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Entropy relations and the application of black holes with the cosmological constant and Gauss–Bonnet term

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ABSTRACT

Based on entropy relations, we derive the thermodynamic bound for entropy and the area of horizons for a Schwarzschild–dS black hole, including the event horizon, Cauchy horizon, and negative horizon (i.e., the horizon with negative value), which are all geometrically bound and comprised by the cosmological radius. We consider the first derivative of the entropy relations to obtain the first law of thermodynamics for all horizons. We also obtain the Smarr relation for the horizons using the scaling discussion. For the thermodynamics of all horizons, the cosmological constant is treated as a thermodynamic variable. In particular, the thermodynamics of the negative horizon are defined well in the $r < 0$ side of space–time. This formula appears to be valid for three-horizon black holes. We also generalize the discussion to thermodynamics for the event horizon and Cauchy horizon of Gauss–Bonnet charged flat black holes because the Gauss–Bonnet coupling constant is also considered to be thermodynamic variable. These results provide further insights into the crucial role played by the entropy relations of multi-horizons in black hole thermodynamics as well as improving our understanding of entropy at the microscopic level.

1. Introduction

In order to understand the entropy of black holes at the microscopic level, the entropy product of multi-horizon black holes has been investigated widely in many previous studies [1–24]. The entropy product is always independent of the mass of black holes, which is universal for many charged and rotating black holes [1–13], black rings, and black strings [14]. In fact, the entropy product, when combined with Cauchy horizon thermodynamics, can be used to determine whether the corresponding Bekenstein–Hawking entropy can be written as a Cardy formula, thereby providing some evidence for a CFT description of the corresponding microstates [14,15]. Therefore, it is important to study the thermodynamics of the Cauchy horizon.

However, the mass-independence of the entropy product fails for some multi-horizon black holes [15–19]. Hence, the entropy sum [12,13,16,20,23] and other thermodynamic relations [16,17,20–22,24] are introduced, which are also mass-independent in some cases and they appear to be universal. In particular, this applies to the relation $T_+S_+ = T_-S_-$, which is closely associated with the mass-independence of the entropy product. This can also be understood well in physical terms by the holographic description, i.e., the thermodynamic method of black hole/CFT (BH/CFT) correspondence [7,25–30]. The thermodynamic relations $T_+S_+ = T_-S_-$ may used as criteria to determine whether there is a two-dimensional CFT dual for black holes in the Einstein gravity theory and other diffeomorphism-invariant gravity theories [7,25–30]. In addition, it has been shown that the thermodynamic relation $T_+S_+ = T_-S_-$ is equivalent to the central charge being the same (i.e. $c_R = c_L$) for some two-horizon black holes. However, the interpretations of other thermodynamic relations are not clear. Thus, the present study focuses on entropy relations and their applications to black hole thermodynamics.

Based on entropy relations, we derive the thermodynamic bound for entropy and the area of horizons for a Schwarzschild–dS black hole, including the event horizon, Cauchy horizon, and negative horizon, which are all geometrically bound and they comprise the cosmological radius. We consider the first derivative of entropy relations to obtain the first law of thermodynamics for all horizons. We also obtain the Smarr relation for horizons using the scaling discussion. For the thermodynamics of all horizons, the cosmolog-

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ical constant is treated as a thermodynamic variable [31–38]). In particular, the thermodynamics of a negative horizon is also well defined on the negative side ($r < 0$). Indeed, there is a singularity for the black hole solution, e.g., located in $r = 0$, so we always select the $r > 0$ side and the existence of black hole horizons avoids the bare singularity. This defines the thermodynamics of positive horizons well. Actually, the thermodynamics of the event horizon [4,7,9,10,15,18,21,24–28,30], Cauchy horizon [4,7,9,10,14,15,18,21,22,24–28,30], and cosmological horizon [39–42] have been studied widely. For the negative horizon, we can choose the $r < 0$ side and the existence of negative horizons also avoids the bare singularity. However, it was shown that the mass-independence of entropy relations may always hold only when the effects of negative horizons are considered [12,13,16,17,20,23,24]. Thus, it is also useful to study the thermodynamics of negative horizons, although the quantum mechanical degrees of freedom remain unclear for the entropy of the negative horizon count (note that they are also unclear for the Cauchy horizon). This formula appears to be valid for three-horizon black holes. We also generalize this discussion to the thermodynamics of the event horizon and the Cauchy horizon of Gauss–Bonnet charged flat black holes because the Gauss–Bonnet coupling constant is also considered to be a thermodynamic variable [35–38,43,44]. These results provide further insights into the crucial roles played by the entropy relations of multi-horizons in black hole thermodynamics and they help us to understand entropy at the microscopic level.

The remainder of this paper is organized as follows. In Section 2, we investigate the entropy relations and their application to a Schwarzschild-dS black hole. In Section 3, we present the entropy relations and their application to a Gauss–Bonnet charged flat black hole. Section 4 gives our conclusions and some discussion.

2. Entropy relations and the application to a Schwarzschild–dS black hole

In this section, we first consider the entropy relations and their application to a four-dimensional Schwarzschild–dS black hole, which is the simplest example of a multi-horizon (A)dS black hole with the line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2 d\varphi^2\right), \tag{2.1}$$

with the horizon function

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{2.2}$$

where M represents the mass of the black hole and $\Lambda = \frac{1}{\ell^2}$ is the cosmological constant. From the roots of the horizon function $f(r)$, we can obtain three black hole horizons [17]

$$r_E = 2\ell \sin\left(\frac{1}{3}\arcsin\left(\frac{3M}{\ell}\right)\right),$$
$$r_C = 2\ell \sin\left(\frac{1}{3}\arcsin\left(\frac{3M}{\ell}\right) + \frac{2\pi}{3}\right),$$
$$r_N = 2\ell \sin\left(\frac{1}{3}\arcsin\left(\frac{3M}{\ell}\right) - \frac{2\pi}{3}\right),$$

where r_E, r_C, and r_N denote the event horizon, cosmological horizon, and negative horizon, respectively. Note that r_N is negative and it is referred to as the "virtual" horizon [17]. In addition, because we focus on a black hole with multi-horizons, we only consider the case with

$$\frac{3M}{\ell} \leq 1, \tag{2.3}$$

which can be viewed as the existence condition for multi-horizon black holes. The entropy of each horizon is

$$S_i = \frac{A_i}{4} = \pi r_i^2 \quad (i = E, C, N). \tag{2.4}$$

The temperatures of the event horizon and negative horizon are

$$T_i = \frac{f'(r_i)}{4\pi} = \frac{\ell^2 - r_i^2}{4\pi \ell^2 r_i} \quad (i = E, N), \tag{2.5}$$

and the Hawking temperature of the cosmological horizon is always selected as the positive one [7]

$$T_C = -T_E|_{r_E \leftrightarrow r_C} = \frac{r_C^2 - \ell^2}{4\pi \ell^2 r_C}, \tag{2.6}$$

where $f'(r)$ denotes the derivative function of $f(r)$ with respect to r.

First, we revisit the thermodynamic relations. For example, the mass-dependence entropy product is [17]

$$S_E S_C S_N = \frac{36\pi^3 M^2}{\Lambda^2} = 36\pi^3 M^2 \ell^4; \tag{2.7}$$

the mass-independence "part" entropy product is [16]

$$S_E S_C + S_E S_N + S_C S_N = \frac{9\pi^2}{\Lambda^2} = 9\pi^2 \ell^4; \tag{2.8}$$

the entropy sum is [12,16]

$$S_E + S_C + S_N = \frac{6\pi}{\Lambda} = 6\pi \ell^2; \tag{2.9}$$

and the mass-independent entropy relation of two positive horizons is [16,17]

$$S_E + S_C + \sqrt{S_E S_C} = 3\pi \ell^2. \tag{2.10}$$

Based on these entropy relations, we can obtain the thermodynamic bound for a Schwarzschild–dS black hole. Because $0 \leq r_E \leq r_C \leq |r_N| \leq 2\ell$, we obtain $0 \leq S_E \leq S_C \leq S_N \leq 4\pi \ell^2$. Thus, from thermodynamic relation (2.10), we obtain

$$0 \leq 3S_E \leq \left(S_E + S_C + \sqrt{S_E S_C}\right) = 3\pi \ell^2 \leq 3S_C,$$

and

$$0 \leq S_C \leq 3\pi \ell^2,$$

which together give

$$0 \leq S_E \leq \pi \ell^2 \leq S_C \leq 3\pi \ell^2.$$

In addition, thermodynamic relation (2.10) also leads to $0 \leq (S_C + S_E) \leq 3\pi \ell^2$; hence, from the entropy sum (2.9), we find that

$$S_N \geq 3\pi \ell^2.$$

Overall, we obtain the entropy bound of the event horizon, the cosmological horizon, and the negative horizon

$$S_E \in \left[0, \pi \ell^2\right], \qquad S_C \in \left[\pi \ell^2, 3\pi \ell^2\right],$$
$$S_N \in \left[3\pi \ell^2, 4\pi \ell^2\right]. \tag{2.11}$$

Furthermore, the area entropy leads to the area bound

$$\sqrt{\frac{A_E}{16\pi}} \in \left[0, \frac{\ell}{2}\right], \qquad \sqrt{\frac{A_C}{16\pi}} \in \left[\frac{\ell}{2}, \sqrt{\frac{3}{4}}\ell\right],$$
$$\sqrt{\frac{A_N}{16\pi}} \in \left[\sqrt{\frac{3}{4}}\ell, \ell\right], \tag{2.12}$$

which are all geometrical bounds of black hole horizons as parameter ℓ is actually the cosmological radius.

Furthermore, we can obtain the first law of thermodynamics from these thermodynamic relations. In fact, the thermodynamics of (A)dS black holes are still open questions. An interesting idea is to treat the cosmological constant as a thermodynamic variable (e.g., see [31–38]). Hence, if we consider the first derivatives of thermodynamic relations (2.7), (2.8), (2.9), we can find

$$S_C S_N \mathrm{d}S_E + S_E S_C \mathrm{d}S_N + S_N S_E \mathrm{d}S_C = 72\pi^3\left(\frac{M}{\Lambda^2}\mathrm{d}M - \frac{M^2}{\Lambda^3}\mathrm{d}\Lambda\right),$$
$$(S_C\mathrm{d}S_E + S_E\mathrm{d}S_C) + (S_C\mathrm{d}S_N + S_N\mathrm{d}S_C) + (S_E\mathrm{d}S_N + S_N\mathrm{d}S_E) = -\frac{18\pi^2}{\Lambda^3}\mathrm{d}\Lambda,$$
$$\mathrm{d}S_E + \mathrm{d}S_C + \mathrm{d}S_N = -\frac{6\pi}{\Lambda^2}\mathrm{d}\Lambda,$$

which lead to

$$\mathrm{d}S_E = -\frac{72\pi^3 M}{(S_E - S_N)(S_C - S_E)\Lambda^2}\mathrm{d}M + \frac{6\pi(12\pi^2M^2 - 3\pi S_E + \Lambda S_E^2)}{(S_E - S_N)(S_C - S_E)\Lambda^3}\mathrm{d}\Lambda,$$
$$\mathrm{d}S_C = \frac{72\pi^3 M}{(S_C - S_N)(S_C - S_E)\Lambda^2}\mathrm{d}M - \frac{6\pi(12\pi^2M^2 - 3\pi S_C + \Lambda S_C^2)}{(S_C - S_N)(S_C - S_E)\Lambda^3}\mathrm{d}\Lambda,$$
$$\mathrm{d}S_N = \frac{72\pi^3 M}{(S_E - S_N)(S_C - S_N)\Lambda^2}\mathrm{d}M - \frac{6\pi(12\pi^2M^2 - 3\pi S_N + \Lambda S_N^2)}{(S_E - S_N)(S_C - S_N)\Lambda^3}\mathrm{d}\Lambda,$$

or equivalently

$$\mathrm{d}M = \frac{\Lambda^2(S_E - S_N)(S_E - S_C)}{72\pi^3 M}\mathrm{d}S_E + \frac{(12\pi^2M^2 - 3\pi S_E + \Lambda S_E^2)}{12\pi^2 M\Lambda}\mathrm{d}\Lambda,$$
$$\mathrm{d}M = -\frac{\Lambda^2(S_C - S_N)(S_E - S_C)}{72\pi^3 M}\mathrm{d}S_C + \frac{(12\pi^2M^2 - 3\pi S_C + \Lambda S_C^2)}{12\pi^2 M\Lambda}\mathrm{d}\Lambda,$$
$$\mathrm{d}M = \frac{\Lambda^2(S_C - S_N)(S_E - S_N)}{72\pi^3 M}\mathrm{d}S_N + \frac{(12\pi^2M^2 - 3\pi S_N + \Lambda S_N^2)}{12\pi^2 M\Lambda}\mathrm{d}\Lambda.$$

If we consider the Hawking temperature (2.5), (2.6), we obtain the first law of thermodynamics for the event horizon, cosmological horizon, and negative horizon of a Schwarzschild–dS black hole

$$\mathrm{d}M = +T_E\mathrm{d}S_E + V_E\mathrm{d}\Lambda, \tag{2.13}$$
$$\mathrm{d}M = -T_C\mathrm{d}S_C + V_C\mathrm{d}\Lambda, \tag{2.14}$$
$$\mathrm{d}M = -T_N\mathrm{d}S_N + V_N\mathrm{d}\Lambda, \tag{2.15}$$

where the thermodynamic potential conjugate to Λ is defined as

$$V_i = \left(\frac{\partial M}{\partial \Lambda}\right)_{S_i} = -\frac{r_i^3}{6} = \frac{(12\pi^2M^2 - 3\pi S_i + \Lambda S_i^2)}{12\pi^2 M\Lambda} \quad (i = E, C, N). \tag{2.16}$$

Furthermore, the Smarr relations for the horizons can be found by scaling arguments. The mass can be viewed as a homogeneous function of the thermodynamic variables S_i and Λ, i.e., $M = M(S_i, \Lambda)$. From the horizon function (2.2), we can find that the mass M scales with $[\text{length}]^1$ and Λ scales with $[\text{length}]^{-2}$. The area entropy (2.4) shows that S_i scales with $[\text{length}]^2$. Then, after rescaling the thermodynamic variables, we can obtain $\lambda^1 M = M(\lambda^2 S_i, \lambda^2\Lambda)$. By taking the first derivative with respect to λ and then setting $\lambda = 1$, we obtain the Smarr relations for the event horizon and negative horizon

$$M = 2(T_E S_E + V_E\Lambda), \tag{2.17}$$
$$M = 2(T_N S_N + V_N\Lambda). \tag{2.18}$$

Note that we select the positive temperature (2.6), rather than the origin negative (opposite) one, and thus the Smarr relation for the cosmological horizon of a Schwarzschild–dS black hole is

$$M = 2(-T_C S_C + V_C\Lambda). \tag{2.19}$$

Finally, we obtain the first law of thermodynamics (2.13)–(2.15), and the Smarr relations (2.17)–(2.19) for the event horizon, cosmological horizon, and negative horizon of a Schwarzschild–dS black hole.

For a Schwarzschild–dS black hole with $\frac{3M}{\ell} > 1$, we can only find one real root of $f(r)$, which is the event horizon, whereas the other two are complex. This case is outside the scope of our discussion because we only consider the thermodynamic laws of horizons, while the thermodynamics of complex horizons are not defined well. However, the four-dimensional uncharged black hole in $f(R)$ gravity [18] has the same line element (2.1) with a different metric function $f(r) = 1 - \frac{2\mu}{r} - \frac{R_0}{12}r^2$, where $R = R_0$ is the constant curvature of the static, spherically symmetric solution. Hence, by following the same procedure, we can obtain similar thermodynamic relations, including the thermodynamic bounds of entropy and area, the first law of thermodynamics, and Smarr relations, where the cosmological constant is $\Lambda_f = \frac{R_0}{4}$ and the cosmological radius is $\ell_f = \frac{2}{\sqrt{R_0}}$. Indeed, we can follow a similar procedure to obtain these results for other black holes with three horizons, e.g., four-dimensional charged static black holes in Einstein–Weyl theory [3] and five-dimensional charged (A)dS black holes in the Gauss–Bonnet gravity [18].

3. Entropy relations and their application to a Gauss–Bonnet black hole

In this section, we use an example of a Gauss–Bonnet black hole to further study the thermodynamic relations and their application. However, we only consider the positive horizons in this case, which have attracted more attention [4,7,9,10,14,15,18,21,22,24–28,30] than the negative ones. We consider the five-dimensional charged asymptotically flat solutions. We take the action as

$$S = \frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{-g}(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{GB}) + \int \mathrm{d}^5x\sqrt{-g}\mathcal{L}_{matter},$$

where

$$\mathcal{L}_0 = -2\Lambda = 0, \qquad \mathcal{L}_1 = R, \qquad \mathcal{L}_{matter} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu},$$
$$\mathcal{L}_{GB} = \alpha\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right).$$

We select the vanishing cosmological constant and rescale the Gauss–Bonnet coupling constant $\tilde{\alpha}$ in the following discussion by

$$\tilde{\alpha} = (d-3)(d-4)\alpha = 2\alpha.$$

The static and charged black hole solution has the form [45–49]

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3^2,$$
$$F = \frac{q}{4\pi r^3}dt \wedge dr,$$

where $d\Omega_3^2$ is the maximally symmetric space in three dimensions and the metric function is

$$V(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left[1 - \sqrt{1 + 4\tilde{\alpha}\left(\frac{2\mu}{r^4} - \frac{q^2}{r^6}\right)}\right]. \tag{3.1}$$

The event horizon r_E and Cauchy horizon r_C are located in the roots of the metric function $V(r)$

$$r_E^2 = \frac{1}{2}(2\mu - \tilde{\alpha}) + \frac{1}{2}\sqrt{(2\mu - \tilde{\alpha})^2 - 4q^2}, \tag{3.2}$$
$$r_C^2 = \frac{1}{2}(2\mu - \tilde{\alpha}) - \frac{1}{2}\sqrt{(2\mu - \tilde{\alpha})^2 - 4q^2}. \tag{3.3}$$

Note that we only consider the positive horizons in the following. We introduce some useful relations: $r_E r_C = q$, $r_E + r_C = \sqrt{2\mu - \tilde{\alpha} + 2q}$, $r_E^2 + r_C^2 = 2\mu - \tilde{\alpha}$. The temperatures, electric potentials, areas, and entropies for horizons are given by

$$T_E = \frac{r_E^2 - r_C^2}{2\pi r_E(2\tilde{\alpha} + r_E^2)}, \qquad T_C = \frac{r_E^2 - r_C^2}{2\pi r_C(2\tilde{\alpha} + r_C^2)}, \tag{3.4}$$
$$\Phi_E = 2\left(\frac{\pi}{4}\right)^{1/3}\frac{Q}{r_E^2}, \qquad \Phi_C = 2\left(\frac{\pi}{4}\right)^{1/3}\frac{Q}{r_C^2}, \tag{3.5}$$
$$A_E = 2\pi^2 r_E^3, \qquad A_C = 2\pi^2 r_C^3, \tag{3.6}$$
$$S_E = \frac{\pi^2 r_E^3}{2}\left(1 + \frac{6\tilde{\alpha}}{r_E^2}\right), \qquad S_C = \frac{\pi^2 r_C^3}{2}\left(1 + \frac{6\tilde{\alpha}}{r_C^2}\right), \tag{3.7}$$

where the ADM mass M and the electric charge Q of the solution are given by

$$M = \frac{3\pi\mu}{4}, \qquad Q = \left(\frac{\pi}{4}\right)^{2/3} q. \tag{3.8}$$

We can find the entropy product [18] and entropy sum

$$S_E S_C = 4\pi^2\left(1 + \frac{12\tilde{\alpha}\mu}{q^2} + \frac{30\tilde{\alpha}^2}{q^2}\right)Q^3,$$
$$S_E + S_C = \frac{\pi^2}{2}\sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu + 5\tilde{\alpha} - q), \tag{3.9}$$

and the area product [18] and area sum

$$A_E A_C = 64\pi^2 Q^3,$$
$$A_E + A_C = 2\pi^2\sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu - \tilde{\alpha} - q). \tag{3.10}$$

The existence of black hole horizons leads to

$$\mu \geq q + \frac{\tilde{\alpha}}{2}. \tag{3.11}$$

By focusing on the area bound of the horizons, we obtain

$$A_E \geq \sqrt{A_E A_C} = 9\pi Q\sqrt{Q}, \qquad A_C \leq \sqrt{A_E A_C} = 9\pi Q\sqrt{Q}.$$

The area sum gives

$$\pi^2\sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu - \tilde{\alpha} - q)$$
$$= \frac{A_E + A_C}{2} \leq A_E \leq A_E + A_C$$
$$= 2\pi^2\sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu - \tilde{\alpha} - q),$$
$$A_C \leq \frac{A_E + A_C}{2} = \pi^2\sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu - \tilde{\alpha} - q).$$

If we consider the above bound together with the existence of black hole horizons, we obtain the area bound of the event horizon and Cauchy horizon

$$A_E \in \left[\pi^2, 2\pi^2\right] \times \sqrt{2\mu - \tilde{\alpha} + 2q}(2\mu - \tilde{\alpha} - q),$$
$$A_C \in \left[0, 9\pi Q\sqrt{Q}\right]. \tag{3.12}$$

We can also obtain the entropy bound, but this is complicated and it is omitted here.

In addition, if we consider the first derivative of the entropy product and sum (3.9), and we perform a small calculation similar to that in the previous section, we can find the first law of thermodynamics for the event horizon and Cauchy horizon of a Gauss–Bonnet charged flat black hole

$$dM = +T_E dS_E + \Phi_E dQ + \Theta_E d\tilde{\alpha}, \tag{3.13}$$
$$dM = -T_C dS_C + \Phi_C dQ + \Theta_C d\tilde{\alpha}. \tag{3.14}$$

Note that we treat the Gauss–Bonnet coupling constant as a thermodynamic variable (e.g., see [35–38,43,44]). Accordingly, the thermodynamic potential conjugate to $\tilde{\alpha}$ is defined as

$$\Theta_E = \left(\frac{\partial M}{\partial \tilde{\alpha}}\right)_{S_E, Q} = \frac{3\pi}{8}(1 - 8\pi\tilde{\alpha} r_E T_E), \tag{3.15}$$
$$\Theta_C = \left(\frac{\partial M}{\partial \tilde{\alpha}}\right)_{S_C, Q} = \frac{3\pi}{8}(1 + 8\pi\tilde{\alpha} r_C T_C). \tag{3.16}$$

Furthermore, the Smarr relation for horizons can be found by scaling arguments. The mass can be viewed as a homogeneous function of the thermodynamic variables S_i and $\tilde{\alpha}$, i.e. $M = M(S_i, \tilde{\alpha})$. From the horizon function (3.1), we can find that mass M scales with $[\text{length}]^2$, the electric charge Q scales with $[\text{length}]^2$, and $\tilde{\alpha}$ scales with $[\text{length}]^2$. The area entropy (3.7) shows that S_i scales with $[\text{length}]^3$. Then, after rescaling the thermodynamic variables, we can obtain $\lambda^2 M = M(\lambda^3 S_i, \lambda^2 Q, \lambda^2\tilde{\alpha})$. By taking the first derivative with respect to λ and then setting $\lambda = 1$, we obtain the Smarr relation for the event horizon

$$M = +\frac{3}{2}T_E S_E + \Phi_E Q + \Theta_E\tilde{\alpha}. \tag{3.17}$$

Note that we select the positive temperature (3.4), rather than the origin negative (opposite) one; thus, the Smarr relation for the Cauchy horizon is

$$M = -\frac{3}{2}T_C S_C + \Phi_C Q + \Theta_C\tilde{\alpha}. \tag{3.18}$$

Finally, we obtain the first law of thermodynamics (3.13), (3.14), and Smarr relations (3.17), (3.18) for the event horizon and the Cauchy horizon of a Gauss–Bonnet charged flat black hole, which are consistent with those described in [18].

4. Conclusions

In this study, based on entropy relations, we obtained the thermodynamic bound of entropy and area for horizons of a Schwarzschild–dS black hole, including the event horizon, Cauchy horizon, and negative horizon, which are all geometrically bound and they comprise the cosmological radius. We considered the first derivatives of the entropy relations and we obtained the first law of thermodynamics for all of the horizons. We also obtained the Smarr relations for the horizons using the scaling discussion. For the thermodynamics of all horizons, the cosmological constant was treated as a thermodynamic variable. In particular, the thermodynamics of negative horizons were also defined well on the negative side ($r < 0$). This formula appears to be valid for three-horizon black holes, e.g., four-dimensional uncharged black holes in $f(R)$ gravity [18], four-dimensional charged static black holes in Einstein–Weyl theory [3], and five-dimensional charged (A)dS black holes in the Gauss–Bonnet gravity [18]. We also generalized the discussion to thermodynamics for the event horizon and Cauchy horizon of Gauss–Bonnet charged flat black holes because the Gauss–Bonnet coupling constant can also be treated as a thermodynamic variable. These results provide further insights into the crucial roles played by the entropy relations of multi-horizons in black hole thermodynamics and they help us to understand entropy at the microscopic level.

In future research, we believe that the validity of this formula holds for general Lovelock gravity and thus more coupling constants can be entered in the laws of black hole thermodynamics. In addition, because the thermodynamics of the negative horizon are introduced, we may also expect to construct a holographic description of the thermodynamics for black holes with three horizons, while the holographic descriptions of thermodynamics for black holes with two horizons can be built well by the thermodynamic method of BH/CFT correspondence [7,25–30].

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References

[1] M. Cvetic, G.W. Gibbons, C.N. Pope, Universal area product formulae for rotating and charged black holes in four and higher dimensions, Phys. Rev. Lett. 106 (2011) 121301, arXiv:1011.0008.

[2] C. Toldo, S. Vandoren, Static nonextremal AdS4 black hole solutions, J. High Energy Phys. 1209 (2012) 048, arXiv:1207.3014.

[3] M. Cvetic, H. Lu, C.N. Pope, Entropy-product rules for charged rotating black holes, Phys. Rev. D 88 (2013) 044046, arXiv:1306.4522.

[4] H. Lu, Y. Pang, C.N. Pope, AdS dyonic black hole and its thermodynamics, J. High Energy Phys. 1311 (2013) 033, arXiv:1307.6243.

[5] D.D.K. Chow, G. Compère, Seed for general rotating non-extremal black holes of $N = 8$ supergravity, Class. Quantum Gravity 31 (2014) 022001, arXiv:1310.1925.

[6] M. Visser, Quantization of area for event and Cauchy horizons of the Kerr–Newman black hole, J. High Energy Phys. 1206 (2012) 023, arXiv:1204.3138.

[7] B. Chen, S.-X. Liu, J.-J. Zhang, Thermodynamics of black hole horizons and Kerr/CFT correspondence, J. High Energy Phys. 1211 (2012) 017, arXiv:1206.2015.

[8] A. Castro, J.M. Lapan, A. Maloney, M.J. Rodriguez, Black hole monodromy and conformal field theory, Phys. Rev. D 88 (2013) 044003, arXiv:1303.0759.

[9] S. Abdolrahimi, A.A. Shoom, Distorted five-dimensional electrically charged black holes, Phys. Rev. D 89 (2) (2014) 024040, arXiv:1307.4406.

[10] H. Lu, Charged dilatonic ads black holes and magnetic $AdS_{D-2} \times R^2$ vacua, J. High Energy Phys. 1309 (2013) 112, arXiv:1306.2386.

[11] M.A. Anacleto, F.A. Brito, E. Passos, Acoustic black holes and universal aspects of area products, arXiv:1309.1486.

[12] J. Wang, W. Xu, X.-H. Meng, The "universal property" of horizon entropy sum of black holes in four dimensional asymptotical (anti-)de-Sitter spacetime background, J. High Energy Phys. 1401 (2014) 031, arXiv:1310.6811.

[13] W. Xu, J. Wang, X.-H. Meng, "Entropy sum" of (A)dS black holes in four and higher dimensions, arXiv:1310.7690.

[14] A. Castro, M.J. Rodriguez, Universal properties and the first law of black hole inner mechanics, Phys. Rev. D 86 (2012) 024008, arXiv:1204.1284.

[15] S. Detournay, Inner mechanics of 3d black holes, Phys. Rev. Lett. 109 (2012) 031101, arXiv:1204.6088.

[16] W. Xu, J. Wang, X.-H. Meng, A note on entropy relations of black hole horizons, Int. J. Mod. Phys. A 29 (18) (2014) 1450088, arXiv:1401.5180.

[17] M. Visser, Area products for black hole horizons, Phys. Rev. D 88 (2013) 044014, arXiv:1205.6814.

[18] A. Castro, N. Dehmami, G. Giribet, D. Kastor, On the universality of inner black hole mechanics and higher curvature gravity, J. High Energy Phys. 1307 (2013) 164, arXiv:1304.1696.

[19] V. Faraoni, A.F.Z. Moreno, Are quantization rules for horizon areas universal?, Phys. Rev. D 88 (4) (2013) 044011, arXiv:1208.3814.

[20] J. Wang, W. Xu, X.-H. Meng, The entropy relations of black holes with multihorizons in higher dimensions, Phys. Rev. D 89 (2014) 044034, arXiv:1312.3057.

[21] P. Pradhan, Area products and mass formula for Kerr–Newman–Taub–NUT space–time, arXiv:1310.7921.

[22] P. Pradhan, Black hole interior mass formula, Eur. Phys. J. C 74 (2014) 2887, arXiv:1310.7126.

[23] Y.-Q. Du, Y. Tian, The universal property of the entropy sum of black holes in all dimensions, arXiv:1403.4190.

[24] W. Xu, J. Wang, X.H. Meng, The new thermodynamic relations of multi-horizons black holes, arXiv:1402.1293.

[25] B. Chen, J.-J. Zhang, Holographic descriptions of black rings, J. High Energy Phys. 1211 (2012) 022, arXiv:1208.4413.

[26] B. Chen, J.-J. Zhang, RN/CFT correspondence from thermodynamics, J. High Energy Phys. 1301 (2013) 155, arXiv:1212.1959.

[27] B. Chen, J.-J. Zhang, Electromagnetic duality in dyonic RN/CFT correspondence, Phys. Rev. D 87 (2013) 081505, arXiv:1212.1960.

[28] B. Chen, Z. Xue, J.-J. Zhang, Note on thermodynamic method of black hole/CFT correspondence, J. High Energy Phys. 1303 (2013) 102, arXiv:1301.0429.

[29] B. Chen, J.-J. Zhang, J.-D. Zhang, D.-L. Zhong, Aspects of warped AdS_3/CFT_2 correspondence, J. High Energy Phys. 1304 (2013) 055, arXiv:1302.6643.

[30] B. Chen, J.-J. Zhang, Thermodynamics in black-hole/CFT correspondence, Int. J. Mod. Phys. D 22 (2013) 1342012, arXiv:1305.3757.

[31] D. Kastor, S. Ray, J. Traschen, Enthalpy and the mechanics of AdS black holes, Class. Quantum Gravity 26 (2009) 195011, arXiv:0904.2765.

[32] B.P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quantum Gravity 28 (2011) 235017, arXiv:1106.6260.

[33] M. Cvetic, G.W. Gibbons, D. Kubiznak, C.N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, Phys. Rev. D 84 (2011) 024037, arXiv:1012.2888.

[34] B.P. Dolan, D. Kastor, D. Kubiznak, R.B. Mann, J. Traschen, Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes, Phys. Rev. D 87 (10) (2013) 104017, arXiv:1301.5926.

[35] W. Xu, H. Xu, L. Zhao, Gauss–Bonnet coupling constant as a free thermodynamical variable and the associated criticality, Eur. Phys. J. C 74 (2014) 2970, arXiv:1311.3053.

[36] H. Xu, W. Xu, L. Zhao, Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions, Eur. Phys. J. C 74 (9) (2014) 3074, arXiv:1405.4143.

[37] W. Xu, L. Zhao, Critical phenomena of static charged AdS black holes in conformal gravity, Phys. Lett. B 736 (2014) 214, arXiv:1405.7665.

[38] N. Altamirano, D. Kubiznak, R.B. Mann, Z. Sherkatghanad, Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume, Galaxies 2 (2014) 89, arXiv:1401.2586.

[39] G. Barnich, Entropy of three-dimensional asymptotically flat cosmological solutions, J. High Energy Phys. 1210 (2012) 095, arXiv:1208.4371.

[40] A. Bagchi, S. Detournay, R. Fareghbal, J. Simon, Holography of 3d flat cosmological horizons, Phys. Rev. Lett. 110 (2013) 141302, arXiv:1208.4372.

[41] M. Riegler, Flat space limit of Cardy formula, arXiv:1408.6931.

[42] R. Fareghbal, A. Naseh, Aspects of flat/CCFT correspondence, arXiv:1408.6932.

[43] D. Kastor, S. Ray, J. Traschen, Smarr formula and an extended first law for Lovelock gravity, Class. Quantum Gravity 27 (2010) 235014, arXiv:1005.5053.

[44] R.G. Cai, L.M. Cao, L. Li, R.Q. Yang, P–V criticality in the extended phase space of Gauss–Bonnet black holes in AdS space, J. High Energy Phys. 1309 (2013) 005, arXiv:1306.6233.

[45] D.G. Boulware, S. Deser, String generated gravity models, Phys. Rev. Lett. 55 (1985) 2656.

[46] J.T. Wheeler, Symmetric Solutions to the Gauss–Bonnet extended Einstein equations, Nucl. Phys. B 268 (1986) 737.
[47] R.G. Cai, Gauss–Bonnet black holes in AdS spaces, Phys. Rev. D 65 (2002) 084014, arXiv:hep-th/0109133.
[48] D.L. Wiltshire, Spherically symmetric solutions of Einstein–Maxwell theory with a Gauss–Bonnet term, Phys. Lett. B 169 (1986) 36.
[49] R.G. Cai, A note on thermodynamics of black holes in Lovelock gravity, Phys. Lett. B 582 (2004) 237, arXiv:hep-th/0311240.