Transverse Emittance Reduction with Tapered Foil

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Abstract

The idea of reducing transverse emittance with tapered energy-loss foil is proposed by J.M. Peterson in 1980s and recently by B. Carlsten. In this paper, we present the physical model of tapered energy-loss foil and analyze the emittance reduction using the concept of eigen emittance. The study shows that, to reduce transverse emittance, one should collimate at least 4% of particles which has either much low energy or large transverse divergence. The multiple coulomb scattering is not trivial, leading to a limited emittance reduction ratio.

I. Introduction

Small transverse emittances are of essential importance for the accelerator facilities generating free electron lasers, especially in hard X-ray region. The idea of reducing transverse emittance with tapered energy-loss foil is recently proposed by B. Carlsten [1], and can be traced back to J.M. Peterson's work in 1980s [2]. Peterson illustrated that a transverse energy gradient can be produced with a tapered energy-loss foil which in turn leads to transverse emittance reduction, and also analyzed the emittance growth from the associated multiple coulomb scattering. However, what Peterson proposed was rather a conceptual than a practical design. In this paper, we build a more complete physical model of the tapered foil based on Ref. [2], including the analysis of the transverse emittance reduction using the concept of eigen emittance and confirming the results by various numerical simulations. The eigen emittance equals to the projected emittance when there is no cross correlation in beam's second order moments matrix [3]. To calculate the eigen emittances, it requires only to know the beam distribution at the foil exit. Thus, the analysis of emittance reduction and the optics design of the subsequent beam line section can be separated. In addition, we can combine the effects of multiple coulomb scattering and transverse energy gradient together in the beam matrix and analyze their net effect. We find that,when applied to an electron linac or electron beam line, the energy spread increase and angular growth due to multiple scattering are not trivial; as a result, the transverse emittance can only be reduced with a limited ratio, e.g. down to about 65% the original value.

The contents of this paper are arranged as follows. In Sec. II, we build the physical model of the tapered foil, derive the transverse eigen emittance and discuss the emittance reduction criteria. In Sec. III, we implement numerical simulations to verify the physical model; and in Sec. IV, we present numerical experiments and subsequent beam line to remove the transverse energy gradient to demonstrate the applicability of such method. Conclusion are given in the last section.

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II. Physical model

Let us consider the electron beam distribution passing through a tapered energy-loss foil, whose geometry is shown in Fig. 1. For convenience, we place the foil where the beam's phase space distributions in three planes are all up-right ellipses. Thus, the second order moments beam matrix (For convenience, only show x, z planes) is

$$\Sigma_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{1}$$

where the 2-by-2 matrices A and B are

$$A = \begin{pmatrix} \sigma_{x0}^2 & 0 \\ 0 & \sigma_{x'0}^2 \end{pmatrix}, B = \begin{pmatrix} \sigma_{z0}^2 & 0 \\ 0 & \sigma_{d0}^2 \end{pmatrix}.$$

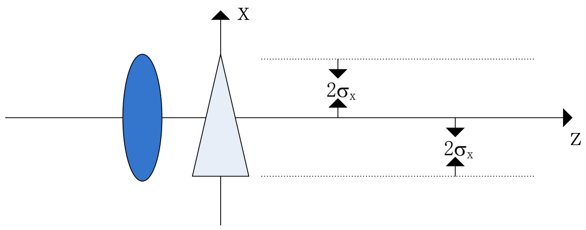

Fig. 1 Geometry of the tapered energy-loss foil.

On exiting the foil, particles at and greater than $+2\sigma_x$ will not lose energy, while particles at x =0 lose energy E_0*ΔF and particles at $-2\sigma_x$ loss energy $2E_0$*ΔF, with E_0 being the beam average energy at the entrance of the foil, ΔF the average energy loss factor of particles with x=0.

Therefore it produces a linear relationship between momentum and x position around the new mean momentum at the exit of the foil, as illustrated in Fig. 2. The correlation can be represented as

$$X_f = M_f X_0, \tag{2}$$

with X = (x, x', z, δ)^T, and M_f is 4-by-4 matrix, with diagonal elements to be 1 and M_f (4,1) = $\Delta F/2\sigma_x$. The slight downward net shift of energy distribution is ignored in Eq. (2).

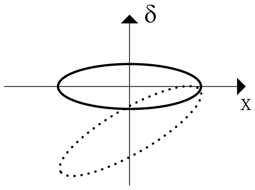

Fig. 2 The beam distribution before (solid line) and after (dashed line) the foil.

The beam matrix after the foil is then calculated by

$$\Sigma = M_f \Sigma_0 M_f^T. \tag{3}$$

Meanwhile, multiple coulomb scattering induces additional energy spread $\Delta\sigma_\delta^2$ and transverse divergence $\Delta\sigma_{x'}^2$. Although there is weak correlation between x' and δ for 100% particles due to coulomb scattering, for the central (~96%) particles the correlations is negligible. Therefore these two effects are taken into account by directly adding $\Delta\sigma_\delta^2$ and $\Delta\sigma_{x'}^2$ to the beam matrix Σ,

$$\Sigma_f = \begin{pmatrix} A_f & C_f \\ C_f & B_f \end{pmatrix}, \tag{4}$$

where the 2-by-2 matrices A_f, B_f, C_f are

$$A_f = \begin{pmatrix} \sigma_{x0}^2 & 0 \\ 0 & \sigma_{x0'}^2 + \Delta\sigma_{x'}^2 \end{pmatrix},$$

$$B_f = \begin{pmatrix} \sigma_{z0}^2 & 0 \\ 0 & \sigma_{\delta 0}^2 + \frac{\Delta F^2}{4} + \Delta\sigma_\delta^2 \end{pmatrix},$$

$$C_f = \begin{pmatrix} 0 & \frac{\Delta F \sigma_{x0}}{2} \\ 0 & 0 \end{pmatrix}.$$

The eigen emittance can be obtained by calculating the eigen values of the $J_4\Sigma_f$ [3], J_4 is the four dimensional unit symplectic matrix,

$$J_4 = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \tag{5}$$

Where the 2-by-2 matrix J is

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{6}$$

The horizontal eigen emittance is in the form

$$\varepsilon_{x,eig} = \frac{1}{2\sqrt{2}}\sqrt{\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}} = \frac{1}{2\sqrt{2}}\sqrt{\mathcal{A}}\sqrt{1 - \sqrt{1 - \mathcal{B}^2/\mathcal{A}^2}}, \tag{7}$$

with $\mathcal{A} = 4(\varepsilon_{x0}^2 + \varepsilon_{z0}^2) + 4(P + Q) + G$,

$\mathcal{B} = 8\sqrt{(\varepsilon_{x0}^2 + P)(\varepsilon_{z0}^2 + Q)}$,

$G = \sigma_{z0}^2 \Delta F^2 \gamma^2, P = \sigma_{x0}^2 \Delta\sigma_{x'}^2 \gamma^2, Q = \sigma_{z0}^2 \Delta\sigma_\delta^2 \gamma^2$.

where G, P and Q are related to the transverse energy gradient, angular growth and energy spread increase, respectively; γ is the Lorenz factor; $\varepsilon_{x0} = \gamma\sigma_{x0}\sigma_{x0'}, \varepsilon_{z0} = \gamma\sigma_{z0}\sigma_{\delta 0}$ are the initial normalized x and z emittances.

Note that P, Q and G are all positive, therefore on the RHS of Eq. (7), $\frac{1}{2\sqrt{2}}\sqrt{\mathcal{A}} \geq \sqrt{\frac{\varepsilon_{x0}^2 + \varepsilon_{z0}^2}{2}}$ $\geq \min(\varepsilon_{x0}, \varepsilon_{z0})$. Both ε_{x0} and ε_{z0} are assumed to be larger than the target x eigen emittance. In other cases, like, ε_{z0} is smaller than the target x eigen emittance, one can make emittance exchange between x and z planes to realize the transverse emittance reduction, which is not the interest of the present study.

To achieve small emittance, it requires

$$(\frac{\mathcal{B}}{\mathcal{A}})^2 = \frac{64(\varepsilon_{x0}^2+P)(\varepsilon_{z0}^2+Q)}{(4(\varepsilon_{x0}^2+\varepsilon_{z0}^2)+4(P+Q)+G)^2} \ll 1, \quad (8)$$

so that

$$\varepsilon_{x,eig} \approx \frac{1}{4}\frac{\mathcal{B}}{\sqrt{\mathcal{A}}} < \min(\varepsilon_{x0},\ \varepsilon_{z0}). \quad (9)$$

Due to coulomb multiple scattering, a few percent (~2%) of particles have large transverse divergence and away from the beam core in x, x' phase space, and similarly in z, δ phase space. In practice, these particles (totally about 4% percent) should be collimated. In what follows, we will show the expressions of G, P and Q for the remaining particles after the 4% collimation.

For the central 98% particles around the mean momentum, the average energy loss factor (denotes as ΔF_{98}) and energy spread increase depend on the relative foil thickness $df = L_f/L_0$, beam energy, and weakly on the foil material. For Carbon foil, it is empirically found that (refer to Sec. III, numerical simulations),

$$\Delta F_{98} = <\frac{\Delta\gamma}{\gamma}> (98\%) \approx \frac{L_f}{L_0}\frac{83}{\gamma} = \frac{83}{\gamma}df, \quad (10)$$

$$\Delta\sigma_\delta^2(98\%) \approx (\frac{35.5}{\gamma}\frac{L_f}{L_0})^2 = (\frac{35.5}{\gamma}df)^2. \quad (11)$$

These two equations are valid for df of 10^{-5} to 10^{-3} and beam energy of 100 MeV to 12 GeV. In the cases of $df > 10^{-3}$, the dependence of ΔF_{98} and $\Delta\sigma_\delta^2$ (98%) on the $1/\gamma$ are no longer linear. Although ΔF_{98} will increase, the $\Delta\sigma_\delta^2$ (98%) will increase more quickly than ΔF_{98} with the increasing df, leading to emittance growth in three planes.

For the central 98% particles, the angular growth using Gaussian approximation [4] is

$$< \theta^2 > (98\%) = (\frac{26.6}{\gamma})^2(1 + 0.038\ln(df))^2 df. \quad (12)$$

We notice that the value of $(1 + 0.038\ln(df))^2$ varies little with df, therefore approximate this term with a constant value 4/9 (it is accurate to 10% for $10^{-6} < df < 10^{-3}$). Then the angular growth approximately linearly depends on df,

$$\Delta\sigma_{x'}^2 = \frac{1}{2} < \theta^2 > \approx \frac{157.4}{\gamma^2}df. \quad (13)$$

Here we make an approximation that $\Delta\sigma_\delta^2$ (98%) and $\Delta\sigma_{x'}^2$ do not depend on x, which greatly simplifies the analysis while does not lead to large disparity between the physical model and real circumstance (see below).

Submit Eqs. (10), (11) and (13) into Eq. (7), we have

$$G \approx 6890\sigma_{z0}^2 df^2, P \approx 157.4\sigma_{x0}^2 df, Q \approx 1260\sigma_{z0}^2 df^2 \approx \frac{G}{5.5}, \quad (14)$$

From Eq. (8), to achieve small x eigen emittance, the most effective way is to increase G and minimize P and Q. However, Q changes monotonically with G, increasing G leads to large Q at the same time, which in turn limits the available minimal $\varepsilon_{x,eig}$ to be about 65% of ε_{x0} (Appendix A).

Let us consider a case that the beam's normalized emittances are 0.7/0.7/1.4 μm, β_x = 1 cm at the foil, rms bunch length σ_z = 0.5 mm. With these parameters, the x eigen emittance with respect to the relative foil thickness *df* and beam energy is shown in Fig. 3. To achieve $\varepsilon_{x,eig}$ = 0.5 μm, it requires energy greater than 1.845 GeV and relative foil thickness *df* about 2×10^{-4}. It is really hard to reduce $\varepsilon_{x,eig}$ close to the theoretical minimal $\varepsilon_{x,eig}$ (~0.455 μm), unless we collimate more particles, in that case, Eqs. (10-13) should be modified, too.

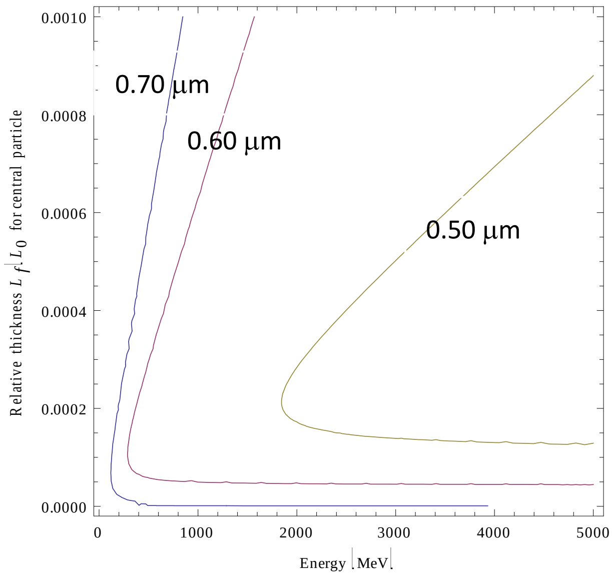

Fig. 3 X eigen emittance with respect to energy and relative foil thickness *df* = L_f/L_0..

III. Numerical Simulations

We verify the above analysis by making plenty of simulations with G4beamline, a particle tracking and simulation program well suited for beam-material interactions [5].

A. Foil with uniform thickness

First, we simulate the case of mono-energetic electron beam with zero initial transverse divergence passing through foil with uniform thickness. The result is shown in Fig. 4. A few percent (~4%) of particles have large divergence and low energy. The mean energy loss factor and transverse divergence of 100% and 98% central particles are calculated for different foil thickness, as shown in Figs. 5. The average energy loss factor of all particles ΔF_{100} approximately equals to *df* [4] and ΔF_{98} changes linearly with *df*. The angular distribution of central 98% particles agrees with Eq. (12) fairly well, however, the simulation results become a little larger than the analytical prediction when *df* is greater than 1×10^{-3}. Varying the beam energy, we observe the linear dependence of ΔF_{98} and $\Delta\sigma_\delta$ (98%) on $/\gamma = E_0/E$ (E_0 is the electron rest mass energy), as shown in Fig. 6, from which, we obtain Eqs. (10) and (11) by fitting the data.

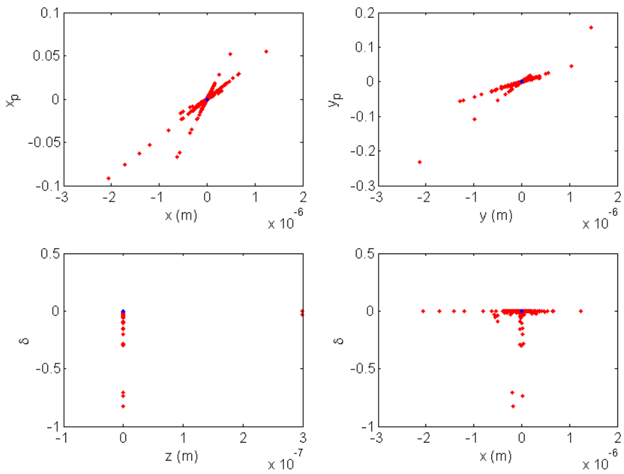

Fig.4 Electron beam distribution after passing through a uniform Carbon foil. Beam energy E = 172 MeV. Foil thickness is L_f = 1.5 μm. Radiation length of Carbon L_0 = 18.8 cm.

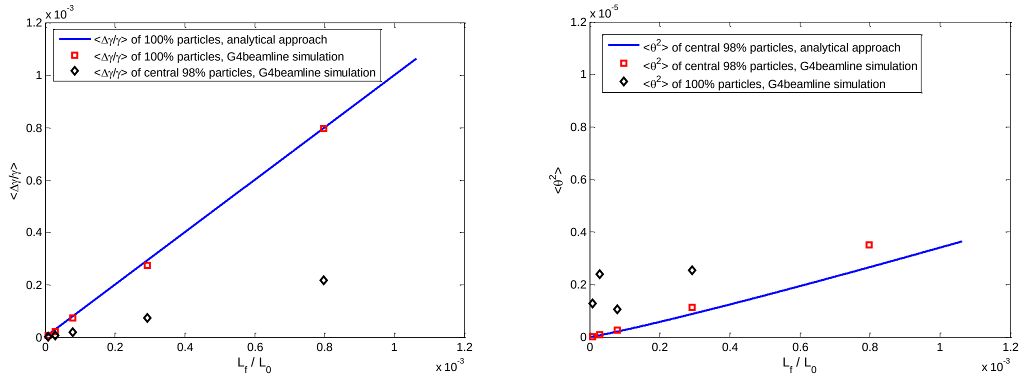

Fig. 5 Average energy loss factor (left) and average angular growth (rigth) with respect to relative foil thickness *df* = L_f/L_0.

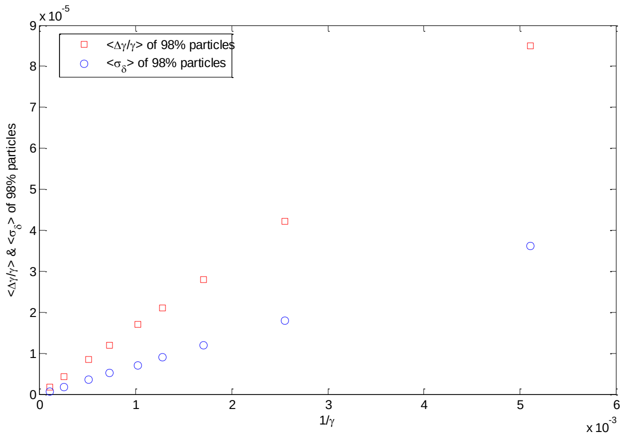

Fig. 6 ΔF_{98} and $\Delta\sigma_\delta$ (98%) with respect to $1/\gamma$, while keeping the relative foil thickness constant, *df* = $L_f/L_0 = 2.2\times10^{-4}$.

B. Tapered foil

We then simulate an electron beam with relative practical parameters passing through a tapered foil (see Fig. 1). The incident beam emittances are 0.7/0.7/1.4 μm, beam energy E = 300 MeV, σ_x = 50 μm, and phase space distributions are all up-right ellipse in three planes. A carbon tapered foil with middle thickness of 100 μm (*df* = 5.3×10^{-4}) is used in the simulation. Figs. 7 presents the average energy loss factor ΔF_{98} and angular growth $<\theta^2>$ of central 98% particles in different x slices. It shows that the physical model (Eqs. (10), (12)) keeps well when counting the x-sliced energy loss and angular growth.

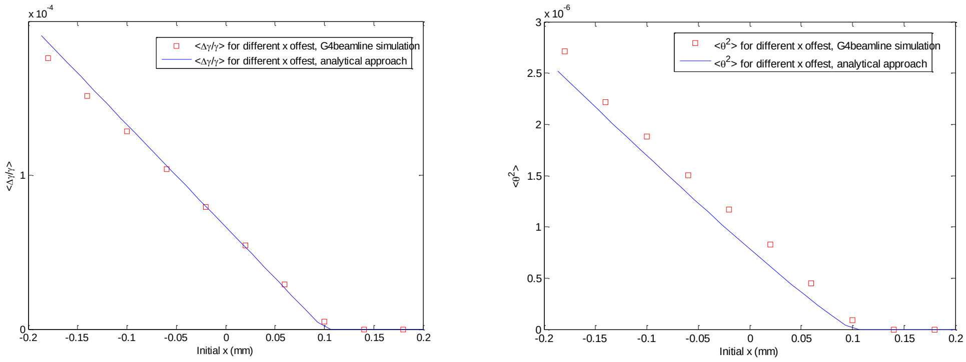

Fig. 7 Average energy loss factor and average angular growth of central 98% particles in different x slices.

IV. Numerical experiment

Here we show a numerical experiment. The main parameters of the beam and the foil are listed in Table 1. We collimate 4% particles with large transverse divergence and energy loss using collimators after the foil. Fig. 8 presents the beam distribution before and a distance after the foil. The x eigen emittance calculated from the beam distribution at the exit of the foil is 0.543 μm, agrees fairly well with that calculated from the physical model 0.498 μm.

Table 1 parameter setting

Parameter	Amplitude	Unit
Beam energy E	2000	MeV
Normalized emittance at foil entrance, $\varepsilon_{x0}/\ \varepsilon_{y0}/\ \varepsilon_{z0}$	0.7/0.7/1.4	μm
β_x/β_y at foil entrance	1/1	cm
Bunch length at foil entrance σ_z	0.5	mm
Relative foil thickness for the central electron, L_f/L_0	2×10^{-4}	
Foil material	Carbon	
X eigen emittance after foil, analytical approach	0.498	μm
X eigen emittance after foil, G4beamline simulation	0.543	μm

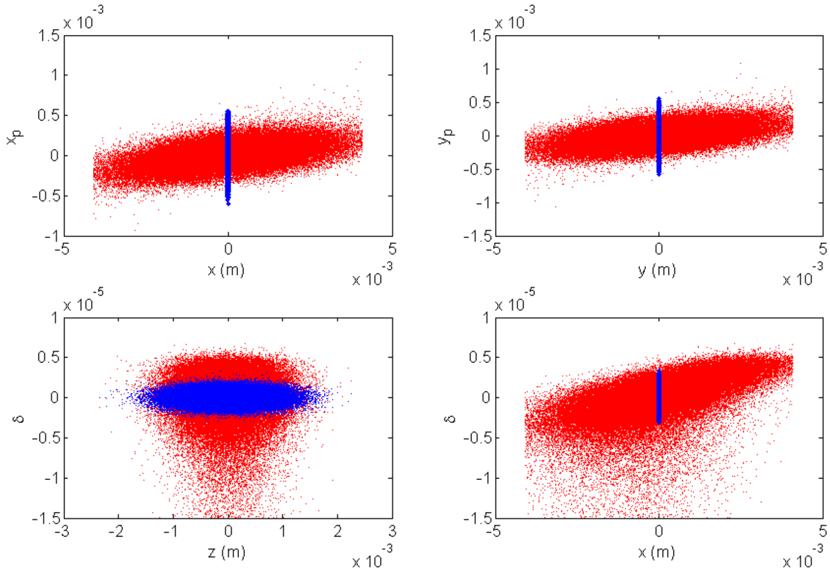

Fig. 8 Beam distribution before (blue plots) and a distance after (red plots) the carbon tapered energy-loss foil.

We also simulate the case of a partially tapered foil, whose thickness varies linearly with x in the range of $[-\sigma_x, \sigma_x]$ and constant out of this range, with the geometry shown in Fig. 9. Consider 96% central particles, the average energy loss factor and angular growth of difference x slice after the foil are shown in Fig. 10. The calculated x eigen emittance from the beam distribution at the exit of the foil is 0.47 μm, 14% smaller than that produced by a purely tapered foil.

Fig. 11 presents a 15.1m long beam line section, including four dipoles and four transverse deflecting cavities to remove the x, δ correlations. Because the beam is partially x, δ correlated, the correlation can not be completely removed, as shown in Fig. 12. Fig. 13 shows the beam distribution and x slice emittances at the exit of this beam line section, which accord with the x eigen emittance calculation result quite well.

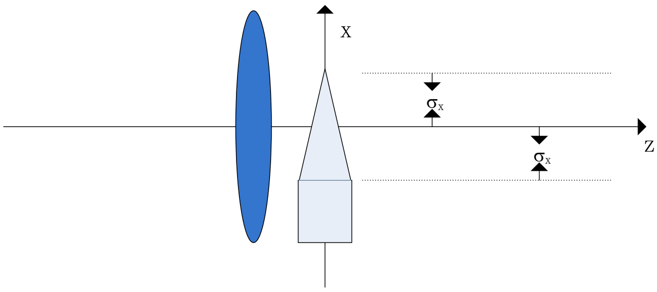

Fig. 9 Geometry of the energy-loss foil.

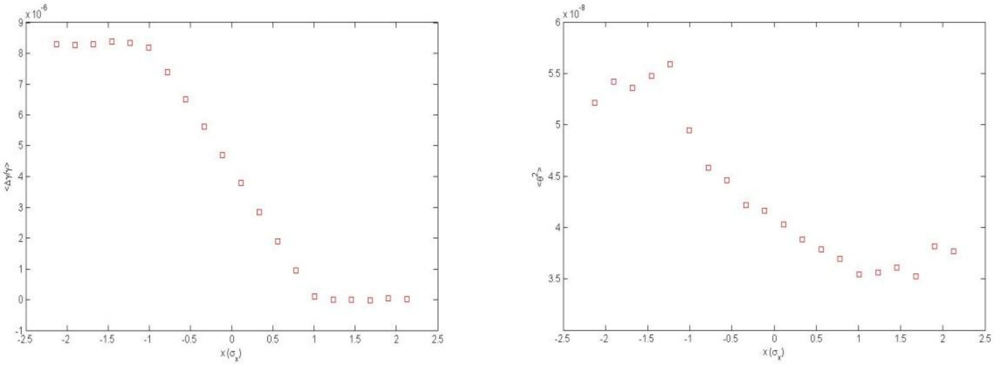

Fig. 10 Average energy loss factor and average angular growth of central 98% particles with respect to different x offset of particles.

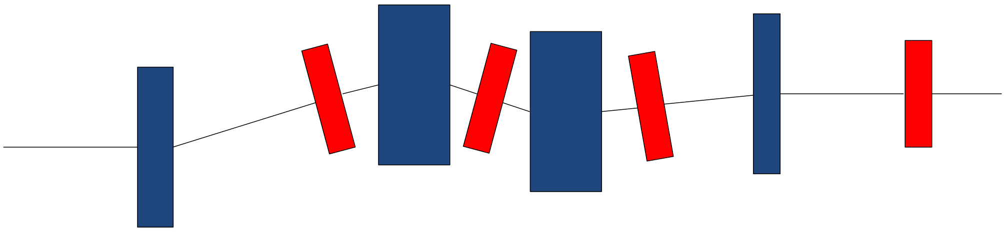

Fig. 11 Beam line section to remove x, δ correlation, including 4 dipoles (blue) and 4 transverse deflecting cavities (red).

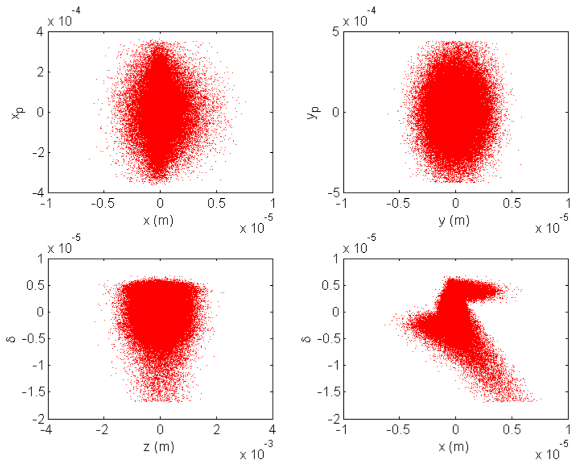

Fig. 12 Beam distribution after the correlation removing beam line section

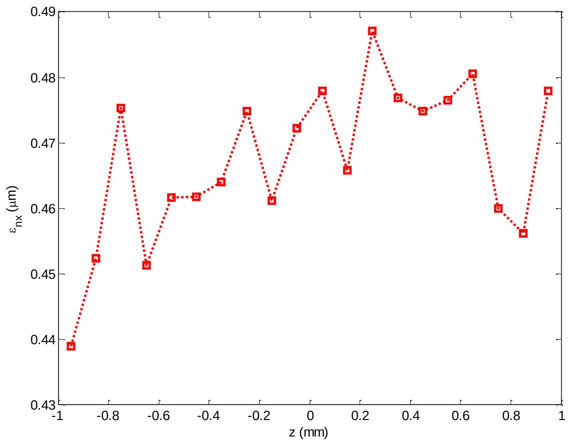

Fig. 13 Normalized x sclice emittance

V. Conclusion

The idea of reducing transverse eigen emittance with tapered energy-loss foil is analyzed in this paper. The multiple coulomb scattering will cause about 4% of particles losing too much energy or having too large transverse divergence, which need to be collimated in practice. For the remaining particles, the average energy loss factor, angular growth and energy spread growth mainly depend on the beam energy and relative foil thickness, and weakly on foil material. We build a physical model of the tapered foil, and find that the energy spread growth due to multiple coulomb scattering is not trivial, but comparable to the transverse energy gradient; as a result, the available minimal x eigen emittance is only about 65% of ε_{x0}. Numerical experiments and subsequent beam line design to remove the transverse energy gradient are presented to demonstrate our analysis.

Acknowledgment
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[4] A.W. Chao and M. Tigner, Handbook of Accelerator Physics and Enginering, 3rd printing, World Scienfific, 2006, p. 242-243, 274-276.
[5]G4beamline website: http://www.muonsinc.com/tic-index.php?page=G4beamline.

Appendix A

Let us rewrite Eq. (8-9),

$$(\frac{\mathcal{B}}{\mathcal{A}})^2 = \frac{64(\varepsilon_{x0}^2+P)(\varepsilon_{z0}^2+Q)}{(4(\varepsilon_{x0}^2+\varepsilon_{z0}^2)+4(P+Q)+G)^2} \ll 1, \quad (A1)$$

$$\varepsilon_{x,eig} \approx \frac{1}{4}\frac{\mathcal{B}}{\sqrt{\mathcal{A}}} < \min(\varepsilon_{x0},\ \varepsilon_{z0}), \quad (A2)$$

and recall that, G, P, Q are related to the transverse energy gradient, angular growth and energy spread increase, respectively; and

$$Q \approx G/5.5. \quad (A3)$$

To achieve small $\varepsilon_{x,eig}$, it is natural to require the contribution of G much larger than P, i.e.

$$\frac{G}{P} = \frac{43.8\ df\ \gamma\ \sigma_z^2}{\varepsilon_{x0}\beta} \gg 1. \quad (A4)$$

For instance, $df = 1\times10^{-4}$, σ_z = 500 μm, β = 1 cm, ε_{x0} = 0.6 μm, it needs $\gamma >> 5.5$.

Divide the numerator and denominator of (A1) with G^2, we then have

$$(\frac{\mathcal{B}}{\mathcal{A}})^2 = \frac{64(\varepsilon_{x0}^2/G+P/G)(\varepsilon_{z0}^2/G+Q/G)}{(4(\varepsilon_{x0}^2/G+\varepsilon_{z0}^2/G)+4(P/G+Q/G)+1)^2} \approx \frac{4(4\varepsilon_{x0}^2/G)(4\varepsilon_{z0}^2/G+4/5.5)}{(4\varepsilon_{x0}^2/G+4\varepsilon_{z0}^2/G+4/5.5+1)^2} \ll 1, \quad (A5)$$

where conditions (A3) and (A4) are used in the deviation.

Fig. A1 shows the value of $\mathcal{B}^2/\mathcal{A}^2$ with respect to $4\varepsilon_{x0}^2/G$ for $1/2 < \varepsilon_{z0}/\varepsilon_{x0} < 10$. There are two cases satisfying $\mathcal{B}^2/\mathcal{A}^2 << 1$,

Case 1, $4\varepsilon_{x0}^2/G << 1$ and $4\varepsilon_{z0}^2/G << 1$

From Fig. A1 and condition (A5),

$$(\frac{\mathcal{B}}{\mathcal{A}})^2 \approx 3.9\varepsilon_{x0}^2/G << 1 \quad (A6)$$

Submit it to (A2), we have

$$\varepsilon_{x,eig} \approx \frac{1}{4}\sqrt{\mathcal{A}}\frac{\mathcal{B}}{\mathcal{A}} \approx 0.49\ \varepsilon_{x0}\sqrt{\frac{\mathcal{A}}{G}} \approx 0.49\ \varepsilon_{x0}\sqrt{1+\frac{4}{5.5}+\frac{4(\varepsilon_{x0}^2+\varepsilon_{z0}^2)}{G}} \geq 0.65\varepsilon_{x0} \quad (A7)$$

Case 2, $4\varepsilon_{x0}^2/G \geq 1$ and $\varepsilon_{z0}/\varepsilon_{x0}$ well above 1

In this case,

$$(\frac{\mathcal{B}}{\mathcal{A}})^2 \approx \frac{4(4\varepsilon_{x0}^2+P)(4\varepsilon_{z0}^2+4G/5.5)}{(4\varepsilon_{x0}^2+4\varepsilon_{z0}^2+9.5G/5.5)^2} \approx \frac{4\varepsilon_{x0}^2}{\varepsilon_{z0}^2} \quad (A8)$$

Submit it to (A2), we have

$$\varepsilon_{x,eig} \approx \frac{1}{4}\sqrt{\mathcal{A}}\frac{\mathcal{B}}{\mathcal{A}} \approx \frac{1}{2}\frac{\varepsilon_{x0}}{\varepsilon_{z0}}\sqrt{\mathcal{A}} = \varepsilon_{x0}\sqrt{1+\frac{\varepsilon_{x0}^2}{\varepsilon_{z0}^2}+\frac{(P+Q)}{\varepsilon_{z0}^2}} \geq \varepsilon_{x0} \quad (A9)$$

In conclusion, the available minimal x eigen emittance will be 65% of ε_{x0}.

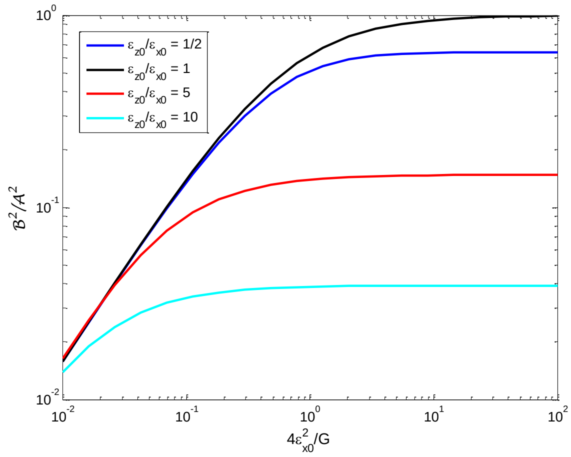

Fig. A1 Ratio of $\mathcal{B}^2/\mathcal{A}^2$ with respect to $4\varepsilon_{x0}{}^2/G$, provided G>>P.