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Plane Waves, Matrix Models and Space-Time Singularities

Thèse

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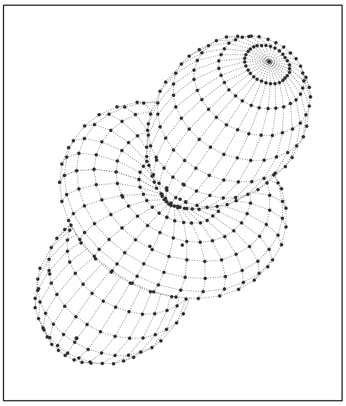

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Planes Waves, Matrix Models and Space-Time Singularities

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Outline

Keywords matrix models, fuzzy sphere, plane wave, Fermi coordinates, quantum evolution, space-time singularity, light-cone gauge, Penrose limit, membrane quantisation

Mots-clés modèles de matrices, sphère floue, onde plane, coordonnés de Fermi, evolution quantique, singularité de l'espace-temps, jauge du cône de lumière, limite de Penrose, quantisation des membranes

As the outcome of four years of learning and research in the string theory group at Neuchâtel, this work does not follow a straight line from beginning to end. It is rather to be understood as a melange of several concepts and techniques which could have been arranged in different order and with different emphasis. Consequently, the reader should not feel forced to follow the chosen path, but is encouraged to pick out his own topics of interest.

Plane waves are one of the major tools employed, introduced in **chapter one** as the result of the Penrose limit of any space-time as well as interesting gravitational backgrounds themselves. Curved, yet plain enough to allow for many detailed calculations they can be considered the next logical step after flat space. We briefly review many of their attractive features, geometry, symmetry and relevance to light-cone quantisation, stressing the use of Brinkmann coordinates. These are Fermi coordinates on plane waves, and in the publication reprinted at the end of the chapter this notion is exploited to construct the geometric expansion of a space-time about the respective plane wave.

Chapter two connects to the first chapter in reviewing the application of the plane wave limit to power-law space-time singularities of the Szekeres-Iyer class, encompassing a wide range of well-known physical solutions. The result is universal: singular homogeneous plane waves, provided that the dominant energy condition (DEC) holds. The following publication builds up on this using functional analytic methods to characterise scalar field probes on the same backgrounds. The criterion of a unique time-evolution classifies singular behaviour of the fields.

Departing from the notion of classical space-time, **chapter three** introduces a new concept: membrane quantisation, a technique to regularise a $U(\infty)$ diffeomorphism subgroup to $U(N)$ matrix theory. We present the basic construction from a new angle detailing gauge-fixing procedure and origin of the gauge field. We also mention the extension to the supersymmetric BMN model and the connection to gauge theory compactification. The fuzzy sphere ground states of the model allow for a fluctuation expansion with complete control over the spectrum. We advocate the use of t'Hooft's R_ξ-gauges and explain the implication of their unphysical gauge parameter.

The BMN matrix model is embedded into the larger context of string theory dualities and M-theory in **chapter four**. Following Blau and O'Loughlin, the models of CSV matrix big-bangs are generalised from flat space to singular homogeneous plane waves, setting the stage for a discussion of fuzzy sphere behaviour in regimes of strong and weak coupling.

A fair amount of background material has been included in this work to strike the balance between an ample introductory text and a rather terse research paper. The advanced reader might want to skip these parts and go straight to the relevant sections.

Notably, this work includes the unabridged reprints of two publications

> [1] M. Blau, D. Frank, and S. Weiss, "Fermi coordinates and Penrose limits," *Class. Quant. Grav.* **23** (2006) 3993–4010, `hep-th/0603109`.

in section 1.6, with a further (unpublished) example of the new techniques in 1.7, and

> [2] M. Blau, D. Frank, and S. Weiss, "Scalar field probes of power-law space-time singularities," *JHEP* **08** (2006) 011, `hep-th/0602207`.

reprinted in section 2.4. Apart from those publications, chapters three and four contain original work not (yet) published in

- section 3.2, mostly a derivation of the membrane matrix model from a perspective complimentary to the usual one found in the literature,
- section 3.5, where we employ the R_ξ-gauges not used before on the BMN matrix model to detail the physical relevance of the one-loop effective potential,
- section 4.3, most prominently the scaling behaviour of the matrix models and
- section 4.4 on fuzzy sphere dynamics in matrix big-bangs.

As the title tells, plane waves and light-cone gauge, (power-law) space-time singularities and fuzzy spheres in matrix models are the main threads running through this work. In its diversity, the present report surely does reflect an important quality of string theory. After many surprises and indeed revolutions the theory has turned into a broad frame-work that allows for the exploration of a wealth of new ideas and theoretical phenomena to sharpen our tools and senses for the experimental results soon to come.

Chapter 1

Plane Waves

1.1 Every Space-Time has a Penrose Limit

Let us begin the story some thirty years ago with the 60th birthday of André Lichnerowicz (1915-1998), a respected French mathematician and physicist known for his work on the frontiers between differential geometry and general relativity. On this occasion, some friends and colleagues of his set up a collection of articles written in his honour, published in Cahen and Flato (eds.) [3].

Amongst the contributors was Roger Penrose with a remarkable observation that every space-time has a plane wave as a limit, the Penrose limit, which is the foundation of the work presented in this chapter.

Starting point of the construction is a general metric brought to the following form

$$ds^2 = 2dUdV + a(U,V,Y^k)\,dV^2 + b_i(U,V,Y^k)\,dY^i dV + g_{ij}(U,V,Y^k)\,dY^i dY^j \tag{1.1.1}$$

Locally, this is always possible by the diffeomorphism degrees of freedom and has a geometric interpretation. The U coordinate is the affine parameter of a bunch of null geodesics (since $\Gamma^{\mu}_{UU} = 0$), which sprouts from a hypersurface through the point U_0 spanned by the transverse V, Y^i. Conversely, we can always construct such a metric by choosing an appropriate bundle of null geodesics as long as those geodesics do not intersect. At the intersection point, the local construction breaks down, so in general and in fact generically the metric (1.1.1) will not be valid globally.

Penrose then proposed an inhomogeneous scaling of the coordinates with parameter λ accompanied by a homogeneous rescaling of the metric

$$(U,V,Y^i) \to (U,\lambda^2 V,\lambda Y^i) \qquad ds^2 \to \lambda^{-2} ds^2 \tag{1.1.2}$$

resulting in the scaled version

$$ds^2 = 2dUdV + \lambda^2 a(U,\lambda^2 V,\lambda Y^k)\,dV^2 + \lambda b_i(U,\lambda^2 V,\lambda Y^k)\,dY^i dV + g_{ij}(U,\lambda^2 V,\lambda Y^k)\,dY^i dY^j$$

In the limit of $\lambda \to 0$ now an expansion of the metric gives the zero order

$$ds^2 = 2dUdV + g_{ij}(U)\,dY^i dY^j \tag{1.1.3}$$

which is a so-called plane wave in Rosen coordinates, by far not unknown at the time. First discovered by Beck in 1925, they have been used since Einstein and Rosen (1937) as models of gravitational waves.

Penrose also presented a nice interpretation of his limit procedure. In his words

> *The concept of tangent space at a point p in a manifold M is, of course, basic to differential geometry. Intuitively, one may envisage smaller and smaller neighbourhoods of p in M which are correspondingly scaled up by larger and larger factors. In the limit we obtain the tangent space T_p to M at p.*

The Penrose limit follows the same principle, although not in the vicinity of a point p, but all along a null geodesic γ. And

> *whereas T_p is, in an essential way, a flat space, the corresponding procedure applied to γ yields a curved space W_γ known as a plane wave.*

Penrose stops at this point, but the interpretation can be taken much further. It is known that the tangent space T_p at a point p can be extended to Riemann coordinates in a neighbourhood of p. These are an expansion in geodesic distance to p with T_p as the lowest order.

A similar coordinate construction also exists as a geodesic expansion not about a point, but about a (null) geodesic: Fermi coordinates. The actual construction Penrose gives slightly clouds this relation with the transverse Y^i not actually being geodesic, in contrast to the coordinates of flat tangent space.

We shall see, however, that true Fermi coordinates do exist that describe the same plane wave space-times in a more geometric fashion. Moreover, combined with the scaling of Penrose they can be extended to higher orders in λ in the vein of the interpretation given by Penrose.

Brinkmann coordinates So let us see what happens if we combine the best of two worlds and construct geodesic coordinates on the plane wave that resulted from the Penrose limit. The transformation is easily found. We focus on the null geodesic γ_U at $V = Y^i = 0$, keeping its affine parameter U as a coordinate. A vielbein can then be used to flatten the metric in the transverse coordinates and render them trivially geodesic

$$u = U \qquad \text{and} \;\; x^a = E^a_i(U)\, y^i \qquad \text{such that} \;\; \delta_{ab} E^a_i(U) E^b_j(U) = g_{ij}(U) \tag{1.1.4}$$

Since the metric is a symmetric matrix but the vielbein is not, there is some leeway in the choice. We shall use it to demand that the vielbein be parallelly transported along the geodesic γ_U, thus fixing it uniquely

$$\text{Parallel transport:} \quad \dot{E}^i_a + \Gamma^i_{uj} E^j_a = 0 \quad \Rightarrow \quad \dot{E}^i_a E_{ib} - E^i_a \dot{E}_{ib} = 0 \tag{1.1.5}$$

The final null coordinate we choose so as to eliminate unwanted $dx^a dU$ cross terms created by the change of transverse coordinates

$$g_{ij}(U) dy^i dy^j = (dx^a)^2 - (\dot{E}_{ia} E^i_b + \dot{E}_{ja} E^j_b) x^b dx^a dU + \delta_{ab} (\dot{E}^a_i E^i_c x^c)(\dot{E}^b_j E^j_d x^d) dU^2 \tag{1.1.6}$$

which by virtue of the parallel transport equation (1.1.5) is indeed possible

$$v = V - \tfrac{1}{2} \dot{E}_{ai} E^i_b x^a x^b \tag{1.1.7}$$

It can be verified that this is also a geodesic coordinate.

Taking stock, we have shifted all information about the curved background into the du^2 component of the metric. The result is a plane wave in Brinkmann coordinates

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + \delta_{ab}dx^a dx^b \qquad \text{where } A_{ab}(u) = \ddot{E}_{ai}E^i_b \tag{1.1.8}$$

with geodesic coordinates v and x^a. Note that in contrast to the U of the Rosen metrics, u is geodesic only at the origin $v = x^a = 0$.

Historically, these metrics have been discovered by Brinkmann in 1923 and are actually older than the Rosen metrics, which have first been discovered by Beck in 1925. In a twist of irony the latter ones have instead dominated the discussion of gravitational waves for some time, leading to much confusion.

1.2 Brinkmann Coordinates, Geometry and Geodesics

In 1913 already, before the publication of the general theory of relativity, Einstein remarked to Max Born that gravitational waves in the weak-field limit exist and travel at the speed of light. Indeed, artificially splitting a general relativistic metric into a sum of flat background Minkowski space and small gravitational fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ the Einstein equations reduce to a wave equation for $h_{\mu\nu}$ at lowest order.

In analogy to electromagnetic waves, the solutions are transversally polarised and do travel at the speed of light, e.g. in the light-like $U = t + z$ direction:

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))\, dy^i dy^j \tag{1.2.1}$$

So a linearised gravitational wave already takes the form of a Rosen metric, although, of course, not fulfilling the non-linear vacuum Einstein equations.

Yet in 1936, after having thoroughly studied Rosen metrics in the full theory, Einstein wrote again to Born stating that *"…gravitational waves do not exist, though they had been assumed a certainty to the first approximation"*.

What had happened? Trying to restrict the general class of Rosen metrics to vacuum Einstein solutions unavoidably leads to coordinate singularities. Mistaking these for physical singularities, Einstein and Rosen (1937) erroneously concluded at first that the metrics themselves must be unphysical.[1]

The problem can be pinned down to a simple example. Consider the apparently singular Rosen metric and its Brinkmann counterpart, the flat metric

$$ds^2_{\text{Rosen}} = 2dUdV + U^2(dy^i)^2 \qquad\qquad ds^2_{\text{Brinkmann}} = 2dudv + (dx^a)^2 \tag{1.2.2}$$

which can be obtained using the vielbein $E(U)_{ia} = U\delta_{ia}$ and therefore $A_{ab}(u) = \ddot{E}_{ai}E^i_b = 0$. But the flat metric is of Brinkmann and Rosen form at the same time! Rosen coordinates therefore are ambiguous enough to contain spurious coordinate singularities.

[1] We urge the reader not aware of this fine anecdote to read the brief account in section 2 of Kennefick [4]

1.2.1 Physical Coordinate Singularities

In stark contrast, there is no such freedom in Brinkmann coordinates. There, the Riemann tensor basically only has one component

$$R_{uaub}(u) = -A_{ab}(u) \tag{1.2.3}$$

which trivially determines the entire degrees of freedom of the metric. Moreover, contracting with δ^{ab}, there is only one non-zero component of the Ricci tensor

$$R_{uu}(u) = -\delta^{ab} A_{ab}(u) = -\operatorname{Tr} A(u) \tag{1.2.4}$$

So we can immediately read off important physical properties from $A_{ab}(u)$:

- the Ricci scalar is always zero $R = 0$,
- plane wave metrics are flat iff $A_{ab}(u) = 0$,
- the subclass of vacuum solutions is simply the one with vanishing trace $\operatorname{Tr} A(u) = 0$,
- they are conformally flat iff $A_{ab}(u) = A(u)\delta_{ab}$, as then the Weyl tensor (the traceless part of the Riemann tensor) vanishes.

Furthermore, all higher curvature invariants of plane waves necessarily vanish. In Brinkmann coordinates the argument is straight-forward. A general scalar curvature invariant is constructed by applying the covariant derivative repeatedly to the Riemann tensor (1.2.3)

$$\nabla_{\mu_1} \dots \nabla_{\mu_n} R_{uaub} \tag{1.2.5}$$

and contracting all indices with the inverse metric. This is simply not possible in a plane wave, where the inverse metric components $g^{uu} = g^{ua} = 0$ vanish and so do the Christoffel symbols $\Gamma^u_{\mu\nu} = 0$ with upper index u. [2]

Now plane waves can still have singularities in the Brinkmann coordinates, but if they do, they are all genuinely physical, as diverging components of the Riemann tensor are measurable. For example the tidal forces on nearby geodesics measured by the geodesic deviation equation necessarily diverge with the Riemann tensor and thus with $A_{ab}(u)$.

1.2.2 Some Hereditary Properties

On top of these restrictions on the Riemann tensor valid for all plane waves, there might of course be more on special subclasses. Notably, since we have introduced plane waves as the result of a Penrose limits of an arbitrary space-time, one might ask whether special properties of the initial metric survive the limiting procedure. If they do, then they are called *hereditary properties*, according to Geroch [7]. Some are immediately obvious, in particular any tensor field constructed from the Riemann tensor and its derivatives that vanishes before taking the limit, also vanishes in the limit. So for example

[2] The argument is also possible in Rosen coordinates using the Penrose scaling (1.1.2) as a constant metric rescaling (homothety). At the fixed point of this homothety, the null geodesic origin $(U, 0, 0)$, the curvature invariants have to be scale invariant. At the same time, they must scale like powers of the inverse metric globally, because the Christoffel symbols and (1.2.5) are homothety invariant. It follows that the curvature invariants must be zero at the origin and, since Rosen coordinates are Killing coordinates (see next section), everywhere. See Schmidt [5] and [6] for more.

- the Penrose limit of a Ricci-flat metric is Ricci-flat,
- the Penrose limit of a conformally flat metric (vanishing Weyl tensor) is conformally flat,
- the Penrose limit of a locally symmetric metric (vanishing covariant derivative of the Riemann tensor) is locally symmetric.

Note, however, that the Penrose limit of an Einstein metric, where the Ricci tensor is proportional to the metric $R_{\mu\nu} = \Lambda g_{\mu\nu}$ cannot remain of the same type, since the scaling properties of $R_{\mu\nu}$ and $g_{\mu\nu}$ are different. In fact, its Penrose limit is Ricci-flat.

All these geometric properties make plane waves in Brinkmann coordinates attractive backgrounds, far beyond their use for the study of gravitational waves. Curved, but still accessible to detailed calculations, they are often the first step to generalising flat-space results and thus play an important role in modern theoretical physics and string theory. Next we shall examine their symmetry structure, which leads to considerable simplifications in particle physics on plane wave backgrounds.

1.3 Symmetries, Symmetric and Homogeneous Plane Waves

1.3.1 Symmetries and Killing Vectors

Plane waves necessarily have a null Killing vector ∂_v, which is the only one obvious in Brinkmann coordinates

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + \delta_{ab}dx^a dx^b \tag{1.3.1}$$

But there are more symmetries, as can be seen in Rosen coordinates, where the metric is independent not only of V but as well of the space-like transverse coordinates y^i

$$ds^2 = 2dUdV + g_{ij}(U)\, dy^i dy^j \tag{1.3.2}$$

Generically, we can go to a variety of different-looking Rosen coordinates by the transformation

$$U = u \qquad y^i = E^i_a x^a \qquad V = v + \tfrac{1}{2}\dot{E}_{ai}E^i_b x^a x^b \qquad \text{where } \dot{E}_{ia}E^a_j - \dot{E}_{ja}E^a_i = 0 \tag{1.3.3}$$

as long as the vielbein $E^i_a(u)$ satisfies the harmonic oscillator equation

$$\ddot{E}_{ai}(u) = A_{ab}(u)E^b_i(u) \tag{1.3.4}$$

So all the transverse directions in any of the different Rosen charts on a particular plane wave must be Killing. Back in Brinkmann coordinates these Killing vectors are[3]

$$\partial_i = \partial_i x^a\, \partial_a + \partial_i v\, \partial_v = E^a_i \partial_a - \dot{E}_{ia}x^a \partial_v \tag{1.3.5}$$

Being coordinate vector fields, the Killing vectors of a particular Rosen chart commute, again by virtue of the symmetry condition (1.1.5)

$$[\partial_i, \partial_j] = (\dot{E}_{ia}E^a_j - \dot{E}_{ja}E^a_i)\partial_v = 0 \tag{1.3.6}$$

[3] using a variation of the symmetry condition (1.1.5) $\dot{E}_{ia}E^a_j = \dot{E}_{ja}E^a_i$

But this is not the end of the story. The system of second order differential equations (1.3.4) has a total of $2d$ linearly independent solutions. Hence to each Rosen chart there must exist a dual Rosen chart with coordinates defined by the remaining d Killing vectors. This corresponds to finding the full set of $2d$ functions with respect to the initial conditions at u_0, say

$$\begin{aligned} \text{Rosen chart: } E_{ia}(u) \equiv Q_{ia}(u): \quad & Q_{ia}(u_0) = \delta_{ia} \qquad \dot{Q}_{ia}(u_0) = 0 \\ \text{Rosen dual: } E_{ka}(u) \equiv P_{ka}(u): \quad & P_{ka}(u_0) = 0 \qquad \dot{P}_{ka}(u_0) = \delta_{ka} \end{aligned} \tag{1.3.7}$$

Keeping the notation of Q and P for the two dual vielbeins, the $2d+1$ Killing vectors are

$$X_i = Q_i^a \partial_a - \dot{Q}_{ia} x^a \partial_v \qquad Y_i = P_i^a \partial_a - \dot{P}_{ia} x^a \partial_v \qquad Z = \partial_v \tag{1.3.8}$$

Their commutator can be identified (1.3.6) as the Wronskian of two such solutions, vanishing within each set of coordinate vectors of one Rosen chart. Between two vectors of different sets the Wronskian still is necessarily constant, guaranteed by equation (1.3.4). As such, it can be evaluated on the initial conditions (1.3.7). Altogether the $2d+1$ Killing vectors form a Heisenberg algebra with a central element Z

$$[X_i, Y_j] = \delta_{ij} Z \qquad \text{all others zero: } [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0 \tag{1.3.9}$$

1.3.2 Hereditary Symmetries

As for properties of tensorial fields, we might ask whether symmetries are also *hereditary properties* of the Penrose limit. Consider a Killing vector ξ of the metric in adapted coordinates before the Penrose limit. Then, under a rescaling of the coordinates, ξ also acquires a dependence on the scaling parameter λ and the $\xi(\lambda)$ remains Killing.

In the limit $\lambda \to 0$, the lowest non-zero and non-singular order of $\xi(\lambda)$

$$\bar{\xi} = \lim_{\lambda \to 0} \lambda^{\Delta_\xi} \xi(\lambda) \tag{1.3.10}$$

singled out by properly adjusting the scaling dimension Δ_ξ, is then a Killing vector of the plane wave limit.

What might happen, however, is that two originally independent Killing vectors ξ_1 and ξ_2 become linearly dependent in the limit if their lowest orders coincide, e.g. $\bar{\xi}_1 = \bar{\xi}_2$. At first glance, one seems to have lost some symmetry. Fortunately this is not true. In this case the limit $\bar{\xi}_-$ of the difference $\xi_- = \xi_1 - \xi_2$ vanishes in the leading term $\bar{\xi}_1 - \bar{\xi}_2 = 0$. So to truly define the Penrose limit of ξ_-, one has to go to the next respective order in λ. This procedure can be repeated until one is left with two linearly independent Killing vectors in the limit, provided that they were independent in the original space-time.

The upshot of this quick argument is that

the number of linearly independent Killing vectors can never decrease in the Penrose limit.

For a more thorough presentation of this sketch as well as the original proof of Geroch [7] see Blau, Figueroa-O'Farrill and Papadopoulos [8]. This publication also shows that the proof extends to the supersymmetric version of the Penrose limit, which is a key to understanding the maximally symmetric plane wave background of IIB string theory as a Penrose limit of the known $AdS_5 \times S^5$ one.

Inherited symmetries might or might not intersect with the class of symmetries we have shown to be present anyway in plane waves. More than these generic symmetries, whether inherited or not, lead to special subclasses of plane waves. In the following we shall discuss two such examples featuring an additional Killing vector with ∂_u components.

1.3.3 Symmetric Plane Waves

The simplest example of a plane wave is one with a constant wave profile A_{ab}

$$ds^2 = 2dudv + A_{ab}x^a x^b du^2 + \delta_{ab}dx^a dx^b \tag{1.3.11}$$

Trivially, it has the additional Killing vector ∂_u. Since A_{ab} is constant, it can be diagonalised once and for all u by a constant rotation of the transverse x^a. It is therefore classified by its set of eigenvalues $\{a_1 \ldots a_d\}$ up to an overall scale, which can always be changed by the boost

$$(u, v, x^c) \to (\lambda u, \lambda^{-1} v, x^c) \qquad \Rightarrow \quad A_{ab} \to \lambda^2 A_{ab} \tag{1.3.12}$$

In the case of diagonal A_{ab}, the vielbein can also be chosen diagonal and the equation (1.3.4) falls apart into individual harmonic oscillator equations

$$y^i = E^i_a(u)x^a \qquad \text{with } \ddot{E}^i_a(u) = a_a E^i_a(u) \tag{1.3.13}$$

each solved by a two parameter solution

$$\begin{aligned} \text{negative eigenvalue } a_a = -\alpha_a^2 : \quad & x^a = (C^a_1 \sin(\alpha_a u) + C^a_2 \cos(\alpha_a u))\ y^a \\ \text{positive eigenvalue } a_a = \alpha_a^2 : \quad & x^a = \left(C^a_1 \exp(\alpha_a u) + C^a_2 \exp(-\alpha_a u)\right)\ y^a \end{aligned} \tag{1.3.14}$$

Take the case of all negative eigenvalues a_a. We then have an example of a pair of dual Rosen metrics as explained above

$$ds^2 = 2dUdV + \sum_a \sin^2(\alpha_a U)\,(dy^a)^2 = 2dUd\tilde{V} + \sum_a \cos^2(\alpha_a U)\,(d\tilde{y}^a)^2 \tag{1.3.15}$$

The non-generic Killing vector $H = \partial_u$ extends the Heisenberg algebra (1.3.9) to an harmonic oscillator algebra with 'Hamiltonian' or 'number operator' H. This is because for the constant A_{ab} at hand, the Rosen coordinates defined by the vielbein $E_{ia} = Q_{ia}$ have a dual given by the derivative $\dot{E}_{ia} = P_{ia}$, also a solution of (1.3.4). Commuting with H therefore swaps X and Y

$$[H, X_i] = Y_i \qquad [H, Y_i] = a_i X_i \tag{1.3.16}$$

The eigenvalues a_i appear here because of (1.3.4): $\dot{P}_{ai} = \ddot{Q}_{ai} = A_{ab}Q^b_i$.

1.3.4 Singular Homogeneous Plane Waves

The second example with extended symmetry we want to give is the class of plane waves with wave-profile proportional to u^{-2}

$$ds^2 = 2dudv + A_{ab}u^{-2}x^a x^b du^2 + \delta_{ab}dx^a dx^b \tag{1.3.17}$$

Because of the u^{-2} factor, they are invariant under the boost (1.3.12)

$$(u, v, x^a) \to (\lambda u, \lambda^{-1} v, x^c) \tag{1.3.18}$$

and hence have a corresponding additional Killing vector

$$u\partial_u - v\partial_v \tag{1.3.19}$$

Again, we can diagonalise these metrics without loss of generality, decoupling (1.3.4)

$$y^i = E^i_a(u)x^a \qquad \text{with} \;\; \ddot{E}^i_a(u) = a_a u^{-2} E^i_a(u) \tag{1.3.20}$$

The equation can be solved by a power-law ansatz $E^i_a(u) = u^{n_a}$. The power is determined by the corresponding A_{ab} eigenvalue

$$n_a(n_a - 1) = a_a \tag{1.3.21}$$

Note the symmetry of this equation: it is a parabola with a minimum of $a_a = -1/4$ at $n_a = 1/2$, thus identifying n_a and $1 - n_a$. This has the remarkable consequence that every plane wave in Rosen coordinates of power-law type and its respective dual

$$ds^2 = 2dUdV + \sum_i U^{2n_i}(dy^i)^2 = 2dUd\tilde{V} + \sum_i U^{2(1-n_i)}(d\tilde{y}^i)^2 \tag{1.3.22}$$

leads to a singular homogeneous plane wave (1.3.17) in Brinkmann coordinates with $a_a = n_a(n_a - 1) \geq -1/4$. In the dual cases $n_a = 0$ and $n_a = 1$ we recover the isometry between flat space and the U^2 Rosen metric (1.2.2).

Power-law plane waves generically arise as Penrose limits of Szekeres-Iyer metrics as approximations of a wide class of space-time singularities. They will feature large parts of chapter two and four.

For $a_a < -1/4$ we can still make the power-law ansatz leading to complex exponents n_a. A real basis can be found

$$y^a = u^{\frac{1}{2} \pm i\phi_a} x^a \qquad \Rightarrow \qquad y^a_1 = u^{\frac{1}{2}} \cos(\phi_a \ln u), \qquad y^a_2 = u^{\frac{1}{2}} \sin(\phi_a \ln u) \tag{1.3.23}$$

but at the expense of elegance, and we shall not make further use of these cases.

1.3.5 More Homogeneous Plane Waves

Both of the examples we have seen can be generalised to a family of homogeneous plane waves each parametrised by a constant symmetric matrix C_{ab} and a constant antisymmetric matrix f_{ab}.

For symmetric plane waves (constant A_{ab}) this generalisation is given by

$$A_{ab}(u) = (e^{uf} C e^{-uf})_{ab} \tag{1.3.24}$$

and for the singular homogeneous ones $A_{ab} \sim u^{-2}$ by

$$A_{ab}(u) = (e^{f \ln u} C e^{-f \ln u})_{ab} u^{-2} \tag{1.3.25}$$

We shall not discuss these any further and refer the reader to Blau and O'Loughlin [9].

1.4 Light-Cone Quantisation and a Covariantly Constant Null Vector

1.4.1 PP-Waves

The modern approach to gravitational plane waves is due to works of Ehlers and Kundt [10].

In analogy to electrodynamic radiation, one focuses on the most important aspect: the existence of a wave-vector. Translated to general relativity, this is metrics which admit a covariantly constant null vector field.

In the following, we derive the most general metric with a covariantly constant null vector field Z. The condition of covariant constancy can be split into the pair

$$\left.\begin{array}{rl}\text{Killing vector field:} & \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 0 \\ \text{gradient vector field:} & \nabla_\mu Z_\nu - \nabla_\nu Z_\mu = 0\end{array}\right\} \nabla_\mu Z_\nu = 0 \text{ covariantly constant} \tag{1.4.1}$$

If Z is nowhere zero, we can take the parameter along its integral curves as a null coordinate v on the manifold such that $Z = \partial_v$. By the Killing equation (1.4.1) the metric is v-independent and since Z is null the particular component $g_{vv} = 0$ vanishes.

The lower-index components of Z are then given by the metric components $Z_\mu = g_{\mu v}$ and the remaining condition in (1.4.1) implies that locally they are the gradient of a scalar field $Z_\mu = g_{\mu v} = \partial_\mu u(x)$. This field is taken as a second coordinate, so in this adapted system $\{u, v, x^\alpha\}$ parts of the metric simplify $2g_{\mu v}dx^\mu dv = 2dudv$

$$ds^2 = 2dudv + H(u, x^c)du^2 + 2A_a(u, x^c)dx^a du + g_{ab}(u, x^c)dx^a dx^b \tag{1.4.2}$$

Amongst these metrics are plane waves in both Rosen and Brinkmann coordinates as special cases and therefore they are form invariant under the coordinate transformation relating the two. More generally, we can eliminate H and A_a in favour of g_{ab} by an internal coordinate transformation of $v \to v + \Lambda(u, x^c)$ which shifts the coefficients H and A_a as in a gauge-transformation

$$H \to H + \tfrac{1}{2}\partial_u \Lambda \qquad \text{and} \quad A_a \to A_a + \partial_a \Lambda \tag{1.4.3}$$

This is reminiscent of the Kaluza-Klein effect, although v is a null direction here.

The converse, flattening the transverse metric g_{ab} at the expense of H and A_a is not possible in arbitrary space-time dimensions. However, there is an important subclass of (1.4.2), the *plane-fronted waves* with *parallel rays*, short *pp-waves* in Brinkmann coordinates

$$ds^2 = 2dudv + H(u, x^c)du^2 + \delta_{ab}dx^a dx^b \tag{1.4.4}$$

'Parallel rays' here denotes the defining feature of a parallel (covariantly constant) null vector whereas 'plane-fronted' says that wave-fronts $u =$ const are flat. The plane waves are the subclass of these with the characteristic function $H(u, x^c) = A_{ab}(u)x^a x^b$ quadratic in x^c.

The pp-waves are interesting for the studies of a wide variety of 'waves' in general relativity, comprising vacuum gravitational waves, but also other kinds of classical radiation propagating along a light-like wave-vector ∂_v. We shall not use them much throughout this work, and instead either work with the general (1.4.2) or specialise on actual plane waves.

1.4.2 String Theory Quantisation

Light-cone gauge quantisation is by now standard element of any introductory text on string theory, see e.g. the Polchinski [11]. The procedure is conveniently carried out on a flat space string background.

One of the remarkable features of plane waves is that the light-cone quantisation technique immediately carries over. Usually, one starts with the Polyakov action of string theory

$$S_{\text{Pol}} = -\tfrac{T}{2}\int d^2\sigma \sqrt{-\gamma}\,\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}\right) \tag{1.4.5}$$

Here $\gamma_{\alpha\beta}$ is the metric on the strings world-sheet, an independent auxiliary field. It does not have a kinetic term and therefore its equations of motion simply constrain the energy-momentum tensor T_{ab} to vanish on-shell

$$0 = T^{ab}(\tau,\sigma) = -\frac{4\pi}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma_{ab}}S_{\text{Pol}} = \frac{1}{\ell^2}\left(\partial^a X^\mu \partial^b X_\mu - \gamma^{ab}\partial_c X^\mu \partial^c X_\mu\right) \tag{1.4.6}$$

The equation is fulfilled for $\gamma_{ab} \sim \partial_a X^\mu \partial_b X_\mu$. Using this classical solution we can integrate out the auxiliary world-sheet metric on the level of the action. This transforms the Polyakov action to the classically equivalent Nambu-Goto action describing the string as a minimal surface in a relativistic background.

But even before enforcing on-shell conditions, the energy-momentum tensor is largely constrained by the symmetries of the theory:

- Target space diffeomorphism invariance
- World sheet diffeomorphism invariance $\Rightarrow \nabla_a T^{ab} = 0$
- Invariance under a Weyl rescaling $\gamma'_{ab} = \exp(\omega)\,\gamma_{ab}$ parameterised by functions $\omega(\tau,\sigma)$, this implies $T^a_a = 0$

The latter is a peculiarity of a two dimensional world-sheet and does not extend to higher dimensional extensions of the Polyakov action, the membrane etc. It carries over to the full quantum string theory without anomaly only in the known critical dimension.

The symmetries are usually employed to bring the world sheet metric to diagonal form

$$\gamma_{ab} = e^{\phi(\tau,\sigma)}\eta_{ab} \tag{1.4.7}$$

In this so-called *conformal gauge* the Polyakov action reads

$$S_{\text{Pol}} = \tfrac{T}{2}\int d^2\sigma\,\left(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu\right) g_{\mu\nu} \tag{1.4.8}$$

and is supplemented by the constraint equations (1.4.6)

$$\left(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu\right) g_{\mu\nu} = 0 \qquad \text{and} \;\; \partial_\tau X^\mu \partial_\sigma X^\nu g_{\mu\nu} = 0 \tag{1.4.9}$$

enforcing conformal flatness also on the induced metric.

Light-cone gauge In the usual flat-space scenario one can now choose light-cone gauge, i.e. identify world-sheet time τ with a target-space null coordinate u. In curved space-time this is not so obvious, we have to find a 'suitable' coordinate u. In particular, this u has to be a solution of the classical equations of motion of the action (1.4.8)

$$0 = \nabla_a \partial^a X^\mu = (\partial_a X^\nu)(\partial^a X^\rho)\, \nabla_\nu \partial_\rho X^\mu \tag{1.4.10}$$

It is not feasible to restrain all coordinates such that the terms in brackets disappear, so we focus on the remaining term. Setting the coordinate $X^\mu = u$ in this equation we get a condition on the vector field Z defined by the flow $Z_\mu = \partial_\mu u$

$$0 = \nabla_\mu \partial_\nu u = \nabla_\mu Z_\nu \tag{1.4.11}$$

to be covariantly constant. This is precisely the defining condition on the grounds of which we have derived the class of metrics (1.4.2)

$$ds^2 = 2dudv + H(u, x^c)du^2 + 2A_a(u, x^c)dx^a du + g_{ab}(u, x^c)dx^a dx^b \tag{1.4.12}$$

Indeed, the particular form of this metric (1.4.12) with $g_{av} = g_{vv} = 0$ leads to $g^{uu} = g^{ua} = 0$ by the definition of the inverse metric $g^{ij} = \varepsilon^{i\, i_1 \dots i_n} \varepsilon^{j\, j_1 \dots j_n} g_{i_1 j_1} \cdots g_{i_n j_n}$. Since the metric is also independent of v, it follows that all Christoffel symbols with upper index u necessarily vanish, $\Gamma^u_{\mu\nu} = 0$. Rewriting the equations of motion (1.4.10)

$$0 = \eta^{ab} \nabla_a \partial_b X^\mu = (\partial_\tau^2 - \partial_\sigma^2) X^\mu + \Gamma^\mu_{\nu\rho}(X)\, (\partial_\tau X^\nu \partial_\tau X^\rho - \partial_\sigma X^\nu \partial_\sigma X^\rho) \tag{1.4.13}$$

we see that they reduce to the wave equation for $X^\mu = u$, and one can choose light-cone gauge with constant p_v exactly as in the flat space case

$$(\partial_\tau^2 - \partial_\sigma^2) u = 0 \qquad \Rightarrow \qquad u = p_v \tau \tag{1.4.14}$$

The equations of motion for the transverse coordinates, setting $X^\mu = X^a$ in (1.4.13), are necessarily independent of v. This is because the only metric component with a v-index, g_{uv}, is constant, and therefore all $\Gamma^\mu_{\rho v}$ vanish.

So the equation of motion for $X^\mu = v$ decouples from the rest. Moreover, it is not independent, but can be derived by the two constraints (1.4.9) and the equations of motion for X^a. It is therefore possible and indeed very simple to eliminate this variable from the action, by going to Hamiltonian variables (at least partially in v, the so-called Routh's procedure).

Light-cone gauge in metrics with a covariantly constant null vector is not only possible and useful in string theory, but also in point particle theories and higher dimensional membrane theories. We shall make more use of its virtues all throughout this work, most notably for membrane quantisation in chapter three.

Plane waves Metrics with a covariantly constant null vector (1.4.12) are the most general backgrounds for light-cone gauge to work (see also Horowitz and Steif [12]). But we can go even further when restricting ourselves to the subclass of plane waves. Indeed, in Brinkmann coordinates the transverse equations of motion (1.4.13) then simplify to

$$(\partial_\tau^2 - \partial_\sigma^2) X_a - p_v^2 A_{ab}(p_v \tau) X^b = 0 \tag{1.4.15}$$

a massive wave equation with time-dependent mass matrix $A_{ab}(p_v\tau)$. Expanding the σ-dependence in Fourier modes

$$X^a(\tau,\sigma) = e^{in\sigma} X^a_n(\tau) \tag{1.4.16}$$

gives decoupled time-dependent harmonic oscillator equations for each mode

$$\ddot{X}^a_n = \left(p_v^2 A^a_b(p_v\tau) - n^2\delta^a_b\right) X^b_n \tag{1.4.17}$$

String theory in plane wave backgrounds therefore reduces to a free field theory of which we have full control. As in flat space, we can explicitly obtain a complete set of solutions to take as a starting point for quantisation in light-cone gauge.

One of the most interesting solutions in this respect is the maximally supersymmetric plane wave background for IIB string theory discovered by Blau, Figueroa-O'Farrill, Hull and Papadopoulos [13].

1.5 Taking Stock and Looking Ahead

Over the course of this article[4] we have accumulated various different notions of plane waves and Penrose limits. The first thing to note is that everything is always based on a single null geodesic, its affine parameter u or U usually taken as a coordinate and the geodesic put at the origin of the respective transverse coordinates.

Brinkmann coordinates on plane waves are *Fermi coordinates* about this null geodesic. All transverse coordinates v and x^a are geodesic themselves, on any hypersurface in space-time labelled by a value of $u = u_0$. ∂_u on the other hand is only geodesic at the origin $v = x^a = 0$, so we can see that the coordinate system on the plane wave is defined as measuring transverse geodesic distance to the defining null geodesic γ.

We have seen that *Rosen coordinates* are Killing coordinates describing a translational invariance in the transverse directions. It follows that, in contrast to Brinkmann coordinates, the defining null geodesic γ at the origin is not alone, but embedded in a bunch of null geodesics, one for any constant value of the transverse coordinates. Rosen coordinates are not uniquely defined, but another set of dual Rosen coordinates is needed to span the whole algebra of linearly independent Killing vectors. This corresponds to embedding the original null geodesic in different bundles of geodesics. The resulting plane wave remains the same.

The geometric object for this kind of behaviour is the *Jacobi field J*, solution to the Jacobi equation

$$\frac{D^2}{du^2} J^\mu + R^\mu_{u\nu u} J^\nu = 0 \tag{1.5.1}$$

A Jacobi field along a (null) geodesic is a vector field describing the distance to a nearby geodesic. This is exactly the situation at hand. The transverse coordinates of the Rosen metric measure the distance to the next U-geodesic of the embedding bundle. As a matter of fact, the restriction of a Killing vector field (here: the transverse Rosen directions) to a geodesic is a Jacobi field in any Riemannian manifold.

[4] Throughout the chapter, we have drawn heavily on the unpublished lecture notes of Blau [14].

Penrose limit Rosen coordinates were also the result of a *Penrose limit* of a general space-time manifold. In short, the prescription is to take a null geodesic, select an embedding bundle of other null geodesics, and blow up the neighbourhood of the original geodesic through a Penrose scaling. This 'blowing up' is essentially constructing a (Rosen) coordinate system via the Jacobi field in the infinitesimal range about the null geodesic. Anything at finite distance is pushed off to infinity, notably the range of the transverse coordinates always becomes infinite in the limit.

In the beginning of this chapter we have constructed the Penrose limit via an adapted coordinate system (1.1.1) with U the affine parameters of the null geodesic bundle. In particular, in the subsequent Penrose scaling prescription, we have relied on the metric components g_{UU} and g_{Ui} to vanish, since they would have dominated the limit.

Now, through the geometric notion of the Jacobi field and the neighbourhood of a null geodesic, it is pretty obvious that this coordinate system is merely convenient and can always be constructed.

Since all Jacobi fields, the coordinate system (1.1.1) and Rosen coordinates rely on an embedding bundle of null geodesics, this description necessarily breaks down at focal points, where neighbouring geodesics suddenly intersect. Usually one restricts the range of U to avoid those points. This breakdown, however, is a superficial coordinate effect, as can be seen when going to Brinkmann coordinates. What truly matters is the one null geodesic at the origin, which survives the limit in its entirety and is only sensitive to physical singularities, through $A_{ab}(u)$.

Higher orders In the light of all these facts, why do we have to take the route of adapted coordinates, the Penrose limit to Rosen coordinates and then transform to Brinkmann? Is there not a direct procedure that gives Brinkmann coordinates from a null geodesic in space-time?

As a related question, the Penrose limit procedure immediately extends to higher orders. Simply do not take it absolutely, but rather expand all terms in the metric (1.1.1) in the scaling parameter λ. The starting point is recovered as $\lambda \to 1$ and the plane wave limit as $\lambda \to 0$.

What are the 'higher orders' of the Brinkmann metric? Can we find a concise prescription, and if, is it equivalent, order by order, to the know one for Rosen coordinates? The following publication addresses these questions, indeed developing a method for directly constructing such an expansion.

It turns out that the construction at the beginning of this chapter of Brinkmann as Fermi coordinates can be immediately generalised. One can construct Fermi coordinates, in the form of a power series of geodesic distance, about the defining null geodesic. Next one applies the same Penrose scaling (1.1.2) to these different coordinates. This results in a different λ expansion, the lowest order of which, however, are Brinkmann coordinates, which, as we know, describe the same manifold as Rosen coordinates.

The now following section 1.6 is an unabridged reprint of Blau, Frank and Weiss [1] published in Class. Quantum Grav. **23** *No 11 (7 June 2006) 3993-4010. Its content represents the joint work of the authors. In an attempt to preserve most of the original structure, no changes have been made to text body and formulae, while layout, section numbering and bibliography have been adapted to integrate into the overall theme.*

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Section 1.6

Fermi Coordinates and Penrose Limits

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Abstract We propose a formulation of the Penrose plane wave limit in terms of null Fermi coordinates. This provides a physically intuitive (Fermi coordinates are direct measures of geodesic distance in space-time) and manifestly covariant description of the expansion around the plane wave metric in terms of components of the curvature tensor of the original metric, and generalises the covariant description of the lowest order Penrose limit metric itself, obtained in [15]. We describe in some detail the construction of null Fermi coordinates and the corresponding expansion of the metric, and then study various aspects of the higher order corrections to the Penrose limit. In particular, we observe that in general the first-order corrected metric is such that it admits a light-cone gauge description in string theory. We also establish a formal analogue of the Weyl tensor peeling theorem for the Penrose limit expansion in any dimension, and we give a simple derivation of the leading (quadratic) corrections to the Penrose limit of $AdS_5 \times S^5$.

1.6.1 Introduction

Following the observations in [13, 16, 17, 18, 19] regarding the maximally supersymmetric type IIB plane wave background, its relation to the Penrose limit of $AdS_5 \times S^5$, and the corresponding BMN limit on the dual CFT side[5], the Penrose plane wave limit construction [21] has attracted a lot of attention. This construction associates to a Lorentzian space-time metric $g_{\mu\nu}$ and a null-geodesic γ in that space-time a plane wave metric,

$$(ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \gamma) \quad \rightarrow \quad d\bar{s}^2_\gamma = 2dx^+dx^- + A_{ab}(x^+)x^a x^b dx^{+2} + \delta_{ab}dx^a dx^b \quad , \tag{1.6.2}$$

the right hand side being the metric of a plane wave in Brinkmann coordinates, characterised by the wave profile $A_{ab}(x^+)$.

The usual definition of the Penrose limit [21, 22, 8] is somewhat round-about and in general requires a sequence of coordinate transformations (to adapted or Penrose coordinates, from Rosen to Brinkmann coordinates), scalings (of the metric and the adapted coordinates) and limits.[6] And even though general arguments about the covariance of the Penrose limit [8] show that there is of course something covariant lurking behind that prescription, after having gone through this sequence of operations one has probably pretty much lost track of what sort of information about the original space-time the Penrose limit plane wave metric actually encodes.

This somewhat unsatisfactory state of affairs was improved upon in [15, 23]. There it was shown that the wave profile $A_{ab}(x^+)$ of the Penrose limit metric can be determined from the original metric without taking any limits, and has a manifestly covariant characterisation as the matrix

$$A_{ab}(x^+) = -R_{a+b+}|_{\gamma(x^+)} \tag{1.6.3}$$

of curvature components (with respect to a suitable frame) of the original metric, restricted to the null geodesic γ. This will be briefly reviewed in section 2.

The aim of the present paper is to extend this to a covariant prescription for the expansion of the original metric around the Penrose limit metric, i.e. to find a formulation of the Penrose limit which is such that

- to lowest order one directly finds the plane wave metric in Brinkmann coordinates, with the manifest identification (1.6.3);
- higher order corrections are also covariantly expressed in terms of the curvature tensor of the original metric.

We are thus seeking analogues of Brinkmann coordinates, the covariant counterpart of Rosen coordinates for plane waves, for an arbitrary metric. We will show that this is provided by Fermi coordinates based on the null geodesic γ. Fermi normal coordinates for *timelike* geodesics are well known and are discussed in detail e.g. in [24, 25]. They are natural coordinates for freely falling observers since, in particular, the corresponding Christoffel symbols vanish along the entire worldline of the observer (geodesic), thus embodying the equivalence principle.

[5] see e.g. [20] for a review and further references.

[6] For sufficiently simple metrics and null geodesics it is of course possible to devise more direct ad hoc prescriptions for finding a Penrose limit.

In retrospect, the appearance of Fermi coordinates in this context is perhaps not particularly surprising. Indeed, it has always been clear that, in some suitable sense, the Penrose limit should be thought of as a truncation of a Taylor expansion of the metric in directions transverse to the null geodesic, and that the full expansion of the metric should just be the complete transverse expansion. The natural setting for a covariant transverse Taylor expansion are Fermi coordinates, and thus what we are claiming is that the precise way of saying "in some suitable sense" is "in Fermi coordinates".

In order to motivate this and to understand how to generalise Brinkmann coordinates, in section 3 we will begin with some elementary considerations, showing that Brinkmann coordinates are null Fermi coordinates for plane waves. Discussing plane waves from this point of view, we will also recover some well known facts about Brinkmann coordinates from a slightly different perspective.

In section 4 we introduce null Fermi coordinates in general, adapting the construction of timelike Fermi coordinates in [25] to the null case. These coordinates $(x^A) = (x^+, x^{\bar{a}})$ consist of the affine parameter x^+ along the null geodesic γ and geodesic coordinates $x^{\bar{a}}$ in the transverse directions. We also introduce the covariant transverse Taylor expansion of a function, which takes the form

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(E^{\mu_1}_{\bar{a}_1} \ldots E^{\mu_n}_{\bar{a}_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} f \right) (x^+)\, x^{\bar{a}_1} \ldots x^{\bar{a}_n} \ , \tag{1.6.4}$$

where E^{μ}_{A} is a parallel frame along γ. As an application we show that the coordinate transformation from arbitrary adapted coordinates (i.e. coordinates for which the null geodesic γ agrees with one of the coordinate lines) to Fermi coordinates is nothing other than the transverse Taylor expansion of the coordinate functions in terms of Fermi coordinates.

In section 5, we discuss the covariant expansion of the metric in Fermi coordinates in terms of components of the Riemann tensor and its covariant derivatives evaluated on the null geodesic. We explicitly derive the expansion of the metric up to quadratic order in the transverse coordinates and show that the result is the exact null analogue of the classical Manasse-Misner result [26] in the timelike case, namely

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ &- \left[R_{+\bar{a}+\bar{b}}\, x^{\bar{a}} x^{\bar{b}} (dx^+)^2 + \tfrac{4}{3} R_{+\bar{b}\bar{a}\bar{c}} x^{\bar{b}} x^{\bar{c}} (dx^+ dx^{\bar{a}}) + \tfrac{1}{3} R_{\bar{a}\bar{c}\bar{b}\bar{d}} x^{\bar{c}} x^{\bar{d}} (dx^{\bar{a}} dx^{\bar{b}}) \right] \\ &+ O(x^{\bar{a}} x^{\bar{b}} x^{\bar{c}}) \end{aligned} \tag{1.6.5}$$

where $(x^{\bar{a}}) = (x^-, x^a)$ and all the curvature components are evaluated on γ. The expansion up to quartic order in the transverse coordinates is given in appendix A.1.

In section 6, we show how to implement the Penrose limit in Fermi coordinates. To that end we first discuss the behaviour of Fermi coordinates under scalings $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$ of the metric. Since Fermi coordinates are geodesic coordinates, measuring invariant geodesic distances, Fermi coordinates will scale non-trivially under scalings of the metric, and we will see that the characteristic asymmetric scaling of the coordinates that one performs in whichever way one does the Penrose limit arises completely naturally from the very definition of Fermi coordinates. Combining this with the expansion of the metric of section 5, we then obtain the desired covariant expansion of the metric around its Penrose limit.

The expansion to $O(\lambda)$, for which knowledge of the expansion of the metric in Fermi coordinates to cubic order is required, reads

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \\ &+ \lambda\left[-2R_{+a+-}\, x^a x^- (dx^+)^2 - \tfrac{4}{3}R_{+bac}\, x^b x^c (dx^+ dx^a) - \tfrac{1}{3}R_{+a+b;c}\, x^a x^b x^c (dx^+)^2\right] \\ &+ O(\lambda^2)\ . \end{aligned} \qquad (1.6.6)$$

where the first line is the Penrose limit metric (1.6.2). In particular, if the characteristic covariantly constant null vector $\partial/\partial x^-$ of (1.6.2) is such that it remains Killing at first order it is actually covariantly constant and the first-order corrected metric is that of a pp-wave which is amenable to a standard light-cone gauge description in string theory [12]. Moreover, in general the above metric is precisely such that it admits a modified light-cone gauge in the sense of [27]. The expansion to $O(\lambda^2)$ is given in appendix A.2.

We illustrate the formalism in section 7 by giving a quick derivation of the second order corrections to the Penrose limit of $AdS_5 \times S^5$. These corrections have been calculated before in other ways [28, 29, 30], and the point of this example is not so much to advocate the Fermi coordinate prescription as the method of choice to do such calculations (even though it is geometrically appealing and transparent in general, and the calculation happens to be extremely simple and purely algebraic in this particular case). Rather, the interest is more conceptual and lies in the precise identification of the corrections that have already been calculated (and subsequently been used in the context of the BMN correspondence) with particular components of the curvature tensor of $AdS_5 \times S^5$.

In section 8 we return to the general structure of the λ-expansion of the metric. The leading non-trivial contribution to the metric is the λ^0-term R_{a+b+} (1.6.3) of the Penrose limit, and higher order corrections involve other frame components of the Riemann tensor, each arising with a particular scaling weight λ^w. In the four-dimensional case it was shown in [31], using the Newman-Penrose formalism, that the complex Weyl scalars Ψ_i, $i = 0, \ldots, 4$ scale as λ^{4-i}. This is formally analogous to the scaling of the Ψ_i as $(1/r)^{5-i}$ with the radial distance, the peeling theorem [32, 33, 34] of radiation theory in general relativity. We will show that the present covariant formulation of the Penrose limit significantly simplifies the analysis of the peeling property in this context (already in dimension four) and, using the analysis in [35, 36, 37] of algebraically special tensors and the (partial) generalised Petrov classification of the Weyl tensor in higher dimensions, allows us to establish an analogous result in any dimension.

We hope that the covariant null Fermi normal coordinate expansion of the metric developed here will provide a useful alternative to the standard Riemann normal coordinate expansion, in particular, but not only, in the context of string theory in plane wave backgrounds and perturbations around such backgrounds.

1.6.2 Lightning Review of the Penrose Limit

The traditional systematic construction of the Penrose limit [21, 22, 8] involves the following steps:

1. First one introduces Penrose coordinates (U, V, Y^k) adapted to the null geodesic γ (see [23] for a general construction), in which the metric takes the form

$$ds^2_\gamma = 2dUdV + a(U,V,Y^k)dV^2 + 2b_i(U,V,Y^k)dY^i dV + g_{ij}(U,V,Y^k)dY^i dY^j\ . \qquad (1.6.7)$$

Here the original null-geodesic γ is the curve $(U,0,0)$ with affine parameter U, embedded into the congruence (U,V_0,Y^i_0) of null geodesics labelled by the constant values (V_0,Y^i_0), $i=1,\ldots,d$, of the transverse coordinates.

2. Next one performs an asymmetric rescaling of the coordinates,

$$(U,V,Y^k)=(u,\lambda^2 v,\lambda y_k)\quad, \tag{1.6.8}$$

accompanied by an overall rescaling of the metric, to obtain the one-parameter family of metrics

$$\begin{aligned}\lambda^{-2}ds^2_{\gamma,\lambda} &= 2dudv+\lambda^2 a(u,\lambda^2 v,\lambda y^k)dv^2+2\lambda b_i(u,\lambda^2 v,\lambda y^k)dy^i dv\\ &\quad +g_{ij}(u,\lambda^2 v,\lambda y^k)dy^i dy^j\quad .\end{aligned} \tag{1.6.9}$$

3. Now taking the combined infinite boost and large volume limit $\lambda\to 0$ results in a well-defined and non-degenerate metric $\bar{g}_{\mu\nu}$,

$$\text{Penrose Limit}:\ d\bar{s}^2_\gamma = \lim_{\lambda\to 0}\lambda^{-2}ds^2_{\gamma,\lambda} \tag{1.6.10}$$

$$= 2dudv+\bar{g}_{ij}(u)dy^i dy^j\quad , \tag{1.6.11}$$

where $\bar{g}_{ij}(u)=g_{ij}(u,0,0)$ is the restriction of g_{ij} to the null geodesic γ. This is the metric of a plane wave in Rosen coordinates.

4. One then transforms this to Brinkmann coordinates $(x^A)=(x^+,x^-,x^a)$, $a=1,\ldots,d$, via

$$(u,v,y^k)=(x^+,x^-+\tfrac{1}{2}\dot{\bar{E}}_{ai}\bar{E}^i{}_b x^a x^b,\bar{E}^k{}_a x^a) \tag{1.6.12}$$

where $\bar{E}^a_{\ i}$ is a vielbein for $\bar{g}_{ij}$, i.e. $\bar{g}_{ij}=\bar{E}^a_{\ i}\bar{E}^b_{\ j}\delta_{ab}$, required to satisfy the symmetry condition $\dot{\bar{E}}_{ai}\bar{E}^i{}_b=\dot{\bar{E}}_{bi}\bar{E}^i{}_a$. In these coordinates the plane wave metric takes the canonical form

$$d\bar{s}^2_\gamma=2dx^+dx^-+A_{ab}(x^+)x^a x^b dx^{+2}+\delta_{ab}dx^a dx^b\quad , \tag{1.6.13}$$

with $A_{ab}(x^+)$ given by [9]

$$A_{ab}=\ddot{\bar{E}}_{ai}\bar{E}^i{}_b\quad . \tag{1.6.14}$$

While this is, in a nutshell, the construction of the Penrose limit metric, the above definition looks rather round-about and non-covariant and manages to hide quite effectively the relation between the original data $(g_{\mu\nu},\gamma)$ and the resulting plane wave metric. In principle taking the Penrose limit amounts to assigning the wave profile A_{ab} to the initial data $(g_{\mu\nu},\gamma)$,

$$(g_{\mu\nu},\gamma)\quad\rightarrow\quad A_{ab}\quad . \tag{1.6.15}$$

This certainly begs the question if there is not a more direct (and geometrically appealing) route from $(g_{\mu\nu},\gamma)$ to A_{ab} which elucidates the precise nature of the Penrose limit and the extent to which it encodes generally covariant properties of the original space-time.

Indeed, as shown in [15, 23], there is. Given the affinely parametrised null geodesic $\gamma=\gamma(u)$, the tangent vector $E^\mu_+=\dot{\gamma}^\mu$ is (by definition) parallel transported along γ. We extend this to a pseudo-orthonormal parallel transported frame $(E^\mu_A) = (E^\mu_+, E^\mu_-, E^\mu_a)$ along γ. Thus, in terms of the dual coframe (E^A_μ), the metric restricted to γ can be written as

$$ds^2|_\gamma = 2E^+E^- + \delta_{ab}E^aE^b \ . \tag{1.6.16}$$

The main result of [15] is the observation that the wave profile $A_{ab}(x^+)$ of the associated Penrose limit metric is nothing other than the matrix

$$A_{ab}(x^+) = -R_{a+b+}|_{\gamma(x^+)} \tag{1.6.17}$$

of frame curvature components of the original metric, evaluated at the point $\gamma(x^+)$.

Modulo constant $SO(d)$-rotations this is independent of the choice of parallel frame and provides a manifestly covariant characterisation of the Penrose limit plane wave metric which, moreover, does not require taking any limits. The geometric significance of $A_{ab}(x^+)$ is that it is the transverse null geodesic deviation matrix along γ [38, Section 4.2] of the original metric,

$$\frac{d^2}{du^2}Z^a = A_{ab}(u)Z^b \ , \tag{1.6.18}$$

with Z the transverse geodesic deviation vector. Since the only non-vanishing curvature components of the Penrose limit plane wave metric $d\bar{s}^2_\gamma$ in Brinkmann coordinates (1.6.13) are

$$\bar{R}_{a+b+} = -A_{ab} \ , \tag{1.6.19}$$

this implies that geodesic deviation along the selected null geodesic in the original space-time is identical to null geodesic deviation in the corresponding Penrose limit plane wave metric and shows that it is precisely this information about tidal forces in the original metric that the Penrose limit encodes (while discarding all other information about the original metric).

Let us now consider higher order terms in the expansion of the original metric about the Penrose limit. To that end we return to (1.6.9) and expand in a power-series in λ. To $O(\lambda)$ one has

$$\begin{aligned} \lambda^{-2}ds^2_{\gamma,\lambda} &= 2dudv + \bar{g}_{ij}(u)dy^idy^j \\ &+ \lambda\left(2\bar{b}_i(u)dy^idv + y^k\bar{g}_{ij,k}(u)dy^idy^j\right) + O(\lambda^2) \end{aligned} \tag{1.6.20}$$

where, as before, an overbar denotes evaluation on the null geodesic, i.e. $\bar{g}_{ij,k}(u) = g_{ij,k}(u,0,0)$ etc. We see that in the expansion of the metric in Penrose coordinates these higher order terms are not covariant (e.g. the $\bar{g}_{ij,k}$ are Christoffel symbols).

This raises the question if there is a different way of implementing the Penrose limit which is such that all terms in the λ-expansion of the metric are covariant expressions in the curvature tensor of the original metric.

A ham-handed way to approach this issue would be to seek a λ-dependent (and analytic in λ) coordinate transformations that extends the transformation from Rosen to Brinkmann coordinates and, applied to the above expansion of the metric, results in order by order covariant expressions. However, first of all this strategy puts undue emphasis on the coordinate transformation that relates Penrose coordinates to the new coordinates, rather than on the expansion of the metric itself. Secondly, even if one happens to find a solution to the problem in this way, in all likelihood one will in the end have discovered a coordinate system that is sufficiently natural to have been discoverable by other, less brute-force, means as well. Indeed, we will see in sections 5 and 6, without having to go through the explicit coordinate transformation from Penrose coordinates, that all this is accomplished by Fermi coordinates adapted to the null geodesic γ.

1.6.3 Brinkmann Coordinates are Null Fermi Coordinates

In this section we will discuss Brinkmann coordinates for plane waves from (what will turn out to be) the point of view of Fermi coordinates. The considerations in this section are elementary, but they serve as a motivation for the subsequent general discussion of Fermi coordinates. Moreover, we find it illuminating to recover some well known facts about Brinkmann coordinates and their relation to Rosen coordinates from this perspective.

First of all, we note that a particular solution of the null geodesic equation in Brinkmann coordinates is the curve $\gamma(u) = (u, 0, 0)$ with affine parameter $u = x^+$ (in the Penrose limit context this is obviously just the original null geodesic γ). Along this curve all the Christoffel symbols of the metric are zero (the a priori non-vanishing Christoffel symbols are linear and quadratic in the x^a and thus vanish for $x^a = 0$). This is the counterpart of the usual statement for Riemann normal coordinates that the Christoffel symbols are zero at some chosen base-point. Here we have a geodesic of such base-points.

Next we observe that the straight lines

$$x^A(s) = (x_0^+, sx^-, sx^a) \tag{1.6.21}$$

connecting a point $(x_0^+, 0, 0)$ on γ to the point (x_0^+, x^-, x^a) are also geodesics. In the standard plane wave terminology these are spacelike or null geodesics with zero lightcone momentum, $p_- = x^{+\prime}(s) = 0$, a prime denoting an s-derivative. Thus the coordinate lines of x^- and x^a are geodesics, while x^+ labels the original null geodesic γ. These are the characteristic and defining properties of null Fermi coordinates.

There is also a Fermi analogue of the Riemann normal coordinate expansion of the metric in terms of the Riemann tensor and its covariant derivatives. In the special case of plane waves we have, combining (1.6.13) with (1.6.19),

$$d\bar{s}^2 = 2dx^+dx^- + \delta_{ab}dx^a dx^b - \bar{R}_{a+b+}(x^+)x^a x^b dx^{+2} \ . \tag{1.6.22}$$

Thus in this case the expansion of the metric terminates at quadratic order.

We can also understand (and rederive) the somewhat peculiar coordinate transformation (1.6.12) from Rosen to Brinkmann coordinates from this point of view. Thus this time we begin with the metric

$$d\bar{s}^2 = 2dudv + \bar{g}_{ij}(u)dy^i dy^j \tag{1.6.23}$$

of a plane wave in Rosen coordinates and introduce a pseudo-orthonormal frame $\bar{E}^\mu_A$,

$$\bar{E}_+ = \partial_u \ \ , \ \ \bar{E}_- = \partial_v \ \ , \ \ \bar{E}_a = \bar{E}^i_a \partial_i \tag{1.6.24}$$

where $\bar{E}^a{}_i(u)$ is a vielbein for $\bar{g}_{ij}(u)$. Demanding that this frame be parallel propagated along the null geodesic congruence, $\bar{\nabla}_u \bar{E}^\mu_A = 0$, imposes the condition

$$\partial_u \bar{E}^i_a + \tfrac{1}{2}\bar{g}^{ij}\partial_u \bar{g}_{jk}\bar{E}^k_a = 0 \quad \Leftrightarrow \quad \dot{\bar{E}}_{ai}\bar{E}^i{}_b = \dot{\bar{E}}_{bi}\bar{E}^i{}_a \ \ , \tag{1.6.25}$$

which is thus the geometric significance of the symmetry condition appearing in the transformation from Rosen to Brinkmann coordinates.

Now we consider geodesics $x^\mu(s)$ emanating from γ, i.e. $(u(0), v(0), y^i(0)) = (u_0, 0, 0)$, with the further initial condition that $x^{\mu\prime}(s=0)$ have no component tangent to γ, i.e. vanishing scalar product with E_-,

$$0 = \bar{g}_{\mu\nu}(u_0) x^{\mu\prime}(0) \bar{E}^\nu_-(u_0) = u'(0) \quad . \tag{1.6.26}$$

Then the Euler-Lagrange equations following from

$$L = u'v' + \tfrac{1}{2}\bar{g}_{ij} y^{i\prime} y^{j\prime} \tag{1.6.27}$$

imply that

1. the conserved lightcone momentum p_v is zero, $p_v = u' = 0$, so that $u(s) = u_0$;
2. the transverse coordinates $y^i(s)$ evolve linearly with s, $y^i(s) = y^{i\prime}(0)s$;
3. the solution for $v(s)$ is $v(s) = v'(0)s + \frac{1}{4}\dot{\bar{g}}_{ij}(u_0) y^{i\prime}(0) y^{j\prime}(0) s^2$.

One now introduces the geodesic coordinates $(x^{\bar{a}}) = (x^-, x^a)$ by the condition that the geodesics be straight lines, i.e. via

$$x^{\bar{a}} = \bar{E}^{\bar{a}}_\mu x^{\mu\prime}(0) s \quad . \tag{1.6.28}$$

Substituting this into the above solution of the geodesic equations one finds

$$y^i(s) = \bar{E}^i_a x^a \quad , \quad v(s) = x^- + \tfrac{1}{4}\dot{\bar{g}}_{ij} \bar{E}^i_a \bar{E}^j_b x^a x^b \quad , \tag{1.6.29}$$

which, together with $u = x^+$, is precisely the coordinate transformation (1.6.12) from Rosen coordinates x^μ to Brinkmann coordinates x^A. Finally we note that, as we will explain in section 4, this transformation can also be regarded as the covariant Taylor expansion of the x^μ in the quasi-transverse variables $x^{\bar{a}}$. Here and in the following we use the terminology that "transverse" refers to the variables x^a and "quasi-transverse" to the variables $(x^{\bar{a}}) = (x^-, x^a)$.

1.6.4 Null Fermi Coordinates: General Construction

We now come to the general construction of Fermi coordinates associated to a null geodesic γ in a space-time with Lorentzian metric $g_{\mu\nu}$. Along γ we introduce a parallel transported pseudo-orthonormal frame E^A_μ,

$$ds^2|_\gamma = 2E^+E^- + \delta_{ab}E^aE^b \quad , \tag{1.6.30}$$

with $E^\mu_+ = \dot{\gamma}^\mu$, the overdot denoting the derivative with respect to the affine parameter. As in the previous section, we now consider geodesics $\beta(s) = (x^\mu(s))$ emanating from γ, i.e. with $\beta(0) = x_0 \in \gamma$, that satisfy

$$g_{\mu\nu}(x_0) x^{\mu\prime}(0) E^\nu_-(x_0) \equiv x^{\mu\prime}(0) E^+_\mu(x_0) = 0 \quad . \tag{1.6.31}$$

In comparison with the standard timelike case, we note that the double role played by the tangent vector E_0 to the timelike geodesic, as the tangent vector and as the vector to which the connecting geodesics $\beta(s)$ should be orthogonal, is in the null case shared among the two null vectors E_+ (the tangent vector) and E_- (providing the condition on $\beta(s)$).

Then the Fermi coordinates $(x^A) = (x^+, x^-, x^a)$ of the point $x = \beta(s)$ are defined by

$$(x^A) = (x^+, x^{\bar{a}} = sE^{\bar{a}}_{\mu}(x_0)x^{\mu\prime}(0)) \tag{1.6.32}$$

where $\gamma(x^+) = x_0$ and $\bar{a} = (-, a)$. We note that these definitions imply that

$$E^{\bar{a}}_{\mu}(x_0)x^{\mu\prime}(0) = \left.\frac{\partial x^{\bar{a}}}{\partial s}\right|_{s=0} = \left.\frac{\partial x^{\bar{a}}}{\partial x^{\mu}}\right|_{s=0} x^{\mu\prime}(0) \tag{1.6.33}$$

and

$$\left.\frac{\partial x^{\mu}}{\partial x^{+}}\right|_{\gamma} = \dot{\gamma}^{\mu} = E^{\mu}_{+} \tag{1.6.34}$$

so that on γ the Fermi coordinates are related to the original coordinates x^μ by

$$\left.\frac{\partial x^{A}}{\partial x^{\mu}}\right|_{\gamma} = E^{A}_{\mu} \quad , \quad \left.\frac{\partial x^{\mu}}{\partial x^{A}}\right|_{\gamma} = E^{\mu}_{A} \quad . \tag{1.6.35}$$

Thus we see that Fermi coordinates are uniquely determined by a choice of parallel pseudo-orthonormal frame along the null geodesic γ. How unique is this choice? Let us first consider the case of timelike Fermi coordinates. In this case, there is a frame (E_0, E_k), $k = 1, \ldots, n = d+1$, with $E_0 = \dot{\gamma}$ tangent to the timelike geodesic. Evidently, therefore, the parallel frame is unique up to constant $SO(d+1)$ rotations of the spatial frame E_k. Consequently, the spatial Fermi coordinates x^k, constructed exactly as above, are unique up to these constant rotations.

In the lightlike case, $SO(d+1)$ is deformed to the semi-direct product of transverse $SO(d)$-rotations of the E_a (which have the obvious corresponding effect on the transverse Fermi coordinates x^a) and the Abelian group $\simeq \mathbb{R}^d$ of null rotations about E_+ which acts as

$$(E_+, E_-, E_a) \mapsto (E_+, E_- - \omega^a E_a - \tfrac{1}{2}\delta_{ab}\omega^a\omega^b E_+, E_a + \omega_a E_+) \quad , \tag{1.6.36}$$

where $(\omega^a) \in \mathbb{R}^d$ are constant parameters. Since the corresponding action on the relevant components $E^{\bar{a}}$ of the dual frame is

$$(E^-, E^a) \mapsto (E^-, E^a + \omega^a E^-) \quad , \tag{1.6.37}$$

this action of constant null rotations on the frame induces the transformation

$$(x^-, x^a) \mapsto (x^-, x^a + \omega^a x^-) \tag{1.6.38}$$

of the Fermi coordinates. Thus null Fermi coordinates are unique up to constant transverse rotations and shifts of the x^a by x^-. This should, in particular, be compared and contrasted with the ambiguity

$$Y^k \mapsto Y'^k = Y'^k(Y^k, V) \tag{1.6.39}$$

of Penrose coordinates (1.6.7), which consists of d functions of $(d+1)$ variables rather than $d(d+1)/2$ constant parameters.

For many (in particular more advanced) purposes it is useful to rephrase the above construction of Fermi coordinates in terms of the Synge world function $\sigma(x, x_0)$ [24, 25]. For a point x in the normal convex neighbourhood of x_0, i.e. such that there is a unique geodesic β connecting x to x_0, with $\beta(0) = x_0$ and $\beta(s) = x$, $\sigma(x, x_0)$ is defined by

$$\sigma(x, x_0) = \tfrac{1}{2}s \int_0^s dt\, g_{\mu\nu}(\beta(t))x^{\mu\prime}(t)x^{\nu\prime}(t) \tag{1.6.40}$$

(this is half the geodesic distance squared between x and x_0). Since, up to the prefactor s, $\sigma(x, x_0)$ is the classical action corresponding to the Lagrangian $L = (1/2)g_{\mu\nu}x^{\mu\prime}x^{\nu\prime}$, standard Hamilton-Jacobi theory implies that

$$\sigma_\mu(x, x_0) \equiv \frac{\partial}{\partial x_0^\mu}\sigma(x, x_0) = -s g_{\mu\nu}(x_0)x^{\nu\prime}(0) \ , \tag{1.6.41}$$

as well as

$$\sigma(x, x_0) = \tfrac{1}{2}g_{\mu\nu}(x_0)\sigma^\mu(x, x_0)\sigma^\nu(x, x_0) \ . \tag{1.6.42}$$

In particular, this way of writing things makes it more transparent that something as innocuous looking as $x^{\mu\prime}(0)$ is actually a bitensor, namely not just a vector at x_0 but also a scalar at x.

Thus we can also summarise the construction (1.6.31,1.6.32) of Fermi coordinates in the following way: given $x_0 \in \gamma$, the condition

$$\sigma^\mu(x, x_0)E^+_\mu(x_0) = 0 \tag{1.6.43}$$

selects those points x that can be connected to x_0 by a geodesic with no initial component along γ. Locally around γ this foliates the space-time into hypersurfaces Σ_{x_0} pseudo-orthogonal to γ. For $x \in \Sigma_{x_0}$, its quasi-transverse Fermi coordinates $x^{\bar{a}}$ are then defined by

$$x^{\bar{a}} = -\sigma^\mu(x, x_0)E^{\bar{a}}_\mu(x_0) \ . \tag{1.6.44}$$

Conversely, for $x \in \Sigma_{x_0}$, the $\sigma^\mu(x, x_0)$ can be expressed in terms of the Fermi coordinates of x (using $E^\mu_A E^A_\nu = \delta^\mu_\nu$) as

$$\sigma^\mu(x, x_0) = E^\mu_A E^A_\nu \sigma^\nu(x, x_0) = -E^\mu_{\bar{a}} x^{\bar{a}} \ . \tag{1.6.45}$$

It now follows from the Hamilton-Jacobi equation (1.6.42) that the geodesic distance squared of a point $x = (x^+, x^-, x^a)$ to $x_0 = (x^+, 0, 0)$ is

$$2\sigma(x, x_0) = \sigma^\mu(x, x_0)E^A_\mu(x_0)E^\nu_A(x_0)\sigma_\nu(x, x_0) = \delta_{ab}x^a x^b \ . \tag{1.6.46}$$

The $\sigma^\mu = \sigma^\mu(x, x_0)$ also appear naturally in the manifestly covariant Taylor expansion of a function $f(x)$ around x_0,

$$f(x) = \sum_{n=0}^\infty (-1)^n \frac{1}{n!} \left(\sigma^{\mu_1} \ldots \sigma^{\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} f\right)(x_0) \ . \tag{1.6.47}$$

This can e.g. be seen by beginning with the ordinary Taylor expansion of $f(x) = f(\beta(s))$, regarded as a function of the single variable s, around $s = 0$, and using the geodesic equation to convert resulting second derivatives of $x^\mu(s)$ into first derivatives. There is an analogous covariant Taylor expansion for higher-rank tensor fields [25] which, in addition to the above component-wise covariant expansion, also involves parallel transport from x_0 to x.

If we want to expand f not around a point x_0 but only in the directions quasi-transverse to a geodesic γ with $\gamma(x^+) = x_0$, we can use the parallel frame to project out the direction tangential to γ. Indeed, for $x \in \Sigma_{x_0}$ we can use (1.6.45) to express σ^μ in terms of the quasi-transverse Fermi coordinates $x^{\bar{a}}$. Plugging this into (1.6.47), one obtains

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(E^{\mu_1}_{\bar{a}_1} \dots E^{\mu_n}_{\bar{a}_n} \nabla_{\mu_1} \dots \nabla_{\mu_n} f \right) (x^+) \, x^{\bar{a}_1} \dots x^{\bar{a}_n} \quad . \tag{1.6.48}$$

This is a Taylor expansion in the quasi-transverse Fermi coordinates $(x^{\bar{a}}) = (x^-, x^a)$, with the full dependence on x^+ retained.

When $f(x)$ is itself a coordinate function, $f(x) = x^\mu$, say, then $\nabla_{\mu_1} f = \delta^\mu_{\mu_1}$ and, for $n \geq 2$,

$$\nabla_{(\mu_1} \dots \nabla_{\mu_n)} f = -\nabla_{(\mu_1} \dots \nabla_{\mu_{n-2}} \Gamma^\mu{}_{\mu_{n-1}\mu_n)} \equiv -\Gamma^\mu{}_{(\mu_1 \dots \mu_n)} \tag{1.6.49}$$

(the covariant derivatives act only on the lower indices) are the generalised Christoffel symbols. Provided that $\{x^\mu\}$ is an adapted coordinate system, in the sense that γ coincides with one of its coordinate lines (Penrose coordinates (1.6.7) are a special case of this), this gives us on the nose the coordinate transformation between such adapted coordinates and Fermi coordinates,

$$x^\mu(x^+, x^{\bar{a}}) = x^\mu(x^+) + E^\mu_{\bar{a}_1}(x^+) x^{\bar{a}_1} - \sum_{n=2}^{\infty} \left(\Gamma^\mu{}_{(\mu_1 \dots \mu_n)} E^{\mu_1}_{\bar{a}_1} \dots E^{\mu_n}_{\bar{a}_n} \right) (x^+) \, x^{\bar{a}_1} \dots x^{\bar{a}_n} \quad . \tag{1.6.50}$$

Thus the coordinate transformation between adapted and Fermi coordinates is nothing other than the quasi-transverse Taylor expansion of the adapted coordinates.

While formally the above equation is correct for an arbitrary coordinate system, it is less explicit if the coordinate system is not adapted since x^+, the coordinate along the geodesic, is then non-trivially related to the x^μ.

In the special case of Rosen coordinates for plane waves, the above expansion is finite and reduces to the standard result (1.6.12,1.6.29). To see this e.g. for the Rosen coordinate v, one calculates

$$v(x^+, x^-, x^a) = v(x^+) + \left(\bar{E}^\mu_{\bar{a}} \partial_\mu v \right) (x^+) \, x^{\bar{a}} + \frac{1}{2} \left(\bar{E}^\mu_{\bar{a}} \bar{E}^\nu_{\bar{b}} \nabla_\mu \partial_\nu v \right) (x^+) \, x^{\bar{a}} x^{\bar{b}} \tag{1.6.51}$$

with all higher order terms vanishing, and uses that on the geodesic $v = 0$, that $\bar{E}^v_- = 1$, $\bar{E}^v_a = 0$ (1.6.24), and that the only non-trivial $\Gamma^v_{\mu\nu}$ is $\Gamma^v_{ij} = -\frac{1}{2}\dot{g}_{ij}$, to find yet again

$$v = x^- + \tfrac{1}{4}\dot{g}_{ij} \bar{E}^i_a \bar{E}^j_b x^a x^b \quad . \tag{1.6.52}$$

1.6.5 Expansion of the Metric in Null Fermi Coordinates

We will now discuss the metric in Fermi coordinates, given by an expansion in the quasi-transverse Fermi coordinates $x^{\bar{a}}$.

First of all it follows from (1.6.30) and (1.6.35) that to zero'th order, i.e. restricted to the null geodesic γ at $x^{\bar{a}} = 0$, the metric is the flat metric.

Moreover, there are no linear terms in the metric, i.e. the Christoffel symbols restricted to γ are zero (the main characteristic of Fermi coordinates in general). To see this, note that the geodesic equation applied to the geodesic straight lines

$$(x^A(s)) = (x^+, x^{\bar{a}}(s) = v^{\bar{a}} s) \tag{1.6.53}$$

implies

$$\frac{d^2}{ds^2} x^A(s) + \Gamma^A{}_{BC} \frac{d}{ds} x^B(s) \frac{d}{ds} x^C(s) = 0 \quad \Rightarrow \quad \Gamma^A{}_{\bar{b}\bar{c}}(x^+, v^{\bar{a}} s)\, v^{\bar{b}} v^{\bar{c}} = 0 \ \ . \tag{1.6.54}$$

Since at $s = 0$ this has to be true for all $v^{\bar{a}}$, we conclude that

$$\Gamma^A{}_{\bar{b}\bar{c}}|_\gamma = 0 \ \ . \tag{1.6.55}$$

Moreover, since the frames E^A_μ are parallel propagated along γ, it follows that in Fermi coordinates

$$\nabla_+ E^A_{\mu=B} = \nabla_+ \delta^A_{\mu=B} = 0 \quad \Rightarrow \quad \Gamma^A{}_{B+}|_\gamma = 0 \ \ . \tag{1.6.56}$$

Together, these two results imply that all Christoffel symbols are zero along γ,

$$\Gamma^A{}_{BC}|_\gamma = 0 \ \ . \tag{1.6.57}$$

To determine the quadratic term in the expansion of the metric, we need to look at the derivatives of the Christoffel symbols. Differentiating (1.6.57) along γ one finds

$$\Gamma^A{}_{BC,+}|_\gamma = 0 \ \ . \tag{1.6.58}$$

From the definition of the Riemann tensor

$$R^A{}_{BCD} = \Gamma^A{}_{BD,C} - \Gamma^A{}_{BC,D} + \Gamma^A{}_{CE}\Gamma^E{}_{BD} - \Gamma^A{}_{DE}\Gamma^E{}_{BC} \tag{1.6.59}$$

it now follows that

$$\Gamma^A{}_{B+,C}|_\gamma = R^A{}_{BC+}|_\gamma \ \ . \tag{1.6.60}$$

To calculate the derivatives $\Gamma^A{}_{\bar{b}\bar{c},\bar{d}}$, we now use the fact all the symmetrised first derivatives of the Christoffel symbols are zero,

$$\Gamma^A{}_{(\bar{b}\bar{c},\bar{d})}|_\gamma = 0 \ \ . \tag{1.6.61}$$

This follows e.g. from applying the Taylor expansion (1.6.50) for adapted coordinates to the Fermi coordinates themselves: all higher order terms in that expansion, whose coefficients are the above symmetrised derivatives of the Christoffel symbols, have to vanish. Incidentally, the required vanishing of the quadratic terms in the expansion (1.6.50) provides another argument for the vanishing (1.6.55) of the $\Gamma^A{}_{\bar{b}\bar{c}}|_\gamma$.

We can now calculate (with hindsight)

$$(R^A{}_{\bar{b}\bar{c}\bar{d}} + R^A{}_{\bar{c}\bar{b}\bar{d}})|_\gamma = (\Gamma^A{}_{\bar{b}\bar{d},\bar{c}} - \Gamma^A{}_{\bar{b}\bar{c},\bar{d}} + \Gamma^A{}_{\bar{c}\bar{d},\bar{b}} - \Gamma^A{}_{\bar{c}\bar{b},\bar{d}})|_\gamma \tag{1.6.62}$$

and use (1.6.61) to conclude that

$$\Gamma^A{}_{\bar{b}\bar{c},\bar{d}}|_\gamma = -\tfrac{1}{3}(R^A{}_{\bar{b}\bar{c}\bar{d}} + R^A{}_{\bar{c}\bar{b}\bar{d}})|_\gamma \quad . \tag{1.6.63}$$

Since we now have all the derivatives of the Christoffel symbols on γ, we equivalently know all the second derivatives $g_{AB,CD}|_\gamma$ of the metric, namely

$$\begin{aligned} g_{AB,C+}|_\gamma &= 0 \\ g_{++,\bar{c}\bar{d}}|_\gamma &= 2R_{+\bar{c}\bar{d}+}|_\gamma \\ g_{+\bar{b},\bar{c}\bar{d}}|_\gamma &= -\tfrac{2}{3}(R_{+\bar{c}\bar{b}\bar{d}} + R_{+\bar{d}\bar{b}\bar{c}})|_\gamma \\ g_{\bar{a}\bar{b},\bar{c}\bar{d}}|_\gamma &= -\tfrac{1}{3}(R_{\bar{c}\bar{a}\bar{d}\bar{b}} + R_{\bar{c}\bar{b}\bar{d}\bar{a}})|_\gamma \quad . \end{aligned} \tag{1.6.64}$$

Thus the expansion of the metric to quadratic order is

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ &- \left[R_{+\bar{a}+\bar{b}}\, x^{\bar{a}}x^{\bar{b}}(dx^+)^2 + \tfrac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) + \tfrac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}})\right] \\ &+ O(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}) \end{aligned} \tag{1.6.65}$$

where all the curvature components are evaluated on the null geodesic. This is the precise null analogue of the Manasse-Misner result [26, 25] in the timelike case, i.e. Fermi coordinates associated to a timelike geodesic.

In the timelike case, the expansion of the metric to fourth order was determined in [39]. The calculations in [39], based on repeated differentiation and expansion of the geodesic and geodesic deviation equations associated to $\gamma(u)$ and $\beta(s)$ and expressing the results in terms of components of the Riemann tensor and its covariant derivatives, are straightforward in principle but somewhat tedious in practice. They can be simplified a bit by using, as we have done above, the symmetrised derivative identities following from (1.6.50) instead of the geodesic deviation equations. Either way, some care is required in translating and adapting the intermediate steps in these calculations to the null case (cf. the comment in appendix A.1). However, as far as we can tell (and we have performed numerous checks), the final results for the expansion of the metric in the timelike and null case are just related by the simple index relabelling $(0,k) \leftrightarrow (+,\bar{a})$, where (x^0, x^k) are the Fermi coordinates in the timelike case, with x^0 proper time along the timelike geodesic. In its full glory, the expansion to quartic order (which we will require later on) is given in appendix A.1.

1.6.6 Covariant Penrose Limit Expansion via Fermi Coordinates

We now come to the heart of the matter, namely the description of the Penrose limit in Fermi coordinates. Let us first investigate how Fermi coordinates transform under scalings of the metric. Thus we consider the scaling

$$g_{\mu\nu} \rightarrow g_{\mu\nu}(\lambda) = \lambda^{-2} g_{\mu\nu} \quad . \tag{1.6.66}$$

First of all we note that γ continues to be a null geodesic for the rescaled metric. The scaling of the metric evidently requires a concomitant scaling of the parallel pseudo-orthonormal frame along γ, $E^A \rightarrow E^A(\lambda)$, which must be such that

$$2\lambda^{-2}E^+E^- + \lambda^{-2}\delta_{ab}E^aE^b = 2E^+(\lambda)E^-(\lambda) + \delta_{ab}E^a(\lambda)E^b(\lambda) \quad . \tag{1.6.67}$$

Consequently, for the transverse components $E^a(\lambda)$ we have (up to rotations)

$$E^a(\lambda) = \lambda^{-1} E^a \ . \tag{1.6.68}$$

In order to determine the transformation of the $E^{\pm}(\lambda)$, we recall that in the construction of the Fermi coordinates the component E_+ is fixed to be the tangent vector to γ, independently of the metric, $E_+^\mu = \dot{\gamma}^\mu$. This requirement determines uniquely

$$E^+(\lambda) = E^+ \quad , \quad E^-(\lambda) = \lambda^{-2} E^- \ , \tag{1.6.69}$$

which is related by a boost to the symmetric choice $E^{\pm}(\lambda) = \lambda^{-1} E^{\pm}$. To determine the Fermi coordinates, we note that

$$\sigma^\mu(x, x_0) = \tfrac{1}{2} s g^{\mu\nu}(x_0) \frac{\partial}{\partial x_0^\nu} \int_0^s dt \ g_{\rho\sigma}(\beta(t)) x^{\rho\prime}(t) x^{\sigma\prime}(t) = -s x^{\mu\prime}(0) \tag{1.6.70}$$

is scale invariant. Thus the Fermi coordinates $x^A(\lambda)$ are

$$\begin{aligned} x^+(\lambda) &= x^+ \\ x^-(\lambda) &= -\sigma^\mu E_\mu^-(\lambda) = \lambda^{-2} x^- \\ x^a(\lambda) &= -\sigma^\mu E_\mu^a(\lambda) = \lambda^{-1} x^a \ . \end{aligned} \tag{1.6.71}$$

Writing this as

$$(x^+, x^-, x^a) = (x^+(\lambda), \lambda^2 x^-(\lambda), \lambda x^a(\lambda)) \ , \tag{1.6.72}$$

we see that here the asymmetric rescaling of the coordinates, which is completely analogous to that imposed "by hand" in Penrose coordinates[7],

$$(U, V, Y^k) = (u, \lambda^2 v(\lambda), \lambda y^k(\lambda)) \tag{1.6.73}$$

arises naturally and automatically from the very definition of Fermi coordinates.

To now implement the Penrose limit,

- one can either start with the expansion (1.6.65,1.6.95) of the unscaled metric in its Fermi coordinates, multiply by λ^{-2} and express the metric in terms of the scaled Fermi coordinates, i.e. make the substitution (1.6.72);
- or one takes the expansion of the rescaled metric in its Fermi coordinates $x^A(\lambda)$ and then replaces in that expansion each $x^A(\lambda)$ by the original x^A.

Which point of view one prefers is a matter of taste and depends on whether one thinks of the scale transformation actively, as acting on space-time, or passively on measuring rods. The net effect is the same.

Let us now look at the effect of this operation on the metric (1.6.65,1.6.95), using the language appropriate to the first point of view to determine the powers of λ with which each term in (1.6.95) appears. There is thus an overall λ^{-2}, and each x^a or dx^a contributes a λ whereas x^-

[7] Here we have explicitly indicated the λ-dependence of the new coordinates that we suppressed for notational simplicity in (1.6.8).

and dx^- gives a λ^2 contribution.[8] The first consequence of this is that the flat metric is of order λ^0, the overall λ^{-2} being cancelled by a λ^2 from either one dx^- or two dx^a's. Moreover, precisely one of the quadratic terms in (1.6.65) also gives a contribution of order λ^0, namely $R_{a+b+}x^a x^b (dx^+)^2$, the λ^{-2} being cancelled by the quadratic term in the x^a's. Thus the metric to order λ^0 is

$$ds^2_{\lambda^0} = 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \ . \tag{1.6.74}$$

Comparison with (1.6.13, 1.6.17) or (1.6.22) shows that this is precisely the Penrose limit along γ of the original metric,

$$ds^2_{\lambda^0} = d\bar{s}^2_\gamma \qquad \text{(Penrose Limit)} \tag{1.6.75}$$

obtained here directly in Brinkmann coordinates.

Moreover the expansion to quartic order in (1.6.95) is sufficient to give us the covariant expansion of the metric around its Penrose limit to order λ^2 (a quintic term would scale at least as $\lambda^{-2}\lambda^5 = \lambda^3$). Explicitly, the $O(\lambda)$ term is

$$ds^2_{\lambda^1} = -2R_{+a+-}\ x^a x^- (dx^+)^2 - \tfrac{4}{3}R_{+bac}\ x^b x^c (dx^+ dx^a) - \tfrac{1}{3}R_{a+b+;c}\ x^a x^b x^c (dx^+)^2 \tag{1.6.76}$$

and the expansion to $O(\lambda^2)$ is given in appendix A.2.

One characteristic property of the lowest order (Penrose limit) metric is the existence of the covariantly constant null vector $\partial_- \equiv \partial/\partial x^-$. We see from the above that ∂_- continues to be null at $O(\lambda)$. Actually this property is guaranteed to persist up to and including $O(\lambda^3)$, since a $(dx^-)^2$-term in the metric will scale at least with a power $\lambda^{-2}\lambda^2\lambda^4 = \lambda^4$ (such a term arises e.g. from the last term in (1.6.65) with $\bar{a} = \bar{b} = -$ and $\bar{c} = c, \bar{d} = d$).

Moreover, we see that ∂_- remains Killing to $O(\lambda)$ provided that $R_{+a+-} = 0$. If that condition is satisfied, actually something more is true. Namely ∂_- remains covariantly constant and the metric is that of a pp-wave (plane-fronted wave with parallel rays), whose general form is

$$ds^2_{\text{pp}} = 2dx^+dx^- + \delta_{ab}dx^a dx^b + A(x^+, x^a)(dx^+)^2 + 2B_b(x^+, x^a)(dx^+ dx^b) \ . \tag{1.6.77}$$

As shown in [12], this is precisely the condition for string theory in a curved background to admit a standard (conformal gauge for the world-sheet metric h_{rs}) light-cone gauge $X^+(\sigma,\tau) = p_-\tau$.

More interestingly, perhaps, in general the metric to $O(\lambda)$ is precisely such that it admits a modified light cone gauge $h^{00} = -1$ and $X^+(\sigma,\tau) = p_-\tau$ [27]. Indeed, the conditions on the metric g_{AB} (we do not consider the conditions on the dilaton) found in [27] in order for X^- to have an explicit representation on the transverse Fock space

$$g_{-+} = 1 \quad , \quad g_{-\bar{a}} = 0 \quad , \quad \partial_-^2 g_{AB} = 0 \ , \tag{1.6.78}$$

(see [40, 41] for a discussion of the case $g_{-+} \neq 1$), and for X^- to be auxiliary, $g_{-\bar{a}} = 0$, are satisfied by the $O(\lambda)$ metric (1.6.74, 1.6.76).

[8] Alternatively, for the counting from the second point of view, one uses the fact that the coordinate components $\mathcal{R}(g)_{\alpha_1\cdots\alpha_n\alpha\beta}$ of the "vertices" $\mathcal{R}(g)_{\bar{a}_1\cdots\bar{a}_n AB}x^{\bar{a}_1}\ldots x^{\bar{a}_n}$ appearing in the expansion of the metric $g_{AB}dx^A dx^B$ scale like the metric, $\mathcal{R}(g(\lambda)) = \lambda^{-2}\mathcal{R}(g)$. This can be checked explicitly for the terms written in (1.6.95) and in general follows from the fact that the expansion of the metric $g_{\mu\nu}(\lambda)$ in its Fermi coordinates $x^A(\lambda)$ must be λ^{-2} times the expansion of $g_{\mu\nu}$ in its Fermi coordinates x^A.

1.6.7 Example: $\mathrm{AdS}_5 \times S^5$

We will now illustrate the formalism introduced above by giving a simple purely algebraic derivation of the Penrose limit expansion of the $\mathrm{AdS}_5 \times S^5$ metric to $O(\lambda^2)$. These terms have been calculated before in different ways [28, 29, 30]. In the present framework, the identification of these corrections with certain components of the curvature tensor of $\mathrm{AdS}_5 \times S^5$ is manifest.

Thus consider the unit (curvature) radius metric[9] of that space-time, a null geodesic γ, with $E_\pm$ the lightcone components of the corresponding parallel frame. Let us consider the case that γ has a non-vanishing component along the sphere (i.e. non-zero angular momentum). Then, due to the product structure of the metric, the components of E_+ along S^5 and AdS_5 are geodesic, and since E_+ is null they are of opposite norm squared α^2. Thus we have the decomposition

$$E_\pm = \tfrac{1}{\sqrt{2}}\alpha^{\pm 1}(E_9 \pm E_0) \tag{1.6.79}$$

where E_0 and E_9 are normalised and geodesic in AdS_5 and S^5 respectively. Without loss of generality we can (and will) assume $\alpha = 1$ because we can either perform a boost now or the coordinate transformation $x^\pm \to \alpha^{\pm 1}x^\pm$ later to achieve this. We now extend E_0 and E_9 to parallel orthonormal frames along γ in AdS_5 and S^5,

$$\begin{aligned} ds^2_{\mathrm{AdS}} &= \eta_{\tilde{A}\tilde{B}}E^{\tilde{A}}E^{\tilde{B}} = -(E^0)^2 + \delta_{\tilde{a}\tilde{b}}E^{\tilde{a}}E^{\tilde{b}} \\ ds^2_S &= \delta_{AB}E^AE^B = \ \ (E^9)^2 + \delta_{ab}E^aE^b \ . \end{aligned} \tag{1.6.80}$$

Here $\tilde{A}, \tilde{B}, \ldots = 0, \ldots, 4$, while $a, b, \ldots = 5, \ldots, 8$ etc. Since both spaces are maximally symmetric, the frame components of the curvature tensor are

$$R_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = -(\eta_{\tilde{A}\tilde{C}}\eta_{\tilde{B}\tilde{D}} - \eta_{\tilde{A}\tilde{D}}\eta_{\tilde{B}\tilde{C}}) \quad , \quad R_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} \ , \tag{1.6.81}$$

and therefore the only non-vanishing frame components in the parallel frame $(E_\pm, E_{\tilde{a}}, E_a)$ along γ are

$$\begin{aligned} &R_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = -(\delta_{\tilde{a}\tilde{c}}\delta_{\tilde{b}\tilde{d}} - \delta_{\tilde{a}\tilde{d}}\delta_{\tilde{b}\tilde{c}}) \qquad R_{abcd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \\ &R_{+a+b} = R_{-a-b} = \ \ R_{+a-b} = \ \ R_{-a+b} = \tfrac{1}{2}\delta_{ab} \\ &R_{+\tilde{a}+\tilde{b}} = R_{-\tilde{a}-\tilde{b}} = -R_{+\tilde{a}-\tilde{b}} = -R_{-\tilde{a}+\tilde{b}} = \tfrac{1}{2}\delta_{\tilde{a}\tilde{b}} \end{aligned} \tag{1.6.82}$$

We now have all the information we need to determine the Penrose limit and the higher order corrections. For the Penrose limit we immediately find, from (1.6.74), the result[10]

$$ds^2_{\lambda^0} = 2dx^+dx^- + dx^2 + d\tilde{x}^2 - \tfrac{1}{2}(x^2 + \tilde{x}^2)(dx^+)^2 \ . \tag{1.6.83}$$

This is of course the standard result [18, 19], namely the maximally supersymmetric BFHP plane wave [13].

On symmetry grounds and/or because the curvature tensors are covariantly constant, all the $O(\lambda)$-corrections (1.6.76) to the Penrose limit are identically zero in this case. Actually, (1.6.82)

[9] We can restrict to unit radius since we have already implemented the large volume limit via the λ-expansion.

[10] Here and in the following we use a short-hand notation, $\tilde{x}^2 = \delta_{\tilde{a}\tilde{b}}x^{\tilde{a}}x^{\tilde{b}}$, $xdx = \delta_{ab}x^a dx^b$, etc.

shows that to any order only even numbers of transverse indices $(a, b, \ldots)$ or $(\tilde{a}, \tilde{b}, \ldots)$ can appear in the expansion of the metric, and thus all odd order corrections $O(\lambda^{2n+1})$ to the metric are identically zero.

For the $O(\lambda^2)$-corrections, displayed in (1.6.98), one finds non-zero contributions from the second, fourth and fifth terms in square brackets as well as from the term quadratic in the Riemann tensor, and one can read off the result

$$\begin{aligned} ds^2 &= 2dx^+dx^- + dx^2 + d\tilde{x}^2 - (x^2+\tilde{x}^2)(dx^+)^2 \\ &+ \lambda^2 \Big[-\tfrac{2}{3}(x^2-\tilde{x}^2)(dx^+dx^-) - \tfrac{1}{3}(x^2dx^2 - (xdx)^2) + \tfrac{1}{3}(\tilde{x}^2d\tilde{x}^2 - (\tilde{x}d\tilde{x})^2) \\ &\qquad + \tfrac{2}{3}x^-(xdx - \tilde{x}d\tilde{x})dx^+ + \tfrac{1}{6}((x^2)^2 - (\tilde{x}^2)^2)(dx^+)^2 \Big] + O(\lambda^4) \end{aligned} \tag{1.6.84}$$

While this may not be the world's nicest metric, at least every term in this metric has a clear geometric interpretation in terms of the Riemann tensor of the original AdS $\times$ S metric. This metric can be simplified somewhat, perhaps at the expense of geometric clarity, by the λ-dependent coordinate transformation

$$x^- = w^-(1 - \frac{\lambda^2}{6}(y^2 - z^2)) \ , \ x^a = y^a(1 - \frac{\lambda^2}{12}y^2) \ , \ x^{\tilde{a}} = z^a(1 + \frac{\lambda^2}{12}z^2) \ , \tag{1.6.85}$$

which has the effect of removing the explicit x^- from the metric and eliminating the radial xdx and $\tilde{x}d\tilde{x}$ terms. Performing only the x^--transformation, and neglecting terms of $O(\lambda^4)$, the metric takes the form

$$\begin{aligned} ds^2 &= 2dx^+dw^- + dx^2 + d\tilde{x}^2 - (x^2+\tilde{x}^2)(dx^+)^2 \\ &+ \frac{\lambda^2}{3} \Big[-3(x^2-\tilde{x}^2)(dx^+dw^-) - (x^2dx^2 - (xdx)^2) + (\tilde{x}^2d\tilde{x}^2 - (\tilde{x}d\tilde{x})^2) \\ &\qquad + ((x^2)^2 - (\tilde{x}^2)^2)(dx^+)^2 \Big] + O(\lambda^4) \ . \end{aligned} \tag{1.6.86}$$

With $w^- \to -2x^-$ and $\lambda \to 1/R$, R the radius, this agrees with the metric found in [28]. The subsequent transformation $(x^a, x^{\tilde{a}}) \to (y^a, z^a)$ leads to the metric

$$\begin{aligned} ds^2 &= 2dx^+dw^- + dy^2 + dz^2 - (y^2+z^2)(dx^+)^2 \\ &\quad + \frac{\lambda^2}{2} \Big[(y^4 - z^4)(dx^+)^2 - 2(y^2-z^2)dx^+dw^- + z^2dz^2 - y^2dy^2 \Big] \ , \end{aligned} \tag{1.6.87}$$

which, with $w^- \to x^-$, is identical to the metric found in [29, 30] (via a coordinate transformation similar to (1.6.85) before taking the Penrose limit) and studied there from the point of view of the BMN correspondence [19].

1.6.8 A Peeling Theorem for Penrose Limits

In section 6 we have seen that the leading non-trivial contribution to the metric in a series expansion in the scaling parameter λ arises at $O(\lambda^0)$ from the R_{a+b+} component of the Riemann tensor. And, more generally, we have essentially already seen (and used) there, although we did not phrase it that way, that under a rescaling

$$g_{\mu\nu} \to g(\lambda)_{\mu\nu} = \lambda^{-2} g_{\mu\nu} \tag{1.6.88}$$

of the metric, effectively the components R_{ABCD} of the Riemann tensor restricted to the null geodesic scale as

$$R_{ABCD}(g(\lambda)) = \lambda^{-2+w_A+w_B+w_C+w_D} R_{ABCD}(g) \tag{1.6.89}$$

where the weights are

$$(w_+, w_-, w_a) = (0, 2, 1) \ . \tag{1.6.90}$$

The resulting scaling weights $w = -2 + w_A + w_B + w_C + w_D$ of the frame components of the Riemann tensor are summarised in the table below.

λ^0	λ^1	λ^2	λ^3	λ^4
R_{a+b+}	R_{+-+a}, R_{+abc}	$R_{+-+-}, R_{+a-b}, R_{+-ab}, R_{abcd}$	R_{+-a-}, R_{-abc}	R_{-a-b}

It is also not difficult to see that the leading scaling weight of a component of the Riemann (Weyl) tensor at a point x not on γ is identical to that on γ,

$$R_{ABCD}(x_0) = O(\lambda^w) \quad \Rightarrow \quad R_{ABCD}(x) = O(\lambda^w) \ . \tag{1.6.91}$$

To be specific, in this equation we let both $R_{ABCD}(x_0)$ and $R_{ABCD}(x)$ refer to frame components at the respective points (since the generalised Petrov classification [35, 36, 37] we will employ below refers to such components), the frame at x being obtained by parallel transport of the standard frame at x_0 along the unique geodesic connecting x and x_0.

The statement (1.6.91) is intuitively obvious since moving away from γ involves more insertions of quasi-transverse coordinates $x^{\bar{a}}$ and thus, upon scaling of the coordinates, higher powers of λ. One can base a formal argument along these lines on the covariant Taylor expansion of a tensor. However, for present purposes it is enough to note that the expansion of a tensor at a point $x = (x^+, \lambda^2 x^-, \lambda x^a)$ around the point $x_0 = (x^+, 0, 0)$ is tantamount to an expansion in non-negative powers of λ. The same is true for the frames and this establishes (1.6.91). This argument also shows that the statement (1.6.91) as such is also valid for Fermi coordinate rather than frame components since they agree at x_0 and differ by higher powers of λ at x.

We will now establish the relation of the above results to the peeling property of the Weyl tensor in the Penrose limit context. This was first analysed in the four-dimensional $d = 2$ case in [31], where it was shown that the complex Weyl scalars Ψ_i, $i = 0, \ldots, 4$ scale as λ^{4-i}, the $O(\lambda^0)$-term Ψ_4 corresponding to the type N Penrose limit components C_{a+b+}.

In higher dimensions $d > 2$, instead of complex Weyl scalars (one complex transverse dimension) one has $SO(d)$-tensors of the transverse rotation group, and the appropriate framework is then provided by the analysis in [35, 36, 37]. There the primary classification of the Weyl tensor (according to principal or Weyl type) is based on the boost weight of a frame component of a tensor under the boost

$$(E_+, E_-) \rightarrow (\alpha^{-1} E_+, \alpha E_-) \tag{1.6.92}$$

Evidently, the individual boost weights b_A are

$$(b_+, b_-, b_a) = (-1, +1, 0) \ . \tag{1.6.93}$$

Comparison with (1.6.90) shows that $b_A = w_A - 1$, and thus the relation between w and the boost weight $b = \sum b_A$ of the Riemann or Weyl tensor is

$$b = \sum_A (w_A - 1) = w - 2 \in \{-2, -1, 0, 1, 2\} \ . \tag{1.6.94}$$

In particular, the characterisation in terms of the scaling weight w is equivalent to that in terms of boost weights, and a component with boost weight b scales as λ^{b+2}.

According to the generalised Petrov classification in [35, 36, 37], the component characterising the alignment property of type N has the lowest boost weight $b = -2$, thus scales as λ^0, as we already know from the Penrose limit, type III has $b = -1$, etc.[11] Thus, generalising the result of [31], we have established that the scaling properties (scaling weights) of the frame components of the Weyl tensor are strictly correlated with their algebraic properties. This can be regarded as a formal analogue, in the Penrose limit context, of the standard peeling theorem [32, 33, 34] of radiation theory in general relativity which describes the algebraic properties of the coefficients of the Weyl tensor in a large distance $1/r$ expansion.

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1.6.9 Higher Order Terms

Expansion of the Metric in Fermi Coordinates to Quartic Order

As mentioned in section 5, the expansion of the metric in null Fermi coordinates follows the pattern of the expansion in the timelike case, determined to quartic order in [39]. Thus one has[12]

$$\begin{aligned} ds^2 \;=\;& 2dx^+dx^- + \delta_{ab}dx^a dx^b \\ & -R_{+\bar{a}+\bar{b}}\, x^{\bar{a}}x^{\bar{b}}(dx^+)^2 - \tfrac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) - \tfrac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}}) \\ & -\tfrac{1}{3}R_{+\bar{a}+\bar{b};\bar{c}}\, x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}(dx^+)^2 - \tfrac{1}{4}R_{+\bar{b}\bar{a}\bar{c};\bar{d}}\, x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}(dx^+dx^{\bar{a}}) - \tfrac{1}{6}R_{\bar{a}\bar{c}\bar{b}\bar{d};\bar{e}}\, x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}(dx^{\bar{a}}dx^{\bar{b}}) \\ & +(\tfrac{1}{3}R_{+\bar{a}A\bar{b}}R^A{}_{\bar{c}+\bar{d}} - \tfrac{1}{12}R_{+\bar{a}+\bar{b};\bar{c}\bar{d}})\, x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}(dx^+)^2 \\ & +(\tfrac{2}{15}R_{+\bar{b}A\bar{c}}R^A{}_{\bar{d}\bar{a}\bar{e}} - \tfrac{1}{15}R_{+\bar{b}\bar{a}\bar{c};\bar{d}\bar{e}})\, x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}(dx^+dx^{\bar{a}}) \\ & +(\tfrac{2}{45}R_{A\bar{c}\bar{a}\bar{d}}R^A{}_{\bar{e}\bar{b}\bar{f}} - \tfrac{1}{20}R_{\bar{a}\bar{f}\bar{b}\bar{c};\bar{d}\bar{e}})\, x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}x^{\bar{f}}(dx^{\bar{a}}dx^{\bar{b}}) \\ & +O(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}x^{\bar{d}}x^{\bar{e}}) \end{aligned} \tag{1.6.95}$$

However, the actual calculation of the fourth and higher order terms requires a closer inspection. For example, to determine the metric at quartic order, one needs to express the third

[11] In comparing with [35, 36, 37], one should note that there the metric decays along the null geodesic (connecting an interior point to conformal infinity) whereas here this decay occurs in the directions quasi-transverse to the null geodesic. Thus their C_{0i0j} correspond to our C_{-a-b} etc.

[12] In the second line, the Manasse-Misner result [26, 25], we have corrected a misprint in [39].

derivatives of the Christoffel symbols in terms of Riemann tensors. One such identity is

$$\begin{aligned}\Gamma^{A}_{++,\bar{a}\bar{b}\bar{c}} &= R^{A}{}_{+(\bar{a}|+;|\bar{b}\bar{c})} + R^{A}{}_{(\bar{a}\bar{b}|+;|\bar{c})+} - R^{A}{}_{+(\bar{a}|B}R^{B}{}_{|\bar{b}|+|\bar{c})} \\ &\quad -3R^{A}{}_{B(\bar{a}|+}R^{B}{}_{|\bar{b}|+|\bar{c})} + 3R^{A}{}_{(\bar{a}|B|\bar{b}|}R^{B}{}_{+|\bar{c})+} - 2R^{A}{}_{(\bar{a}|\bar{p}|\bar{b}|}R^{\bar{p}}{}_{+|\bar{c})+} \end{aligned} \tag{1.6.96}$$

As written, this identity is correct both in the null and (with the substitution $(+,\bar{a}) \to (0,k)$) in the timelike case, whereas the expression given in [39, eq.(33)],

$$\begin{aligned}\Gamma^{\mu}_{00,klm} &= R^{\mu}{}_{0(k|0;|lm)} + R^{\mu}{}_{(kl|0;|m)0} - R^{\mu}{}_{0(k|\kappa}R^{\kappa}{}_{|l|0|m)} \\ &\quad -3R^{\mu}{}_{\kappa(k|0}R^{\kappa}{}_{|l|0|m)} + R^{\mu}{}_{(k|p|l|}R^{p}{}_{0|m)0} \end{aligned} \tag{1.6.97}$$

is valid only in the timelike case (where it agrees with (1.6.96)).

Expansion around the Penrose Limit to $O(\lambda^2)$

The covariant Fermi coordinate expansion of the Penrose limit to $O(\lambda^2)$ is

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \delta_{ab}dx^a dx^b - R_{a+b+}x^a x^b (dx^+)^2 \\ &+ \lambda\left[-2R_{+a+-}\, x^a x^-(dx^+)^2 - \tfrac{4}{3}R_{+bac}\, x^b x^c (dx^+dx^a) - \tfrac{1}{3}R_{+a+b;c}\, x^a x^b x^c (dx^+)^2\right] \\ &+ \lambda^2\Big[-R_{+-+-}\, x^- x^-(dx^+)^2 - \tfrac{4}{3}R_{+b-c}\, x^b x^c(dx^+dx^-) - \tfrac{4}{3}R_{+-ac}\, x^- x^c (dx^+dx^a) \\ &\quad -\tfrac{4}{3}R_{+ba-}\, x^b x^-(dx^+dx^a) - \tfrac{1}{3}R_{acbd}\, x^c x^d (dx^a dx^b) - \tfrac{2}{3}R_{+a+-;c}\, x^a x^- x^c (dx^+)^2 \\ &\quad -\tfrac{1}{3}R_{+a+b;-}\, x^a x^b x^-(dx^+)^2 - \tfrac{1}{4}R_{+bac;d}\, x^b x^c x^d (dx^+dx^a) \\ &\quad +(\tfrac{1}{3}R_{+aAb}R^{A}_{c+d} - \tfrac{1}{12}R_{+a+b;cd})\, x^a x^b x^c x^d (dx^+)^2\Big] \\ &\quad +O(\lambda^3) \end{aligned} \tag{1.6.98}$$

Determining the expansion to $O(\lambda^3)$ would require knowledge of the quintic terms in the expansion of the metric in Fermi coordinates.

The entire section 1.6 is an unabridged reprint of Blau, Frank and Weiss [1] published in Class. Quantum Grav. **23** *No 11 (7 June 2006) 3993-4010.*

1.7 Second Example: T^{11}

In addition to the example of $AdS_5 \times S^5$ given in the publication, we would like to present the Fermi-Penrose expansion of the Einstein manifold T^{11}.

The motivation for treating $AdS_5 \times S^5$ is clear – it is one of only two (other than flat space) maximally supersymmetric backgrounds of the type IIB string. Moreover, its Penrose limit found by Blau, Figueroa-O'Farrill, Hull and Papadopoulos in [13] preserves this supersymmetry thus constituting the only other such background.

String theory on $AdS_5 \times S^5$ is also important in the light of the AdS/CFT conjecture. Since this conjecture is presumed to also hold on more general AdS-spaces, other examples were investigated. The simplest extensions of this, according to Gubser [42], are given by trading the S^5 in for Einstein manifolds T^{pq}. These are 5-dimensional coset spaces $SU(2) \times SU(2)/U(1)$, the simplest of which is T^{11} preserving $N = 1$ supersymmetry.

In analogy to the $AdS_5 \times S^5$ example, we shall therefore construct the Fermi-Penrose expansion of $\mathbb{R} \times T^{pq}$ up to second order (which can be extended straight-forwardly to $AdS_5 \times T^{pq}$).

Start with the quite general form of the metric with the set of parameters $\{a, b, c, d\}$ and $\{p, q\}$

$$\begin{aligned} ds^2_{\mathrm{T}11} &= -d^2 dt^2 + c^2(dr - p\cos y_1 dy_2 - q\cos y_3 dy_4)^2 \\ &\quad + a^2(dy_1^2 + \sin^2 y_1 dy_2^2) + b^2(dy_3^2 + \sin^2 y_3 dy_4^2) \end{aligned} \tag{1.7.1}$$

This metric as a sum of squares has the 'natural' vielbein $g_{\mu\nu} = E^a_\mu E^b_\nu \eta_{ab}$

$$E^a_\mu dx^\mu = \left(d\, dt, c(dr - p\cos y^1 dy^2 - q\cos y^3 dy^4), a dy^1, a\sin y^1 dy^2, b dy^3, b\sin y^3 dy^4\right) \tag{1.7.2}$$

In this form, the vielbein is not parallelly transported along the geodesic r, but serves as the initial condition to the parallel transport equation. Solving this gives a vielbein that rotates as we move along r with the two angular velocities $\alpha = \left(\frac{c^2 p}{2a^2}\right)$ and $\beta = \left(\frac{c^2 q}{2b^2}\right)$

$$\begin{aligned} E^a_\mu dx^\mu = \Big(&d\, dt, c(dr - p\cos y^1 dy^2 - q\cos y^3 dy^4), \\ &a\cos(\alpha r) dy^1, -a\sin(\alpha r)\sin y^1 dy^2, b\cos(\beta r) dy^3, -b\sin(\beta r)\sin y^3 dy^4\Big) \end{aligned} \tag{1.7.3}$$

We change from (t, r) to light-cone variables (u, v). Taking the Fermi-Penrose limit we get the associated plane wave in Brinkmann coordinates with constants $A = \frac{cp}{2a^2}$ and $B = \frac{cq}{2b^2}$

$$ds^2|_{\lambda^0} = 2dx^+ dx^- - \tfrac{1}{2}\left(A^2\, x^2 + B^2\, \tilde{x}^2\right) du^2 + dx^2 + d\tilde{x}^2 \tag{1.7.4}$$

The first order terms all vanish, but at second order we get

$$\begin{aligned} ds^2|_{\lambda^2} &= -\tfrac{2}{3}\left(A^2 x^2 + B^2 \tilde{x}^2\right) dx^+ dx^- + \tfrac{1}{6}\left(A^2 x^2 + B^2 \tilde{x}^2\right)^2 (dx^+)^2 \\ &\quad + \tfrac{2}{3} A^2\, x^-\, (x dx) dx^+ + \tfrac{2}{3} B^2\, x^-\, (\tilde{x} d\tilde{x}) dx^+ \\ &\quad + \left(A\, (\varepsilon_{ab}\, x^a dx^b) + B\, (\varepsilon_{ab}\, \tilde{x}^a d\tilde{x}^b)\right)^2 - \frac{1}{3a^2}(\varepsilon_{ab}\, x^a dx^b)^2 - \frac{1}{3b^2}(\varepsilon_{ab}\, \tilde{x}^a d\tilde{x}^b)^2 \end{aligned} \tag{1.7.5}$$

This shall serve as a further demonstration of the formalism we have developed. Einstein spaces will not play a role in what is to come in the following chapters.

Chapter 2

Scalar Field Evolution across Spacetime Singularities

Apart from the promising prospect of having gravitational waves around, Einstein's theory has opened up another completely new line of study: Space-time singularities. On the observational side, they seem to be and remain hidden by the *Cosmic Censorship Hypothesis*, which states that singularities should not be naked by nature, but dressed with an event horizon effectively shielding them from view. On purely theoretical grounds, however, we can take a lot more liberty and subject singularities to tests with a variety of probes.

In tackling these problems, one encounters the first road-block already when trying to pin down the definition of a singularity. Clearly, one can get easily deceived by mere coordinate singularities that would disappear in another chart. The apparent remedy is to focus on invariant quantities like points of infinite curvature or tidal forces between geodesics. But also this approach is not without pitfalls: A point might be qualified singular, but located at infinite geodesic distance and thus effectively beyond life-time experience.

Bringing geodesics and thus a kind of probes into play, one might take the next step and ask whether a point is singular in itself or just in the eye of the beholder. Other probes might experience different behaviour and either compensate or exacerbate background influences, e.g. by infinite excitation of internal modes. Prime example, the first-quantised string is well-defined on geodesically incomplete orbifolds, but singular on some otherwise regular space-times, as demonstrated by Horowitz and Steif [12].

Due to all its complexity, in this work we shall not treat the issue of space-time singularities exhaustively in any way. Rather, we will present two aspects of probing power-law singularities in this chapter and then come back to singularities in the context string theory dualities in chapter four.

First, we shall present the results of Blau, Borunda, O'Loughlin and Papadopoulos [23], in a way the basis for our later work. The authors analysed null geodesics near a certain class of power-law singularities in Szekeres-Iyer metrics, which approximate a host of physical metrics close to their singularity. Taking the Penrose limit with respect to these null geodesics they found a universal behaviour in the vicinity of the singularity: All resulting plane waves were of the singular homogeneous type (1.3.4)

$$ds^2 = 2dudv + \sum_a n_a(n_a - 1)u^{-2}(x^a)^2 du^2 + \delta_{ab}dx^a dx^b \tag{2.0.1}$$

provided that the space-time satisfies the Dominant Energy Condition (DEC).

We will build up on this interesting result in the second part of this chapter culminating in the publication of Blau, Frank and Weiss [2]. On the same Szekeres-Iyer backgrounds, we now use scalar field probes. Applying methods from functional analysis to be explained below, the probe can either show uniquely determined evolution across the singularity or not. According to the criterion of Horowitz and Marolf [43] one would reject the latter case as singular.

In chapter four we shall come back to homogeneous singular plane waves in the context of strings and M-theory: Matrix big-bangs. Models of matrix quantum mechanics dual to string theory on the respective background open up a refreshingly new way of thinking about singularities. Particularly intriguing is the resolution of space-time coordinates into non-commuting matrix objects near the singular point.

2.1 Szekeres-Iyer Metrics and Singular Homogeneous Plane Waves

2.1.1 Szekeres-Iyer metrics

In the context of studying the Cosmic Censorship Hypothesis, Szekeres and Iyer [44] (see also Szekeres and Celerier [45] as well as Blau, Borunda, O'Loughlin, Papadopoulos [23] for a summary) have proposed a large class of *metrics with power-law type singularities*. By construction those metrics are spherically symmetric and in four dimensions encompass a large variety of physically relevant solutions to the Einstein equations. Examples are the Friedmann-Robertson-Walker metrics, Schwarzschild and models of collapsing dust (Tolman-Bondi metrics).

Metric In general $d+2$ dimensions these metrics can be brought to the following form

$$ds^2_{\mathrm{SI}} = -e^{A(U,V)}dUdV + e^{B(U,V)}d\Omega_d^2 \tag{2.1.1}$$

however, close to a singularity described by $x(U,V)=0$ they take on the simpler limit form

$$ds^2_{\mathrm{SI}} = -x^p dUdV + x^q d\Omega_d^2 \tag{2.1.2}$$

as can be seen by expanding $A(U,V) = p\ln x(U,V) + \textit{regular terms}$ and similarly for B and q. Spherically symmetric, the transverse dimensions are written as a d-sphere with colatitude θ and so on

$$d\Omega_d^2 = d\theta^2 + \sin^2\theta d\Omega_{d-1}^2 \tag{2.1.3}$$

The residual isometries $U \to U'(U)$ and $V \to V'(V)$ can be used to make $x(U,V)$ linear in these coordinates with three cases corresponding to three different types of singularities designated throughout this chapter by the parameter $\eta = \{1,0,-1\}$ as in [23]

$$x(U,V) = \begin{cases} \pm(U+V): & \text{spacelike singularity } (\eta=1) \\ \pm(U-V): & \text{timelike singularity } (\eta=-1) \\ \pm U \text{ or } \pm V: & \text{null singularity } (\eta=0) \end{cases} \tag{2.1.4}$$

In the case $\eta=0$ this is obvious, while in the cases $\eta=\pm1$ this can be seen by going from light-cone to coordinates $x=(U+\eta V)$ and $y=(U-\eta V)$

$$ds^2_{\mathrm{SI}} = \eta x^p(dy^2 - dx^2) + x^q d\Omega_d^2 \tag{2.1.5}$$

with $\eta=\pm1$ interchanging the role of time and space coordinate.

Null Geodesics The geodesic equations with affine parameter u for the metric (2.1.5) are (taking θ, the colatitude of the d-sphere, as varying and all other angles constant)

$$2x\ddot{x} + p(\dot{x}^2 + \dot{y}^2) + \eta q x^{q-p}\dot{\theta}^2 = 0 \tag{2.1.6}$$

$$x\ddot{y} + p\dot{y}\dot{x} = 0 \text{ and } x\ddot{\theta} + q\dot{\theta}\dot{x} = 0 \tag{2.1.7}$$

The two equations in the second line have the same structure and can be integrated after multiplication with x^{p-1} (x^{q-1}, respectively). This leads to the conserved momenta P and L

$$(x^p\dot{y}) = \text{const} = P, \qquad (x^q\dot{\theta}) = \text{const} = L \tag{2.1.8}$$

The remaining equation is then automatically solved by imposing the null condition on the tangent vector of the geodesic

$$\dot{x}^\mu\dot{x}^\nu g_{\mu\nu} = \dot{x}^2 - x^{-2p}P^2 - \eta x^{-q-p}L^2 = 0 \tag{2.1.9}$$

While this equation can be integrated, the general solution is of less interest as in using the metric (2.1.5) one is already in the limit of $x \to 0$. Consequently, one of the terms $x^{-2p}P^2$ or $x^{-q-p}L^2$ dominates the behaviour even in the general case of non-zero P^2 and L^2.

2.1.2 Penrose-Fermi limit

Before taking into account that $x \to 0$, we can calculate the Penrose limit of the SI-metric (2.1.5) about an arbitrary geodesic (2.1.6). We proceed as outlined in Blau, Frank and Weiss [1] (reprinted in chapter one of the present work): first constructing an orthonormal frame parallelly transported along the geodesic and then extracting from the Riemann tensor on the geodesic the characteristic information on the resulting Brinkmann metric.

The obvious component of the frame is E_+, the tangent vector on the geodesic

$$E_+ = \dot{x}\partial_x + \dot{y}\partial_y + \dot{\theta}\partial_\theta \tag{2.1.10}$$

Rotational symmetry allows us to take the geodesic to lie in the (x, y, θ) plane and so the frame in the transverse directions (i.e. all angles but θ) is the standard frame of the $(d-1)$-sphere with radius $r = x^{-q/2}$

$$E_{\tilde{a}} = \frac{1}{x^{-q/2}\sin\theta} e_{\tilde{a}} \tag{2.1.11}$$

where $e_{\tilde{a}}$ with $\tilde{a} = \{2 \dots d\}$ is an orthonormal frame for the $d\Omega^2_{d-1}$ (or just $d\phi^2$ for $d = 2$).

The missing space-like frame component can be found by using the orthonormality conditions and fulfilling the parallel transport equation along the geodesic

$$\frac{d}{du}E^\mu_a + \Gamma^\mu_{\nu\rho}E^\nu_a\dot{x}^\rho = 0 \tag{2.1.12}$$

E_- is then determined algebraically by the frame's orthonormality (actually, we would only need it for higher orders of the Penrose-Fermi expansion which we shall not compute here). So with the definition of $h(u) = \frac{1}{2}p\int^u x(u')^{-q/2-1}du'$ and $A(u) \equiv P^{-1}\left(h(u)\dot{x} - x^{-q/2}\right)$, $B(u) \equiv P^{-1}\left(L^2h(u)^2 - \eta x^p\right)$

$$E_1 = L\,A(u)\,\partial_x + L\,h(u)\,x^{-p}\,\partial_y + \eta\left(A(u)\,\dot{x}x^p - Ph(u)x^{-p}\right)\partial_\theta \tag{2.1.13}$$

$$2PE_- = \left(\dot{x}B(u) - 2L^2h(u)A(u)\right)\partial_x - Px^{-p}B(u)\,\partial_y - L\left(x^{-q}B(u) - 2\eta x^{p-q/2}A(u)\right)\partial_\theta$$

Hence we dispose of a parallelly transported orthonormal frame on the geodesic and we can calculate the relevant components of the Riemann tensor in frame coordinates $A_{ab} \equiv -E^{\rho}_{a} E^{\sigma}_{b} R_{u\rho u\sigma}|_{\gamma}(u)$

$$\begin{aligned} A_{11}(u) &= -\tfrac{1}{4}P^2 q(q-2p-2)x(u)^{-2(p+1)} - \eta\tfrac{1}{4}L^2 p(q+2)x(u)^{-(p+q+2)} \\ A_{\tilde{a}\tilde{a}}(u) &= -\tfrac{1}{4}P^2 q(q-2p-2)x(u)^{-2(p+1)} - \eta\tfrac{1}{4}L^2 q(p+2)x(u)^{-(p+q+2)} - L^2 x(u)^{-2q} \end{aligned} \tag{2.1.14}$$

and zero off-diagonal components. This determines the Brinkmann metric entirely

$$ds^2_{\mathrm{BM}} = 2dudv + A_{ab}(u)x^a x^b du^2 + (dx^a)^2 \tag{2.1.15}$$

As in the first chapter, the construction extends immediately to higher orders. Since we are dealing with a metric in the near-singularity limit, we shall refrain from doing so and instead analyse the limit following [23].

2.1.3 Near singularity: Homogeneous Plane Waves

Let us return to the null-geodesic condition on $x(u)$

$$\dot{x}^2 = x^{-2p}P^2 + \eta x^{-q-p}L^2 \tag{2.1.16}$$

Trivially, we can enforce either P- or L-term to dominate by setting the respective other to zero.

But since in using the approximate SI-metric we are already in a $x \to 0$ limit close to the singularity, therefore, depending on the exponents in (2.1.16), one term dominates the behaviour and we can neglect the other.

Accordingly, we distinguish two cases in the following analysis, which both lead to valid null geodesics running into the singularity for $\eta = 1$ (space-like).

For $\eta = -1$ (time-like), only the first (P-dominated) behaviour can occur. Otherwise, due to the square in (2.1.16), the singularity is protected by an angular momentum barrier and cannot be reached by a null geodesic.

Behaviour 1: Dominating P^2-term ($p > q$ for generic non-zero P and L):

$$x(u) = (cu)^{1/(p+1)} \quad \text{with } c = P(p+1) \tag{2.1.17}$$

requires positive exponent $p > -1$ if we want the geodesics to reach the singularity at finite u.

Putting this solution into the general Penrose limit we have found, we can neglect the $\sim\ x^{-(p+q+2)}$ term with respect to the $\sim\ x^{-2(p+1)}$ one

$$\begin{aligned} A_{11} &= -a(a-1)u^{-2} \qquad \text{with } a = \frac{q}{2(p+1)} \\ A_{\tilde{a}\tilde{a}} &= -a(a-1)u^{-2} - L^2\left(P(p+1)u\right)^{-4a} \end{aligned} \tag{2.1.18}$$

Since we are in the range $p > q$ and therefore $p+1 > q$, we can neglect the L^2 in $A_{\tilde{a}\tilde{a}}$ and in the limit arrive at a homogeneous singular plane wave, always.

Behaviour 2: Dominating L^2-term ($p < q$ for generic non-zero P and L):

$$x(u) = (cu)^{2/(p+q+2)} \qquad \text{with } c = \tfrac{1}{2}L(p+q+2) \tag{2.1.19}$$

where $p+q > -2$ and $\eta = -1$ for the singularity to be at finite u.

In the Penrose limit this time we neglect the respective other term $\sim x^{-2(p+1)}$ as compared to $\sim x^{-(p+q+2)}$ and on the solution (2.1.19) we get

$$\begin{aligned} A_{11} &= b(b-1)u^{-2} \qquad \text{with } b = \frac{p}{p+q+2} \\ A_{\tilde{a}\tilde{a}} &= a(a-1)u^{-2} - L^2\left(\tfrac{1}{2}L(p+q+2)u\right)^{-4a} \qquad \text{with } a = \frac{q}{p+q+2} \end{aligned} \tag{2.1.20}$$

In contrast to the former case, this time it is possible that the second term in $A_{\tilde{a}\tilde{a}}$ cannot be neglected, that happens when $p - q > -2$. Later in this chapter we shall, however, see that this is excluded for Szekeres-Iyer metrics satisfying the Dominant Energy Condition (DEC).

Universal power-law behaviour Remarkably, in all other circumstances we arrive at the same limit: Homogeneous singular plane waves. These are plane waves of power-law type in Rosen coordinates

$$ds^2 = 2dUdV + \sum_i U^{2n_i}(dy^i)^2 = 2dUd\tilde{V} + \sum_i U^{2(1-n_i)}(d\tilde{y}^i)^2 \tag{2.1.21}$$

characterised by a universal u^{-2} profile in Brinkmann coordinates.

$$ds^2 = 2dudv + \sum_a n_a(n_a-1)u^{-2}(x^a)^2du^2 + \delta_{ab}dx^adx^b \tag{2.1.22}$$

We have already come across them in chapter one on plane waves with additional symmetry (1.3.4).

This finding is the cornerstone of the conjecture presented by Blau, Borunda, O'Loughlin and Papadopoulos [23] stating that

> *Penrose limits of (in some sense physically reasonable) space-time singularities are singular homogeneous plane waves with wave profile $A_{ab}(u) \sim u^{-2}$.*

Moreover, it is already the proof of the conjecture for Szekeres-Iyer metrics incorporating a wide class of physically reasonable singularities (including the Schwarzschild black hole), all spherically symmetric, however. Probing some isolated examples of anisotropic space-times further spurred confidence in the conjecture (see references in [23], notably Kunze [46] for the Kasner metric), but is not proven in general for the lack of an all-encompassing class of metrics in the spirit of Szekeres-Iyer.

In [23] it was also noted that the frequencies ω_a appearing in the (Brinkmann) wave profile $A_{ab}(u) = -\omega_a^2\delta_{ab}u^{-2}$ are bounded from above by $\omega_a^2 \leq \frac{1}{4}$. This is in accord with the plane waves being of power-law type in Rosen coordinates (2.1.21), as Brinkmann wave profiles with $\omega_a^2 > \frac{1}{4}$ are *not* of this type. Instead, the latter would lead to imaginary exponents and have a more complicated real Rosen form.

The conjecture and proof for Szekeres-Iyer metrics also puts singular homogeneous plane waves into the spotlight as a stage for discussion of other probes near singularities, for example strings. Papadopoulos, Russo and Tseytlin [47] have already discussed these power-law plane waves and shown string theory to be exactly solvable on them.

In chapter four of the present work we will therefore come back to singular homogeneous plane waves in the context of matrix big-bangs. We will see that string theory dualities allow for an alternative description near the singularity in terms of matrix models thus providing a new handle on the backgrounds and objects on them.

Still, we have left a backdoor in the conjecture in not being specific about the term *physically reasonable*, and also have already used it to discard a certain range of space-like singularities. In the following section we will try to offer a remedy with the proposal of [23] of using the (not undisputed) Dominant Energy Condition to determine physical viability.

2.2 Energy Conditions: Dominant, Strong and Weak

Not all conceivable kinds of matter with energy-momentum tensor $T_{\mu\nu}$ are considered physical. A set of *energy conditions* tries to make this notion precise, see e.g. the text-book of Wald [48]. For example the *Weak Energy Condition (WEC)* excludes tachyonic matter by demanding that the energy density as seen by an observer with time-like world-line ξ^μ be positive:

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0. \tag{2.2.1}$$

The *Dominant Energy Condition (DEC)* includes this relation in requiring $P^\mu = -T^\mu_\nu \xi^\nu$ to be time-like or null, saying that the energy flow of the matter as seen by observer ξ^μ should occur with speed of light at most.

For diagonalisable energy-momentum tensors T^μ_ν, the condition simplifies in the basis of eigenvectors. Denoting by $-\rho$ the eigenvalue of the time-like eigenvector and by P_α the space-like eigenvalues, the DEC is equivalent to

$$\rho \geq |P_\alpha| \tag{2.2.2}$$

In the case of equality at least once we say that matter is *extremal* and the *strict* DEC is violated.

For the sake of completeness, we also mention the *Strong Energy Condition (SEC)*

$$R_{\mu\nu}\xi^\mu\xi^\nu = 8\pi(T_{\mu\nu}\xi^\mu\xi^\nu + \tfrac{1}{2}T) \geq 0 \tag{2.2.3}$$

saying, by Einstein's equation, that the stresses are not so large as to make the respective curvature negative. Note that the SEC does *not* imply the WEC, the name is merely to suggest that it is considered an in some sense *physically* stronger condition.

Szekeres-Iyer metrics Now, by the equivalence of curvature and energy we may ask for what range of parameters p and q the Szekeres-Iyer metrics describe non-extremal equations of state. The Einstein tensor of the Szekeres-Iyer metrics $ds^2 = \eta x^p dy^2 - \eta x^p dx^2 + x^q dz^i dz^j$

$$\begin{aligned}
G^y_y &= \tfrac{1}{8}\eta\ (4dq + 2dpq - d(d+1)q^2)x^{-p-2} - \tfrac{1}{2}d(d-1)x^{-q} \\
G^x_x &= \tfrac{1}{8}\eta\ (d(1-d)q^2 - 2dpq)x^{-p-2} - \tfrac{1}{2}d(d-1)x^{-q} \\
G^i_j &= \tfrac{1}{8}\eta\ (4p - 4q + 4dq + d(1-d)q^2)\ \delta^i_j\ x^{-p-2} - \tfrac{1}{2}(d-1)(d-2)\ \delta^i_j\ x^{-q}
\end{aligned} \tag{2.2.4}$$

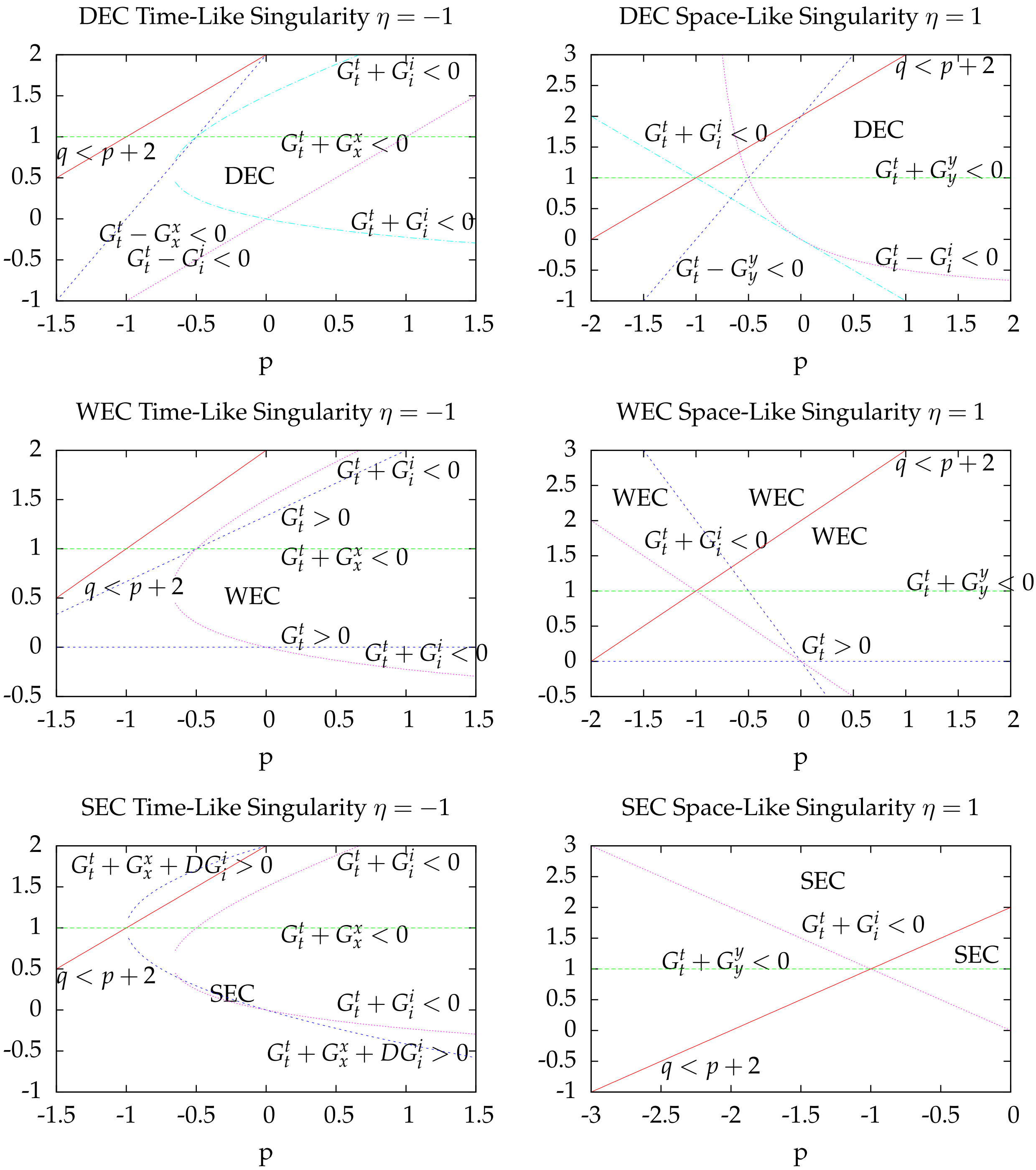

Figure 2.1: *Restrictions of the various energy conditions on the parameter space (p, q) in four space-time dimensions (d=2). In each case the constraints are drawn as lines, and the sector of validity is marked by DEC, SEC or WEC.*

is diagonal, so to test for extremal equations of state we consider $\rho - P_\alpha = 0$ in the vicinity of the singularity $x \to 0$. In the case of time-like singularities ($\eta = -1$) the coordinate y plays the role of time in the equations (thus $\rho \equiv -G^y_y$, $P^x \equiv -G^x_x$ and $P^i \equiv -G^i_i$), for space-like singularities x tracks the time flow.

In both space- and time-like cases, we get a first condition on the parameters p and q by regarding $G^x_x - G^y_y$. For $q > p+2$ the second (x^{-q}) term in equation (2.2.4) dominates as $x \to 0$ and is the only meaningful quantity as the Szekeres-Iyer metrics are already the result of a limiting process. So for $q > p+2$ we get extremal equations of state from $G^x_x - G^y_y = 0$ for both the space and the time-like case. $q \leq p+2$ is thus a necessary condition for the strict DEC to hold and in the following we shall only consider this situation.

From $\rho = G^t_t \geq |P_\alpha|$ we obtain four different constraints on p and q. The results are presented graphically in figure 2.1, where we have also included pictures of the WEC and SEC regions of validity.

The question of actual physical relevance of the energy conditions is disputed and merely an attempt at characterising physical matter. It is not conclusive, and indeed physically interesting counterexamples have been found.

Without entering this discussion, we simply remark the fact stated in Blau, Borunda, O'Loughlin, Papadopoulos [23] that when demanding for the DEC to hold then the critical region $p - q > -2$ in the case $\eta = 1$ (space-like singularity) is excluded from the physical range. Now this is the region for which the second term in $A_{\bar{a}\bar{a}}$ in (2.1.19) was found not to be negligible. In all other cases, close to singularities of the Szekeres-Iyer type the Penrose limit was shown to yield plane waves of power-law type (singular homogeneous plane waves).

2.3 Methods From Functional Analysis

We have reviewed the results of the publication of Blau, Borunda, O'Loughlin, Papadopoulos [23], using geodesics to probe singularities in the very general class of Szekeres-Iyer metrics. According to their findings, physical singularities seem to exhibit a *universal power-law behaviour* about null geodesics close to the singularity.

This remarkable observation was based on null geodesics and immediately raises the question of what might happen when using different probes? The pursuit of this question culminated in the publication we present at the end of this chapter. Putting scalar field probes in a Szekeres-Iyer metric and studying their time-evolution indeed proved a good stage for studying this universality.

The results were obtained using rather different methods of functional analysis. The next section is therefore devoted to introducing the necessary concepts in order to get a feel for the matter. Three examples of operators on a Hilbert space will build up the notion of the problematics of self-adjointness and self-adjoint extensions. We hope this complements the analysis of the publication.

Although we try to be brief on topics that will be covered later, for a stringent line of reasoning some things necessarily overlap. Readers familiar with the uniqueness problem of self-adjoint extensions of operators and its implications on physics can safely skip forward to the actual publication.

2.3.1 Hilbert Space and Time-Evolution for Scalar Fields

Time-like Killing vector Consider a metric with time-like Killing vector ξ^μ and $a(x)^2 = -\xi^\mu \xi_\mu$

$$ds^2 = -a(x)^2 dt^2 + h_{ij}(x) dx^i dx^j \tag{2.3.1}$$

According to the symmetry of the Killing vector, we can split the wave equation for a scalar field ψ moving in this metric into

$$\partial_t^2 \psi = a|h|^{-\frac{1}{2}} \, \partial_i \left(a|h|^{\frac{1}{2}} \, h^{ij} \, \partial_j \right) \psi \equiv -A\psi \tag{2.3.2}$$

Hence the new operator A can be seen as the 'square of the Hamiltonian' governing the time-evolution of the scalar field ψ on a Hilbert space of functions. It is this operator and its functional analytic properties that our analysis of scalar field probes in the Szekeres-Iyer metrics is based on.

Most certainly, we want this operator to be symmetric $\langle \psi_1 | A\psi_2 \rangle = \langle \psi_1 A | \psi_2 \rangle$, and this requirement determines the measure of the scalar product of functions of the Hilbert space

$$\langle \psi_1 | \psi_2 \rangle = -\int d^{d-1}x \, a^{-1} \sqrt{|h|} \, \psi_1^* \psi_2 \tag{2.3.3}$$

Spherical symmetry If we add spherical symmetry then the metric takes the form

$$ds^2 = -a(r)^2 dt^2 + b(r)^2 dr^2 + c(r)^2 d\Omega_d^2 \tag{2.3.4}$$

In this case the operator A of before behaves like $\partial_r(\rho(r)\partial_r\psi)$

$$\partial_t^2 \psi = -A\psi \equiv ab^{-1}c^{-d} \partial_r (ab^{-1}c^d \, \partial_r \psi) + a^2 c^{-2} \partial_\Omega^2 \psi \tag{2.3.5}$$

and contains terms linear in the derivatives $\partial_r \psi$. By a suitable unitary transformation

$$\psi(r) = \lambda(r)\phi(r) \qquad \Rightarrow \; \partial_t^2 \phi = -(\lambda^{-1} A \lambda)\phi = \tilde{A}\phi \sim \partial_r^2 \phi + V_{\text{eff}}\phi \tag{2.3.6}$$

we can trade these linear terms for an effective potential.

The scalar product of the Hilbert space is again defined by requiring symmetry of A, but this time the redefinition of $\psi = \lambda(r)\phi$ gives an integration measure

$$\langle \psi_1 | \psi_2 \rangle = \int a^{-2} \sqrt{|g|} \, dr d\Omega_d \, \lambda^2 \phi_1 \phi_2 = \int dr d\Omega_d \, a^{-2} b^2 \phi_1 \phi_2 \tag{2.3.7}$$

which is flat for $a = \pm b$, e.g. for Szekeres-Iyer metrics. We therefore have two equivalent descriptions of the same physics, either in terms of the Hilbert space $(\psi, \langle \cdot | \cdot \rangle_\psi)$ or the more convenient $(\phi, \langle \cdot | \cdot \rangle_\phi)$. Considering Szekeres-Iyer metrics, we will derive conditions on A to be (essentially) self-adjoint and guarantee a unique generator of time-evolution using the latter picture, henceforth dropping the tilde on $\tilde{A}$.

Unitary Time-Evolution and Self-adjointness Let us return to the time evolution of fields in either picture. In contrast to what is usual in quantum mechanics, the time derivative appears squared in the evolution equation, so we need to formally take the square root of A

$$i\partial_t\phi = (A)^{1/2}\phi \tag{2.3.8}$$

This operation is not uniquely defined unless A is *essentially* self-adjoint, i.e. has a *unique* self-adjoint extension. In contrast to the definition of a singularity as seen by a classical point particle, Horowitz and Marolf [43] defined a field theory on curved background as singular if unique time evolution cannot be guaranteed. To study essential self-adjointness of A, we must answer to symmetry and positivity of A and uniqueness on the boundary. Symmetry of A can be shown by partial integration and is automatically guaranteed since it was used for defining our scalar product.

Still, possible ambiguity can arise in the choice of boundary conditions. Following the exhaustive presentation in the textbook of Reed and Simon [49], we discuss self-adjointness along the lines of three exemplary operators. The examples will be momentum, Laplacian on the half-line and finally a radial Hamiltonian with a potential. At the end of the next section, we will be ready to examine the Szekeres-Iyer metrics in the vicinity of their singular point.

2.3.2 Uniqueness of Self-Adjoint Extensions

This section is to sharpen the senses of the reader for the problem of boundary conditions in self-adjoint extensions of operators, following the textbook of Reed and Simon [49]. Only the statements of the third subsection will be vital for the following discussion of Szekeres-Iyer metrics.

Deficiency Indices and U(1) Ambiguity in Self-Adjoint Extensions

We would now like to comment on the uniqueness of self-adjoint extensions. Consider an operator H, i.e. a linear map from a Hilbert space $\mathcal{H} = (\phi, \langle\phi_1|\phi_2\rangle)$ to itself $\mathcal{H} \to \mathcal{H}$ with a certain domain of definition $\mathcal{D}(H)$. An extension H' of H is defined on a larger domain $\mathcal{D}(H)$ but coincides with H on its original domain $H'\phi \equiv H\phi \quad \forall\phi \in \mathcal{D}(H)$.

An operator H is called *closed* if its graph, i.e. the pair of points $(\phi, H\phi) \in \mathcal{H} \times \mathcal{H}$ is closed. This means that every convergent series $\lim_{n\to\infty}(\phi_n, H\phi_n)$ with $\phi_n \in \mathcal{D}(H)$ has a limit (ϕ, ψ) with $\phi \in \mathcal{D}(H)$ and $\psi = H\phi$. A non-closed operator is closable if it has a closed extension, and the smallest such extension is called its *closure*.

An operator H with domain dense in $\mathcal{H}$ is called *symmetric* if it acts the same to the left and to the right on all functions from its domain.

$$\langle\phi_1|H\phi_2\rangle = \langle H\phi_1|\phi_2\rangle \quad \forall\phi_1, \phi_2 \in \mathcal{D}(H) \tag{2.3.9}$$

We have already used this definition in the setup. The difference to the non-trivial question of self-adjointness lies in the domains, the boundary conditions of integration.

The adjoint of H is defined by

$$\langle\psi|H\phi\rangle = \langle H^*\psi|\phi\rangle = \langle\xi|\phi\rangle \quad \forall\phi \in \mathcal{D}(H) \tag{2.3.10}$$

and since H is dense in $\mathcal{H}$ we conclude that for any given value of the scalar product on the left-hand side the function ξ on the right-hand side is uniquely determined by its scalar product with $\phi \in \mathcal{D}(H)$. On the other hand, ψ need not be part of $\mathcal{D}(H)$ itself. Therefore the adjoint of a symmetric H is an extension of H, $\mathcal{D}(H) \subseteq \mathcal{D}(H^*)$. If the domains are equal, the operator H is called self-adjoint. Symmetric operators always have closed extensions.

In order to see this, consider for example on the Hilbert space $L^2(0,1)$ the derivative operator T with special boundary conditions

$$T = i\partial_x \qquad \mathcal{D}(T) = \{\phi \in \mathcal{H} |\ \phi(0) = \phi(1) = 0\}. \tag{2.3.11}$$

By integration by parts, we get its adjoint T^* which takes the same algebraic form for vanishing boundary terms. This is true for any function from $\mathcal{D}(T)$, as they are all identically zero at the boundary. But since the boundary condition is automatically fulfilled for $\phi \in \mathcal{D}(T)$ in equation (2.3.10), we can now let T^* act on any function of $L^2(0,1)$ without restrictions, so its domain is larger than the one of T. As an example, the function $\psi(x) = e^{ikx}$ is in the domain of T^*, but not of T.

When relaxing the conditions on $\phi \in \mathcal{D}(T)$ and thus extending the domain of T we need to take care when partially integrating to find the adjoint T^*. In order to still satisfy the boundary conditions, we now have to restrict the domain of T^*. We deduce that for an extension H' of a symmetric operator H

$$H \subseteq H' \subseteq H'^* \subseteq H^* \tag{2.3.12}$$

and the extension H' is self-adjoint for equality $H' = H'^*$.

Relaxing the boundary conditions of the example (2.3.11) to

$$T' = i\partial_x \qquad \mathcal{D}(T') = \{\phi \in \mathcal{H} |\ \phi(0) = e^{i\alpha}\phi(1)\}. \tag{2.3.13}$$

with α an arbitrary but fixed parameter, we obtain the adjoint again by integration by parts

$$\int_0^1 \psi^*(i\partial_x\phi)\, dx = -i\psi^*\phi\Big|_0^1 + \int_0^1 (i\partial_x\psi^*)\phi\, dx \tag{2.3.14}$$

Vanishing of the boundary term restricts the domain of the adjoint to functions satisfying

$$(\psi^*(1)\phi(1) - \psi^*(0)\phi(0)) = 0 \quad \Rightarrow \quad \psi^*(1)e^{i\alpha} = \psi^*(0) \tag{2.3.15}$$

This is exactly the same condition as for T' in equation (2.3.13), so $T'^* = T'$ and the operator is self-adjoint. So we found a self-adjoint extension of T, but it is by far not unique. Any choice of the parameter α will give a different self-adjoint extension.

Whether the self-adjoint extension of an operator is unique or not can be made more precise by the notion of *deficiency indices*. Note that the exponentials $\phi = e^{\pm cx}$ for real c are eigenfunctions of the adjoint operator T^* with boundary conditions as in (2.3.11) with imaginary eigenvalues. Those functions are *not* in the domain of T nor of its extensions T'_α, T'^*_α. In general the kernels

$$\mathcal{K}_\pm = \ker(H^* \pm i) \tag{2.3.16}$$

are called *deficiency subspaces* ($\mathcal{K}_+, \mathcal{K}_-$) of A and their dimensions (n_+, n_-) *deficiency indices*. The indices might be any non-negative integer or infinity.

A symmetric operator H has self-adjoint extensions iff its deficiency indices $n_+ = n_- = n$ are equal and the different extensions are parametrised by $U(n)$, the partial isometries from $\mathcal{K}_+ \to \mathcal{K}_-$. The self-adjoint extensions can then be constructed by extending the domain

$$\mathcal{D}(H') = \{\phi_0 + \phi_+ + U\phi_+ \mid \phi_0 \in \mathcal{D}(H), \phi_+ \in \mathcal{K}_+\} \tag{2.3.17}$$

If the indices are both zero $n_+ = n_- = 0$, a unique self-adjoint extension exists and the operator is called *essentially self-adjoint*.

Returning to the example above, solutions to $T^*\phi_\pm = \pm i\phi_\pm$ are given by exponentials $\phi_\pm \sim e^{\mp x}$ and span the deficiency subspaces $\mathcal{K}_\pm$. The isometries $U_\gamma : \mathcal{K}_- \to \mathcal{K}_+$ are given by $\phi_- \to \gamma\phi_+$ and the domain of T can be extended accordingly

$$\mathcal{D}(T') = \{\phi_0 + \beta(\phi_+ + \gamma\phi_+) \mid \phi_0 \in \mathcal{D}(T), \beta \in \mathbb{C}\} \tag{2.3.18}$$

The parameter γ with $|\gamma| = 1$ in this procedure is fixed but arbitrary and reflects the $U(1)$ freedom that was formerly expressed by α. The two correspond by $e^{i\alpha} = \phi(1)/\phi(0) = \frac{1+\gamma e}{e+\gamma}$.

Laplace operator, motion on a half line

An antilinear map $C : \mathcal{H} \to \mathcal{H}\,(C(\alpha\phi + \beta\psi) = \bar{\alpha}C\phi + \bar{\beta}C\psi)$ is called a *conjugation* if it is norm-preserving and squares to the identity $C^2 = 1$.

Von Neumann's theorem states that given a conjugation that commutes with a symmetric operator H and maps its domain of definition onto itself $C : \mathcal{D}(H) \to \mathcal{D}(H)$ the deficiency indices of H are equal. This is clear since then $C(H + i) = (H - i)C$ and C maps the deficiency subspaces $\mathcal{K}_\pm \to \mathcal{K}_\mp$ onto each other.

Let H be the operator $-\partial^2$ on the half-line $L^2(0, \infty)$. Since H commutes with complex conjugation, its deficiency indices must be equal. Consider solutions $H^*\phi + i\phi = 0$. Both

$$\phi_{1/2}(x) = e^{\pm\sqrt{i}x} \tag{2.3.19}$$

are solutions with the same eigenvalue, but only $\phi_2 = \exp(-\sqrt{i}x)$ is in $L^2(0, \infty)$. So the deficiency indices of H are $(1, 1)$ and H can be made self-adjoint by relaxing the boundary conditions to

$$\mathcal{D}(H_a) = \{\psi | \psi \in AC^2[0, \infty], \partial\psi(0) + a\psi(0) = 0\} \tag{2.3.20}$$

with a arbitrary.

The symmetry of these boundary conditions can also be seen by partial integration

$$\int_0^\infty u\partial^2 v = u\partial v\Big|_0^\infty - \partial u \cdot v\Big|_0^\infty + \int_0^\infty \partial^2 u \cdot v \tag{2.3.21}$$

The physical interpretation of these conditions is the following: Consider a quantum particle on the half line with time evolution generated by H. Now the plane waves $e^{\pm ikx}$ are not in $\mathcal{D}(H_a)$ near zero, as they do not satisfy the boundary conditions. The linear combinations $e^{ikx} + \alpha e^{ikx}$ with $\alpha = (ik - a)/(ik + a)$, on the other hand, do satisfy the boundary conditions. Their physical interpretation is that of an incoming particle reflected at zero undergoing a change of phase

$\alpha = (ik - a)/(ik + a)$. Hence different choices of a in H_a and thus different self-adjoint extensions of H correspond to different reflections, different physics.

In the following we are going to take a closer look at Laplacians with an added potential that arise from spherically symmetric configurations. In expressing those operators in spherical coordinates we end up with the radial Laplacian on the half-line, cutting out zero.

Physically we are going to find that a repulsive potential $V(x)$ strong enough to protect the cut-out point at zero from the particle's motion gives a self-adjoint operator, as then we do not have to care about boundary conditions.

The Limit-Point / Limit-Circle Criterion

The actual operators of our interest are of the form $H = -\partial_r^2 + V(r)$, defined on the half-line with a potential arising from spherically symmetric configurations as in (2.3.6).

Their eigenvalue equation $H\phi(r) = E\phi(r)$ has two solutions, but they are not necessarily square-integrable. However, in the vicinity of the critical points zero and infinity it can be shown that if for *one* choice of E both solutions are in L^2, then for *all* E both solutions are in L^2 (Theorem X.6 of Reed and Simon [49]).

The potential $V(x)$ is said to be in the *limit circle case* at zero (resp. infinity) if both solutions to any eigenvalue are square-integrable at zero (resp. infinity), otherwise it is in the *limit point case*. The theorem then tells us that if this holds for one choice of E, it holds for any.

According to Theorem X.7 of Reed and Simon [49] (Weyl's limit point/limit circle criterion), H is essentially self-adjoint if, at the boundary points zero and infinity, $V(x)$ is in the limit point case, i.e. *not* both solutions to one eigenvalue are square integrable.

Furthermore, Theorem X.10 in Reed and Simon [49] states for potentials $V(x)$ continuous and *positive* near $x = 0$: If $V(x) \geq 3/4x^{-2}$ then it is in the limit point case (H essentially self-adjoint) at zero, else it is in the limit circle case (H not essentially self-adjoint).

2.3.3 Szekeres-Iyer Metrics

Finally we come back to Szekeres-Iyer metrics, which we restrict to time-like singularities $\eta = 1$

$$ds^2 = -x^p dt^2 + x^p dx^2 + x^q d\Omega_d^2 \tag{2.3.22}$$

They are examples of the spherically symmetric metrics with a time-like Killing-vector (2.3.4) with $a^2(x) = b^2(x) \equiv x^p$ and $c^2(x) \equiv x^q$. We can therefore proceed as before in calculating the time-evolution operator. After the unitary transformation to a flat scalar product Hilbert space, we obtain the symmetric operator A free from linear derivatives ∂_x. We also split off the angular part, turning $\partial_\Omega^2 Y$ into its eigenvalues $l(l + d - 1)Y$, where Y denotes the spherical harmonics in the transverse d dimensions. Finally, with parameter $s \equiv dq/4$, the wave equation can be written as

$$-\partial_t^2\phi = -\partial_x^2\phi + s(s-1)x^{-2}\phi + l(l+d-1)x^{p-q}\phi \equiv A\phi \tag{2.3.23}$$

Clearly we have now the case of an operator with a kinetic term and a potential as prepared in the last part of the previous subsection. Please remember that only for a potential $V(x) \geq 3/4x^{-2}$ we are in the limit point case and the operator A is essentially self-adjoint.

Considering equation (2.3.23) with potential

$$V(x) = s(s-1)x^{-2} + l(l+d-1)x^{p-q} \tag{2.3.24}$$

for non-vanishing angular momentum we have to distinguish between the cases $p-q < -2$, $p-q = -2$ and $p-q > -2$. The possibility of vanishing angular momentum is included in the discussion of the first case.

$\mathbf{p-q=-2}$**:** First, we consider the case of equally strong first and second (centrifugal) term in the potential. The condition on the prefactor $c \equiv s(s-1) + l(l+d-1)$ is $c \geq 3/4$.
This can be seen by solving the differential equation (2.3.23) with the ansatz $\phi = x^\alpha$. One obtains two solutions x^{α_1}, x^{α_2} with $\alpha_1 = (1+\sqrt{1+4c})/2$ and $\alpha_2 = (1-\sqrt{1+4c})/2$. The first one is always square integrable near $x = 0$, the latter if and only if $\alpha_2 > -1/2$, ie. for $c < 3/4$.
In the 'worst case' of vanishing angular momentum $l = 0$ the exponents of the solutions are simply $\alpha_1 = s$ and $\alpha_2 = 1-s$. This translates into the valid ranges $s \leq -1/2$ or $s \geq 3/2$, i.e. $q \leq -2/d$ or $q \geq 6/d$ for A to be essentially self-adjoint.

$\mathbf{p-q>-2}$**:** In this case the centrifugal (second) term can be neglected compared to the leading x^{-2} term in the potential as $x \to 0$. The discussion reduces to the first case without centrifugal term ($l = 0$).

$\mathbf{p-q<-2}$**:** In this case the centrifugal (second) term exceeds the first term in the potential as $x \to 0$ and for small $x < \epsilon$ the potential will be greater $V(x) > x^{-2}$. The centrifugal barrier protects the singularity at zero. For vanishing angular momentum we refer to case one with $l = 0$.

This completes our introduction to the functional analytic methods that lay the ground-work for the publication presented in the next section. According to the criterion of Horowitz and Marolf [43] we can now define a space-time as singular on the basis of whether or not a scalar field on it has a uniquely defined time-evolution, i.e. governed by an essentially self-adjoint operator at the singularity.

In order not to create too much overlap with next section, we spare the physical reflection of the results for later. Instead, in the following we present the complementary topic of the Friedrichs extension that rounds up the discussion but is not vital to an understanding of the matter.

2.3.4 Energy and the Friedrichs extension

Consider the spherically symmetric metric of before

$$ds^2 = -a(r)^2 dt^2 + b(r)^2 dr^2 + c(r)^2 d\Omega^2. \tag{2.3.25}$$

where the motion of a scalar field is governed by the Lagrangian

$$\mathcal{L} = g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi = -a^{-2}(\partial_t\psi)^2 + b^{-2}(\partial_r\psi)^2 + c^{-2}h^{ij}\nabla_i\psi\nabla_j\psi \tag{2.3.26}$$

The energy-momentum tensor is defined as the variation of the action $S = \int_M \sqrt{-g} d^{d+2}x\, \mathcal{L}$

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu\psi\nabla_\nu\psi - \tfrac{1}{2}g_{\mu\nu}\nabla^\lambda\psi\nabla_\lambda\psi \tag{2.3.27}$$

and is conserved in the sense of $\nabla_\nu T^{\mu\nu} = 0$. So in particular the energy-momentum density $P_\mu = T_{\mu\nu}\xi^\nu$ along a Killing vector is a conserved quantity and we obtain by Stoke's theorem

$$\int_{\mathcal{M}} \sqrt{-g}\nabla_\mu T^{\mu\nu}\xi_\nu \, d^{d+2}x = \int_{\partial\mathcal{M}} \sqrt{g|_{\partial\mathcal{M}}}\, T^{\mu\nu}\xi_\nu n_\mu \, d^{d+1}x = 0, \tag{2.3.28}$$

where $g|_{\partial\mathcal{M}}$ is the pull-back of the metric and n_μ the unit normal on the boundary $\partial\mathcal{M}$. With our space-time foliated by the Killing vector ξ, we define its boundary by a cylinder with top and bottom a slice of the foliation, respectively, and the rim pushed away to infinity. Then the surface-integral turns into a difference between the top and the bottom slice, which must equal each other. Hence the conserved quantity associated with the Killing vector can be written as an integral over the space-like slice and is interpreted as the energy.

With respect to the time-like Killing vector $\xi = \partial_t$ we obtain the energy-density in the coordinates of above

$$T_{tt} = \tfrac{1}{2}(\partial_t\psi)^2 + \tfrac{1}{2}a^2b^{-2}(\partial_r\psi)^2 + \tfrac{1}{2}a^2c^{-2}(\partial_\Omega\psi)^2 \tag{2.3.29}$$

Integration over the space-like slice with the appropriate measure $\int bc^d\sqrt{h}\, d^{d+1}x$ and the unit normal $n^\mu = a^{-1}\xi^\mu$ gives an expression for the energy of a particle at a given time

$$E = \int_S bc^d\sqrt{h}\, d^{d+1}x \; T^{\mu\nu}\xi_\nu n_\mu = \int a^{-1}bc^d\sqrt{h}\, d^{d+1}x \; \left(\tfrac{1}{2}a^2b^{-2}(\partial_r\psi)^2 + \tfrac{1}{2}a^2c^{-2}(\partial_\Omega\psi)^2\right) \tag{2.3.30}$$

Due to the additional factor of a^{-1} brought in by the unit normal, the integration over the time-slice can be seen as an integral with measure $\int a^{-1}bc^d\sqrt{h}\, d^{d+1}x$. The result is the same as the expectation values of the operator A of before (2.3.5) in this measure, as partial integration reveals

$$E = \tfrac{1}{2}\int d^{d+1}x \; \left(\psi\partial_r(ab^{-1}c^d\partial_r\psi) + abc^{d-2}\sqrt{h}\psi\partial_\Omega^2\psi\right) \tag{2.3.31}$$

We therefore have a natural expression of the particle's energy on the derived Hilbert space of before with flat measure (when $a = b$) expressed as a quadratic form on the space of fields.

This can be turned into a new guideline for the construction of self-adjoint extensions. Whereas before we only sought for just any self-adjoint extension of A, we can now ask questions about the energy quadratic form $E(\phi,\psi) = \langle\phi|A|\psi\rangle$.

It turns out that for A positive and symmetric the quadratic form $E(\phi,\psi) = \langle\phi|A|\psi\rangle$ is closable and its closure is the quadratic form of a *unique* self-adjoint extension of A. This extension is called the *Friedrichs extension*.

Some facts about the Friedrichs extension (see Reed and Simon [49])

- Of course, for an essentially self-adjoint operator the Friedrichs extension is the only self-adjoint extension.
- The lower bound of the spectrum of the Friedrichs extension is the lower bound of the closed quadratic form E.
- For C symmetric and $A = C^2$ densely defined, the Friedrichs extension of A^2 is given by C^*C (cf. Laplacian $\Delta = (-i\nabla)\cdot(i\nabla)$).

The requirement of a continuous energy quadratic form is very physical in nature, as every limit of a series of wave-functions should have its energy given by the limit of the series of energies.

The criterion of Horowitz and Marolf [43] characterised a naked singularity as physical if it is endowed with a unique time-evolution by an essentially self-adjoint operator. The unique properties of the Friedrichs extension and its identification with the energy, however, might justify an additional allowance. Even for a range of possible self-adjoint extensions, unique time-evolution can be provided by picking out the more physical Friedrichs extension.

The now following section 2.4 is an unabridged reprint of Blau, Frank and Weiss [2] published in J. High Energy Phys. JHEP08(2006)011. Its content represents the joint work of the authors. In an attempt to preserve most of the original structure, no changes have been made to text body and formulae, while layout, section numbering and bibliography have been adapted to integrate into the overall theme.

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Section 2.4

Scalar Field Probes of Power-Law Space-Time Singularities

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Abstract We analyse the effective potential of the scalar wave equation near generic space-time singularities of power-law type (Szekeres-Iyer metrics) and show that the effective potential exhibits a universal and scale invariant leading inverse square behaviour $\sim x^{-2}$ in the "tortoise coordinate" x provided that the metrics satisfy the strict Dominant Energy Condition (DEC). This result parallels that obtained in [23] for probes consisting of families of massless particles (null geodesic deviation, a.k.a. the Penrose Limit). The detailed properties of the scalar wave operator depend sensitively on the numerical coefficient of the x^{-2}-term, and as one application we show that timelike singularities satisfying the DEC are quantum mechanically singular in the sense of the Horowitz-Marolf (essential self-adjointness) criterion. We also comment on some related issues like the near-singularity behaviour of the scalar fields permitted by the Friedrichs extension.

2.4.5 Introduction

The study of scalar field propagation in non-trivial curved (and possibly singular) backgrounds is of fundamental importance in a variety of contexts including quantum field theory in curved backgrounds, cosmology, the stability and quasi-normal mode analysis of black hole metrics etc.

Typically, this is studied within the context of a particular metric or class of metrics. For certain purposes, however, only the knowledge of the leading behaviour of the metric near a horizon or the singularity is required. In that case, one can attempt to work with a general parametrisation of the metric near that locus and, in this way, ascertain which features of the results that have been obtained previously for particular metrics are special features of those metrics or valid more generally.

In particular, practically all explicitly known metrics with singularities are of "power-law type" [44] in a neighbourhood of the singularity (instead of showing, say, some non-analytic behaviour). In the spherically symmetric case, the leading behaviour of generic metrics with such singularities of power-law type is captured by the 2-parameter family

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \tag{2.4.32}$$

of Szekeres-Iyer metrics [44, 45, 50]. The singularity, located in these coordinates at $x = 0$, is timelike for $\eta = -1$ and spacelike for $\eta = +1$. This class of metrics thus provides an ideal laboratory for investigating the behaviour of particles, fields, strings, ... in the vicinity of a generic singularity of this type.

A first investigation along these lines was performed in [15, 23] in the context of the Penrose Limit, i.e. of probing a space-time via the geodesic deviation of families of massless particles. There it was shown that the plane wave Penrose limits,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \ , \tag{2.4.33}$$

of metrics with singularities of power-law type have a universal u^{-2}-behaviour near the singularity, $A_{ab}(u) \sim u^{-2}$, provided that the near-singularity stress-energy (Einstein) tensor satisfies the strict dominant energy condition (DEC). This behaviour, which is precisely such that it renders the plane wave metric scale invariant [9], had previously been observed in various particular examples and is thus now understood to be a general feature of this class of singularities.

It is then natural to wonder whether a similar universality result can be established in other circumstances or for other kinds of probes and if, analogously, some energy condition plays a role in establishing this. If one considers e.g. the Klein-Gordon equation $\Box\phi = 0$ for scalar fields, it is not difficult to see [51, 52] that under certain conditions the scalar effective potential V_{eff} for general metrics with singularities of power-law type displays an inverse square behaviour, $V_{\text{eff}}(x) \sim x^{-2}$, near the singularity. This observation was then used in [52] to study the quasi-normal modes for black holes with generic singularities of this type.

The purpose of this note is to study other aspects and consequences of this universality. In particular, we will first show that the results obtained in [23], namely the scale invariant inverse square behaviour of the wave profile $A_{ab}(u)$, as well as a crucial [47, 9] lower bound on the coefficients, have a precise and rather striking analogue in the case of a scalar field. Schematically, this analogy can be expressed as

$$\text{strict DEC} \quad \Rightarrow \quad \begin{cases} A_{ab}(u) \to c_a \delta_{ab} u^{-2} & \text{scale invariant} \quad (c_a \geq -1/4) \\ V_{\text{eff}}(x) \to c x^{-2} & \text{scale invariant} \quad (c \geq -1/4) \end{cases} \tag{2.4.34}$$

Once again this shows that this inverse square behaviour, that had been observed before in various specific examples in a variety of contexts, is a general feature of a large class of space-time singularities. The precise statements are derived in sections 2.2 and 2.3 and discussed in section 2.4, while sections 2.5 and 2.6 deal with minor variations of this theme.

We hasten to add that if such an inverse square behaviour were universally true without any further qualifications then it would probably have to be true on rather trivial (dimensional) grounds alone. What makes the results obtained here and in [23] more interesting is that a priori in either case a more singular behaviour can and does occur and is only excluded provided that some further (e.g. positive energy) condition is imposed.

The significance of the x^{-2}-behaviour is that (as anticipated in (2.4.34)), the corresponding Schrödinger operator $-\partial_x^2 + cx^{-2}$, to which we will have reduced the Klein-Gordon operator, defines a scale invariant (c is dimensionless) "conformal quantum mechanics" [53] problem. Thus, here and in [23] we find a rather surprising emergence of scale invariance in the near-singularity limit. One implication of this scale invariance in the plane wave case, discussed in [54], is that it leads to a Hagedorn-like behaviour of string theory in this class of backgrounds that is quite distinct from that in plane wave backgrounds with, say, a constant profile and more akin to that in Minkowski space. It would be interesting to explore other consequences of this near-singularity scale invariance.

This class of scale invariant models has recently also appeared and been discussed in various other related settings, most notably in the analysis of the near-horizon (rather than the near-singularity) properties of black holes, see e.g. [55, 56, 57, 58, 59], where the emergence of scale invariance can largely be attributed to the near-horizon AdS geometry, as well as in quantum cosmology [60].

Having reduced the Klein-Gordon operator to the Schrödinger operator $-\partial_x^2 + cx^{-2}$ (after a separation of variables and a unitary transformation), one can then turn to a more detailed spectroscopy of the Szekeres-Iyer metrics by analysing the properties of this operator. Indeed, as is well known, the inverse square potential is a critical borderline case in the sense that the functional analytic properties of this operator depend in a delicate way on the numerical value of the coefficient c. This value, in turn, depends on the dimension d (number of transverse dimensions) and the Szekeres-Iyer parameter q (it turns out to be independent of p, while the corresponding coefficients c_a in the Penrose limit case typically depend on (p, q) and d).

As one application, we will analyse the Horowitz-Marolf criterion [43] for general singularities of power-law type. Horowitz and Marolf defined a static space-time to be quantum mechanically non-singular (with respect to a certain class of test fields) if the evolution of a probe wave packet is uniquely determined by the initial wave packet (as would be the case in a globally hyperbolic space-time) without having to specify boundary conditions at the classical singularity. This criterion can be rephrased as the condition that the (spatial part of the) Klein-Gordon operator be essentially self-adjoint (and thus have a unique self-adjoint extension).

While such a necessarily only semi-classical analysis is certainly not a substitute for a full quantum gravitational analysis, it nevertheless has its virtues since one can learn what kind of problems persist, can arise or can be resolved when passing from test particles to test fields.

Intuitively one might expect a classical singularity with a sufficiently "positive" (in an appropriate sense) matter content to remain singular even when probed by non-stringy test objects other than classical point particles. This line of thought was one of the motivations for analysing the Horowitz-Marolf criterion in this framework, and we will indeed be able to show (section 3.4) that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

A second issue we will briefly address is that of the allowed near-singularity behaviour of the scalar fields for a given self-adjoint extension (section 3.5). A priori, one might perhaps expect a sufficiently repulsive singularity to be regular in the Horowitz-Marolf sense simply because the corresponding unique self-adjoint extension forces the scalar field to be zero at the singularity, thus in a sense again excluding the singularity from the space-time. It is also possible, however, and potentially more interesting, to have a self-adjoint extension with scalar fields that actually probe the singularity in the sense that they are allowed to take on non-zero values there. We propose to call such singularities "hospitable", establish once again a relation, albeit not a strict correlation, with the DEC, and show among other things that, in a suitable sense, half of the Horowitz-Marolf regular power-law singularities are hospitable whereas the others are not.

2.4.6 Universality of the Effective Scalar Potential for Power-Law Singularities

Geometric Set-Up

Even though we will ultimately be interested in the properties of the scalar wave (Klein-Gordon) equation $(\Box - m^2)\phi = 0$ in the Szekeres-Iyer metrics (2.4.32), to set the stage it will be convenient to begin the discussion in the more general setting of metrics with a hypersurface orthogonal Killing vector. The general set-up here and in section 3.1 is modelled on the approach of [61] (with minor adaptations to allow for both timelike and spacelike singularities).

We begin with the n-dimensional metric

$$ds^2 = \eta a^2 dy^2 + h_{ij} dx^i dx^j \tag{2.4.35}$$

where a and h_{ij} are independent of y, $\xi = \partial_y$ is a hypersurface orthogonal Killing vector with norm $\xi^\mu \xi_\mu = \eta a^2$, and thus timelike (spacelike) for $\eta = -1$ ($\eta = +1$). Correspondingly we assume that the metric h_{ij} induced on the hypersurfaces $\Sigma_y \cong \Sigma$ of constant y is Riemannian (Lorentzian) for $\eta = -1$ ($\eta = +1$).

Denoting the covariant derivatives with respect to the metric h_{ij} by D_i, the wave operator

$$\Box \equiv \frac{1}{\sqrt{-\det g}} \partial_\mu \sqrt{-\det g}\, g^{\mu\nu} \partial_\nu \tag{2.4.36}$$

is easily seen to take the form

$$\Box = a^{-2}(\eta \partial_y^2 + a D^i a D_i) \ . \tag{2.4.37}$$

Thus the massive wave equation $(\Box - m^2)\phi = 0$ can be written as

$$\partial_y^2 \phi = -A\phi \ , \tag{2.4.38}$$

where A is the operator

$$A = \eta a D^i a D_i - \eta a^2 m^2 \quad . \tag{2.4.39}$$

Assuming now spherical symmetry, the metric takes the warped product form

$$ds^2 = \eta a(x)^2 dy^2 - \eta b(x)^2 dx^2 + c(x)^2 d\Omega_d^2 \tag{2.4.40}$$

where $d\Omega_d^2$, $d = n-2$, denotes the standard metric on the d-sphere S^d. It will be apparent from the following that the assumption of spherical symmetry could be relaxed - we will only use the warped product form of the metric in an essential way.

We could fix the residual x-reparametrisation invariance by introducing the "area radius" $r = c(x)$ as a new coordinate. However, for the following it will be more convenient to choose the gauge $a(x) = b(x)$ (i.e. x is a "tortoise coordinate" for $\eta = -1$ respectively "conformal time" for $\eta = +1$),

$$ds^2 = \eta a(x)^2(-dx^2 + dy^2) + c(x)^2 d\Omega_d^2 \quad . \tag{2.4.41}$$

Then the operator A is

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2}\Delta_d - \eta a^2 m^2 \quad , \tag{2.4.42}$$

where $\sigma(x) = c(x)^d$ and Δ_d denotes the Laplacian on S^d.

To put A into standard Schrödinger form, we transform from the functions $\phi(x)$ to the half-densities (cf. (2.4.83)) $\tilde{\phi}(x) = \sigma^{1/2}\phi(x)$. The corresponding unitarily transformed operator $\tilde{A}$ is

$$\begin{aligned} \tilde{A} &= \sigma^{1/2} A \sigma^{-1/2} = -\partial_x^2 + V + \eta a^2 c^{-2}\Delta_d - \eta a^2 m^2 \\ V(x) &= \sigma(x)^{-1/2}(\partial_x^2 \sigma(x)^{1/2}) \quad . \end{aligned} \tag{2.4.43}$$

After the usual separation of variables in the y-direction,

$$\tilde{\phi}(y, x, \theta^a) = \mathrm{e}^{-iEy}\, \tilde{\phi}(x, \theta^a) \quad , \tag{2.4.44}$$

and the decomposition into angular spherical harmonics $Y_{\ell\vec{m}}(\theta^a)$, with

$$\begin{aligned} -\Delta_d Y_{\ell\vec{m}}(\theta^a) &= \ell_d^2 Y_{\ell\vec{m}}(\theta^a) \\ \ell_d^2 &= \ell(\ell + d - 1) \quad , \end{aligned} \tag{2.4.45}$$

the Klein-Gordon equation for the metric (2.4.41) reduces to a standard one-dimensional time-independent Schrödinger equation

$$\left[-\partial_x^2 + V_{\mathrm{eff},\ell}(x)\right]\tilde{\phi}(x) = E^2 \tilde{\phi}(x) \tag{2.4.46}$$

($\tilde{\phi}(x) = \tilde{\phi}_{E,\ell,\vec{m}}(x)$) with effective scalar potential

$$V_{\mathrm{eff},\ell}(x) = V(x) - \eta a(x)^2(\ell_d^2 c(x)^{-2} + m^2) \quad . \tag{2.4.47}$$

The Effective Scalar Potential for Power-Law Singularities

The leading behaviour of generic (spherically symmetric) metrics with singularities of power-law type[1], i.e. metrics of the general form [44]

$$ds^2 = -dt^2 + [t-\tau(r)]^{2a} f(r,t)^2 dr^2 + [t-\tau(r)]^{2b} g(r,t)^2 d\Omega_d^2 \ , \tag{2.4.48}$$

with f and g functions of r and t that are regular and non-vanishing at the location $t = \tau(r)$ of the singularity, is captured by the 2-parameter family of Szekeres-Iyer metrics [44, 45] (see also [23] and the generalisation to string theory backgrounds discussed in [50])

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \ . \tag{2.4.49}$$

The Kasner-like exponents $p, q \in \mathbb{R}$ characterise the behaviour of the geometry near the singularity at $x = 0$. This singularity is timelike for $\eta = -1$ (x is a radial coordinate) and spacelike for $\eta = +1$ (with x a time coordinate). In particular, these metrics possess the hypersurface orthogonal Killing vector ∂_y, and are already in the "tortoise" form (2.4.41), with $a(x)^2 = x^p$ and $c(x)^2 = x^q$. Thus we can directly read off the effective scalar potential from the results of the previous section.

From (2.4.43), we deduce, with $\sigma(x) = x^{dq/2}$, that

$$V(x) = s(s-1)x^{-2} \qquad s = \frac{dq}{4} \ . \tag{2.4.50}$$

Thus, from (2.4.47) we find (see also [52])

$$V_{\text{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{2.4.51}$$

We are interested in the leading behaviour of this potential as $x \to 0$ (subdominant terms can in any case not be trusted as we have only kept the leading terms in the metric (2.4.49)). For the time being we will consider the massless case $m^2 = 0$ (see section 2.5 for $m^2 \neq 0$).

Provided that $s(s-1) \neq 0$, which term in (2.4.51) dominates depends on p and q. When $q < p+2$, one finds

$$q < p+2: \qquad V_{\text{eff},\ell}(x) \to s(s-1)x^{-2} \ . \tag{2.4.52}$$

The two salient features of this potential are the inverse square behaviour and a coefficient c that is bounded from below by $-1/4$,

$$c = s(s-1) \geq -\frac{1}{4} \ , \tag{2.4.53}$$

with equality for $s = 1/2$, i.e. $q = 2/d$.

As mentioned in the introduction, the significance of the x^{-2}-behaviour is that it defines a scale invariant "conformal quantum mechanics" [53] problem, discussed more recently in related contexts e.g. in [55, 56, 57, 58, 59, 60]. Moreover, for practical purposes [52, 65] the virtue of the

[1]Such metrics encompass practically all explicitly known singular spherically symmetric solutions of the Einstein equations like the Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, etc. On the other hand, this class of metrics does prominently *not* include the BKL metrics [62, 63] describing the chaotic oscillatory approach to a spacelike singularity. Whether or not such a behaviour occurs depends in a delicate way on the matter content, see e.g. [64] and references therein.

x^{-2} (as opposed to a more singular) behaviour is that it leads to a standard regular-singular differential operator.

The significance of the bound on c is that in this range the operator $-\partial_x^2 + c/x^2$ is positive, as can be seen by writing

$$-\partial_x^2 + s(s-1)x^{-2} = (\partial_x + sx^{-1})(-\partial_x + sx^{-1}) = (-\partial_x + sx^{-1})^\dagger(-\partial_x + sx^{-1}) \ . \tag{2.4.54}$$

When $q = p+2$, the metric is conformally flat, both terms in (2.4.51) contribute equally, and one again finds the x^{-2}-behaviour

$$q = p+2: \qquad V_{\text{eff},\ell}(x) \to cx^{-2} \ , \tag{2.4.55}$$

where now

$$c = s(s-1) - \eta\ell_d^2 \ . \tag{2.4.56}$$

Thus in this case c is still bounded by $-1/4$ for timelike singularities, while c can become arbitrarily negative for sufficiently large values of ℓ_d^2 for $\eta = +1$.

Once $q > p+2$, the second term in (2.4.51) dominates (for $\ell_d^2 \neq 0$), and one finds the more singular leading behaviour

$$q > p+2: \qquad V_{\text{eff},\ell}(x) \to -\eta\ell_d^2\, x^{-2-a} \quad a > 0 \ . \tag{2.4.57}$$

Examples of metrics with $q \leq p+2$ are the Schwarzschild and Friedmann-Robertson-Walker (FRW) metrics and indeed, as we will recall below, all metrics satisfying the strict Dominant Energy Condition.

In particular, for the $(d+2)$-dimensional (positive or negative mass) Schwarzschild metric, one has

$$\text{Schwarzschild}: \qquad p = \frac{1-d}{d} \qquad q = \frac{2}{d} \ , \tag{2.4.58}$$

as is readily seen by expanding the metric near the singularity and going to tortoise coordinates. Thus the Schwarzschild metric has $s = 1/2$ and c takes on the d-independent extremal value $c = -1/4$, as observed before e.g. in [65, 52] in related contexts.

For decelerating cosmological FRW metrics, with cosmological scale factor (in comoving time) $\sim t^h$, $0 < h < 1$,

$$h = \frac{2}{(d+1)(1+w)} \ , \tag{2.4.59}$$

with w the equation of state parameter, $P = w\rho$, one finds [15, 23]

$$\text{FRW}: \qquad p = q = \frac{2h}{1-h} \ , \tag{2.4.60}$$

as can be seen by going to conformal time. A routine calculation shows that the above result (2.4.52) for the purely x-dependent part of the effective potential (with x conformal time) is actually an exact result, and not an artefact of the near-singularity Szekeres-Iyer approximation.

It remains to discuss the case when $q < p+2$, so that the first term in (2.4.51) would be dominant, but the coefficient $s(s-1) = 0$. When $s = 0$, then one has $q = 0$ and this is generally interpreted [44] as corresponding not to a true central singularity (as the radius of the transverse sphere remains constant as $x \to 0$) but as a shell crossing singularity.

The other possibility is $s = 1$, i.e. $q = 4/d$. This is a case in which (because of the cancellation of the leading terms) subleading corrections to the metric (2.4.49) can become relevant and should be retained. An example of metrics with $s = 1$ is provided by FRW metrics with a radiative equation of state. Using (2.4.60), one has

$$q = \frac{4}{d} \Leftrightarrow h = \frac{2}{d+2} \Leftrightarrow w = \frac{1}{d+1} \quad , \tag{2.4.61}$$

which is precisely the equation of state parameter for radiation. However, as follows from the remark above, in this special case the vanishing of the effective potential for $p = q$ is actually an exact result.

The Significance of the (Strict) Dominant Energy Condition

We have seen that generically the leading behaviour of the scalar effective potential near a singularity of power-law type is either $\sim x^{-2}$ or $\sim x^{p-q}$. We will now recall from [44, 23] that the latter behaviour can arise only for metrics violating the strict Dominant Energy Condition (DEC). While there is nothing particularly sacrosanct about the DEC, and other energy conditions could be considered, the DEC appears to play a privileged role in exploring and understanding the (p, q)-plane of Szekeres-Iyer metrics.

The *Dominant Energy Condition* on the stress-energy tensor T^μ_ν (or Einstein tensor G^μ_ν) [38] requires that for every timelike vector v^μ, $T_{\mu\nu}v^\mu v^\nu \geq 0$, and $T^\mu_\nu v^\nu$ be a non-spacelike vector. This may be interpreted as saying that for any observer the local energy density is non-negative and the energy flux causal.

The Einstein tensor of Szekeres-Iyer metrics is diagonal, hence so is the corresponding stress-energy tensor. In this case, the DEC reduces to

$$\rho \geq |P_i| \quad , \tag{2.4.62}$$

where $-\rho$ and P_i, $i = 1, \ldots, d+1$ are the timelike and spacelike eigenvalues of T^μ_ν respectively. We say that the *strict* DEC is satisfied if these are strict inequalities and we will say that the matter content (or equation of state) is "extremal" if at least one of the inequalities is saturated.

Now it follows from the explicit expression for the components

$$\begin{aligned} G^x_x &= -\tfrac{1}{2}d(d-1)x^{-q} - \tfrac{1}{8}\eta dq((d-1)q + 2p)x^{-(p+2)} \\ G^y_y &= -\tfrac{1}{2}d(d-1)x^{-q} + \tfrac{1}{8}\eta dq(2p + 4 - (d+1)q)x^{-(p+2)} \end{aligned} \tag{2.4.63}$$

of the Einstein tensor that for $q > p+2$ the relation between $-\rho$ and the radial pressure P_r (identified with G^x_x and G^y_y - which is which depends on the sign of η) becomes extremal as $x \to 0$ [44, 23],

$$q > p+2: \qquad G^x_x - G^y_y \to 0 \quad \Leftrightarrow \quad \rho + P_r \to 0 \ . \tag{2.4.64}$$

Put differently, $q \leq p+2$ is a necessary condition for the strict DEC to hold, and thus for metrics satisfying the strict DEC the leading behaviour of the effective potential is always $V_{\text{eff},\ell}(x) \to cx^{-2}$.

As an aside, we note that it follows from (2.4.63) that precisely those metrics that satisfy the physically more reasonable (non-negative pressure) and more common extremal near-singularity equation of state $\rho = +P_r$ have $q = 2/d$, i.e. $s = 1/2$, leading to the critical value $c = -1/4$ frequently found in applications (to e.g. Schwarzschild-like geometries).

Comparison with Massless Point Particle Probes (the Penrose Limit)

In the previous section we have established that

1. for metrics with singularities of power-law type satisfying the strict DEC the leading behaviour of the scalar effective potential near the singularity is

$$V_{\text{eff},\ell}(x) \to cx^{-2} \tag{2.4.65}$$

2. this class of potentials is singled out by its scale invariance;

3. the corresponding coefficient c of the effective potential is bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.

These observations bear a striking resemblance to the results obtained recently in [23] in the study of plane wave Penrose limits

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \ , \tag{2.4.66}$$

of space-time singularities. Namely, it was shown in [23] that

1. Penrose limits of metrics with singularities of power-law type show a universal u^{-2}-behaviour near the singularity,

$$A_{ab}(u) \to c_a \delta_{ab} u^{-2} \ , \tag{2.4.67}$$

   provided that the strict DEC is satisfied;

2. such plane waves are singled out [9] by their scale invariance, reflected e.g. in the isometry $(u, v) \to (\lambda u, \lambda^{-1} v)$ of the metric (2.4.66, 2.4.67);

3. the coefficients c_a (related to the harmonic oscillator frequency squares by $c_a = -\omega_a^2$) are bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.[2]

The similarity of these two sets of statements is quite remarkable because the objects these statements are made about are rather different. For example, the potential is that of a one-dimensional motion on the half line in one case, and that of a d-dimensional harmonic oscillator (with time-dependent frequencies) in the other.

The analogy with the above statements about scalar effective potentials is brought out even more clearly if one reinterprets [15, 23] the Penrose limit in terms of null geodesic deviation in the original space-time. Then this result can be rephrased as the statement that the leading

[2]One significance of this bound on the c_a is that in this range one can consider the possibility to extend the string modes across the singularity at $u = 0$ [47].

behaviour of the geometry as probed by a family of massless point particles near a singularity is that of a plane wave with a u^{-2} geodesic effective potential. The analogy with the results of the previous section should now be apparent.

One minor difference between the results obtained here and those of [23] is that in the case of Penrose limits the strict DEC needed to be invoked only in the case of spacelike singularities, $\eta = +1$, timelike singularities always giving rise to plane waves with a u^{-2}-behaviour. This should be regarded as an indication (cf. the discussion in [23, Section 4.4]) that scalar waves are better probes of timelike singularities than massless point particles.

Massive Scalar Fields and Geodesic Incompleteness

The simple above analysis can evidently be generalised in various ways, e.g. by considering other kinds of probes. We will briefly comment on the two most immediate generalisations, namely massive and non-minimally coupled scalar fields.

We begin with a massive scalar for which the effective potential is

$$V_{\mathrm{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{2.4.68}$$

For the mass term to be relevant (dominant) as $x \to 0$ it is clearly necessary that $p < -2$ and $q < 0$. Intuitively one might expect a mass term to be irrelevant at short distances near a singularity. This expectation is indeed borne out: as we will now show, for metrics satisfying the above inequalities the would-be singularity at $x = 0$ is at infinite affine distance for causal geodesics so that such space-times are actually causally geodesically complete.

Null geodesics were analysed in [23]. Here we generalise this to causal geodesics. In terms of the conserved angular and y-momentum L and P, the geodesic equation for the metric (2.4.49) reduces to

$$\dot{x}^2 = P^2 x^{-2p} + \eta L^2 x^{-p-q} + \eta\epsilon x^{-p} \ , \tag{2.4.69}$$

where $\epsilon = 0$ ($\epsilon = 1$) for null (timelike) geodesics respectively.

For $\eta = -1$, if the first term in (2.4.69) is sub-dominant the geodesic effective potential is repulsive (e.g. via the angular momentum barrier) and the geodesics will not reach $x = 0$. Thus generic timelike geodesics will reach $x = 0$ only if (p, q) lie in the positive wedge bounded by the lines $p = 0$ and $p = q$. Radial null geodesics do not feel any repulsive force, and solving

$$\dot{x}^2 \sim x^{-2p} \quad \Rightarrow \quad x(u) \sim \begin{cases} u^{1/(p+1)} & p \neq -1 \\ \exp u & p = -1 \end{cases} \tag{2.4.70}$$

shows that $x = 0$ is reached at a finite value of the affine parameter only for $p > -1$. We thus conclude that Szekeres-Iyer metrics with $\eta = -1$ and $p \leq -1$ are causally geodesically complete. In particular, therefore, the mass term in the scalar effective potential is sub-dominant for metrics with honest timelike power-law singularities, and for all such metrics the scalar effective potential has the same leading behaviour as in the massless case.

For $\eta = +1$, the situation is more complex as all three terms in (2.4.69) are positive. If the first term dominates, either because of suitable inequalities satisfied by (p, q) or, for any (p, q), because one is considering radial null geodesics, the analysis and conclusions are identical to the above. Namely, $x = 0$ is at finite affine distance for $p > -1$. Analogously, if the second term dominates (e.g. for angular null geodesics) one finds the condition $p+q>-2$, and if the third term dominates one has $p>-2$. Since one needs $p<-2$ for the mass term to dominate in the scalar effective potential, only the second case is possible. But then the condition $p+q>-2$, with $p<-2$, implies $q>0$, so that the angular momentum term in the effective potential dominates the mass term.

We thus conclude that, for both $\eta=+1$ and $\eta=-1$, the mass term is always subdominant for metrics that are causally geodesically incomplete at $x=0$.

As an aside we note that the Szekeres-Iyer metrics for which the mass term does dominate ($p<-2$ and $q<0$), in addition to being non-singular, also necessarily violate the strict DEC.

Non-Minimally Coupled Scalar Fields

We will now very briefly also consider a non-minimally coupled scalar field

$$(\Box-\xi R)\phi=0 \ . \tag{2.4.71}$$

The Ricci scalar of the Szekeres-Iyer metric (2.4.49) is

$$R=d(d-1)x^{-q}-\tfrac{1}{4}\eta(4p+4qd-d(d+1)q^2)x^{-(p+2)} \ , \tag{2.4.72}$$

where once again only the leading order term should be trusted and retained. Thus the new effective potential

$$V^{\xi}_{\mathrm{eff},\ell}(x)=V_{\mathrm{eff},\ell}(x)-\eta\xi x^p R \tag{2.4.73}$$

is again a sum of two terms, proportional to x^{-2} and x^{p-q} respectively, so that the dominant behaviour is still $\sim x^{-2}$ provided that the metric does not violate the strict DEC. For $q<p+2$ and the conformally invariant coupling

$$\xi=\xi_*=\frac{d}{4(d+1)} \ , \tag{2.4.74}$$

one finds

$$V^{\xi_*}_{\mathrm{eff},\ell}(x)=\frac{(p-q)d}{4(d+1)}x^{-2}=(p-q)\xi_* x^{-2} \ . \tag{2.4.75}$$

Note that with this conformally invariant coupling the coefficient c now depends on $p-q$ rather than on q. The appearance of $(p-q)$ could have been anticipated since for $p=q$ the Szekeres-Iyer metric is conformal to an x-independent metric, and hence a conformal coupling cannot generate an x-dependent effective potential. Note also that for the conformal coupling (and, indeed, generic values of ξ) the coefficient c is no longer bounded by $-1/4$ so that the Schrödinger operator is no longer necessarily bounded from below.

2.4.7 Self-Adjoint Physics of Power-Law Singularities

In the previous section we have determined the leading behaviour of the scalar wave operators near a power-law singularity. In this section we will now study various aspects of these operators.

Functional Analysis Set-Up

In order to analyse the properties of the wave operator, we will need to equip the space of scalar fields with a Hilbert space structure. We will be pragmatic about this and introduce the minimum amount of structure necessary to be able to say anything of substance.

We thus return to the discussion of section 2.1, now being more specific about the spaces of functions the various operators appearing there act on [61], beginning with the operator A introduced in (2.4.39),

$$A = \eta a D^i a D_i - \eta a^2 m^2 \quad . \tag{2.4.76}$$

Since $D^i D_i$ is symmetric (formally self-adjoint) with respect to the natural spatial density $\sqrt{-\eta \det h}$ induced on the slices Σ of constant y by the metric (2.4.35), the operator A is symmetric with respect to the scalar product

$$\begin{aligned}(\phi_1, \phi_2) &= \int d^{n-1}x \, \sigma \phi_1^* \phi_2 \\ \sigma &= a^{-1}\sqrt{-\eta \det h} = \eta \sqrt{-\det g} g^{yy} \quad ,\end{aligned} \tag{2.4.77}$$

on $D(A) = C_0^\infty(\Sigma)$,

$$(A\phi_1, \phi_2) = (\phi_1, A\phi_2) \quad . \tag{2.4.78}$$

Moreover, for $\eta = -1$ the operator A is positive,

$$\eta = -1 \quad \Rightarrow \quad (\phi, A\phi) \geq 0 \quad . \tag{2.4.79}$$

We are thus led to introduce the Hilbert space $L^2(\Sigma, \sigma d^{n-1}x)$ of functions on Σ square integrable with respect to the above scalar product.

Passing to spherically symmetric metrics (2.4.40) in the tortoise gauge (2.4.41), A takes the form (2.4.42)

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2} \Delta_d - \eta a^2 m^2 \quad , \tag{2.4.80}$$

where $\sigma(x) = c(x)^d$. Since A is symmetric with respect to the scalar product (2.4.77), the unitarily transformed operator

$$\tilde{A} = \sigma^{1/2} A \sigma^{-1/2} \quad , \tag{2.4.81}$$

acting on the half-densities

$$\tilde{\phi}(x) = \sigma(x)^{1/2}\phi(x) \quad , \tag{2.4.82}$$

is symmetric with respect to the corresponding "flat" ($\sigma(x) \to 1$) scalar product

$$\langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle := \int dx \, d\Omega \, \tilde{\phi}_1^* \tilde{\phi}_2 = (\phi_1, \phi_2) \quad . \tag{2.4.83}$$

We now assume that the metric develops a singularity at some value of x, where e.g. the area radius goes to zero, $r \equiv c(x) \to 0$, which we may as well choose to happen at $x = 0$. Thus we consider $x \in (0, \infty)$ and take $\Sigma = \mathbb{R}^{n-1} \backslash \{0\}$, parametrised by x and the angular coordinates.

Then the initial domain of $\tilde{A}$ is $D(\tilde{A}) = C_0^\infty(\mathbb{R}^{n-1}\backslash\{0\})$ or $\tilde{D}(\tilde{A}) = C_0^\infty(\mathbb{R}_+) \otimes C^\infty(S^d)$, which are dense in the unitarily transformed Hilbert space

$$L^2(\mathbb{R}^{n-1}\backslash\{0\}, dx\, d\Omega) \cong L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) \ . \tag{2.4.84}$$

Decomposing the second factor into eigenspaces of the Laplacian Δ_d on S^d,

$$L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) = \bigoplus_{\ell=0}^{\infty} L_\ell \ , \tag{2.4.85}$$

and defining $\tilde{D}_\ell = \tilde{D} \cap L_\ell$, one has

$$\tilde{A}|_{\tilde{D}_\ell} = \tilde{A}_\ell \otimes \mathbb{I} \ , \tag{2.4.86}$$

where

$$\tilde{A}_\ell = -\partial_x^2 + V_{\text{eff},\ell}(x) \tag{2.4.87}$$

with $V_{\text{eff},\ell}(x)$ given in (2.4.47).

Questions about the original operator A can thus be reduced to questions about the family $\{\tilde{A}_\ell\}$ of standard Schrödinger-type operators. For example, to show that A is essentially self-adjoint on $D(A)$ it is sufficient to prove that, for each ℓ, $\tilde{A}_\ell$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$.

While one can analyse this question of self-adjointness just as readily for $\eta = +1$ as for $\eta = -1$, the physical significance of this condition in the case of spacelike singularities is not clear to us. Thus we will focus on static space-times with timelike singularities in the following and set $\eta = -1$. An extension of the general formalism to stationary non-static space-times is developed in [66].

We conclude this section with a comment on the choice of Hilbert space structure. The L^2 Hilbert space introduced above is certainly a natural choice, but not the only one possible. Based on physical requirements such as the finiteness of the energy of scalar field probes, other (Sobolev) Hilbert space structures have been proposed in the literature - see e.g. [67, 68]. The energy is, by definition,

$$E[\phi] = \int_\Sigma \sqrt{h} d^{n-1}x \ T_{\mu\nu}(\phi)\xi^\mu n^\nu \ , \tag{2.4.88}$$

where $T_{\mu\nu}(\phi)$ is the stress energy tensor of the scalar field, $\xi = \partial_y$ is the timelike Killing vector, and n the unit normal to Σ. In the present case this reduces to

$$E[\phi] = \int_\Sigma \sigma d^{n-1}x \ T_{yy} \ , \tag{2.4.89}$$

which identifies T_{yy} as the energy density with respect to the measure $\sigma d^{n-1}x$ employed above [68]. For a minimally coupled complex scalar field one has

$$T_{yy} = \frac{1}{2}\left[\partial_y\phi^*\partial_y\phi + a^2 h^{ij}\partial_i\phi^*\partial_j\phi\right] \ . \tag{2.4.90}$$

Thus, with an integration by parts (certainly allowed for $\phi \in D(A)$) the energy can be written as

$$\begin{aligned} E[\phi] &= \int_\Sigma \sigma d^{n-1}x \ \left(\partial_y\phi^*\partial_y\phi + \phi^* A\phi\right) \\ &= (\partial_y\phi, \partial_y\phi) + (\phi, A\phi) \ . \end{aligned} \tag{2.4.91}$$

For a comparison of the two definitions (2.4.89) and (2.4.91) of the energy and the role of boundary terms, see e.g. the discussion in [69] and the comment in section 3.5 below. Adopting the expression (2.4.91) as the definition of the energy suggests introducing a Sobolev structure on the space of scalar fields using the quadratic form

$$Q_A(\phi) = (\phi, A\phi) \tag{2.4.92}$$

associated to A, via [67, 68]

$$||\phi||^2_{H^1} = (\phi,\phi) + Q_A(\phi) \ , \tag{2.4.93}$$

thus enforcing the condition that the energy be finite. For present purposes we simply note that at least for the Friedrichs extension A_F of A, based on the closure of the quadratic form $Q_A(\phi)$ with respect to the L^2 norm, the resulting potential energy $Q_{A_F}(\phi)$ is finite (and positive) by definition without having to invoke Sobolev spaces (see also the discussion in [70, 71]).[3] We will use specifically this extension in the discussion of section 3.5 below.

Essential Self-Adjointness and the Horowitz-Marolf Criterion

The spatial part A of the wave operator is real and symmetric (with respect to an appropriate scalar product on a C_0^∞ domain of A), and as such has self-adjoint extensions, each leading to a well defined (and reasonable [70]) time-evolution. If the self-adjoint extension is not unique, however, i.e. if the operator is not essentially self-adjoint, then also the corresponding time-evolution is not uniquely determined. Thus the Horowitz-Marolf criterion [43] (unique time-evolution without having to impose boundary conditions at the singularity) amounts to the condition that the operator A be essentially self-adjoint.

To test for essential self-adjointness [72, 49], one can e.g. use [43] the standard method of Neumann deficiency indices or the Weyl limit point – limit circle criterion (employed in this context in [73]). Roughly speaking, in order for A to be essentially self-adjoint the (effective) potential $V_{\text{eff},\ell}$ appearing in the operator $\tilde{A}_\ell$ has to be sufficiently repulsive near $x = 0$ to prevent the waves $\tilde{\phi}$ from leaking into the singularity.

Concretely, in the present case, where we only have control over the operator A near the singularity at $x = 0$, the criteria for the operator $\tilde{A}_\ell$ to be essentially self adjoint on $C_0^\infty(\mathbb{R}_+)$ at $x = 0$ boil down to the following elementary conditions on the effective potential $W \equiv V_{\text{eff},\ell}$ [72, 49]:

- If
$$W(x) \geq \frac{3}{4}x^{-2} \tag{2.4.94}$$
near zero, then $-\partial_x^2 + W(x)$ is essentially self-adjoint at $x = 0$.
- If for some $\epsilon > 0$
$$W(x) \leq \left(\frac{3}{4} - \epsilon\right)x^{-2} \tag{2.4.95}$$
(in particular also if $W(x)$ is decreasing) near $x = 0$, then $-\partial_x^2 + W(x)$ is not essentially self-adjoint at $x = 0$.

[3] Working with such a Sobolev space structure is certainly possible but also complicates the determination of self-adjoint extensions of A, since e.g. studying the closure of A now involves studying the sixth order operator A^3, arising from the term $||A\phi||^2_{H^1} = (A\phi, A\phi) + (A\phi, A^2\phi)$ in the operator norm.

The significance of the factor 3/4 can be appreciated by looking at the critical (and relevant for us) case of an inverse square potential

$$W(x) = s(s-1)x^{-2} \ . \tag{2.4.96}$$

In this case the leading behaviour of the two linearly independent solutions of the equation

$$\left(-\partial_x^2 + W(x)\right)\tilde{\phi}_\lambda(x) = \lambda\tilde{\phi}_\lambda(x) \tag{2.4.97}$$

near $x = 0$ is given by the two linearly independent solutions of the equation

$$\left(-\partial_x^2 + W(x)\right)\tilde{\phi}_0(x) = 0 \ , \tag{2.4.98}$$

namely

$$\tilde{\phi}_0 \sim x^s \quad \text{or} \quad \tilde{\phi}_0 \sim x^{1-s} \tag{2.4.99}$$

Thus both solutions are square integrable near $x = 0$ when $2s > -1$ and $2(1-s) > -1$, or

$$-\frac{1}{2} < s < \frac{3}{2} \quad \Leftrightarrow \quad s(s-1) < \frac{3}{4} \ , \tag{2.4.100}$$

and in this range of $c = s(s-1)$ the potential is limit circle and the self-adjoint extension is not unique. Conversely, it follows that for $c \geq 3/4$ the solutions of equation (2.4.97) for $\lambda = \pm i$ (which are necessarily complex linear combinations of the two linearly independent real solutions) are not square-integrable near $x = 0$. Thus the deficiency indices are zero and the operator is essentially self-adjoint for $c \geq 3/4$.

Even when there are two normalisable solutions, all is not lost however, as it may be indicative of the possibility (or even necessity) to continue the fields and/or the metric through the singularity [65]. Evidently, such an analytic continuation requires some thought (to say the least) in the case of Szekeres-Iyer metrics with generic (non-rational) values of p and q.

The Horowitz-Marolf Criterion for Power-Law Singularities

In the case at hand, timelike singularities of power-law type, the effective potential is given by (2.4.51) with $\eta = -1$ and $s = qd/4$. We had already seen in section 2.5 that the mass term is never dominant at $x = 0$ and we can therefore also set $m^2 = 0$. Thus the operator of interest is

$$\begin{aligned} \tilde{A}_\ell &= -\partial_x^2 + V_{\text{eff},\ell}(x) \\ V_{\text{eff},\ell}(x) &= s(s-1)x^{-2} + \ell_d^2 x^{p-q} \ , \end{aligned} \tag{2.4.101}$$

It is now straightforward to determine for which values of (p,q) the classical singularities at $x = 0$ become regular or remain singular when probed by scalar waves. First of all, we will show that we can reduce the analysis to the case $\ell = 0$:

- For $q < p+2$, the first term in the potential is dominant and independent of ℓ. Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is essentially self-adjoint. As we know from (2.4.94), this condition is satisfied iff $s(s-1) \geq 3/4$.

- For $q > p+2$, the operators $\tilde{A}_\ell$ for $\ell \neq 0$ are essentially self-adjoint by the criterion (2.4.94). Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is.

- In the borderline case $q = p+2$, for $\ell \neq 0$ we have

$$\ell \neq 0 \quad \Rightarrow \quad s(s-1) + \ell_d^2 \geq 3/4 \tag{2.4.102}$$

(with equality only for $s = 1/2$ and $\ell = d = 1$). Even in this case, therefore, all the $\tilde{A}_\ell$ with $\ell \neq 0$ are essentially self-adjoint and only $\tilde{A}_{\ell=0}$ needs to be examined.

We can thus conclude that the operator A is essentially self-adjoint iff $s(s-1) \geq 3/4$ and that, in view of (2.4.100), it fails to be essentially self-adjoint for

$$A \text{ not e.s.a.} \quad \Leftrightarrow \quad -\frac{1}{2} < s < \frac{3}{2} \quad \Leftrightarrow \quad -\frac{2}{d} < q < \frac{6}{d} \ . \tag{2.4.103}$$

The Significance of the (Strict) Dominant Energy Condition

While this has been rather straightforward, one of the virtues of the present approach, based on using a class of metrics appropriate for a generic singularity of power-law type, is that it allows us to draw a general conclusion regarding the relation between the Horowitz-Marolf criterion and properties of the matter (stress-energy) content of the space-time near the singularity.

Indeed, as we will now show, whenever the matter content of the near-singularity space-time is sufficiently "positive" (in the sense of the strict DEC, as it turns out), the space-time remains singular according to the Horowitz-Marolf criterion, i.e. when probed with scalar waves.

We can deduce from (2.4.63) that metrics with timelike power-law singularities satisfying the strict DEC lie in a bounded region inside the strip $0 < q < 2/d$ [23]. Indeed, for $q < p+2$ only the second terms in (2.4.63) are relevant, and one finds

$$\rho - P_r = \tfrac{1}{4} dq(2 - dq)\, x^{-(p+2)} \ . \tag{2.4.104}$$

Thus one has

$$\rho - P_r > 0 \quad \Leftrightarrow \quad 0 < q < \frac{2}{d} \ . \tag{2.4.105}$$

In particular, therefore, it follows from (2.4.103) that for such metrics the operator A is not essentially self-adjoint and we can draw the general conclusion that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

Even though metrics with $q = 2/d$, say, like negative mass Schwarzschild, still satisfy the bound (2.4.103), thus remain singular while obeying an extremal equation of state, we cannot strengthen the above statement to include general metrics with extremal equations of state. This can be seen e.g. from examples in [43] and is due to the fact that extremal metrics can also be found elsewhere in the (p, q)-plane, in particular in the region $q > p+2$, while violating the bound (2.4.103).

The Friedrichs Extension and "Hospitable" Singularities

In the previous section we have discussed self-adjoint extensions of (the spatial part A of) the Klein-Gordon operator. We have not discussed, however, what these self-adjoint extensions imply about the behaviour of the allowed scalar fields ϕ (those in the domain of the self-adjoint extension of A) near the singularity at $x = 0$.

It is certainly possible that self-adjointness can be achieved by allowing only scalar fields that vanish at the singularity. In some sense, then, the singularity remains excluded from the space-time and is not probed directly by the scalar field ϕ. We will see that this is indeed what happens in (in a precise sense) one half of the cases in which there is a unique self-adjoint extension.

However, it is a priori also possible (and perhaps more interesting) to have a well-defined time-evolution (which we take to mean "defined by some self-adjoint extension" [70]) with scalar fields that are permitted to be non-zero at the singularity. In that case, the singularity would be probed more directly by the scalar field, and one might then perhaps like to define a classical singularity to be "hospitable" (for a scalar field), if there is a self-adjoint extension which allows the scalar fields to take non-zero values at the locus of the singularity. We will see that this possibility is indeed realised as well, not only for the other half of the essentially self-adjoint cases, but also for e.g. the Friedrichs extension A_F of the operator A in a certain range of parameters for which A is not essentially self-adjoint.

To address these issues, we need to determine the domain of definition of the relevant self-adjoint extension of $\tilde{A}_0 = -\partial_x^2 + cx^{-2}$ for $c = s(s-1) \in [-1/4, \infty)$. For $\tilde{A}_0$ essentially self-adjoint, i.e. $c \geq 3/4$, this can be done by explicitly determining the domain of the closure $\bar{A}_0$ of the operator $\tilde{A}_0$. While we have done this (see also [74]), alternatively, for all $c \geq -1/4$, one can determine the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$, constructed from the closure of the associated quadratic form. For $c \geq 3/4$, such that $\tilde{A}_0$ is essentially self-adjoint, its unique self-adjoint extension of course agrees with the Friedrichs extension. Precisely this problem has been addressed and solved in [75], and instead of reinventing the wheel here we can draw on the results of that reference to analyse the issue at hand.

The main result of [75] of interest to us is their Theorem 6.4. Applied to the operator $\tilde{A}_0$, this theorem[4] states that the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$ is

$$\begin{aligned} D(\tilde{A}_F) = \{f \in L^2(0,\infty) : \quad & f(0) = 0, f \in A(0,\infty), \partial_x f \in L^2(0,\infty), \\ & x^{-1} f \in L^2(0,\infty), (-\partial_x^2 + cx^{-2}) f \in L^2(0,\infty)\} \end{aligned} \tag{2.4.106}$$

where $A(0,\infty)$ denotes the space of absolutely continuous functions. In [75], this result was established for $c > 0$. As far as we can see, this result is correct, as it stands, also for $-1/4 < c < 0$. We will comment on the special case $c = -1/4$ below.

We will now extract from this result some restrictions on the behaviour of f near $x = 0$ (assuming that we can model the leading behaviour of f as $x \to 0$ by some power of x):

1. From the condition $x^{-1} f \in L^2$ we learn that $f(x) \sim x^{\frac{1}{2}+\epsilon}$ for some $\epsilon > 0$. Then the conditions $f(0) = 0$ and $\partial_x f \in L^2$ are also satisfied.

[4] Actually, in [75] a more general operator, including in particular a non-zero harmonic oscillator term Bx^2, was studied. However, this term serves only to regularise the wave functions at infinity. Since we are concerned with the behaviour at $x = 0$, this term is of no consequence for the present considerations.

2. The remaining condition $(-\partial_x^2 + cx^{-2})f \in L^2$ can be satisfied in one of two ways. Either both terms separately are in L^2 or f lies in the kernel of the operator (as $x \to 0$). In the former case, we find the condition $f(x) \sim x^{\frac{3}{2}+\epsilon}$ with $\epsilon > 0$. In the latter case, since the two functions in the kernel are x^s and x^{1-s}, with (as usual) $c = s(s-1)$, we now need to distinguish several cases:

   (a) $c > 3/4$: this means that $s > 3/2$ or $s < -1/2$. The solution x^s with $s > 3/2$, i.e. $f(x) \sim x^{\frac{3}{2}+\epsilon}$, yields nothing new. The solution x^{1-s} with $s > 3/2$ (or, equivalently, the solution x^s with $s < -1/2$) is ruled out by condition 1.

   (b) $c = 3/4$: this means that $s = 3/2$ or $s = -1/2$. In this case, we can allow $x^{3/2}$ and thus relax the domain to include functions $f(x) \sim x^{\frac{3}{2}+\epsilon}$, now with $\epsilon \geq 0$.

   (c) $-1/4 < c < 3/4$: thus $-1/2 < s < 3/2$ and $s \neq 1/2$. Thus the solution x^s is adjoined to the functions $\{x^{\frac{3}{2}+\epsilon}\}$ for $s > 1/2$, and the solution x^{1-s} for $s < 1/2$.

It remains to discuss the special value $c = -1/4$ or $s = 1/2$ which is not covered by the formulation of the domain in (2.4.106). This is the minimal allowed value of interest to us ($c = s(s-1)$ with s real), and also the minimal value for which the operator remains positive (and thus has a Friedrichs extension). In this case, the two solutions are $x^s = x^{\frac{1}{2}}$ and $x^{\frac{1}{2}} \log x$, and we checked that, as expected, the domain of the Friedrichs extension includes $x^{1/2}$. This can also be deduced e.g. from [76], which moreover illustrates nicely some of the weirdness of non-Friedrichs extensions.

The above discussion shows that the two definitions (2.4.89) and (2.4.91) of the energy, a priori differing by boundary terms due to the integration by parts, agree for the Friedrichs extension for $c > -1/4$ and differ only by a finite term for $c = -1/4$. The issue of boundary terms for more general domains is discussed in [69].

Returning to the original question of determining the behaviour of the allowed scalar fields in the domain of the self-adjoint extension of the spatial part A of the Klein-Gordon operator, we need to now undo the transformation $\phi \to \tilde{\phi}$ from the initial scalar fields ϕ to the half-densities $\tilde{\phi}$ that we performed in section 2.1 to put A into the form of a standard Schrödinger operator.

This transformation back from $\tilde{\phi}$ to ϕ is accomplished by multiplication by x^{-s}. Now the upshot of the above discussion is that the lowest power of x appearing in the domain of $\tilde{A}_F$ is

$$\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon} & \text{for } s > 3/2 \text{ or } s < -1/2 \\ x^s & \text{for } 1/2 \leq s \leq 3/2 \\ x^{1-s} & \text{for } -1/2 \leq s \leq 1/2. \end{cases} \tag{2.4.107}$$

Evidently these functions are, in particular, positive powers of x. Thus they, and therefore all functions in the domain, tend to zero for $x \to 0$, consistent with the condition $f(0) = 0$ in (2.4.106). However this is not necessarily true for the transformed functions, for which one has ($\delta = \delta(s) > 0$ is a positive real number depending on s)

$$\phi_{\min} = x^{-s}\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon-s} = x^{-\delta} & \text{for } s > 3/2 \\ x^0 = 1 & \text{for } 1/2 \leq s \leq 3/2 \\ x^{1-2s} = x^{\delta} & \text{for } -1/2 \leq s < 1/2 \\ x^{\frac{3}{2}+\epsilon-s} = x^{2+\delta} & \text{for } s < -1/2 \end{cases} \tag{2.4.108}$$

The final result is the simple statement that a ϕ in the domain of the Friedrichs extension A_F of A necessarily goes to zero for $s < 1/2$, ϕ can be non-zero (but remains bounded) for $1/2 < s \leq 3/2$, and can become increasingly singular for large $s > 3/2$.

Note that this statement is not invariant under $s \to 1 - s$. Indeed, while the operator $-\partial_x^2 + s(s-1)x^{-2}$ has this invariance, and therefore also statements about its essential self-adjointness are symmetric under $s \to 1 - s$ (as we have seen), the unitary transformation between ϕ and $\tilde{\phi}$ depends linearly on s and thus leads to a behaviour of the original scalar fields ϕ that does not have this symmetry.

Once again we find a pleasing relation with the DEC, since the watershed happens exactly at $s = 1/2 \Leftrightarrow q = 2/d$ which, as we have seen, corresponds to $\rho = P_r$. Timelike singularities satisfying the strict DEC have $0 < q < 2/d$ (2.4.105), thus $0 < s < 1/2$. Moreover, metrics with $s \leq -1/2$ have a unique self-adjoint extension ($c \geq 3/4$), thus are regular in the Horowitz-Marolf sense, but are not "hospitable" in the sense described above, while those with $s \geq 3/2$ are.

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Chapter 3

Fuzzy Sphere Solutions in the BMN Matrix Model

3.1 Supermembranes

String theory is a unique theory in many respects. Its extraordinary infinite dimensional conformal symmetry allows for consistent quantisation, and has spurred an enormous interest in exploring more exotic conformal theories in the high-energy field, sometimes even completely without space-time interpretation.

Yet string theory is based on the Nambu-Goto action of actual strings moving on a general relativistic background and leads to space-time (super-)gravity in a certain limit. Also in this direction the action has immediate generalisations as its basic formulation is insensitive to changing space-time as well as internal world-sheet dimensions. Pictorially, this is going from strings tracing out a 1+1 dimensional world-sheet in time to 2+1 dimensional membranes and even higher p-branes. But since without the extended symmetry of the 1+1 case no consistent quantisation frame-work is known, interest has flared up and subsided again several times in last century's physics.

Originally, Dirac [77] had put forward the idea of describing the electron fundamentally as a membrane in 1962, even before string theory came to bloom in the seventies. As an attempt to finding alternative theories to the divergency-ridden quantum electrodynamics he had long rejected, however, the proposal never really caught on.

Interest was renewed in the eighties, when Hughes, Liu and Polchinski [78] showed that κ-*symmetry* crucial to the projection of supersymmetry from target-space to the world-sheet of the superstring could be generalised. κ-symmetry depends strongly on the dimensions of the Γ-matrices and therefore greatly restricts the allowed combinations of dimensionalities. The *brane scan* summarises these constraints into a table of the valid theories, a brane extension of the well-known fact that the superstring can only be formulated in 3,4,6 and 10 dimensions classically. For the supermembrane, this sequence is 4,5,7 and 11, and includes the highest possible space-time dimension in the scan[1] with the supermembrane by Bergshoeff, Sezgin and Townsend [79].

[1] By considering vector and tensor fields on the world-volume, other combinations become possible, notably a 5-brane in 11 dimensions and the famous D-branes of string theory.

Incidentally, 11 is also the highest dimensionality for a supergravity theory – the one coupling to the supermembrane. Now just as the 10 dimensional supergravities can be understood from dimensional reduction of 11d, the classical string action is a compactification of the membrane (Duff, Howe, Iname and Stelle [80]), therefore raising the outstanding question whether a full 11 dimensional quantum completion of the membrane might be achieved, with 11d supergravity and 10d string theory as subsectors.

A milestone is surely the discovery of the membrane-quantisation technique by de Wit, Hoppe and Nicolai [81], based on an original gauge-fixing and re-writing of the theory. In this scheme, the diffeomorphisms are all fixed but for the area-preserving ones. They can be seen as a $U(\infty)$ gauge group and regularisation is achieved by cutting it down to $U(N)$ sectors. While not a complete quantisation of the full theory, it still brought new insight being one of the compelling examples of relations between gravity and large N gauge theories.

Nowadays, membranes have grown more important than ever since with the third 'string revolution' it was realised that quantum string theory must contain high energy states beyond the realm of perturbative expansion. They take the form of D-branes and other extended objects and the search for their quantum description might ultimately lead to a unified 11 dimensional theory, a formulation of 'M-theory', see chapter four of this work.

Whether the quest will succeed or not, membranes and matrix models are surely of interest in their own right and the main focus of this chapter. After gathering some facts about the general action and κ-symmetry, we will explain (bosonic) membrane quantisation in some detail. We will then turn to most symmetric case next to flat space, the BMN-model, and also explain its parallel origin from super-Yang-Mills theory. Moving on to its non-trivial classical vacua, fuzzy spheres of radius 1, we expand about them to quadratic order thus obtaining the complete perturbative spectrum. Finally we compute the Coleman-Weinberg one-loop effective potential to comment on stability of fuzzy spheres of different radii.

3.1.1 The Membrane Action in Nambu-Goto and Polyakov form

The dynamics of an extended object sweeping out a d dimensional world-sheet with tension T in D dimensional target space can be described by minimising an action given by its world-volume determinant

$$S_{\text{DNG}} = -T \int d^d\sigma \sqrt{-h}, \quad \text{with } h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \tag{3.1.1}$$

We will call it the Dirac-Nambu-Goto action, though strictly speaking Nambu and Goto formulated this action for the string ($d = 1+1$) and Dirac for the membrane ($d = 2+1$) only. We have already encountered this action in chapter one on light-cone gauge in string theory. At the level of the classical equations of motion it is equivalent to the Polyakov action

$$S_{\text{Polyakov}} = -\tfrac{T}{2} \int d^d\sigma \sqrt{-\gamma}\, \left(\gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} - (d-2)\right) \tag{3.1.2}$$

The latter is formulated in terms of an auxiliary field, the world-sheet metric $\gamma_{\alpha\beta}$, which is equal[2] to the induced metric *on-shell* $\gamma_{\alpha\beta} = h_{\alpha\beta}$. On integrating out the auxiliary metric, one recovers Dirac-Nambu-Goto.

[2]up to a conformal factor in the special case $d = 1+1$ of the string

The $(d-2)$-term vanishes in the case of string theory, and expresses the absence of scale invariance for higher dimensional embedded objects such as the $d = 3$ membrane. Also, the number of independent metric components on the world-sheet rises faster with the dimension d than the diffeomorphism degrees of freedom, respectively. All in all, we have a vastly reduced symmetry for the membrane theory that does not allow for the same amount of control that we have on string theory.

Just like the string couples to an antisymmetric $B_{\alpha\beta}$-field, the theory also naturally couples to a d-form field pulled back to the world-sheet by adding the action

$$S_{\text{form}} = \frac{T}{d!} \int d^d\sigma \varepsilon^{\alpha_1 \dots \alpha_d} \partial_{\alpha_1} X^{\mu_1} \cdots \partial_{\alpha_d} X^{\mu_d} C_{\mu_1 \dots \mu_d} \tag{3.1.3}$$

3.1.2 Supersymmetry and Kappa-Symmetry

Membrane theory can be promoted to a supersymmetric theory in a straight-forward way, replacing target space by curved superspace with coordinates $Z^{\mathcal{M}} = (X^\mu, \theta^M)$. We then have to express the metric of gravity in terms of the supervielbein $E_{\mathcal{M}}^{\mathcal{R}}$ and pull back to the world-sheet by $\partial_\alpha Z^{\mathcal{M}} E_{\mathcal{M}}^{\mathcal{R}}$-terms. To avoid confusion with the various greek and latin indices used throughout this work already for (bosonic) gravity, we adopt the non-standard notation here of indices $\mathcal{M} = (\mu, M)$ for curved and $\mathcal{R} = (r, R)$ for flat superspace. Thus fermions always have capital indices.

Also the d-form gets extended to a super-d-form and in the fermionic components comprises the terms (powers of θ and Γ) necessary to close the algebra of supersymmetry transformations. The supermembrane action then reads in full

$$\begin{aligned} S = -\tfrac{T}{2} \int d^d\sigma \sqrt{-\gamma} \, \big(& \gamma^{\alpha\beta} (\partial_\alpha Z^{\mathcal{M}} E_{\mathcal{M}}^r)(\partial_\beta Z^{\mathcal{N}} E_{\mathcal{N}}^s) \eta_{rs} - (d-2) \big) \\ & - \frac{2}{d!} \varepsilon^{\alpha_1 \dots \alpha_d} \, (\partial_{\alpha_1} Z^{\mathcal{M}_1} E_{\mathcal{M}_1}^{\mathcal{R}_1}) \cdots (\partial_{\alpha_d} Z^{\mathcal{M}_d} E_{\mathcal{M}_d}^{\mathcal{R}_d}) \, C_{\mathcal{R}_1 \dots \mathcal{R}_d} \end{aligned} \tag{3.1.4}$$

Note that this action is only supersymmetric in target-space, with superdiffeomorphisms, tangent space Lorentz symmetry and d-form gauge invariance. The world-sheet is unchanged, and all α-indices remain purely bosonic. On top of that, the superstring has another crucial symmetry: invariance under the following transformation generated by a space-time spinor and worldvolume scalar $\kappa(\sigma)$

$$\delta Z^{\mathcal{M}} E_{\mathcal{M}}^r = 0, \qquad \delta Z^{\mathcal{M}} E_{\mathcal{M}}^R = (\kappa(1+\Gamma))^R \tag{3.1.5}$$

where Γ is the completely antisymmetric contraction on the world-sheet

$$\Gamma = \frac{(-1)^{d(d-3)/4}}{d! \sqrt{\gamma}} \varepsilon^{\alpha_1 \dots \alpha_d} \, (\partial_{\alpha_1} Z^{\mathcal{M}_1} E_{\mathcal{M}_1}^{r_1}) \cdots (\partial_{\alpha_d} Z^{\mathcal{M}_d} E_{\mathcal{M}_1}^{r_d}) \, \Gamma_{[r_1 \dots r_d]} \tag{3.1.6}$$

Since this Γ is traceless and $\Gamma^2 = 1$, the matrices $\frac{1}{2}(1 \pm \Gamma)$ in (3.1.5) serve as projection operators allowing to gauge away half of the fermionic degrees of freedom. This in turn makes it possible to match the number of fermionic and bosonic degrees of freedom *on the world-sheet*. It can be verified that κ-symmetry is the vital ingredient to promoting target-space supersymmetry to d-dimensional world-volume supersymmetry.

Counting fermionic and bosonic degrees of freedom, it is clear that this cannot hold for all combinations of world-sheet d and target space D dimensions. In fact, as shown by Achucarro, Evans, Townsend and Wiltshire [82], imposing this κ-symmetry requires a certain Γ-matrix identity to hold (see Duff [83]) which poses severe restrictions on the dimensionalities. The only combinations allowed are summarised in the *brane scan*

$$\text{world-sheet } d \quad \begin{array}{lcl} d=2 & : & D=3,4,6,10 \\ d=3 & : & D=4,5,7,11 \\ d=4 & : & D=6,8 \\ d=5 & : & D=9 \\ d=6 & : & D=10 \end{array} \quad \text{space-time } D \tag{3.1.7}$$

recovering the known set for the classical superstring. κ-symmetry will be put to use in section 3.3.2 in the derivation of the BMN model. Until then we shall suspend the discussion of fermions for a presentation of the quantisation of the bosonic membrane.

In the account so far we have largely drawn on the excellent review on supermembranes by Duff [83] and refer the reader to this source for more information and further references.

3.2 Membrane Quantisation

There is a very unique and original method found by de Wit, Hoppe and Nicolai [81] of quantising the $d = 3$ membrane in a certain gauge. It is possible to rewrite the theory such that quantisation amounts to discretising the 2-dimensional space-like part of the smooth surface by projecting it onto an infinite dimensional matrix-algebra that can in turn be regularised.

The key of this method is to gauge-fix the world-sheet diffeomorphisms of the membrane action up to a remaining $U(\infty)$ symmetry. As we will see, it is possible to separate this $U(\infty)$ from the rest such that the action takes the shape of a Yang-Mills theory, the $U(\infty)$ gauge field acting by Poisson brackets. Quantisation regularises the $U(\infty)$ as the infinite limit of a $U(N)$ group. The latter has a matrix representation, so technically quantisation follows the line of replacing Poisson brackets of fields by commutators of matrices.

The original formalism due to de Wit, Hoppe and Nicolai [81] and extended to curved backgrounds by Kim and Park [84] starts from the Dirac-Nambu-Goto action. In a subtle procedure *all* world-sheet diffeomorphisms are fixed in light-cone gauge. The $U(\infty)$ gauge field is subsequently re-introduced as the Lagrange multiplier enforcing part of the gauge.

While this method has some advantages in connecting easily with the Hamiltonian formulation of light-cone gauge fixing, we shall take a complementary road here. Starting from the Polyakov action, it is possible to carve out the unfixed $U(\infty)$ gauge theory content explicitly by algebraic manipulations. With the world-sheet metric we already have an auxiliary field that makes the internal diffeomorphisms manifest and indeed the $U(\infty)$ gauge field is contained in its off-diagonal components.

In the following we will work out the algebraic manipulations as well as light-cone gauge fixing and the implied conditions on the target-space metric. We will see how the gauge field arises explicitly and exactly which degrees of freedom it transports, the remaining $U(\infty)$ group of time-dependent area-preserving diffeomorphisms. Regularisation of $U(\infty)$ by a $U(N)$ matrix representation then follows in a straight-forward manner.

3.2.1 Membrane to Matrix Action – Bosonic Part

Algebraic manipulations As announced in the outline we start with the Polyakov action

$$S = \tfrac{T}{2} \int d^3\sigma \, \frac{1}{\sqrt{-\hat{\gamma}}} \left(\hat{\gamma}\, \hat{\gamma}^{\alpha\beta}\, \hat{h}_{\alpha\beta} - \hat{\gamma}\right) \tag{3.2.1}$$

In anticipation of light-cone gauge to be imposed later on, we foliate the 3 dimensional world-sheet along a time-like direction $\alpha = 0$ into 2d space-like surfaces. We denote 3d quantities by hatted symbols $\hat{\gamma}_{\alpha\beta}$ with indices $\alpha, \beta = \{0, 1, 2\}$ and the respective dimensional reduction by latin-indexed symbols γ_{ab} where $a, b = \{1, 2\}$. For the sake of legibility, we quickly pass to the reduced quantities and never mix the different index types on the same side of any equation.

The reduced γ_{ab} is defined by $\gamma_{ab} \equiv \hat{\gamma}_{ab}$. In this spirit and in slight abuse of notation we also define $\hat{\gamma}_{00} \equiv \gamma_{00}$ and further down $\hat{h}_{00} \equiv h_{00}$ and $\hat{h}_{0a} \equiv h_{0a}$. Of course with upper indices $\hat{\gamma}^{ab} \neq \gamma^{ab}$ and we also have to distinguish between the 3d determinant $\hat{\gamma}$ and the 2d γ

$$\hat{\gamma}_{\alpha\beta} = \begin{pmatrix} \gamma_{00} & u_a \\ u_b & \gamma_{ab} \end{pmatrix} \qquad \Rightarrow \qquad \hat{\gamma} = \gamma_{00}\gamma + u_c \varepsilon^{ca} \gamma_{ab} \varepsilon^{bd} u_d \tag{3.2.2}$$

Instead of calculating $\hat{\gamma}$ explicitly, we will base all subsequent manipulations on the formula[3]

$$6\hat{\gamma} = \hat{\gamma}_{\alpha\beta}(\hat{\gamma}\hat{\gamma}^{\alpha\beta}) = \hat{\gamma}_{\alpha_1\beta_1}(\hat{\gamma}_{\alpha_2\beta_2}\hat{\gamma}_{\alpha_3\beta_3}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\beta_1\beta_2\beta_3}) \tag{3.2.3}$$

The expression in brackets $(\hat{\gamma}\hat{\gamma}^{\alpha\beta})$ is called the adjugate matrix (determinant times inverse). Plugging this into the Polyakov action allows us to split up terms into time- and space-like indices. Taking the respective multiplicities from index permutations into account we get

$$\begin{aligned} S = \tfrac{T}{2} \int d\tau d^2\sigma \, \frac{1}{\sqrt{-\hat{\gamma}}} \Big(&\tfrac{1}{2} h_{00}(\gamma_{ab}\gamma_{cd}\varepsilon^{0ac}\varepsilon^{0bd}) + 2h_{0a}(u_b\gamma_{cd}\varepsilon^{0bc}\varepsilon^{0da}) \\ &+ h_{ab}(\gamma_{00}\gamma_{cd} - u_c u_d)\varepsilon^{0ca}\varepsilon^{0db} - \hat{\gamma}\Big) \end{aligned} \tag{3.2.4}$$

In the h_{00}-term, we remark a lower dimensional version of the determinant formula (3.2.3)

$$2\gamma = \gamma_{ab}\gamma_{cd}\varepsilon^{ac}\varepsilon^{bd} \tag{3.2.5}$$

The second summand we rewrite using the definition of a new field A_a given in terms of u_a

$$A_a = \tfrac{1}{\gamma} u_b \varepsilon^{bc} \gamma_{ca} \qquad \Leftrightarrow \qquad u_a = \gamma_{ab}\varepsilon^{bc} A_c \tag{3.2.6}$$

The definition does not seem well motivated at this point, but we will see shortly that the gauge field A of the $U(\infty)$ we want to work out is contained entirely in the off-diagonal components u_a of the world-sheet metric. In the new variables this link is more clear-cut, and we will just have to to gauge fix the components $A_a \sim \partial_a A$ to reveal the final connection A.

We also put the definitions $h_{00} \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$ and $h_{0a} \equiv \partial_a X^\mu \dot{X}^\nu g_{\mu\nu}$ into the action

$$\begin{aligned} S = \tfrac{T}{2} \int d\tau d^2\sigma \, \frac{1}{\sqrt{-\hat{\gamma}}} \Big(&\dot{X}^\mu(\dot{X}^\nu\gamma + 2\gamma A_c \varepsilon^{ca}\partial_a X^\nu) g_{\mu\nu} \\ &+ h_{ab}(\gamma_{00}\gamma_{cd}\varepsilon^{ca}\varepsilon^{db} - u_c u_d \varepsilon^{ca}\varepsilon^{db}) - \hat{\gamma}\Big) \end{aligned} \tag{3.2.7}$$

[3] or, for a d-dimensional matrix γ_{ab} in general $d!\gamma = \gamma_{a_1b_1} \ldots \gamma_{a_db_d}\varepsilon^{a_1\ldots a_d}\varepsilon^{b_1\ldots b_d}$.

All kinetic terms are assembled in the first line. We complete them to a square by adding and subtracting $\pm\gamma\ A_c\varepsilon^{ca}\ A_d\varepsilon^{db}\ h_{ab}$, i.e. adding to the upper line and subtracting from the lower.

Thus we have simplified the kinetic term at the expense of the h_{ab}-term in the second line which only seems to get ever more complicated. However, also the latter reduces drastically on using the *purely space-like subset* of the equations of motion $\gamma_{ab} = h_{ab}$

$$h_{ab}(\gamma_{00}\gamma_{cd}\varepsilon^{ca}\varepsilon^{db} - u_d u_c\varepsilon^{ca}\varepsilon^{db} - \gamma A_c\varepsilon^{ca}\ A_d\varepsilon^{db}) = 2(\gamma_{00}\gamma + u_c\varepsilon^{ca}\gamma_{ab}\varepsilon^{bd}u_d) = 2\hat{\gamma} \tag{3.2.8}$$

The remaining determinant $2\hat{\gamma}$ cancels partly against the $-\hat{\gamma}$-term in the Polyakov action.

All in all, we can bring the Polyakov action into the following form by using the space-like equations of motion $h_{ab} = \gamma_{ab}$ and otherwise purely algebraic transformations

$$\begin{aligned} S_{\text{Pol}} = \tfrac{T}{2}\int d\tau d^2\sigma\ &\frac{\gamma}{\sqrt{-\hat{\gamma}}}(\dot{X}^\mu + A_a\varepsilon^{ab}\partial_b X^\mu)(\dot{X}^\nu + A_c\varepsilon^{cd}\partial_d X^\nu)g_{\mu\nu} \\ &- \frac{\sqrt{-\hat{\gamma}}}{2\gamma}\ (\partial_a X^\mu\varepsilon^{ab}\partial_b X^\nu)g_{\mu\rho}g_{\nu\sigma}(\partial_c X^\rho\varepsilon^{cd}\partial_d X^\sigma) \end{aligned} \tag{3.2.9}$$

Light-cone and other gauges We will now choose light-cone gauge $\tau \equiv X^+$ on the most general target-space metric (where metric components can only depend on $g_{\mu\nu} \equiv g_{\mu\nu}(X^+, X^i)$)

$$ds^2 = g_{++}dX^+dX^+ + 2g_{+-}dX^+dX^- + 2g_{+i}dX^+dX^i + g_{ij}dX^i dX^j \tag{3.2.10}$$

on which consistent light-cone quantisation is possible, see also Kim and Park [84]. Up to a conformal factor, this is the metric (1.4.2) we have used back in chapter one for light-cone gauge in string theory. Most notably, X^- is explicitly Killing.

While this gauge fixes $\tau \to \tau' = \tau$ completely, time-dependent diffeomorphisms of the remaining $\sigma^a \to \sigma'^a(\tau, \sigma^b)$ are still possible. We can fix those as well by demanding for the conjugate momentum P_- to be conserved in time[4]

$$P_- = \frac{\gamma}{\sqrt{-\hat{\gamma}}}g_{+-} = \omega(\sigma^a) \tag{3.2.11}$$

This is achieved by choosing $\sigma^a \to \sigma'^a(\tau, \sigma^b)$ such that the transformed space-like determinant γ cancels the time-dependence of the remaining g_{+-} and the quotient $\hat{\gamma}/\gamma$ (which itself is independent of this transformation).

The whole procedure renders the action polynomial in the dynamical variables, enforcing a gauge in which the unwanted prefactors $\gamma/\sqrt{-\hat{\gamma}}$ in the action cancel. This in turn restricts A_a by the equations of motion for X^-, such that in the most general case we get

$$\partial_a(\varepsilon^{ab}\omega A_b) = 0 \quad \Rightarrow \quad A_a = \tfrac{1}{\omega}\partial_a A \tag{3.2.12}$$

X^- can be eliminated as a cyclic variable by Routh's procedure.[5] In Poisson-bracket notation

$$\{X, Y\} \equiv \tfrac{1}{\omega}\varepsilon^{ab}\partial_a X\partial_b Y \tag{3.2.13}$$

the final gauge-fixed action reads

$$\begin{aligned} S = \tfrac{1}{2}T\int d\tau d^2\sigma\ \omega(\sigma^a)\Big(&\big(\dot{X}^i + \{A, X^i\}\big)\big(\dot{X}^j + \{A, X^j\}\big)\tfrac{g_{ij}}{g_{+-}} + \\ &+ \tfrac{g_{++}}{g_{+-}} + \tfrac{2g_{+i}}{g_{+-}}\big(\dot{X}^i + \{A, X^i\}\big) - \tfrac{1}{2}\ g_{+-}\{X^i, X^j\}\{X^k, X^l\}g_{ik}g_{jl}\Big) \end{aligned} \tag{3.2.14}$$

[4] The density $\omega(\sigma^a)$ can be chosen to be any constant by purely space-like diffeomorphisms $\sigma^a \to \sigma'^a(0, \sigma^b)$.

[5] Routh's procedure is partly passing to Hamiltonian variables in the respective cyclic ones only

The remaining time-dependent area-preserving diffeomorphisms So which gauge-freedom is propagated by A? To be precise, only the transformation of $\omega(\sigma)$ had to be fixed in the last step. The ω transforms as a density of weight 1 exactly like $\sqrt{\gamma}$, since the quotient $\hat{\gamma}/\gamma$ is invariant after fixing light-cone gauge and $g_{+-}(\tau, X^i)$ transforms as a scalar.

Consider the action of an infinitesimal coordinate transformation $\sigma^a \to \bar{\sigma}^a = \sigma^a + \epsilon^a$ on ω

$$\omega(\sigma) = \bar{\omega}(\bar{\sigma}) + \bar{\omega}\partial_a \epsilon^a = \bar{\omega}(\sigma) + \partial_a(\bar{\omega}\epsilon^a) \tag{3.2.15}$$

and demand invariance

$$\sigma^a \to \bar{\sigma}^a = \sigma^a + \epsilon^a \qquad \text{with } \partial_a(\omega\epsilon^a) = 0 \quad \Rightarrow \quad \epsilon^a = \tfrac{1}{\omega}\varepsilon^{ab}\partial_b \Lambda \tag{3.2.16}$$

In doing so, we have demanded $\delta\omega = 0$, which is the functional variation $\delta\omega = \bar{\omega}(\sigma) - \omega(\sigma)$ in contrast to the total variation $\tilde{\delta}\omega = \bar{\omega}(\bar{\sigma}) - \omega(\sigma)$. The former is equal to the Lie derivative and has group structure in the sense that the commutator of two δs yields another δ. In particular, under this transformation the fields X^i enjoy the transformation property we want

$$X(\sigma) = \bar{X}(\sigma + \epsilon) = \bar{X}(\sigma) + \tfrac{1}{\omega}\partial_a \bar{X}\varepsilon^{ab}\partial_b\Lambda \equiv \bar{X}(\sigma) + \{\bar{X}, \Lambda\} \tag{3.2.17}$$

while $\tilde{\delta}X = 0$ just vanishes.

We can now calculate the transformation of A by its definition from $A_{0a} = \frac{1}{\gamma}\gamma_{0b}\varepsilon^{bc}\gamma_{ca}$, which has the structure of a 2-tensor density of weight -1.

$$A_{0a}(\sigma) = \bar{A}_{bd}(\bar{\sigma}(\sigma))\dot{\bar{\sigma}}^d\partial_a\bar{\sigma}^b + \bar{A}_{0b}(\bar{\sigma}(\sigma))\partial_a\bar{\sigma}^b - \bar{A}_{0a}(\bar{\sigma}(\sigma))\partial_b\bar{\sigma}^b \tag{3.2.18}$$

with the intermediate definition of $A_{da} \equiv \frac{1}{\gamma}\gamma_{db}\varepsilon^{bc}\gamma_{ca}$ such that $A_{da}\varepsilon^{dc}\partial_c\dot{\Lambda} = \delta^c_a\partial_c\dot{\Lambda}$.

$$A_{0a}(\sigma) = \bar{A}_{ca}(\sigma)\dot{\epsilon}^c + \bar{A}_{0a}(\sigma) + \partial_b\bar{A}_{0a}(\sigma)\epsilon^b + A_{0b}(\sigma)\partial_a\epsilon^b - \bar{A}_{0a}(\sigma)\partial_b\epsilon^b \tag{3.2.19}$$

Therefore A transforms by virtue of the definition $A_{0a} = \frac{1}{\omega}\partial_a A$ like

$$\begin{aligned}\partial_a A &= \partial_a\bar{A} + \partial_a\dot{\Lambda} + \partial_b(\tfrac{1}{\omega}\partial_a\bar{A})\varepsilon^{bc}\partial_c\Lambda + \partial_b\bar{A}\varepsilon^{bc}\partial_a(\tfrac{1}{\omega}\partial_c\Lambda) - \partial_a\bar{A}\partial_b\tfrac{1}{\omega}\varepsilon^{bc}\partial_c\Lambda \\ &= \partial_a\bar{A} + \partial_a\dot{\Lambda} + \partial_a\{\bar{A}, \Lambda\}\end{aligned} \tag{3.2.20}$$

This is exactly the behaviour required for a gauge theory of the group $U(\infty)$ of the remaining time-dependent area-preserving diffeomorphisms

$$\begin{aligned}X(\tau,\sigma) &= \bar{X}(\tau,\sigma) + \{\bar{X}, \Lambda\} \\ A(\tau,\sigma) &= \bar{A}(\tau,\sigma) + \dot{\Lambda} + \{\bar{A}, \Lambda\}\end{aligned} \tag{3.2.21}$$

The gauge field thus comes from the off-diagonal components of the world-sheet metric in the Polyakov action, and its transformation property is determined by the remaining unfixed world-sheet diffeomorphisms, the time-dependent area-preserving ones.

3.2.2 Form Action

Let us briefly comment on the form action that is optional in the bosonic case, but required for the supersymmetric version. The 11 dimensional membrane couples to a 3-form potential by

$$\mathcal{L}_{\text{form}} = -\tfrac{1}{3!}\varepsilon^{\alpha\beta\gamma}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\partial_\gamma X^\rho\,C_{\mu\nu\rho}(X) \tag{3.2.22}$$

In order for light-cone quantisation to work, we have to constrain the form accordingly. The light-cone momentum P_- acquires an additional term

$$P_- = \omega(\sigma) - \tfrac{1}{2}\partial_+(C_{-\mu\nu}\,\partial_a X^\mu \varepsilon^{ab}\partial_b X^\nu) \tag{3.2.23}$$

which has to vanish, as well as the contribution to the X^- equations of motion

$$\partial_a(C_{\mu-\nu}\dot{X}^\mu)\varepsilon^{ab}\partial_b X^\nu - \tfrac{1}{2}\partial_- C_{\mu\nu\rho}\dot{X}^\mu \partial_a X^\nu \varepsilon^{ab}\partial_b X^\rho = 0 \tag{3.2.24}$$

A form that does not depend on X^- nor has any components in that direction

$$C_{\mu\nu\rho} = C_{\mu\nu\rho}(X^+, X^i) \quad \text{and} \quad C_{-\mu\nu} = 0 \tag{3.2.25}$$

satisfies these constraints and is therefore perfectly acceptable in this formalism.

The remaining potential Lagrangian then reads

$$\mathcal{L}_{\text{form}} = -\tfrac{1}{3!}\omega(\sigma)\,\left(C_{+jk}(\tau, X^l)\{X^j, X^k\} + C_{ijk}(\tau, X^l)\,\dot{X}^i\{X^j, X^k\}\right) \tag{3.2.26}$$

3.2.3 Quantisation

What we call membrane quantisation now is actually a regularisation of the classical theory: The $U(\infty)$ gauge group of the action (3.2.14) gets cut down to a smaller $U(N)$. This is a highly non-trivial process, and while formally the limit can be recovered by sending $N \to \infty$, there are many subtle issues like regarding the topology of the membrane.

Formally, the regularisation is performed by replacing Poisson-brackets of coordinate fields with commutators of matrix valued objects $\{X, Y\} \to -i[X, Y]$ and the space-like integral with a trace $\int d^2\sigma \to \frac{1}{N}\mathrm{Tr}$. This results in a sector of the theory with $P_- = \omega = N/R$. The regularised action reads

$$S = \frac{1}{4\pi^2 \ell_p^3 R}\int d\tau\ \mathrm{Tr}\left(\tfrac{1}{2}DX^i\,DX^j\,\tfrac{g_{ij}}{g_{+-}} + \tfrac{g_{++}}{2g_{+-}} + \tfrac{g_{+i}}{g_{+-}}\,DX^i - \tfrac{1}{4}\,g_{+-}[X^i, X^j][X^k, X^l]g_{ik}g_{jl}\right) \tag{3.2.27}$$

with $DX \equiv \dot{X} - i[A, X^i]$. Still, the term membrane quantisation is justified, since the resulting one-dimensional matrix theory is straight-forwardly quantised to matrix quantum mechanics.

What seems like a very specific, ad-hoc prescription, in fact taps into the vast field of non-commutative geometry. Quite generally, one can try to discretise space by replacing the algebra of coordinate functions with a non-commutative algebra, just like we have done here. A wealth of literature, also introductory, is available on this discipline, but we will not pursue the abstract mathematical direction in this work.

The original publication of de Wit, Hoppe and Nicolai [81] carries on to discuss in some detail how coordinates on a spherical membrane translate into non-commutative matrix generators of SU(2), preserving their respective bracket algebra. This can, however, be generalised to membranes of arbitrary topology, all apparently encoded in the same matrix quantum mechanics. We shall therefore proceed on the level of the action, where there is a lot to say still, before entering a discussion of ground-states and spectra. Later on we will take a look up close at fuzzy spheres, the regularisations of spherical membranes.

3.3 The BMN Matrix Model

3.3.1 BMN: The Membrane on the Maximally Supersymmetric Hpp-wave

Originally, the membrane quantisation scheme had been put forward by de Wit, Hoppe and Nicolai [81] in flat space and including fermions. We have not treated fermions so far, and shall not do so in all generality, but will present a short sketch of how to obtain the relevant terms further down. Here, we would just like to quote the flat space matrix model Lagrangian

$$\mathcal{L}_{\mathrm{BFSS}} = \tfrac{1}{2}(DX^i)^2 + \tfrac{1}{4}[X^i, X^j]^2 + \tfrac{i}{2}\psi^\dagger D\psi - \tfrac{1}{2}\psi^\dagger\Gamma^i[X^i, \psi] \tag{3.3.1}$$

Complying with the saying that things are never called after their true inventors, we shall call this action the BFSS model, after Banks, Fischler, Shenker and Susskind [85]. Next chapter we shall explain their important conjecture that justifies the naming.

One might think this was the easiest resulting matrix model, but at closer inspection it exhibited some severe obstructions soon after its discovery: The commutator potential vanishes for commuting matrices and thus allows for so-called flat directions. Hence one has to deal with infinitely many zero-energy states communicating via long spikes. The resulting vacuum structure seemed not treatable and only much later it was realised that this could be interpreted as states of a multi-particle system (of D0-branes, see next chapter) instead.

But we have seen that we can formulate membrane quantisation for a quite general class of backgrounds. Of those, cases preserving a certain amount of supersymmetry are of most interest. Apart from flat space, there are only three solutions to 11 dimensional supergravity preserving maximal supersymmetry, as classified by Figueroa-O'Farril and Papadopoulos [86]. They are $AdS_4 \times S^7$, $AdS_7 \times S^4$ and the Hpp-wave

$$ds^2_{\mathrm{pp}} = 2dx^+dx^- - (\tfrac{\mu}{3})^2\big((x^a)^2 + \tfrac{1}{4}(x^i)^2\big)(dx^+)^2 + (dx^a)^2 + (dx^i)^2 \text{ and } F_{+abc} = \mu \tag{3.3.2}$$

where the transverse coordinates fall into two classes labelled by $a, b, c \in \{1\dots3\}$ and $i, j, k \in \{4\dots9\}$ making for a total of nine.

This is a plane wave in Brinkmann coordinates, more precisely an Hpp-wave, a plane wave with parallel rays (pp) equipped with a homogeneous (H) four-form flux, see Figueroa-O'Farril and Papadopoulos [87]. Generic Hpp-waves are solutions to 11 dimensional supergravity and already preserve half of the supersymmetry. The special maximally symmetric case can be obtained as the Penrose limit of either of the two $AdS \times S$ backgrounds (see Blau, Figueroa-O'Farrill, Hull and Papadopoulos [18]), a procedure which was shown to preserve the supersymmetry of the original background. Note that the Hpp-wave includes flat space for the special value of the 'mass' parameter $\mu = 0$, whereas all other cases $\mu \neq 0$ are isometric.

Of the maximally supersymmetric ones, only this background exhibits the x^- structure required for membrane quantisation. The procedure then leads to the Berenstein-Maldacena-Nastase (BMN) matrix quantum mechanics [19]

$$\begin{aligned}\mathcal{L}_{\mathrm{BMN}} = \tfrac{1}{2}(DX^I)^2 - \tfrac{1}{2}(\tfrac{\mu}{3})^2(X^a)^2 - \tfrac{1}{8}(\tfrac{\mu}{3})^2(X^i)^2 - i(\tfrac{\mu}{3})\varepsilon^{abc}X^aX^bX^c + \tfrac{1}{4}[X^I, X^J]^2 \\ + \tfrac{i}{2}\psi^\dagger D\psi - \tfrac{1}{2}\psi^\dagger\Gamma^I[X^I, \psi] + \tfrac{i}{2}(\tfrac{\mu}{4})\psi^\dagger\Gamma^{123}\psi\end{aligned} \tag{3.3.3}$$

where we have used the collective index $I = (a, i)$ to shorten notation in this one formula.

The mass deformation with respect to flat space actually helps the analysis and leads to well-defined fuzzy sphere vacua. We shall present an analysis of the ground states and a perturbative expansion about them later in this chapter. Before that, let us take a closer look at the fermion terms and the connection of the model to maximally supersymmetric gauge theory.

3.3.2 Fermions

The fermionic aspect of the supersymmetric background is well understood in the important cases and has been subject of numerous publications. Flat space has been treated in the original paper of de Wit, Hoppe and Nicolai [81]. Extensions to curved backgrounds can be found e.g. in de Wit, Peeters and Plefka [88] (up to second order in the fermionic coordinates θ) and de Wit, Peeters, Plefka and Sevrin [89] (for coset spaces, notably the AdS$\times$S backgrounds). We shall not repeat the lengthy calculations here, but just sketch the procedure, mostly following the account in Dasgupta, Sheikh-Jabbari and van Raamsdonk [90].

Fermionic Light-Cone Gauge The 11 dimensional supermembrane is formulated in superspace with fermionic coordinates in form of a 32-component spinor θ in addition to the bosonic ones. Whereas this makes supersymmetry manifest in target space, it can only be transferred to the membrane world-sheet if the additional κ-symmetry holds. Imposing κ-symmetry effectively halves the number of fermionic degrees of freedom. In practice, this allows to choose a fermionic light-cone gauge

$$\hat{\Gamma}^{+}\theta = 0 \tag{3.3.4}$$

which in a suitably chosen representation of the 32×32 $\hat{\Gamma}$-matrices

$$\hat{\Gamma}^{+} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \qquad \hat{\Gamma}^{-} = \sqrt{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \qquad \hat{\Gamma}^{i} = \sqrt{2}\begin{pmatrix} \Gamma^i & 0 \\ 0 & -\Gamma^i \end{pmatrix} \tag{3.3.5}$$

reduces θ to a 16-component Majorana spinor ψ

$$\theta = 2^{-1/4}\begin{pmatrix} 0 \\ \psi \end{pmatrix} \qquad \bar{\theta} = 2^{-1/4}\begin{pmatrix} -\psi^T \\ 0 \end{pmatrix} \tag{3.3.6}$$

that appears in the BMN model action.

The supervielbein for the Hpp-wave What makes the treatment of fermions complicated is the fact that the background we are dealing with is given in terms of the on-shell supergravity component fields, that is the vielbein e^r_μ and spin connection ω with the non-zero components[6]

$$e^{-}_{-} = e^{+}_{+} = 1 \qquad e^{i}_{j} = \delta^{i}_{j} \qquad e^{-}_{+} = \tfrac{1}{2}g_{++} \qquad \omega^{-i}_{+} = \tfrac{1}{2}\partial_i F^2 \qquad (\text{and } i \leftrightarrow a) \tag{3.3.7}$$

the gravitino $\psi_\mu = 0$ and the antisymmetric gauge tensor field $C_{\mu\nu\rho}$ (with field strength F_{+abc}). The supermembrane on the other hand is formulated in off-shell superspace with coordinates $Z^{\mathcal{M}} = (X^\mu, \theta^M)$ where the geometry is encoded in the supervielbein $E^{\mathcal{R}}_{\mathcal{M}}$, the (dependent) spin-connection $\Omega^{\mathcal{RS}}_{\mathcal{M}}$ and the antisymmetric tensor $B_{\mathcal{MNP}}$. In a slightly non-standard fashion, we

[6] Note that this implies a *light-cone* metric $\eta_{+-} = 1$ and $\eta_{++} = \eta_{--} = 0$ in flat tangent space.

shall denote indices in curved superspace by $\mathcal{M} = (\mu, M)$ and in flat tangent superspace by $\mathcal{R} = (r, R)$. Fermion indices will always be capital letters in the following.

One therefore needs to first derive an expression of the supervielbein as an expansion of θ in terms of the on-shell component fields. In practice, this can be done by a method called 'gauge completion' and to first order it leads to the on-shell formulation of supergravity from superspace.

For the supermembrane, however, one also needs higher orders. Up to second order in θ the procedure has been carried out in de Wit, Peeters, and Plefka [88] (see also references therein). Using a different method, an expression valid to all orders has been found by de Wit, Peeters, Plefka and Sevrin [89] for coset spaces (notably the AdS×S backgrounds in the title of the paper and the Hpp-wave).

For the Hpp-wave in fermionic light-cone gauge $\hat{\Gamma}^+\theta = 0$ it can be shown (Dasgupta, Sheikh-Jabbari and van Raamsdonk [90]) that the matrix $\mathcal{M}^2$ characterising higher orders of θ in [89] vanishes, such that the result agrees with that of [88] (gravitino $\psi_\mu = 0$)

$$\begin{aligned} E_M^R &= \delta_M^R & E_\mu^R &= (D_\mu\theta)^R = -\tfrac{1}{4}\omega_\mu^{rs}(\hat{\Gamma}_{rs}\theta)^R + e_\mu^s(T_s^{tuvw}\theta)^R F_{tuvw} \\ E_M^r &= -(\bar{\theta}\hat{\Gamma}^r)_M & E_\mu^r &= e_\mu^r + \bar{\theta}\hat{\Gamma}^r D_\mu\theta \end{aligned} \tag{3.3.8}$$

where $T_r^{stuv} = \frac{1}{2!3!4!}(\hat{\Gamma}_r^{stuv} - 8\delta_r^{[s}\hat{\Gamma}^{tuv]})$.

The gauge $\hat{\Gamma}^+\theta = 0$ we have chosen simplifies the vielbein components E_μ^r considerably

$$\bar{\theta}\hat{\Gamma}^r D_\mu\theta = \bar{\theta}\hat{\Gamma}^r(-\tfrac{1}{4}\omega^{rs}\hat{\Gamma}_{rs} + e_\mu^s T_s^{tuvw}F_{tuvw})\theta = \tfrac{i}{4}\mu\ \delta_\mu^+\delta_-^r\ \psi^T\Gamma^{123}\psi \tag{3.3.9}$$

Hence the whole effect on the vielbein components with purely bosonic indices E_μ^r lies in adding a term to the E_+^- parts. Since $e_+^- = \frac{1}{2}g_{++}$, this amounts to simply shifting g_{++} by

$$g_{++} \to g_{++} + \tfrac{i}{2}\mu\ \psi^T\Gamma^{123}\psi \tag{3.3.10}$$

in (3.3.7) and thus in the matrix Lagrangian we have calculated for the bosonic membrane.

This term is in fact the only μ-dependent fermion term that enters the calculation. It is the analogue of the Myers term $\mu\varepsilon_{abc}X^aX^bX^c$ of the bosons and required for supersymmetry of the resulting matrix theory. The other two fermion terms, the kinetic one from the Polyakov action and the three-vertex interaction from the Chern-Simons term are also present in flat space $\mu = 0$ and their derivation is well established.

Fermion terms from the Polyakov action Let us turn back to the Polyakov action

$$S_{\text{Polyakov}} = \frac{T}{2}\int d^3\sigma\ \frac{1}{\sqrt{-\hat{\gamma}}}(\hat{\gamma}\ \hat{\gamma}^{\alpha\beta}\ \hat{h}_{\alpha\beta} - \hat{\gamma}) \tag{3.3.11}$$

and see how the fermion kinetic term arises. Whereas before we have only treated the induced metric from the bosonic space-time coordinates $\hat{h}_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}$, now, in the supersymmetric case, we have to plug in the pullback of the supervielbein $\partial_\alpha X^\mu E_\mu^r + \partial_\alpha\theta^M E_M^r$

$$\hat{h}_{\alpha\beta} = (\partial_\alpha X^\mu E_\mu^r)(\partial_\beta X^\nu E_\nu^s)\eta_{rs} + 2(\partial_\alpha X^\mu E_\mu^r)(\partial_\beta\theta^M E_M^s)\eta_{rs} + (\partial_\alpha\theta^M E_M^r)(\partial_\beta\theta^N E_N^s)\eta_{rs} \tag{3.3.12}$$

The first summand describes the standard bosonic membrane of before plus the fermionic 'Myers'-term in the E_+^--components

$$(\partial_\alpha X^\mu E_\mu^r)(\partial_\beta X^\nu E_\nu^s)\eta_{rs} = (\partial_\alpha X^\mu)g_{\mu\nu}(\partial_\beta X^\nu) + 2(\partial_\alpha X^+)E_+^-\eta_{-+}(\partial_\beta X^+) \tag{3.3.13}$$

Clearly one can see how the second term with the $(++)$ curved index structure of $E_+^-\eta_{-+}$ can be absorbed by the shift of g_{++} in (3.3.10).

For the other two terms, note that when the $E_M^r = -(\bar{\theta}\hat{\Gamma}^r)_M$ is contracted with θ^M, due to our gauge choice only the non-diagonal Γ^- gives a non-zero contribution. The last summand in (3.3.12) is therefore zero, since $\eta_{--} = 0$.

The second summand in (3.3.12), finally, yields the fermion kinetic term.

$$2\hat{\gamma}\hat{\gamma}^{\alpha\beta}(\partial_\alpha X^\mu E_\mu^r)(\partial_\beta \theta^M E_M^s)\eta_{rs} = 2\hat{\gamma}\hat{\gamma}^{\alpha\beta}(\partial_\alpha X^+ e_+^+)\eta_{+-}(\partial_\beta \theta^M E_M^-) = 2\hat{\gamma}\hat{\gamma}^{0\beta}(\bar{\theta}\hat{\Gamma}^-\partial_\beta\theta) \tag{3.3.14}$$

Now splitting $\hat{\gamma}\hat{\gamma}^{0\beta}$ into $\hat{\gamma}\hat{\gamma}^{00} = \gamma$ and $\hat{\gamma}\hat{\gamma}^{0b} = \gamma A_c \varepsilon^{ca}$ we arrive at

$$2\hat{\gamma}\hat{\gamma}^{\alpha\beta}(\partial_\alpha X^\mu E_\mu^r)(\partial_\beta \theta^M E_M^s)\eta_{rs} = 2\gamma\psi^T\dot{\psi} + 2\gamma\psi^T(A_c\varepsilon^{cb}\partial_b\psi) \tag{3.3.15}$$

the fermion gauge kinetic term.

Fermion terms from the Chern-Simons term The remaining fermion terms work exactly the same way as in the flat-space case. They stem from the super-gauge field, and due to the fermionic light-cone gauge the only part that remains is the

$$\tfrac{i}{2}\psi^T\Gamma^a\{X^a,\psi\} + \tfrac{i}{2}\psi^T\Gamma^i\{X^i,\psi\} \tag{3.3.16}$$

3.3.3 Reduction of the Fermions

Starting point of the BMN-Lagrangian was 11 dimensional superspace sporting the corresponding 32 component spinors. In even dimensions a chiral representation can always be chosen such as to break them apart into two 16 component Weyl spinors ψ. In 11 dimensions this is not obvious, and only happens in our model by virtue of the κ-symmetry, which effectively reduces the fermion content. The motion of the fermions is governed by the BMN Lagrangian

$$\mathcal{L}_{\text{ferm}} = \tfrac{i}{2}\psi^\dagger D\psi - \tfrac{1}{2}\psi^\dagger\Gamma^a[X^a,\psi] - \tfrac{1}{2}\psi^\dagger\Gamma^i[X^i,\psi] + \tfrac{i}{2}(\tfrac{\mu}{4})\psi^\dagger\Gamma^{123}\psi \tag{3.3.17}$$

The odd Γ_0-matrix gives the scalar product between the Weyl-fermion and its complex conjugate in the kinetic term, while the other nine Clifford matrices couple scalars and fermions.

In 10 dimensions the Weyl and the Majorana condition can be imposed independently, so an additional reality condition restrains the number of components of our Weyl spinor from 16 to 8. In again a chiral basis of the 9 dimensional Clifford algebra this constraint would relate the upper and lower 8 components of the spinor simply by complex conjugation.

In our model, however, we have the additional mass deformation term $\psi^\dagger\Gamma^{123}\psi$. So instead, we would like to emphasise the related explicit breaking of the symmetry SO(9) down to SO(3)×SO(6). On the boson level, the SO(3) interchanges the three modes X^a, $a = \{1,2,3\}$

while SO(6) acts on the remaining X^i, $i = \{4\dots 9\}$. On the fermion level, this symmetry translates into the isomorphic Lie algebra

$$\mathrm{SO}(9) \to \mathrm{SO}(3) \times \mathrm{SO}(6) \sim \mathrm{SU}(2) \times \mathrm{SU}(4) \tag{3.3.18}$$

Therefore we choose a slightly different basis to emphasise this decomposition of spinors. The 16 components of ψ fall into the 2×4 components ψ^{AI} and the $\bar{2} \times \bar{4}$ components $\bar{\psi}^{AI}$. The Γ-matrices decompose as

$$\Gamma^a = \begin{pmatrix} -\sigma^a \times 1 & 0 \\ 0 & \sigma^a \times 1 \end{pmatrix} \qquad \Gamma^i = \begin{pmatrix} 0 & 1 \times \rho^i \\ 1 \times (\rho^i)^\dagger & 0 \end{pmatrix} \tag{3.3.19}$$

where the three σ^a are the $SU(2)$ Pauli matrices and the six Γ-matrices ρ^i carry $SU(4)$ indices and satisfy the algebra

$$\rho^i (\rho^j)^\dagger + (\rho^j)^\dagger \rho^i = 2\delta^{ij} \tag{3.3.20}$$

of the off-diagonal blocks of the corresponding $SO(9)$ Γ-matrices in chiral representation.

We can rotate the ρ^i such that the charge conjugation matrix B defined by $\Gamma^* = B\Gamma B^{-1}$ (the one imposing the Majorana constraint) takes the form

$$B = \begin{pmatrix} 0 & \varepsilon_{AB} \times \delta_{IJ} \\ -\varepsilon_{AB} \times \delta_{IJ} & 0 \end{pmatrix} \tag{3.3.21}$$

By virtue of the Majorana constraint $\psi^\dagger = B\psi$ we can relate the two blocks of $\psi = (\psi^{AI}, \varepsilon_{BC}\delta_{JK}\psi^{\dagger CK})$ to each other by pulling indices on the two 8 component spinors with the metric $\varepsilon_{AB}\delta_{IJ}$.

The Γ^{123}-matrix in this representation is diagonal, with eigenvalues $-i$ $(+i)$ for the upper (lower) half spinor components. The terms in the Lagrangian decompose

$$\begin{aligned}
\tfrac{i}{2}\psi^\dagger D\psi &\to i\psi^{\dagger AI} D\psi_{AI} \\
-\tfrac{1}{2}\psi^\dagger \Gamma^a [X^a, \psi] &\to -\psi^{\dagger AI}\sigma^a_{AB}\delta_{IJ}[X^a, \psi^{BJ}] \\
-\tfrac{1}{2}\psi^\dagger \Gamma^i [X^i, \psi] &\to \tfrac{1}{2}\psi^{\dagger AI}(\rho^i)^J_I \varepsilon_{AB}\delta_{JK}[X^i, \psi^{\dagger BK}] - \tfrac{1}{2}\psi^{AI}(\rho^{\dagger i})^J_I \varepsilon_{AB}\delta_{JK}[X^i, \psi^{BK}] \\
\tfrac{i}{2}(\tfrac{\mu}{4})\psi^\dagger \Gamma^{123}\psi &\to -\tfrac{\mu}{4}\psi^{\dagger AI}\psi_{AI}
\end{aligned} \tag{3.3.22}$$

The fermion Lagrangian is the sum of these four terms.

3.4 Supersymmetry and Gauge Theoretic Origin of the BMN Model

So far we have only treated the BMN model (3.3.3) as a regularisation of the supermembrane on the maximally supersymmetric plane wave, which included the BFSS matrix model (3.3.1) of flat space. But just as much as one can understand the supergravity background as the mass deformation of 11 dimensional Minkowski space, one can derive the BMN matrix quantum mechanics as the mass deformation of the BFSS model on the gauge theory side also.

In the flat space BFSS case, the matrix model of the supermembrane can be obtained from the $N = 1$ superconformal Yang-Mills theory in $D = 10$ dimensions. The procedure is fairly standard – a compactification of the 9 space-like directions on a 9-torus and a truncation of the resulting spectrum to the lowest (massless) Kaluza-Klein modes yields the BFSS result. Considering the fact that the mass deformation of the Hpp-wave background only affects three of the nine transverse dimensions, we can certainly compactify $N = 1$, $D = 10$ SYM theory (transversely) on a six-torus to obtain $N = 4$, $D = 4$ SYM theory in the same standard way.

How to proceed from there, however, is not obvious. Kim and Park [84] have taken toroidal compactification all the way down to the one dimensional (BFSS) $N = 16$ SYM quantum mechanics and then investigated all possible mass deformations compatible with supersymmetry. This approach resulted in a classification of all the mass deformed theories by the amount of supersymmetry they preserve. It was found that there is only one preserving the full $N = 16$ supersymmetry, and this is the BMN model, consistent with the fact that the Hpp-wave is the unique maximally supersymmetric mass deformation of 11 dimensional Minkowski space.

$N = 4$ **SYM on** $R \times S^3$ Alternatively, one can obtain the BMN model in a more geometrical fashion by a compactification of the $N = 4, D = 4$ SYM on an $R \times S^3$ background, i.e. a three-sphere in the space-like dimensions.

The procedure is described in the publication Kim, Klose and Plefka [91] and in the Ph.D.-thesis of Klose [92]. For more detailed information on the symmetry structure see also the two consecutive publications by Ishiki, Takayama and Tsuchiya [93] and [94] (with Shimasaki), or, independently, Okuyama [95]. See also references therein, notably Blau [96] for more general SYM compactifications on curved spaces with Killing spinors. We shall give a short sketch of the procedure referring the reader to the mentioned references for calculational detail.

The $N = 4, D = 4$ SYM action with *curved* indices μ, ν the $R \times S^3$

$$\begin{aligned} \mathcal{L}_{\text{SYM}} = \frac{2}{g_{YM}} \int d^4x \sqrt{-g} \text{Tr} \Big(&- \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - D_\mu X^i D^\mu X^i - \tfrac{1}{12} R (X^i)^2 + \tfrac{1}{4} [X^i, X^j]^2 \\ &-2i \lambda_A^\dagger \sigma^\mu D_\mu \lambda^A + \rho_i^{AB} \lambda_A^\dagger i\sigma^2 [X^i, \lambda_B^*] - (\rho_i^\dagger)_{AB} (\lambda^A)^T i\sigma^2 [X^i, \lambda^B] \end{aligned} \tag{3.4.1}$$

requires an additional scalar mass term $\frac{1}{12} R (X^i)^2$ with respect to the flat action, since in D dimensions the conformally invariant wave operator is $\Box - \frac{D-2}{4(D-1)} \mathcal{R}$ (with $\mathcal{R}$ the Ricci scalar).

The theory has a superconformal symmetry with algebra $SU(2, 2|4)$, generated by the conformal Killing spinors of the background. The bosonic subgroup of this group is $SO(2, 4) \times SO(6)$. The four dimensional conformal group $SO(2, 4)$ acts non-trivially on the curved space-time manifold, while $SO(6)$ is the R-symmetry group rotating the transverse scalars X^i. $SO(6)$ acts on the fermions as representations of the isometric $SU(4) \sim SO(6)$. This is reflected by the decomposition of the Γ-matrices according to $SO(9, 1) \to SO(3, 1) \times SO(6) \sim SU(2) \times SU(4)$ described in the previous section.

On the fermionic side, the group has 32 supercharges as the 32 real components of a 10 dimensional Majorana-spinor. As in the last section, this spinor can be split into two fermionic Weyl spinor supercharges with 16 degrees of freedom Q_L and Q_R. This split will be used further down to truncate the field spectrum by reducing the symmetry group $SU(2, 2|4)$ to $SU(4|2)$ with only Q_L acting.

Harmonic Expansion Reflecting the symmetry of the theory, all fields can now be expanded in spherical harmonics on S^3. The three-sphere can be written as a coset space $G/H = SO(4)/SO(3)$ where $SO(4) \sim SU_L(2) \times SU_R(2)$ is generated by two sets of $SU(2)$ generators, left

J^i_L and right J^i_R. The diagonal subgroup $H = SO(3)$, which is the isometry group of the sphere's tangent space, is generated by the sum $L^i = J^i_L + J^i_R$.

All representations of the sphere's isometry group $SO(4) \sim SU_L(2) \times SU_R(2)$ are given by a tuple of copies of the standard $SO(3)$ spherical harmonics $(Y^L_{lm}, Y^R_{\tilde{l}\tilde{m}})$. The set of all these functions is too big still, since it has yet to be modded by the tangent space group H and therefore comes with fields of all possible spins on the three-sphere.

The map to the diagonal subgroup is given by the Clebsch-Gordan coefficients for adding the right and left spin representations. These can be used now to select the appropriate set of spherical harmonics describing the scalars, spinors and vectors of respective spins 0, $\frac{1}{2}$ and 1 in the tangent space of the sphere. The Clebsch-Gordan coefficients impose a triangular inequality on the left and right spins

$$|J_L - J_R| \leq L \leq J_L + J_R \tag{3.4.2}$$

so that a complete set of one scalar Y^1_{kI}, two spinor $Y^{2\pm}_{kI}$ and three vector Y^{3a}_{kI} families of spherical harmonics is obtained. They are orthonormal harmonic polynomials and thus diagonalise the Laplace operator on the three-sphere (amongst other properties)

$$\nabla^2 Y^1_{kI} = -\tfrac{1}{R^2} k(k+2) Y^1_{kI} \tag{3.4.3}$$

$$\nabla^2 Y^{2\pm}_{kI} = -\tfrac{1}{R^2}(k(k+3) + \tfrac{3}{4}) Y^{2\pm}_{kI} \qquad \not\nabla Y^{2\pm}_{kI} = \pm\tfrac{i}{R}(k+\tfrac{3}{2}) Y^{2\pm}_{kI} \tag{3.4.4}$$

$$\nabla^2 Y^{3\pm}_{kI} = -\tfrac{1}{R^2}(k(k+4)+2) Y^{3\pm}_{kI} \tag{3.4.5}$$

We have only included two vector families in the list, $Y^{3\pm}_{kI}$. The third one is given by the gradient of the scalar family $Y^{3,0}_{kI,\mu} = \nabla_\mu Y^1_{kI}$ and is the only one with non-zero divergence $\nabla^\mu Y^{3,0}_{kI,\mu} \neq 0$. We eliminate it without further ado by choosing Coulomb gauge.

Truncation In principle, thus, the theory can be reformulated in terms of an infinite tower of Kaluza-Klein-states. But where in compactification on a torus the spectrum can consistently be truncated to the massless mode(s), the situation here is not so clear. In fact 'consistent' means to pick out a subsector of the theory the fields of which can never produce excitations outside that sector in the full interacting theory. Only then it is possible to set all other fields to zero once and for all without changing the physics of the theory (at least on a classical level).

To check consistency, the full equations of motions have to be derived, truncated and then used to back-determine a truncated action. We shall refrain from doing so and refer the interested reader to the literature [91].

Still, we can motivate the solution on symmetry grounds. In fact, the truncation is not immediately obvious because the ensemble of states does not decompose into finite dimensional representations of the symmetry group – the whole tower of states of different masses and spins forms one single infinite dimensional superrepresentation of the superconformal group. This can be seen by starting with the lowest weight state $(0,0,6)$ and working all the way up through the whole representation by acting repeatedly with the two supercharges $Q_L = (\frac{1}{2}, 0, 4)$ and $Q_R = (0, \frac{1}{2}, \bar{4})$. Figure 3.1 shows how one traverses the tower of states.

However, if one takes into account only half of the supercharges Q_L (say), then the tower does decompose into finite dimensional multiplets of the reduced symmetry group $SU(4|2)$. It is therefore legal to focus on the low energy sector, the multiplet to the lower left of the diagram. From the figure we see that the lowest multiplet of the mode expansion of the fields consists of

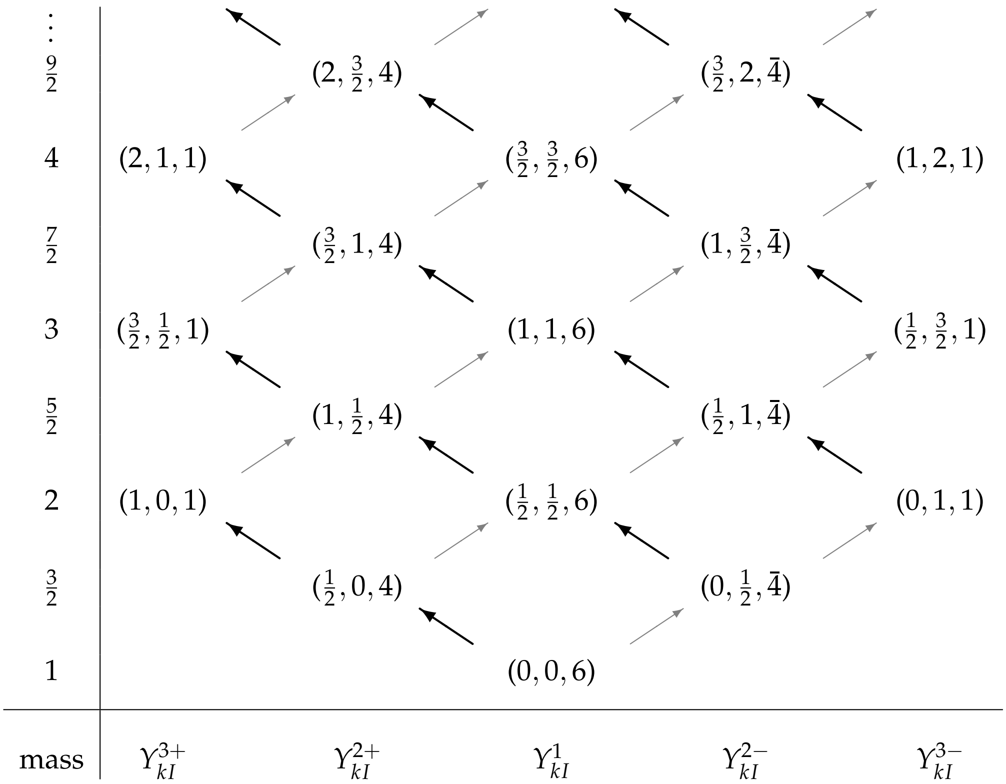

Figure 3.1: *The Kaluza-Klein tower [93]. Acting with* $Q_L = (\frac{1}{2}, 0, 4)$ *moves up-left along the arrows and with* $Q_R = (0, \frac{1}{2}, \bar{4})$ *up-right. Restricting to* Q_L *breaks the full infinite* $SU(2,2|4)$ *representation into finite* $SU(2|4)$ *ones.*

- a set of 6 scalars X^i from the mode expansion of the transverse scalars in terms of Y^1_{kI},
- a spinor ψ with $SU(4)$ index from the lowest fermion modes Y^{2+}_{kI} and
- a vector X^a as coefficients of the gauge field A_a expansion w.r.t Y^{3+}_{kI}, neutral under $SO(6)$.

Their respective harmonic functions are given in geometric terms by a constant, two Killing spinors and three Killing vectors. These are the zero-modes of the compactification and the field content of the low energy limit: the BMN model.

Since we already know these modes diagonalise the Laplacian and thus provide additional mass-terms, it is straight-forward to obtain the BMN-model from $R \times S^3$ SYM restricted to this sector. The only term that is not obvious is the Myers term $\varepsilon_{abc} X^a X^b X^c$. It can be derived from the three-vertex terms in the Yang-Mills action $\nabla^\mu A^\nu [A_\mu, A_\nu]$ using the vector spherical harmonics property [93] $\varepsilon_{abc} \nabla_a Y^{3+}_{JM,b} = -2(J+1) Y^{3+}_{JM,c}$.

We want to stress again that we have skipped the most important and non-trivial part of the calculation: justifying the consistency of this truncation.

3.5 Fuzzy Spheres

3.5.1 Classical Fuzzy Sphere Solution of the BMN Lagrangian

The bosonic potential of the BMN Lagrangian can be written as a sum of square terms

$$V_{\text{BMN}} = \tfrac{1}{4}\Big([X^a, X^b] - i\tfrac{\mu}{3}\varepsilon^{abc}X^c\Big)^2 - \tfrac{1}{8}(\tfrac{\mu}{3}X^i)^2 + \tfrac{1}{2}[X^a, X^i]^2 + \tfrac{1}{4}[X^i, X^j]^2 \tag{3.5.1}$$

For classical supersymmetric solutions each of the terms has to vanish independently, so the mass term for the transverse directions $i \in \{4\dots 9\}$ forces us to take $X^i = 0$.

The $a \in \{1\dots 3\}$ potential on the other hand allows also for non-trivial vacua

$$X^a = \tfrac{\mu}{3}J^a \quad \text{with } [J^a, J^b] = i\varepsilon^{abc}J^c \tag{3.5.2}$$

where the matrices J^a fulfil the commutation relations of generators $SU(2)$.

Solutions of this kind lift in a straight-forward way to spherical membranes expressed in coordinate functions (x^1, x^2, x^3). Taken as functions of the angular coordinates $\sigma^1 = \theta$, $\sigma^2 = \phi$

$$x^1 = \sin\theta\cos\phi \qquad x^2 = \sin\theta\sin\phi \qquad x^3 = \cos\theta \tag{3.5.3}$$

they fulfil the same $SU(2)$ commutation relations with respect to the Poisson bracket. When going to matrices of finite size N via membrane quantisation, the spherical membranes get regularised to irreducible $SU(2)$ matrix representations, the so-called *fuzzy spheres*.

With the same ease fuzzy spheres lift up on the gauge theory side to gauge field configurations in Super-Yang-Mills on $R \times S^3$. In this context one can compute their associated electric and magnetic field shedding light on their physical interpretation from a different perspective. For example Popov [97] describes time-dependent fuzzy-sphere 'bounce' solutions as dyons in Yang-Mills.

In the BMN model the trivial $X^a = 0$ solution along with the fuzzy spheres exhaust already the list of supersymmetric classical vacua. This discrete set of solutions is a very appealing particularity of the BMN model and in stark contrast to the massless $\mu = 0$ BFSS model. In the latter, commuting sets of matrices always have zero potential and can get arbitrarily large at no cost of energy. These flat directions or moduli spaces that complicate the analysis do not arise in the massive case.

However, a vacuum solution can be any representation of matrix dimension N, not necessarily irreducible. Each reducible representations can be written as a direct sum of k irreducible ones with dimensions $\{N_1 \dots N_k\}$ such that $\sum_k N_k = N$. Therefore the possible representations for a given matrix size N can be labelled by the set of all partitions of N and each BMN vacuum is interpreted as a multi-particle state of a collection of k fuzzy spheres.

Apart from these vacua, a number of other solutions to the BMN model exist, and can be classified as multiplets of the Lie superalgebra $SU(4|2)$. Some of these are BPS multiplets which preserve a certain number of supersymmetries. We shall not enter a discussion of the full spectrum here, which has been treated exhaustively e.g. in Dasgupta, Sheikh-Jabbari and van Raamsdonk [98] and in Kim and Park [99].

Returning to the fuzzy spheres, we can connect the trivial vacuum to any fuzzy sphere solution by introducing a radial variable $\beta(t)$

$$X^a = \tfrac{\mu}{3}\beta(t)J^a \tag{3.5.4}$$

Restricted to this solution with all other fields zero, the equations of motion reduce to the radial

$$\ddot{\beta} + 2(\tfrac{\mu}{3})^2 \beta(\beta - 1)(\beta - \tfrac{1}{2}) = 0 \tag{3.5.5}$$

A quartic 'Mexican Hat' potential with the trivial vacuum at $\beta = 0$ and the fuzzy sphere at $\beta = 1$. In between the two, we find another static solution: an unstable maximum at $\beta = \frac{1}{2}$.

With the intention of further exploring this instability, we will expand the Lagrangian about a fuzzy sphere background (3.5.4) and take a look at the resulting fluctuations. We start the discussion by Faddeev-Popov gauge fixing the action expanded about a general background B^a in order to get rid of some off-diagonal terms and in the following subsection then focus on fuzzy spheres.

3.5.2 R_ξ Gauges and Ghosts

In this section we expand about a general background $X^a = B^a + Y^a$. Later on we will be interested in fuzzy spheres $B^a = (\frac{\mu}{3})\beta J^a$. We will be rather explicit in this section, since we use a more general ξ-gauge than is commonly employed in this context. The advantage is an additional parameter which allows to quickly identify unphysical content of the theory.

The gauge-kinetic part of the BMN-Lagrangian expanded as $X^a = B^a + Y^a$

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}(DX^a)^2 = \tfrac{1}{2}\dot{X}^2 - i\dot{X}^a[A, X^a] - \tfrac{1}{2}[A, X^a]^2 \\
&= \tfrac{1}{2}(\dot{B}^a)^2 + \dot{B}^a\dot{Y}^a + \tfrac{1}{2}(\dot{Y}^a)^2 \\
&\quad - i\dot{B}^a[A, B^a] - i\dot{B}^a[A, Y^a] - i\dot{Y}^a[A, B^a] - i\dot{Y}^a[A, Y^a] \\
&\quad - \tfrac{1}{2}[A, B^a]^2 - [A, B^a][A, Y^a] - \tfrac{1}{2}[A, Y^a]^2
\end{aligned} \tag{3.5.6}$$

requires gauge fixing in order to render functional integration over the fields well-defined. We will follow the Faddeev-Popov method well adapted to practical calculations. The degrees of freedom

$$\delta A = \partial\Lambda - i[A, \Lambda] \qquad \delta X^a = -i[X^a, \Lambda] \tag{3.5.7}$$

are fixed by the gauge-condition

$$F = \partial A + i\xi[B^a, X^a] \tag{3.5.8}$$

designed to cancel a specific term in the kinetic Lagrangian.

The parameter ξ was used by t'Hooft in the R_ξ-gauges to smoothly choose between different gauges in order to show the renormalisability of spontaneously broken gauge theories. This choice incorporates a family of gauge choices of which any one can be selected by assigning a value to the arbitrary parameter ξ at any step in the procedure. As a result of gauge independence, ξ-dependent terms must always cancel out in physical amplitudes. Conversely, we can use ξ to pin down unphysical quantities in the calculations[7]. In particular, ξ-dependent mass terms hint at an over-complete spectrum still with unphysical degrees of freedom.

The Faddeev-Popov determinant is given in terms of auxiliary scalar anticommuting fields (ghosts) η and $\bar{\eta}$

$$\mathcal{L} = -\bar{\eta}\frac{\delta F}{\delta\Lambda}\eta = -\bar{\eta}\partial^2\eta + i\bar{\eta}\partial[A, \eta] - \xi\bar{\eta}[B^a, [X^a, \eta]] \tag{3.5.9}$$

[7] and, of course, to check for errors

and has to be added to the Lagrangian as the 'Jacobian' which changes functional integration to single out gauge orbits. Choosing one point of these gauge orbits is done by adding the gauge fixing part

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{2\xi}F^2 = -\tfrac{1}{2\xi}\dot{A}^2 - i\dot{A}[B^a, X^a] + \tfrac{1}{2}\xi[B^a, X^a]^2 \\ &= -\tfrac{1}{2\xi}\dot{A}^2 - i\dot{A}[B^a, Y^a] + \tfrac{1}{2}\xi[B^a, Y^a]^2 \end{aligned} \tag{3.5.10}$$

to the Lagrangian. Due to our well-designed gauge choice, the ξ-independent term here cancels an unwanted single derivative term in the kinetic Lagrangian

$$\begin{aligned}\mathcal{L} &= \ldots - i\dot{B}^a[A, Y^a] - i\dot{Y}^a[A, B^a] - i\dot{A}[B^a, Y^a] + \ldots \\ &= \ldots + i\dot{B}^a[Y^a, A] - i(B^a[\dot{Y}^a, A] + B^a[Y^a, \dot{A}]) + \ldots \\ &= \ldots + 2i\dot{B}^a[A, Y^a] + \ldots \end{aligned} \tag{3.5.11}$$

For a properly normalised kinetic term, we will rescale the gauge field A in the following $A \to -i\sqrt{\xi}A$. All in all the gauge-fixed kinetic and ghost Lagrangian reads

$$\begin{aligned}\mathcal{L} = &\tfrac{1}{2}(\dot{B}^a)^2 + \dot{B}^a\dot{Y}^a - \sqrt{\xi}\dot{B}^a[A, B^a] \\ &+ \tfrac{1}{2}(\dot{Y}^a)^2 + \tfrac{1}{2}\dot{A}^2 + \dot{\bar{\eta}}\dot{\eta} \\ &+ \tfrac{1}{2}\xi[A, B^a]^2 + \tfrac{1}{2}\xi[B^a, Y^a]^2 + 2\sqrt{\xi}\dot{B}^a[A, Y^a] - \xi[\bar{\eta}, B^a][B^a, \eta] \\ &- \sqrt{\xi}\dot{Y}^a[A, Y^a] + \xi[A, B^a][A, Y^a] + \sqrt{\xi}\dot{\bar{\eta}}[A, \eta] + \xi[\bar{\eta}, B^a][Y^a, \eta] + \tfrac{1}{2}\xi[A, Y^a]^2 \end{aligned} \tag{3.5.12}$$

3.5.3 The Perturbative Spectrum of the Fuzzy Sphere

We are now ready to expand the BMN-Lagrangian about the fuzzy sphere background $B^a = (\frac{\mu}{3})\beta J^a$ (3.5.4) we have discussed before. It turns out that the expansion can be done conveniently in terms of *fuzzy spherical harmonics* Y^N_{lm} (we shall often drop the superscript N in what follows).

In close analogy to the classical spherical harmonic functions, which are eigenfunctions of the Laplace operator in spherical coordinates, they form a complete $N \times N$ matrix basis with the same symmetry structure. Most importantly, they diagonalise the Casimir operator $[J^a, [J^a, Y_{lm}]] \sim Y_{lm}$, the matrix cousin of the angular Laplacian. Their construction and important properties are summarised briefly in the appendix to this chapter.

Since the fuzzy sphere generators J^a are just a matrix representation of $SU(2)$, it is no surprise that in the expansion of the BMN-action about this vacuum mass terms come in shape of such Casimir operators. The fuzzy spherical harmonics will then serve to diagonalise the mass matrix and as a complete matrix basis obtain full control over the perturbative spectrum.

On top of the ordinary *scalar* spherical harmonics, tensor versions can be constructed. In the following, we are going to make use of scalar, spinor and vector spherical harmonics for the different field expansions in the matrix theory, starting with the most basic ones, the scalars.

Transverse scalars The Lagrangian of the six transverse scalars reads

$$\mathcal{L} = \tfrac{1}{2}(DX^i)^2 - \tfrac{1}{2}\tfrac{1}{4}(\tfrac{\mu}{3}X^i)^2 + \tfrac{1}{2}[X^a, X^i]^2 + \tfrac{1}{4}[X^i, X^j]^2 \tag{3.5.13}$$

As announced, we plug the expansion $X^a = B^a + Y^a$ into the potential, where $B^a = (\frac{\mu}{3})\beta J^a$ with J^a the fuzzy sphere generators. The transverse $X^i = Y^i$ are fluctuations only and zero on the background. Focusing on their second order terms

$$\mathcal{L} = \tfrac{1}{2}(\dot{Y}^i)^2 - \tfrac{1}{2}\tfrac{1}{4}(\tfrac{\mu}{3})^2(Y^i)^2 - \tfrac{1}{2}(\tfrac{\mu}{3})^2\beta^2\big(Y^i[J^a,[J^a,Y^i]]\big) \tag{3.5.14}$$

Thus the mass eigenstates are those Y^i that diagonalise the Casimir operator $[J^a,[J^a,Y^i]]$, which is why we expand $Y^i = X^i_{lm}Y_{lm}$ in its eigenbasis, the scalar fuzzy spherical harmonics Y_{lm}

$$\mathcal{L} = N\sum_{lm} \tfrac{1}{2}(\dot{X}^i_{lm})^2 - \tfrac{1}{2}(\tfrac{\mu}{3})^2\big(\tfrac{1}{4} + j(j+1)\beta^2\big)(X^i_{lm})^2 \tag{3.5.15}$$

Gauge field and ghosts For gauge field and ghosts we computed the second order Lagrangian during gauge fixing. The relevant terms are

$$\mathcal{L} = \tfrac{1}{2}\dot{A}^2 - \tfrac{1}{2}(\tfrac{\mu}{3})\xi\beta^2 A[J^a,[J^a,A]] + \dot{\bar{\eta}}\dot{\eta} - (\tfrac{\mu}{3})\beta^2\xi\bar{\eta}[J^a,[J^a,\eta]] \tag{3.5.16}$$

where we have already rescaled $A \to -i\sqrt{\xi}A$ to normalise its kinetic term and expanded about the fuzzy sphere. Again we expand A_{jm} and η_{jm} in scalar spherical harmonics

$$\mathcal{L} = N\sum_{jm}\Big(\tfrac{1}{2}\dot{A}^2_{jm} - (\tfrac{\mu}{3})^2\beta^2\xi j(j+1)A^2_{jm} + \dot{\bar{\eta}}_{jm}\dot{\eta}_{jm} - (\tfrac{\mu}{3})^2\beta^2\xi j(j+1)\bar{\eta}_{jm}\eta_{jm}\Big) \tag{3.5.17}$$

just to find equal gauge dependent masses for both families. This does not come as a surprise: the spectrum does contain unphysical propagating modes, manifest for example in the ghosts. The same gauge dependent mass term will also appear in the vector boson mode X_{jjm} and lead to the necessary cancellations in physical amplitudes. This intricate cancellation is an intrinsic and well-known effect in gauge theories and can be exploited to verify physical content in calculations, see Frank [100] for a detailed calculational example.

Vector bosons The remaining fields X^a form $SO(3)$ vectors and have to be expanded in vector spherical harmonics. Starting with the relevant parts of the BMN Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(DX^a)^2 - \tfrac{1}{2}(\tfrac{\mu}{3}X^a)^2 - i\tfrac{\mu}{3}\varepsilon_{abc}X^aX^bX^c + \tfrac{1}{4}[X^a,X^b]^2 \tag{3.5.18}$$

we expand the fields as usual about the fuzzy sphere background $X^a = (\frac{\mu}{3})\beta J^a + Y^a$.
To first order in Y^a we get

$$\mathcal{L} = -(\tfrac{\mu}{3})^3\beta(1 - 3\beta + 2\beta^2)J^cY^c \tag{3.5.19}$$

with zeros, i.e. no tadpoles at the classical radii $\beta \in \{0, \frac{1}{2}, 1\}$.
The second order terms are

$$\mathcal{L} = -\tfrac{1}{2}(\tfrac{\mu}{3})^2(Y^a)^2 + \tfrac{1}{2}(\tfrac{\mu}{3})^2(\beta^2 - 3\beta)i\varepsilon_{abc}[J^a,Y^b]Y^c + \tfrac{1}{2}(\tfrac{\mu}{3})^2\beta^2(\varepsilon_{abc}[Y^b,J^c])^2 \tag{3.5.20}$$

Due to the vector nature of the Y^a there is not only the Casimir operator acting on them. We have to simultaneously diagonalise the operator $\varepsilon_{abc}[J^b, Y^c_{jlm}]$, the reason for expanding in vector spherical harmonics. Again, see the appendix to this section for a short summary of their properties. So $Y^a = X_{jlm} Y^a_{jlm}$

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{2}(\tfrac{\mu}{3})^2 N \sum_{jm} \big(&(1 - 3\beta + 2\beta^2) X^2_{jjm} \\ &+ (1 + (\beta^2 - 3\beta)(j+1) + \beta^2 (j+1)^2) X^2_{j-1,jm} \\ &+ (1 - (\beta^2 - 3\beta) j + \beta^2 j^2) X^2_{j+1,jm} \big)\end{aligned} \tag{3.5.21}$$

gives the mass eigenvalues. For example at the radii $\beta = 1, \frac{1}{2}$

$$(\beta = 1) \quad \mathcal{L} = -\tfrac{1}{2}(\tfrac{\mu}{3})^2 N \sum_{jm} \big(j^2 X^2_{j-1,jm} + (j+1)^2 X^2_{j+1,jm} \big) \tag{3.5.22}$$

$$(\beta = \tfrac{1}{2}) \quad \mathcal{L} = -\tfrac{1}{8}(\tfrac{\mu}{3}) N \sum_{jm} \big((j^2 - 3j) X^2_{j-1,jm} + (j^2 + 5j + 4) X^2_{j+1,jm} \big) \tag{3.5.23}$$

Clearly we can see $X_{j-1,jm}$ exhibit tachyonic modes for the unstable $\beta = \frac{1}{2}$ in the lower js before the positive term takes over. This means we have got 4 tachyons, one for $j = 1$ and three for $j = 2$, the only tachyons we will find in the spectrum at the unstable classical radius $\beta = \frac{1}{2}$.
There is an additional unphysical (ξ-dependent) mass term from gauge fixing that we have neglected so far

$$\mathcal{L} = \tfrac{1}{2}\xi [B^a, Y^a]^2 = \tfrac{1}{2}(\tfrac{\mu}{3})^2 \xi j(j+1) \beta^2 X^2_{jjm} Y^2_{jm} \tag{3.5.24}$$

Only the X_{jjm} fields are affected here because of the vector spherical harmonics property $[J^a, Y^a_{jlm}] = \sqrt{l(l+1)}\delta_{jl} Y_{lm}$ relating this mode to the scalar spherical harmonics.
Because of this switch from vector to scalar spherical harmonics, we have to examine the orthonormality relations

$$\mathrm{Tr}\; Y^a_{jlm} Y^a_{j'l'm'} = (-1)^{j-l+m+1} \mathrm{Tr}\; (Y^a_{jl,-m})^\dagger Y^a_{j'l'm'} = (-1)^{j-l+m+1} N \delta_{jj'} \delta_{ll'} \delta_{-m,m'} \tag{3.5.25}$$

we have tacitly been using. The minus signs there also show up in front of the kinetic term, they only contribute to overall normalisation. In the special gauge fixing term on the other hand, the vector spherical harmonic gets converted into a scalar spherical harmonic. Those obey another condition which gives a relative minus sign to this one mass term:

$$\mathrm{Tr}\; Y_{lm} Y_{l'm'} = (-1)^m \mathrm{Tr}\; (Y_{l,-m})^\dagger Y_{l'm'} = -(-1)^{m+1} N \delta_{ll'} \delta_{-m,m'} \tag{3.5.26}$$

So the additional gauge dependent mass term for the X_{jjm} scalars reads:

$$\mathcal{L} = \tfrac{1}{2}\xi [B^a, Y^a]^2 = -\tfrac{1}{2}(\tfrac{\mu}{3})^2 N \xi j(j+1) \beta^2 X^2_{jjm} \tag{3.5.27}$$

With this minus sign all gauge dependent mass terms are equal as they should be.

Fermions Finally, we expand the fermion Lagrangian

$$\begin{aligned}\mathcal{L} = i\psi^{\dagger AI} D\psi_{AI} - (\tfrac{\mu}{3})\beta \psi^{\dagger AI} \sigma^a_{AB} \delta_{IJ} [J^a, \psi^{BJ}] + \tfrac{1}{2}\psi^{\dagger AI} (\rho^i)_I^J \varepsilon_{AB} \delta_{JK} [X^i, \psi^{\dagger BK}] \\ - \tfrac{i}{2}\psi^{AI} (\rho^{\dagger i})_I^J \varepsilon_{AB} \delta_{JK} [X^i, \psi^{BK}] - \tfrac{\mu}{4}\psi^{\dagger AI}\psi_{AI}\end{aligned} \tag{3.5.28}$$

Field	$\times$	$\text{mass}^2/(\frac{\mu}{3})^2$	$j_{\min}\dots(N-1)$	$-m_{\max}\dots m_{\max}$
X_{jjm}	$1\times(2j+1)$	$(1-3\beta+2\beta^2)+\xi j(j+1)\beta^2$	$j_{\min}=1$	$m_{\max}=j$
$X_{j-1,jm}$	$1\times(2j-1)$	$(1+(\beta^2-3\beta)(j+1)+\beta^2(j+1)^2)$	$j_{\min}=1$	$m_{\max}=(j-1)$
$X_{j+1,jm}$	$1\times(2j+3)$	$(1-(\beta^2-3\beta)j+\beta^2 j^2)$	$j_{\min}=0$	$m_{\max}=(j+1)$
η_{jm}	$1\times(2j+1)$	$\xi j(j+1)\beta^2$	$j_{\min}=0$	$m_{\max}=j$
A_{jm}	$1\times(2j+1)$	$\xi j(j+1)\beta^2$	$j_{\min}=0$	$m_{\max}=j$
X_{jm}	$6\times(2j+1)$	$\frac{1}{4}+j(j+1)\beta^2$	$j_{\min}=0$	$m_{\max}=j$
$\psi_{j+\frac{1}{2},jm}$	$4\times(2j+2)$	$(\frac{3}{4}+j\beta)^2$	$j_{\min}=0$	$m_{\max}=(j+\frac{1}{2})$
$\psi_{j-\frac{1}{2},jm}$	$4\times(2j)$	$(\frac{3}{4}-(j+1)\beta)^2$	$j_{\min}=1$	$m_{\max}=(j-\frac{1}{2})$

Table 3.1: *All fluctuations' masses. The multiplicities of each spherical harmonic modes can be found in the appendix. For the complex 1-dimensional fermionic modes do not forget a multiplicity of 4 each because of the additional $SU(4)$ index.*

So in order to find the mass eigenstates we decompose the spinors into spinor spherical harmonics, diagonalising the operator $\sigma^a_{AI\ BJ}[J^a,\psi^{BJ}]$. To second order

$$\mathcal{L}=i\psi^\dagger D\psi-(\tfrac{\mu}{3})\beta\psi^\dagger\sigma^a[J^a,\psi]-(\tfrac{\mu}{4})\psi^\dagger\psi \tag{3.5.29}$$

we find a fermion mass spectrum of

$$\mathcal{L}=-N\sum_{jm}(\tfrac{\mu}{3})(\tfrac{3}{4}+j\beta)\psi^\dagger_{j+\frac{1}{2},jm}\psi_{j+\frac{1}{2},jm}+(\tfrac{\mu}{3})\big(\tfrac{3}{4}-(j+1)\beta\big)\psi^\dagger_{j-\frac{1}{2},jm}\psi_{j-\frac{1}{2},jm} \tag{3.5.30}$$

3.5.4 The effective potential for the fuzzy sphere's radius

The effective potential An interesting application of the mass spectrum we have found is the computation of the one-loop effective potential, or Coleman-Weinberg potential. While we give a brief motivation of the matter in the following paragraph, all we have to say is fairly textbook knowledge, presented in detail for example in Peskin and Schroeder [101].

The derivation of the effective potential is rooted in the deep similarities between quantum field theory and statistical mechanics in the path integral formalism. In analogy to the Helmholtz free energy $F(H)$ of a magnetic system, one defines the energy functional $E[J]$ from the partition function (we take ϕ as a placeholder for any fields in the theory)

$$Z[J]=e^{-iE[J]}=\int\mathcal{D}\phi\ \exp\left(i\int\delta^d x\,(\mathcal{L}[\phi]+J\phi)\right) \tag{3.5.31}$$

By a Legendre transformation, we can change variables from the fields sources J to its classical value $\phi_{\text{cl}}=\langle\phi\rangle_J$, weighted average over all possible fluctuations in the presence of a source J.

The resulting quantity, the analogue of the Gibbs free energy, is known as the effective action

$$\Gamma[\phi_{\rm cl}] = -E[J] - \int d^d x J \phi_{\rm cl} \tag{3.5.32}$$

Since $\phi_{\rm cl}$ and J are conjugate variables $\delta\Gamma[\phi_{\rm cl}]/\delta\phi_{\rm cl} = -J$, it follows that when setting the external sources J to zero we can find the stable quantum states as the extrema of this $\Gamma[\phi_{\rm cl}]$.

While still in the functional formalism we can thus find all solutions including space-time dependent solitons or instantons. Concentrating on constant vacuum expectation values $\phi_{\rm cl}$ independent of x, we can reduce the extensive quantity $\Gamma[\phi_{\rm cl}]$ to an ordinary function $\Gamma[\phi_{\rm cl}] = -(VT)V_{\rm eff}(\phi_{\rm cl})$ by dividing out space-time volume (VT). The constant $\phi_{\rm cl}$ is then found by simply deriving the effective potential $V_{\rm eff}$

$$\frac{\partial}{\partial \phi_{\rm cl}} V_{\rm eff}(\phi_{\rm cl}) = 0 \tag{3.5.33}$$

First order computation of the effective potential and the tadpole issue An explicit formula for the the effective potential can be obtained order by order in perturbation theory, see e.g. [101]. To zero order, we simply have the classical potential of the Lagrangian to determine the vacuum expectation value. To next order, one would take into account fluctuations about $\phi_{\rm cl}$ (not necessarily constant in the perturbation expansion of this paragraph), but also counterterms in the Lagrangian and the sources

$$\phi = \phi_{\rm cl} + \eta \qquad J = J_1 + \delta J \qquad \mathcal{L} = \mathcal{L}_1 + \delta\mathcal{L} \tag{3.5.34}$$

We borrow the notation of [101], where the Lagrangian is split into a piece $\mathcal{L}_1$ depending on renormalised parameters and another one containing the counterterms $\delta\mathcal{L}$. By introducing δJ, in principle we have an additional counterterm which serves an important purpose: J_1 is defined as the solution to the classical field equation with respect to $\mathcal{L}_1$

$$\left.\frac{\delta \mathcal{L}_1}{\delta \phi}\right|_{\phi=\phi_{\rm cl}} + J_1 = 0 \tag{3.5.35}$$

The δJ now ensures the original definition of $\phi_{\rm cl} = \langle \phi \rangle_J$, determined order by order. What does this mean? Potentially, tadpole diagrams might contribute to the expectation value of the fluctuations $\langle \eta \rangle$ and thus shift the value of $\langle \phi \rangle$ away from $\phi_{\rm cl}$. The additional counterterm δJ is set to exactly cancel these tadpole contributions, so that we can simply neglect them in the following. This issue is important to us, since tadpole terms do appear in our fluctuation expansion about the fuzzy sphere away from its classical solutions (3.5.19).

Otherwise neglecting counterterms, one can proceed with the perturbative expansion of $Z[J]$, $E[J]$ and, by Legendre transform, of $\Gamma[\phi_{\rm cl}]$. The result (implicitly discarding connected diagrams)

$$\Gamma[\phi_{\rm cl}] = \int d^d x \mathcal{L}_1[\phi_{\rm cl}] + \tfrac{i}{2} \log\det\left(-\frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi}\right) + \dots \tag{3.5.36}$$

is evaluated by the usual tricks of Wick rotation and dimensional regularisation [101]

$$\log\det(\partial^2 + m^2) = (VT) \int \frac{d^d k}{(2\pi)^d} \log(-k^2 + m^2) = -i(VT) \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} (m^2)^{-d/2} \tag{3.5.37}$$

Ghost and fermion determinants can be computed in a similar fashion, with a different prefactor due to their anticommuting nature. For fermions, they always come in pairs of plus and minus the same mass term, hence they combine into a Klein Gordon operator roughly like

$$\log(k+im)+\log(k-im)=\log(k^2-m^2) \tag{3.5.38}$$

Our fuzzy sphere calculation was done in a one (time) dimensional model, so the effective potential takes a slightly unfamiliar form as compared to the usual four dimensional case. Altogether, the one-loop corrections to the potential for the fuzzy sphere's radius boil down to a simple sum of the masses of the fluctuations

$$V_{\text{eff}}=\frac{1}{2}\sum_{\text{bosons}} m-\sum_{\text{ghosts}} m-\frac{1}{2}\sum_{\text{fermions}} |m| \tag{3.5.39}$$

With the mass spectrum of all fluctuations about a static fuzzy sphere of radius β already calculated and the formula (3.5.39) at hand, it is easy to generate the various plots displayed in figures 3.2 and 3.3. Taken at face value, however, the results seem completely unphysical, and indeed the interpretation of the effective potential is a subtle issue.

Gauge dependence First of all, the potential can only be considered physical on the *classical solutions* of the system. In our case, these are the three possible values of $\beta=\{0,\frac{1}{2},1\}$. With the class of R_ξ-gauges we have fixed a gauge labelled by an unphysical parameter ξ. The effective potential depends on this parameter through the masses of the unphysical modes in the spectrum, some scalars, ghosts and the gauge field, see table 3.1. But, and this is the important point, the gauge dependent masses do always add up to zero in the extrema. There, the classical solution $(1+(\beta^2-3\beta)+\beta^2)=0$ holds, so the X_{jjm} become massless up to pure gauge. We can then easily take the square root, and for each mode (j,m) add up the respective three equal gauge dependent masses

$$\frac{1}{2}m(X_{jjm})+\frac{1}{2}m(A_{jm})-m(\eta_{jm})=0 \tag{3.5.40}$$

The gauge dependence of the effective potential is no surprise, but a well-known effect and discussed for example by Dolan and Jackiw [102] for the case of scalar electrodynamics. A naive way out of the dilemma seems to be the choice of a somehow distinguished gauge, like the unitary, i.e. manifestly ghost free gauge $\xi\to\infty$, where unphysical modes decouple with infinite mass. This hope is treacherous, however, as in this limit the theory is unrenormalisable and a physically relevant discussion would necessarily require the inclusion of higher orders. As emphasised in [102], when working with the R_ξ gauges the physical information must be extracted before the regulator ξ can be sent to its limit.

Maxwell construction In its maximum the potential is gauge-independent as well, but must be considered with care. The key point is that in the very definition of the effective potential V_{eff} we have restricted fields $\phi_{\text{cl}}(x)$ to space-time independent constants ϕ_{cl}. As discussed in [101], this does not always result in the true minimum energy configuration for a given expectation value. Stressing again the analogy to statistical mechanics, a real unrestrained system would in this case develop Weiss domains as in a ferromagnet, or bubbles as in boiling liquid. By adopting such a patchwork groundstate, the system would interpolate the expectation values of

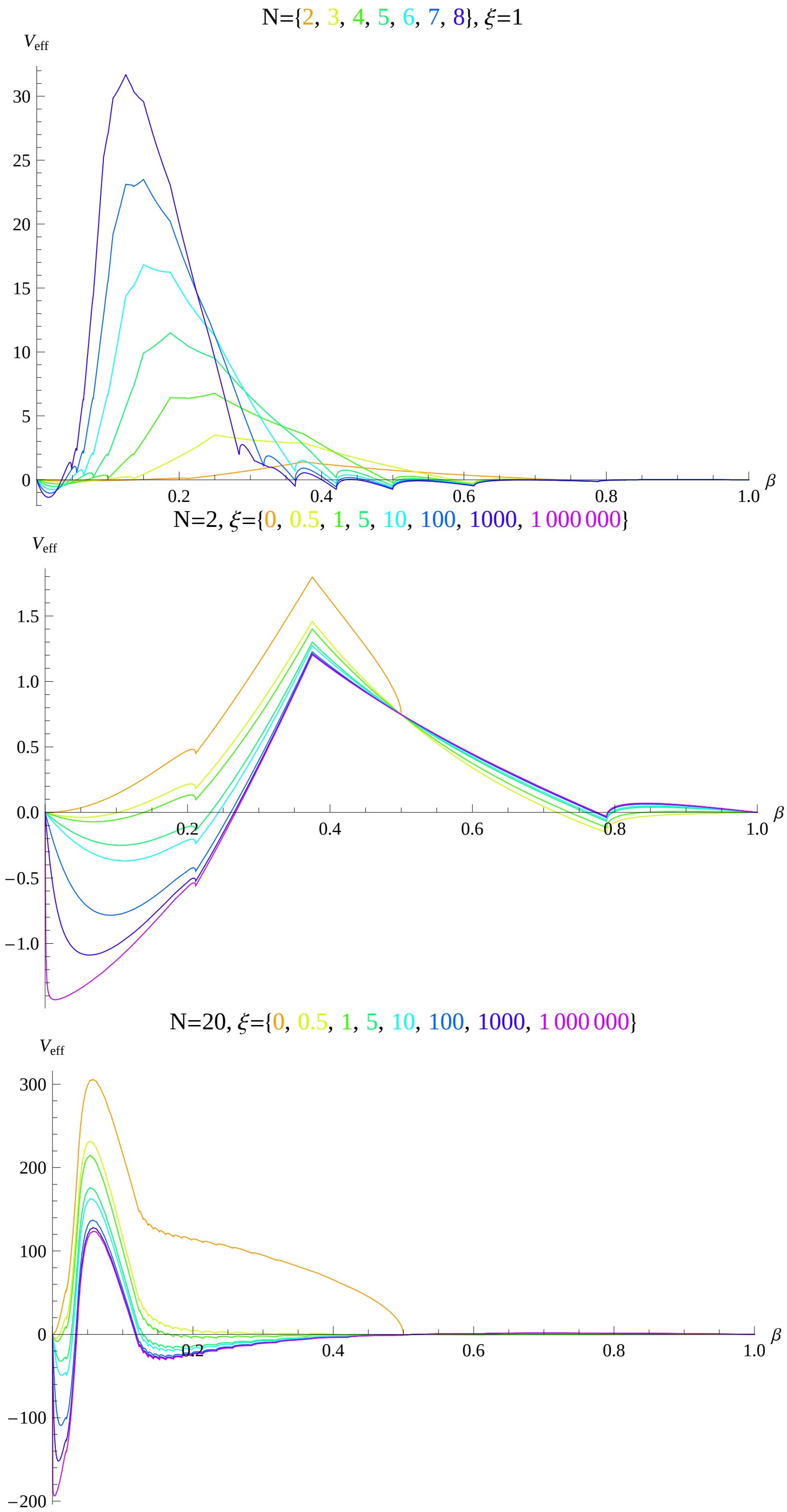

Figure 3.2: *Plots of the real part of the effective potential with different matrix sizes N and unphysical gauge parameters ξ. The potential is gauge independent at the physical solutions $\beta = \{0, \frac{1}{2}, 1\}$ but cannot be taken seriously at the maximum $\beta = \frac{1}{2}$ due to the Maxwell construction. Supersymmetry makes it vanish at the ends $\beta = \{0, 1\}$. Otherwise, it shows strong gauge dependence and even further, unphysical minima. See the main text for a complete discussion.*

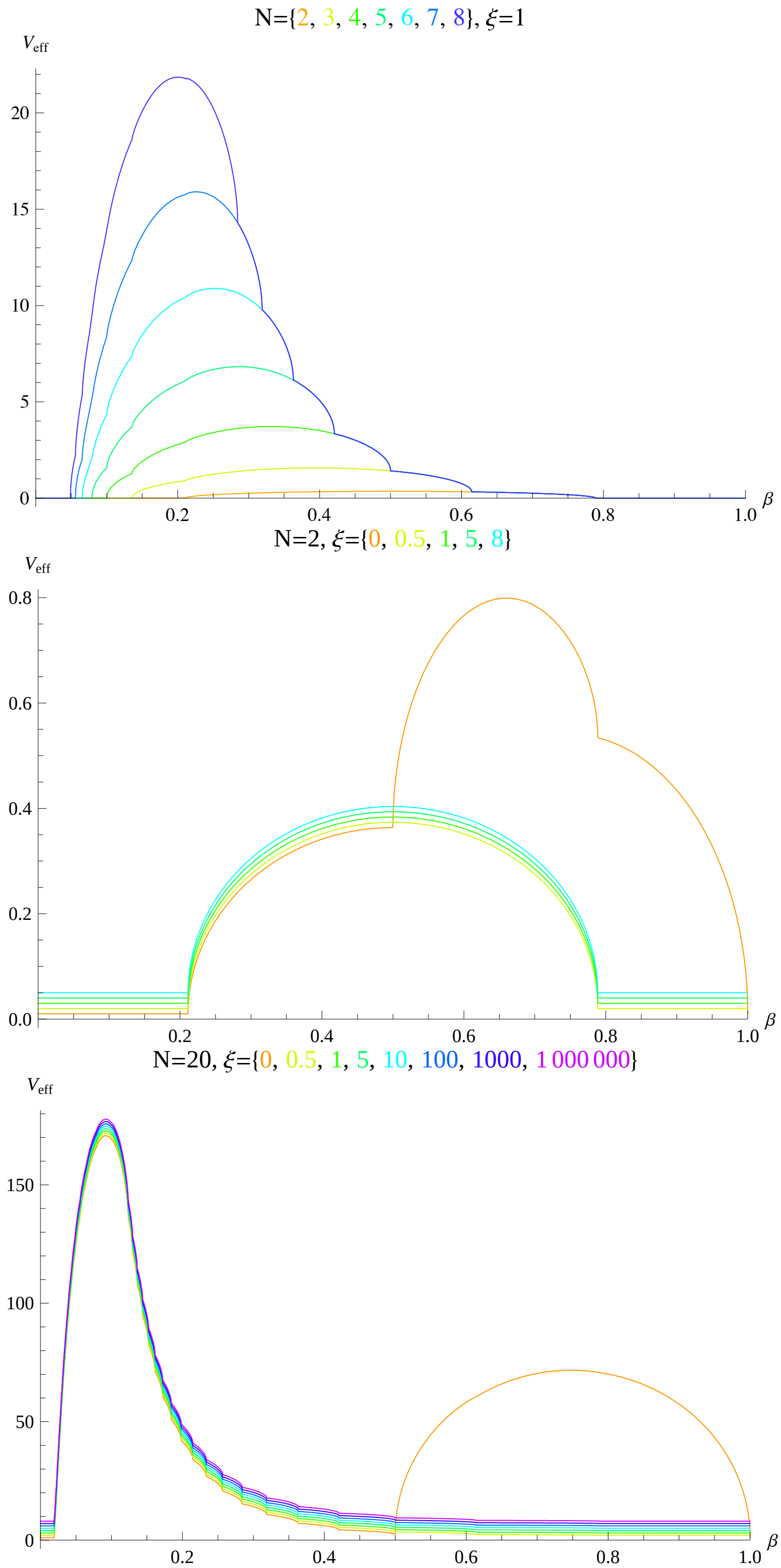

Figure 3.3: *Plots of the imaginary part of the effective potential with different matrix sizes N and unphysical gauge parameters ξ. The lines in the second and third plot are displayed with a slight offset, so identical graphs can still be seen. The imaginary part comes from tachyonic masses in the spectrum and shows that the system would like to fall into another state.*

several constant minima to preserve the overall expectation value of an in-between maximum. Yet its total energy would be lower than the one of the globally constant configuration.

The accepted way out is the *Maxwell construction* of simply drawing a straight line between two minima, cutting off any unphysical in-between maxima. This also takes account of the effect that a system would tunnel from any local minimum in the potential down to an absolute one, in spite of potential barriers. The effective potential is thus a *convex* function of $\phi_{\rm cl}$, which is a well-established exact result for the Gibbs free energy in thermodynamics.

Our fuzzy sphere system is not just restrained to time-independence, but also artificially to one given fuzzy sphere matrix solution with the radius β the only left variable. The system has no chance to fall into any other matrix state, like for example a block-diagonal solution of two smaller fuzzy spheres. The quantum fluctuations, on the other hand, do not know such constraints and might drive the system into any other conceivable direction of the full matrix system. Such instabilities are expressed by tachyons, modes of negative square mass, which therefore add an imaginary part to the sum (3.5.39). Indeed, we can see that the spectrum about the unstable $\beta = \frac{1}{2}$ vacuum acquires an imaginary part, which suggests a possible third way out in the full nonperturbative theory.

Symmetry The one-loop effective potential has been used to demonstrate the effect of radiatively induced vacuum expectation values by Coleman and Weinberg [103], surely the acme of spontaneous symmetry breaking. A massless scalar field with quartic interactions, for example, obtains a mass term by one-loop corrections shifting the minimum of its potential away from zero.[8] Does the effective potential for the fuzzy sphere therefore distinguish one of the two vacua $\beta = \{0, 1\}$? The answer is no. The effect of Coleman and Weinberg is possible because no symmetry protects the theory from developing a mass. A symmetry of the full quantum theory, on the other hand, is of course respected also by radiative corrections, see [101]. In our case, this symmetry is supersymmetry. The fuzzy sphere is a BPS state, and the unbroken generators guarantee its zero energy also at one-loop. Consequently, the effective potential shown in figure 3.2 vanishes at $\beta = \{0, 1\}$.

3.5.5 Another Direction

As a short and illustrative example of how strongly we have restrained our matrix system, we add another degree of freedom to our solution. At the unstable fuzzy sphere with radius $\beta = 1/2$ we have four tachyonic fluctuations $X_{j-1,jm}$, one for $j = 1$ and three for $j = 2$. The former actually points into the radial direction as the fuzzy sphere generators are expressed in vector spherical harmonics as $J^a = -\sqrt{(N+1)(N-1)/4}\; Y^a_{010}$.

From the next tachyons in line, the Y^i_{12m} modes, we can form three hermitian linear combinations, all rescaled by $\sqrt{(N+1)(N-1)/4}$ to match the normalisation of the generators

$$T^a_z = \sqrt{\tfrac{(N+1)(N-1)}{4}}\, Y^a_{120} \qquad T^a_x \sim \tfrac{1}{\sqrt{2}}(Y^a_{12-1} - Y^a_{121}) \qquad T^a_y \sim \tfrac{1}{\sqrt{2}} i (Y^a_{12-1} + Y^a_{121}) \tag{3.5.41}$$

Trying the linear combination $X^a \sim bJ^a + cT^a_z$ (not caring about normalisation) in our matrix

[8] Actually, as pointed out in [103], in this illustrative example the minimum is well outside the range of perturbation theory. The same effect, however, does also occur in massless scalar electrodynamics and leads to spontaneous symmetry breaking well within the range of validity.

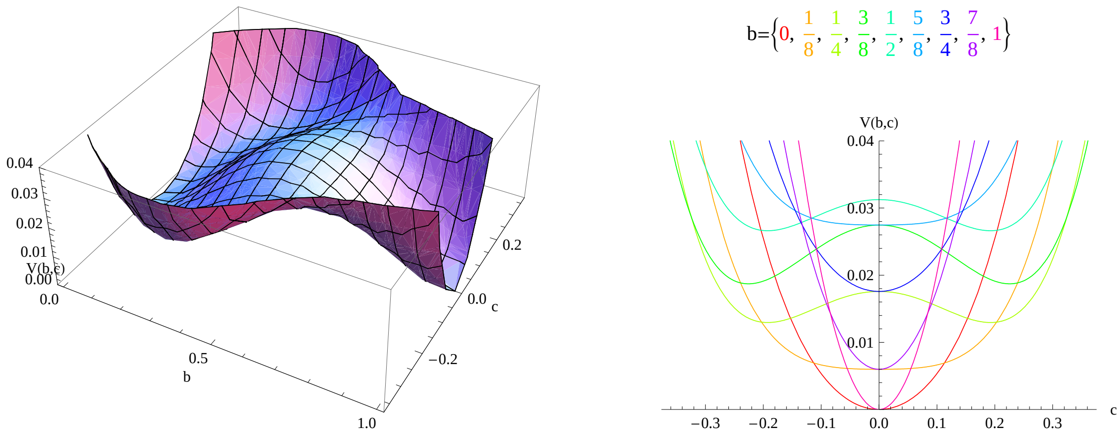

Figure 3.4: *On the left hand side, there is a 2d plot of $V(b,c)$ in the limit $N \to \infty$ against $b = 0..1$ and $c = -0.4..0.4$. The right-hand side graph is an overlay of cuts $b = const$, plotting $V(b,c)$ against c.*

Lagrangian, we have to deal with commutator terms $[Y^a_{120}, Y^b_{120}]$. They are resolved by

$$[Y^a_{j_1 l_1 m_1}, Y^b_{j_2 l_2 m_2}] = \sqrt{N(2l_1+1)(2l_2+1)} \left(\begin{array}{cc|c} l_1 & 1 & j_1 \\ m_1 - a & a & m_1 \end{array} \right) \left(\begin{array}{cc|c} l_2 & 1 & j_2 \\ m_2 - b & b & m_2 \end{array} \right) \tag{3.5.42}$$
$$\sum_l (-1)^l \left((-1)^{2l_1 - 1 + N} - (-1)^{2l_2 - 1 + N} (-1)^{l_1 + l_2 - l} \right)$$
$$\left(\begin{array}{cc|c} l_1 & l_2 & l \\ m_1 - a & m_2 - b & m_1 - a + m_2 - b \end{array} \right) \left\{ \begin{array}{ccc} l_1 & l_2 & l \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{array} \right\} Y_{l, m_1 - a + m_2 - b}$$

where a, b are indices into a rotational basis $e_a = \{e_{-1}, e_0, e_1\}$. The brackets denote Clebsch-Gordan coefficients and the braces Wigner-6j symbols.

In our case, there are only two non-zero contributions on the right hand side, proportional to Y_{1m} and Y_{3m}, respectively. The resulting potential as a function of the two 'radii' b and c is

$$V(b,c) \sim \left(\tfrac{1}{2} b^2 (b^2 - 1) + \tfrac{1}{2} c^2 (1 - 9b + 12b^2) - \left(\frac{243}{10(N^2-4)} - \frac{27N^2}{8(N^2-4)} \right) c^4 \right) \tag{3.5.43}$$

The first term of order c^0 is the ordinary potential and the second term c^2 is the mass we have found before for the tachyon. The c^3 term is absent, because Tr $\varepsilon^{abc} T^a_z T^b_z T^c_z = 0$. Interestingly, the c^4 term approaches the finite value $\frac{27}{8}$ as $N \to \infty$.

We can plainly see how the system now has more freedom to respond than just the original radial direction and might now take different classical paths, e.g. rolling sideways in the potential. Quantum fluctuations as in the one-loop effective potential put pressure in these additional directions, but not only. For the full unrestrained system, we will not be able to say conclusively in which superposition of states the system will finally settle when a radius $\beta = 1/2$ fuzzy sphere decays.

3.5.6 Time-dependence

Along a somewhat complementary line, we can also consider non-static fuzzy spheres with a time-dependent radius. The most prominent such solution is the so-called bounce of a fuzzy sphere starting at the maximum of the potential, rolling down-hill through the minimum and bouncing off the potential wall on the other side to return to its starting point after an infinite amount of time. Solving (3.5.5), the time-dependent radius reads

$$\beta(t) = \frac{1}{\sqrt{2}\cosh(\frac{1}{\sqrt{2}}(\frac{\mu}{3})t)} + \frac{1}{2} \tag{3.5.44}$$

The solution is used for example in Popov [97] (see also references therein). Up-lifted to Yang-Mills theory on $R \times S^3$ it describes a dyonic configuration and allows to calculate the respective electric and magnetic field for further study.

While our focus has been on static fuzzy spheres, in the fluctuation expansion we also have the time-dependent terms under control. Using $B^a = (\frac{\mu}{3})\beta(t)J^a$ in equation (3.5.12), an additional second order term arises mixing gauge field A and fluctuations Y^a. Expanding in spherical harmonics using $[J^a, Y^a_{jlm}] = \sqrt{l(l+1)}\delta_{jl}Y_{lm}$

$$\mathcal{L} = 2\sqrt{\xi}(\tfrac{\mu}{3})\ \dot{\beta}(t)\ J^a[A, Y^a] = 2iN(\tfrac{\mu}{3})\sqrt{\xi j(j+1)}\ \dot{\beta}(t)\ A_{jm}X_{jjm} \tag{3.5.45}$$

Note the additional factor of i with respect to the kinetic terms of A_{jm} and X_{jjm}. This comes from a rescaling of the fields such that the kinetic terms have the right signs, something we have encountered before (3.5.26). The mixing of the two fields slightly complicates the matter, but can be resolved by arranging each A_{jm} and X_{jjm} into a two-by-two matrix to be diagonalised.

There are also further tadpole terms in (3.5.12) to be taken into consideration.

Due to our limited means, we shall not tap into the rich subject of time-dependent solutions here. The one-loop potential we have considered is already subtle in the static case, and not easily generalised beyond that. Also, the gauge dependence complicates the matter, and it is not clear what physical information one wants to extract from (3.5.45).

Instead, we shall explore the wealthy subject of matrix models in the M-theory and string theory duality context. The next chapter is devoted to an introduction to matrix big-bang models and the study of fuzzy sphere dynamics on a classical level in there.

3.A Appendix: Fuzzy Spherical Harmonics

We summarise some facts about fuzzy spherical harmonics following the appendix of Das, Michelson, Shapere [104].

Scalar spherical harmonics The ordinary spherical harmonics $Y_{lm}(\varphi, \theta)$ are a basis of functions on the 2-sphere, coordinatised by the angles φ and θ. They are an infinite collection of $m = \{-l \dots l\}$ dimensional irreducible representations of $SU(2)$ numbered by $l = \{0 \dots \infty\}$. The generators J^i, $(i = 1, 2, 3)$ of $SU(2)$ in this space are given by the derivative operator $i\nabla^i$, which is more conveniently cast into the linear combination $(J^+ = J^1 + iJ^2, J^- = J^1 - iJ^2, J^3)$ of raising and lowering operators and the diagonal element J^3 with eigenvalues m. The quantum numbers l come from the eigenvalues of the (quadratic) Casimir operator $(J^i)^2 = -\Delta$. This is summarised by the algebraic properties

$$
\begin{aligned}
&\int d\Omega\, Y^\dagger_{lm}(\varphi,\theta) Y_{l'm'}(\varphi,\theta) = \delta_{ll'}\delta_{mm'} && Y^\dagger_{lm}(\varphi,\theta) = (-1)^m Y_{l,-m}(\varphi,\theta)\\
&-i\nabla^3 Y_{lm}(\varphi,\theta) = m Y_{lm}(\varphi,\theta) && -\Delta Y_{lm}(\varphi,\theta) = l(l+1) Y_{lm}(\varphi,\theta)\\
&i\nabla^\pm Y_{lm}(\varphi,\theta) = \sqrt{(l\mp m)(l\pm m+1)}\, Y_{l,m\pm1}(\varphi,\theta)
\end{aligned}
\tag{3.A.1}
$$

While this is an infinite dimensional function basis, it naturally falls into layers of irreducible representations l. It is possible to consistently truncate this algebra by introducing an upper cutoff N such that $l = \{0 \dots N-1\}$ only. This renders the basis finite dimensional, with a total number $N^2 = \sum_0^{N-1}(2l+1)$ of so-called *fuzzy* spherical harmonics Y^N_{lm}. Naturally, we can therefore find an N^2 dimensional basis of $N \times N$ matrices to represent them under the action of the corresponding matrix representation of the $SU(2)$ generators J^i given by

$$
\begin{aligned}
&J^3_{ab} = \tfrac{1}{2}(N+1-2a)\delta_{ab}, \quad J^\pm_{ab} = -\left|\sum_{n=1}^{a} N+1-2n\right|^{1/2} \delta_{a,b\pm1}, \quad a,b = \{1\dots N\}\\
&\text{such that} \quad [J^+, J^-] = 2J^3 \quad [J^3, J^\pm] = \pm J^\pm
\end{aligned}
\tag{3.A.2}
$$

The fuzzy spherical harmonics can then be constructed recursively by

$$
\begin{aligned}
&Y^N_{ll} = n_l\,(J^+)^l \qquad \text{with numerical factor } n_l = (-1)^l \sqrt{\frac{N(2l+1)!(N-l-1)!}{l!(N+l)!}}\\
&Y^N_{lm} = n'_l\,[J^-, Y^N_{l,m+1}] \qquad \text{with } n'_l = ((l+m+1)(l-m))^{-1/2}
\end{aligned}
\tag{3.A.3}
$$

with the usual range of parameters $j = \{0\dots N-1\}$ and $m = \{-j \dots +j\}$, so for $N=3$, say, we have $1+3+5 = 9$ modes.
So we have a complete $N \times N$ basis of eigenmatrices of J^3 and $(J^i)^2$ with the properties

$$
\begin{aligned}
&\mathrm{Tr}\, Y^{N\dagger}_{lm} Y^N_{l'm'} = N\delta_{ll'}\delta_{mm'} && Y^{N\dagger}_{lm} = (-1)^m Y^N_{l,-m}\\
&[J^3, Y^N_{lm}] = m Y^N_{lm} && [J^i,[J^i, Y^N_{lm}]] = l(l+1) Y^N_{lm}\\
&[J^\pm, Y^N_{lm}] = \sqrt{(l\mp m)(l\pm m+1)}\, Y^N_{l,m\pm1}
\end{aligned}
\tag{3.A.4}
$$

Up to normalisation the generators are equal to the basis elements $J^3 \sim Y^N_{10}$ and $J^\pm = Y^N_{1,\pm1}$.

Vector spherical harmonics Given two representations of $SU(2)$ we can construct a product representation and then decompose it into a sum via Clebsch-Gordan coefficients. Take a standard real three dimensional vector space for example, with the generators of rotation $J^a_{ab} = i\varepsilon^a_{bc}$. The representation on the product space with the (fuzzy) spherical harmonics

$$
\vec{Y}_{jlm} = \sum_{m'=-1}^{1} \left(\begin{matrix} l & 1 \\ m-m' & m' \end{matrix} \middle| \begin{matrix} j \\ m \end{matrix} \right) Y_{l,m-m'}\, \vec{e}_{m'}
\tag{3.A.5}
$$

is called the (fuzzy) *vector spherical harmonics*. Due to the nature of the Clebsch-Gordan coefficient, the difference between j and l can only ever take three values $j = \{l-1, l, l+1\}$, so they are falling into three series. Making use of the rotational basis $\vec{e}_0 = \vec{e}_3$ and $\vec{e}_\pm = \sqrt{1/2}\,\vec{e}_1 \pm i\vec{e}_2$

$$\begin{aligned}
\vec{Y}^N_{jjm} &= (j(j+1))^{-1/2}[\vec{J}, Y^N_{jm}], \qquad (3.A.6)\\
\vec{Y}^N_{j,j-1,m} &= (2j(2j-1))^{-1/2}\Big(\sqrt{(j-m)(j-m-1)}\,Y^N_{j-1,m+1}\,\vec{e}_- +\\
&\qquad \sqrt{(j+m)(j+m-1)}\,Y^N_{j-1,m-1}\,\vec{e}_+ + \sqrt{2(j^2-m^2)}\,Y^N_{j-1,m}\,\vec{e}_0\Big)\\
\vec{Y}^N_{j,j+1,m} &= (2(j+1)(2j+3))^{-1/2}\Big(\sqrt{(j+m+1)(j+m+2)}\,Y^N_{j+1,m+1}\,\vec{e}_- +\\
&\qquad \sqrt{(j-m+1)(j-m+2)}\,Y^N_{j+1,m-1}\,\vec{e}_+ - \sqrt{2(j+1)^2-m^2}\,Y^N_{j+1,m}\,\vec{e}_0\Big)
\end{aligned}$$

with the following multiplicities for the three series

$$\begin{aligned}
\vec{Y}_{jjm}:&\ j=\{1\ldots N-1\},\ m=\{-j\ldots+j\}\\
\vec{Y}_{j-1,jm}:&\ j=\{1\ldots N-1\},\ m=\{-(j-1)\ldots+(j-1)\}\\
\vec{Y}_{j+1,jm}:&\ j=\{0\ldots N-1\},\ m=\{-(j+1)\ldots+(j+1)\} \qquad (3.A.7)
\end{aligned}$$

so that the total number always adds up to $3N^2$ as required. Take $N=3$ for example with $3+5=8$ $Y_{j-1,jm}$-modes, $3+5+7=15$ $Y_{j+1,jm}$-modes and $1+3=4$ Y_{jjm}-modes; which makes a total of 27 matrices.

The basis satisfies a number of identities, notably each $\vec{Y}^N_{jlm}$ transforms as a vector under the action of $\vec{J}$. Dropping the never-changing superscript N and using index notation $\vec{Y}^N_{jlm} = Y^a_{jlm}\vec{e}_a$

$$\begin{aligned}
&\mathrm{Tr}\, Y^{\dagger c}_{jlm} Y^c_{j'l'm'} = N\delta_{jj'}\delta_{ll'}\delta_{mm'} && Y^{\dagger c}_{jlm} = (-1)^{j-l+m+1} Y^c_{jl,-m}\\
&[J^a,[J^a,Y^b_{jlm}]] = l(l+1)Y^b_{jlm} && [J^a, Y^a_{jlm}] = \sqrt{l(l+1)}\delta_{jl} Y_{lm}\\
&\varepsilon_{abc}[J^b, Y^c_{jlm}] = iY^a_{llm}\delta_{jl} + i(l+1)Y^a_{l-1,lm}\delta_{j,l-1} - ilY^a_{l+1,lm}\delta_{l+1,j} && (3.A.8)
\end{aligned}$$

Spinor spherical harmonics In exactly the same way, via Clebsch-Gordan coefficients, we can construct *spinor spherical harmonics* $\vec{S}^N_{jlm}$. So for the two dimensional spinor space we get two series

$$S^N_{j+\frac{1}{2},j,m} = (2j+1)^{-\frac{1}{2}}\begin{pmatrix}\sqrt{j+m+1/2}\,Y_{j,m-\frac{1}{2}}\\ \sqrt{j-m+1/2}\,Y_{j,m+\frac{1}{2}}\end{pmatrix} \qquad \begin{cases} j=\{0\ldots N-1\}\\ m=\{-(j+\frac{1}{2})\ldots+(j+\frac{1}{2})\}\end{cases} \qquad (3.A.9)$$

$$S^N_{j+\frac{1}{2},j,m} = (2j+1)^{-\frac{1}{2}}\begin{pmatrix}-\sqrt{j-m+1/2}\,Y_{j,m-\frac{1}{2}}\\ \sqrt{j+m+1/2}\,Y_{j,m+\frac{1}{2}}\end{pmatrix} \qquad \begin{cases} j=\{1\ldots N-1\}\\ m=\{-(j-\frac{1}{2})\ldots+(j-\frac{1}{2})\}\end{cases}$$

Again the index ranges ensure a total of $2N^2$ matrices, so for $N=3$ we have $2+4+6=12$ $S_{j+\frac{1}{2},jm}$-modes and $2+4=6$ $\psi_{j-\frac{1}{2},jm}$-modes; a total of 18 modes.

The spinor matrix basis has the following properties, notably it diagonalises the operator

$\sigma^a[J^a, S^N_{jlm}]$ where σ^a are the three Pauli matrices. Suppressing N

$$
\begin{aligned}
&\mathrm{Tr}\; S^\dagger_{jlm} S^{j'l'm'} = N\delta_{jj'}\delta_{ll'}\delta_{mm'} && \sigma^2 S^\dagger_{j\pm\frac{1}{2}} = (-1)^{m+\frac{1}{2}\pm\frac{1}{2}} S_{j\pm\frac{1}{2},j,-m} \\
&[J^a,[J^a,S_{jlm}]] = l(l+1)S_{jlm} && \\
&\sigma^a[J^a,S_{jlm}] = \left(lS_{l+\frac{1}{2},lm}\delta_{j,l+\frac{1}{2}} - (l+1)S_{l-\frac{1}{2},lm}\delta_{j,l-\frac{1}{2}}\right) && \qquad (3.A.10)
\end{aligned}
$$

Chapter 4

Fuzzy Spheres in Plane Wave Matrix Big Bangs

In the previous chapter we have introduced the 11 dimensional supermembrane in a rather self-contained way. This is only half of the story, however, and already the scan in the beginning of last chapter of the combinations of target-space and world-sheet dimensions that are compatible with supersymmetry hints at deeper roots. In fact, on symmetry grounds alone one can recover the allowed dimensions superstrings can live in (classically) and also see the importance of the maximally allowed 11 dimensions. This picture is nowadays also included in string theory. With the 'third string revolution' came the insight that strings can only be a part of the story. Higher dimensional objects on which open strings would end have to be accorded dynamical features, although due to their high masses they are not directly accessible by perturbative means.

Given the existence of these non-perturbative objects, branes, the efforts have been concerted to find an all-engulfing mother theory, M-theory, incorporating all 11 dimensions and branes in a unified way. Even though the theory has not yet manifested itself, over the years overwhelming evidence in favour of its existence has been gathered. All string theories as well as other theories like matrix models seem to be interwoven by a network of dualities that realises them as limits of one another in the outreaches of their parameter space. One prominent example is T-duality, where type IIB string theory is understood as the dual description of IIA compactified on a small circle.

A vital contribution to this picture of M-theory came from Witten [105] who found evidence of a hidden Kaluza-Klein structure in the IIA spectrum. In this way IIA theory is seen as the compactification of M-theory on a circle of radius $R = \ell_s g_s$. The proposed particle objects, D0-branes, have masses expected by compactification $M_{\text{D0}}^2 = 1/(\ell_s g_s)^2$. Apart from general considerations, a concrete D0-brane action has been suggested on the grounds of a certain kinematical limit of M-theory. The action takes the form of a matrix model with diagonal matrices corresponding to classically aligned point particles. General matrices with off-diagonal elements do not have a simple interpretation, hinting at a deconstruction of the structure of space-time itself in this sector.

Recently, an interesting model has been proposed by Craps, Sethi and Verlinde (CSV) [106] that combines many of these aspects of M-theory into the dynamics of one single system. In the following we are going to introduce this model and its generalisation, the plane wave matrix big-bangs of Blau and O'Loughlin [107].

We set the stage by presenting some dualities in the next section, then introduce matrix models in the string theory context and the technique of *discrete light-cone quantisation* (DLCQ). This will enable us to explain the big-bang models, leading to an investigation of dynamical fuzzy spheres in these settings.

4.1 M-Theory, String Theory and Dualities

4.1.1 D-branes

A simple argument tells us that the string might be the fundamental, but by far not the only object present in string theory. Consider type I strings, open strings with either Neumann or Dirichlet boundary conditions in target-space. The latter ones define hypersurfaces, D-branes, in space which the strings' ends can never leave. After the discovery of a string duality which can exchange boundary conditions, it became apparent that they cannot be just excluded once and for all from the theory.

D-brane DBI action Although these objects must be highly massive in order to not appear in perturbative string theory, they must still be dynamic, as can be deduced by their interaction with open strings: An open string with both ends on the same D-brane certainly has a massless gauge field in its spectrum. In its transverse components this gives rise to fluctuations of the brane, essentially the same mechanism that generates gravity as string fluctuations on a (Minkowski) background.

Along the brane, these fluctuations tell us that the gauge field must also appear in the brane action. Indeed, the conditions on the D-branes implied by conformal invariance of the open string are equivalent to the equations of motion of the DBI-action, the action on the world-sheet of a single D-brane sporting such a gauge field[1]

$$S = -T_p \int d^{p+1}\sigma e^{-\phi} \sqrt{\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi \ell_s^2 F_{\alpha\beta})} \tag{4.1.1}$$

The action resembles the one of the membrane of last chapter in its pull-back of the metric $G_{\alpha\beta}$ to the world-sheet. It is extended by coupling to the antisymmetric NS-NS $B_{\alpha\beta}$-field and the field strength $F_{\alpha\beta}$ of the $U(1)$ gauge field on the brane. Also note the dilaton coupling $e^{-\phi}$.

While the string is the natural source of NS-NS gauge fields, this is not the case for the R-R gauge fields in the spectrum. Their sources remained mysterious until in a celebrated publication [108] Polchinski identified them as the Dp-branes, coupling naturally to the R-R $(p+1)$-form potentials by an integral running over the brane's world-volume

$$\int d^{p+1}\sigma \; C_{p+1} \tag{4.1.2}$$

At the time, this insight came as a relief, for string dualities frequently interchange NS-NS and R-R states, so the existence and identification of all sources is crucial. Reversing the argument reveals the spectrum of branes present in type II theories from the known R-R spectrum. In type IIA string theory, these are the 'even' D-branes: (point particle) D0-branes, (membrane)

[1] The action is correct only for slowly varying field strengths, i.e. up to derivatives of $F_{\alpha\beta}$.

D2, D4, D6, and D8. In IIB there are the 'odd' (instanton solution) D(-1), (D-strings) D1, D3, D5, D7 and space-time filling D9.

As also explained in [108], boundaries of the string world-sheet reflect the left- and right-moving currents associated with supersymmetry into one another, leaving a linear combination invariant. D-branes seen as such boundaries must therefore be BPS-states in flat space, partly breaking and partly preserving supersymmetry. The power of symmetry therefore ensures that qualitative aspects of these objects are true on all energy scales and one has not fallen prey to a discussion of mere artefacts in a certain regime.

Non-abelian extension The action (4.1.1) was written down for a single Dp-brane with a $U(1)$ gauge group. However, it can be extended to N Dp-branes (each of the same p) by completely general reasoning about the interactions with strings. A string ending on two separate Dp-branes introduces a mixing of the two branes' $U(1)$ gauge groups. The stretched strings produce a force, an interaction carried by their massive degrees of freedom between the branes.

Now assemble the two branes' gauge-fields into the diagonal entries of a two-by-two matrix, where the interaction is accounted for by the off-diagonal components. As long as the branes have distinct positions, the full symmetry group of the matrix is broken into $U(1) \times U(1)$ by the vacuum expectation value of the interactions, a kind of Higgs-mechanism.[2] Approaching the branes, now, we slowly restore the full $U(2)$ (or, in general, $U(N)$) symmetry for coincident branes.

As the world-volume action itself contains no information regarding its states, the action must be the same in all cases: the DBI-action (4.1.1) for non-commuting $U(N)$ matrices. But while in its abelian version the DBI action is manageable, the general non-abelian case poses severe problems as to gauge-fixing, supersymmetry and operator ordering in calculations. Already the generalisation of the trace in its very definition is not without ambiguity: The proposition of using a 'symmetric trace', symmetrising over gauge indices and thus effectively ignoring commutators, to date still leaves open questions.

Super Yang-Mills Fortunately, in many interesting cases it suffices to consider the low energy limit of the DBI-action: Yang-Mills theory. In the abelian case, this can be found by expanding the square root in (4.1.1) about a metric background to second order in the fields.

$$S = T_p \int d^{p+1}\sigma \left(\tfrac{1}{2}\partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{1}{4g_{YM}^2} F_{\alpha\beta}F^{\alpha\beta} \right) \tag{4.1.3}$$

This looks like a Yang-Mills theory with scalar fields. In this limit it is easy to generalise the action to the non-abelian case, simply change the gauge group to $U(N)$ and add appropriate commutator terms as dictated by symmetry. Also the supersymmetric extension of the bosonic part is implicit: All in all, the Dp-brane action truncated in this way always takes the shape of 10 dimensional $U(N)$ Super Yang-Mills compactified to $p+1$ dimensions.

This shall suffice as a very rough introduction to D-branes. We will follow the truncation procedure in detail later this chapter to derive the matrix model of D1-branes 'dual' to IIA strings in a certain plane wave background. But first we will clarify the notion of duality this involves.

[2] although the symmetry breaking is explicit and not dynamical

4.1.2 M-Theory and a Network of Dualities

Amongst the many dualities in string theory, we would like to explain a chain relating four string theories to M-Theory compactified on a torus, as depicted in figure 4.1.

S-duality Historically, the torus made his first appearance in the basement of the 'house' 4.1, as an S-duality relating two IIB theories by inverting their coupling. This remarkable self-duality is based on the fact that there are in fact two kinds of strings present in IIB: the perturbative fundamental F1-strings and massive D1-strings with tensions

$$\left.\begin{aligned} T_{\rm F1} &= \tfrac{1}{2\pi\ell_s^2} \\ T_{\rm D1} &= \tfrac{1}{2\pi g_s \ell_s^2} \end{aligned}\right\} \Rightarrow T_{\rm F1}/T_{\rm D1} = g_s \tag{4.1.4}$$

But this means that whereas at weak coupling the F1 string is perturbative and the D1 non-perturbative, the situation inverts at strong coupling and the two objects change role!

What seems a rather superficial observation gets more profound in connection with an $SL(2,\mathbb{R})$ symmetry of type IIB supergravity, the low energy limit of the IIB strings. Combine the dilaton with the R-R one-form C_0 into a single field

$$\tau = C_0 + ie^{-\phi} \tag{4.1.5}$$

Then IIB supergravity is invariant under $SL(2,\mathbb{R})$

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1 \tag{4.1.6}$$

where the metric in Einstein frame as well as the 4-form field remain the same, while the NS two form field strength H_3 and the R-R two form field strength F_3 together transform as a tuple

$$\begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \tag{4.1.7}$$

Hence the F1-string as the NS-NS source and the D1-brane as the R-R source get mixed at the low energy level. This low energy symmetry can be extended to an $SL(2,\mathbb{Z})$ symmetry of the full IIB string theory: Consider a system of p F1- and q D1-strings, see either the textbook Polchinski [109] or the original publication by Witten [110]. Bringing the two kinds of strings together, they condense into a lower energy bound state. This can be shown to saturate a BPS condition and the preserved amount of supersymmetry stabilises these objects, validating their use for arguments beyond the perturbative regime.

The stable bound states exist for every tuple (p,q) if p and q are relatively prime. Otherwise, if $(p,q) = (kp', kq')$, the system factorises into k subsystems. So to every transformation of the 2-form coupling of the fundamental string

$$\int_M B_2 \rightarrow \int_M dB_2 + cC_2 \tag{4.1.8}$$

there exists a supersymmetric (d,c) string, a bound state with the correct quantum numbers, provided that d and c are relatively prime. This serves as the new fundamental object of the S-dual theory.

Of course there is more to say about the $SL(2,\mathbb{Z})$ symmetry of the type IIB string, like its appearance in the formula for the BPS bound on the bound string states, or its action on the higher branes present. But these few facts shall suffice for motivation and we press on with the matter of relating this symmetry to M-theory compactification.

T-duality It turns out that the toroidal $SL(2,\mathbb{Z})$ symmetry of IIB indeed has a higher dimensional origin, but not within IIB itself. The key lies in the connection to M-theory via IIA string theory, so what we do is a T-duality up both 'walls' of the house 4.1 relating the two S-dual IIBs to a (different) IIA each.

T-duality appears in the expanded spectrum of the free string at the attempt of compactification on a circle R. An ordinary point particle compactified develops a Kaluza-Klein tower of massive Fourier-modes along the circle, with a mass gap of $M \sim 1/R$ inversely proportional to the radius. On shrinking the circle away, $R \to 0$, the massive modes decouple and one is left with the zero-modes of the tower, a consistent truncation of the theory to a lower-dimensional point particle.

For strings, on the other hand, the situation is different. The expansion solving the equations of motion for the closed bosonic string $\partial_z \partial_{\bar{z}} X^\mu = 0$ (where $z = e^{\tau - i\sigma}$ and $\bar{z}$ the complex conjugate) is well known

$$X^\mu = x^\mu - i\sqrt{\frac{\ell_s^2}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\ell_s^2}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + i\sqrt{\frac{\ell_s^2}{2}}\sum_{m\neq 0}\left(\frac{\alpha_m^\mu}{m}z^{-m} + \frac{\tilde{\alpha}_m^\mu}{m}\bar{z}^{-m}\right) \quad (4.1.9)$$

If we demand invariance of X^μ under $\sigma \to \sigma + 2\pi$ this leads to an identification $\alpha_0^\mu = \tilde{\alpha}_0^\mu$ of left- and right-moving modes. This is only true for a non-compact direction, whereas for a circle in the direction $X^{25} \sim X^{25} + 2\pi R$, say, the condition is relaxed to integer m multiples $\alpha_0^\mu = \tilde{\alpha}_0^\mu + \frac{\sqrt{2}}{\ell_s} mR$. The momentum $p^\mu = \frac{1}{\sqrt{2}\ell_s}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$ in the $\mu = 25$ direction becomes quantised

$$\alpha_0^{25} = \frac{\ell_s}{\sqrt{2}}\left(\frac{n}{R} + m\frac{R}{\ell_s^2}\right), \qquad \tilde{\alpha}_0^{25} = \frac{\ell_s}{\sqrt{2}}\left(\frac{n}{R} - m\frac{R}{\ell_s^2}\right) \quad (4.1.10)$$

The Kaluza-Klein spectrum of a string state therefore exhibits a two-fold tower, the ordinary one of excited modes n and another one of sectors numbered by m. The m is interpreted as the *winding number* of how many times the closed string winds around the compact dimension.

Now it turns out that these towers are dual to each other: The mass of a Kaluza Klein state $(\alpha_0^{25})^2 + (\tilde{\alpha}_0^{25})^2$ is invariant under a sign flip of $\tilde{\alpha}_0^{25}$ and thus under the change $n \leftrightarrow m$ and $R \leftrightarrow \ell^2/R$. This inverse radius behaviour has an astonishing consequence – as we shrink the circle and widen the mass gap, the gap in stretching energy between different winding modes narrows. In the limit, these towers interchange and instead of reducing the dimensionality of the theory, we have gone from one string theory to another: from IIA to IIB.

In can be shown that this T-duality extends to the whole spectrum of IIA/B superstrings and also to type I open strings and the non-perturbative objects 'imported' from them, D-branes. On open type I strings, T-duality trades Neumann for Dirichlet boundary conditions and vice versa thus taking Dp-branes on which Dirichlet boundary strings end into D(p±1)-branes. Interpreting T-duality geometrically leads to the same picture of taking a wrapped Dp-brane into D(p-1)-brane, thus exchanging the stack of even branes in IIA with the odd ones in IIB. The same exchange is effected on the R-R forms (even in IIB, odd in IIA), to which the D-branes are charges.

This is a short summary of the T-duality procedure. Much more could be said, also in equations. We refer the reader to the standard textbook of Polchinski [109] for more. The effects on some parameters in the theories are summarised in figure 4.1, not in all generality, but adapted to the case we ultimately want to consider of a particular class of plane wave M-theory metrics.

M-Theory: ℓ_p

$$x^{11} \sim x^{11} + 2\pi R_{11} \qquad x^9 \sim x^9 + 2\pi R_9$$

$$ds_{11}^2 = e^{-\frac{2}{3}\phi_{11A}} \left(ds_{\text{ref}}^2 + g_{99}(dx^9)^2\right) + e^{4/3\phi_{11A}}(dx^{11})^2$$

M2-brane: tension $T_{\text{M2}} = \frac{1}{4\pi^2 \ell_p^3}$ $\begin{cases} \text{wrapped} \rightarrow \text{IIA string} \\ \text{transverse} \rightarrow \text{D2 brane} \end{cases}$

on $R_{11} = \ell_{11} g_{11A}$ on $R_9 = \ell_9 g_{9A}$

IIA String Theory on R_{11} $\longleftarrow$ 9/11 flip $\longrightarrow$ **IIA String Theory on R_9**

$\ell_{11}^2 = \frac{\ell_p^3}{R_{11}}$

$g_{11A}^2 = \left(\frac{R_{11}}{\ell_p}\right)^3 = \left(\frac{R_{11}}{\ell_{11}}\right)^2$

$ds_{11A}^2 = ds_{\text{ref}}^2 + g_{99}(dx^9)^2$

dilaton $e^{2\phi_{11A}}$

$\ell_9^2 = \frac{R_{11}}{R_9}\ell_{11}^2$

$g_{9A}^2 = \left(\frac{R_9}{R_{11}}\right)^3 g_{11A}^2$

$e^{2\phi_{9A}} = e^{-\phi_{11A}} g_{99}^{3/2}$

$ds_{9A}^2 = e^{\frac{2}{3}(\phi_{9A}-\phi_{11A})}(ds_{\text{ref}}^2 + e^{2\phi_{11A}}(dx^{11})^2)$

D0-brane mass $M^2 = \frac{1}{R_9^2} = \frac{1}{(\ell_9 g_9)^2}$

D2-brane tension $T_{\text{D2}} = \frac{1}{4\pi^2 \ell_9^3 g_9}$

T on R_9 T: $R \rightarrow \ell_s^2/R$, $g_s^2 \rightarrow g_s^2 \ell_s^2/R$ T on R_{11}

IIB String Theory on R_{11} $\longleftarrow$ S-dual $\longrightarrow$ **IIB String Theory on R_9**

$g_{11B}^2 = \frac{\ell_{11}^2}{R_9^2} g_{11A}^2 = \left(\frac{R_{11}}{R_9}\right)^2$

$e^{2\phi_{11B}} = g^{99} e^{2\phi_{11A}}$

$ds_{11B}^2 = ds_{\text{ref}}^2 + g^{99}(dx^9)^2$

$\ell_{11}^2 = \left(\frac{R_9}{R_{11}}\right)\ell_9^2$

$g_{11B} = \left(\frac{R_{11}}{R_9}\right) = g_{9B}^{-1}$

$e^{2\phi_{11B}} = e^{-2\phi_{9B}}$

$ds_{11B}^2 = e^{\phi_{11B}} ds_{9B}^2$

$g_{9B}^2 = \frac{\ell_9^2}{R_{11}^2} g_{9A}^2 = \left(\frac{R_9}{R_{11}}\right)^2$

$e^{2\phi_{9B}} = e^{-2\phi_{11A}} g_{99}$

$ds_{9B}^2 = e^{\phi_{9B}}(ds_{\text{ref}}^2 + g^{99}(dx^{11})^2)$

D1-brane tension $T_{\text{D1}} = \frac{1}{2\pi \ell_9^2 g_9^B}$

Figure 4.1: *M-Theory compactification on a torus. Although this somewhat mars the structural symmetry, complying with the conventions the metric is depicted with reference to the IIA$_{11}$ string frame metric ds^2_{ref} (without the dx^9 direction). Missing quantities can be inferred by the inherent symmetry of the diagram.*

IIA as an M-theory compactification The way from IIB to IIA is well understood, and even though only limited information is available on M-theory, we will be able to make the link on both sides up the 'roof' in 4.1 by connecting IIA to 11 dimensions.

On the low energy supergravity scale this connection is obvious – 10 dimensional IIA supergravity is the compactification on a circle of the highest, 11 dimensional, supergravity. While IIA supergravity is completed by full IIA string theory, no quantum complete theory incorporating 11 dimensional supergravity as a limit is known.

However, clues as to how this compactification picture extends to the full IIA superstring led Witten to postulate the existence of such an 11 dimensional completion, dubbed M-theory. One of the clues is the spectrum of D0-particles in IIA theory, as BPS states prone to non-perturbative effects. They each have masses $M_{\mathrm{D0}} = 1/(g_s \ell_s)$ and are assumed to form bound states of integer multiples $M_{n\mathrm{D0}} = nM_{\mathrm{D0}}$. But this is the Kaluza-Klein spectrum of an 11 dimensional point particle compactified on a circle of radius $R = g_s \ell_s$! The decompactification limit is sending $R \to \infty$, and that is the strong coupling limit $g_s \to \infty$ of IIA. M-theory is therefore defined to be this as of yet unknown strong coupling limit of IIA.

Another clue comes from an observation by Duff, Howe, Inami and Stelle [80]. The membrane action we have treated exhaustively in last chapter gives, on compactifying and wrapping the membrane simultaneously, back the ordinary IIA string action. While the membrane, in this context called M2-brane, does not feature a quantum complete action, it should certainly be part of M-theory in some way. This is exploited in re-interpreting the membrane quantisation of last chapter: as assembling the M2-brane from D0-particles.

The M-theory torus All steps we have (briefly) reviewed together comprise figure 4.1, at least at the level of coupling constants and other basic numerical quantities.

Start with M-theory which should somehow incorporate the 11 dimensional metric ds_p^2 (diagonal here for our purposes) and a fundamental length scale ℓ_p (p for Planck-length). Compactified along two different circles, R_{11} and R_9, say, we arrive at two different IIA theories, IIA_{11} and IIA_9. The quantities at this level, string lengths ℓ_{11} and ℓ_9, string couplings g_{11}^A and g_9^A, as well as the string metrics can all be determined by 11 dimensional quantities and compactification radii.

T-duality on each side on the respective other circle, R_9 for IIA_{11} and R_{11} for IIA_9, leads to the two different IIB theories. Those are now related by S-duality, and although in the picture we stick to a simple strong-weak duality and not the full $SL(2,\mathbb{Z})$, it becomes clear now where the toroidal symmetry comes from: It is the remainder of the two different compactifications of M-theory on a torus!

The focus on coupling constants does not do justice to how intricately strings and branes are interwoven in this duality picture. For illustration, take the M2-brane that should appear in M-theory and trace its way along the dualities. Wrapped around the compactified R_{11} on the left hand side this object gives the fundamental string F1 of IIA_{11} and of IIB_{11}. On the right hand side, the M2 is transverse to the circle R_9 of compactification, and gives a D2-brane in IIA_9. T-duality changes this D2 into a D1-string in IIB_9, which is, as we know, S-dual to the F1 string in IIB_9, completing the tour of the dualities.

4.1.3 DLCQ and D0-branes

BFSS conjecture and infinite momentum frame Building up on our introduction to D-branes earlier, regard the low-energy limit of the D0-brane action, the compactification of 10 dimensional Super Yang-Mills on a 9-torus

$$S \sim \frac{1}{g_{YM}} \int d\tau \ \mathrm{Tr} \ \left(\tfrac{1}{2}(\dot{X}^i)^2 + \tfrac{1}{4}[X^i, X^j]^2 + \tfrac{i}{2}\psi^\dagger \dot{\psi} - \tfrac{1}{2}\psi^\dagger \Gamma_i [\psi, X^i] \right) \tag{4.1.11}$$

As we have motivated, the objects in the theory are D0-branes, properly described by diagonal matrices and connected by stringy interactions in the off-diagonal components.

The model describes D0-branes in IIA string theory, but these objects are also supposed to be part of M-theory. How do we have to modify the action, which is intrinsically linked to maximally supersymmetric Yang-Mills in 10 dimensions, to describe particles in the 11 dimensions of M-theory?

The point is, according to Banks, Fischler, Shenker and Susskind (BFSS) [85], that we don't. We just have to change the interpretation of the model and take the nine transverse dimensions as transverse to the two dimensions of a light-cone, naturally resulting in an 11 dimensional theory in the infinite momentum frame (IMF).

The IMF is a limit procedure applied to a theory (M-theory) compactified on a space-like circle, X^{11}, say, with radius R^{11}. The momentum $p_{11} \sim N/R^{11}$ becomes quantised, splitting into sectors labelled by N. It is then argued that as $N \to \infty$ all modes with vanishing or negative momentum p_{11} decouple. But the only objects with non-vanishing p_{11} in IIA string theory are D0-branes. Therefore in the IMF they must decouple from the rest and the conjecture of BFSS [85] states that

> *M-theory in the infinite momentum frame (IMF) is exactly described by the $N \to \infty$ limit of the D0-brane matrix quantum mechanics (4.1.11).*

Discrete Light Cone Quantisation It is curious how a theory on Minkowski space has all of a sudden turned into matrix quantum mechanics, which is obviously a Galilean system. However, this follows a general scheme: In the light-cone frame the Galilean group is a subgroup of the Poincare group.

Start with a generical physical system with mass M^2 in flat space. The on-shell condition in light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 + X^{11})$ can be solved for the light-cone Hamiltonian $H \equiv P_+$

$$2P_+P_- - P_i^2 = M^2 \qquad \Rightarrow \qquad H = \frac{P_i^2}{2P_-} + \frac{M^2}{2P_-} \tag{4.1.12}$$

The expression takes a form reminiscent of Galilean mechanics, with μ and p the total mass and momentum of the system and U the Galilean invariant internal energy.

$$H = \frac{p^2}{2\mu} + U \tag{4.1.13}$$

The reason is a Galilean subgroup of the full Poincare group where the generators are identified in the following way

- $H = P_+$ is identified with Galilean time translations
- P_- is Galilean invariant mass μ (a central charge of the algebra)
- Transverse rotations and translations translate one to one
- Galilean boosts B_i are given by a sum of a Lorentz boost K_{0i} in a transverse direction i and a rotation of X^i into the light-cone variable X^{11} by L_{i11} simultaneously: $B_i = \frac{1}{\sqrt{2}}(K_{0i} + L_{i11})$

The generator of boosts within the light-cone K_{011} does not belong to this subgroup, but from its commutation relations we can rederive the form of the Galilean invariant internal energy U

$$\left.\begin{array}{r}[K_{011}, H] = H \\ [K_{011}, P_-] = -P_-\end{array}\right\} \Rightarrow HP_- \text{ is boost invariant} \tag{4.1.14}$$

So $U = \frac{M^2}{2P_-}$ as before and H scales like $1/P_-$ on rescaling all Galilean masses P_- in the system.
Now discrete light-cone quantisation (DLCQ) is defined by compactifying the theory on a circle of radius R in the light-like direction X^-. This should be understood primarily as discretising the spectrum of P_-

$$P_- = \frac{N}{R} \tag{4.1.15}$$

Since P_- is conserved, the system splits into a number N of different sectors. This led Susskind [111] to a refinement of the BFSS conjecture:

> *The DLCQ of M-theory is given exactly by the $U(N)$ matrix model (4.1.11) for finite N and the full theory is recovered in the limit $N \to \infty$.*

Seiberg-Sen In the limit $N \to \infty$ the DLCQ, i.e. compactification on a light-like circle $x^- \sim x^- + 2\pi R$, is expected to become identical to the IMF limit, compactification on a vanishing space-like circle $x'^9 \sim x'^9 + 2\pi\varepsilon R$, $\varepsilon \to 0$.
Exploiting this fact, Seiberg [112] and Sen [113] (see also Blau and O'Loughlin [107]) have proposed to realise the relation between the two compactifications as an infinite boost of the coordinates that transforms the two circles into one another.

$$x^\pm = e^{\mp\beta} x'^\pm \qquad \text{with } e^{-\beta} = \tfrac{1}{\sqrt{2}}\varepsilon \tag{4.1.16}$$

On the level of energies and momenta, we can see how the boost acts on the primed coordinates

$$\frac{N}{\varepsilon R} = p'_9 = \tfrac{1}{\sqrt{2}}(p'_+ - p'_-) = \tfrac{1}{\sqrt{2}}(e^{-\beta} p_+ - e^{\beta} p_-) \to -\frac{1}{\varepsilon} p_- \tag{4.1.17}$$

where p_9 is quantised on the space-like circle and $p_- = \frac{N}{R}$. The boost therefore blows up the energy E' and the momentum p'_9 simultaneously, at the same time shrinking away their difference. Expanding the square-root in the energy formula accordingly we recover the familiar

$$E' = \sqrt{(p'_9)^2 + (p'_i)^2 + M^2} \quad \Rightarrow \quad H'_{\text{lc}} = E' - p'_9 = \frac{(p'_i)^2}{2p'_9} + \frac{M^2}{2p'_9} + O((p'_9)^{-3}) \tag{4.1.18}$$

While definitely the two different compactifications should give the same result at $N \to \infty$, Seiberg and Sen also recovered the relation at finite N. This can be done by focusing on the infinitely small difference $E' - p_9'$ by scaling all relevant scales in the theory, energies by a factor of ε^{-1} and lengths by ε. Thus the DLCQ Hamiltonian can be realised at finite N by

$$H_{\text{DLCQ}} = \lim_{\varepsilon\to 0} \varepsilon H'_{\text{lc}} = \lim_{\varepsilon\to 0}\left(\varepsilon \frac{(p_i')^2}{2p_9'} + \varepsilon \frac{M^2}{2p_9'}\right) \; = \; \frac{p_i^2}{2p_-} + \frac{M^2}{2p_-} \tag{4.1.19}$$

Note that the infinite boost and subsequent rescaling together correspond to a Penrose limit

$$(x^+, x^-, x^i) = (y^+, \varepsilon^2 y^-, \varepsilon y^i) \tag{4.1.20}$$

which suggests that the procedure can be generalised to plane waves.

A skew house We can apply this technique now to the diagram 4.1, tracing out the path of the M2-brane as we did before. Nothing changes on the left-hand side, the wrapped membrane gives the IIA and by T-duality the IIB fundamental string.

But on the right-hand side we now get a concrete picture of the branes involved: According to the DLCQ prescription, we compactify on a light-like circle (as the limit of a small space-like circle). This describes M-theory exactly as the matrix model of D0-branes and again we can see the M2-brane transverse to the circle of compactification in this model: Reversing the line of argument of the membrane quantisation technique, we can tell that in the matrix model these extended objects must appear as bound states of D0-branes. T-duality finally changes D0- to D1-branes, which means unfolding a compactified dimension in the D0-brane matrix model to obtain matrix strings.

This matrix model therefore, captures the same physics in another coupling regime as IIA string theory. In the following we shall present the model of Craps, Sethi and Verlinde (CSV) [106] that makes use of the M-theory house to dynamically trace out a path between the two regimes of IIA fundamental string and D1-brane matrix strings.

4.2 Plane Wave Matrix Big-Bangs I: Derivation

4.2.1 The CSV-model

We have seen so far how dualities interconnect different string theories and limits of M-theory, respectively. In particular, we have introduced matrix models as the DLCQ of M-theory.

Now Craps, Sethi and Verlinde (CSV) [106] have proposed a model, called the matrix big-bang model, that incorporates all the salient features of the diagram 4.1 in one dynamical model. Consider the (flat) IIA string theory metric and coupling

$$ds^2 = -2dy^+dy^- + (dy^i)^2 \qquad e^{2\phi} = e^{-3y^+} \tag{4.2.1}$$

The dynamics of the linear dilaton drive the model from a strong-coupling phase near $y^+ = -\infty$ all the way through a non-controllable phase into a weakly coupled perturbative IIA string theory regime at late times.

With its coupling singularity at $y^+ = -\infty$ the model is one of the rare example of time- or rather null-dependent singular backgrounds in string theory that are somehow accessible in calculations. In the strong coupling phase, it has the interesting aspect of the resolving the coordinates close to the singularity by non-commutative matrix objects.

Recently, the model was generalised to a class of plane wave matrix big-bangs by Blau and O'Loughlin [107]. Before presenting these models in detail, let us collect a few peculiarities about the CSV model, which we shall need in the following.

DLCQ in the CSV model The Seiberg-Sen procedure boosted a space-like circle to a light-like one, all within the same two dimensional subspace spanned by the light-cone coordinates. On a flat background with constant dilaton, this was unproblematic.

CSV [106], however, have to implement the limit in a slightly different way, because in their model a null linear dilaton prevents them from compactifying the null direction y^+, which unfurls only in the limit and therefore cannot serve as a dilaton parameter. So they proposed a clever way of circumventing this problem: By using a third coordinate transverse to the light-cone for the space-like compactification. The necessary Lorentz transformation to rotate the light-like into a transverse circle

$$(y^+, y^-, y^9) \sim (y^+, y^- + 2\pi R, y^9 + 2\pi\varepsilon R) \;\leftrightarrow\; (y'^+, y'^-, y'^9) \sim (y'^+, y'^-, y'^9 + 2\pi\varepsilon R) \tag{4.2.2}$$

is a null-rotation

$$\begin{aligned} y^+ &= y'^+ & y^- &= y'^- + \varepsilon^{-1} y'^9 + \tfrac{1}{2}\varepsilon^{-2} y'^+ \\ y^9 &= y'^9 + \varepsilon^{-1} y'^+ & y^i &= y'^i \end{aligned} \tag{4.2.3}$$

Note, how neatly this integrates into the diagram 4.1. By definition of the background, CSV start out in IIA_{11} string theory on the left-hand side of the diagram 4.1, that is M-theory compactified space-like on R_{11}. This direction is non-flat and therefore gives rise to the dilaton.

The null-rotation, which is a combination of an ordinary rotation and a boost, can be seen as something like a skew 9-11 flip: Instead of up-lifting along R_{11} and then compactifying again light-like along the other branch (i.e. on a vanishing space-like circle R_9), it directly takes the transverse circle R_9 of IIA_{11} and rotates it into the null circle on the right-hand side of figure 4.1.

As in Seiberg-Sen, CSV proceed with a boost and subsequent uniform rescaling, which together comprise the Penrose rescaling

$$(y^+, y^-, y^i) \to (y^+, \gamma^2 y^-, \gamma y^i) \tag{4.2.4}$$

The scaling is necessary in order to arrive at a well-defined limit of the light-cone energy. Null rotation and Penrose scaling together have the effect

$$E_{\mathrm{lc}} = \tfrac{1}{\sqrt{2}} \left(\left(1 + \frac{\alpha^2}{2\gamma^2}\right) E' - \left(1 - \frac{\alpha^2}{2\gamma^2}\right) p'_{11} + \left(\sqrt{2}\frac{\alpha}{\gamma}\right) p'_9 \right) \tag{4.2.5}$$

which for the choice $\alpha = \sqrt{2}\gamma$ reduces immediately to the light-cone Hamiltonian

$$H_{\mathrm{DLCQ}} = E' - p'_9 \tag{4.2.6}$$

as in Seiberg-Sen.[3] Since we are talking about physical energies, the question arises whether one is allowed to seemingly arbitrarily rescale the coordinates. However, the flat space coordinates are trivially geodesic, and the Penrose scaling can be understood as a uniform rescaling plus a boost isometry. Therefore the procedure corresponds to a simple change of the scale of physical energies.

4.2.2 A plane wave background

We now have all the ingredients to implement the plane wave matrix big-bang model as in Blau and O'Loughlin [107]. Their observation was that the linear dilaton background (4.2.1) of the CSV model lifts up to a very particular plane wave in 11 dimensions

$$ds^2_{\text{CSV } 11} = -2d\hat{y}^+ d\hat{y}^- + \hat{y}^+ \sum_i (dy^i)^2 + (\hat{y}^+)^{-2}(dy^{11})^2 \tag{4.2.7}$$

suggesting the extension to the generic class of Rosen plane waves with power-law behaviour

$$ds^2_{11} = -2d\hat{y}^+ d\hat{y}^- + \sum_i (\hat{y}^+)^{2n_i}(d\hat{y}^i)^2 + (\hat{y}^+)^{2b}(d\hat{y}^{11})^2 \tag{4.2.8}$$

They are of singular homogeneous form in Brinkmann coordinates (see section 1.3.4)

$$ds^2_{11} = -2d\hat{z}^+ d\hat{z}^- + \left(\sum_i n_i(n_i-1)(\hat{z}^i)^2 + b(b-1)(\hat{z}^{11})^2\right)(\hat{z}^+)^{-2}(d\hat{z}^+)^2 + (dz^i)^2 + (dz^{11})^2 \tag{4.2.9}$$

from which we can read off the Einstein equation $R_{++} = 0$

$$\sum_i n_i(n_i-1) + b(b-1) = 0 \tag{4.2.10}$$

Due to the homogeneity it takes an algebraic form and severely constrains the range of all the parameters n_i and in particular b

$$-1 \le b \le 2 \tag{4.2.11}$$

Going down from 11 dimensions (4.2.8) to the string frame metric

$$ds^2_{11} = e^{-2\phi/3}\, ds^2_{\text{st}} + e^{4\phi/3} d\hat{y}^2_{11} \tag{4.2.12}$$

we find a plane wave multiplied by a conformal factor

$$ds^2_{\text{st}} = -2(\hat{y}^+)^b\, d\hat{y}^+ d\hat{y}^- + \sum_i (\hat{y}^+)^{2n_i+b}(d\hat{y}^i)^2 \qquad e^{2\phi} = (\hat{y}^+)^{3b} \tag{4.2.13}$$

The factor can be absorbed into a redefinition of $\hat{y}^+$ such that $dy^+ = (\hat{y}^+)^b d\hat{y}^+$. There are two possibilities:

[3] This choice is very illustrative for the procedure. However, the original CSV choice is another one, which at the level of fluctuations about a fixed string background amounts to the same Hamiltonian. We shall make use of this in the following plane wave case. See Blau and O'Loughlin [107] for further details.

- **For $\mathbf{b} \neq -\mathbf{1}$** this integrates to $y^+ \sim (\hat{y}^+)^{b+1}$ and leads to the background

$$ds^2_{\rm st} = -2dy^+dy^- + \sum_i (y^+)^{2m_i}(dy^i)^2 \qquad e^{2\phi} = \big((b+1)y^+\big)^{3b/(b+1)} \tag{4.2.14}$$

The Einstein equations[4] $R_{++} + 2\partial_+\partial_+\phi = 0$ again constrain the redefined parameters m_i

$$2m_i = \frac{2n_i + b}{b+1} \quad \Rightarrow \quad \sum_i m_i(m_i - 1) + \frac{3b}{b+1} = 0 \tag{4.2.15}$$

leading to the same range for b as before in (4.2.11).

- **For $\mathbf{b} = -\mathbf{1}$** the integration gives $y^+ = \log \hat{y}^+$ and one recovers the linear dilaton of the CSV model. Since b is a limit case of the range (4.2.11), all other parameters are restricted by the (vacuum) Einstein equations (4.2.10) to $n_i = 1/2$, the CSV model given by (4.2.7) and (4.2.1)

$$ds^2_{\rm st} = -2dy^+dy^- + \sum_i (dy^i)^2 \qquad e^{2\phi} = e^{-3y^+} \tag{4.2.16}$$

Trying to generalise the CSV procedure to these singular homogeneous plane wave backgrounds, we will see that to each step in CSV there is an exact analogue.

The coordinate transformations To implement the CSV procedure, we first have to generalise the coordinate transformations from flat space to plane waves, starting with the null rotations. In fact any plane wave

$$ds^2 = -2dy^+dy^- + g_{ij}(y^+)dy^idy^j \tag{4.2.17}$$

has such an isometry (using $h^{ik}(y^+) = \int^{y^+} g^{ik}(u)\, du$)

$$\begin{aligned} y^+ &= y'^+ & y^- &= y'^- + \varepsilon^{-1}y'^9 + \tfrac{1}{2}\varepsilon^{-2}h^{99}(y'^+) \\ y^9 &= y'^9 + \varepsilon^{-1}h^{99}(y'^+) & y^i &= y'^i + \varepsilon h^{9i}(y'^+) \end{aligned} \tag{4.2.18}$$

generated by the hidden translational Killing vectors of its dual Rosen metric

$$P^{(i)} = y^{(i)}\partial_{y^-} + h^{(i)k}(y^+)\partial_{y^k} \tag{4.2.19}$$

The isometry implements exactly the required change of coordinates, rotating an almost light-like circle of compactification in coordinates y into a small space-like circle in y'

$$(y^+, y^-, y^9) \sim (y^+, y^- + 2\pi R, y^9 + 2\pi\varepsilon R) \;\leftrightarrow\; (y'^+, y'^-, y'^9) \sim (y'^+, y'^-, y'^9 + 2\pi\varepsilon R) \tag{4.2.20}$$

Of course, written suggestively as a Penrose rescaling the next step in the CSV procedure is automatically valid for plane waves, which are themselves the result of a Penrose limit. However, the scaling was necessary to obtain the proper limit of physical energies. This cannot be

[4] Note the partial derivative since the Christoffel $\Gamma^{\mu}_{++} = 0$ in Rosen coordinates

done by any arbitrary scaling prescription, but makes sense in the CSV setting as a combined boost and uniform flat coordinate rescaling.

Blau and O'Loughlin pointed out that this combination is also possible in the power-law plane waves under consideration, due to a certain scale-invariance. This is slightly obscure in Rosen coordinates, but obvious in Brinkmann coordinates on the same plane waves. Consider

$$ds^2 = -2dy^+dy^- + g_{ij}(y^+)dy^i dy^j = -2dz^+dz^- + A_{ab}(z^+)z^a z^b (dz^+)^2 + (dz^a)^2 \tag{4.2.21}$$

Precisely for the power-law plane waves in Brinkmann coordinates $A_{ab} = n_a(n_a - 1)(z^+)^{-2}\delta_{ab}$ we can identify a boost isometry (1.3.18)

$$(z^+, z^-, z^i) \to (\lambda^{-1}z^+, \lambda z^-, z^i) \tag{4.2.22}$$

and a uniform rescaling homothety of the coordinates

$$(z^+, z^-, z^i) \to (\lambda z^+, \lambda z^-, \lambda z^i) \tag{4.2.23}$$

which together give the known Penrose rescaling (homothety) of plane wave metrics.

In Rosen coordinates the same mechanism works, where transverse Rosen coordinates get boosted by $y^i \to \lambda^{m_i} y^i$ and rescaled by $y^i \to \lambda^{1-m_i} y^i$. Of course this is not a uniform rescaling any more, but since Brinkmann coordinates are Fermi-coordinates measuring physical geodesic distance, we should really base our debate on them. In Brinkmann (geodesic) coordinates the split into boost-isometry and uniform rescaling therefore provides the correct physical picture for a discussion of the energies.

Energies The result of the transformations on the light-cone energy is similar to the CSV case

$$\begin{aligned} E_{\mathrm{lc}} &= i\partial'_+ - \frac{\alpha}{\gamma} g^{99}\partial'_9 + \frac{\alpha^2}{2\gamma^2} g^{99}\partial'_- \\ &= \frac{1}{\sqrt{2}}\left(\left(1 + \frac{\alpha^2}{2\gamma^2}\, g^{99}\right)E' + \left(1 - \frac{\alpha^2}{2\gamma^2}\, g^{99}\right)p'_{11} + \frac{\sqrt{2}\alpha}{\gamma}\, g^{99} p'_9\right) \end{aligned} \tag{4.2.24}$$

and in the case of constant $g^{99}(y^+)$ we can make the same choice $\alpha = \sqrt{2}\gamma$ leading to the light-cone Hamiltonian

$$H_{\mathrm{DLCQ}} = E' - p'_9 \tag{4.2.25}$$

For non-trivial $g^{99}(y^+)$, obviously there can be no such easy choice of constants. However, in the next subsection we shall focus on fluctuations about a fixed string background. In this setting y^+ and y^9 are gauge-fixed without fluctuations $\delta p_- = \delta p_9 = 0$. At this level potentially disturbing quantities are set to zero and the light-cone energy is the same in primed and unprimed coordinates.

This will be good enough for our analysis, so following Blau and O'Loughlin [107] we will set $\alpha = \gamma$ in the following without the factor of $\sqrt{2}$ which was necessary to obtain the Seiberg-Sen Hamiltonian exactly.

4.2.3 TS Dualities

With the coordinate transformations as necessary ingredients, we are now in a position to complete the journey in the diagram from type IIA string theory to its DLCQ matrix model limit. In principle we know how to take the DLCQ of M-theory. First compactify on a light-like circle to the D0-brane sector of IIA string theory. A subsequent T-duality unwraps the IIA_9 D0-branes into IIB_9 D1-branes, in this DLCQ limit described by the matrix string action.

CSV in their paper, however, wanted to find the matrix string dual of perturbative IIA string theory. This can be understood in the diagram 4.1 by starting on the left-hand side with IIA_{11}, lifting up to M-theory and then following the DLCQ path down the right.

CSV described a simpler, equivalent path arriving at the same IIB matrix string dual by a T- and subsequent S-duality (going down, then right in the the diagram). With the coordinate transformation described above, we shall start to detail this route now, expressing all quantities in each step in terms of the original IIA parameters.

Consider type IIA string theory with length ℓ_s and coupling g_s, understood as a compactification of M-theory on a space-like circle of radius $R_{11} = \ell_s g_s$. Since we are working with a non-constant dilaton, we shall distinguish between the constant coupling g_s, i.e. the normalisation of the dilaton, and the dilaton field e^{ϕ}. Hence the radius R_{11} is constant, and also the duality transformations in the diagram 4.1 act on g_s and ℓ_s for the most part.

We assume a further almost null circle in the theory and apply the coordinate transformations described above to mutate it into a small (constant) space-like circle of radius $R_9 = \varepsilon R$. As explained before, on top of this we have to apply a scaling of the coordinates, which then affects all length and mass scales in the theory. All in all this results in a scaled IIA'_{11} with

$$\text{IIA}'_{11}: \quad \begin{array}{rr} \text{string length}: \ell_s'^2 = \varepsilon^2 \ell_s^2 & \text{string coupling}: g_s' = g_s \\ \text{metric}: ds'^2 = \varepsilon^2 (ds^2 + g_{99}(dy^9)^2) & \text{dilaton}: e^{2\phi'} = e^{2\phi} \\ \text{M-theory circle}: R_{11}' = \varepsilon \ell_s g_s & \text{transverse circle}: R_9' = \varepsilon^2 R \end{array} \tag{4.2.26}$$

Performing T-duality along the small space-like circle $R_9' = \varepsilon^2 R$ we arrive at IIB''_{11}

$$\text{IIB}''_{11}: \quad \begin{array}{rr} \text{string length}: \ell_s''^2 = \varepsilon^2 \ell_s^2 & \text{string coupling}: g_s'' = \dfrac{\ell_s g_s}{\varepsilon R} \\ \text{metric}: ds''^2 = \varepsilon^2 (ds^2 + g^{99}(dy^9)^2) & \text{dilaton}: e^{2\phi''} = g^{99} e^{2\phi} \\ \text{M-theory circle}: R_{11}'' = \varepsilon \ell_s g_s & \text{transverse circle}: R_9'' = \dfrac{\ell_s^2}{R} \end{array} \tag{4.2.27}$$

A final S-duality leads us to $\text{II}\tilde{\text{B}}_9$

$$\text{II}\tilde{\text{B}}_9: \quad \begin{array}{rr} \text{string length}: \tilde{\ell}^2 = \varepsilon \ell_s^2 \dfrac{R_{11}}{R} & \text{string coupling}: \tilde{g}_s = \dfrac{\varepsilon R}{\ell_s g_s} \\ \text{metric}: \tilde{ds}^2 = \varepsilon^2 e^{\tilde{\phi}} (ds^2 + g^{99}(dy^9)^2) & \text{dilaton}: e^{2\tilde{\phi}} = g_{99} e^{-2\phi} \\ \text{transverse circle}: \tilde{R}_{11} = \varepsilon R_{11} & \text{M-theory circle}: \tilde{R}_9 = \dfrac{\ell_s^2}{R} \end{array} \tag{4.2.28}$$

With the parameters of $\text{II}\tilde{\text{B}}_9$ given in terms of the starting point IIA_{11}, we will now expand the standard DBI-action of D1-branes in the small parameter ε to arrive at the matrix string action.

On top of the energy debate before we also see clearly that in the final $\mathrm{I\tilde{I}B}_9$ the string length $\tilde{\ell}_s$ as well as the coupling $\tilde{g}_s$ are proportional to ε. It is important that both are vanishingly small to render the truncation of the full theory to the D1-brane sector valid.

4.2.4 The Matrix Model from DBI-Action Truncation

Start with the DBI action of the IIB_9 D1-branes written in all IIB_9 inherent quantities with tildes

$$S = -\frac{1}{2\pi\tilde{\ell}_s^2\tilde{g}_s}\int d\tau d\sigma\; e^{-\tilde{\phi}}\sqrt{-\det(\partial_\alpha\tilde{y}^\mu\partial_\beta\tilde{y}^\nu\tilde{g}_{\mu\nu} + 2\pi\tilde{\ell}_s^2 F_{\alpha\beta})} \tag{4.2.29}$$

Rewriting the action in the original IIA_{11} parameters (no tilde), the dilaton coupling precisely cancels the dilaton pre-factor the metric has acquired during the S-duality. Furthermore, $\tilde{\ell}_s^2\tilde{g}_s = \varepsilon^2\ell_s^2$ and the ε^2 in there cancels against the metrics ε^2 from scaling. So

$$S = -\frac{1}{2\pi\ell_s^2}\int d\tau d\sigma\;\sqrt{-\det(\partial_\alpha y^\mu\partial_\beta y^\nu g_{\mu\nu} + \partial_\alpha y^9\partial_\beta y^9 g^{99} + 2\pi\frac{\ell_s^3 g_s}{R}\sqrt{g^{99}}e^{\phi}F_{\alpha\beta})} \tag{4.2.30}$$

We now choose a string background solution in our Rosen coordinates, which we define by

$$y^+ = a\tau \qquad y^9 = b\sigma \tag{4.2.31}$$

and adapt the y^- to fulfil the constraints from fixing light-cone gauge $g_{\mu\nu}\partial_\alpha y^\mu\partial_\beta y^\nu \sim \eta_{\alpha\beta}$

$$-a\partial_\sigma y^- = 0 \qquad -2a\,\partial_\tau y^- + g^{99}b^2 = 0 \tag{4.2.32}$$

This returns us the background with small fluctuations Y^i and A_α

$$y^+ = a\tau \qquad y^9 = b\sigma \qquad y^- = \frac{b^2}{2a}h^{99}(a\tau) + \frac{b}{a}Y^9 \qquad y^i = Y^i \qquad A_\alpha = A_\alpha \tag{4.2.33}$$

where as before $h^{ij}(u) \equiv \int^u g^{ij}(u')du'$. Calculating the 2×2 determinant in the action up to second order in the fluctuations we find

$$\begin{aligned} S = -\frac{1}{2\pi\ell^2}\int d\tau d\sigma\;\Big((g^{99})^2b^4 &+ 2g^{99}b^3\partial_\tau Y^9 + b^2(\partial_\sigma Y^9)^2 \\ &+ g^{99}b^2 g_{ij}(\partial_\sigma Y^i\partial_\sigma Y^j - \partial_\tau Y^i\partial_\tau Y^j) + 4\pi^2\frac{\ell_s^6 g_s^2}{R^2}g^{99}e^{2\phi}F_{\tau\sigma}^2\Big)^{1/2} \end{aligned} \tag{4.2.34}$$

We now expand the square root about $(g^{99})^2b^4$. There is no need to go beyond linear order, except in one term

$$\sqrt{(g^{99})^2b^4 + 2g^{99}b^3\partial_\tau Y^9} = g^{99}b^2 + b\partial_\tau Y^9 - \tfrac{1}{2}g_{99}(\partial_\tau Y^9)^2 + \dots \tag{4.2.35}$$

Of these terms we shall only keep the last summand, the first two being constant and a total derivative that we can integrate out. All in all we arrive at the action

$$S = -\frac{1}{2\pi\ell^2}\int d\tau d\sigma\;\left(\tfrac{1}{2}g_{99}\partial_\alpha Y^9\partial_\beta Y^9\eta^{\alpha\beta} + \tfrac{1}{2}g_{ij}\partial_\alpha Y^i\partial_\beta Y^j\eta^{\alpha\beta} + 2\pi^2\frac{\ell_s^6 g_s^2}{b^2R^2}e^{2\phi}F_{\tau\sigma}^2\right) \tag{4.2.36}$$

We choose b such that $y^9 \sim y^9 + 2\pi\tilde{R}_9$ with $\tilde{R}_9 = \ell_s^2/R$ matches $\sigma \sim \sigma + 2\pi\ell_s$, therefore $b = \ell_s/R$. Note how the fluctuations of y^- we have called Y^9 behave as if they actually were fluctuations of y^9, which itself has been completely gauge-fixed.

4.3 Plane Wave Matrix Big-Bangs II: Properties

4.3.1 Rosen and Brinkmann Matrix Models

All along the derivation of the matrix model we were allowed to change between Rosen and Brinkmann coordinates at will. Indeed, we have chosen to perform the DBI-action truncation in Rosen coordinates – a reasonable choice, since T-duality is easier with the transverse Rosen coordinate y^9 being Killing. Nevertheless, we can also do the expansion in Brinkmann coordinates z^μ about the string background

$$z^+ = \tau, \quad z^9 = b\, e(\tau)\sigma, \quad z^- = \tfrac{1}{2}b^2\, h^{99}(\tau) + \tfrac{1}{2}b^2\, e(\tau)\partial_\tau e(\tau)\, \sigma^2 + b\, Y^9, \quad z^i = Z^i, \tag{4.3.1}$$

The z^- has been chosen deliberately to fulfil the Brinkmann light-cone gauge constraints

$$\partial_\sigma z^- = b^2 e(\tau)\, \partial_\tau e(\tau)\, \sigma, \quad 2\, \partial_\tau z^- = A^{99}(\tau)\, (be(\tau)\sigma)^2 + (b\, \partial_\tau e(\tau)\, \sigma)^2 + (be(\tau))^2 \tag{4.3.2}$$

T-duality has replaced the metric component $g_{99} = (y^+)^{2m_9}$ by its inverse g^{99}, so we will also denote the corresponding Brinkmann quantity with upstairs indices $A^{99} = m_9(m_9+1)(z^+)^{-2}$. As before the constant $b = \ell_s/R$, but we have also included an $e(\tau)$ in the definition of z^9. Comparing with y^9 we see that this is just the normal way of changing plane wave coordinates via a vielbein $e(\tau)^2 = g^{99}$ where $A^{99} = \ddot{e}(\tau)/e(\tau)$, respectively. In the case of power-law plane waves at hand we have explicitly $e(\tau) = \tau^{m_9}$, which is the natural vielbein on the one hand and enforced by the light-cone gauge constraints on the other. The $h^{99}(\tau)$ remains the same function, now defined as $h^{99}(\tau) = \int^\tau e(\tau')^2 d\tau'$.

Returning to the DBI-action, we can now again calculate the 2×2 determinant and do the same expansion of the square root to second order in the fluctuations. In the transverse coordinates this is trivial: Just set all the g_{ij} of before to unity, and bring in another $A_{ij}Z^iZ^j$ term of the Brinkmann metrics instead. Also a second order term, it trivially leads to a mass term in the resulting matrix action truncation.

More interesting is what happens to the Y^9 fluctuation we have deliberately *not* called Z^9

$$S = -\frac{1}{2\pi\ell^2}\int d\tau d\sigma\, \Big((be)^4 + 2b^3e^2\partial_\tau Y^9 + b^2(\partial_\sigma Y^9)^2 - (be)^2 A_{ij}Z^iZ^j + (be)^2((\partial_\sigma Z^i)^2 - (\partial_\tau Z^i)^2)\Big) \tag{4.3.3}$$

Note that the A^{99} is effectively off the board! Instead we have factors of $e(\tau)^2 = g^{99}$ in front of almost every term. Expanding the square root with focus on the non-trivial factors only

$$\sqrt{(be)^4 + 2b^3e^2\partial_\tau Y^9} = b^2e^2 + b\partial_\tau Y^9 - \tfrac{1}{2}e^{-2}(\partial_\tau Y^9)^2 + \tfrac{1}{2}e^{-2}(\partial_\sigma Y^9)^2 + \dots \tag{4.3.4}$$

we see that all Y^9 terms take the familiar Rosen shape. Hence in our definition of Y^9 we have effectively switched from Brinkmann to Rosen coordinates

$$S = -\frac{1}{2\pi\ell^2}\int d\tau d\sigma\, \Big(\tfrac{1}{2}g_{99}\partial_\alpha Y^9\partial_\beta Y^9\eta^{\alpha\beta} + \tfrac{1}{2}\delta_{ij}\partial_\alpha Z^i\partial_\beta Z^j\eta^{\alpha\beta} - \tfrac{1}{2}A_{ij}(\tau)Z^iZ^j\Big) \tag{4.3.5}$$

This does not mean, however, that we have to bear with this hybrid form of the action. In general, the translation from general relativistic coordinate transformations to the Yang-Mills action resulting from this procedure is far from obvious, even more so when going over from

$U(1)$ to non-commuting $U(N)$ matrices. Fortunately the particular change from Rosen to Brinkmann by a τ-only dependent vielbein carries over to field theory and amounts to a simple field redefinition $Y^9 = e(\tau)Z^9$. This results in a full Brinkmann model with all fields treated on the same footing

$$S = -\frac{1}{2\pi\ell^2}\int d\tau d\sigma\ \left(\tfrac{1}{2}\delta_{\hat{a}\hat{b}}\partial_\alpha Z^{\hat{a}}\partial_\beta Z^{\hat{b}}\eta^{\alpha\beta} - \tfrac{1}{2}A_{\hat{a}\hat{b}}(\tau)Z^{\hat{a}}Z^{\hat{b}}\right) \tag{4.3.6}$$

where indices are allowed to run over the full range including the 9-index. Keep in mind, however, that we have only treated cases without mixing terms, i.e. $A_{9i} = g_{9i} = 0$, and the validity of the truncations is not completely clear for non-constant plane waves, except for the homogeneous singular ones.

As an aside, note that the effect of the field redefinition also occurs in a much simpler example, the harmonic oscillator, see Blau, O'Loughlin [114]

$$\mathcal{L}_{\text{ho1}} = \dot{x}^2 - m^2x^2 \tag{4.3.7}$$

Just like the Brinkmann matrix action, by the transformation $y = ex$ with the 'vielbein' $e = \sin(mt)$ its mass term can be converted into an action with time-dependent coupling in the kinetic term, reminiscent of the Rosen matrix action

$$\mathcal{L}_{\text{ho2}} = \sin^2(mt)\ \dot{y}^2 \tag{4.3.8}$$

The two formulations therefore, despite their different appearance, capture the same physics.

4.3.2 Non-Abelian Extension, Coupling and Membrane Quantisation

As we said, the DBI-action is not without issues in the non-Abelian case. Fortunately, for most practical calculations it suffices to work with the truncated action, Yang-Mills, which can easily be generalised to more complicated gauge groups. This is the road we have taken here and we shall extend the Yang-Mills actions we have obtained to the non-Abelian cases by adding the scalar quartic potential in the canonical way.

Just how the scalars couple is seen most easily by regarding the flat space matrix string model as the eight-fold compactification of 10 dimensional Super-Yang-Mills in its non-Abelian $U(N)$ version. By the Kaluza-Klein mechanism, a gauge field mode becomes a scalar field $A_i \to Z^i/g_{YM}$ rescaled by g_{YM}^{-1} for a properly normalised kinetic term. This transfers the coupling of the Yang-Mills term $\sim g_{YM}^{-2}F^2$ in front of the respective potential term, so for the non-Abelian matrix string action we infer from (4.2.36)

$$S = \frac{1}{2\pi\ell^2}\int d\tau d\sigma\ \left(-\tfrac{1}{2}g_{ij}D_\alpha Y^i D_\beta Y^j\eta^{\alpha\beta} + \tfrac{1}{4}g_{YM}^2[Y^i, Y^j][Y^k, Y^l]g_{ik}g_{jl} - \tfrac{1}{4}g_{YM}^{-2}F_{\alpha\beta}^2\right)$$
$$\text{with } g_{YM} = \frac{2\pi}{\ell_s^2 g_s}e^{-\phi} \tag{4.3.9}$$

The same mechanism also works for the D0-brane matrix action. With the membrane quantisation of last chapter, however, we have yet another derivation at our disposition which by construction directly results in the $U(N)$ version of the very same D0-brane action.

In fact, in the membrane quantisation scheme only the non-Abelian $U(N)$ action really makes sense, with $U(N)$ the regularisation of the original $U(\infty)$ algebra of the membranes coordinates.

As an added bonus, by straight-forward calculation we get the $e^{-2\phi}$-dilaton coupling of the commutator terms that we had to put in by hand before.

Recall the final matrix action (3.2.27) for the most general target-space metric $g_{\mu\nu}(X^+, X^i)$ the derivation allowed for

$$S = \frac{1}{4\pi^2 \ell_p^3 R} \int d\tau \ \mathrm{Tr} \left(\tfrac{1}{2} DX^i \, DX^j \, \tfrac{g_{ij}}{g_{+-}} + \tfrac{g_{++}}{2g_{+-}} + \tfrac{g_{+i}}{g_{+-}} \, DX^i - \tfrac{1}{4} \, g_{+-} [X^i, X^j][X^k, X^l] g_{ik} g_{jl} \right) \tag{4.3.10}$$

In the given scenario the 11 dimensional metric reads in terms of the string frame metric

$$ds_{11}^2 = e^{-2/3\phi} ds_{\mathrm{st}}^2 + e^{4/3\phi} dx_{11}^2 \tag{4.3.11}$$

The string-frame metrics in Rosen- or Brinkmann form we have used have $g_{+-}^{(\mathrm{st})} = -1$ and $g_{+i}^{(\mathrm{st})} = 0$ in common, so in these cases the action reads

$$S = -\frac{1}{4\pi^2 \ell_s^4 g_s^2} \int d\tau \ \left(\tfrac{1}{2} DX^i \, DX^j \, g_{ij}^{(\mathrm{st})} + \tfrac{1}{2} g_{++}^{(\mathrm{st})} + \tfrac{1}{4} \, e^{-2\phi} g_{ik}^{(\mathrm{st})} g_{jl}^{(\mathrm{st})} [X^i, X^j][X^k, X^l] \right) \tag{4.3.12}$$

Thus the $e^{-2\phi}$-coupling of the commutator potential is recovered (compare Das and Michelson [115] in special cases).

4.3.3 Scale Invariance

Before discussing singularities and dynamical features of the model, we would like to direct the reader's attention to a peculiar property of the full matrix model with the commutator terms included. The action in the power-law plane waves we consider

$$\begin{aligned} S_{\mathrm{BC}} = \int d^2\sigma \mathrm{Tr} \big(& - \tfrac{1}{4} g_{YM}^{-2} \eta^{\alpha\gamma} \eta^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} - \tfrac{1}{2} \eta^{\alpha\beta} \delta_{ab} D_\alpha Z^a D_\beta Z^b \\ & + \tfrac{1}{4} g_{YM}^2 \delta_{ac} \delta_{bd} [Z^a, Z^b][Z^c, Z^d] + \tfrac{1}{2} \sum_a m_a (m_a - 1) \tau^{-2} \delta_{ab} Z^a Z^b \big) \end{aligned} \tag{4.3.13}$$

exhibits a certain scaling behaviour, which comes from uncompactified 10 dimensional Yang-Mills on the one hand and from the particular scaling of the homogeneous plane wave mass term on the other.

Consider a rescaling of the 'world-sheet' coordinates $\sigma^\alpha \to \lambda^{\omega_\tau} \sigma^\alpha$. Then, in the current form of the Yang-Mills action where the coupling g_{YM} appears just as a prefactor, the gauge field A_α must scale like ∂_α in order to achieve a homogeneous overall scaling of the action.

Since the matrix string action without the mass term can be understood as a compactification of Yang-Mills on a torus, the scaling behaviour carries over from one theory to another. When determining the scaling coefficients, however, we have to keep in mind that the Z^a are a rescaled version of the gauge field components $Z^a \sim A^a / g_{YM}$ – rescaled such that the coupling appears in front of their commutator term. In our power-law plane wave background, we have the coupling determined by the (algebraic) Einstein equation (4.2.15)

$$g_{YM}^2 \sim e^{-2\phi} \sim \tau^{-3b/(b+1)} \qquad \text{with} \quad \frac{-3b}{b+1} = \sum_a m_a (m_a - 1) \tag{4.3.14}$$

So the action scales homogeneously, even for non-trivial coupling, with the coefficients

$$\begin{aligned}
&(\tau,\sigma) \to \lambda^{\omega_\tau}(\tau,\sigma) \quad \text{and } A \to \lambda^{-\omega_\tau} A \\
&\quad Z^a \to \lambda^{\omega_Z} Z^a, \quad \omega_Z = -\tfrac{1}{2}(\sum_a m_a(m_a-1)+2)\,\omega_\tau \\
&\quad S \to \lambda^{\omega_S} S, \quad \omega_S = 2\omega_Z
\end{aligned} \tag{4.3.15}$$

Interestingly, this scaling also works for the mass term in the action, since the double derivative of the kinetic term and the τ^{-2}-mass inherently scale the same way. The deeper reason behind is that the singular homogeneous plane wave backgrounds, the basis for this action, show a boost invariance (1.3.18) as already mentioned in chapter one, section 1.3.4.

By virtue of this scaling, any particular solution $Z^a(\tau,\sigma)$ gives rise to a 1-parameter family of solutions $\lambda^{\omega_Z} Z^a(\lambda^{\omega_\tau}\tau, \lambda^{\omega_\tau}\sigma)$.

The most interesting case is that of scale invariance of the action, i.e.

$$\omega_S = 0 \quad \Leftrightarrow \quad \sum_a m_a(m_a-1) = -2. \tag{4.3.16}$$

A similar invariance arises in massless ϕ^4 theory, where it can be exploited to parametrise instanton solutions, see Zinn-Justin [116].

This special parameter characterises the border-line case corresponding to the (reduced) matrix string model with dilaton parameter $b=2$. This model is dual to the matrix big-bang model of CSV [106], in the sense that the 11d metric is isometric to CSV, but compactified along a different Killing direction (Blau and O'Loughlin [107]). In this special model therefore exists a conserved Noether current, and accordingly a conserved charge. We shall not exploit this any further, but just note that in the case of the fuzzy sphere solutions with time-only dependent radius $Z^a \sim cr(\tau)J^a$ we will consider the next section this leads to the conserved charge

$$Q^\tau = 2\pi\big((\tau-\tfrac{1}{2})\dot r^2(\tau) - \tfrac{1}{8\tau^2}r^2 - \tfrac{c^2}{4\tau^2}r^4(\tau)\big) \tag{4.3.17}$$

Finally, even though we have not been paying much attention to the fermions throughout this chapter, the scaling behaviour also extends to the two fermion terms $\psi D\psi$ and $g_{YM}\psi Z^a\Gamma^a\psi$ in the action necessary for supersymmetry. The conditions that they scale identically to the other terms enforces the scaling

$$\omega_f = \tfrac{1}{2}(\sum_a m_a(m_a-1)+1)\,\omega_\tau. \tag{4.3.18}$$

4.4 Singularity Properties and Time-Dependent Fuzzy Spheres

4.4.1 Strong and Weak Coupling Ranges

While during the derivation of the matrix model we did not care much about the models parameters and their ranges, let us come back to the discussion started in section 4.2.2. Recall that for the class of homogeneous singular plane waves (4.2.8) we are discussing, the Einstein equation reduces to a purely algebraic constraint on the parameters (4.2.15)

$$\frac{3b}{b+1} = -\sum_a m_a(m_a-1) \tag{4.4.1}$$

Depending on different values of the dilaton parameter between $-1 \le b \le 2$ (the other values being excluded), several behaviours can be distinguished both at the singularity and at infinity.

At the singularity In the CSV model ($b = -1$), even though the IIA string frame metric is flat, we do have a singularity at $y^+ \to -\infty$. This can be seen either as a coupling singularity of the dilaton $\sim \exp(-3y^+)$ or as a singularity in the more physical Einstein frame metric. Whereas in the IIA the singular point is located at infinite geodesic distance, in the up-lift the 11 dimensional geodesic parameter goes exponentially $y^+ = \log \hat{y}^+$. Here, the singularity occurs at finite geodesic distance in the resulting homogeneous singular plane wave background.

For the power-law behaviour of plane waves with $b > -1$, the singularity is at finite geodesic distance $y^+ = 0$ in 11 and 10 dimensions like-wise. According to the sign of the dilaton

$$e^{2\phi} \sim (y^+)^{3b/(b+1)} \tag{4.4.2}$$

there are two distinguished cases

$$\begin{aligned} \text{strong coupling singularity:} &\quad -1 < b < 0 \\ \text{weak coupling singularity:} &\quad 0 < b \leq 2 \end{aligned} \tag{4.4.3}$$

with the first one alike to the CSV model.

At infinity Again in the CSV-model, the string coupling approaches zero at infinity $y^+ \to \infty$, suitable for a description in terms of perturbative IIA string theory. The same happens for the first range $-1 < b < 0$ (strong coupling singularity).

For $0 < b \leq 2$ the coupling explodes at infinity, making a perturbative treatment impossible. The situation can be remedied by adding a linear term to the logarithmic dilaton, compatible with the Einstein-dilaton equation, as has been done in Papadopoulos, Russo and Tseytlin [47]. This, however, does not lift up to a homogeneous singular plane wave anymore, and we shall not try to extend our analysis to these cases.

Anyway, homogeneous singular plane waves arise most prominently as Penrose limits of the Szekeres-Iyer metrics, as explained in chapter two. As such, they should be taken as a power-law approximation to the true space-time metric in the vicinity of the singularity. In the following, therefore, we shall rather concentrate on the behaviour near the singularity, where the system is described by the gauge theory we have derived.

At this point it is not without interest to regard the behaviour of the most prominent class of matrix model solutions in the two regimes: Fuzzy spheres.

4.4.2 Time-Dependent Fuzzy Sphere Solutions

An interesting class of solutions to the classical equations of motion of the plane wave matrix string model is given by the fuzzy spheres we have already encountered in chapter three in the context of the BFSS and BMN matrix models of D0-branes. These configurations are composed of the three constant generators J^a of an SU(2) matrix representation times a time-(τ)-dependent radius

$$Z^a(\tau) \sim c\, r(\tau) J^a \quad \text{where} \quad [J^a, J^b] = i\varepsilon^{abc} J^c \tag{4.4.4}$$

for some proportionality constant c that takes care of normalisation and that we can use later in the analysis of the radial equation of motion. All other fields are set to zero. The solutions are independent of σ, so in the context of matrix string theory they actually rather correspond to fuzzy cylinders. According to the derivation given of the D1-brane action they are trivially related to the corresponding fuzzy spheres in the D0-brane action by T-duality. Solutions of this kind have been studied extensively for example in Das and Michelson [115], [117] and in [104] with Shapere.

For simplicity of the analysis we focus on the case of

$$\textit{all equal} \quad m_a = \mu \quad \forall a \tag{4.4.5}$$

throughout the remainder of this chapter.

The generators J^a resolve the commutators reducing the equations of motion to a single non-linear differential radial equation for $r(\tau)$.

$$\ddot{r}(\tau) - \mu(\mu-1)\tau^{-2}\, r(\tau) + c^2\tau^{8\mu(\mu-1)}\, r(\tau)^3 = 0 \tag{4.4.6}$$

The gauge theory coupling g_{YM}, given by the inverse of the time-dependent string coupling, appears in this formula as

$$g_{YM}^2 \sim e^{-2\phi} \sim \tau^{-\frac{3b}{b+1}} = \tau^{8\mu(\mu-1)} \tag{4.4.7}$$

and the range of strong gauge (i.e. weak string) coupling in terms of the parameter μ is

$$\begin{aligned} \text{strong gauge coupling:} &\quad 0 < \mu < 1 \\ \text{weak gauge coupling:} &\quad \mu < 0 \text{ or } \mu > 1 \end{aligned} \tag{4.4.8}$$

The equations are symmetric under $\mu \to 1-\mu$ with a maximum value of $b = 2$ for $\mu = \frac{1}{2}$.

Our aim is to discuss fuzzy-sphere behaviour now in the context of weak and strong gauge coupling. In principle, one would expect a gauge system at weak coupling to develop highly non-Abelian solutions, since the cost of the commutator potential is negligible as compared to the kinetic and mass term. Vice versa, at strong gauge coupling the system will fall into an Abelian phase of commuting matrix solutions.

When restrained to inherently non-Abelian fuzzy spheres, this behaviour of the system should manifest itself in the study of the resulting radial equation. Since the complete solution is beyond reach, we shall treat the non-linear $r(\tau)^3$-term as a small 'perturbation' at first, and focus on the fundamental system of solutions to the linear part of the equation

$$r_1(\tau) = \tau^{\mu} \tag{4.4.9}$$

$$r_2(\tau) = \tau^{1-\mu} \tag{4.4.10}$$

But is it really justified to call the non-linear term small here? A comparison between the linear (mass) term $\tau^{-2}\, r(\tau)$ and the non-linear $\tau^{8\mu(\mu-1)}\, r(\tau)^3$ shows that the coupling of the former always comes with a smaller (equal, in the limit case) exponent, since $8\mu(\mu-1) \geq -2$.

Also, on the two branches of the solution for any value of μ the non-linear term is sub-dominant

$$r_1(\tau) = \tau^{\mu} \quad \rightarrow \quad 8\mu(\mu-1) + 3\mu > -2 + \mu \qquad \text{always} \tag{4.4.11}$$

$$r_2(\tau) = \tau^{1-\mu} \quad \rightarrow \quad 8\mu(\mu-1) + 3(1-\mu) > -2 + (1-\mu) \qquad \text{always} \tag{4.4.12}$$

We will therefore use the fundamental system to discuss the behaviour of fuzzy spheres as $\tau \to 0$, even in the case of strong (infinite) coupling, since the mass term always dominates.

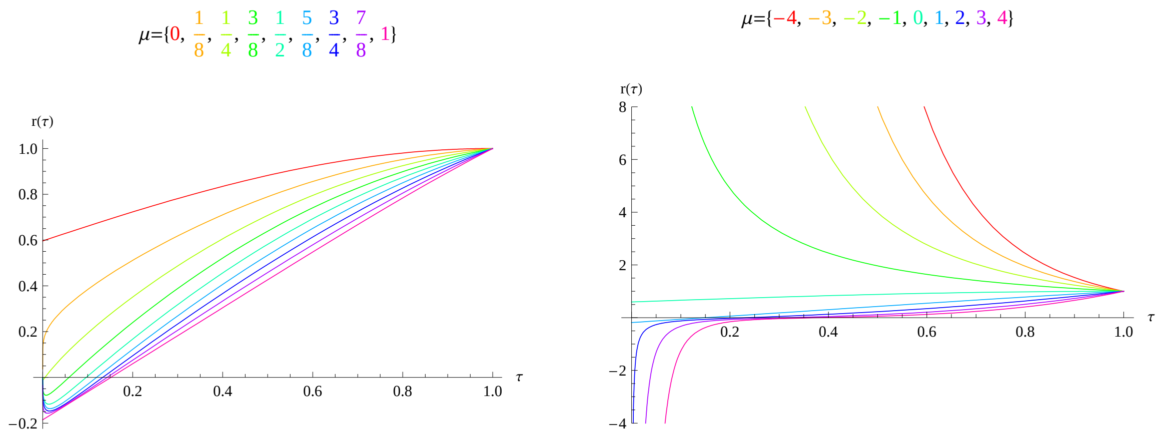

Figure 4.2: *The numerical solutions for the homogeneous plane wave fuzzy spheres. Evaluation was started at $\tau = 1$ with initial conditions $r(1) = r_1(1)$, $r'(1) = r_1'(1)$ (with $r_1(\tau) = \tau^\mu$) and done backwards in time towards the singular point $\tau \to 0$. The spheres shrink continously back to the origin for different μ in $0 < \mu < 1$ (left) and diverge outside that range (right). In the limit cases $\mu = 0$ and $\mu = 1$ the radius goes to finite values.*

This fortunate circumstance justifies the perturbation analysis in all cases we will consider in the following, i.e. both at weak and strong coupling.

At **weak gauge coupling** now, i.e. outside the parameter range $0 < \mu < 1$, one of the two solutions goes to zero as $\tau \to 0$, while the other one, respectively, explodes. Generic solutions will be formed by linear combinations of the fundamental system, so it would require tremendous fine-tuning to model fuzzy spheres that do not explode at the origin, if it is possible at all.

On the other hand, both solutions are well-behaved as $\tau \to 0$ in the **strong gauge coupling** parameter range $0 < \mu < 1$. In contrast to the former case, this range seems to provide a linear fundamental system stable enough to send generic solutions to zero as $\tau \to 0$. Confirmed by our analysis, one could say that the system would like to fall into its Abelian phase, but when restrained to inherently non-Abelian fuzzy spheres tends to avoid the pressure of the commutator potential by shrinking the radius away at the origin.

It might look surprising at first glance that the linear solutions do reflect the behaviour of the coupling of the quartic potential. The underlying reason is Einstein equation (4.2.15), tying the dilaton to the plane wave mass term, thus relating their behaviour in a non-trivial way.

Numerical evaluation with Mathematica of the full non-linear radial equation for certain parameters sheds light on the problem from a different angle. Due to the singular nature at $\tau \to 0$, we start these checks with fuzzy spheres of finite radius at time $\tau = 1$ and evaluate back towards $\tau \to 0$ in time, to see if the solution explodes or not.

Of course numerical evaluation towards a singular point is problematic, but the resulting plots seem to confirm the analysis of above. Again, inside the range $0 < \mu < 1$ (strong gauge coupling) the numerical solutions smoothly approach zero as $\tau \to 0$, which is illustrated in figure 4.2. Outside that range (weak gauge coupling) it was not possible to model initial conditions for the same behaviour and the radius would always blow up numerically as $\tau \to 0$.

In fact, one can see that the numerical solution stays very close to the linear solution (or the particular linear combination) that served as initial condition at $\tau = 1$. This in turn confirms once more the earlier treatment of the non-linear term as a perturbation.

Take the function $r_1(\tau) = \tau^\mu$ as a starting point[5], as was done in the illustrations. Then for $\mu < 0$ ($r_1(\tau)$ exploding) the ratio between numerical and the exploding linear solution stayed well within 1 ± 0.1 for all checks performed ($0 \le \tau \le 1$, $\mu = \{-1 \ldots -8\}$), behaviour actually getting better as μ decreases.

For $\mu > 1$ ($r_1(\tau)$ converging), the numerical solution still follows the converging linear one at first, but of course the divergence in the other branch weighs in heavily when approaching $\tau \to 0$.

To summarise, one can say that the behaviour of fuzzy sphere objects differs largely between models of strong and weak gauge coupling at the singularity. Physically most interesting might be the case of weak gauge/strong string coupling at the singularity $\tau \to 0$, where, as in Das and Michelson [115], weakly coupled gauge theory at the origin allows for large (and thus largely non-Abelian) fuzzy spheres. These objects become smaller at late times when the theory goes into perturbative weakly coupled string theory.

4.5 Concluding Remarks

The way we have strayed from the original line of studies of singularities is quite remarkable: In chapter two we had started with the presentation of geodesic probes of power-law metrics, followed by a subsequent analysis of scalar field probes on the same backgrounds. The classical next step would have been to work out stringy probes along the lines of e.g. Horowitz and Steif [12]. Instead, embedding classical string theory into the bigger framework of the duality network and M-theory, we went for something completely different: Resolving perturbative strings into non-Abelian objects close to a singularity by switching to a dual gauge theory description – a tantalising idea.

The backgrounds we have considered were the singular homogeneous plane waves, here narrowed down to 11 dimensional vacuum solutions. A reasonable choice, as they were found in chapter two to approximate a wide class of 'physical' singularities. But since we now work in a framework that is conceptually different, and not only on a technical level, we should ask whether it also extends beyond plane waves. Could gauge theory serve to resolve all singularities in principal by introducing non-commutativity and a certain graininess of space-time?

A related question is how to possibly extend the DLCQ procedure to derive matrix theory from M-theory, with focus on the Penrose limit employed. Is this the Penrose limit of an already plane wave background, and thus trivial, or is the DLCQ a lossy process projecting any metric on its plane wave limit? If so, and given the fact that with the Fermi-Penrose expansion of chapter one we are in complete control of the background, can we extend the DLCQ to higher orders and thus to a lossless perturbative expansion?

Of course, to date only sectors of the duality network are accessible this way, and care must be taken to justify the emphasis of the respective part. Whether and how singularities are resolved in general will have to wait until the formulation of the all-conclusive answer: M-theory.

[5] Symmetry $\mu \to 1 - \mu$ here interchanges which linear solution is being used as initial condition.

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Bibliography

[1] M. Blau, D. Frank, and S. Weiss, "Fermi coordinates and Penrose limits," *Class. Quant. Grav.* **23** (2006) 3993–4010, `hep-th/0603109`.

[2] M. Blau, D. Frank, and S. Weiss, "Scalar field probes of power-law space-time singularities," *JHEP* **08** (2006) 011, `hep-th/0602207`.

[3] M. Cahen and M. Flato, "Differential Geometry,". Reidel, Dordrecht, Netherlands 1976.

[4] D. Kennefick, "Controversies in the History of the Radiation Reaction problem in General Relativity," `gr-qc/9704002`.

[5] H.-J. Schmidt, "Lectures on mathematical cosmology," `gr-qc/0407095`.

[6] H.-J. Schmidt, "Why do all the curvature invariants of a gravitational wave vanish?," `gr-qc/9404037`.

[7] R. P. Geroch, "Limits of spacetimes," *Commun. Math. Phys.* **13** (1969) 180–193.

[8] M. Blau, J. M. Figueroa-O'Farrill, and G. Papadopoulos, "Penrose limits, supergravity and brane dynamics," *Class. Quant. Grav.* **19** (2002) 4753, `hep-th/0202111`.

[9] M. Blau and M. O'Loughlin, "Homogeneous plane waves," *Nucl. Phys.* **B654** (2003) 135–176, `hep-th/0212135`.

[10] J. Ehlers and W. Kundt, "Exact solutions of the gravitational field equations,".

[11] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,". Cambridge, UK: Univ. Pr. (1998) 402 p.

[12] G. T. Horowitz and A. R. Steif, "Strings in strong gravitational fields," *Phys. Rev.* **D42** (1990) 1950–1959.

[13] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," *JHEP* **01** (2002) 047, `hep-th/0110242`.

[14] M. Blau, "Plane waves and Penrose limits,". unpublished.

[15] M. Blau, M. Borunda, M. O'Loughlin, and G. Papadopoulos, "Penrose limits and spacetime singularities," *Class. Quant. Grav.* **21** (2004) L43, `hep-th/0312029`.

[16] R. R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond- Ramond background," *Nucl. Phys.* **B625** (2002) 70–96, `hep-th/0112044`.

[17] R. R. Metsaev and A. A. Tseytlin, "Exactly solvable model of superstring in plane wave Ramond-Ramond background," *Phys. Rev.* **D65** (2002) 126004, `hep-th/0202109`.

[18] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "Penrose limits and maximal supersymmetry," *Class. Quant. Grav.* **19** (2002) L87–L95, `hep-th/0201081`.

[19] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," *JHEP* **04** (2002) 013, `hep-th/0202021`.

[20] D. Sadri and M. M. Sheikh-Jabbari, "The plane-wave / super Yang-Mills duality," *Rev. Mod. Phys.* **76** (2004) 853, `hep-th/0310119`.

[21] R. Penrose, "Any space-time has a plane wave as a limit,". in *Differential geometry and relativity*, Reidel, Dordrecht (1976) pp. 271–275.

[22] R. Gueven, "Plane wave limits and T-duality," *Phys. Lett.* **B482** (2000) 255–263, `hep-th/0005061`.

[23] M. Blau, M. Borunda, M. O'Loughlin, and G. Papadopoulos, "The universality of Penrose limits near space-time singularities," *JHEP* **07** (2004) 068, `hep-th/0403252`.

[24] J. L. Synge, "Relativity: The General theory,". North-Holland, Amsterdam, 1960.

[25] E. Poisson, "The motion of point particles in curved spacetime," *Living Rev. Rel.* **7** (2004) 6, `gr-qc/0306052`.

[26] F. Manasse and C. Misner, "Fermi normal coordinates and some basic concepts in differential geometry,". J. Math. Phys. 4 (1963) 735–745.

[27] R. E. Rudd, "Light cone gauge quantization of 2-D sigma models," *Nucl. Phys.* **B427** (1994) 81–110, `hep-th/9402106`.

[28] A. Parnachev and A. V. Ryzhov, "Strings in the near plane wave background and AdS/CFT," *JHEP* **10** (2002) 066, `hep-th/0208010`.

[29] J. Callan, Curtis G. *et al.*, "Quantizing string theory in AdS(5) x S**5: Beyond the pp- wave," *Nucl. Phys.* **B673** (2003) 3–40, `hep-th/0307032`.

[30] J. Callan, Curtis G., T. McLoughlin, and I. Swanson, "Holography beyond the Penrose limit," *Nucl. Phys.* **B694** (2004) 115–169, `hep-th/0404007`.

[31] K. E. Kunze, "Asymptotic behaviour of curvature and matter in the Penrose limit," `gr-qc/0411115`.

[32] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[33] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond.* **A270** (1962) 103–126.

[34] E. Newman and R. Penrose, "An Approach to gravitational radiation by a method of spin coefficients," *J. Math. Phys.* **3** (1962) 566–578.

[35] A. Coley, R. Milson, V. Pravda, and A. Pravdova, "Classification of the Weyl tensor in higher-dimensions," *Class. Quant. Grav.* **21** (2004) L35–L42, `gr-qc/0401008`.

[36] R. Milson, A. Coley, V. Pravda, and A. Pravdova, "Alignment and algebraically special tensors in Lorentzian geometry," *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 41–61, `gr-qc/0401010`.

[37] A. Pravdova, V. Pravda, and A. Coley, "A note on the peeling theorem in higher dimensions," *Class. Quant. Grav.* **22** (2005) 2535–2538, `gr-qc/0505026`.

[38] S. W. Hawking and G. F. R. Ellis, "The Large scale structure of space-time,". Cambridge University Press, Cambridge, 1973.

[39] W.-Q. Li and W.-T. Ni, "Expansions of the affinity, metric and geodesic equations in Fermi normal coordinates about a geodesic,". J. Math. Phys. 20 (1979) 1925-1929.

[40] R. R. Metsaev and A. A. Tseytlin, "Superstring action in AdS(5) x S(5): kappa-symmetry light cone gauge," *Phys. Rev.* **D63** (2001) 046002, `hep-th/0007036`.

[41] R. R. Metsaev, C. B. Thorn, and A. A. Tseytlin, "Light-cone superstring in AdS space-time," *Nucl. Phys.* **B596** (2001) 151–184, `hep-th/0009171`.

[42] S. S. Gubser, "Einstein manifolds and conformal field theories," *Phys. Rev.* **D59** (1999) 025006, `hep-th/9807164`.

[43] G. T. Horowitz and D. Marolf, "Quantum probes of space-time singularities," *Phys. Rev.* **D52** (1995) 5670–5675, `gr-qc/9504028`.

[44] P. Szekeres and V. Iyer, "Spherically symmetric singularities and strong cosmic censorship," *Phys. Rev.* **D47** (1993) 4362–4371.

[45] M.-N. Celerier and P. Szekeres, "Timelike and null focusing singularities in spherical symmetry: A solution to the cosmological horizon problem and a challenge to the cosmic censorship hypothesis," *Phys. Rev.* **D65** (2002) 123516, `gr-qc/0203094`.

[46] K. E. Kunze, "T-duality and Penrose limits of spatially homogeneous and inhomogeneous cosmologies," *Phys. Rev.* **D68** (2003) 063517, `gr-qc/0303038`.

[47] G. Papadopoulos, J. G. Russo, and A. A. Tseytlin, "Solvable model of strings in a time-dependent plane-wave background," *Class. Quant. Grav.* **20** (2003) 969–1016, `hep-th/0211289`.

[48] R. M. Wald, "General Relativity,". Chicago, Usa: Univ. Pr. (1984) 491p.

[49] M. Reed and B. Simon, "Methods of Modern Mathematical Physics. 2. Fourier Analysis, Selfadjointness,". Academic Press, New York (1972).

[50] G. Papadopoulos, "Power-law singularities in string theory and M-theory," *Class. Quant. Grav.* **21** (2004) 5097–5120, `hep-th/0404172`.

[51] M. Blau and M. O'Loughlin. *unpublished* (2004).

[52] S. Das and S. Shankaranarayanan, "High frequency quasi-normal modes for black-holes with generic singularities," *Class. Quant. Grav.* **22** (2005) L7, `hep-th/0410209`.

[53] V. de Alfaro, S. Fubini, and G. Furlan, "Conformal Invariance in Quantum Mechanics," *Nuovo Cim.* **A34** (1976) 569.

[54] M. Blau, M. Borunda, and M. O'Loughlin, "On the Hagedorn behaviour of singular scale-invariant plane waves," *JHEP* **10** (2005) 047, `hep-th/0412228`.

[55] P. Claus *et al.*, "Black holes and superconformal mechanics," *Phys. Rev. Lett.* **81** (1998) 4553–4556, `hep-th/9804177`.

[56] G. W. Gibbons and P. K. Townsend, "Black holes and Calogero models," *Phys. Lett.* **B454** (1999) 187–192, `hep-th/9812034`.

[57] T. R. Govindarajan, V. Suneeta, and S. Vaidya, "Horizon states for AdS black holes," *Nucl. Phys.* **B583** (2000) 291–303, `hep-th/0002036`.

[58] D. Birmingham, K. S. Gupta, and S. Sen, "Near-horizon conformal structure of black holes," *Phys. Lett.* **B505** (2001) 191–196, `hep-th/0102051`.

[59] H. E. Camblong and C. R. Ordonez, "Black hole thermodynamics from near-horizon conformal quantum mechanics," *Phys. Rev.* **D71** (2005) 104029, `hep-th/0411008`.

[60] B. Pioline and A. Waldron, "Quantum cosmology and conformal invariance," *Phys. Rev. Lett.* **90** (2003) 031302, `hep-th/0209044`.

[61] R. M. Wald, "Dynamics in nonglobally hyperbolic, static space-times," *J. Math. Phys.* **21** (1980) 2802–2805.

[62] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology," *Adv. Phys.* **19** (1970) 525–573.

[63] V. a. Belinsky, I. m. Khalatnikov, and E. m. Lifshitz, "A General Solution of the Einstein Equations with a Time Singularity," *Adv. Phys.* **31** (1982) 639–667.

[64] T. Damour, M. Henneaux, and H. Nicolai, "Cosmological billiards," *Class. Quant. Grav.* **20** (2003) R145–R200, `hep-th/0212256`.

[65] A. Giveon, B. Kol, A. Ori, and A. Sever, "On the resolution of the time-like singularities in Reissner-Nordstroem and negative-mass Schwarzschild," *JHEP* **08** (2004) 014, `hep-th/0401209`.

[66] I. Seggev, "Dynamics in Stationary, Non-Globally Hyperbolic Spacetimes," *Class. Quant. Grav.* **21** (2004) 2651–2668, `gr-qc/0310016`.

[67] A. Ishibashi and A. Hosoya, "Who's afraid of naked singularities? Probing timelike singularities with finite energy waves," *Phys. Rev.* **D60** (1999) 104028, `gr-qc/9907009`.

[68] J. G. Stalker and A. S. Tahvildar-Zadeh, "Scalar waves on a naked-singularity background," *Class. Quant. Grav.* **21** (2004) 2831–2848, `gr-qc/0401011`.

[69] G. W. Gibbons, S. A. Hartnoll, and A. Ishibashi, "On the stability of naked singularities," *Prog. Theor. Phys.* **113** (2005) 963–978, `hep-th/0409307`.

[70] A. Ishibashi and R. M. Wald, "Dynamics in non-globally-hyperbolic static spacetimes. II: General analysis of prescriptions for dynamics," *Class. Quant. Grav.* **20** (2003) 3815–3826, `gr-qc/0305012`.

[71] A. Ishibashi and R. M. Wald, "Dynamics in non-globally hyperbolic static spacetimes. III: anti-de Sitter spacetime," *Class. Quant. Grav.* **21** (2004) 2981–3014, `hep-th/0402184`.

[72] M. Reed and B. Simon, "Methods of Modern Mathematical Physics. 1. Functional Analysis,". Academic Press, New York (1972).

[73] D. A. Konkowski, T. M. Helliwell, and C. Wieland, "Quantum singularity of Levi-Civita spacetimes," *Class. Quant. Grav.* **21** (2004) 265–272, `gr-qc/0401009`.

[74] H. Falomir, M. A. Muschietti, and P. A. G. Pisani, "On the resolvent and spectral functions of a second order differential operator with a regular singularity," *J. Math. Phys.* **45** (2004) 4560–4577, `math-ph/0404034`.

[75] A. B. von Keviczky, N. Saad, and R. L. Hall, "Friedrichs extensions of Schrödinger operators with singular potentials," `math-ph/0312027`.

[76] K. Kirsten, P. Loya, and J. Park, "The very unusual properties of the resolvent, heat kernel, and zeta function for the operator $-d^2/dr^2 - 1/(4r^2)$," *J. Math. Phys.* **47** (2006) 043506, `math-ph/0511002`.

[77] P. A. M. Dirac, "An Extensible model of the electron," *Proc. Roy. Soc. Lond.* **A268** (1962) 57–67.

[78] J. Hughes, J. Liu, and J. Polchinski, "Supermembranes," *Phys. Lett.* **B180** (1986) 370.

[79] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Supermembranes and eleven-dimensional supergravity," *Phys. Lett.* **B189** (1987) 75–78.

[80] M. J. Duff, P. S. Howe, T. Inami, and K. S. Stelle, "Superstrings in D = 10 from supermembranes in D = 11," *Phys. Lett.* **B191** (1987) 70.

[81] B. de Wit, J. Hoppe, and H. Nicolai, "On the quantum mechanics of supermembranes," *Nucl. Phys.* **B305** (1988) 545.

[82] A. Achucarro, J. M. Evans, P. K. Townsend, and D. L. Wiltshire, "Super p-Branes," *Phys. Lett.* **B198** (1987) 441.

[83] M. J. Duff, "Supermembranes," `hep-th/9611203`.

[84] N. Kim and J.-H. Park, "Massive super Yang-Mills quantum mechanics: Classification and the relation to supermembrane," *Nucl. Phys.* **B759** (2006) 249–282, `hep-th/0607005`.

[85] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A conjecture," *Phys. Rev.* **D55** (1997) 5112–5128, `hep-th/9610043`.

[86] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten- and eleven- dimensional supergravities," *JHEP* **03** (2003) 048, `hep-th/0211089`.

[87] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory," *JHEP* **08** (2001) 036, `hep-th/0105308`.

[88] B. de Wit, K. Peeters, and J. Plefka, "Superspace geometry for supermembrane backgrounds," *Nucl. Phys.* **B532** (1998) 99–123, `hep-th/9803209`.

[89] B. de Wit, K. Peeters, J. Plefka, and A. Sevrin, "The M-theory two-brane in AdS(4) x S(7) and AdS(7) x S(4)," *Phys. Lett.* **B443** (1998) 153–158, `hep-th/9808052`.

[90] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, "Matrix perturbation theory for M-theory on a PP-wave," *JHEP* **05** (2002) 056, `hep-th/0205185`.

[91] N. Kim, T. Klose, and J. Plefka, "Plane-wave matrix theory from N = 4 super Yang-Mills on R x S**3," *Nucl. Phys.* **B671** (2003) 359–382, `hep-th/0306054`.

[92] T. Klose, "Plane-Wave Matrix Theory: Gauge Theoretical Origin and Planar Integrability," *Ph. D. Thesis* (2005). unpublished.

[93] G. Ishiki, Y. Takayama, and A. Tsuchiya, "N = 4 SYM on R x S**3 and theories with 16 supercharges," *JHEP* **10** (2006) 007, `hep-th/0605163`.

[94] G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, "Embedding of theories with SU(2|4) symmetry into the plane wave matrix model," *JHEP* **11** (2006) 089, `hep-th/0610038`.

[95] K. Okuyama, "N = 4 SYM on R x S(3) and pp-wave," *JHEP* **11** (2002) 043, `hep-th/0207067`.

[96] M. Blau, "Killing spinors and SYM on curved spaces," *JHEP* **11** (2000) 023, `hep-th/0005098`.

[97] A. D. Popov, "Bounces/Dyons in the Plane Wave Matrix Model and SU(N) Yang-Mills Theory," `0804.3845`.

[98] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, "Protected multiplets of M-theory on a plane wave," *JHEP* **09** (2002) 021, `hep-th/0207050`.

[99] N. Kim and J.-H. Park, "Superalgebra for M-theory on a pp-wave," *Phys. Rev.* **D66** (2002) 106007, `hep-th/0207061`.

[100] D. Frank, "Deconstructed Dimensions – Natural Symmetry Breaking from Networks of Non-linear Sigma Models," *Diploma Thesis* (2003). unpublished.

[101] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory,". Reading, USA: Addison-Wesley (1995) 842 p.

[102] L. Dolan and R. Jackiw, "Gauge invariant signal for gauge symmetry breaking," *Phys. Rev.* **D9** (1974) 2904.

[103] S. R. Coleman and E. J. Weinberg, "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking," *Phys. Rev.* **D7** (1973) 1888–1910.

[104] S. R. Das, J. Michelson, and A. D. Shapere, "Fuzzy spheres in pp-wave matrix string theory," *Phys. Rev.* **D70** (2004) 026004, `hep-th/0306270`.

[105] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, `hep-th/9503124`.

[106] B. Craps, S. Sethi, and E. P. Verlinde, "A matrix big bang," *JHEP* **10** (2005) 005, `hep-th/0506180`.

[107] M. Blau and M. O'Loughlin, "DLCQ and Plane Wave Matrix Big Bang Models," `0806.3255`.

[108] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `hep-th/9510017`.

[109] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond,". Cambridge, UK: Univ. Pr. (1998) 531 p.

[110] E. Witten, "Bound states of strings and p-branes," *Nucl. Phys.* **B460** (1996) 335–350, `hep-th/9510135`.

[111] L. Susskind, "Another conjecture about M(atrix) theory," `hep-th/9704080`.

[112] N. Seiberg, "Why is the matrix model correct?," *Phys. Rev. Lett.* **79** (1997) 3577–3580, `hep-th/9710009`.

[113] A. Sen, "D0 branes on T(n) and matrix theory," *Adv. Theor. Math. Phys.* **2** (1998) 51–59, `hep-th/9709220`.

[114] M. Blau and M. O'Loughlin, "Multiple M2-Branes and Plane Waves," `0806.3253`.

[115] S. R. Das and J. Michelson, "pp wave big bangs: Matrix strings and shrinking fuzzy spheres," *Phys. Rev.* **D72** (2005) 086005, `hep-th/0508068`.

[116] J. Zinn-Justin, "Quantum field theory and critical phenomena," *Int. Ser. Monogr. Phys.* **77** (1989) 1–914.

[117] S. R. Das and J. Michelson, "Matrix membrane big bangs and D-brane production," *Phys. Rev.* **D73** (2006) 126006, `hep-th/0602099`.