thermodynamic parameters $\epsilon_R = 0$ and $P_R \sim 1/\sqrt{r_c - r_H}$ in proximity of the Rindler horizon, see Section 4.6.1. Such behavior was to be expected due to the universal form of the near-horizon region of a nonextremal black hole. The effective temperature (4.95) clearly diverges as Σ_c approaches the horizon but this divergence evidently arises

only from the time coordinate rescaling. In other words, in rescaling time so that the induced metric on Σ_c is flat, the effective pressure and temperature diverge as the hypersurface approaches the horizon. If one worked instead with the original coordinate t as in (4.17) the pressure would remain finite but the induced metric (the background metric for the fluid) would, as expected, become null in this limit.

4.5.2 AdS black brane hydrodynamics on Σ_c at first order

We will now consider the case of hydrodynamic perturbations on the AdS black brane background (4.90) in boosted and rescaled coordinates (4.53) with (4.91)

$$ds^2 = -2 \frac{L}{r_c \sqrt{h_c(r_H)}} u_a dx^a dr + \frac{r^2}{r_c^2} \left(h_{ab} - \frac{h(r, r_H)}{h_c(r_H)} u_a u_b \right) dx^a dx^b. \quad (4.102)$$

The equation of state (4.40) sets a relation between the pressure and energy density. As a matter of fact their explicit expressions (4.92) and (4.93) only depend on the horizon radius r_H through h_c . Hence we have $d+1$ independent fluid parameters: r_H which is related to the temperature T of the dual fluid through (4.95) and the fluid velocity u_a with the usual constraint $u_a u^a = -1$. Such parameters need to be promoted to be slowly varying functions $r_H(x)$ and $u_a(x)$ of the field theory coordinates x_a on Σ_c .

The general formulae developed in Section 4.4 comes now at hand. What we need to do is to compute the solutions to Eq. (4.65-4.68) at first order in a hydrodynamic expansion. The bulk stress tensor is given by $\tilde{T}_{\mu\nu}^{(1)} = \frac{2}{d} \Lambda g_{\mu\nu}^{(1)}$ and the Ricci tensor $\hat{R}_{\mu\nu}^{(1)}$ can be straightforwardly derived (see details in Appendix A) from the generic seed metric form (4.53). Given that in our case G is independent of the fluid parameters due to (4.91), we have $DG = D^\perp G = 0$ which is a considerable simplification. Moreover, anytime that a field theory derivative ∂_a hits the horizon radius parameter r_H through derivatives of F and λ , we can use the fluid stress tensor conservation equations at leading order (4.43) to trade $\partial_a r_H$ with derivatives of the fluid velocity, which is equivalent to take the yet to be derived solution $g_{ab}^{(1)}$ to be on-shell

$$\frac{\partial_a r_H}{r_H} = \left(\frac{1}{d} \theta u_a - \delta(r_c) a_a \right), \quad (4.103)$$

where

$$\delta(r_c) = \frac{2(1 - (r_H/r_c)^{d+1})}{2 + (d-1)(r_H/r_c)^{d+1}} = \frac{2h_c}{2h_c + (d+1)(1-h_c)}. \quad (4.104)$$

Hence solutions to Eq. (4.65-4.68) with Dirichlet boundary conditions $\gamma^{(1)}(r_c) = \alpha^{(1)}(r_c) = \beta_a^{(1)}(r_c) = \tilde{\gamma}_{ab}^{(1)}(r_c) = 0$, a regularity condition on the horizon and

Landau gauge conditions (4.81-4.82) are

$$\gamma^{(1)} = \frac{2L}{r_c \sqrt{h_c}} r \left(1 - \frac{r}{r_c} \right) \theta, \quad (4.105)$$

$$\alpha^{(1)} = \frac{L}{d h_c^{3/2}} \frac{r}{r_c} \left(-(d+1) \left(1 - \frac{r_c^d}{r^d} \right) + (d-1)h + \left(2 - (d+1) \frac{r_c^d}{r^d} \right) h_c \right) \theta, \quad (4.106)$$

$$\beta_a^{(1)} = \frac{L \delta(r_c)}{r_c^2 h_c^{3/2}} r \left(r \left(1 - \frac{r_c^{d+1}}{r^{d+1}} \right) - r_c \left(1 - \frac{r_c^d}{r^d} \right) h_c \right) a_a, \quad (4.107)$$

$$\tilde{\gamma}_{ab}^{(1)} = \frac{2L}{r_c r_H} \sqrt{h_c} r^2 \left(k(r) - k(r_c) + \frac{1}{(d+1)} \log h/h_c \right) \sigma_{ab}, \quad (4.108)$$

where we have defined

$$k(r) = \frac{r_H}{r} {}_2F_1 \left(1, \frac{1}{d+1}, 1 + \frac{1}{d+1}, \frac{r_H^{d+1}}{r^{d+1}} \right), \quad (4.109)$$

and ${}_2F_1$ is a Hypergeometric function.

The fluid stress-energy tensor at first order (4.83) can now be written in the familiar form as

$$T_{ab}^{F(1)} = -2\eta(r_c) \sigma_{ab} - \zeta(r_c) \theta h_{ab}, \quad (4.110)$$

where the shear and bulk viscosity are given respectively by

$$\eta(r_c) = \frac{r_H^d}{r_c^d}; \quad \zeta(r_c) = 0. \quad (4.111)$$

Hence, although the fluid is non conformal due to the non conformal equation of state (4.40) giving a non zero trace for the stress energy tensor, the bulk viscosity is vanishing at each radial slice Σ_c . Moreover with the entropy density (4.24), the universal shear viscosity over entropy ratio bound, as in (2.69), is recovered at each hypersurface Σ_c

$$\frac{\eta(r_c)}{s(r_c)} = \frac{1}{4\pi}, \quad (4.112)$$

confirming the results found previously for the non relativistic fluid dual to a finite cutoff hypersurface in AdS gravity [126] and the relativistic version of it [125], see also [157]. Alternative derivations using RG flows can be found in [109, 158, 159], see also [52] for a derivation using linear response theory.

4.5.3 Relation to the conformal fluid at infinity

The solution that we just found in the previous section gives a first order correction to AdS black brane background with a Dirichlet boundary condition on Σ_c . As

discussed for the thermodynamic properties around (4.99), in order to connect such solution to the one with Dirichlet boundary conditions imposed on the conformally flat metric at infinity as in standard AdS/CFT correspondence, it is first necessary to redefine the field theory coordinates $y^a = \Omega^{-1}x^a$ with $\Omega = r_c/L$. The leading order metric (4.90) in boosted coordinates now reads

$$ds^2 = -2 \frac{u_a}{\sqrt{h_c}} dy^a dr + \frac{r^2}{L^2} \left(h_{ab} - \frac{h}{h_c} u_a u_b \right) dy^a dy^b, \quad (4.113)$$

so that when pushing the cutoff hypersurface to the conformal boundary $r_c \rightarrow \infty$ we have $h_c \rightarrow 1$ and the metric is the usual AdS black brane metric, see *e.g.* [86]. The rescaling acts on the derivatives as well $\partial_a^y = \Omega \partial_a^x$ with u_a unchanged, hence we have

$$\theta^x = \frac{L}{r_c} \theta^y; \quad a_a^x = \frac{L}{r_c} a_a^y; \quad \sigma_{ab}^x = \frac{L}{r_c} \sigma_{ab}^y. \quad (4.114)$$

The first order metric perturbations as derived in (4.105-4.108), taking into account $dx^a dx^b \rightarrow r_c^2/L^2 dy^a dy^b$, are now rescaled to

$$\gamma^{y(1)} = \frac{r_c}{L} \gamma^{(1)}; \quad \alpha^{y(1)} = \frac{r_c}{L} \alpha^{(1)}; \quad \beta_a^{y(1)} = \frac{r_c}{L} \beta_a^{(1)}; \quad \tilde{\gamma}_{ab}^{y(1)} = \frac{r_c}{L} \tilde{\gamma}_{ab}^{(1)}, \quad (4.115)$$

and in the limit $r_c \rightarrow \infty$ they become

$$\gamma^{y(1)} \rightarrow 2r \theta^y, \quad (4.116)$$

$$\alpha^{y(1)} \rightarrow \frac{r}{d} (d-1)(h-1) \theta^y \quad (4.117)$$

$$\beta_a^{y(1)} \rightarrow -r a_a^y, \quad (4.118)$$

$$\tilde{\gamma}_{ab}^{y(1)} \rightarrow 2 \frac{r^2}{r_H} \left(k(r) - k(\infty) + \frac{\log h}{(d+1)} \right) \sigma_{ab}^y = 2 \frac{r^2}{r_H} H(r/r_H) \sigma_{ab}^y, \quad (4.119)$$

where asymptotically $H(x) \sim \frac{1}{x} - \frac{1}{(d+1)} \frac{1}{x^{d+1}}$. Now, we can always perform a diffeomorphism of the radial coordinate r such that

$$r \rightarrow r - \frac{1}{d} \theta^y, \quad (4.120)$$

and working to first order in gradients this results in a shift of the scalar quantities

$$\gamma^{y(1)} \rightarrow \gamma^{y(1)} - 2r \theta^y; \quad \alpha^{y(1)} \rightarrow \alpha^{y(1)} + \frac{r \theta^y}{d} (2 - (d-1)(h-1)). \quad (4.121)$$

Such transformation applied to the first order hydrodynamic solution (4.116-4.119) and using the definition (4.63) of the metric perturbation, results into the first order nonlinear correction of the AdS black brane metric

$$\begin{aligned} g_{\mu\nu}^{(1)} dy^\mu dy^\nu &= 2 \frac{r^2}{r_H} H(r/r_H) \sigma_{ab}^y dy^a dy^b + \frac{2}{d} r \theta^y u_a u_b dy^a dy^b + \\ &\quad - 2r a_{(a}^y u_{b)} dy^a dy^b, \end{aligned} \quad (4.122)$$

which exactly coincides with the one derived in [86].

The shear and bulk viscosity of the conformal fluid at infinity can be obtained directly from (4.111) together with a rescaling by a factor $(r_c/L)^{d-1}$ following again the prescription (4.15) for the dual fluid stress tensor

$$\begin{aligned}\eta^{\text{CFT}} \sigma_{ab}^y &= \lim_{r_c \rightarrow \infty} \left(\frac{r_c}{L} \right)^{d-1} \eta(r_c) \frac{r_c}{L} \sigma_{ab}^y = \frac{r_H^d}{L^d} \sigma_{ab}^y, \\ \zeta^{\text{CFT}} &= \lim_{r_c \rightarrow \infty} \left(\frac{r_c}{L} \right)^{d-1} \zeta(r_c) = 0.\end{aligned}\tag{4.123}$$

Hence, we have shown that the stress-energy tensor at finite cutoff reproduces the usual AdS/CFT results in (2.69) as the cutoff is taken to the asymptotic boundary.

4.5.4 UV field theory interpretation

The Dirichlet boundary condition on a finite cutoff hypersurface Σ_c necessarily leads to a non-Dirichlet boundary condition at the boundary at infinity. The aim of this section is to explore the interpretation of the fluid on the cutoff surface as a state in a deformation of the ultraviolet conformal field theory on the boundary. The strategy is to extrapolate the solution (4.115) to the asymptotic boundary by sending $r \rightarrow \infty$ and change to Fefferman-Graham coordinates. Using the standard AdS/CFT dictionary one can thereby show how the original CFT on the boundary has been deformed.

As we have derived in the previous section, the black brane metric in Eddington-Finkelstein coordinates up to order one in the hydrodynamic expansion is

$$\begin{aligned}ds^2 &= -2 \frac{u_a}{\sqrt{h_c}} dy^a dr + g_{ab}(y, r) dy^a dy^b, \\ g_{ab}^{(0)}(y, r) &= r^2 (h_{ab} - h/h_c u_a u_b), \\ g_{ab}^{(1)}(y, r) &= \alpha^{(1)} \frac{1}{d} \theta u_a u_b + 2 \beta^{(1)} a_{(a} u_{b)} + \tilde{\gamma}^{(1)} \sigma_{ab} + \gamma^{(1)} \frac{1}{d} \theta h_{ab},\end{aligned}\tag{4.124}$$

where the coefficients in $g_{ab}^{(1)}$ are given by (4.115) with (4.105-4.108). In writing (4.124) we have implicitly introduced an additional notational redefinition with respect to the previous section

$$\alpha^{y(1)} \rightarrow \alpha^{(1)} \frac{1}{d} \theta; \quad \beta_a^{y(1)} \rightarrow \beta^{(1)} a_a; \quad \tilde{\gamma}_{ab}^{y(1)} \rightarrow \tilde{\gamma}^{(1)} \sigma_{ab}; \quad \gamma^{y(1)} \rightarrow \gamma^{(1)},\tag{4.125}$$

and we have set $L = 1$ for convenience and all the fluid parameters are assumed to depend on y^a . In order to interpret such metric as a deformation of the CFT at infinity we first need to bring (4.124) into Fefferman-Graham form

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} G_{ab}(z, \rho) dz^a dz^b.\tag{4.126}$$

Such metric can be expanded near the asymptotic boundary located at $\rho = 0$

$$G_{ab}(z, \rho) = G_{ab}(z) + G_{ab}^{(2)}(z)\rho^2 + \dots, \quad (4.127)$$

and the leading order contribution $G_{ab}(z)$ is conventionally interpreted according to AdS/CFT as the fixed source associated to the dual stress tensor. In what follows we want to show that in the case at hand, where the Dirichlet boundary condition has been imposed on Σ_c , the asymptotic boundary value $G_{ab}(z)$ depends on the hydrodynamic expansion and in particular it is non flat beyond the leading order. More precisely, since the hydrodynamic expansion has been performed in Eddington-Finkelstein coordinates y rather than the Fefferman-Graham ones z , at the end of the game we will be actually interested in the quantity $G_{ab}(z(y))$. So let us start playing.

Leading order in a hydrodynamic expansion. In order to find the Fefferman-Graham form (4.126) of the black brane solution (4.124) we need to solve the following equations order by order for the variables $\rho(r, y)$, $z^a(r, y)$ and the metric $G_{ab}(\rho, z)$

$$\begin{aligned} (\partial_r \rho)^2 + G_{ab}(z, \rho) \partial_r z^a \partial_r z^b &= 0, \\ (\partial_r \rho) (\partial_a \rho) + G_{cd}(z, \rho) \partial_r z^c \partial_a z^d &= -\frac{u_a}{\sqrt{h_c}} \rho^2, \\ (\partial_a \rho) (\partial_b \rho) + G_{cd}(z, \rho) \partial_a z^c \partial_b z^d &= \rho^2 g_{ab}(y, r), \end{aligned} \quad (4.128)$$

with

$$\begin{aligned} \rho(y, r) &= \rho^{(0)}(y, r) + \rho^{(1)}(y, r) + \dots, \\ z^a(y, r) &= z^{(0)a}(y, r) + z^{(1)a}(y, r) + \dots, \\ G_{ab}(z, \rho) &= G_{ab}^{(0)}(z, \rho) + G_{ab}^{(1)}(z, \rho) + \dots \end{aligned} \quad (4.129)$$

At leading order this is given by

$$\begin{aligned} \rho^{(0)}(y, r) &= \frac{1}{r} \left(\frac{2}{1 + \sqrt{h(r)}} \right)^{\frac{2}{d+1}} = \frac{1}{r} \sqrt{A(\rho(r))}, \\ z^{(0)a}(y, r) &= y^a + k(r) u^a, \end{aligned} \quad (4.130)$$

$$G_{ab}^{(0)}(z, \rho) = A(\rho) \left(h_{ab} - \frac{h(r(\rho))}{h_c} u_a u_b \right); \quad (4.131)$$

where

$$A(\rho) = \left(1 + \frac{1}{4} (r_H \rho)^{d+1} \right)^{\frac{4}{d+1}}, \quad (4.132)$$

and

$$k(r) = \frac{\sqrt{h_c}}{r} {}_2F_1 \left(1, \frac{1}{d+1}, 1 + \frac{1}{d+1}, \frac{r_H^{d+1}}{r^{d+1}} \right); \quad \partial_r k(r) = -\frac{\sqrt{h_c}}{r^2 h(r)}. \quad (4.133)$$

The inverse transformation is given by

$$r^{(0)}(\rho) = \frac{1}{\rho} \sqrt{A(\rho)}; \quad y^{(0)a}(z, \rho) = z^a - k(r(\rho)) u^a. \quad (4.134)$$

It is now straightforward to perform the near-boundary expansion (4.127) of the leading order Fefferman-Graham metric just derived (4.131) to get

$$G_{ab}^{(0)}(y) = \eta_{ab} + (1 - 1/h_c) u_a u_b, \quad (4.135)$$

and it is clear that at this order in hydrodynamic expansion there is no difference between $u_a(z)$ and $u_a(y)$. Note that the deformation in the background metric appears to arise already at zeroth order in the hydrodynamic expansion. This was to be expected because, as discussed in Section 4.3.2, we started from a seed metric in which the time Killing vector is normalized to one at the cutoff hypersurface which implies that its norm at infinity is not canonical. One can therefore rescale the coordinates so that the zeroth order term in (4.135) is flat; this is achieved by rescaling the direction parallel to the velocity by a factor of $\sqrt{h_c}$ but leaving the directions perpendicular to the velocity unchanged.

First order in a hydrodynamic expansion. Interesting results come at first order. Life is more complicated now since we have to consider that the just derived zeroth order Fefferman-Graham metric (4.131) depends on the Fefferman-Graham coordinates z through the fluid velocities $u_a(z)$ and on the Eddington-Finkelstein coordinates y through $r_H(y)$

$$G_{ab}^{(0)}(z, \rho) = A(y, \rho) \left(h_{ab}(z) - \frac{h(y, r(\rho))}{h_c(y)} u_a(z) u_b(z) \right). \quad (4.136)$$

Hence, it is useful to first perform a Taylor expansion in the Fefferman-Graham coordinates z in order to obtain expressions in y only

$$\begin{aligned} u^a(z) &= u^a(y) + k(r(\rho)) a^a(y), \\ G_{ab}^{(0)}(z, \rho) &= G_{ab}^{(0)}(y, \rho) + u^c(y) k(r(\rho)) \partial_c G_{ab}^{(0)}(y, \rho) = \\ &= A(\rho) \left(h_{ab}(y, r(\rho)) - \frac{h(y, r(\rho))}{h_c(y)} u_a(y) u_b(y) \right) + \\ &\quad + 2k(r(\rho)) A(\rho) \left(1 - \frac{h(y, r(\rho))}{h_c(y)} \right) a_{(a}(y) u_{b)}(y). \end{aligned} \quad (4.137)$$

We can now use these expressions into the equations (4.128) to derive the first order hydrodynamic corrections in the expansions of the coordinate transformations and

the Fefferman-Graham metric (4.129). Parameterizing them in the following way

$$\begin{aligned}
 z^{(1)a}(y, r) &= \frac{1}{d} l(r) u^a \theta + m(r) a^a, \\
 \rho^{(1)}(y, r) &= \frac{1}{r} \sqrt{A} X^{(1)}(y, r) \frac{1}{d} \theta, \\
 G_{ab}^{(1)}(y, \rho) &= \phi(y, r(\rho)) \frac{1}{d} \theta u_a u_b + \psi(y, r(\rho)) \frac{1}{d} \theta h_{ab} + \\
 &\quad + \Gamma(y, r(\rho)) \sigma_{ab} + 2 \Sigma(y, r(\rho)) a_{(a} u_{b)},
 \end{aligned} \tag{4.138}$$

and making use of the on-shell equations (4.103), we can derive

$$\begin{aligned}
 l'(r) &= \frac{1}{r} \sqrt{\frac{h_c}{h}} k' + k' k - 2k' \frac{1}{r} \frac{\sqrt{h_c}}{h} + \frac{1}{2} k' k (d+1) \frac{(1-h_c)}{h_c} + \frac{1}{r^2} k' \frac{h_c}{h} \alpha^{(1)}, \\
 m'(r) &= \frac{1}{r} k' \frac{(1-\sqrt{h_c})}{\sqrt{h_c}} \delta(r_c) + \frac{1}{r^2} k' \beta^{(1)} + k k', \\
 X^{(1)}(y, r) &= k + \frac{1}{r \sqrt{h_c h}} \left(\frac{r_c^d r}{r_H^{d+1}} (1-h_c) - 1 \right),
 \end{aligned} \tag{4.139}$$

and

$$\begin{aligned}
 \phi(y, r) &= 2 A k \frac{h}{h_c} - 2 \frac{A}{r} \frac{1}{\sqrt{h_c}} + A k h \frac{(1-h_c)}{h_c^2} (d+1) + \frac{1}{r^2} A \alpha^{(1)} - 2 A X^{(1)} \frac{h}{h_c}, \\
 \psi(y, r) &= -2 A k + \frac{1}{r^2} A \gamma^{(1)} + 2 A X^{(1)}, \\
 \Sigma(y, r) &= \frac{A k h}{h_c} (1 - \delta(r_c)) + \frac{A \delta(r_c)}{r \sqrt{h_c}} - A k h \frac{(1-h_c)}{2 h_c^2} (d+1) \delta(r_c) + \frac{1}{r^2} A \beta^{(1)}, \\
 \Gamma(y, r) &= -2 A k + \frac{A}{r^2} \tilde{\gamma}^{(1)}.
 \end{aligned} \tag{4.140}$$

Notice that all the above solutions reduce to the expressions given earlier in [160] after using the diffeomorphism which redefines (4.121) and then sending $r_c \rightarrow \infty$.

Finally we can extract the metric at asymptotic infinity using the expressions (4.115) with (4.105-4.108) into (4.140). Performing the near boundary expansion (4.127) we can read off

$$\begin{aligned}
 G_{ab}(y) &= \eta_{ab} - \frac{2}{r_c h_c^{1/2}} \frac{1}{d} \theta u_a u_b + \frac{\delta(r_c)}{r_c h_c} a_{(a} u_{b)} + \\
 &\quad - 2 \left(k(r_c) + \frac{1}{(d+1)} \frac{\sqrt{h_c}}{r_H} \ln h_c \right) \sigma_{ab} + \dots,
 \end{aligned} \tag{4.141}$$

where we included the zeroth order contribution (4.135) and we also performed the rescaling in the u_a directions by factors of $\sqrt{h_c}$ to restore a flat metric at leading order as already discussed. Dots in the last line denote higher orders contributions

to the hydrodynamic expansion. As anticipated the metric $G_{ab}(y)$ characterizes the background metric for the dual field theory on the asymptotic boundary or, equivalently, the source for the dual stress energy tensor. Whenever r_c is finite and hence $h_c \neq 1$, the metric $G_{ab}(y)$ is clearly not flat at nontrivial order in a hydrodynamic expansion. Imposing a Dirichlet boundary condition on the finite cutoff surface Σ_c therefore translates into making a specific deformation of the original CFT defined on asymptotic infinity: a non-flat background metric for the field theory fluid. Hence in terms of such CFT the fluid lives in a dynamical background metric, namely a metric which depends on the fluid velocity and temperature; a similar interpretation was given in [125]. When $r_c \rightarrow \infty$ the boundary metric (4.141) is just η_{ab} as expected.

4.6 Other examples

In this section we are going to discuss other possible examples of the bulk stress-energy tensor setting the stage for their more general exploration.

4.6.1 Vanishing cosmological constant: $\Lambda = 0$

For the case $\Lambda = 0$ the equation of state (4.40) degenerates and allows two possible nontrivial solutions

$$\epsilon = 0; \quad \epsilon = -\frac{2d}{d-1}P. \quad (4.142)$$

Rindler fluid. The first solution is the well-known case of Rindler geometry as discussed extensively in literature, see *e.g.* [112]. Einstein equations (4.26-4.28) with the redefinitions (4.21) solved by

$$g(r) = \text{const.}; \quad f(r) = r, \quad (4.143)$$

giving, after rescaling the coordinates to get as usual a metric in the form (4.18), the Rindler geometry in rescaled Eddington-Finkelstein coordinates

$$ds^2 = -\frac{r}{r_c} dt^2 + dx_i dx^i + 2 \frac{1}{\sqrt{r_c}} dt dr. \quad (4.144)$$

The null surface $r = 0$ acts as an horizon for accelerated observers and the pressure (4.34) and energy density (4.35) can be computed knowing that $G(r) = 1$, $F(r) = r/r_c$ and $\lambda = 1/\sqrt{r_c}$

$$\epsilon = 0; \quad p = \frac{1}{\sqrt{r_c}}. \quad (4.145)$$

Hydrodynamic solutions nearby the Rindler geometry have been found in [112] up to second order. In our setting such a geometry is important since it arises in the near-horizon limit of the AdS black brane metric. We have seen for thermodynamic configurations in Section 4.5.1 how the near-horizon limit gives the values of the pressure and energy density above (4.100) and (4.101). In the same way, also the transport properties in AdS in Section 4.5.2 in the limit $r_c \rightarrow r_H$ give

$$\eta(r_H) = 1; \quad \zeta(r_H) = 0 \quad (4.146)$$

with the usual shear viscosity over entropy universal bound being satisfied as $s = 1/4\pi$, in agreement with Rindler fluid transport properties found in [112].

Taub geometry. The second solution in (4.142) is a negative energy equation of state. It can be realized within the other possible solution to Einstein equations with

$$g(r) = r^2; \quad f(r) = \frac{c_3}{r^{d-1}}, \quad (4.147)$$

The resulting metric in the rescaled coordinates is then

$$ds^2 = -\frac{r_c^{d-1}}{r^{d-1}} dt^2 + \frac{r^2}{r_c^2} dx_i dx^i + 2\sqrt{r_c^{d-1}} dt dr, \quad (4.148)$$

which is of the type of Taub geometry, namely a vacuum, homogeneous but anisotropic solution of Einstein gravity first found in four bulk spacetime dimensions in [161]. There is a curvature singularity at $r = 0$ which is timelike and naked. One can associate a pressure and energy density given by

$$\epsilon = -2dr_c^{(d-3)/2}; \quad P = (d-1)r_c^{(d-3)/2}, \quad (4.149)$$

to such solution, but as noticed in [113] there is way to associate a causal horizon to such geometry. Moreover given the thermodynamic relation $P + \epsilon = Ts$ such geometry would give a negative temperature. Hence the interpretation of this geometry in terms of a dual field theory at finite temperature might be problematic.

4.6.2 Positive cosmological constant: $\Lambda > 0$

Let us now turn to the case in which the bulk stress-energy tensor is simply given by a positive cosmological constant $\Lambda = \frac{d(d+1)}{2l^2}$. The spacetime metric (4.16) solving

Einstein equations (4.26-4.28) with the redefinitions (4.21) can be obtained from the AdS black brane solution (4.90) through a Wick rotation of the time coordinate $t \rightarrow -it$, of the radial coordinate $r \rightarrow ir$ and of the AdS radius $L^2 \rightarrow -l^2$, namely

$$ds^2 = \frac{r^2}{l^2} (h(r)dt^2 + dx_i dx^i) + 2 dt dr \quad \text{with} \quad h(r) = 1 - \left(\frac{r_H}{r}\right)^{d+1}. \quad (4.150)$$

Such metric represents a de Sitter brane in the so-called inflationary patch; de Sitter is not static so one should not expect the ansatz to produce a spacetime foliated by timelike hypersurfaces. In fact the induced metric on hypersurfaces of constant r is positive definite and the unit normal vector is now timelike. One can still write down the Brown-York tensor on the foliating hypersurface Σ_c , but it cannot be interpreted as a perfect fluid stress-energy tensor since the hypersurface is spacelike. This would only be the case after the analytic continuation mentioned above, but this would trivially reproduce all the results discussed in Section 4.5.

4.6.3 Fluids for which $\mathcal{T}_{rr} = 0$

There are cases for which the matter bulk stress-energy tensors compatible with the static ansatz (4.16) will in addition satisfy $\mathcal{T}_{rr} = 0$, as we have seen for *e.g.* the case of a simple cosmological constant. The significance is that if $\mathcal{T}_{rr}$ vanishes, then one can immediately integrate the (rr) Einstein equation (4.26) with (4.21) to obtain

$$g(r) = (c_1 r + c_2)^2. \quad (4.151)$$

Again vanishing c_1 is generically not consistent with the other Einstein equations unless the bulk stress-energy tensor is zero. Hence, as we did for the negative cosmological case, we are going to assume $c_1 \neq 0$ and absorb both constants into a redefinition of the origin and scale of the radial coordinate, $(c_1 r + c_2) \rightarrow r$. Integrating the remaining Einstein equations (4.27) one obtains the solution for $f(r)$ in terms of the bulk stress tensor component $\mathcal{T}_{tr}$

$$f(r) = \frac{c_3}{r^{d-1}} + \frac{2}{d r^{d-1}} \int^r dr' \mathcal{T}_{r't}(r')^d. \quad (4.152)$$

The class of stress energy tensors for which $\mathcal{T}_{rr} = 0$ includes in particular gauge fields. In fact a vector field stress energy tensor is expressed in terms of the field strength $F_{\mu\nu}$ as

$$\mathcal{T}(F)_{\mu\nu} = 2 \left(F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} g_{\mu\nu} \right) + m^2 \left(A_{\mu} A_{\nu} - \frac{1}{2} A^{\rho} A_{\rho} g_{\mu\nu} \right), \quad (4.153)$$

then the metric ansatz together with the antisymmetry of $F_{\mu\nu}$ forces the (rr) components in the first term to vanish. The second term involves the mass parameter

of the vector field and the vector potential A_μ ; the (rr) components vanish if $m^2 = 0$ (i.e. it is a gauge field) or $A_r = 0$. However the latter is generically not implied by the symmetries of the equilibrium static solution, which permit a non-zero $F_{tr}(r)$.

Consider the case in which the bulk stress energy tensor consists of a cosmological constant and a gauge field. The gauge field equation give

$$F_{tr} = \frac{q}{r^d}, \quad (4.154)$$

with the conserved charge being proportional to q and the general solution for $f(r)$ hence becomes

$$f(r) = \frac{c_3}{r^{d-1}} - \frac{2}{d(d+1)}\Lambda r^2 + \frac{2q^2}{d(d-1)r^{2(d-1)}}. \quad (4.155)$$

For negative cosmological constant we therefore recover AdS charged branes, as expected.

For $\Lambda = 0$ the solution with $c_3 > 0$ describes what might be called a charged Taub fluid: the metric is not asymptotically flat and has a naked singularity at $r = 0$. For $c_3 < 0$ $f(r)$ is positive for $0 < r < r_H$ and negative for $r > r_H$ where $f(r_H) = 0$. In the inner region hypersurfaces of constant r are timelike, but there is a naked singularity and the region is bounded by a horizon. In the outer region hypersurfaces of constant r are spacelike, and both t and r are null coordinates as $r \rightarrow \infty$.

One can understand the relationship of the latter solution to the regions inside a Reissner-Nordstrom black hole as follows. Consider four-dimensional black holes (the generalization to $d > 2$ being straightforward). Start from the metric in ingoing coordinates

$$ds^2 = -\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)dv^2 + 2dv dR + R^2 d\Omega_2^2. \quad (4.156)$$

Now zoom into the neighborhood of a point on the two sphere, which without loss of generality can be chosen to be the north pole, by letting $\theta = \epsilon x$ with $\epsilon \ll 1$ i.e.

$$d\theta^2 + \sin^2 \theta d\phi^2 \approx \epsilon^2(dx^2 + x^2 d\phi^2). \quad (4.157)$$

In addition scale the radial coordinate such that $r = \epsilon R$ remains finite and the time coordinate such that $t = v/\epsilon$ stays fixed, and also holding fixed

$$2m \equiv 2M\epsilon^3; \quad q \equiv Q\epsilon^2. \quad (4.158)$$

Under such rescalings one can see immediately using

$$R_\pm = M \pm \sqrt{M^2 - Q^2} \quad (4.159)$$

that the outer horizon at R_+ is pushed to infinity (in the r coordinate) while the inner horizon at R_- remains at a finite value of r . The resulting metric is

$$ds^2 = - \left(\frac{q^2}{r^2} - \frac{2m}{r} \right) dt^2 + 2dt dr + r^2(dx^2 + x^2 d\phi^2), \quad (4.160)$$

which is the $d = 2$ case of the metric given above. As discussed above this metric covers the region between an outer horizon, an inner horizon and the singularity.

For $\Lambda > 0$, the solution with $c_3 > 0$ describes a charged solution with a singularity at $r = 0$ and a horizon at a finite value of $r = r_H$. The hypersurfaces of constant r are only timelike in the region $r < r_H$. The solution with $c_3 \leq 0$ is more interesting: whilst the behavior of $f(r)$ at very small r and very large r is unchanged, the function can pass through zero more than once in the intermediate region, corresponding to inner and outer horizons.

4.6.4 Fluids for which $\mathcal{T}_{rr} \neq 0$

Many common matter Lagrangians induce stress energy tensors which are compatible with the static ansatz (4.16) but have $\mathcal{T}_{rr} \neq 0$. In such cases the (rr) Einstein equation (4.26) does not decouple and one cannot in general immediately solve for $g(r)$ (and hence for the other defining functions); the Einstein and matter field equations remain coupled.

Examples can be found within a class of Lagrangians which have recently received considerable attention in the context of AdS/CMT: (neutral) scalars coupled to vector fields, so-called Einstein-Maxwell-Dilaton models. Expressing the matter action for a single such scalar ϕ coupled to a vector field A_μ as

$$S_m = - \int d^{d+2}x \sqrt{-g} \left(\frac{1}{2}(\partial\phi)^2 + V(\phi) + \frac{1}{4}e^{\alpha\phi}F^2 + \frac{1}{2}e^{\beta\phi}m^2A^2 \right), \quad (4.161)$$

with the scalar potential and the parameters (α, β, m^2) defining the model, then the equations of motion are known to admit Lifshitz, hyperscaling violating Lifshitz solutions and other charged dilatonic black holes for various choices of these parameters. The matter stress energy tensor is

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{4}(\partial\phi)^2 g_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu} + \\ &+ \frac{1}{2}e^{\alpha\phi}(F_{\mu\rho}F_\nu^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}g_{\mu\nu}) + \frac{1}{2}m^2e^{\beta\phi}(A_\mu A_\nu - \frac{1}{2}A^\rho A_\rho g_{\mu\nu}), \end{aligned} \quad (4.162)$$

and the matter field equations are

$$\begin{aligned} \square\phi &= V'(\phi) + \frac{1}{4}\alpha e^{\alpha\phi}F^2 + \frac{1}{2}\beta m^2 e^{\beta\phi}A^2; \\ \nabla_\mu(e^{\alpha\phi}F^{\mu\nu}) &= 2m^2 e^{\beta\phi}A^\nu, \end{aligned} \quad (4.163)$$

with ∇_μ the covariant derivative. Consistency with the static, spatially homogeneous ansatz requires

$$\phi = \phi(r); \quad A_\mu dx^\mu = a(r) \left(dt - \frac{dr}{f(r)} \right) \quad (4.164)$$

but then $\mathcal{T}_{rr} \neq 0$ whenever $\phi(r) \neq 0$ and/or $m^2 a(r) \neq 0$. The metric plus matter is characterized by four functions but the equations of motion are coupled and non-linear so cannot be solved analytically in general. For example, in the case of the pure massive vector (no scalar field) an exact solution at zero temperature with Lifshitz scaling symmetry is known, see e.g. [162], but corresponding finite temperature blackened solutions have only been found numerically, see for example [163, 164, 165, 166]. The zero temperature Lifshitz solution can be written in our coordinate system as

$$f(r) = \frac{r^2}{z^2}; \quad g(r) = \left(\frac{r}{z} \right)^{2/z}; \quad a(r) = 2 \frac{\sqrt{z-1}}{z^2} r; \quad m^2 = \frac{d^2}{2z^2}. \quad (4.165)$$

In this case the (rr) Einstein equation can be integrated to give an analytic solution for the function $g(r)$, but the latter is no longer given by $g(r) \propto r^2$. The usual form of the Lifshitz metric, i.e.

$$ds^2 = \frac{d\rho^2}{\rho^2} + \left(-\frac{d\tau^2}{\rho^{2z}} + \frac{dx^i dx_i}{\rho^2} \right), \quad (4.166)$$

is obtained by the redefinitions

$$r = z\rho^{-z}; \quad d\tau = \left(dt - \frac{d\rho}{\rho} \right). \quad (4.167)$$

4.7 Discussion and Outlook

In this Chapter we have presented a construction of generic $(d+2)$ -dimensional near equilibrium metrics corresponding to the hydrodynamic regime of putative $(d+1)$ -dimensional holographic fluids associated with timelike hypersurfaces foliating a general bulk spacetime, *i.e.* with general bulk stress energy tensor. Using the method of Hamiltonian holographic renormalization we gave a prescription for the fluid stress-energy tensor in the case of (conformally) flat Dirichlet boundary

conditions on these timelike hypersurfaces. Our prescription is consistent with standard holographic results in the limit where the timelike hypersurface is taken to the conformal boundary. The resulting stress tensor is proportional to the Brown-York stress tensor of the corresponding hypersurface plus certain boundary terms. These boundary terms are in principle uniquely defined when the hypersurfaces are taken to the asymptotic boundary and represent the necessary counterterms to ensure the on-shell action to be finite. On a finite cutoff the above mentioned boundary terms cannot be fixed uniquely but we have shown that in the hydrodynamic regime they only provide a redefinition of the thermodynamic quantities without affecting the thermodynamic relation nor hydrodynamics.

Another possible source of boundary terms is the UV part of the spacetime between the finite cutoff and the boundary at infinity which we have not considered here. In the spirit of Wilsonian holographic renormalization described in Section 2.2.2, this part of the geometry is dual to the contribution of high energy degrees of freedom which can be integrated-out giving rise to a boundary effective action. Our results in this Chapter are related instead to the IR region of the spacetime. We have not computed the on-shell S^{IR} action but we know what is the momentum conjugate to the metric $\gamma_{ab}(r_c)$ on the finite cutoff. It would be interesting to see how, in the hydrodynamic regime, our results can be matched to the local contributions coming from the so called UV part of spacetime as derived in Chapter 3. This would represent an improvement to the membrane paradigm coupling (3.106) expressed for linearized gravitational perturbations. In this way one could handle the dissipative contributions coming from the event horizon in a fully nonlinear way.

In this Chapter relatively to earlier works (see for example [125, 157]) we have clarified a number subtleties. In particular, we have emphasized the fact that different coordinate systems give physically distinct fluids on timelike hypersurfaces obtained at any given radial cutoff. At leading order in the hydrodynamic expansions we can simply perform coordinate transformations and relate the pressures and energy densities but the hydrodynamic expansions are taken about different hypersurfaces and in particular with respect to different dual field theory spacetime coordinates; hence out of equilibrium we are dealing with physically different fluids. In the case of pure AdS gravity this subtlety does not arise with a flat or conformally flat Dirichlet boundary condition on the finite cutoff due to the fact that the conformal factor does not depend on the field theory coordinates but the issues discussed here would be relevant for dealing with hydrodynamics for cases such as AdS R-charged black holes (obtained as decoupling limits of rotating D3-branes, see [167]). One of the conclusions of [125, 157]) was that the fluid changes from a relativistic to non-relativistic fluid as the radial coordinate decreases. Here

we found that the near horizon description is the Rindler fluid of [111, 112], which indeed can be viewed as a non-relativistic fluid.

After discussing classes of spacetimes with a general bulk stress tensor at thermodynamic equilibrium, we concentrated on the specific case of Einstein gravity in AdS and verified the consistency of our prescription for the fluid stress tensor with standard holographic results when the timelike hypersurface is taken to the conformal boundary of AdS. Having at our disposal the holographic dictionary at conformal infinity we gave a precise interpretation of the fluid on the cutoff hypersurface in terms of a specific deformation of the UV CFT. The resulting UV fluid can be thought of as living in a non-flat background, depending on the fluid velocity and temperature.

Finally, we have also explored the near-horizon limit of the cutoff AdS fluid, which up to first order in a gradient expansion is effectively a Rindler fluid. We will show in Chapter 5 what are the differences of this holographically defined fluid near the horizon and the membrane fluid discussed in Section 2.2.1.

Recently an interesting connection between asymptotically flat spacetimes and asymptotically AdS black holes has emerged [102, 103]: it has been shown that asymptotically AdS black holes compactified on tori correspond to certain asymptotically flat Schwarzschild black branes and the holographic dictionary for the stress energy tensor has been derived through generalized dimensional reduction. It would be interesting to see how our construction would fit into this framework, and also how our construction can be applied to blackfolds [168, 169] which interpolate between asymptotically AdS and asymptotically flat regions.

Appendix A: Details of the hydrodynamic expansion

General hydrodynamic construction of Section 4.4.

The zeroth order Christoffel symbols for the metric (4.53) are

$$\begin{aligned}
 \Gamma_{rr}^{(0)r} &= 0, & \Gamma_{rr}^{(0)a} &= 0, & \Gamma_{ra}^{(0)r} &= \frac{1}{2}\lambda F' u_a, \\
 \Gamma_{ab}^{(0)r} &= -\frac{1}{2}\lambda^2 F (G' h_{ab} - F' u_a u_b), \\
 \Gamma_{rb}^{(0)a} &= \frac{1}{2} \frac{G'}{G} h_b^a, & \Gamma_{ab}^{(0)c} &= -\frac{1}{2}\lambda u^c (G' h_{ab} - F' u_a u_b), \\
 \text{with } \Gamma_{\mu r}^{(0)\mu} &= \frac{1}{G^{d/2}} \partial_r G^{d/2}; & \Gamma_{\mu a}^{(0)\mu} &= 0.
 \end{aligned} \tag{4.168}$$

The linearized Christoffel symbols can be formally computed to all orders using the definition (4.62) and (4.168) to express explicitly $\nabla_\mu^{(0)}$ giving

$$\begin{aligned}
 \delta\Gamma_{rr}^{(n)\mu} &= 0, \\
 \delta\Gamma_{ra}^{(n)r} &= \frac{1}{2}\lambda u^c \partial_r g_{ca}^{(n)} - \frac{1}{2}\lambda \frac{G'}{G} u^c h_a^d g_{cd}^{(n)}, \\
 \delta\Gamma_{ab}^{(n)r} &= -\frac{1}{2}\lambda^2 F \partial_r g_{ab}^{(n)} + \frac{1}{2}\lambda^2 (G' h_{ab} - F' u_a u_b) u^c u^d g_{cd}^{(n)}, \\
 \delta\Gamma_{rb}^{(n)a} &= \frac{1}{2} \frac{1}{G} h^{ac} \partial_r g_{cb}^{(n)} - \frac{1}{2} \frac{G'}{G^2} h^{ac} h_b^d g_{cd}^{(n)}, \\
 \delta\Gamma_{ab}^{(n)c} &= -\frac{1}{2}\lambda u^c \partial_r g_{ab}^{(n)} + \frac{1}{2} \frac{\lambda}{G} (G' h_{ab} - F' u_a u_b) h^{ce} u^f g_{ef}^{(n)}, \\
 \text{with } \delta\Gamma_{\rho r}^{(n)\rho} &= \frac{1}{2} \partial_r \left(\frac{1}{G} h^{cd} g_{cd}^{(n)} \right); & \delta\Gamma_{\rho a}^{(n)\rho} &= 0.
 \end{aligned} \tag{4.169}$$

The linearized Ricci tensor to all orders can be computed using the definition (4.61) and the result (4.169)

$$\delta R_{rr}^{(n)} = -\frac{1}{2} h^{ab} \partial_r^2 \left(\frac{1}{G} g_{ab}^{(n)} \right) - \frac{1}{2} \frac{G'}{G} h^{ab} \partial_r \left(\frac{1}{G} g_{ab}^{(n)} \right); \tag{4.170}$$

$$\begin{aligned}
 \lambda^{-1} \delta R_{ra}^{(n)} &= \frac{1}{2G^{d/2}} u^b \partial_r \left(G^{d/2} \partial_r g_{ab}^{(n)} \right) + \frac{1}{4} F' u_a h^{cd} \partial_r \left(\frac{1}{G} g_{cd}^{(n)} \right) + \\
 &\quad - \frac{1}{2G^{d/2}} u^c h_a^d \partial_r \left(\frac{G'}{G} G^{d/2} g_{cd}^{(n)} \right);
 \end{aligned} \tag{4.171}$$

$$\begin{aligned}
 \lambda^{-2} \delta R_{ab}^{(n)} &= -\frac{1}{4} F (G' h_{ab} - F' u_a u_b) h^{cd} \partial_r \left(\frac{1}{G} g_{cd}^{(n)} \right) - \frac{1}{2G^{d/2}} \partial_r \left(G^{d/2} F \partial_r g_{ab}^{(n)} \right) + \\
 &\quad + \frac{G' F}{G} h_{(a}^c \partial_r g_{cb)}^{(n)} + \frac{F' G'}{G} u_{(a} h_b^c u^d g_{cd}^{(n)} - F' u_{(a} u^c \partial_r g_{cb)}^{(n)} + \\
 &\quad - \frac{1}{2} \frac{F G'^2}{G^2} h_{(a}^c h_b^d g_{cd}^{(n)} - \frac{1}{2} \frac{G'^2}{G} h_{ab} u^c u^d g_{cd}^{(n)} + \\
 &\quad + \frac{1}{2G^{d/2}} u^c u^d \partial_r \left(G^{d/2} (G' h_{ab} - F' u_a u_b) g_{cd}^{(n)} \right).
 \end{aligned} \tag{4.172}$$

Such expressions can now be used in (4.59) together with (4.63) to derive Eq. (4.65-4.68) in the main text.

General first order hydrodynamics.

Christoffel symbols up to first order obtained from the seed metric (4.53)

$$\begin{aligned}
 \Gamma_{rr}^{(1)\mu} &= 0, \\
 \Gamma_{ra}^{(1)r} &= \frac{1}{2} \lambda u_a F' + \frac{1}{2} a_a - \frac{1}{2} D_a^\perp \ln \lambda, \\
 \Gamma_{ab}^{(1)r} &= -\frac{1}{2} \lambda^2 F (G' h_{ab} - F' u_a u_b) - \lambda K_{ab} G + u_{(a} a_{b)} \lambda F + \\
 &\quad + u_{(a} \lambda D_{b)}^\perp F - \frac{1}{2} \lambda u_a u_b D F - \frac{1}{2} \lambda h_{ab} D G + \lambda F u_{(a} \partial_{b)} \ln \lambda, \\
 \Gamma_{rb}^{(1)a} &= \frac{G'}{2G} h_b^a + \frac{1}{G\lambda} \Omega_b^a + \frac{1}{2G\lambda} a^a u_b - \frac{1}{2\lambda G} u_b D^{\perp a} \ln \lambda, \\
 \Gamma_{ab}^{(1)c} &= -u^c K_{ab} + u^c u_{(a} a_{b)} - \frac{1}{2} \lambda u^c (G' h_{ab} - F' u_a u_b) + \\
 &\quad - \frac{(G - F)}{G} (2u_{(a} \Omega_{b)}^c + a^c u_a u_b) + \frac{1}{G} h_{(a}^c \partial_{b)} G + \\
 &\quad - \frac{1}{2G} h_{ab} D^{\perp c} G + \frac{1}{2G} u_a u_b D^{\perp c} F + u^c u_{(a} \partial_{b)} \ln \lambda.
 \end{aligned} \tag{4.173}$$

and useful contractions

$$\begin{aligned}
 \Gamma_{\rho r}^{(1)\rho} &= \frac{1}{G^{d/2}} \partial_r G^{d/2}, \\
 \Gamma_{\rho a}^{(1)\rho} &= \frac{1}{G^{d/2}} \partial_a G^{d/2} + u_a D \ln \lambda - D_a^\perp \ln \lambda.
 \end{aligned} \tag{4.174}$$

The Ricci tensor components at first order are

$$\hat{R}_{rr}^{(1)} = 0, \tag{4.175}$$

$$\begin{aligned}
 \hat{R}_{ra}^{(1)} &= u_a \left(\frac{G'}{2G} \partial_c u^c + \frac{d}{2G} D G' - \frac{d G'}{4G^2} D G \right) + \frac{d}{4} \frac{G'}{G} a_a + \\
 &\quad - (d-1) \frac{1}{2G} D_a^\perp G' + (d-1) \frac{G'}{2G^2} D_a^\perp G - \frac{d}{4} \frac{G'}{G} D_a^\perp \ln \lambda,
 \end{aligned} \tag{4.176}$$

$$\begin{aligned}
 \lambda^{-1} \hat{R}_{ab}^{(1)} &= u_a u_b \left(\frac{1}{2} F' \partial_c u^c - \frac{d G'}{4G} D F + \frac{d F'}{4G} D G - \frac{1}{2} d \frac{G' F}{G} D \ln \lambda \right) + \\
 &\quad + h_{ab} \left(-\frac{1}{2} G' \partial_c u^c - D G' - \frac{(d-2)}{2} \frac{G'}{G} D G \right) + u_{(a} D_{b)}^\perp F' + \\
 &\quad + u_{(a} a_{b)} \left(F' + \frac{(d-2)}{2} \frac{G' F}{G} \right) + \frac{(d-2)}{2} \frac{G'}{G} u_{(a} D_{b)}^\perp F + \\
 &\quad - \frac{d}{2} G' K_{ab} + u_{(a} D_{b)}^\perp \ln \lambda \left(F' + \frac{1}{2} (d-2) \frac{G' F}{G} \right).
 \end{aligned} \tag{4.177}$$

AdS black brane hydrodynamics of Section 4.5.

Equations of motion for the AdS black brane nonlinear perturbations at first order are

$$\partial_r^2 \left(\frac{r_c^2}{r^2} \gamma^{(1)} \right) + \frac{2}{r} \partial_r \left(\frac{r_c^2}{r^2} \gamma^{(1)} \right) = 0, \quad (4.178)$$

$$\begin{aligned} d h_c \frac{r}{L^2} \alpha^{(1)'} + h_c \frac{d(d-1)}{L^2} \alpha^{(1)} + \left(\frac{(d+1)}{L^2} + \frac{(d-2)}{L^2} h \right) \gamma^{(1)} + \\ - \frac{r}{L^2} \left((d-1)h + \frac{1}{2} r h' \right) \gamma^{(1)'} - \frac{1}{2} \frac{r^2}{L^2} h \gamma^{(1)''} - 2d \frac{r}{r_c L} \sqrt{h_c} \theta = 0, \end{aligned} \quad (4.179)$$

$$\beta_a^{(1)''} + \frac{(d-2)}{r} \beta_a^{(1)'} - 2 \frac{(d-1)}{r^2} \beta_a^{(1)} - \frac{d}{r} \frac{L}{r_c \sqrt{h_c}} \delta(r_c) a_a = 0, \quad (4.180)$$

$$\partial_r \left(r^{d+2} h(r) \partial_r \left(\tilde{\gamma}_{ab}^{(1)} \frac{L^2}{r^2} \right) \right) + 2d \sqrt{h_c} \frac{L^3 r^{d-1}}{r_c} \sigma_{ab} = 0. \quad (4.181)$$

The general solution to such equations is

$$\begin{aligned} \gamma^{(1)} &= (\gamma_0 r + \gamma_1 r^2) \theta, \\ \beta_a^{(1)} &= r \left(\beta_0 r + \frac{\beta_1}{r^d} - \frac{L}{r_c \sqrt{h_c}} \delta(r_c) \right) a_a, \\ \alpha^{(1)} &= \frac{\alpha_0}{r^{d-1}} \theta + \frac{L}{d h_c^{3/2}} \frac{r}{r_c} ((d-1)(h-1) + 2(h_c-1)) \theta, \\ \tilde{\gamma}_{ab}^{(1)} &= \frac{1}{2} \frac{L \sqrt{h_c}}{r_c r_H} r^2 (4k(r) + \tilde{\gamma}_0 + \tilde{\gamma}_1 \ln h(r)) \sigma_{ab}, \end{aligned} \quad (4.182)$$

where $k(r)$ is given by (4.109) and $\gamma_0, \gamma_1, \beta_0, \beta_1, \alpha_0, \tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are integration constants. Dirichlet boundary conditions fix some of them to

$$\begin{aligned} \gamma_1 &= -\frac{\gamma_0}{r_c}, & \beta_1 &= r_c^d \left(-r_c \beta_0 + \frac{L}{r_c \sqrt{h_c}} \delta(r_c) \right), \\ \alpha_0 &= -\frac{(d+1)}{d} \frac{L r_c^{d-1}}{h_c^{3/2}} (h_c - 1), & \tilde{\gamma}_0 &= -4k(r_c) - \tilde{\gamma}_1 \ln h_c. \end{aligned} \quad (4.183)$$

Landau gauge conditions (4.81-4.82) fix the other integration constants to

$$\begin{aligned} \gamma_0 &= \frac{2L}{r_c \sqrt{h_c}}, \\ \beta_0 &= -\frac{L(d h_c(2 + (d-1)\delta(r_c)) - (d+1)(2 + d\delta(r_c)))}{2(d+1)r_c^2 h_c^{3/2}} \delta(r_c), \end{aligned} \quad (4.184)$$

and requiring regularity on the horizon r_H gives

$$\tilde{\gamma}_1 = \frac{4}{d+1}, \quad (4.185)$$

leading to the solutions (4.105-4.65).

Appendix B: The Bianchi identities

The Bianchi identities at order n are

$$(\nabla_\mu G^{\mu\nu})^{(n)} = \sum_{k=0}^n \nabla_\mu^{(n-k)} G^{(k)\mu\nu} = 0. \quad (4.186)$$

Using the Einstein equations at each order and the conservation of the bulk stress energy tensor at order n , $(\nabla_\mu T^{\mu\nu})^{(n)} = 0$, the following identity holds

$$\nabla_\mu^{(0)} G^{(n)\mu\nu} = \nabla_\mu^{(0)} \mathcal{T}^{(n)\mu\nu} \quad \text{or} \quad \nabla_\mu^{(0)} R^{(n)\mu\nu} = \nabla_\mu^{(0)} \bar{\mathcal{T}}^{(n)\mu\nu}. \quad (4.187)$$

Knowing that

$$\begin{aligned} \lambda^{-2} \delta R^{(n)} &= -\frac{1}{2} \frac{F}{G^{d/2}} \partial_r \left(G^{d/2} \partial_r \left(\frac{1}{G} g_{cd}^{(n)} h^{cd} \right) \right) - \frac{1}{2} F' \partial_r \left(\frac{1}{G} g_{cd}^{(n)} h^{cd} \right) + \\ &\quad - \frac{1}{2G^{d/2+1}} \partial_r \left(G^{d/2} F \partial_r g_{cd}^{(n)} h^{cd} \right) + \frac{1}{G^{d/2}} \partial_r \left(G^{d/2} \partial_r g_{cd}^{(n)} u^c u^d \right) + \\ &\quad + \frac{d}{2G^{d/2+1}} \partial_r \left(G^{d/2} G' g_{cd}^{(n)} u^c u^d \right) - \frac{d}{2} \frac{G'^2}{G^2} g_{cd}^{(n)} u^c u^d + \frac{G' F}{2G^2} \partial_r g_{cd}^{(n)} h^{cd}, \end{aligned} \quad (4.188)$$

and using the leading order equations of motion (4.26) one can show from (4.187) that the following expressions are identically satisfied

$$\begin{aligned} \lambda F \delta R_{ar}^{(n)} u^a + \delta R_{ab}^{(n)} u^a u^b &= \left(-\frac{1}{d} \mathcal{T}^{(0)} + \lambda \mathcal{T}_{re}^{(0)} u^e \right) g_{ab}^{(n)} u^a u^b, \\ \lambda F \delta R_{ar}^{(n)} h_c^a + \delta R_{ab}^{(n)} u^b h_c^a &= \left(-\frac{1}{d} \mathcal{T}^{(0)} + \frac{1}{d} \frac{1}{G} \mathcal{T}_{ef}^{(0)} h^{ef} \right) g_{ab}^{(n)} u^b h_c^a. \end{aligned} \quad (4.189)$$

Using

$$\lambda F \bar{\mathcal{T}}_{ar}^{(n)} + \bar{\mathcal{T}}_{ab}^{(n)} u^b = \lambda F \mathcal{T}_{ar}^{(n)} + \mathcal{T}_{ab}^{(n)} u^b - \frac{1}{d} \mathcal{T}^{(0)} g_{ab}^{(n)} u^b - \frac{1}{d} \sum_{k=1}^{n-1} g_{ab}^{(k)} u^b \mathcal{T}^{(n-k)}, \quad (4.190)$$

integrating (4.189) and evaluating them on Σ_c gives

$$\left(\lambda (\hat{R}_{ar}^{(n)} - \mathcal{T}_{ar}^{(n)}) + (\hat{R}_{ab}^{(n)} - \mathcal{T}_{ab}^{(n)}) u^b \right) \Big|_{\Sigma_c} = f_a^{(n)}(x) \Big|_{\Sigma_c}, \quad (4.191)$$

where $f_a^{(n)}(x)$ arises as an integration constant. The Gauss-Codazzi equations on Σ_c at order n are then given by

$$\begin{aligned} \nabla^b T_{ab}^{\text{BY}(n)} \Big|_{\Sigma_c} &= -2R_{a\mu}^{(n)} n^\mu \Big|_{\Sigma_c} = -2(\hat{R}_{a\mu}^{(n)} + \delta R_{a\mu}^{(n)}) \Big|_{\Sigma_c} n^\mu = \\ &= -2 \left(\lambda \hat{R}_{ar}^{(n)} + \hat{R}_{ab}^{(n)} u^b \right) \Big|_{\Sigma_c} = -2n^\mu \mathcal{T}_{a\mu}^{(n)} - 2f_a^{(n)}(x) \Big|_{\Sigma_c}. \end{aligned} \quad (4.192)$$

As discussed around (4.42), conservation of the fluid stress tensor requires that $\mathcal{T}_{a\mu} n^\mu$ vanishes to all orders, which in turn requires that $\mathcal{T}_{a\mu}^{(n)} n^\mu = 0$ to all orders n since n^μ does not change due to the required Dirichlet boundary conditions. If the fluid is not conserved then $\mathcal{T}_{a\mu}^{(n)} n^\mu \neq 0$ characterizes this non-conservation. In both cases the integration constant arising from integrating the Bianchi identities is therefore zero for the fluid stress energy tensor to satisfy the required conservation equation.

5

THE MEMBRANE PARADIGM

ON A MEMBRANE COVERING THE INTERIOR OF A BLACK HOLE AND HOW
ACCURATE IT IS FROM AN ASYMPTOTIC OBSERVER POINT OF VIEW

The membrane paradigm is a simplified point of view on black hole physics. Its relevance in the flow of arguments of this thesis is depicted by its role in appropriately neglecting the interior of a black hole. In this Chapter, based on parts of [2] and on [3] we are going to question the limits of applicability of this approximation scheme. The lessons turn out to be of general relevance beyond the holographic point of view.

5.1 Introduction

In Section 2.2.1 of the Introduction we reviewed the membrane paradigm from a historical point of view, emphasizing its multiple formulations. One of those, the most famous one, is what we called the membrane fluid, where the membrane is thought to behave as a nonrelativistic fluid with simple physical properties as discussed around (2.46). However, we also noticed that this fluid has some pathological properties such as a negative bulk viscosity and moreover it is not clear how to couple this part of the spacetime to the exterior and make use of this approximation. In Chapter 4 we identified another type of fluid living on a hypersurface close to the horizon of a black hole assuming a holographic-like dictionary there, that is a Rindler fluid. We will show in Section 5.3 how these two pictures are different from one another and how the holographic Rindler fluid is richer in many ways, since, for example, it encodes the regular boundary condition

on the event horizon. In fact one could also take the Rindler fluid as an improved, nonlinear definition of the membrane paradigm itself as it has been suggested in [157].

We also discussed a more modern interpretation of the membrane paradigm as a simple boundary condition (2.51) mimicking the ingoing behavior of the fields at the horizon or at a stretched horizon. One might think that one can always smoothly connect the generic membrane response on a stretched horizon (2.51) to the ingoing behavior on the horizon itself (2.50). We will show in Section 5.2 that this turns out to be almost always the case. We will present a general argument showing that there is a domain of frequencies for which the membrane paradigm effectively reproduces the outgoing behavior on the horizon instead. The regime of applicability does not comprise, for example, the short-living gapped quasinormal modes, those which decay very fast after the equilibrium of a black hole has been perturbed. Our argument is general since it relies only on the near-horizon region of a nonextremal, non necessarily AdS, black hole.

We will test our general argument in the context of the AdS/CFT correspondence. As we saw in Section 2.2.1, the definition of the membrane paradigm as a boundary condition is particularly useful when it comes to couple it to the UV spacetime. The latter transmits the information to the boundary at infinity where the dual field theory correlators can be holographically read off. By coupling the membrane paradigm to the UV spacetime we will be able to see whether all the correct dual observables are reproduced at infinity having replaced the IR part of the spacetime with a simple boundary condition. We will focus our attention on studying quasinormal modes as they are better suited to illustrate our arguments, given that the limitations on the membrane paradigm are found for a certain domain of the frequencies of the probe fields. In Section 5.2.1 we will explicitly show that the gapped quasinormal modes in a black brane in AdS_3 cannot be reproduced at infinity and in section 5.2.2 we will show that the opposite is true for hydrodynamic modes in AdS_5 . However, the latter case will exhibit some complications due to the fact that here the bulk fields transform nontrivially under some local symmetry. For this reason the solutions in the external UV region depend also on certain gapless degrees of freedom, namely the Goldstone bosons of a spontaneously broken symmetry as we discussed extensively in Chapter 3. In particular one of those corresponds to a non hydrodynamic degree of freedom and needs to be discarded when seeking for the hydrodynamic regime on a stretched horizon.

5.2 Regime of applicability

We start by investigating in detail the scalar field ϕ case and focus on its universal near-horizon behavior in the presence of any non extremal black hole. The expansion has been given already in eq. (2.48), and we write it here again for completeness

$$\begin{aligned} \phi = e^{-i\omega t + i\vec{k} \cdot \vec{x}} & \left\{ c_{\text{out}}(1-u)^{i\omega/4\pi T} \left(1 + \alpha_1(1-u) + \dots \right) + \right. \\ & \left. c_{\text{in}}(1-u)^{-i\omega/4\pi T} \left(1 + \beta_1(1-u) + \dots \right) \right\}, \end{aligned} \quad (5.1)$$

where T is the temperature of the black hole, u is the radial coordinate, $u = 1$ is the rescaled horizon radius and α_i and β_j depend on the number of dimensions, the mass of the field and its momentum. If we use this expansion in the membrane paradigm boundary condition

$$4\pi T(1-u) \left. \frac{\partial_u \phi}{i\omega \phi} \right|_{u=u_\delta} = \sigma \quad (5.2)$$

we readily obtain a relation between the outgoing and the ingoing modes

$$\frac{c_{\text{out}}}{c_{\text{in}}} = (1-u_\delta)^{-i\tilde{\omega}} \frac{(1-\sigma)\tilde{\omega} + \beta_1(2i + (1-\sigma)\tilde{\omega})(1-u_\delta) + \dots}{(1+\sigma)\tilde{\omega} + \alpha_1(-2i + (1+\sigma)\tilde{\omega})(1-u_\delta) + \dots} \quad (5.3)$$

where we have rescaled $\tilde{\omega} = \omega/2\pi T$. Using $\sigma = 1$ and keeping only the leading order terms, eq. (5.3) reduces to

$$c_{\text{out}}/c_{\text{in}} = (1-u_\delta)^{1-i\tilde{\omega}} \times \frac{i\beta_1}{\tilde{\omega}}. \quad (5.4)$$

It is easy to see that for values of $\tilde{\omega}$ such that $\mathcal{I}m(\tilde{\omega}) > -1$, this formula has the desirable effect, *i.e.* it leads to $|c_{\text{out}}/c_{\text{in}}| \ll 1$ for $u_\delta \rightarrow 1$, and the infalling behavior of the field is recovered on the horizon. However for $\mathcal{I}m(\tilde{\omega}) < -1$ it effectively leads to outgoing boundary conditions $|c_{\text{out}}/c_{\text{in}}| \gg 1$ as $u_\delta \rightarrow 1$ instead. Note that this holds no matter how close to the event horizon the membrane is.

The root of this discrepancy is that $\sigma = \pm 1$ is formally correct only at the event horizon and away from it is slightly different. This “flow” of the membrane conductivity can be constructed perturbatively in the $1 - u_\delta$ expansion allowing a general form for $\sigma = \sigma(\tilde{\omega}, \tilde{k}, u_\delta)$ in (2.51). Eq. (5.3) implies that we may be able to cover a somewhat wider range of imaginary parts of the frequency if we include a finite number of these corrections in σ . It is clear though that in order to cover

the whole complex frequency plane the value of the *exact* membrane conductivity at any $u_\delta \neq 1$ is required, which is equivalent to knowing the whole solution with the ingoing boundary condition at the event horizon. However, this defies the purpose of introducing the membrane paradigm on a stretched horizon.

In holography, solutions of the equations of motion for a scalar satisfying ingoing boundary conditions at the event horizon encode the retarded two-point function of a dual scalar operator in the thermal state. Conversely, imposing the outgoing boundary conditions leads to the advanced two-point function. This implies that using the membrane paradigm on the stretched horizon with $\sigma = 1$ only yields a good approximation to the retarded Green's function if $\mathcal{I}m(\tilde{\omega}) > -1$, whereas for $\mathcal{I}m(\tilde{\omega}) < -1$ we obtain instead an approximation to the advanced Green's function. This means that, at best, the membrane paradigm will reveal only a few of the lowest lying quasinormal modes if any. In particular only hydrodynamic modes are reproduced, namely those which vanish $\tilde{\omega} \rightarrow 0$ when $\tilde{k} = k/2\pi T \rightarrow 0$.

We will now illustrate and confirm our general findings with two explicit examples within holography.

5.2.1 Gapped quasinormal modes in a BTZ₃ black brane

Consider the exactly soluble case of a massless scalar field in $2 + 1$ dimensions in the Banados-Teitelboim-Zanelli (BTZ) black brane background in Einstein gravity with a negative cosmological constant [133, 127]. The metric is

$$ds^2 = \frac{du^2}{4u^2 f(u)} - \frac{(2\pi T)^2}{u} f(u) dt^2 + \frac{(2\pi T)^2}{u} dx^2, \quad (5.5)$$

where $f(u) = 1 - u$, the horizon is at $u = 1$ and we set the AdS radius to unity. We will work in Fourier space where the scalar field exhibiting the same symmetries of the background can be parametrized as

$$\phi(t, x, u) = \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t + ikx} \phi(\omega, k, u). \quad (5.6)$$

For simplicity, we will set the momentum k to zero. The near boundary expansion of the the scalar field¹ is

$$\phi(u) = \phi^{(0)} + (\phi^{(1)} - \frac{1}{4} \tilde{\omega}^2 \phi^{(0)} \log u) u + \dots, \quad (5.7)$$

and it is given in terms of two coefficients. One is the Dirichlet boundary condition interpreted holographically as the source $\phi^{(0)}$ of the dual operator $\mathcal{O}$ and the other

¹This scalar is dual to an operator $\mathcal{O}$ of conformal dimension $\Delta = 2$ in the dual (1+1)-dimensional conformal field theory.

$\phi^{(1)}$ is fixed as a function of $\phi^{(0)}$ after requiring a membrane boundary condition (5.2) on a stretched horizon. The retarded/advanced Green's function can be computed using the holographic prescription (2.72) and the result is

$$G_{R/A} / \left(\frac{2\pi^2 T^2}{k_3^2} \right) = \tilde{G}_{R/A} = \frac{\phi^{(1)}}{\phi^{(0)}} - \frac{1}{2} \tilde{\omega}^2, \quad (5.8)$$

where k_3 is proportional to the Newton constant in $2+1$ dimensions and $\tilde{G}$ is the zero temperature retarded/advanced Green's function. As we discussed in Section 2.2.2 of the Introduction, quasinormal modes appear as poles of G_R . When the membrane is taken to be at the event horizon, which is equivalent to impose the usual ingoing boundary condition in holography, the retarded Green's function reads

$$\tilde{G}_R = \frac{i\tilde{\omega}}{2} - \frac{\tilde{\omega}^2}{4} - \frac{\gamma\tilde{\omega}^2}{2} - \frac{\tilde{\omega}^2}{2} \psi_0 \left(1 - \frac{i\tilde{\omega}}{2} \right) \quad (5.9)$$

with γ the Euler constant and $\psi_0(x) = \Gamma'(x)/\Gamma(x)$, and diverges (with single pole singularities) at

$$\tilde{\omega} = -2in \quad \text{for } n = 1, 2, \dots, \quad (5.10)$$

which are all gapped quasinormal modes of the scalar field in a BTZ background in AdS_3 as discussed in (2.73).

However, if we move the membrane slightly away from the horizon these quasinormal modes are no longer captured. In fact relying on an intermediate step as described in Section 2.2.1, we solve the scalar field equation of motion for the configuration obeying Dirichlet boundary conditions² at $u = 0$ and $u = u_\delta$,

$$\phi(u = 0) = \phi^{(0)} \quad \text{and} \quad \phi(u = u_\delta) = \phi_\delta, \quad (5.11)$$

and subsequently use eq. (5.2) to express ϕ_δ in terms of $\phi^{(0)}$. This, in turn, determines $\phi^{(1)}$, which is enough to evaluate eq. (5.8). The results are summarized in Fig. 5.1 and nicely confirm the general expectations obtained above, as we indeed see that the retarded Green's function is not well approximated for $\mathcal{I}m(\tilde{\omega}) < -1$ and the advanced Green's function is not well approximated for $\mathcal{I}m(\tilde{\omega}) > 1$. This, in particular, implies that none of the quasinormal modes in this setup are captured by the membrane paradigm, unless it is taken exactly to be at the event horizon.

We repeated the same calculation in $4+1$ dimensions and found similar behavior. This is an example where the implementation of the membrane paradigm on a stretched horizon is strictly necessary, since exact analytic solutions are not available and one needs to perform numerical calculations. Numerics requires in fact

²For a massive field, we would typically demand that at $u = 0$ the leading fall-off of the solution is fixed.

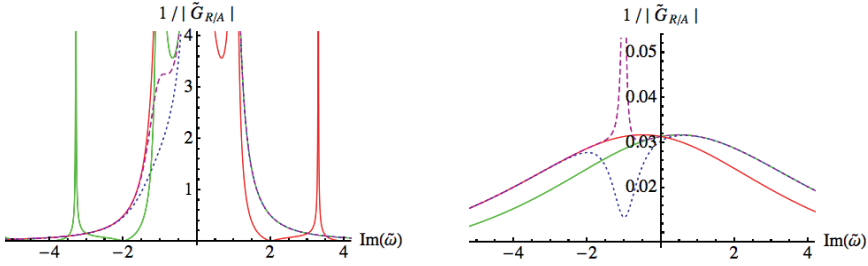


Figure 5.1: Absolute value of the inverse of the retarded/advanced Green's function (green/red) and the membrane paradigm approximations of the former at $u_\delta = 0.9$ (blue) and $u_\delta = 0.999$ (magenta) as a function of $\Im(\tilde{\omega})$ for $\Re(\tilde{\omega}) = 0$ (left) and $\Re(\tilde{\omega}) = 5$ (right). One can clearly see that the stretched horizon approximation works for $\Im(\tilde{\omega}) > -1$, whereas for $\Im(\tilde{\omega}) < -1$ it leads to the advanced Green's function, in line with the approximation in eq. (5.4). Zeros of the green curve correspond to the locations of the quasinormal modes, as given by eq. (5.10), and lie beyond the range of applicability of the membrane paradigm.

to have a very small but finite deviation from the exact location of the horizon radius where differential equations are singular. Only after assuming a nontrivial membrane coupling $\sigma = \sigma(\omega, k)$ we are able to recover the lowest lying gapped quasinormal modes, but as we anticipated this procedure goes beyond the membrane approximation³.

5.2.2 Hydrodynamic modes in an AdS_5 black brane

Let us now consider gravitational perturbations $\delta h_{\mu\nu}(t, \vec{x}, u)$ of a 4+1 dimensional black brane in AdS

$$ds^2 = \frac{du^2}{u^2 f(u)} - \frac{(2\pi T)^2}{u} f(u) dt^2 + \frac{(2\pi T)^2}{u} d\vec{x}^2, \quad (5.12)$$

where $f(u) = 1 - u^2$, the horizon is at $u = 1$ and we have set the AdS radius $L = 2$. Actually we have already shown in Chapter 2 that the membrane paradigm approach on the stretched horizon is a good approximation to compute long-wavelength quasinormal modes for such perturbations⁴. In fact in Section 3.5.1 we correctly derived the sound and shear dispersion relations up to second order in a hydrodynamic expansion. However, for completeness and clearness of

³This approach would be equivalent, for example, to [170] where (asymptotic) gapped quasinormal modes are found from an increasingly accurate near-horizon expansion of the field.

⁴Earlier approaches to this problem include [171, 93, 52, 27, 139].

the exposition, we will give here a shorter version of the same findings highlighting the presence of an additional gapless non hydrodynamic mode on the stretched horizon.

Sound channel perturbations with momentum in the x -direction are the interesting ones since they include the above mentioned additional mode. The non-vanishing metric variations in a radial gauge $\delta h_{\mu u} = 0$ are

$$\delta h_{tt}, \delta h_{xt}, \delta h_{xx} \text{ and } \delta h_{\alpha\alpha} = \frac{1}{2}(\delta h_{yy} + \delta h_{zz}), \quad (5.13)$$

which in Fourier space are

$$\delta h_{\mu\nu}(t, x, u) = \int \frac{d\omega dk}{(2\pi)^2} \delta h_{\mu\nu}(\omega, k, u) e^{-i\omega t + ikx}. \quad (5.14)$$

The standard approach [135] in dealing with the gravitational perturbations (5.13) is to introduce the linearized gauge-invariant variable

$$\begin{aligned} Z(u) = & 2k^2 f(u) H_{tt}(u) + 4\omega k H_{xt}(u) + 2\omega^2 H_{xx}(u) + \\ & + H_{\alpha\alpha}(u) (k^2(1+u^2) - \omega^2), \end{aligned} \quad (5.15)$$

where we redefined for convenience $H_{ab} := |g^{ac}| \delta h_{cb}$ and $g^{\mu\nu}$ is the inverse of the black brane metric (5.12). Using all linearized Einstein equations one obtains a decoupled second order ordinary differential equation for Z , and therefore the problem of finding the retarded stress tensor correlator in the sound channel is completely analogous to the scalar field case studied in the previous section. Hence, to test the membrane paradigm we once more impose the universal relation (5.2) on a stretched horizon u_δ with ϕ simply replaced by Z .

A generic solution for Z can be found analytically order by order in a hydrodynamic expansion

$$\begin{aligned} Z(u) = & c_{\text{in}}(1-u^2)^{-i\tilde{\omega}/2} (X_0(u) + \lambda X_1(u) + \dots) + \\ & + c_{\text{out}}(1-u^2)^{i\tilde{\omega}/2} (Y_0(u) + \lambda Y_1(u) + \dots), \end{aligned} \quad (5.16)$$

where λ is a bookkeeping parameter counting powers of $\tilde{\omega} \ll 1$ and $\tilde{k} \ll 1$. To leading order in λ the solution reads

$$X_0(u) = Y_0(u) = \frac{k^2(1+u^2) - 3\omega^2}{5k^2 - 3\omega^2}, \quad (5.17)$$

which, together with the membrane paradigm (5.2), give a relation between the outgoing and ingoing coefficients c_{out} and c_{in} on the stretched horizon u_δ analogue of eq. (5.3)

$$\frac{c_{\text{out}}}{c_{\text{in}}} = (1-u_\delta)^{-i\tilde{\omega}} \frac{(1-\sigma)}{2i\tilde{\omega}(\sigma+1)} + (1-u_\delta)^{1-i\tilde{\omega}} \frac{b(\tilde{\omega}, \tilde{k}, \sigma)}{(\sigma+1)^2 \tilde{\omega} (3\tilde{\omega}^2 - 2\tilde{k}^2)} + \dots, \quad (5.18)$$

where b is an analytic function of $\tilde{\omega}$, $\tilde{k}$ and the membrane coupling σ . One can clearly see that for small $\tilde{\omega}$ and $\tilde{k}$ one indeed obtains from (5.18) a very small ratio of $c_{\text{out}}/c_{\text{in}}$ unless $\tilde{\omega} = \pm\sqrt{2/3}\tilde{k}$. With some work, one can also determine the approximate retarded Green's function, and from its poles one obtains two branches of solutions, which for small enough δ read

$$\tilde{\omega} = \pm\sqrt{\frac{1}{3}}\tilde{k} + \mathcal{O}(\tilde{k}^2) \quad (5.19)$$

and

$$\tilde{\omega} = \pm\sqrt{\frac{2}{3}}\tilde{k} + \mathcal{O}(\tilde{k}^2). \quad (5.20)$$

The mode (5.19) is just the standard hydrodynamic sound wave, whereas the second one is spurious, as it does not solve the linearized equations of hydrodynamics of the underlying microscopic theory and ceases to exist when one imposes the ingoing boundary condition at the event horizon. Moreover, the presence of a pole in the second term in eq. (5.18) implies that the solution with the membrane paradigm boundary condition on a stretched horizon and with ingoing boundary conditions on the event horizon are not smoothly connected to each other for $\tilde{\omega} = \pm\sqrt{2/3}\tilde{k}$. Hence, the mode (5.20) has to be discarded. This yields one more model-dependent restriction on the allowed frequencies for the membrane paradigm on the stretched horizon.

The same conclusions can be obtained at the level of Goldstone bosons. We have showed in Section 2.1.2 of the Introduction that hydrodynamics can be recasted as a theory of gapless degrees of freedom, namely Goldstones of certain spontaneously broken symmetries. In fact the sound wave dispersion relation (5.19) is naturally associated to the so-called longitudinal Goldstone. Here we will show that also (5.20) can be naturally associated to a Goldstone boson, though non hydrodynamical.

In calculating the above leading order dispersion relations (5.19) and (5.20) we kept σ arbitrary and the result did not depend on the value of σ . This suggests that both modes are not an intrinsic property of the membrane. We should then be able to obtain the same result considering the UV part of the spacetime alone. Indeed, the emergence of gapless modes in the holographic context can be thought of as Goldstone bosons arising due to spontaneous symmetry breaking by the classical solution with double-Dirichlet boundary conditions, one on the conformal boundary and one on the stretched or event horizon as discussed in Chapter 3. Such Goldstones are non-local Wilson-line like objects already encountered in (3.97)

$$\pi_t = \int_0^{u_\delta} H_{tu}(u)du, \quad \pi_x = \int_0^{u_\delta} H_{xu}(u)du, \quad (5.21)$$

where f_δ is a shortcut for $f(u_\delta)$. In radial gauge $H_{\mu u} = 0$, where these perturbations are redefined as $\partial_u H_{uu} := u\sqrt{f(u)}\delta h_{uu}$ and $H_{au} := 2\pi T |g^{ac}|\delta h_{cu}$, combinations (5.21) arise as nontrivial boundary conditions $\tilde{H}_{ab}^\delta$ say on the stretched horizon

$$\tilde{H}_{tt}^\delta = 2i\tilde{\omega}\pi_t - \frac{(1+u_\delta^2)}{2f_\delta}H_{uu}(u_\delta), \quad (5.22a)$$

$$\tilde{H}_{xt}^\delta = i\tilde{\omega}\pi_x - i\tilde{k}f_\delta\pi_t, \quad (5.22b)$$

$$\tilde{H}_{xx}^\delta = -2i\tilde{k}\pi_x + \frac{1}{2}\sqrt{f_\delta}H_{uu}(u_\delta), \quad (5.22c)$$

$$\tilde{H}_{\alpha\alpha}^\delta = \frac{1}{2}\sqrt{f_\delta}H_{uu}(u_\delta), \quad (5.22d)$$

once Dirichlet boundary conditions on the other boundary have been fixed. The gravitational constraint equations, ua - and uu -components of the Einstein equations, turn out to be dynamical equations for the quantities (5.21) and for H_{uu} . In the limit where the stretched horizon is very close to the event horizon, at leading order in $\tilde{k}$ and $\tilde{\omega}$ and after solving for $H_{uu}(u_\delta)$, the remaining constraint equations are

$$(3\tilde{\omega}^2 - \tilde{k}^2)\pi_x = 0, \quad (3\tilde{\omega}^2 - 2\tilde{k}^2)\hat{\pi}_t = 0, \quad (5.23)$$

where $\hat{\pi}_t = \sqrt{f_\delta}\pi_t$. This near-horizon redefinition would be natural if equations (5.23) would follow from an action principle in which the Goldstone bosons appear quadratically⁵. Eq. (5.23) for the longitudinal and timelike Goldstone, to leading order in δ , directly lead to the sound waves (5.19) and the spurious mode (5.20) respectively. Note however that in the strict horizon limit the π_t Goldstone decouples from the dynamics and one is only left with the hydrodynamic sound wave excitation.

Hence in short we have seen that the membrane paradigm is capable in revealing hydrodynamic modes, though in the specific case of gravitational perturbations another mode is present in the spectrum on the stretched horizon. Such a mode can be naturally associated to the timelike Goldstone π_t which decouples in the near-horizon limit. Therefore, in the case the strict horizon limit is inaccessible and one is forced to deal with quantities defined on the stretched horizon, the additional spurious mode has to be discarded when aiming in uncovering only the hydrodynamic spectrum of the dual field theory. In principle one could try to remove the spurious mode already from the start by making the cutoff u_δ to be $\tilde{k}$ and $\tilde{\omega}$ dependent, but superficially this would require significant fine-tuning and would go beyond the scope of the simple membrane response approximation.

⁵We have seen explicitly in Chapter 3 that this is indeed the case.

5.3 The membrane fluid

In this Section we first derive the nonrelativistic Navier-Stokes equations (2.46) governing the dynamics of an event horizon. In particular we make manifest their origin as a degenerate subset of Einstein equations with no information on the regular behavior of the horizon surface. Subsequently we make an explicit comparison between the just derived membrane fluid with recent developments of fluid/gravity duality on finite cutoffs, such as a Rindler fluid discussed in Chapter 4. It will be clear that the latter approach is richer in many ways than the *ad hoc* rewriting of Einstein equations (2.46).

5.3.1 Derivation of the membrane fluid equations

Spacelike foliation of a hypersurface $\mathcal{H}$

Consider a generic $d + 1$ -dimensional spacelike, timelike or null hypersurface $\mathcal{H}$ embedded in a $(d + 2)$ -dimensional spacetime with metric g and further foliate it by d -dimensional spacelike hypersurfaces S_τ such that $\mathcal{H} = \bigcup_{\tau \in \mathbb{R}} S_\tau$, as in [172]. The metric q induced by the spacetime metric g onto S_τ is positive definite and in components can be expressed

$$q_{\mu\nu} = g_{\mu\nu} + l_\mu k_\nu + k_\mu l_\nu, \quad (5.24)$$

where (l, k) is a pair of null vectors normal to S_τ satisfying

$$l \cdot l = 0; \quad k \cdot k = 0; \quad l \cdot k = -1. \quad (5.25)$$

Such pair is uniquely defined provided we introduce the evolution vector h

$$h = l - Ck; \quad h \cdot h = 2C, \quad (5.26)$$

whose defining properties are that of being tangent to $\mathcal{H}$, orthogonal to S_τ at any point in $\mathcal{H}$ and

$$\mathcal{L}_h \tau = h^\mu \partial_\mu \tau = 1, \quad (5.27)$$

where $\mathcal{L}_h$ is the Lie derivative along h . The property (5.27) shows that any point of S_τ is transported to $S_{\tau+\delta\tau}$ by the vector $\delta\tau h$ and $\mathcal{L}_h$ becomes the evolution operator along $\mathcal{H}$. The character of h gives the character of the hypersurface $\mathcal{H}$, in particular if $C < 0$, then $\mathcal{H}$ and h are timelike.

The vectors (l, k) are co-linear to the pair $(\tilde{l}, \tilde{k})$ with parameters A and B respectively

$$l = A\tilde{l}; \quad k = B\tilde{k}, \quad (5.28)$$

where $(\tilde{l}, \tilde{k})$ are defined as the null normal vectors to two families of null hypersurfaces generated by outgoing and ingoing light rays orthogonally from S_τ

$$\underline{\tilde{l}} = -du; \quad \underline{\tilde{k}} = -dv, \quad (5.29)$$

with u and v parameterizing the dual-null foliations⁶ in the dual-null formalism of *e.g.* [173].

A natural additional vector that can be constructed out of (l, k) is

$$m = l + Ck; \quad m \cdot m = -2C; \quad m \cdot h = 0. \quad (5.30)$$

Notice that if h is timelike, m is necessarily spacelike and it defines the normal vector to the hypersurface $\mathcal{H}$, which, together with h spans the orthogonal space to S_τ .

Given a generic tensor T on the bulk spacetime, one can canonically define another tensor of the same covariance type on the subspace S_τ using the projector q_b^a defined out of (5.24)

$$(q^*T)_{j_1 \dots j_n}^{i_1 \dots i_m} := q_{\mu_1}^{i_1} \dots q_{\mu_m}^{i_m} q_{j_1}^{\nu_1} \dots q_{j_n}^{\nu_n} T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_m}. \quad (5.31)$$

A tensorial field T for which $q^*T = T$ is said to be tangent to the surface S_τ . For any such tangent tensorial fields T we can define the covariant derivative $\mathcal{D}$ on the spacelike surfaces S_τ

$$\mathcal{D}T := q^*\nabla T, \quad (5.32)$$

where ∇ is the covariant derivative on the $(d+2)$ -dimensional bulk spacetime.

Extrinsic geometry of a spacelike surface S_τ

Given a vector field v orthogonal to the spacelike surface S_τ , the deformation tensor along this field is defined as

$$\Theta^{(v)} := q^*(\nabla v); \quad \Theta_{ij}^{(v)} = q_i^\mu q_j^\nu \nabla_\mu v_\nu, \quad (5.33)$$

which measures the variation of the metric in S_τ when the surface S_τ is displaced along v ⁷. The deformation tensor is symmetric and can be decomposed into a traceless symmetric shear tensor and a trace contribution

$$\sigma_{ij}^{(v)} := \Theta_{ij}^{(v)} - \frac{1}{d} \theta^{(v)} q_{ij}; \quad \theta^{(v)} := q^{ij} \Theta_{ij}^{(v)}, \quad (5.34)$$

⁶Such construction always exists and it is uniquely defined when $\mathcal{H}$ is spacelike or timelike. In the case such hypersurface is null there is freedom in choosing the null foliation outside $\mathcal{H}$.

⁷From the definition of the Lie derivative $*q(\mathcal{L}_v q) = *q \nabla_v q + 2\Theta^{(v)}$ and using the idempotent property of the projector $*q \nabla_v q = 0$ we have that $\Theta^{(v)} = \frac{1}{2} *q(\mathcal{L}_v q)$.

the latter measuring the change of the area element in S_τ when displaced by v .

The variation of the normal fields to S_τ with respect to each other is instead contained in the normal fundamental forms, which for the pair (l, k) can be written as

$$\Omega^{(l)} := \frac{1}{k \cdot l} k \cdot \nabla_q l; \quad \Omega_i^{(l)} = \frac{1}{k_\mu l^\mu} k_\nu q_i^\rho \nabla_\rho l^\nu; \quad (5.35)$$

$$\Omega^{(k)} := \frac{1}{k \cdot l} l \cdot \nabla_q k; \quad \Omega_i^{(k)} = \frac{1}{k_\mu l^\mu} l_\nu q_i^\rho \nabla_\rho k^\nu, \quad (5.36)$$

with the relation

$$\Omega^{(k)} = -\Omega^{(l)}. \quad (5.37)$$

Kinematics of a spacelike surface S_τ

Covariant derivatives of the normal vectors l and k can be decomposed using quantities defined on the spacelike surface S_τ after using the projector (5.24)

$$\nabla_\mu l_\nu = \Theta_{\mu\nu}^{(l)} + \Omega_\mu^{(l)} l_\nu - l_\mu \nabla_k l_\nu - \nu_{(l)} k_\mu l_\nu, \quad (5.38)$$

$$\nabla_\mu k_\nu = \Theta_{\mu\nu}^{(k)} - \Omega_\mu^{(l)} k_\nu - k_\mu \nabla_l k_\nu - \nu_{(k)} l_\mu k_\nu, \quad (5.39)$$

where we have used equations (5.28-5.29) to derive

$$\nabla_l l = \nu_{(l)} l; \quad \nu_{(l)} := \mathcal{L}_l \ln A; \quad (5.40)$$

$$\nabla_k k = \nu_{(k)} k; \quad \nu_{(k)} := \mathcal{L}_k \ln B, \quad (5.41)$$

where $\nu_{(l)}$ and $\nu_{(k)}$ are the inaffinity parameters of the null vector fields. Combining (5.38-5.39) and using the definitions (5.26) and (5.30) we can derive the covariant derivatives for m and h

$$\begin{aligned} \nabla_\mu m_\nu &= \Theta_{\mu\nu}^{(m)} + \Omega_\mu^{(l)} h_\nu - l_\mu \nabla_k l_\nu - \nu_{(l)} k_\mu l_\nu + \\ &\quad + (\nabla_\mu C) k_\nu - C k_\mu \nabla_l k_\nu - C \nu_{(k)} l_\mu k_\nu, \end{aligned} \quad (5.42)$$

$$\begin{aligned} \nabla_\mu h_\nu &= \Theta_{\mu\nu}^{(h)} + \Omega_\mu^{(l)} m_\nu - l_\mu \nabla_k l_\nu - \nu_{(l)} k_\mu l_\nu + \\ &\quad - (\nabla_\mu C) k_\nu + C k_\mu \nabla_l k_\nu + C \nu_{(k)} l_\mu k_\nu, \end{aligned} \quad (5.43)$$

where we also used the properties

$$\Theta^{(m)} = \Theta^{(l)} + C \Theta^{(k)}; \quad \Theta^{(h)} = \Theta^{(l)} - C \Theta^{(k)}, \quad (5.44)$$

which follow straightforwardly from definition (5.33).

Dynamics of a spacelike surface S_τ

Dynamical equations for the extrinsic geometry quantities can be obtained by inserting the Ricci identity for a generic vector field v^μ

$$R^\mu_{\nu\rho\sigma}v^\nu = (\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)v^\mu, \quad (5.45)$$

in the Einstein equations

$$R_{\mu\nu} = \bar{\mathcal{T}}_{\mu\nu}, \quad (5.46)$$

where $\bar{\mathcal{T}}_{\mu\nu}$ is the traceless bulk stress energy tensor. There are as many equations as many independent ways to project the Einstein equations along m , h and q . Projection along *e.g.* q and m results in a dynamical equation for the normal fundamental form $\Omega_\alpha^{(l)}$ along the evolution vector h as previously derived in [172]

$$\begin{aligned} q^* \mathcal{L}_h \Omega_i^{(l)} + \theta^{(l)} \Omega_\alpha^{(l)} = & -\mathcal{D}_j \sigma_i^{(m)j} + \frac{(d-1)}{d} \mathcal{D}_i \theta^{(m)} + \\ & + \mathcal{D}_i (\nu_{(l)} + C\nu_{(k)}) - \theta^{(k)} \mathcal{D}_i C + q^* \bar{\mathcal{T}}_{i\mu} m^\mu. \end{aligned} \quad (5.47)$$

Another example is given by the evolution equation for $\theta^{(m)}$ along m by projecting twice along m

$$\begin{aligned} \nabla_m \theta^{(m)} + \mathcal{L}_m (\nu_{(l)} + C\nu_{(k)}) + \Theta_{ij}^{(m)} \Theta^{(m)ij} - \mathcal{L}_l \nu_{(l)} - \nu_{(l)} \theta^{(l)} + \\ - 2C \Omega^{(l)i} \mathcal{D}_i \ln A - (\theta^{(k)} + \nu_{(k)}) \mathcal{L}_m C - C \nabla_\mu (\nabla_k l^\mu + \nabla_l k^\mu) + \\ + 4C \Omega^{(l)i} \Omega_i^{(l)} - 2C \nu_{(l)} \nu_{(k)} - C^2 \nu_{(k)} \theta^{(k)} - C^2 \nabla_k \nu_{(k)} + \\ + \bar{\mathcal{T}}_{\mu\nu} m^\mu m^\nu = 0, \end{aligned} \quad (5.48)$$

or the evolution equation for $\theta^{(m)}$ along h by projecting Einstein equations along h and m

$$\begin{aligned} \nabla_h \theta^{(m)} + \mathcal{L}_h (\nu_{(l)} + C\nu_{(k)}) + \Theta_{ij}^{(h)} \Theta^{(m)ij} - \mathcal{L}_l \nu_{(l)} - \nu_{(l)} \theta^{(l)} + \\ - (\theta^{(k)} + \nu_{(k)}) \mathcal{L}_h C + C \nabla_\mu (\nabla_k l^\mu - \nabla_l k^\mu) + C \nu_{(k)} \theta^{(k)} + \\ + C^2 \nabla_k \nu_{(k)} + \bar{\mathcal{T}}_{\mu\nu} m^\mu h^\nu = 0. \end{aligned}$$

An example of a tensorial equation is the evolution equation of the shape tensor $\Theta_{ij}^{(m)}$ which is

$$\begin{aligned} q^* \mathcal{L}_h \Theta_{ij}^{(m)} = & \Theta_{ik}^{(m)} \Theta_j^{(h)k} + \nu_{(l)} \Theta_{ij}^{(l)} + C \Omega_i \mathcal{D}_j \ln A + \\ & + \Theta_{ij}^{(k)} \mathcal{L}_h C - C \nabla_\mu (\nabla_k l_\nu - \nabla_l k_\nu) q_i^\mu q_j^\nu + \\ & - C^2 \nu_{(k)} \Theta_{ij}^{(k)} + h^\rho R_{\mu\nu\rho\sigma} m^\nu q_i^\sigma q_j^\mu. \end{aligned} \quad (5.49)$$

All the details of the derivations can be found in Appendix A.

The null limit

The equations derived above are valid for a generic hypersurface $\mathcal{H}$. The limit in which the hypersurface $\mathcal{H}$ is null such as the horizon of a black hole can be achieved by sending

$$C \rightarrow 0. \quad (5.50)$$

This implies that the two former orthogonal vectors h and m now coincide $h, m \rightarrow l$ and the null vector l plays the double role of being tangent and normal to the null hypersurface $\mathcal{H}$. In this way the system of equations derived above degenerates down to the usual dynamical equations for null hypersurfaces, see *e.g.* [174]. Equations (5.47) become the so called Damour-Navier-Stokes equations

$$q^* \mathcal{L}_l \Omega_i^{(l)} + \theta^{(l)} \Omega_i^{(l)} = -\mathcal{D}_j \sigma_i^{(l)j} + \frac{(d-1)}{d} \mathcal{D}_i \theta^{(l)} + \mathcal{D}_i \nu_{(l)} + q^* \bar{\mathcal{T}}_{i\mu} l^\mu, \quad (5.51)$$

which are equivalent to Navier-Stokes equations (2.46) after identifying the normal fundamental form $\Omega^{(l)}$ with the surface momentum density $\mathcal{P}$, the affinity parameter $\nu_{(l)}$ with the pressure P and the external force f with a suitable projection of the bulk stress energy tensor $\bar{\mathcal{T}}$

$$\mathcal{P}_i = -\Omega_i^{(l)}; \quad P = \nu_{(l)}; \quad f_i = q^* \bar{\mathcal{T}}_{i\mu} l^\mu, \quad (5.52)$$

and the bulk and shear viscosity as in (2.47). Equations (5.48-5.49) degenerate to the so-called null Raychaudhuri equation

$$\nabla_l \theta^{(l)} + \Theta_{ij}^{(l)} \Theta^{(l)ij} - \nu_{(l)} \theta^{(l)} + \bar{\mathcal{T}}_{\mu\nu} l^\mu l^\nu = 0, \quad (5.53)$$

and the tensorial equation (5.49) becomes the tidal force equation for $\Theta_{ij}^{(l)}$

$$q^* \mathcal{L}_l \Theta_{ij}^{(l)} = \Theta_{ik}^{(l)} \Theta_j^{(l)k} + \nu_{(l)} \Theta_{ij}^{(l)} + q^* (l^\nu R_{j\mu\nu} l^\mu). \quad (5.54)$$

5.3.2 Membrane fluid Vs Rindler fluid

Let us here illustrate the properties of the membrane fluid equations (2.46):

- they are intrinsically nonrelativistic,
- they do not contain information about thermodynamic equilibrium,
- they model a fictitious intrinsically first order dissipative fluid with a negative bulk viscosity (2.47),
- they are not decoupled but part of a bigger set of equations which collectively defines the evolution of the event horizon. The remaining equations such as the null Raychaudhuri eq. (5.53) and the tidal force eq. (5.54) do not have a natural interpretation in terms of fluid quantities,

- they are obtained without any specific information on the horizon, besides it being a null degenerate surface through the limit (5.50) of equations (5.47).

All these properties of the membrane fluid equations (2.46) have to be compared with the properties of another type of fluid which can be associated with a holography inspired approach to a region in the proximity of the horizon of a nonextremal black hole, that is the Rindler fluid. Here, as opposed to conventional fluid/gravity duality where the dual fluid lives on the boundary of spacetime, the fluid lives on a timelike hypersurface very close to the horizon, *i.e.* the stretched horizon. We refer to Chapter 4 for a more detailed explanation and let us highlight some of the relevant properties of the holographic Rindler fluid equations:

- they are fully relativistic,
- the existence of a thermodynamic equilibrium is assumed by the presence of an equilibrium seed metric,
- dissipative behavior is obtained after perturbing the thermodynamic solution around equilibrium. This formalism systematically incorporates in principle all orders in a hydrodynamic expansion,
- all components of the Einstein equations are needed to work out the fluid equations,
- transport properties are derived after explicitly solving Einstein equations providing regular boundary conditions on the horizon.

Let us conclude here by saying that the membrane fluid and the Rindler fluid are substantially different. On one side the membrane fluid equations (2.46) are just a rewriting of a subset of Einstein equations (5.51) with suitable *ad hoc* identifications (5.52) in order to formally resemble Navier-Stokes equations. Fluid/gravity duality approach for Rindler fluids enables instead to systematically account for the low energy behavior of the dual field theory on a stretched horizon. Of course one could easily use the explicit form of the horizon metric and some derivative expansion in the membrane fluid equations (2.46), but this would be equivalent of taking the holographic point of view and would for example spoil the original structure of the membrane fluid equations. In other words membrane fluid dynamics is incomplete in accounting for the near-horizon region of a black hole. A better way to model the near-horizon IR spacetime is the Rindler fluid dynamics which can be eventually coupled to the external spacetime.

5.4 Discussion and Outlook

In this Chapter we have studied the range of validity of the membrane paradigm as a particular boundary condition imposed on a stretched horizon, and which is supposed to represent the response of the interior of a black hole to external perturbations. Though we mostly worked in the context of the AdS/CFT correspondence, we expect our results to hold in more general gravitational setups as they rely on generic properties of horizons. We found that the membrane paradigm works very well except for the case of gapped quasinormal modes of probe fields in a black hole background. The spectrum of hydrodynamic quasinormal modes is correctly reproduced instead as long as one neglects the contribution on the stretched horizon of an additional non hydrodynamic mode.

So far the membrane boundary condition of the form (5.2) has been assumed, but it would be nice to derive it from a variational principle point of view as an action modeling the IR physics. This same question is linked to the desire of finding a general action principle for dissipative fluid dynamics and would require the use of the full Schwinger-Keldysh formalism. Moreover, recent developments showed that at a quantum level the black hole horizon is nothing like a regular surface but it might develop a firewall [175]. Hence, it would be interesting to understand what are the implications of a quantum reasoning to the membrane paradigm, in particular how modifications of the horizon boundary conditions translate from the boundary field theory point of view.

In this Chapter we have also emphasized the differences between the membrane fluid and the holographic Rindler fluid interpretation of a near-horizon region of the spacetime. In particular we have showed in Chapter 4 how through a holographic construction one can obtain fluid equations up to arbitrary order in a gradient expansion in contrast to the membrane fluid which is only a suitable rewriting of certain components of Einstein equations into a fluid-like fashion. We concluded that the holographic Rindler fluid dynamics could be an alternative better nonlinear definition of the membrane paradigm as it can be coupled to the external part of the spacetime. It would be nice to check how this coupling happens for example by matching the IR Rindler fluid dynamics to the UV part of the spacetime discussed in Chapter 3.

Appendix A: Details on the d+1+1 foliation of the spacetime

Knowing (5.27), (5.28) and (5.29) it is possible to show the following identities

$$\mathcal{D}_i B = 0; \quad \mathcal{D}_i (C/A) = 0 \quad \text{on } \mathcal{H}. \quad (5.55)$$

These can then be used to derive the useful relations

$$q_i^\nu \nabla_k l_\nu = -\Omega_i^{(l)} + \mathcal{D}_i \ln A; \quad q_i^\nu \nabla_l k_\nu = \Omega_i^{(l)}. \quad (5.56)$$

Let us first consider

$$\begin{aligned} (\nabla_j \nabla_i - \nabla_i \nabla_j) m^j &= \nabla_j \Theta_i^{(m)j} + h^j \nabla_j \Omega_i^{(l)} + \Omega_i^{(l)} \theta^{(h)} + \Omega_i^{(l)} \nu_{(l)} + \\ &\quad - C \nu_{(k)} \Omega_i^{(l)} + \Theta_i^{(h)j} \Omega_j^{(l)} - \Theta_i^{(l)j} \mathcal{D}_j \ln A + \\ &\quad + m_i \Omega^{(l)j} \Omega_j^{(l)} - l_i \Omega^{(l)j} \mathcal{D}_j \ln A - l_i \nabla_j (\nabla_k l^j) + \\ &\quad + m_i \nu_{(l)} \nu_{(k)} - k_i \mathcal{L}_l \nu_{(l)} - \nu_{(l)} \nabla_l k_i - \nu_{(l)}^2 k_i + \\ &\quad - \nu_{(l)} k_i \theta^{(l)} + (\theta^{(k)} + \nu_{(k)}) \nabla_i C - C k_i \nabla_j (\nabla_l k^j) + \\ &\quad - \nu_{(k)} l_i \mathcal{L}_k C - C l_i \mathcal{L}_k \nu_{(k)} - C \nu_{(k)} \nabla_k l_i - C \nu_{(k)}^2 l_i + \\ &\quad - C \nu_{(k)} \theta^{(k)} l_i - \nabla_i \theta^{(m)} - \nabla_i (\nu_{(l)} + C \nu_{(k)}) + \\ &\quad - \Theta_i^{(k)j} \mathcal{D}_j C + \nu_{(k)} l_i \nabla_k C, \end{aligned} \quad (5.57)$$

where we repeatedly used (5.42). By further projecting on q , using (5.56-5.55) and the relations

$$q_i^\nu \nabla_\mu \Theta_\nu^{(m)\mu} = \mathcal{D}_\mu \Theta_i^{(m)\mu} + \Theta_i^{(m)\mu} \mathcal{D}_\mu \ln A, \quad (5.58)$$

$$q_i^\nu \mathcal{L}_h \Omega_\nu^{(l)} = q_i^\nu h^\mu \nabla_\mu \Omega_\nu^{(l)} + \Theta_i^{(h)\mu} \Omega_\mu^{(l)}, \quad (5.59)$$

as well as Einstein equations (5.46) we get the generalized Damour-Navier-Stokes equation (5.47). Equations (5.48-5.49) can be derived similarly.

To derive equation (5.49) we consider

$$h^\rho \nabla_\rho \nabla_\mu m_\nu q_i^\mu q_j^\nu = h^\rho (R_{\nu\sigma\rho\mu} m^\sigma + \nabla_\mu \nabla_\rho m_\nu) q_i^\mu q_j^\nu, \quad (5.60)$$

and repeatedly make use of (5.42-5.43), (5.56) and

$$q_i^\mu q_j^\nu \mathcal{L}_h \Theta_{\mu\nu}^{(m)} = q_i^\mu q_j^\nu \nabla_h \Theta_{\mu\nu}^{(m)} + \Theta_{i\rho}^{(m)} \Theta_j^{(h)\rho} + \Theta_{j\rho}^{(m)} \Theta_i^{(h)\rho}. \quad (5.61)$$

NOTATION

η_{ab} flat metric on the boundary spacetime

$(-1, +, \dots, +)$ mostly plus signature

$c = \hbar = 1$ Planck units

$2k_{(d+1)}^2 = 16\pi G_N^{(d+2)}$ Gravitational coupling constant, sometimes set to 1

$(d+2)$ bulk spacetime dimensions

$(d+1)$ boundary spacetime dimensions

d boundary space dimensions

$\mu, \nu, \dots$ bulk spacetime indices with $\mu = 1, \dots, d+2$

$m, n, \dots$ bulk space indices with $m = 1, \dots, d+1$

$a, b, \dots$ boundary spacetime indices with $a = 1, \dots, d+1$

$i, j, \dots$ boundary space indices with $i = 1, \dots, d$

$x^\mu = (t, x^i, r)$ bulk spacetime coordinates

r or u bulk radial coordinate

$x^a = (t, x^i)$ also sometimes $x = x^a$, boundary spacetime coordinates

$A_{(a}B_{b)} = \frac{1}{2}(A_aB_b + A_bB_a)$ symmetric combination

$A_{[a}B_{b]} = \frac{1}{2}(A_aB_b - A_bB_a)$ antisymmetric combination

$A_{\langle a}B_{b\rangle} = \frac{1}{2}(A_aB_b + A_bB_a - \frac{1}{\text{dim}}A_cB^c\delta_{ab})$ symmetric traceless combination

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ partial derivatives

BIBLIOGRAPHY

- [1] J. de Boer, M. P. Heller, and N. Pinzani-Fokeeva, “Effective actions for relativistic fluids from holography,” *Submitted to JHEP* (2015) , [arXiv:1504.07616 \[hep-th\]](#).
- [2] N. Pinzani-Fokeeva and M. Taylor, “Towards a general fluid/gravity correspondence,” *Phys.Rev.* **D91** no. 4, (2015) 044001, [arXiv:1401.5975 \[hep-th\]](#).
- [3] J. de Boer, M. P. Heller, and N. Pinzani-Fokeeva, “Testing the membrane paradigm with holography,” *Phys.Rev.* **D91** no. 2, (2015) 026006, [arXiv:1405.4243 \[hep-th\]](#).
- [4] F. Bigazzi, A. L. Cotrone, D. Musso, N. P. Fokeeva, and D. Seminara, “Unbalanced Holographic Superconductors and Spintronics,” *JHEP* **1202** (2012) 078, [arXiv:1111.6601 \[hep-th\]](#).
- [5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*. Course of Theoretical Physics, Vol 6. Pergamon Press, 1959.
- [6] P. Kovtun, “Lectures on hydrodynamic fluctuations in relativistic theories,” *J.Phys.* **A45** (2012) 473001, [arXiv:1205.5040 \[hep-th\]](#).
- [7] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, “A Theory of first order dissipative superfluid dynamics,” [arXiv:1105.3733 \[hep-th\]](#).
- [8] S. Bhattacharyya, “Constraints on the second order transport coefficients of an uncharged fluid,” *JHEP* **1207** (2012) 104, [arXiv:1201.4654 \[hep-th\]](#).
- [9] J. S. Schwinger, “Brownian motion of a quantum oscillator,” *J.Math.Phys.* **2** (1961) 407–432.
- [10] L. Keldysh, “Diagram technique for nonequilibrium processes,”

- Zh.Eksp.Teor.Fiz.* **47** (1964) 1515–1527.
- [11] C. R. Galley, “Classical Mechanics of Nonconservative Systems,” *Phys.Rev.Lett.* **110** no. 17, (2013) 174301, [arXiv:1210.2745 \[gr-qc\]](#).
 - [12] S. Endlich, A. Nicolis, R. A. Porto, and J. Wang, “Dissipation in the effective field theory for hydrodynamics: First order effects,” *Phys.Rev.* **D88** (2013) 105001, [arXiv:1211.6461 \[hep-th\]](#).
 - [13] N. Andersson and G. Comer, “A covariant action principle for dissipative fluid dynamics: From formalism to fundamental physics,” *Class.Quant.Grav.* **32** no. 7, (2015) 075008, [arXiv:1306.3345 \[gr-qc\]](#).
 - [14] S. Grozdanov and J. Polonyi, “Viscosity and dissipative hydrodynamics from effective field theory,” *Phys.Rev.* **D91** no. 10, (2015) 105031, [arXiv:1305.3670 \[hep-th\]](#).
 - [15] C. R. Galley, D. Tsang, and L. C. Stein, “The principle of stationary nonconservative action for classical mechanics and field theories,” [arXiv:1412.3082 \[math-ph\]](#).
 - [16] S. Grozdanov and J. Polonyi, “Dynamics of the electric current in an ideal electron gas: a sound mode inside the quasi-particles,” [arXiv:1501.06620 \[hep-th\]](#).
 - [17] P. Kovtun, G. D. Moore, and P. Romatschke, “Towards an effective action for relativistic dissipative hydrodynamics,” *JHEP* **1407** (2014) 123, [arXiv:1405.3967 \[hep-ph\]](#).
 - [18] M. Harder, P. Kovtun, and A. Ritz, “On thermal fluctuations and the generating functional in relativistic hydrodynamics,” [arXiv:1502.03076 \[hep-th\]](#).
 - [19] B. Carter, “Elastic Perturbation Theory in General Relativity and a Variation Principle for a Rotating Solid Star,” *Comm. Mat. Phys.* **305** (1973) 261–286.
 - [20] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, “Null energy condition and superluminal propagation,” *JHEP* **0603** (2006) 025, [arXiv:hep-th/0512260 \[hep-th\]](#).
 - [21] N. Andersson and G. Comer, “Relativistic fluid dynamics: Physics for many different scales,” *Living Rev.Rel.* **10** (2007) 1, [arXiv:gr-qc/0605010 \[gr-qc\]](#).
 - [22] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, “Effective field theory for

- hydrodynamics: thermodynamics, and the derivative expansion,” *Phys.Rev.* **D85** (2012) 085029, [arXiv:1107.0731 \[hep-th\]](#).
- [23] J. Bhattacharya, S. Bhattacharyya, and M. Rangamani, “Non-dissipative hydrodynamics: Effective actions versus entropy current,” *JHEP* **1302** (2013) 153, [arXiv:1211.1020 \[hep-th\]](#).
 - [24] F. M. Haehl, R. Loganayagam, and M. Rangamani, “Effective actions for anomalous hydrodynamics,” *JHEP* **1403** (2014) 034, [arXiv:1312.0610 \[hep-th\]](#).
 - [25] G. Ballesteros, “The effective theory of fluids at NLO and implications for dark energy,” *JCAP* **1503** no. 03, (2015) 001, [arXiv:1410.2793 \[hep-th\]](#).
 - [26] S. Endlich, A. Nicolis, and J. Wang, “Solid Inflation,” *JCAP* **1310** (2013) 011, [arXiv:1210.0569 \[hep-th\]](#).
 - [27] D. Nickel and D. T. Son, “Deconstructing holographic liquids,” *New J.Phys.* **13** (2011) 075010, [arXiv:1009.3094 \[hep-th\]](#).
 - [28] I. Low and A. V. Manohar, “Spontaneously broken space-time symmetries and Goldstone’s theorem,” *Phys.Rev.Lett.* **88** (2002) 101602, [arXiv:hep-th/0110285 \[hep-th\]](#).
 - [29] A. Nicolis, R. Penco, and R. A. Rosen, “Relativistic Fluids, Superfluids, Solids and Supersolids from a Coset Construction,” *Phys.Rev.* **D89** no. 4, (2014) 045002, [arXiv:1307.0517 \[hep-th\]](#).
 - [30] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, *et al.*, “Towards hydrodynamics without an entropy current,” *Phys.Rev.Lett.* **109** (2012) 101601, [arXiv:1203.3556 \[hep-th\]](#).
 - [31] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, *et al.*, “Constraints on Fluid Dynamics from Equilibrium Partition Functions,” *JHEP* **1209** (2012) 046, [arXiv:1203.3544 \[hep-th\]](#).
 - [32] S. Bhattacharyya, “Entropy current and equilibrium partition function in fluid dynamics,” *JHEP* **1408** (2014) 165, [arXiv:1312.0220 \[hep-th\]](#).
 - [33] S. Bhattacharyya, “Entropy Current from Partition Function: One Example,” *JHEP* **1407** (2014) 139, [arXiv:1403.7639 \[hep-th\]](#).
 - [34] F. M. Haehl, R. Loganayagam, and M. Rangamani, “Adiabatic hydrodynamics: The eightfold way to dissipation,” [arXiv:1502.00636 \[hep-th\]](#).
 - [35] F. M. Haehl, R. Loganayagam, and M. Rangamani, “The eightfold way to

- dissipation,” [arXiv:1412.1090 \[hep-th\]](#).
- [36] S. Hawking, “Black hole explosions,” *Nature* **248** (1974) 30–31.
 - [37] L. Ferrarese and H. Ford, “Supermassive black holes in galactic nuclei: Past, present and future research,” *Space Sci.Rev.* **116** (2005) 523–624, [arXiv:astro-ph/0411247 \[astro-ph\]](#).
 - [38] K. Riles, “Gravitational Waves: Sources, Detectors and Searches,” *Prog.Part.Nucl.Phys.* **68** (2013) 1–54, [arXiv:1209.0667 \[hep-ex\]](#).
 - [39] **Perimeter Institute for Theoretical Physics** Collaboration, A. E. Broderick *et al.*, “The Event Horizon of M87,” *Astrophys.J.* **805** no. 2, (2015) 179, [arXiv:1503.03873 \[astro-ph.HE\]](#).
 - [40] H. R. S. and R. R., “Lines of force of a point charge near a Schwarzschild black hole,” *Phys.Rev.* **D 8** (1973) 3259.
 - [41] R. Znajek, “The electric and magnetic conductivity of a Kerr hole,” *Mon.Not.Roy.Astron.Soc.* **185** (1978) 833.
 - [42] T. Damour, “Black Hole Eddy Currents,” *Phys.Rev.* **D18** (1978) 3598–3604.
 - [43] T. Damour, “Quelques proprietes mecaniques, electromagnetiques, thermodynamiques et quantiques des trous noirs,” *These de Doctorat d’Etat, Universite Pierre et Marie Curie, Paris VI* (1979) .
 - [44] T. Damour, “Surface effects in black hole physics,” *Proceedings of the second Marcel Grossman Meeting on General Relativity, Ed. R. Ruffini,(North Holland-Amsterdam)* (1982) 587.
 - [45] D. Macdonald and K. S. Thorne, “Black Hole Electrodynamics: an absolute-space/universal-time formulation ,” *Mon. Not. R. astr. Soc.* **198** (1982) 345–382.
 - [46] R. H. Price and K. S. Thorne, “Membrane viewpoint on black holes:properties and evolution of the stretched horizon,” *Phys. Rev.* **D33** (1986) 915–941.
 - [47] R. F. Penna, R. Narayan, and A. Sadowski, “General Relativistic Magnetohydrodynamic Simulations of Blandford-Znajek Jets and the Membrane Paradigm,” [arXiv:1307.4752 \[astro-ph.HE\]](#).
 - [48] K. S. Thorne, R. Price, and D. Macdonald, *Black Holes: The Membrane Paradigm*. The Silliman Memorial Lecture Series. Yale University Press, 1986.

- [49] S. Bhattacharyya, A. De, S. Minwalla, R. Mohan, and A. Saha, “A membrane paradigm at large D ,” [arXiv:1504.06613 \[hep-th\]](#).
- [50] R. Emparan, T. Shiromizu, R. Suzuki, K. Tanabe, and T. Tanaka, “Effective theory of Black Holes in the $1/D$ expansion,” [arXiv:1504.06489 \[hep-th\]](#).
- [51] R. Emparan, R. Suzuki, and K. Tanabe, “The large D limit of General Relativity,” *JHEP* **1306** (2013) 009, [arXiv:1302.6382 \[hep-th\]](#).
- [52] N. Iqbal and H. Liu, “Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm,” *Phys.Rev.* **D79** (2009) 025023, [arXiv:0809.3808 \[hep-th\]](#).
- [53] A. Einstein and N. Rosen, “The Particle Problem in the General Theory of Relativity,” *Phys. Rev.* **48** (Jul, 1935) 73–77.
<http://link.aps.org/doi/10.1103/PhysRev.48.73>.
- [54] M. Parikh and F. Wilczek, “An Action for black hole membranes,” *Phys.Rev.* **D58** (1998) 064011, [arXiv:gr-qc/9712077 \[gr-qc\]](#).
- [55] C. Herzog and D. Son, “Schwinger-Keldysh propagators from AdS/CFT correspondence,” *JHEP* **0303** (2003) 046, [arXiv:hep-th/0212072 \[hep-th\]](#).
- [56] G. 't Hooft, “Dimensional reduction in quantum gravity,” [arXiv:gr-qc/9310026 \[gr-qc\]](#).
- [57] L. Susskind, “The World as a hologram,” *J.Math.Phys.* **36** (1995) 6377–6396, [arXiv:hep-th/9409089 \[hep-th\]](#).
- [58] R. Bousso, “The Holographic principle,” *Rev.Mod.Phys.* **74** (2002) 825–874, [arXiv:hep-th/0203101 \[hep-th\]](#).
- [59] J. Bekenstein, “Black holes and the second law,” *Lett.Nuovo Cim.* **4** (1972) 737–740.
- [60] J. D. Bekenstein, “Black holes and entropy,” *Phys.Rev.* **D7** (1973) 2333–2346.
- [61] J. D. Bekenstein, “Generalized second law of thermodynamics in black hole physics,” *Phys.Rev.* **D9** (1974) 3292–3300.
- [62] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv.Theor.Math.Phys.* **2** (1998) 231–252, [arXiv:hep-th/9711200 \[hep-th\]](#).

- [63] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” *Phys.Rept.* **323** (2000) 183–386, [arXiv:hep-th/9905111](#) [[hep-th](#)].
- [64] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS / CFT correspondence,” [arXiv:hep-th/0201253](#) [[hep-th](#)].
- [65] A. Zaffaroni, *Introduction to the AdS/CFT correspondence*.
<http://laces.web.cern.ch/Laces/LACES09/notes/dbranes/lezioniLosanna.pdf>.
- [66] J. McGreevy, “Holographic duality with a view toward many-body physics,” *Adv.High Energy Phys.* **2010** (2010) 723105, [arXiv:0909.0518](#) [[hep-th](#)].
- [67] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” *Class.Quant.Grav.* **26** (2009) 224002, [arXiv:0903.3246](#) [[hep-th](#)].
- [68] S. A. Hartnoll, “Horizons, holography and condensed matter,” [arXiv:1106.4324](#) [[hep-th](#)].
- [69] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. A. Wiedemann, “Gauge/String Duality, Hot QCD and Heavy Ion Collisions,” [arXiv:1101.0618](#) [[hep-th](#)].
- [70] K. Skenderis, “Lecture notes on holographic renormalization,” *Class. Quant. Grav.* **19** (2002) 5849–5876, [hep-th/0209067](#).
- [71] S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys.Lett.* **B428** (1998) 105–114, [arXiv:hep-th/9802109](#) [[hep-th](#)].
- [72] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150](#).
- [73] J. W. York, “Role of Conformal Three-Geometry in the Dynamics of Gravitation,” *Phys. Rev. Lett.* **28** (Apr, 1972) 1082–1085.
<http://link.aps.org/doi/10.1103/PhysRevLett.28.1082>.
- [74] G. W. Gibbons and S. W. Hawking, “Action integrals and partition functions in quantum gravity,” *Phys. Rev. D* **15** (May, 1977) 2752–2756.
<http://link.aps.org/doi/10.1103/PhysRevD.15.2752>.
- [75] J. de Boer, E. P. Verlinde, and H. L. Verlinde, “On the holographic renormalization group,” *JHEP* **0008** (2000) 003, [arXiv:hep-th/9912012](#) [[hep-th](#)].

- [76] I. Papadimitriou and K. Skenderis, “AdS / CFT correspondence and geometry,” *Proceedings Strasbourg 2003, AdS/CFT correspondence*, ed. O. Biquard 73, [hep-th/0404176](#).
- [77] I. Papadimitriou, “Holographic renormalization as a canonical transformation,” *JHEP* **1011** (2010) 014, [arXiv:1007.4592 \[hep-th\]](#).
- [78] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” *JHEP* **07** (1998) 023, [arXiv:hep-th/9806087](#).
- [79] M. Henningson and K. Skenderis, “Holography and the Weyl anomaly,” *Fortsch.Phys.* **48** (2000) 125–128, [arXiv:hep-th/9812032 \[hep-th\]](#).
- [80] V. Balasubramanian and P. Kraus, “A stress tensor for Anti-de Sitter gravity,” *Commun. Math. Phys.* **208** (1999) 413–428, [arXiv:hep-th/9902121](#).
- [81] S. de Haro, S. N. Solodukhin, and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” *Commun. Math. Phys.* **217** (2001) 595–622, [arXiv:hep-th/0002230](#).
- [82] J. D. Brown and J. W. J. York, “Quasilocal energy and conserved charges derived from the gravitational action,” *Phys.Rev.* **D47** (1993) 1407–1419, [arXiv:gr-qc/9209012 \[gr-qc\]](#).
- [83] G. Policastro, D. T. Son, and A. O. Starinets, “The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma,” *Phys.Rev.Lett.* **87** (2001) 081601, [arXiv:hep-th/0104066 \[hep-th\]](#).
- [84] G. Policastro, D. T. Son, and A. O. Starinets, “From AdS / CFT correspondence to hydrodynamics,” *JHEP* **0209** (2002) 043, [arXiv:hep-th/0205052 \[hep-th\]](#).
- [85] G. Policastro, D. T. Son, and A. O. Starinets, “From AdS / CFT correspondence to hydrodynamics. 2. Sound waves,” *JHEP* **0212** (2002) 054, [arXiv:hep-th/0210220 \[hep-th\]](#).
- [86] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” *JHEP* **0802** (2008) 045, [arXiv:0712.2456 \[hep-th\]](#).
- [87] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, “Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions,” *JHEP* **0812** (2008) 116, [arXiv:0809.4272 \[hep-th\]](#).

- [88] D. T. Son and A. O. Starinets, “Viscosity, Black Holes, and Quantum Field Theory,” *Ann.Rev.Nucl.Part.Sci.* **57** (2007) 95–118, [arXiv:0704.0240 \[hep-th\]](#).
- [89] M. Rangamani, “Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence,” *Class.Quant.Grav.* **26** (2009) 224003, [arXiv:0905.4352 \[hep-th\]](#).
- [90] A.S.Eddington, “A Comparison of Whitehead’s and Einstein’s Formulae ,” *Nature* **113** no. 192, (1924) .
- [91] D. Finkelstein, “Past-Future Asymmetry of the Gravitational Field of a Point Particle,” *Phys. Rev.* **110** (May, 1958) 965–967.
<http://link.aps.org/doi/10.1103/PhysRev.110.965>.
- [92] A. Buchel and J. T. Liu, “Universality of the shear viscosity in supergravity,” *Phys.Rev.Lett.* **93** (2004) 090602, [arXiv:hep-th/0311175 \[hep-th\]](#).
- [93] A. O. Starinets, “Quasinormal spectrum and the black hole membrane paradigm,” *Phys.Lett.* **B670** (2009) 442–445, [arXiv:0806.3797 \[hep-th\]](#).
- [94] S. C. Huot, S. Jeon, and G. D. Moore, “Shear viscosity in weakly coupled $N = 4$ super Yang-Mills theory compared to QCD,” *Phys.Rev.Lett.* **98** (2007) 172303, [arXiv:hep-ph/0608062 \[hep-ph\]](#).
- [95] P. B. Arnold, G. D. Moore, and L. G. Yaffe, “Transport coefficients in high temperature gauge theories. 2. Beyond leading log,” *JHEP* **0305** (2003) 051, [arXiv:hep-ph/0302165 \[hep-ph\]](#).
- [96] H. Song, S. A. Bass, U. Heinz, T. Hirano, and C. Shen, “200 A GeV Au+Au collisions serve a nearly perfect quark-gluon liquid,” *Phys.Rev.Lett.* **106** (2011) 192301, [arXiv:1011.2783 \[nucl-th\]](#).
- [97] S. Kachru, X. Liu, and M. Mulligan, “Gravity duals of Lifshitz-like fixed points,” *Phys.Rev.* **D78** (2008) 106005, [arXiv:0808.1725 \[hep-th\]](#).
- [98] D. Son, “Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry,” *Phys.Rev.* **D78** (2008) 046003, [arXiv:0804.3972 \[hep-th\]](#).
- [99] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” *Phys.Rev.Lett.* **101** (2008) 061601, [arXiv:0804.4053 \[hep-th\]](#).
- [100] R. C. Costa, “Holographic Reconstruction and Renormalization in Asymptotically Ricci-flat Spacetimes,” *JHEP* **1211** (2012) 046,

- arXiv:1206.3142 [hep-th].
- [101] M. Gary, D. Grumiller, and R. Rashkov, “Towards non-AdS holography in 3-dimensional higher spin gravity,” *JHEP* **1203** (2012) 022, arXiv:1201.0013 [hep-th].
 - [102] M. M. Caldarelli, J. Camps, B. Gouteraux, and K. Skenderis, “AdS/Ricci-flat correspondence and the Gregory-Laflamme instability,” *Phys.Rev.* **D87** (2013) 061502, arXiv:1211.2815 [hep-th].
 - [103] M. Caldarelli, J. Camps, B. Gouteraux, and K. Skenderis, “AdS/Ricci-flat correspondence,” arXiv:1312.7874 [hep-th].
 - [104] R. Caldeira Costa, “Aspects of the zero Λ limit in the AdS/CFT correspondence,” *Phys.Rev.* **D90** no. 10, (2014) 104018, arXiv:1311.7339 [hep-th].
 - [105] S. de Haro, K. Skenderis, and S. N. Solodukhin, “Gravity in warped compactifications and the holographic stress tensor,” *Class.Quant.Grav.* **18** (2001) 3171–3180, arXiv:hep-th/0011230 [hep-th].
 - [106] W. Li and T. Takayanagi, “Holography and Entanglement in Flat Spacetime,” *Phys.Rev.Lett.* **106** (2011) 141301, arXiv:1010.3700 [hep-th].
 - [107] A. Strominger, “The dS / CFT correspondence,” *JHEP* **0110** (2001) 034, arXiv:hep-th/0106113 [hep-th].
 - [108] E. Witten, “Quantum gravity in de Sitter space,” arXiv:hep-th/0106109 [hep-th].
 - [109] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, “Wilsonian Approach to Fluid/Gravity Duality,” *JHEP* **1103** (2011) 141, arXiv:1006.1902 [hep-th].
 - [110] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, “From Navier-Stokes To Einstein,” *JHEP* **1207** (2012) 146, arXiv:1101.2451 [hep-th].
 - [111] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, “The Holographic fluid dual to vacuum Einstein gravity,” *JHEP* **1107** (2011) 050, arXiv:1103.3022 [hep-th].
 - [112] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, “The relativistic fluid dual to vacuum Einstein gravity,” *JHEP* **1203** (2012) 076, arXiv:1201.2678 [hep-th].
 - [113] C. Eling, A. Meyer, and Y. Oz, “The Relativistic Rindler Hydrodynamics,”

- JHEP* **1205** (2012) 116, [arXiv:1201.2705 \[hep-th\]](#).
- [114] A. Meyer and Y. Oz, “Constraints on Rindler Hydrodynamics,” *JHEP* **1307** (2013) 090, [arXiv:1304.6305 \[hep-th\]](#).
 - [115] G. Chirco, C. Eling, and S. Liberati, “Higher Curvature Gravity and the Holographic fluid dual to flat spacetime,” *JHEP* **1108** (2011) 009, [arXiv:1105.4482 \[hep-th\]](#).
 - [116] C. Eling, A. Meyer, and Y. Oz, “Local Entropy Current in Higher Curvature Gravity and Rindler Hydrodynamics,” *JHEP* **1208** (2012) 088, [arXiv:1205.4249 \[hep-th\]](#).
 - [117] D.-C. Zou, S.-J. Zhang, and B. Wang, “Holographic charged fluid dual to third order Lovelock gravity,” *Phys.Rev.* **D87** no. 8, (2013) 084032, [arXiv:1302.0904 \[hep-th\]](#).
 - [118] I. Bredberg and A. Strominger, “Black Holes as Incompressible Fluids on the Sphere,” *JHEP* **1205** (2012) 043, [arXiv:1106.3084 \[hep-th\]](#).
 - [119] T.-Z. Huang, Y. Ling, W.-J. Pan, Y. Tian, and X.-N. Wu, “From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime,” *JHEP* **1110** (2011) 079, [arXiv:1107.1464 \[gr-qc\]](#).
 - [120] R. Nakayama, “The Holographic Fluid on the Sphere Dual to the Schwarzschild Black Hole,” [arXiv:1109.1185 \[hep-th\]](#).
 - [121] D. Anninos, T. Anous, I. Bredberg, and G. S. Ng, “Incompressible Fluids of the de Sitter Horizon and Beyond,” *JHEP* **1205** (2012) 107, [arXiv:1110.3792 \[hep-th\]](#).
 - [122] C. Niu, Y. Tian, X.-N. Wu, and Y. Ling, “Incompressible Navier-Stokes Equation from Einstein-Maxwell and Gauss-Bonnet-Maxwell Theories,” *Phys.Lett.* **B711** (2012) 411–416, [arXiv:1107.1430 \[hep-th\]](#).
 - [123] C.-Y. Zhang, Y. Ling, C. Niu, Y. Tian, and X.-N. Wu, “Magnetohydrodynamics from gravity,” *Phys.Rev.* **D86** (2012) 084043, [arXiv:1204.0959 \[hep-th\]](#).
 - [124] V. Lysov, “On the Magnetohydrodynamics/Gravity Correspondence,” [arXiv:1310.4181 \[hep-th\]](#).
 - [125] D. Brattan, J. Camps, R. Loganayagam, and M. Rangamani, “CFT dual of the AdS Dirichlet problem : Fluid/Gravity on cut-off surfaces,” *JHEP* **1112** (2011) 090, [arXiv:1106.2577 \[hep-th\]](#).
 - [126] R.-G. Cai, L. Li, and Y.-L. Zhang, “Non-Relativistic Fluid Dual to

- Asymptotically AdS Gravity at Finite Cutoff Surface,” *JHEP* **1107** (2011) 027, [arXiv:1104.3281](#) [[hep-th](#)].
- [127] D. T. Son and A. O. Starinets, “Minkowski space correlators in AdS / CFT correspondence: Recipe and applications,” *JHEP* **0209** (2002) 042, [arXiv:hep-th/0205051](#) [[hep-th](#)].
- [128] J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory Principles and Applications*. Cambridge Monographs on Mathematical Physics, 2006.
- [129] K. Skenderis and B. C. van Rees, “Real-time gauge/gravity duality,” *Phys.Rev.Lett.* **101** (2008) 081601, [arXiv:0805.0150](#) [[hep-th](#)].
- [130] K. Skenderis and B. C. van Rees, “Real-time gauge/gravity duality: Prescription, Renormalization and Examples,” [arXiv:0812.2909](#) [[hep-th](#)].
- [131] D. Birmingham, I. Sachs, and S. N. Solodukhin, “Conformal field theory interpretation of black hole quasinormal modes,” *Phys.Rev.Lett.* **88** (2002) 151301, [arXiv:hep-th/0112055](#) [[hep-th](#)].
- [132] E. Berti, V. Cardoso, and A. O. Starinets, “Quasinormal modes of black holes and black branes,” *Class.Quant.Grav.* **26** (2009) 163001, [arXiv:0905.2975](#) [[gr-qc](#)].
- [133] M. Banados, C. Teitelboim, and J. Zanelli, “The Black hole in three-dimensional space-time,” *Phys.Rev.Lett.* **69** (1992) 1849–1851, [arXiv:hep-th/9204099](#) [[hep-th](#)].
- [134] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, “Relativistic viscous hydrodynamics, conformal invariance, and holography,” *JHEP* **0804** (2008) 100, [arXiv:0712.2451](#) [[hep-th](#)].
- [135] P. K. Kovtun and A. O. Starinets, “Quasinormal modes and holography,” *Phys.Rev.* **D72** (2005) 086009, [arXiv:hep-th/0506184](#) [[hep-th](#)].
- [136] L. Susskind and E. Witten, “The Holographic bound in anti-de Sitter space,” [arXiv:hep-th/9805114](#) [[hep-th](#)].
- [137] V. Balasubramanian and P. Kraus, “Space-time and the holographic renormalization group,” *Phys.Rev.Lett.* **83** (1999) 3605–3608, [arXiv:hep-th/9903190](#) [[hep-th](#)].
- [138] I. Heemskerk and J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” *JHEP* **1106** (2011) 031, [arXiv:1010.1264](#) [[hep-th](#)].

- [139] T. Faulkner, H. Liu, and M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” *JHEP* **1108** (2011) 051, [arXiv:1010.4036 \[hep-th\]](#).
- [140] S.-J. Sin and Y. Zhou, “Holographic Wilsonian RG Flow and Sliding Membrane Paradigm,” *JHEP* **1105** (2011) 030, [arXiv:1102.4477 \[hep-th\]](#).
- [141] V. Balasubramanian, M. Guica, and A. Lawrence, “Holographic Interpretations of the Renormalization Group,” *JHEP* **1301** (2013) 115, [arXiv:1211.1729 \[hep-th\]](#).
- [142] K. Wilson and J. Kogut, “The renormalisation group and the ϵ expansion,” *Physics Reports* **12** no. 2, (1974) 75–199.
- [143] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, “Effective field theory for massive gravitons and gravity in theory space,” *Annals Phys.* **305** (2003) 96–118, [arXiv:hep-th/0210184 \[hep-th\]](#).
- [144] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” *Prog.Theor.Phys.* **113** (2005) 843–882, [arXiv:hep-th/0412141 \[hep-th\]](#).
- [145] R. Arnowitt, S. Deser, and C. W. Misner, “Dynamical Structure and Definition of Energy in General Relativity,” *Physical Review* **116** (Dec., 1959) 1322–1330.
- [146] J. Hartle and S. Hawking, “Wave Function of the Universe,” *Phys.Rev.* **D28** (1983) 2960–2975.
- [147] M. Crossley, P. Glorioso, H. Liu, and Y. Wang, “Off-shell hydrodynamics from holography,” [arXiv:1504.07611 \[hep-th\]](#).
- [148] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, *et al.*, “Local Fluid Dynamical Entropy from Gravity,” *JHEP* **0806** (2008) 055, [arXiv:0803.2526 \[hep-th\]](#).
- [149] I. Booth, M. P. Heller, and M. Spalinski, “Black Brane Entropy and Hydrodynamics,” *Phys.Rev.* **D83** (2011) 061901, [arXiv:1010.6301 \[hep-th\]](#).
- [150] T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, “Emergent quantum criticality, Fermi surfaces, and AdS(2),” *Phys.Rev.* **D83** (2011) 125002, [arXiv:0907.2694 \[hep-th\]](#).
- [151] T. Faulkner and J. Polchinski, “Semi-Holographic Fermi Liquids,” *JHEP*

- 1106** (2011) 012, [arXiv:1001.5049 \[hep-th\]](#).
- [152] V. Balasubramanian, B. Czech, B. D. Chowdhury, and J. de Boer, “The entropy of a hole in spacetime,” *JHEP* **1310** (2013) 220, [arXiv:1305.0856 \[hep-th\]](#).
- [153] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer, and M. P. Heller, “Bulk curves from boundary data in holography,” *Phys.Rev.* **D89** no. 8, (2014) 086004, [arXiv:1310.4204 \[hep-th\]](#).
- [154] S. Hassan and R. A. Rosen, “Bimetric Gravity from Ghost-free Massive Gravity,” *JHEP* **1202** (2012) 126, [arXiv:1109.3515 \[hep-th\]](#).
- [155] R. M. Wald, “Black hole entropy is the Noether charge,” *Phys.Rev.* **D48** (1993) 3427–3431, [arXiv:gr-qc/9307038 \[gr-qc\]](#).
- [156] C. Adam, T. Klöhn, C. Naya, J. Sanchez-Guillen, R. Vazquez, *et al.*, “Baryon chemical potential and in-medium properties of BPS skyrmions,” [arXiv:1504.05185 \[hep-th\]](#).
- [157] R. Emparan, V. E. Hubeny, and M. Rangamani, “Effective hydrodynamics of black D3-branes,” *JHEP* **1306** (2013) 035, [arXiv:1303.3563 \[hep-th\]](#).
- [158] S. Kuperstein and A. Mukhopadhyay, “The unconditional RG flow of the relativistic holographic fluid,” *JHEP* **1111** (2011) 130, [arXiv:1105.4530 \[hep-th\]](#).
- [159] S. Kuperstein and A. Mukhopadhyay, “Spacetime emergence via holographic RG flow from incompressible Navier-Stokes at the horizon,” *JHEP* **1311** (2013) 086, [arXiv:1307.1367 \[hep-th\]](#).
- [160] I. Kanitscheider and K. Skenderis, “Universal hydrodynamics of non-conformal branes,” *JHEP* **04** (2009) 062, [arXiv:0901.1487 \[hep-th\]](#).
- [161] A. H. Taub, “Empty Space-Times Admitting a Three Parameter Group of Motions,” *Annals of Mathematics* **53** (1951) 472.
- [162] M. Taylor, “Non-relativistic holography,” [arXiv:0812.0530 \[hep-th\]](#).
- [163] U. H. Danielsson and L. Thorlacius, “Black holes in asymptotically Lifshitz spacetime,” *JHEP* **0903** (2009) 070, [arXiv:0812.5088 \[hep-th\]](#).
- [164] G. Bertoldi, B. A. Burrington, and A. Peet, “Black Holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent,” *Phys.Rev.* **D80** (2009) 126003, [arXiv:0905.3183 \[hep-th\]](#).
- [165] G. Bertoldi, B. A. Burrington, and A. W. Peet, “Thermodynamics of black

- branes in asymptotically Lifshitz spacetimes,” *Phys.Rev.* **D80** (2009) 126004, [arXiv:0907.4755 \[hep-th\]](#).
- [166] G. Bertoldi, B. A. Burrington, A. W. Peet, and I. G. Zadeh, “Lifshitz-like black brane thermodynamics in higher dimensions,” *Phys.Rev.* **D83** (2011) 126006, [arXiv:1101.1980 \[hep-th\]](#).
- [167] J. Erdmenger, M. Rangamani, S. Steinfurt, and H. Zeller, “Hydrodynamic Regimes of Spinning Black D3-Branes,” *JHEP* **1502** (2015) 026, [arXiv:1412.0020 \[hep-th\]](#).
- [168] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, “World-Volume Effective Theory for Higher-Dimensional Black Holes,” *Phys.Rev.Lett.* **102** (2009) 191301, [arXiv:0902.0427 \[hep-th\]](#).
- [169] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, “Blackfolds in Supergravity and String Theory,” *JHEP* **1108** (2011) 154, [arXiv:1106.4428 \[hep-th\]](#).
- [170] A. Nunez and A. O. Starinets, “AdS / CFT correspondence, quasinormal modes, and thermal correlators in N=4 SYM,” *Phys.Rev.* **D67** (2003) 124013, [arXiv:hep-th/0302026 \[hep-th\]](#).
- [171] P. Kovtun, D. T. Son, and A. O. Starinets, “Holography and hydrodynamics: Diffusion on stretched horizons,” *JHEP* **0310** (2003) 064, [arXiv:hep-th/0309213 \[hep-th\]](#).
- [172] E. Gourgoulhon, “A Generalized Damour-Navier-Stokes equation applied to trapping horizons,” *Phys.Rev.* **D72** (2005) 104007, [arXiv:gr-qc/0508003 \[gr-qc\]](#).
- [173] S. Hayward, “General laws of black hole dynamics,” *Phys.Rev.* **D49** (1994) 6467–6474.
- [174] E. Gourgoulhon and J. L. Jaramillo, “A 3+1 perspective on null hypersurfaces and isolated horizons,” *Phys.Rept.* **423** (2006) 159–294, [arXiv:gr-qc/0503113 \[gr-qc\]](#).
- [175] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, “Black Holes: Complementarity or Firewalls?,” *JHEP* **1302** (2013) 062, [arXiv:1207.3123 \[hep-th\]](#).

SAMENVATTING

Fluïda

Het water in de Amsterdamse grachten, de lucht in de atmosfeer en bijvoorbeeld honing hebben tenminste een gemeenschappelijke eigenschap: het zijn allemaal *fluïda*. Een fluïdum kan betrekking hebben op een vloeistof, een gas of een plasma. Zij hebben de neiging de ruimte waarin zij zijn besloten te vullen, in tegenstelling tot vaste stoffen, die hun eigen vorm behouden.

Het vakgebied dat zich bezighoudt met de wiskundige beschrijving van bewegende fluïda wordt aangeduid als de *vloeistofdynamica* of de *stromingsleer*. Een belangrijke aanname in de theoretische beschrijving van fluïda is dat de bijdrage van individuele deeltjes kan worden verwaarloosd ten opzichte van het collectieve gedrag van het medium, waarvoor een effectieve beschrijving kan worden gegeven. Een voorbeeld van dit principe ziet men in de beschrijving van de golven die ontstaan als een steen in het water valt. De vloeistofdynamica beschrijft de golven en niet de bewegingen van de individuele watermoleculen; dat zou onnodig lastig zijn. De stromingsleer geeft een goede en efficiënte beschrijving in tal van toepassingen, bijvoorbeeld bij het voorspellen van het weer, het beschrijven van de oceaanstromingen of bij de studie van luchtstromen die aan een oppervlak grenzen, zoals de lucht die langs de vleugel van een vliegtuig stroomt.

Men zou kunnen veronderstellen dat het gedrag van fluïda universeel zou moeten zijn wanneer het individuele gedrag van deeltjes kan worden genegeerd. Dat is gedeeltelijk waar, in de zin dat ieder fluïdum dat in equilibrium is golfverschijnselen laat zien wanneer het wordt verstoord. Daarnaast zien we dat deze golven in water een heel andere snelheid hebben dan in bijvoorbeeld honing. Dus ook al geeft de vloeistofdynamica een algemene beschrijving van een fluïdum, een aantal

eigenschappen van een fluïdum moet worden gespecificeerd. Een voorbeeld van zo een intrinsieke eigenschap is de *viscositeit*. Viscositeit is de mate waarin een fluïdum de beweging van een aangrenzend object doorgeeft, ofwel hoe dissipatief een fluïdum is. In een viscoos fluïdum, zoals honing, zullen aangrenzende delen een relatief vergelijkbare snelheid hebben. Water heeft een lage viscositeit, zodat aangrenzende delen relatief verschillende snelheden kunnen hebben doordat de beweging niet efficiënt wordt doorgegeven door het medium. Dit soort intrinsieke eigenschappen volgen niet uit de effectieve vloeistofdynamische beschrijving, omdat ze voortkomen uit de specifieke interacties tussen de individuele deeltjes. In de vloeistofdynamische beschrijving moeten de parameters van deze eigenschappen van tevoren worden gegeven. Deze parameters kunnen worden bepaald door middel van experimenten of op basis van berekeningen. Deze berekeningen moeten worden gedaan met behulp van een fundamentele beschrijving van de deeltjes en hun interacties. In sommige gevallen kan dit worden gedaan, maar wanneer de interacties tussen de deeltjes sterk zijn, zijn deze berekeningen erg moeilijk. De meeste rekentechnieken werken juist goed in het regime waarin de interacties tussen de deeltjes relatief klein zijn.

Naast de bovengenoemde voorbeelden van bekendere vloeistoffen komen er ook meer exotische vloeistoffen voor in bijvoorbeeld de relativistische Heavy Ion Collider (RHIC) in Brookhaven in de Verenigde Staten en in de Large Hadron Collider (LHC) in Genève in Zwitserland. In beide laboratoria worden experimenten gedaan waarbij bundels atomen met grote snelheden op elkaar botsen. De energie is daarbij dermate hoog dat de atomen na de botsing oplossen in hun elementaire bestanddelen, de zogenaamde quarks en gluonen. Gedurende een fractie van een seconde, voordat de elementaire deeltjes weer samengaan, gedragen zij zich als een sterk gekoppelde vloeistof bij een zeer hoge temperatuur en een zeer hoge dichtheid. Deze exotische vloeistof wordt ook wel *quark-gluon plasma* (QGP) genoemd⁸. De omstandigheden die in deze experimenten worden gecreëerd zijn vergelijkbaar met de omstandigheden vlak na de Oerknal, toen ons universum nog te heet was voor de vorming van atomen en het quark-gluon plasma dominant was. Het experimenteren met en het beschrijven van een quark-gluon plasma geeft ons dus een inkijkje in een zeer vroeg stadium van het heelal en kan nuttig zijn bij het verklaren van waarnemingen. Echter, als we de vloeistofdynamische beschrijving willen vergelijken met waarnemingen is het noodzakelijk dat de parameters van de intrinsieke eigenschappen van de vloeistof, het quark-gluon plasma, bekend zijn. Aangezien het quark-gluon plasma een sterk gekoppelde vloeistof is, is de

⁸Een plasma is een toestandsvorm die binnen de definitie van fluïda valt. Een verschil tussen een vloeistof of een gas en een plasma is dat het plasma bestaat uit geladen deeltjes, zoals bij een elektromagnetisch plasma. De deeltjes in het quark-gluon plasma hebben ook een lading, maar dan met betrekking tot de quantum-chromodynamische interacties.

berekening van bijvoorbeeld de viscositeitscoëfficiënt erg moeilijk.

Fluïda in zwaartekracht

Vloeistofgedrag speelt ook een rol in de context van de zwaartekracht en zwarte gaten. Zwarte gaten werden al een eeuw geleden voorspeld, maar zijn niet direct waargenomen. Er wordt verondersteld dat er zich een zwart gat bevindt in het midden van ieder melkwegstelsel. De belangrijkste eigenschap van een zwart gat is dat niets aan de zwaartekracht kan ontsnappen wanneer de zogenaamde gebeurtenissenhorizon is overschreden; zelfs licht niet. De gebeurtenissenhorizon kan veranderen met de tijd. Een zwart gat wordt groter wanneer het materie absorbeert. Als we iets in een zwart gat zouden kunnen gooien, dan zou de geometrie en in het bijzonder de gebeurtenissenhorizon beginnen te fluctueren en zouden zwaartekrachtsgolven worden gecreëerd, op vergelijkbare manier als waarop golven ontstaan wanneer we een steen in het water gooien. Deze fluctuaties zouden zich met een bepaalde snelheid voortplanten totdat het systeem terugkeert naar een evenwichtstoestand. Men zou kunnen suggereren dat een zwart gat, dat objecten die de gebeurtenissenhorizon overschrijden opslokt, een natuurlijk dissipatief object is. Dit gedrag komt ook voor in de vloeistofdynamica. Deze observatie heeft geleid tot pogingen om deze analogie preciezer te maken.

De eerste pogingen leidden in de jaren tachtig tot de formulering van het zogenaamde *membraan paradigma*. Het membraan paradigma stelt dat we een generiek zwart gat kunnen modelleren door het zwarte gat te vervangen met een membraan, dat aan nog een aantal verder te specificeren eigenschappen voldoet. In één specifieke formulering van het membraan paradigma gedraagt het membraan zich als een vloeistof, die ook wel *membraan fluïdum* wordt genoemd. Deze vloeistof heeft een negatieve viscositeit, wat een opmerkelijke eigenschap is, want dat zou namelijk betekenen dat het membraan energie levert aan de vloeistofstroom. Het is de vraag of we kunnen spreken van een echte fysieke vloeistof, of dat het vloeistofmodel simpelweg een handig model is van een gebeurtenissenhorizon. In dit proefschrift betogen we dat het membraan niet als een echte vloeistof kan worden gezien; het concept van een membraan als vloeistof is dus misleidend. Het raamwerk van de holografische zwaartekracht, dat we hieronder introduceren, biedt een beter perspectief om de relatie tussen vloeistofgedrag en zwaartekracht te verklaren.

Holografie, ontwikkeld in het midden van de jaren negentig, betreft het idee dat een zwaartekrachtstheorie holografisch is. Dat wil zeggen dat we een zwaartekrachtstheorie ook kunnen modelleren op een holografisch scherm. Een zwaartekrachtstheorie in drie dimensies kunnen we bijvoorbeeld wiskundig coderen op een tweedimensionaal hologram, waarbij het model op het holografische scherm geen zwaartekracht kent. Dit idee is precies gemaakt voor een specifiek type geometrie, namelijk de zogenaamde Anti de Sitter (AdS) ruimtetijd. Het is mogelijk om een *holografisch woordenboek* te construeren tussen de twee totaal verschillende gedaantes: de zwaartekrachtstheorie in de AdS ruimtetijd en het hologram dat bestaat uit een theorie zonder zwaartekracht op een ruimtetijd met één dimensie minder. Met andere woorden, er is een recept waarmee we het hologram kunnen (de-)coderen. Grootheden die gedefinieerd zijn in de zwaartekrachtstheorie kunnen worden gerelateerd aan grootheden die gedefinieerd zijn in de theorie met één dimensie minder. Een belangrijke eigenschap van holografie is dat het niet alleen een equivalentie geeft tussen twee theoriën, maar ook een *dualiteit*: als het ene model een sterke koppeling heeft, dan heeft het andere model een zwakke koppeling en andersom. Dat is handig, omdat we op deze manier via holografie in staat zijn om een sterk gekoppelde theorie te beschrijven in termen van een zwak gekoppelde theorie.

Er zijn zwarte gaten in AdS ruimtetijd die door middel van het holografische woordenboek geassocieerd kunnen worden aan een duale theorie die sterk gekoppeld is. Deze duale modellen beschrijven deeltjes die zich in een thermische toestand bevinden, waarbij de temperatuur eindig is. Een interessante observatie is dat fluctuaties van deze zwarte gaten kunnen worden gerelateerd aan het vloeistofgedrag van de deeltjes in het duale model. Dit collectieve gedrag van de deeltjes in het duale model lijkt bovendien erg op het gedrag van het quark-gluon plasma. De intrinsieke eigenschappen van de vloeistof kunnen nu expliciet berekend worden, door de berekening te doen in de zwaartekrachtstheorie en het resultaat met behulp van het holografische woordenboek uit te drukken in termen van het duale model. De viscositeit blijkt bijvoorbeeld positief en zeer klein te zijn, wat overeenkomt met de verwachtingen over het quark-gluon plasma. Holografie biedt dus niet alleen een raamwerk om vloeistofeigenschappen te verklaren, maar kan ook gebruikt worden om problemen op te lossen in sterk gekoppelde theoriën.

Motivatatie en de belangrijkste resultaten

Dit proefschrift is een reis in de wereld van de begrippen die we hierboven introduceerden. Wij zijn zowel geïnteresseerd in wat de zwaartekracht ons kan vertellen over de vloeistofdynamica, als in hoe de vloeistofdynamica ons kan helpen bij het begrijpen holografie.

Zo zijn er vele formuleringen van de vloeistofdynamica die allemaal dezelfde resultaten zouden moeten geven. Sommige van deze formuleringen zijn oud en bekend, maar er zijn ook nieuwere formuleringen waarvan niet duidelijk is hoe betrouwbaar ze zijn. In dit proefschrift dragen we hieraan bij doordat we voor zo een minder conventionele formulering van de vloeistofdynamica vloeistofgedrag vinden voor bepaalde zwarte gaten in AdS ruimtetijd. We bevestigen dat in de situaties die wij beschouwen bepaalde kenmerken van deze formuleringen correct zijn. Via holografie worden de eigenschappen van het duale model en het vloeistofgedrag beschreven, zodat het relatief makkelijk is om expliciete berekeningen te doen. Anderzijds zeggen onze resultaten ook iets over holografie zelf, aangezien we het holografische woordenboek uitbreiden met de onconventionele formuleringen van de vloeistofdynamica.

Als we veronderstellen dat elke zwaartekrachtstheorie holografisch is, is het noodzakelijk om de techniek van holografie te ontwikkelen voor geometrieën die algemener zijn dan alleen de AdS ruimtetijd. In dit proefschrift stellen we een holografisch woordenboek op dat geldig is in het regime van vloeistofgedrag, voor een aantal typen zwarte gaten in algemenere ruimtetijden die niet noodzakelijkerwijs van het Anti de Sitter type zijn. We doen dat met name op vlakken in het inwendige van de ruimtetijd, terwijl men gewoonlijk juist de rand van de ruimtetijd gebruikt, die zich op oneindige afstand bevindt. We laten zien dat wanneer dit oppervlak dichtbij de gebeurtenishorizon van een zwart gat is, de duale holografische vloeistof zich juist heel anders gedraagt dan het membraan fluïdum. Zo zijn er geen vreemde verschijnselen als een negatieve viscositeit. Daarmee beantwoorden we een aantal openstaande vragen met betrekking tot de vloeistofbeschrijving van het membraan. In het bijzonder concluderen we dat een holografische vloeistofbeschrijving van de nabije omgeving van de horizon beter werkt dan de beschrijving door middel van het membraan fluïdum.

Gelukkig hoeven we het membraan paradigma niet helemaal los te laten. Het membraan paradigma geeft een andere en meer algemene formulering. In dit proefschrift laten we zien dat deze formulering een goede benadering is zolang we het inwendige van het zwarte gat buiten beschouwing kunnen laten, zolang we in het vloeistofregime zijn. Er zijn echter ook andere regimes dan het vloeistofregime. Na het verstoren van een zwart gat ontstaan er fluctuaties. De snelste fluctuaties

verspreiden zich zeer snel. Andere fluctuaties zijn persistenter en kunnen worden geassocieerd met het vloeistofgedrag. Het begrijpen van de structuur van deze fluctuaties is belangrijk in bijvoorbeeld het astronomisch onderzoek naar zwarte gaten. Na het aanbrengen van een verstoring keert ieder zwart gat op zijn eigen specifieke manier terug naar een evenwichtstoestand; een zwart gat heeft zijn eigen specifieke set van kortdurende en langdurende trillingen. Deze trillingen kunnen in principe worden waargenomen en zij geven daarmee eigenlijk een vingerafdruk van het zwarte gat. In dit proefschrift geven we ook een algemeen argument waarmee we laten zien dat het membraan paradigma gedeeltelijk faalt in het beschrijven van de hierboven genoemde snelle fluctuaties van een zwart gat. Het membraan paradigma werkt dan alleen als wordt aangenomen het membraan overal precies samenvalt met de gebeurtenissenhorizon.

RIASSUNTO

Fluidi

L'acqua che scorre nei canali di Amsterdam, l'aria che circola nell'atmosfera che ci circonda e per esempio il caffè che beviamo ogni mattina hanno una proprietà in comune: queste sostanze sono dette *fluidi*. A differenza dei solidi, che hanno una forma definita, i fluidi sono per definizione capaci di assumere la forma del recipiente in cui sono contenuti e sono capaci di scorrere o meglio fluire. Pertanto, diversamente dal linguaggio comune, in fisica il termine fluido si riferisce ad entrambe le fasi della materia, quella liquida e quella gassosa.

Il modello teorico che studia il comportamento di tali fluidi è la *fluidodinamica* (o *idrodinamica*), che è una descrizione *effettiva* cioè basata sull'ipotesi fondamentale in cui il fluido è visto come un mezzo continuo che si comporta in modo collettivo, senza considerare il contributo di tutte le particelle infinitesime che lo compongono. Per esempio, per sapere quali sono le correnti create nell'aria dopo una giornata di alta o bassa pressione, non è necessario sapere come si muovono tutte le particelle che la compongono, che francamente sarebbe un compito molto complicato, ma è sufficiente sapere la velocità, la temperatura ed altri parametri delle correnti iniziali per predire come cambierà il tempo in futuro. La fluidodinamica è proprio in grado di prevedere questi fenomeni, infatti non solo viene applicata alla meteorologia ma anche per esempio allo studio del moto dell'aria interagente con superfici solide come l'ala di un aereo. Questo diventa utile per capire come deve essere costruito l'aereo stesso affinché il suo moto nell'aria sia più efficiente e per l'appunto più fluido.

Anche se la fluidodinamica è una descrizione generica valida per tutti i fluidi, sappiamo già che non tutti i fluidi si comportano allo stesso modo. Ad esempio,

ponendo del miele su un cucchiaino e rovesciandolo, si può osservare che questo resiste molto al movimento tendendo a rimanere molto compatto e a cascare molto lentamente, mentre se si ripete la stessa operazione con l'acqua, questa scivola via subito. Queste proprietà sono dovute ai particolari tipi di interazioni che avvengono tra le particelle che compongono un fluido e ogni fluido avrà chiaramente tipi di particelle, e quindi tipi di interazione, differenti. Questi specifici effetti si possono incorporare nella descrizione della fluidodinamica con dei semplici parametri come la *viscosità*. Un fluido molto viscoso, come il miele, tende a muoversi lentamente mentre un fluido poco viscoso come l'acqua tende a scorrere facilmente. La viscosità misura quindi la capacità di un fluido di dissipare l'energia di movimento e convertirla in calore, più un fluido è viscoso e più è *dissipativo*. La viscosità e altri parametri simili possono essere misurati sperimentalmente oppure calcolati da un punto di vista teorico se si ha a disposizione una descrizione completa del comportamento delle particelle individuali, quindi conoscendo le leggi matematiche fondamentali che le descrivono. In alcuni casi determinare questi parametri risulta essere facile, ma ci sono altri casi, in cui le particelle sono *fortemente interagenti* dove i calcoli teorici diventano molto difficili se non impossibili. La maggior parte dei nostri strumenti teorici attuali sono infatti più adatti al regime opposto dove le particelle sono debolmente interagenti. Infatti quando le particelle interagiscono poco passa molto tempo prima che una incontri l'altra, quindi per un certo periodo in prima approssimazione si può pensare che la particella sia libera o isolata che risulta essere una semplificazione notevole.

Oltre ai fluidi comuni che abbiamo incontrato fino ad ora, ne esistono di altri più esotici che si possono trovare per esempio al Relativistic Heavy Ion Collider (RHIC) a Brookhaven, negli Stati Uniti oppure presso il Large Hadron Collider (LHC) di Ginevra, in Svizzera. In questi due laboratori ci sono stati e sono attualmente in corso, esperimenti che realizzano collisioni fra due fasci di atomi contrapposti e accelerati ad altissima energia. L'energia risulta essere così alta che, dopo l'impatto, gli atomi si disgregano nei loro costituenti elementari, i cosiddetti quark e gluoni. Per un lasso di tempo molto breve, prima di raffreddarsi e di ricombinarsi in altri atomi, queste particelle si comportano collettivamente come un fluido fortemente interagente ad alta temperatura e ad alta densità, chiamato *Plasma di Quark e Gluoni* ⁹. L'ambiente ricreato in questi esperimenti riproduce le caratteristiche dell'universo pochi istanti dopo la grande esplosione che creò l'universo, il cosiddetto Big Bang. In quel momento l'ambiente circostante e l'universo stesso era così caldo ed energetico che gli atomi ancora non si erano formati e il plasma di

⁹Il plasma si comporta come un fluido, ma a differenza dai liquidi o dai gas, le particelle che lo compongono sono cariche. In particolare, le particelle del plasma di quark e gluoni sono cariche rispetto alla forza cromodinamica, che è la forza che per esempio tiene insieme i protoni e neutroni nel nucleo di un atomo.

quark gluoni dominava la scena. Quindi, analizzare il comportamento di questo plasma da un punto di vista sperimentale ci permette di capire l'universo nelle sue primissime fasi di vita e ci aiuta a spiegare perché l'universo sia esattamente così come lo osserviamo. Tuttavia, per poter fare qualsiasi predizione da poter confrontare con i risultati sperimentali per mezzo del modello teorico di cui siamo a disposizione, la fluidodinamica, bisogna conoscere il valore dei parametri intrinseci di questo plasma, che abbiamo detto essere fortemente ingrediente, quindi proprio in quel regime dove i calcoli risultano essere difficili.

Fluidi in gravità

In modo del tutto inaspettato, i fluidi si possono trovare anche in ambito gravitazionale ed in particolar modo nel contesto dei buchi neri. Questi ultimi sono oggetti celesti che popolano il nostro universo e che presumibilmente si trovano al centro di ogni galassia. I buchi neri, come le comuni stelle, sono oggetti massivi capaci di attrarre altri oggetti che passano nelle loro vicinanze. La forza di attrazione dei buchi neri è però talmente elevata che anche la luce viene attratta, e una volta oltrepassata una certa superficie detta *orizzonte degli eventi* non ne può più uscire. Curiosamente tale superficie è dinamica, cioè può evolvere nel corso del tempo. Infatti se gettassimo qualcosa dentro un buco nero, il suo orizzonte degli eventi, e tutto lo spazio circostante, o meglio lo *spaziotempo*¹⁰, inizierebbe ad oscillare in un modo molto simile alle onde che vengono create in acqua dopo averci gettato un sasso. Inoltre, dato che l'orizzonte degli eventi è una superficie oltre la quale la materia e l'energia vengono perse completamente, i buchi neri assumono naturalmente un carattere dissipativo. Tutto questo particolare comportamento assomiglia moltissimo ad un fluido, infatti ci sono stati molti sviluppi per rendere tali idee più concrete.

La prima volta in cui queste idee hanno preso piede è stato negli anni '80 nel contesto del cosiddetto *paradigma di membrana*. Questo modello teorico è una versione semplificata del buco nero che viene visto come una membrana, cioè una superficie dotata di semplici caratteristiche fisiche come per esempio la proprietà di

¹⁰Lo spaziotempo può essere pensato come un tessuto che può essere distorto da una massa, un po' come quando saliamo su un tappeto elastico e questo si deforma con il nostro peso. Quando questa massa si muove e cambia nel tempo, si possono creare delle onde di spaziotempo che si propagano: le onde gravitazionali.

condurre elettricità. In particolare, in una delle molte formulazioni del paradigma, questa membrana si comporta come un fluido, che d'ora in poi chiameremo *fluido di membrana*. Tuttavia, questo fluido risulta avere delle proprietà patologiche come per esempio una viscosità negativa, cioè muovendo parti del fluido verrebbe prodotta energia anziché dispersa. Quindi anche se il paradigma di membrana risulta essere una semplificazione notevole nello studio dei buchi neri come li vediamo per esempio dalla terra da dove tutto quello che si trova dietro l'orizzonte degli eventi è comunque inaccessibile, la sua interpretazione come fluido forse non è la più conveniente. Infatti il modo migliore in cui si può vedere il comportamento di fluido in gravità è nel contesto della gravità olografica.

L'*olografia* è un concetto sviluppatosi intorno alla metà degli anni '90. L'idea fondamentale sta nel fatto che per ogni teoria gravitazionale, una teoria che descrive lo spaziotempo come un sistema dinamico, si possa associare una teoria di particelle ma in una dimensione inferiore. In un certo senso la dimensione aggiuntiva e la gravità stessa possono essere viste come un ologramma matematicamente codificato su una qualche superficie di dimensione inferiore nel linguaggio di una teoria di particelle dove la gravità è assente. In questo modo l'olografia stabilisce una equivalenza tra due teorie completamente diverse che ora possono e devono essere considerate come due facce di una unica teoria sottostante. Basta sapere come decodificare l'ologramma per andare, per esempio, dalla descrizione geometrica ad una descrizione particellare. La cosa più importante e sorprendente è che questa equivalenza avviene in modo molto speciale, cioè quando una teoria è debolmente interagente l'altra risulta essere fortemente interagente e vice versa, per cui le due teorie sono dette *duali* tra loro. Questo fatto ha conseguenze molto profonde, dato che, quando risulta conveniente, si può usare la teoria debolmente interagente per calcolare quantità relative alla teoria fortemente interagente che non si sarebbe in grado di derivare normalmente.

Un esempio concreto di gravità olografica è stato realizzato per una classe di spaziotempo specifica, quelli con curvatura negativa¹¹ chiamati *anti-de Sitter*. In questo caso infatti è stato stabilito un preciso *dizionario olografico*, cioè una ricetta in grado di tradurre in modo esplicito quantità espresse nel linguaggio geometrico in quantità definite nel linguaggio della teoria delle particelle e vice versa. Ad esempio, uno spazio di anti-de Sitter vuoto stabilisce la condizione di non avere nessuna particella nella teoria duale. Un buco nero nello spazio di anti-de Sitter corrisponde invece ad avere un insieme di particelle in equilibrio termico. Analogamente, i buchi neri leggermente perturbati, quindi con una geometria leggermente diversa e dinamica, corrispondono al comportamento fluidodinamico della teoria

¹¹Un esempio di spazio con curvatura negativa è la sella di cavallo, mentre uno spazio con curvatura positiva è la sfera.

duale. Inoltre tale fluido è un parente stretto del plasma di quark e gluoni fortemente accoppiato discusso in precedenza. Adesso però, le sue proprietà intrinseche, come la viscosità, possono essere calcolate facilmente analizzando la geometria dei buchi neri in spazio di anti-de Sitter. Pertanto, non solo l'olografia rende esplicito come il comportamento fluidodinamico può essere realizzato in gravità, ma può anche essere usata come strumento teorico per studiare certe teorie di particelle fortemente interagenti mediante la geometria gravitazionale duale che è in generale più facile da gestire.

Motivazione e riassunto dei risultati principali

In questa tesi faremo un viaggio tra i concetti di cui sopra. Saremo interessati sia a quello che la gravità può insegnarci sulla fluidodinamica, così come nel modo in cui la fluidodinamica possa aiutarci a comprendere l'olografia in sé.

Per esempio, abbiamo stabilito che l'idrodinamica è una descrizione valida che comprende il comportamento dinamico di ogni fluido, tuttavia tale descrizione non è unica. Vi è una formulazione convenzionale, abbastanza datata e consolidata, e altre formulazioni più recenti che sono motivate dal desiderio di riscrivere la teoria più sistematicamente, in un linguaggio che richiede possibilmente meno principi fondamentali. Tuttavia, questi nuovi approcci alla fluidodinamica sono meno studiati e non è chiaro quanto siano affidabili. In questa tesi aiutiamo a chiarire in parte tali questioni per mezzo dell'olografia. Utilizzando un buco nero leggermente perturbato in spazio di anti-de Sitter riusciamo a mostrare come il comportamento di fluido viene alla luce proprio in una di queste formulazioni non convenzionali. Siamo quindi in grado di confermare, almeno nel nostro esempio, alcune caratteristiche di queste formulazioni.

In questa tesi dimostriamo inoltre l'esistenza del comportamento fluidodinamico in gravità attraverso un modo leggermente diverso da quello convenzionale di realizzare l'olografia. Solitamente la teoria delle particelle si dice essere codificata su una superficie a dimensione inferiore che si trova sul bordo dello spaziotempo di anti-de Sitter. Qui invece consideriamo questa superficie da qualche parte all'interno dello spazio, senza specificare nessun tipo di geometria in cui si trova e tenendola generale fino ad un certo punto. In questo modo siamo in grado di generalizzare il dizionario olografico nel regime di fluido per spaziotempo che non sono neces-

sariamente anti-de Sitter. Inoltre in questa tesi spostiamo tale superficie verso l'orizzonte degli eventi di un buco nero per vedere se il fluido olografico così ottenuto si comporta similmente al fluido di membrana discusso sopra. Mostriamo che invece i due tipi di fluido sono molto diversi tra loro chiarificando alcune vecchie questioni collegate con l'interpretazione di fluido del paradigma di membrana.

Tuttavia, il paradigma di membrana non deve essere completamente scartato dato che può essere definito in un altro modo. Anche in questo caso l'interno del buco nero viene completamente dimenticato e l'unica informazione che viene necessariamente mantenuta è il fatto che tutto quello che circonda il buco nero nelle sue vicinanze un giorno o l'altro cadrà nel buco nero stesso oltrepassando l'orizzonte degli eventi. In questa tesi ci chiediamo se tale approssimazione ai buchi neri funziona sempre e dimostriamo che è così ad eccezione di alcuni casi particolari. Per esempio, ci sono certe perturbazioni dello spaziotempo di un buco nero che non possono essere descritte da un regime fluidodinamico della teoria duale. Per essere più precisi quando si perturba un buco nero prima del regime di fluido, vi è un'altra configurazione molto complessa in cui non solo ci sono onde che si propagano, ma anche perturbazioni che decadono molto velocemente. L'olografia è in generale in grado di catturare questo tipo di fenomeno, mentre il paradigma di membrana risulta leggermente più restrittivo.

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*Forse, davvero, ci piace, si ci piace di piú
Oltrepassare in volo, in volo piú in la'
Meglio del perdersi in fondo all'immobile
Meglio del sentirsi forti nel labile.
Forse, sicuro, é il bene piú radioso che c'è
Lieve svenire per sempre persi dentro di noi
Meglio del perdersi in fondo all'immobile
Meglio del sentirsi forti nel labile.
Forse, davvero, ci piace, si ci piace di piú
Lieve, Marlene Kuntz*